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VARIATIONS ON THE THEME OF THE CONLEY CONJECTURE

A dissertation submitted in partial satisfaction of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

MATHEMATICS

by

Doris Hein

June 2012

The Dissertation of Doris Hein is approved:

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2012

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Abstract

Variations on the Theme of the Conley Conjecture

by

Doris Hein

We prove a generalization of the Conley conjecture: Every Hamiltonian diffeomorphism of a closed symplectic manifold has infinitely many periodic orbits if the first Chern class vanishes on the second fundamental group. In particular, this removes the rationality condition from similar theorems by Ginzburg and Gürel. The proof in the irrational case involves several new ingredients including the definition and the properties of the filtered Floer homology for Hamiltonians on irrational manifolds. For this proof, we develop a method of localizing the filtered Floer homology for short action intervals using a direct sum decomposition. One of the summands only depends on the behavior of the Hamiltonian in a fixed open set and enables us to use tools from more restrictive cases in the proof of the Conley conjecture. We also prove the Conley conjecture for cotangent bundles of oriented, closed manifolds, and Hamiltonians, which are quadratic at infinity, i.e., we show that such Hamiltonians have infinitely many periodic orbits.

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Chapter 1

Introduction

The main topic of this thesis is Hamiltonian dynamics on symplectic manifolds. We are particularly interested in the minimal number of periodic orbits guaranteed by the topology of the underlying symplectic manifold. On many manifolds, it is known that every Hamiltonian system has infinitely many periodic orbits and we extend the class of manifolds for which this result is established. More concretely, we remove the rationality condition from the analogous theorem in [GG2] for closed manifolds. We also prove the existence of infinitely many periodic orbits for a certain class of Hamiltonians on cotangent bundles in the original setting of Hamiltonian mechanics.

Classical Hamiltonian mechanics on cotangent bundles is the historical origin of modern symplectic dynamics. In this physical setting, the Hamiltonian is a time-dependent function on the phase space, i.e., on the cotangent bundle of the configuration space (a manifold). The base B describes the set of possible positions of the particle and the fiber coordinate in the cotangent bundle T^*B is the momentum. The Hamiltonian is then the function that describes the total energy of the system, i.e. the sum of kinetic and potential energy depending on the position and the momentum. The cotangent bundle T^*B carries a natural symplectic structure given by the non-degenerate 2-form $\omega = dp \wedge dq$. Here, q is the position on the base B and p is the fiber coordinate, i.e., the momentum. In modern symplectic geometry, the role of cotangent bundles with its natural symplectic structure is taken by general symplectic manifolds. The Hamiltonian is now a time-dependent function on such a symplectic manifold (M^{2n}, ω).

By the non-degeneracy of the symplectic form ω, the Hamiltonian function induces the Hamiltonian vector field via the equation $i_{X_H}\omega = -dH$. The flow of this vector field and its orbits are the topics of Hamiltonian dynamics. An important step in understanding the flow, we mostly focus on periodic orbits with integer periods. These periodic orbits can also be described as fixed points of the time-1-map, the Hamiltonian diffeomorphism, and its iterations. To study the orbits of Hamiltonian diffeomorphisms, we can restrict our attention to Hamiltonians which are one-periodic in time, i.e., we can view the Hamiltonian as a function $H\colon S^1 \times M \to \mathbb{R}$. For a general Hamiltonian, we can reparametrize time and obtain a Hamiltonian which is one-periodic in time and gives rise to the same Hamiltonian diffeomorphism. The main theorem of this thesis establishes the existence of infinitely many periodic orbits of Hamiltonian systems under certain conditions on the symplectic manifold and, in the cotangent bundle case, on the class of considered Hamiltonians.

Providing a lower bound for the number of periodic orbits of a Hamiltonian system in terms of the topology of the manifold, Conley and Zehnder proved the Arnold conjecture in [CZ1], i.e., they proved that every Hamiltonian diffeomorphism has at least as many periodic orbits as the minimal number of critical points of a function on the underlying manifold. Shortly after this proof, in 1984, Conley conjectured in [Co] that on certain symplectic manifolds, every Hamiltonian system has infinitely many periodic orbits. Originally stated for tori, the Conley conjecture has by now been established in a variety settings both in Hamiltonian dynamics on symplectic manifolds and also in the analogous form in Lagrangian mechanics.

In the framework of classical Lagrangian mechanics on tangent bundles, the Conley conjecture has been studied e.g. by Long, Lu and Mazzucchelli in [Lo1, Lu, Ma] for different classes of Lagrangians. The similarity between the problems in Hamiltonian and Lagrangian dynamics can also be seen on the level of the proofs, although the methods utilized in [Lo1, Lu, Ma] are quite different from the Floer homological techniques used here. The class of Hamiltonians on a cotangent bundle which are quadratic at infinity considered in this thesis includes the Hamiltonians of classical mechanics (see Example 2.1.2) and the convex quadratic Hamiltonians used in [Lo1, Lu], but does not include all Tonelli Hamiltonians which are used in [Ma]. Namely, Tonelli Hamiltonians are assumed to satisfy a convexity condition and to have superlinear (but not necessarily quadratic) growth everywhere on the cotangent bundle. In this thesis, we consider Hamiltonians, which we only require to have quadratic growth in the fibers outside a compact set.

In the Hamiltonian setting, the existence of infinitely many periodic orbits has been studied mainly on closed manifolds. From the original Conley conjecture concerning Hamiltonian systems on tori, the assumptions on the symplectic manifold have been relaxed to more general settings. However, the Conley conjecture does not hold on all closed symplectic manifolds. For example, there exist Hamiltonians on S^2 for which the Hamiltonian diffeomorphism has exactly two fixed points and no other periodic orbits, see Example 2.2.2.

The original conjecture, i.e., the existence of infinitely many periodic orbits for all Hamiltonian systems on tori, was proved by Conley and Zehnder in [CZ2] in the non-degenerate case. The general case including degenerate Hamiltonians was established on all surfaces other than S^2 by Franks and Handel in [FH]. The proof of the general case on tori by Hingston in [Hi] fully established the original Conley conjecture for all Hamiltonians and presented ideas which opened the door to further generalizations.

The non-degenerate case was already generalized by Salamon and Zehnder in 1992 in [SZ] to the case of symplectically aspherical manifolds, i.e., symplectic manifolds (M, ω) such that $\omega|_{\pi_2(M)} = 0$ and $c_1(M)|_{\pi_2(M)} = 0$. The main point in this proof is that the existence of only finitely many periodic orbits contradicts the non-degeneracy of the Hamiltonian.

The existence of a degenerate periodic orbit under the assumption of only finitely many periodic orbits was further investigated by Hingston. She studied this degenerate periodic orbit in detail and used its properties in [Hi] to prove the degenerate Conley conjecture on tori. Ginzburg in [Gi2] introduced the notion of symplectically degenerate maximum for a periodic orbit with these properties. Again the geometric characterization of this special degenerate orbit played an important role in proving the general case of the Conley conjecture for symplectically aspherical manifolds. This proof also utilized the squeezing method developed in [GG1], which relies on properties of the filtered Floer homology of different Hamiltonians and uses special Hamiltonians whose periodic orbits are known. The combination of this squeezing method with the properties of symplectically degenerate maxima has proved to be useful for establishing the Conley conjecture in more general cases. Ginzburg and Gürel used these ideas to establish the Conley conjecture on closed symplectically rational manifolds M with $c_1(M)|_{\pi_2(M)} = 0$ in [GG2]. They also showed in [GG4] that the Conley conjecture holds if the symplectic manifold is negative monotone, i.e., the symplectic form and the first Chern class of the manifold are related by $\omega|_{\pi_2(M)} = \lambda c_1(M)|_{\pi_2(M)}$ for $\lambda < 0$.

The Conley conjecture has also been investigated in other situations than classical mechanics and Hamiltonians on closed manifolds. For instance, similar results are known for Hamiltonian diffeomorphisms with displaceable support; see, e.g., [FS, Gü, HZ, Sc, Vi1]. Here the manifold M is required to be symplectically aspherical, but not necessarily closed.

It is likely that the conditions on the closed manifold can be further relaxed and there are several ideas about what more general settings for the Conley conjecture should be. For instance, Chance and McDuff conjectured that a Hamiltonian diffeomorphism has infinitely many periodic orbits whenever certain Gromov-Witten invariants of the underlying symplectic manifold vanish. A conjecture by Gürel specifies this as the condition that the minimal Chern number is larger than the dimension of the symplectic manifold. This conjecture would be implied by two conjectures of Ginzburg; the first being that there are infinitely many periodic orbits whenever there are more periodic orbits than guaranteed by the Arnold conjecture. This result is known for S^2, see [Fr, LeC] and also [BH, CKRTZ, Ke], but open in the general case. The second conjecture by Ginzburg is that it may be sufficient that the Hamiltonian has a non-elliptic orbit to guarantee that it has infinitely many periodic orbits.

In this thesis, we will establish the Conley conjecture in two new settings. More concretely, we generalize the case of a closed manifold by dropping all requirements on the symplectic form and only assuming that $c_1(M)|_{\pi_2(M)} = 0$. In the non-compact case, we do not work in the Lagrangian setting but with Hamiltonian systems on the cotangent bundle T^*B for a closed manifold B where the Hamiltonians are quadratic at infinity.

The main tool for proving results of the type of the Conley conjecture is a variational approach to Hamiltonian mechanics. The periodic orbits of a Hamiltonian diffeomorphism with fixed period are characterized as critical points of an action functional. For the action functional, Floer developed a homology theory analogous to Morse theory, the Floer theory. For closed symplectic manifolds, the Floer homology has been shown to be independent of the Hamiltonian and isomorphic to the singular homology of the underlying manifold. If the manifold is not closed, the class of Hamiltonians needs to be restricted in order to obtain compactness results for the moduli spaces. For the case of a cotangent bundle considered in this thesis, we will use the class of Hamiltonians introduced and studied by Abbondandolo and Schwarz. They proved in [AS] that for Hamiltonians which are quadratic at infinity, the Floer homology is defined and isomorphic to the homology of the loop space of the base, see also [Se, Vi3] for similar results for different classes of Hamiltonians. In particular, we will use the isomorphisms in both cases to show that the Floer homology in a certain degree is non-zero.

In determining a lower bound on the number of periodic orbits using Floer homological methods, the main difficulty arises from two aspects of the definition of Floer homology. The first one is the fact that Floer homology can detect orbits of a fixed period, but does not distinguish between simple and iterated orbits. In counting the number of periodic orbits, we must therefore find a way to exclude iterated orbits from being counted as new orbits.

The second difficulty comes from the definition of the action functional. For exact symplectic forms, the action functional can be defined unambiguously for a loop by fixing a primitive. In the general case, the action functional used to define the Floer homology is only defined for capped loops on the manifold, i.e. loops equipped with embedded disks spanned by the loop. These disks are used to trivialize the tangent bundle along the loop and needed to have not only the Hamiltonian action but also the grading of Floer homology by the Conley-Zehnder index of capped periodic orbits well-defined.

In this thesis, we do not impose any conditions on the symplectic form as has been done in previous work on the Conley conjecture. This may result in possibly very large Floer homology groups, since we need to involve each periodic orbit together with different cappings. Depending on the second homotopy group of the symplectic manifold, the number of different cappings of one periodic orbit can be very large and becomes an issue in the definition of the action filtration of Floer homology for degenerate Hamiltonians, see Section 4.2. We will give an example of a symplectic manifold where the space of cappings leads to a dense set of critical values of the action functional. In this case, which was not covered in any of the former proofs, the filtered Floer homology requires a more careful definition which also affects certain choices in the proof of the Conley conjecture.

To overcome the issues of iterations and recappings of periodic orbits, we need to construct special tools that distinguish periodic orbits. The difficulty arising from iteration already was an issue in proving the Conley conjecture in more restrictive settings, e.g. in [Gi2, Hi, SZ]. In this thesis, we use similar ideas and study the behavior of the action and the grading under iteration. These tools have proved to be very powerful in Hamiltonian mechanics, establishing not only results concerning the number of periodic orbits, but also information about the minimal period of these orbits.

For the non-degenerate case of the Conley conjecture, the properties of Floer homology and its grading are sufficient. The result then follows by a similar argument as the one used in the symplectically aspherical case in [SZ] proving that there are either infinitely many periodic orbits or a degenerate orbit. In the degenerate case, the issue is more involved. The main point for this case is to study the degenerate periodic orbit whose existence is guaranteed by the argument for the non-degenerate case. This orbit has special properties, which gave rise to the notion of symplectically degenerate extrema. The geometric characterization of symplectically degenerate extrema and a localization of the problem are the key parts of the proof in our settings.

The notion of a symplectically degenerate maximum was introduced in [Gi2] and used in [Gi2, Hi] for the first proofs of the degenerate Conley conjecture on manifolds other than surfaces. In this thesis, the existence of a symplectically degenerate maximum cannot be guaranteed in the case of the cotangent bundle. Therefore we need to consider a symplectically degenerate minimum in order to prove the Conley conjecture in both our settings. For the case of a closed manifold, both a symplectically degenerate maximum or minimum could be used. We will discuss the definition of both symplectically degenerate maxima and minima and compare their properties. In particular, we show that all results on symplectically degenerate maxima established in previous proof of the Conley conjecture also hold in a slightly modified form for symplectically degenerate minima and can be applied in a similar fashion to prove the Conley conjecture in our settings.

To overcome the problem of a large number of cappings of a periodic orbits, we will localize the filtered Floer homology in order to apply tools from the symplectically aspherical case. In order to achieve this goal, we will discuss in detail how filtered Floer homology is defined using a direct limit construction in the irrational case and which properties carry over to the limit. To localize the problem, we establish a direct sum decomposition of filtered Floer homology for small action intervals. This decomposition is similar to the one used in [GG2] for symplectically rational manifolds. Under certain additional assumptions, this direct sum decomposition is compatible with the limit definition of filtered Floer homology. To meet these assumptions, we need to modify some of the choices in the proof of the Conley conjecture. This provides a method to localize the filtered Floer homology and use the Darboux theorem to study one of the summands in detail, which has the same properties as the filtered Floer homology in the aspherical case. For this summand, we can then use the computations done in [Gi2] to generalize the proof of the Conley conjecture.

The main point of the proof is then to show that the filtered Floer homology for a large iteration is non-zero in a degree which does not contain any iterated orbits. Thus there is a simple periodic orbit with this period and this degree. Using this, the proof of the Conley conjecture relies on analyzing the indices and actions of periodic orbits and their iterations. The fact that this Floer homology group is non-zero also implies the generalization of results by Ginzburg and Gürel in [GG2] concerning the behavior of the action and the index of capped periodic orbits under iteration. More concretely, we show that the differences between actions and indices for iterations of a capped periodic orbit and a sequence of geometrically distinct orbits are bounded. This theorem by Ginzburg and Gürel is proved in the case of a rational symplectic manifold using the analogous result on the filtered Floer homology group being non-zero. Since we generalize this theorem in this thesis, we also obtain a generalization of the bounded gap theorem.

In Chapter 2, we discuss the considered settings and state the results. In Chapter 3, we introduce basic definitions in symplectic geometry in symplectic geometry and Hamiltonian mechanics setting the stage for the definition of Floer homology. Our main tool, the Floer homology and its properties, is discussed in

Chapter 4. In particular, we explain the action filtration in the irrational case and study homomorphisms between Floer homology groups for different Hamiltonians in this chapter. Using Theorem 2.2.6, we prove the existence of infinitely many periodic orbits and also a statement on the minimal periods in Chapter 5. Before we can prove Theorem 2.2.6, we need to establish our new tools needed to generalize the existing results. For this matter, we define symplectically degenerate minima in Chapter 6. We characterize them using their geometric properties and compare them to symplectically degenerate maxima. In Chapter 7, we construct the direct sum decomposition for small action intervals used to localize the problem and generalize the concept of local Floer homology. Using these notions, we finally prove Theorem 2.2.6 in Chapter 8.

Chapter 2

Settings and main results

This thesis has two types of results. The first and most important is the generalization of the Conley conjecture, which concerns the number of periodic orbits of a Hamiltonian system and also contains information about the periods. The second result follows from an auxiliary result used in the proof of the Conley conjecture and concerns the behavior of Hamiltonian actions and mean indices of periodic orbits under iteration. The first section will be devoted to introducing the considered symplectic manifolds and classes of Hamiltonians before we state the theorems more rigorously and discuss details of the proofs.

2.1 The Settings

In this section, we introduce the manifolds and the classes of Hamiltonians we are going to work with. The two settings are a class of closed symplectic manifolds and a special class of Hamiltonians on cotangent bundles. We will describe both settings separately.

2.1.1 Quadratic Hamiltonians on cotangent bundles

As the first of our two settings, we consider a closed, oriented manifold B and let $M = T^*B$ be the cotangent bundle of B with its canonical symplectic structure ω. Since this is a non-compact manifold, we need to restrict the class of Hamiltonians in order to obtain compactness results needed for our main tool, the Floer homology, to be defined. A natural class of Hamiltonians to work with, from both technical and conceptual point of view, is that of Hamiltonians quadratic at infinity, cf. [AS]. More concretely, we use the class of Hamiltonians defined by requiring Hamiltonians $H\colon S^1 \times M \to \mathbb{R}$ to satisfy the following conditions:

(H1) there exist constants $h_0 > 0$ and $h_1 \geq 0$, such that

$$dH(t,q,p)\left(p\frac{\partial}{\partial p}\right) - H(t,q,p) \geq h_0\|p\|^2 - h_1 \text{ and}$$

(H2) there exists a constant $h_2 \geq 0$, such that

$$\|\nabla_q H(t,q,p)\| \leq h_2(1+\|p\|^2) \text{ and } \|\nabla_p H(t,q,p)\| \leq h_2(1+\|p\|).$$

We will call Hamiltonians on the cotangent bundle *admissible*, if they satisfy these conditions. This class of Hamiltonians has originally been defined and studied in [AS].

Remark 2.1.1 *The choice of a class of Hamiltonians is important from at least a technical point of view. For a Hamiltonian H in this class, it has been proven in [AS] that the Floer homology is well-defined and independent of the choice of H in this class. Furthermore, the results in [AS] imply that* $\mathrm{HF}_{-n}(H) \neq 0$. *The restrictions on the class of Hamiltonians are imposed to ensure these properties; see also [Se, SW, Vi3] for similar results for different classes of Hamiltonians. One can expect a result similar to Theorem 2.2.1 to hold for other classes of Hamiltonians. For example, our argument also goes through for Hamiltonians which, at infinity, are autonomous and fiberwise convex and have superlinear growth; see Remark 2.2.3. For the classes of convex and quadratic Hamiltonians or Tonelli Hamiltonians, analogous theorems were proven by means of Lagrangian methods in [Lo1, Lu, Ma]. It is not clear if the Floer homology is defined for Tonelli Hamiltonians, since the Hamiltonian flow is not automatically complete. Note, however, that the Floer homology is defined if the Hamiltonian is autonomous and Tonelli at infinity. But even then, the invariance of filtered Floer homology has not been established, cf. [Se, Vi3].*

The class of Hamiltonians defined by the above conditions (H1) and (H2) is natural from the historical perspective: These conditions on the Hamiltonian imply that H grows quadratically at infinity, i.e., for some suitable constant h_3, the

Hamiltonian H satisfies the growth condition

$$H(t,q,p) \geq \frac{1}{2}h_0\|p\|^2 - h_3. \tag{2.1.1}$$

This inequality shows that our class of Hamiltonians includes an important example, which relates to the origins of symplectic geometry in classical mechanics.

Example 2.1.2 *In particular, the above growth conditions on the Hamiltonian hold for all conservative Hamiltonians describing systems from classical mechanics on B, i.e., Hamiltonians of the form $H(t,p,q) = \frac{1}{2}\|p\|^2 + V(q)$. More generally, one could also use Hamiltonians of the form $H(t,p,q) = \frac{1}{2}\|p\|_t^2 + V(t,p,q)$, where in every fiber the function V is constant outside a compact set in $M = T^*B$. The metric $\|\cdot\|_t$ and V can be chosen to be time-dependent as long as both are periodic in time. Our conditions on the Hamiltonian are also satisfied for periodic in time electro-magnetic Hamiltonians, i.e., the Hamiltonians describing the motion of a charge in an exact magnetic field and a conservative force field; see, e.g., [Gi1]. In this thesis, we only work with time-periodic Hamiltonians and thus need to assume the magnetic field, the metric and the potential to be periodic in time.*

2.1.2 Closed symplectic manifolds with vanishing first Chern class

Let now M be a closed symplectic manifold, i.e. a compact manifold with no boundary. Such manifolds can be equipped with an almost complex structure J and thus have similar properties to complex manifolds. In particular, the Chern class of such symplectic manifold is well-defined. For technical reasons, we assume that the first Chern class vanishes over the second fundamental group, i.e., $c_1(M)|_{\pi_2(M)} = 0$. In particular, all Calabi-Yau manifolds fall into this case. Due to compactness of the manifold M, we do not need to restrict the class of Hamiltonians in this setting. For a Hamiltonian H to be *admissible* in this setting, we only require H to be one-periodic in time, i.e., a smooth function $H\colon S^1 \times M \to \mathbb{R}$. See also Section 3.1 for a more detailed discussion on symplectic manifolds and the conditions assumed here.

In the case of a closed symplectic manifold, some restrictions have to be imposed on the manifold for our main theorems to hold, cf. Example 2.2.2 below. For more restrictive classes of closed symplectic manifolds than the one considered here, the results of this thesis have been established in [Gi2, GG2]. Our setting is a strict generalization of the settings considered there, since we do not impose any conditions on the symplectic form. See Example 3.1.1 for an example of a closed symplectic manifold in the class considered here, where the previous proofs of analogous results do not apply.

2.2 The Conley conjecture

Our first result establishes the existence of infinitely many periodic orbits for admissible Hamiltonians in both settings described above. More detailed, we have the following result:

Theorem 2.2.1 (The Conley Conjecture) *Assume that M is one of the symplectic manifolds described above and let $H\colon S^1 \times M \to \mathbb{R}$ be an admissible Hamiltonian. Then the time-one-map φ_H of the Hamiltonian flow of H has infinitely many periodic orbits. Furthermore, if the φ_H has only finitely many fixed points, then there are simple periodic orbits of arbitrarily large period.*

The following example shows that the condition that $c_1(M)|_{\pi_2(M)} = 0$ in the closed case cannot be dropped completely, even though further generalizations than this condition may be possible.

Example 2.2.2 *Let $M = S^2$ be the unit sphere in $\mathbb{R}^3$ and let H be a scaled height function, i.e. $H(x, y, z) = kz$. If k/π is irrational, the only periodic orbits of φ_H are the fixed points at the poles and thus φ_H does not have infinitely many periodic orbits.*

This a generalization of a conjecture Conley stated in 1984 in [Co] for the case that $M = T^{2n}$ is a torus. It has been proven for weakly non-degenerate Hamiltonian diffeomorphisms of tori in [CZ2] and of symplectically aspherical manifolds in [SZ]. In [FH], the conjecture was proven for all Hamiltonian diffeomorphisms of surfaces other than S^2. In its original form, as stated in [Co] for all Hamiltonian diffeomorphisms of tori, the conjecture was established in [Hi] and the case of an arbitrary closed, symplectically aspherical manifold was settled in [Gi2]. This proof was extended to closed symplectically rational manifolds M with $c_1(M)|_{\pi_2(M)} = 0$ in [GG2] and to negative monotone symplectic manifolds, i.e. to manifolds with

$\omega|_{\pi_2(M)} = \lambda c_1(M)|_{\pi_2(M)}$ for $\lambda < 0$, in [GG4]. The papers of Hingston and Ginzburg provided and studies the concept of symplectically degenerate extrema also utilized here for dealing with the degenerate case of the Conley conjecture for compact manifolds; see [Gi2, GG2, He2, Hi].

Our argument also uses recent results on the Floer homology of Hamiltonians on cotangent bundles proven e.g. in [AS, Se, Vi3]. In this thesis, we use the class of Hamiltonians introduced in [AS] and use their result to guarantee that Floer homology, the most important tool for the proofs, is well-defined, cf. [He1].

The Lagrangian version of the conjecture has been considered, e.g., in [Lo1, Lu, Ma]. The similarity between the problems can also easily be seen on the level of the proofs, although the methods utilized in [Lo1, Lu, Ma] are quite different from the Floer homological techniques used here. Our class of Hamiltonians quadratic at infinity includes the Hamiltonians of classical mechanics (see Example 2.1.2) and convex quadratic Hamiltonians used in [Lo1, Lu], but does not include all Tonelli Hamiltonians. Namely, Tonelli Hamiltonians are assumed to satisfy a convexity condition and to have superlinear (but not necessarily quadratic) growth, while Hamiltonians quadratic at infinity only need to have quadratic growth outside a compact set.

Remark 2.2.3 *Our proof of Theorem 2.2.1 can also be used for a more general class of non-compact manifolds than the cotangent bundle considered here. Namely, let (M, ω) be a Liouville domain and let H be a Hamiltonian, which is autonomous and depends only on the "radial variable" at infinity and has superlinear growth. For such Hamiltonians, the Floer homology is defined and isomorphic to the symplectic homology $\mathrm{SH}(M)$; see [Se]. This homology is a unital algebra with unit in degree $-n$ in our degree conventions. (Strictly speaking, the cohomology $(\mathrm{HF}(H))^*$ is such an algebra.) Hence, $\mathrm{HF}_{-n}(H) = \mathrm{SH}_{-n}(M) \neq 0$ if and only if $\mathrm{SH}(M) \neq 0$. In this case, our proof of Theorem 2.2.1 in Chapter 5 goes through and the Conley conjecture holds in this general setting. Strictly speaking, this is not a generalization of Theorem 2.2.1 since on cotangent bundles most Hamiltonians which are quadratic at infinity are not in the above class.*

By a similar argument as the ones used in the preceding proofs of this type of results in [Hi, Gi2, GG2], the assumption of only finitely many periodic orbits guarantees the existence of a special degenerate orbit, a so-called symplectically degenerate minimum, see the proof in Section 5.1. Thus it suffices to prove Theorem 2.2.1 in the presence of a symplectically degenerate minimum.

Definition 2.2.4 *An isolated capped k-periodic orbit $\bar{x}$ of a k-periodic Hamiltonian H is called a symplectically degenerate minimum of H if*

$$\Delta_H(\bar{x}) = 0 \text{ and } \mathrm{HF}_{-n}(H, \bar{x}) \neq 0.$$

In this definition, $\Delta_H(\bar{x})$ denotes the mean index of $\bar{x}$ and $\mathrm{HF}_*(H, \bar{x})$ is the local Floer homology of H at $\bar{x}$.

We refer to Section 3.3 for more details on the mean index and Section 4.3 for the definition of the local Floer homology. See also [GG3] for details on the local Floer homology and to [SZ] for the definition of the mean index.

Theorem 2.2.5 (Degenerate Conley Conjecture) *Let M be one of the above manifolds and H an admissible Hamiltonian on M. Assume that φ_H has only finitely many fixed points and, furthermore, that H has a symplectically degenerate minimum. Then the Hamiltonian diffeomorphism φ_H generated by H has simple periodic orbits of arbitrarily large period.*

Similarly to the arguments in [Gi2, GG2] used for other classes of manifolds, this theorem implies the Conley conjecture as stated in Theorem 2.2.1. We will prove this theorem, and also show how it implies Theorem 2.2.1, in Chapter 5. The argument will mainly rely on properties of Floer homology. In particular, the proof of Theorem 2.2.5 is based on a Floer theoretical argument establishing

Theorem 2.2.6 *Let M be one of the following:*

- *M is a closed symplectic manifold, which is weakly monotone and let H be a Hamiltonian on M or*
- *$M = T^*B$ is the cotangent bundle of a closed manifold B and H is an admissible Hamiltonian on M as defined in Section 2.1.1.*

Assume that $\bar{x}$ is a symplectically degenerate minimum of H and that $\mathcal{A}_H(x) = c$. Then for every sufficiently small $\epsilon > 0$ there exists some $k_\epsilon \in \mathbb{N}$ such that

$$\mathrm{HF}_{-n-1}^{(kc-\epsilon,\, kc-\delta_k)}(H^{(k)}) \neq 0 \text{ for all } k > k_\epsilon \text{ and some } \delta_k \in (0, \epsilon).$$

In this theorem, $H^{(k)}$ denotes the one-periodic Hamiltonian H viewed as a k-periodic function for some integer k, see also Section 3.2. As in [Gi2, GG2], this theorem implies Theorem 2.2.5 and thus also Theorem 2.2.1. The main new point in the proof of Theorem 2.2.6 is moving from a symplectically degenerate maximum to a symplectically degenerate minimum in Chapter 6 and establishing a direct sum decomposition in filtered Floer homology in the irrational case in Chapter 7.

2.3 Action and index gap

Theorem 2.2.6 can also be used to control the behavior of actions and mean indices of periodic orbits, cf. [GG2]. To state the results, we need to introduce some notation. We call the difference $\mathcal{A}_{H^{(k)}}(\bar{x}) - \mathcal{A}_{H^{(k)}}(\bar{y})$ the *action gap* between the two capped k-periodic orbits $\bar{x}$ and $\bar{y}$. Similarly, the *mean index gap* between the two orbits is the difference $\Delta_{H^{(k)}}(\bar{x}) - \Delta_{H^{(k)}}(\bar{y})$. Both can be zero, even for geometrically distinct orbits x and y. The *action-index gap* between $\bar{x}$ and $\bar{y}$ is the vector in $\mathbb{R}^2$ whose components are the action gap and the mean index gap.

Recall also that an increasing sequence of integers $\nu_1 < \nu_2 < \dots$ is called *quasi-arithmetic* if the differences $\nu_{i+1} - \nu_i$ are bounded by a constant, which is independent of i.

Theorem 2.3.1 (Bounded gap theorem) *Let H be a Hamiltonian on a closed symplectic manifold (M^{2n}, ω) with first Chern number $N \geq 2n$ such that all periodic orbits of φ_H are isolated.*

Then there exists a capped one-periodic orbit $\bar{x}$ of H, a quasi-arithmetic sequence of iterations ν_i, and a sequence of capped ν_i-periodic orbits $\bar{y}_i$, geometrically distinct from $\bar{x}^{\nu_i}$, such that the sequence of action–index gaps

$$\big(\mathcal{A}_{H^{(\nu_i)}}(\bar{x}^{\nu_i}) - \mathcal{A}_{H^{(\nu_i)}}(\bar{y}_i), \Delta_{H^{(\nu_i)}}(\bar{x}^{\nu_i}) - \Delta_{H^{(\nu_i)}}(\bar{y}_i)\big)$$

is bounded.

This is a generalization of a result in [GG2] where the theorem was proved in the case of a symplectically rational manifold. As in the rational case, this theorem implies the following corollary in the generalized situation.

Corollary 2.3.2 *Let M and H be as in Theorem 2.3.1. Then there exists a quasi-arithmetic sequence of iterations ν_i and sequences of geometrically distinct ν_i-periodic orbits $\bar{z}_i$ and $\bar{z}'_i$ such that the sequence of action–index gaps between $\bar{z}_i$ and $\bar{z}'_i$ is bounded.*

In the case of a closed, rational symplectic manifold, the analog of Theorem 2.3.1 was proved in [GG2, section 4] using the version of Theorem 2.2.6 for this class of manifolds. We omit the proofs of Theorem 2.3.1 and Corollary 2.3.2 in this thesis, as the argument in the present version follows the same path as the argument in [GG2], utilizing the generalization of Theorem 2.2.6 to our class of manifolds.

Since we do not assume rationality of the symplectic manifold (M, ω), we cannot use the rationality constant to obtain an action bound to show that the orbits $\bar{y}_i$ and the iterations of $\bar{x}$ are geometrically distinct. For this purpose, we now use bounds on the mean index, which could also have been used in the rational case. Apart from this necessary change, the argument in [GG2] carries over to our case word for word.

Chapter 3

Preliminaries

In this chapter we will set the notation used throughout the thesis and discuss details of the symplectic manifolds considered in the theorems.

3.1 Symplectic manifolds

Let (M, ω) be a closed symplectic manifold of dimension $2n$, i.e., a manifold equipped with a non-degenerate 2-form ω. Here, we will only discuss the structures and properties of symplectic manifolds needed in the proofs. For more details on symplectic manifolds and in particular for proofs of the results mentioned here, see e.g. [MS1, MS2].

An almost complex structure J on M is called *compatible* with the symplectic form ω, if $\omega(\cdot, J\cdot)$ is a Riemannian metric on M. For every symplectic manifold (M, ω), the space of compatible almost complex structures is non-empty and contractible. Similarly, a Riemannian metric g is compatible with ω if it is of the form $g(\cdot,\cdot) = \omega(\cdot, J\cdot)$ for some almost complex structure J. From now on, we will always assume the Riemannian metric on M to be defined in this fashion.

For our purposes, the most important structures on a symplectic manifold (M, ω) are the symplectic form and the first Chern class $c_1(M) \in H^2(M)$ and the relation between them.

Both 2-forms define a homomorphism $\pi_2(M) \to \mathbb{R}$ via integration over spheres in M. The subgroup $\langle c_1(M), \pi_2(M)\rangle \subset \mathbb{R}$ is discrete and the positive generator N of this subgroup is the *minimal Chern number*. When this subgroup is zero, we set $N = \infty$. A symplectic manifold M is called monotone, if $[\omega]\,|_{\pi_2(M)} = \lambda c_1(M)|_{\pi_2(M)}$ for some non-negative $\lambda \in \mathbb{R}$. Throughout this thesis, we assume that M is *weakly monotone*, i.e. M is monotone or $N \geq n-2$.

The manifold M is called *rational*, if the image of ω is also a discrete subgroup of $\mathbb{R}$. For closed rational symplectic manifolds, all our theorems were proved in [GG2]. In this thesis, we will focus on the new methods needed to generalize the results.

Let us now construct an example of an irrational symplectic manifold with vanishing first Chern class, which falls into the setting of Section 2.1.2, but where the result of Theorem 2.2.1 is not covered in [GG2].

Example 3.1.1 (An irrational symplectic manifold with $c_1 = 0$) *In order to construct such a manifold, we first need a symplectic manifold X with vanishing first Chern class and sufficiently large second homology $H^2(X)$. There are several examples of such manifolds, e.g., we could take X to be a $K3$-surface. To give a more concrete example, take X to be the $K3$-surface given by a non-singular quartic on $\mathbb{CP}^3$ with the canonical symplectic structure ω coming from the complex structure of $\mathbb{CP}^3$. This symplectic form on M gives rise to the symplectic form $(\omega,\,\omega)$ on the product $M = X \times X$. A generic perturbation of this product symplectic structure is irrational, i.e., a symplectic structure of the form $(\omega, \alpha\omega)$ is irrational for generic $\alpha \in \mathbb{R}$.*

3.2 Hamiltonian flows and capped periodic orbits

A Hamiltonian is a smooth function $H\colon S^1 \times M \to \mathbb{R}$ with $S^1 = \mathbb{R}/\mathbb{Z}$, i.e., all considered Hamiltonians H on M are assumed to be one-periodic in time and we will set $H_t(x) = H(t,\,x)$. A one-periodic Hamiltonian H is of course also k-periodic for any integer k. For our argument it is sometimes crucial to keep track of the period we are interested in. If a one-periodic Hamiltonian H is viewed as k-periodic, we refer to it as the *kth iteration* of H and denote it by $H^{(k)}$. In particular, the function $H^{(k)}$ is a function $H^{(k)}\colon S^1_k \times M \to \mathbb{R}$, where $S^1_k = \mathbb{R}/k\mathbb{Z}$.

As the symplectic form ω is non-degenerate, the equation

$$i_{X_H}\omega = -dH$$

gives rise to a well-defined Hamiltonian vector field X_H. For a fixed choice of an almost complex structure J which is compatible with ω and the Riemannian metric $g(\cdot,\cdot) = \omega(\cdot, J\cdot)$, we find that $JX_H = \nabla H$, where ∇H is the gradient of H with respect to g. Denote the flow of this vector field by φ_H^t.

The composition $\varphi_H^t \circ \varphi_K^t$ of two Hamiltonian flows is again Hamiltonian and generated by the function

$$(K\#H)_t = K_t + H_t \circ \varphi_K^{-t}.$$

In general, this function need not be one-periodic in time, even if both H and K are one-periodic Hamiltonians. But $K\#H$ will be one-periodic if both are one-periodic and, in addition, K generates a loop of Hamiltonian diffeomorphisms. This will always be the case in this thesis.

The time-1-map of the flow of the Hamiltonian vector field X_H is called a Hamiltonian diffeomorphism and denoted by φ_H. The fixed points of φ_H are in one-to-one correspondence with one-periodic orbits of φ_H^t. A fixed point of φ_H (or a periodic point of higher period) is called *contractible*, if the corresponding periodic flow line of X_H is contractible. In this thesis, we only work with contractible periodic orbits and every periodic orbit is assumed to be contractible, even if this is not explicitly stated.

Let $x\colon S_T^1 \to M$ be a contractible loop, where $S_T^1 = \mathbb{R}/T\mathbb{Z}$ is the circle of circumference T. A *capping* of x is defined to be a map $u\colon D^2 \to M$ such that $u|_{\partial D^2} = x$, where ∂D^2 is identified with S_T^1. Two cappings are called *equivalent* if the integrals over the symplectic form ω and the first Chern class $c_1(M)$ over the two capping discs agree.

We refer to the pair $(x, [u])$ of a contractible loop x which is equipped with an equivalence class of cappings $[u]$ as a *capped loop* and denote it by $\bar{x}$. In the symplectically aspherical case, all cappings are equivalent. In particular, this is true for the setting of $M = T^*B$ described above. In our case of a closed manifold however, the cappings introduce new difficulties in the proof, which are addressed later in Section 4.2 and Chapter 7.

3.3 Hamiltonian action and the mean index

The periodic orbits of a Hamiltonian systems can be characterized as critical point of a functional on the space of capped loops. In this section, we will describe this action functional and also the mean index and the Conley-Zehnder index used for the grading of Floer homology discussed further in Chapter 4.

The capped T-periodic orbits are the critical points of the Hamiltonian action functional which is defined by

$$\mathcal{A}_H(\bar{x}) = -\int_u \omega + \int_{S^1_T} H_t(x(t))\, dt \tag{3.3.1}$$

on the space of capped loops in M. This space is a covering space of the space of contractible loops. The set of critical values of the action is denoted by $\mathcal{S}(H)$ and called the *action spectrum* of H.

Definition 3.3.1 *A T-periodic orbit is called non-degenerate, if the linearized return map $d\varphi_H^T$ does not have one as an eigenvalue. Following [SZ], we call an orbit weakly non-degenerate if at least one eigenvalue is not equal to one and strongly degenerate otherwise.*

For the remaining part of this section, we will focus on the case of capped one-periodic orbits of a Hamiltonian H which only has non-degenerate periodic orbits. The non-degeneracy of all orbits is a generic condition on H and we call such Hamiltonians *non-degenerate*. All definitions and properties carry over word-for-word to capped T-periodic orbits.

Apart from the action, we also need the mean index $\Delta_H(\bar{x})$ of a capped periodic orbit $\bar{x}$. Roughly speaking, this index is the rotation number of the eigenvalues of $d\varphi_H^t$ along the orbit. For a rigorous definition and properties of the mean index we refer the reader to [Lo2, SZ]. Strictly speaking, the mean index is defined only on capped periodic orbits with a choice of trivialization of TM over the capping disk. For a fixed choice of a capping, the space of trivializations is contractible. Thus we only need to specify the capping to have the mean index well-defined. In [SZ], the mean index is defined in the symplectically aspherical case, when everything is independent of the capping. A list of properties of the mean index, taking the dependence of the capping into account, can also be found in [GG2].

Both the mean index and the Hamiltonian action of a capped periodic orbit depend on the equivalence class of the capping u of the loop x. More concretely let $A \in \pi_2(M)$ be an embedded 2-sphere and denote by $\bar{x}\#A$ the recapping of $\bar{x}$ by attaching A to the capping disk. Then the above formula for the Hamiltonian action shows that the action of the recapped orbit is related to the action of the orbit with the original capping by

$$\mathcal{A}_H(\bar{x}\#A) = \mathcal{A}_H(\bar{x}) - \int_A \omega.$$

For the mean index, we only mention that the mean index $\Delta(\bar{x})$ depends on the capping via

$$\Delta_H(\bar{x}\#A) = \Delta_H(\bar{x}) - 2c_1(A),$$

where $c_1(A)$ is the pairing of the first Chern class $c_1(M)$ with the sphere A.

The kth iteration of a capped orbit $\bar{x}$ carries a natural capping given by the k-fold covering of D^2 and with that capping it is denoted by $\bar{x}^k$. The mean index and the action both are homogeneous with respect to iteration and the action and mean index of iterations are given as

$$\mathcal{A}_{H^{(k)}}(\bar{x}^k) = k\mathcal{A}_H(\bar{x}) \text{ and } \Delta_{H^{(k)}}(\bar{x}^k) = k\Delta_H(\bar{x}). \tag{3.3.2}$$

Related to the mean index, we can also assign an integer-valued index to capped periodic orbits which will be used as a grading for the Floer homology. Up to sign we define the Conley-Zehnder index as in [Sa, SZ], to which we refer for more details on the definition. In this thesis, we use the normalization and conventions from [Gi2], i.e., the grading is defined such that for a non-degenerate maximum $\bar{x}_0$ with trivial capping of an autonomous Hamiltonian with small Hessian at x_0 we have $\mu_{\mathrm{CZ}}(\bar{x}_0) = n$.

The Conley-Zehnder index is closely related to the mean index. More concretely, the Conley-Zehnder index grows roughly linearly with the order of iteration and the mean index can be viewed as the linear part, i.e., an alternative definition of the mean index is $\Delta_H(\bar{x}) = \lim_{k\to\infty} \mu_{\mathrm{CZ}}(\bar{x}^k)/k$. Furthermore, the difference between the Conley-Zehnder index and the mean index is bounded in terms of the dimension of the symplectic manifold. Namely, we have

$$\|\Delta_H(\bar{x}) - \mu_{\mathrm{CZ}}(\bar{x})\| \le n \tag{3.3.3}$$

for all capped periodic orbits of H and the inequality is strict if the orbit $\bar{x}$ is non-degenerate. This bound of the Conley-Zehnder index in terms of the mean index and the linearity of the mean index in the order if iteration will play a crucial role in the proofs.

Chapter 4

Floer homology

In this thesis, we are interested in the periodic orbits of a Hamiltonian. One of the most important tools to study the periodic orbits is the Hamiltonian Floer homology, which is an analog of Morse homology defined for the action functional of a non-degenerate Hamiltonian.

In this section, we define the (Hamiltonian) Floer homology and discuss the properties used in the proofs. From now on, we always assume that M is equipped with an almost complex structure J compatible with ω and the corresponding Riemannian metric g discussed in Section 3.1. This almost complex structure can also be one-periodic in time, but we omit the time-dependence in the notation.

As mentioned above, we phrase all definitions in terms of one-periodic orbits. All constructions carry over word for word to capped T-periodic orbits for arbitrary T with the same properties.

4.1 Definition of Floer homology

The Floer homology is defined analogously to the Morse homology for the action functional defined in (3.3.1) on the infinite-dimensional space of capped loops. We refer the reader to Floer's original papers [Fl1, Fl2, Fl3, Fl4] or to, e.g., [HS, MS2, Sa, SZ] for further references and introductory accounts of the construction of (Hamiltonian) Floer homology. Throughout this thesis, we use the notation and conventions of [Gi2, GG2], where the grading convention is that of [Gi2], which is also described above.

For this section, we always assume that the Hamiltonian is non-degenerate. Many results establishing the existence of infinitely many periodic orbits of Hamiltonian systems were proved first under this additional assumption and generalized later to the degenerate case. The main tools, Floer homology and the Conley-Zehnder index, are constructed in the non-degenerate case and extend by continuity to all Hamiltonians. For filtered Floer homology, the extension is slightly more involved and requires a direct limit construction, see Section 4.2.

The Floer chain groups are generated as vector spaces over $\mathbb{Z}_2$ by the capped one-periodic orbits of H and the boundary operator is defined analogously to the boundary operator of Morse homology. The gradient flow lines are replaced by maps $u\colon S^1 \times \mathbb{R} \to M$ that solve the Floer equation

$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} = -\nabla H_t(u), \tag{4.1.1}$$

the so-called Floer trajectories. The energy of a Floer trajectory u is defined by

$$E(u) = \int_{-\infty}^{\infty} \int_{S^1} \left\| \frac{\partial u}{\partial s} \right\|^2 \, dt \, ds.$$

For a non-degenerate Hamiltonian H, Floer trajectories with finite energy converge to periodic orbits x and y as s goes to $\pm\infty$ The boundary operator ∂ counts Floer trajectories converging to periodic orbits y and x as $s \to \pm\infty$. As the orbits x and y enter the Floer chain groups multiple times with different cappings, we also require Floer trajectories connecting the capped orbits $\bar{x}$ and $\bar{y}$ to be compatible with the cappings in the sense that $[(\text{capping of } \bar{x})\#u] = [\text{capping of } \bar{y}]$. For such Floer trajectories, the energy is given by

$$E(u) = \mathcal{A}_H(\bar{x}) - \mathcal{A}_H(\bar{y}).$$

If the almost complex structure J satisfies certain generic regularity requirements, see e.g. [MS2, Sa], the space of such solutions is a smooth compact manifold and carries a natural $\mathbb{R}$-action by a shift in the s-direction. Denote the quotient of the solution space by this action, the moduli space, with $\mathcal{M}(\bar{x}, \bar{y}, J)$.

The dimension of the moduli space $\mathcal{M}(\bar{x}, \bar{y}, J)$ is $\mu_{\text{CZ}}(\bar{x}) - \mu_{\text{CZ}}(\bar{y}) - 1$. Now the boundary operator is defined by

$$\partial \bar{x} = \sum_y \#\mathcal{M}(\bar{x}, \bar{y}, J) \cdot \bar{y}$$

where the sum runs over all $\bar{x}$ such that $\mu_{\mathrm{CZ}}(\bar{y}) = \mu_{\mathrm{CZ}}(\bar{x}) - 1$ and $\#\mathcal{M}(\bar{x}, \bar{y}, J)$ is counted modulo 2. For this index difference, the moduli space is zero-dimensional and due to compactness results for $\mathcal{M}(\bar{x}, \bar{y}, J)$, the number of elements in this space is finite.

In the case of a closed manifold M, the necessary compactness results for the moduli space $\mathcal{M}(\bar{x}, \bar{y}, J)$ are by now well-known. For more details and proofs of these results, we refer, e.g., to [MS2, Sa].

For our class of Hamiltonians on the cotangent bundle, it is shown in [AS] that the conditions (H1) and (H2) on the Hamiltonian H imply that the Floer trajectories are uniformly bounded in the C^0-norm, if the almost complex structure J is sufficiently close to the standard almost complex structure on $M = T^*B$. Using this bound, the compactness of moduli spaces follows as in the case of a closed manifold. Similar bounds on Floer trajectories for different classes of Hamiltonians on non-closed symplectic manifolds are established in, e.g., [Se, Vi3].

Studying the moduli spaces for index difference two, it is well-established that this definition of ∂ indeed defines a chain boundary operator satisfying $\partial \circ \partial = 0$ and the Floer homology is defined in the usual way, see, e.g. [Sa].

4.1.1 Homotopy maps

For two non-degenerate Hamiltonians H^0 and H^1, we could now ask how their Floer homologies are related. It turns out that the Floer homology is independent of the Hamiltonian and only depends on the underlying symplectic manifold. Namely, a homotopy from H^0 to H^1 induces a homomorphism of chain complexes which gives an isomorphism between the Floer homology groups $\mathrm{HF}_*(H^0)$ and $\mathrm{HF}_*(H^1)$. To define this homomorphism, denote by H^s the homotopy from H^0 to H^1 such that $H^s = H^0$ for $s \leq 0$ and $H^s = H^1$ for $s \geq 1$. Then the map between the Floer homologies of H^0 and H^1 is given similarly to the Floer boundary operator using a version the Floer equation (4.1.1) with the homotopy H^s on the right hand side, i.e., counting solutions of

$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} = -\nabla H^s_t(u). \tag{4.1.2}$$

Similarly to the definition of the Floer boundary operator, solutions to equation (4.1.2) converge to a capped periodic solution $\bar{x}$ of H^0 as $s \to -\infty$ and a capped periodic solution $\bar{y}$ of H^1 as $s \to +\infty$. The moduli space $\mathcal{M}(\bar{x}, \bar{y}, H^s, J)$ of solutions is finite if $\mu_{\mathrm{CZ}}(\bar{y}) = \mu_{\mathrm{CZ}}(\bar{x})$. (The necessary compactness result in the cotangent bundle case follows again from a uniform bound on the solutions in the C^0 topology which is established in [AS].)

Now we define the map $\Phi_{H^0,H^1}\colon \mathrm{HF}_*(H^0) \to \mathrm{HF}_*(H^1)$ by

$$\Phi(\bar{x}) = \sum_y \#\mathcal{M}(\bar{x}, \bar{y}, H^s, J) \cdot \bar{y}. \tag{4.1.3}$$

It is shown in [Sa, SZ] that this map is a natural isomorphism and independent of the choice of the homotopy. These properties of Φ_{H^0,H^1} also imply that we can concatenate homotopies and find for three Hamiltonians H^0, H^1, H^2 that

$$\Phi_{H^0,H^2} = \Phi_{H^1,H^2} \circ \Phi_{H^0,H^1}. \tag{4.1.4}$$

Since the homotopy maps provide an isomorphism of Floer homology for different Hamiltonians, we could ask now what the Floer homology on a given symplectic manifold is. The answer to this question is different in our two different settings.

In the setting of a closed manifold, there is an isomorphism relating the Floer homology of a Hamiltonian to the homology of the symplectic manifold. More concretely, we have

$$\mathrm{HF}_*(K) \cong H_{*+n}(M) \otimes \Lambda \tag{4.1.5}$$

for the Novikov ring Λ, which is needed to take different cappings into account. For a more detailed definition of the Novikov ring, see e.g. [GG2, HS, Us1]. This isomorphism is a standard fact about Floer homology, see e.g. [MS2, Sa] and can be established using a Hamiltonian which is a C^2-small Morse function on M. For such a Hamiltonian H, the periodic orbits are the critical points of H and the Floer homology is (up to a shift in the grading) equal to the Morse homology and thus equal to the singular homology of M.

For the cotangent bundle setting and our class of admissible Hamiltonians, it is proven in [AS] that there exists an isomorphism

$$\mathrm{HF}_*(H) \cong H_{-*}(\Lambda_0 B), \tag{4.1.6}$$

where $\Lambda_0 B$ is the space of contractible loops on the base B. See also [Se, SW, Vi3] for similar results for somewhat different classes of Hamiltonians.

4.2 Filtered Floer Homology

In this section, we give a definition of filtered Floer homology for degenerate Hamiltonians on symplectically irrational manifolds. We will also discuss the compatibility of the homotopy maps with the action filtration and properties of homotopy maps between filtered Floer homology.

4.2.1 Action filtration and homotopy maps revisited

As the action decreases along Floer trajectories of a non-degenerate Hamiltonian H, we can also define the Floer chain complex with a restriction of the action interval. Namely, we only consider orbits with action in an interval (a, b) if the endpoints a and b of the action interval are not in the action spectrum $\mathcal{S}(H)$. This complex gives rise to the *filtered Floer homology* $\mathrm{HF}_*^{(a,\,b)}(H)$.

The action filtration is respected by homotopy maps defined in (4.1.3) if the homotopy is monotone decreasing, i.e. if $\partial H_t^s/\partial s \leq 0$ for all $t \in S^1$. Thus we can use the homotopy maps again to extend also filtered Floer homology by continuity to degenerate Hamiltonians if the manifold is rational. The rationality assumption is needed, since in this case small perturbations do not change the capped periodic orbits with action in (a, b) and the construction does not depend on the choice of a sufficiently small perturbation. In the irrational case, we will discuss the details of filtered Floer homology for degenerate Hamiltonians in Section 4.2.2 and focus on the rational case here.

By construction of filtered Floer homology for non-degenerate Hamiltonians and a standard diagram chasing argument, we have a long exact sequence of filtered Floer homology groups

$$\cdots \to \mathrm{HF}_*^{(a,\,b)}(K) \to \mathrm{HF}_*^{(a,\,c)}(K) \to \mathrm{HF}_*^{(b,\,c)}(K) \to \mathrm{HF}_{*-1}^{(a,\,b)}(K) \to \cdots \qquad (4.2.1)$$

for any non-degenerate Hamiltonian K with $a, b, c \notin \mathcal{S}(K)$.

Let us now study some special cases of homotopy maps of filtered Floer homology in the rational case before we define the filtered Floer homology in the irrational case. We will then discuss which properties carry over to the irrational case. Assume for now that the symplectic manifold is rational.

The homotopy map for monotone homotopies Hamiltonians is not necessarily an isomorphism of filtered Floer homology. This can be easily seen by considering a homotopy which shifts the Hamiltonian H^0 by an additive constant to $H^1 = H^0+1$ and a small action interval, i.e., we consider the homotopy $H^s = H^0 + f(s)$ with $f(s) = 0$ for $s < 0$ and $f(s) = 1$ for $s > 1$. Then all Hamiltonians in the homotopy have the same periodic orbits and the homotopy map is the identity map. The action of a one-periodic orbit $\bar{x}$ get also shifted up by $f(s)$. If the action interval (a, b) is chosen such that $0 \neq \bar{x} \in \mathrm{HF}_*^{(a,\,b)}(H^0)$ and $\mathcal{A}_{H^0}(\bar{x}) \geq b - 1$, then $\mathcal{A}_{H^1}(\bar{x}) = \mathcal{A}_{H^0}(\bar{x}) + 1 \notin (a,\ b)$ and thus $\Psi_{H^0,H^1}(\bar{x}) = 0$.

In some special cases, however, the homotopy map does induce an isomorphism of filtered Floer homology. For example, consider a (not necessarily decreasing) homotopy H^s from H^0 to H^1 for which the endpoints a, b of the action interval are not in the action spectrum of H^s for all s. Such a homotopy induces an isomorphism of filtered Floer homology and we find

$$\mathrm{HF}_*^{(a,\,b)}(H^0) \cong \mathrm{HF}_*^{(a,\,b)}(H^1). \tag{4.2.2}$$

The isomorphism is constructed by breaking the homotopy into a composition of nearly constant homotopies, see [BPS, Gi2, Vi2] and depends on the homotopy, but not on other choices in the construction. In general, this map does not agree with the homotopy map Ψ_{H^0,H^1}. For a monotone decreasing homotopy, however, the isomorphism in (4.2.2) does coincide with Ψ_{H^0,H^1}.

A particular case of a homotopy with $a, b \notin \mathcal{S}(H^s)$ for all s, which will be used later in the proof, is the following

Definition 4.2.1 *A homotopy H^s is called isospectral if the action spectrum $\mathcal{S}(H^s)$ is independent of s.*

As in [Gi2, GG2], one particular case of an isospectral homotopy is of interest for us. For $t \in S^1$ and $s \in [0, 1]$, let η_s^t be a family of loops of Hamiltonian diffeomorphisms based at id, i.e., $\eta_s^0 = id$ for all s. In other words, η_s^t is a based homotopy from the loop η_0^t to the loop η_1^t. Let G_t^s be a family of one-periodic Hamiltonians generating these loops and let H be a fixed one-periodic Hamiltonian. Then $H^s := G^s \# H$ is an isospectral homotopy, provided that G^s are suitably normalized. For details on the normalization, we refer to [Gi2, GG2].

Consider now a homotopy H^s such that the endpoints a, b of the action interval are not in the action spectrum $\mathcal{S}(H^s)$ for all s. If a third Hamiltonian K is less than all Hamiltonians in the homotopy, i.e., if $K \leq H^s$ for all s, the diagram

$$\begin{array}{ccc} \mathrm{HF}_*^{(a,\,b)}(H^0) & \xrightarrow{\cong} & \mathrm{HF}_*^{(a,\,b)}(H^1) \\ & \underset{\cong}{\searrow} & \downarrow \\ & & \mathrm{HF}_*^{(a,\,b)}(K) \end{array} \tag{4.2.3}$$

is commutative for homotopy maps given by homotopies from H^0 and H^1 to K and the horizontal map being the isomorphism (4.2.2). For this diagram to commute, it is essential that $K \leq H^s$ holds for all Hamiltonians in the homotopy and not just for the endpoints H^0 and H^1, since we need to break the homotopy H^s into a composition of small homotopies with the same properties for the proof.

4.2.2 Filtered Floer homology in the irrational case

In the case of an irrational manifold, the action filtration of Floer homology for degenerate Hamiltonians cannot be unambiguously defined simply by continuity as the resulting groups depend very sensitively on the non-degenerate perturbation. We thus use the following construction for the filtered Floer homology, which in the case of a rational manifold gives the same homology groups as the above extension by continuity.

Let H be a fixed Hamiltonian on M. To define $\mathrm{HF}_*^{(a,\,b)}(H)$, consider perturbations K of H with the following properties:

(P1) the Hamiltonian K is non-degenerate;

(P2) the boundary values a and b of the action interval are not in the action spectrum $\mathcal{S}(K)$ of K;

(P3) we have $K \geq H$.

For the remaining part of this section we will always assume the above properties whenever we speak of perturbations K of a Hamiltonian H.

The set of such perturbations is partially ordered by $K^1 \leq K^0$ whenever $K_t^1(x) \leq K_t^0(x)$ for all $x \in M$ and $t \in S^1$. Consider a monotone decreasing homotopy K^s from K^0 to K^1. By condition (P1), both perturbations K^0 and K^1 are non-degenerate. Thus we have an induced monotone homotopy map between the Floer homology groups, which are well-defined by (P1) and (P2). In this case, the homotopy map is still a homomorphism, but it need not be an isomorphism. Those monotone homotopy maps give rise to transition maps $\mathrm{HF}_*^{(a,b)}(K) \to \mathrm{HF}_*^{(a,b)}(\tilde{K})$ whenever $K \geq \tilde{K}$. Then we can define the filtered Floer homology of H by

$$\mathrm{HF}_*^{(a,b)}(H) = \varinjlim \mathrm{HF}_*^{(a,b)}(K)$$

as the direct limit of filtered Floer homology groups for perturbations satisfying the conditions (P1)-(P3).

Remark 4.2.2 *If H is non-degenerate and a and b are not in the action spectrum $\mathcal{S}(H)$, this definition yields the ordinary filtered Floer homology of H, as H can be viewed as the trivial perturbation of itself. Due to the non-degeneracy, H itself satisfies the conditions required from the considered perturbations K. Thus the set $\{H\}$ is cofinal and is sufficient to define the limit.*

The maps of the exact sequence (4.2.1) commute with the monotone homotopy maps. Hence, for the limit function H, the analog sequence

$$\cdots \to \mathrm{HF}_*^{(a,b)}(H) \to \mathrm{HF}_*^{(a,c)}(H) \to \mathrm{HF}_*^{(b,c)}(H) \to \mathrm{HF}_{*-1}^{(a,b)}(H) \to \cdots \tag{4.2.4}$$

is also exact, since exactness is compatible with the limit construction, when the homotopy maps used in the limit commute with the maps of the exact sequence.

In the definition of the filtered Floer homology as a limit, we can also restrict the family of perturbations by requiring other properties in addition to (P1)-(P3). The restricted family of perturbations is sufficient to define the limit if they form a cofinal set, i.e. for any perturbation satisfying (P1)-(P3) we can find a smaller one with the additional properties. The limit then does not depend on the perturbations that do not have the additional properties.

In particular, we will later consider a cofinal set of perturbations for which the filtered Floer homology splits into a direct sum decomposition that is compatible with the monotone homotopy maps. Then we will also have a direct sum decomposition of the limit, see Section 7.1 and in particular Remark 7.1.5.

We can also define monotone homotopy maps for homotopies starting at a degenerate Hamiltonian H. Due to condition (P3), the monotone homotopy map for a homotopy from any perturbation K factors through all perturbations closer to the limit function H than K. Then we define the monotone homotopy map from $\mathrm{HF}^{(a,b)}(H)$ as the limit of monotone homotopy maps from the perturbations. The resulting map, still called monotone homotopy map, has the same properties as the usual homotopy maps. If the endpoint of the homotopy is a degenerate Hamiltonian, it follows from the definition of the direct limit and property (4.1.4) that the homotopy map into the limit homology is also well-defined.

4.3 Local Floer homology

In this section, we define a local version of Floer homology providing an invariant for isolated periodic orbits. The local Floer homology groups can be seen as building blocks for the Floer homology of non-degenerate Hamiltonians. In this section, we briefly recall the definition and basic properties of local Floer homology following mainly [Gi2, GG2], although this notion goes back to the original work of Floer (see, e.g., [Fl4, Fl5]) and has been revisited a number of times since then; see, e.g., [Po, Section 3.3.4].

Let $\bar{x}$ be an isolated capped one-periodic orbit of a Hamiltonian H. Pick a sufficiently small tubular neighborhood U of $\bar{x}$ and consider a non-degenerate C^2-small perturbation $\tilde{H}$ of H supported in U. Then the degenerate periodic orbit $\bar{x}$ splits into a finite number of periodic orbits of $\tilde{H}$ equipped with a capping which is a perturbation of the capping of $\bar{x}$. Every (anti-gradient) Floer trajectory u connecting two one-periodic orbits of $\tilde{H}$ lying in U is also contained in U, provided that $\|\tilde{H} - H\|_{C^2}$ and $supp(\tilde{H} - H)$ are small enough. Thus, by the compactness and gluing theorems, every broken anti-gradient trajectory connecting two such capped orbits also lies entirely in U. The vector space (over $\mathbb{Z}_2$) generated by one-periodic orbits of $\tilde{H}$ in U is a complex with (Floer) differential defined in the standard way. The continuation argument (see, e.g., [SZ]) shows that the homology of this complex is independent of the choice of $\tilde{H}$ and of the almost complex structure. We refer to the resulting homology group $\mathrm{HF}_*(H, \bar{x})$ as the *local Floer homology* of H at $\bar{x}$. The proofs of these facts are very similar to the proof of the direct sum decomposition in Chapter 7. In fact, local Floer homology is a special case of that direct sum, see Remark 7.1.2.

Example 4.3.1 *Assume that $\bar{x}$ is a non-degenerate one-periodic orbit of a Hamiltonian H with $\mu_{CZ}(\bar{x}) = k$. Then $\mathrm{HF}_l(H, \bar{x}) = \mathbb{Z}_2$ when $l = k$ and $\mathrm{HF}_l(H, \bar{x}) = 0$ otherwise.*

The local Floer homology groups can be seen as building blocks of filtered Floer homology for small action intervals. Namely, let $c \in \mathbb{R}$ be such that all capped one-periodic orbits $\bar{x}_i$ of H with action c are isolated. (As a consequence, there are only finitely many orbits with action close to c.) Then, if $\epsilon > 0$ is small enough,

$$\mathrm{HF}_*^{(c-\epsilon,\, c+\epsilon)}(H) = \bigoplus_i \mathrm{HF}_*(H, \bar{x}_i). \tag{4.3.1}$$

In particular, if all capped one-periodic orbits $\bar{x}$ of H are isolated and $\mathrm{HF}_k(H, \bar{x}) = 0$ for some k and all $\bar{x}$ with $\mu_{\mathrm{CZ}}(\bar{x}) = k$, we have $\mathrm{HF}_k(H) = 0$ by the long exact sequence (4.2.4) of filtered Floer homology.

We define the *support* of $\mathrm{HF}_*(H, \bar{x})$ as the collection of integers k such that $\mathrm{HF}_k(H, \bar{x}) \neq 0$. Clearly, the group $\mathrm{HF}_*(H, \bar{x})$ is finitely generated and hence supported in a finite range of degrees, namely in $[\Delta_H(\bar{x}) - n,\, \Delta_H(\bar{x}) + n] \cap \mathbb{Z}$. As the mean index grows linearly with the order of iteration, the support of local Floer homology will also change with iteration and provide an important tool for the proof of the Conley conjecture.

Chapter 5

Proofs of Theorem 2.2.1 and Theorem 2.2.5

In this Chapter, we will prove the Conley conjecture, i.e., the existence of infinitely many periodic orbits for Hamiltonian systems, in both our settings. Postponing the technical proof of Theorem 2.2.6 to Chapter 8, we will assume this result in this chapter and use it to prove Theorem 2.2.1 and Theorem 2.2.5. The arguments in this chapter follow the proofs in [He1, He2] and mainly use the properties of Floer homology and the behavior of the mean index under iteration discussed above.

5.1 Proof of Theorem 2.2.1

In this section, we proof the Conley conjecture in both settings described Section 2.1. To do this, we show that there exists a symplectically degenerate minimum whenever the number of one-periodic orbits is finite. Then Theorem 2.2.1 follows from Theorem 2.2.5 which is established in Section 5.2.

The key to the proof are the isomorphisms (4.1.5) and (4.1.6) relating the Floer homology groups to the homology of the underlying manifold. Using these isomorphisms, we can show

Lemma 5.1.1 *In both settings considered in Theorem 2.2.1, we have* $\mathrm{HF}_{-n}(K) \neq 0$ *for all admissible Hamiltonians* K.

Proof We consider the two settings separately, since we need to use a different isomorphisms depending on the setting.

For closed manifolds, we use the isomorphism (4.1.5). In this case, the lemma follows from the basic fact that for all closed symplectic manifolds $H_0(M) \neq 0$ and therefore also $\mathrm{HF}_{-n}(K) = H_0(M) \neq 0$ for all Hamiltonians K.

For the cotangent bundle setting whith $M = T^*B$ and our class of Hamiltonians, we use the isomorphism (4.1.6) established in [AS]. The base manifold B is contained in the space $\Lambda_0 B$ of contractible loops in B via constant loops. Therefore also have isomorphisms

$$\mathrm{HF}_*(H) \cong H_{-*}(\Lambda_0 B) \cong H_{-*}(B) \oplus H_{-*}(\Lambda_0 B, B).$$

Since we assume that B is closed and orientable, we have $H_n(B) \neq 0$. Then the above isomorphism in degree $* = -n$ reads $\mathrm{HF}_{-n}(K) \cong H_n(\Lambda_0 B) \neq 0$ and thus implies the lemma. □

This lemma applies for all admissible Hamiltonians independent of the period considered in the construction of Floer homology. In particular, the lemma holds for iterated Hamiltonians $K = H^{(k)}$.

To proof the theorem, assume now that there are only finitely many one-periodic orbits. If H does not have a symplectically degenerate minimum, all one-periodic orbits with non-zero local Floer homology in degree $-n$ have non-zero mean index. The main point of the proof will be the behavior of the mean index under iteration and the support of local Floer homology.

The mean index $\Delta_{H^{(k)}}(\bar{x}^k)$ grows linearly with iteration by (3.3.2), and is related to the Conley-Zehnder index grading the Floer homology via (3.3.3). This shows that for every one-periodic orbit $\bar{x}$ with non-zero mean index, the mean index of a sufficiently large iteration is larger than n or less than $-3n$.

Using the inequality (3.3.3), this implies that the Conley-Zehnder index of this iteration is not in the interval $[-2n, 0]$ and thus the support of local Floer homology, i.e., the degrees in which the local Floer homology is non-zero, shifts away from this interval. Thus for a sufficiently large order k of iteration of the Hamiltonian H, the local Floer homology $\mathrm{HF}_{-n}(H^{(k)}, \bar{x}^{(k)})$ becomes zero for all iterated one-periodic orbits with mean index $\Delta_H(\bar{x}) \neq 0$. It follows then by a standard argument using (4.3.1) that for the kth iteration of H we have $0 = \mathrm{HF}_{-n}(H^{(k)})$, see e.g. [Gi2, GG2, Hi, SZ].

This contradicts Lemma 5.1.1 and thus proves the existence of a symplectically degenerate minimum if the number of one-periodic orbits is finite. In the presence of a symplectically degenerate minimum, the theorem follows from Theorem 2.2.5, which also implies the statement concerning the periods of the periodic orbits.

5.2 Proof of Theorem 2.2.5

In this section, we will show how the degenerate Conley conjecture follows from Theorem 2.2.6, which is proved in Chapter 8. This will also complete the proof of Theorem 2.2.1.

Recall that we assume the existence of a symplectically degenerate minimum, i.e., a capped one-periodic orbit $\bar{x}$ satisfying

$$\Delta_H(\bar{x}) = 0 \text{ and } \mathrm{HF}_{-n}(H, \bar{x}) \neq 0,$$

and that the number of one-periodic orbits is finite. We also recall that Theorem 2.2.6 asserts that the filtered Floer homology group

$$\mathrm{HF}_{-n-1}^{(kc-\epsilon,\, kc-\delta_k)}(H^{(k)}) \neq 0 \tag{5.2.1}$$

for sufficiently large k and some constants $\epsilon > 0$ and $\delta_k \in (0, \epsilon)$, where c is the action of the symplectically degenerate minimum $\bar{x}$. We will use this to prove that there are simple periodic orbits of all sufficiently large prime periods. In particular, this implies the existence of infinitely many distinct periodic orbits.

Arguing by contradiction, we assume that the assertion of Theorem 2.2.5 is false, i.e., that for sufficiently large periods all periodic orbits are iterated. Let k be a sufficiently large prime. We choose k to be prime to ensure that all k-periodic orbits are either simple or iterated one-periodic orbits. In particular, assume k to be so large that every k-periodic orbit is an iterated one-periodic orbit and (5.2.1) holds.

Since the Floer chain groups are generated by periodic orbits, Theorem 2.2.6 implies that every Hamiltonian with a symplectically degenerate minimum has a capped periodic orbit in degree $-n-1$. In particular, this holds for all iterations of our Hamiltonian H. By the inequality (3.3.3) relating the mean index to the Conley-Zehnder index, this means that there exists a k-periodic orbit y_k of H with

$$-2n-1 \leq \Delta_{H^{(k)}}(y_k) \leq -1.$$

The mean index of an iterated capped orbit growths linearly with iteration. Thus for a one-periodic orbit, the mean indices of the iterations are zero whenever the one-periodic orbit has mean index zero. For a one-periodic orbit with non-zero mean index, the mean index of a sufficiently large iteration is not in the interval $[-2n-1, -1]$. The assumption that there are only finitely many one-periodic orbits implies that for sufficiently large k, no kth iteration of a one-periodic orbit has mean index in $[-2n-1, -1]$. Thus the orbit y_k cannot be an iterated one-periodic orbit in contradiction to the choice of k. This shows the desired result, since we have found a simple k-periodic orbit y_k for all sufficiently large primes k.

Chapter 6

Symplectically degenerate extrema

In this chapter, we will study symplectically degenerate extrema and their geometric characterization. Even though we use a symplectically degenerate minimum in the proof of Theorem 2.2.6, the symplectically degenerate maximum was studied first and used to prove similar results. The existence of a symplectically degenerate minimum was established in Chapter 5 and only enters the proof of Theorem 2.2.6 via the geometric properties discussed here. The discussion of symplectically degenerate maxima follows [Gi2]. We will omit the details of the proof and only point out which details need to be changed. The results on symplectically degenerate minima are also discussed in [He1].

6.1 Symplectically degenerate maxima

In this section, we state some geometric properties of symplectically degenerate maxima, which are defined analogously to symplectically degenerate minima which were already defined in Section 2.2 and will be discussed later.

Definition 6.1.1 *An isolated capped k-periodic orbit $\bar{x}$ of a k-periodic Hamiltonian H is called a symplectically degenerate maximum of H if*

$$\Delta_H(\bar{x}) = 0 \text{ and } \mathrm{HF}_n(H, \bar{x}) \neq 0.$$

Remark 6.1.2 *To prove Theorem 2.2.1 in the case of a closed manifold* M*, both a symplectically degenerate minimum or a symplectically degenerate maximum can be used. In this case, the isomorphism 4.1.5 give both* $\mathrm{HF}_n(H) \cong H_{2n}(M) \neq 0$ *and also* $\mathrm{HF}_{-n}(H) \cong H_0(M) \neq 0$*. The former is used in [Gi2, GG2, He2] to prove the existence of a symplectically degenerate maximum by an analogous argument to one in Chapter 5.*

In the case of the cotangent bundle $M = T^*B$*, the transition to a symplectically degenerate minimum is necessary to prove Theorem 2.2.1. The Floer homology for a Hamiltonian on the cotangent bundle is isomorphic to the homology of the free loop space* $\Lambda_0 B$ *of the base manifold up to a sign change in degree. A symplectically degenerate maximum would have degree* $n > 0$ *in the Floer chain groups and would therefore necessarily be zero in homology, since* $H_{-n}(\Lambda_0 B) = 0$*. Thus we would get the analog of Lemma 5.1.1 in degree* n *for closed manifolds, but not for our class of Hamiltonians on cotangent bundles.*

For the formulation of the geometric characterization of a symplectically degenerate maximum we first need to define the norm of a tensor with respect to a coordinate system. By definition, on a finite-dimensional vector space, the norm $\|v\|_\Xi$ of a tensor v with respect to a coordinate system Ξ is the norm of v with respect to the inner product for which Ξ is an orthonormal basis. For a coordinate system ξ on a manifold M near a point x_0, the natural coordinate basis in $T_{x_0}M$ is denoted by ξ_{x_0}.

Proposition 6.1.3 ([GG2, GG3]) *Let* $\bar{x}$ *be a symplectically degenerate maximum of a Hamiltonian* H *and let* $x_0 = x(0) \in M$*. Then there exists a sequence of contractible loops* γ_i *of Hamiltonian diffeomorphisms such that* $x(t) = \gamma_i^t(x_0)$*, i.e each loop* η_i *sends* x_0 *to* x*. Furthermore, the Hamiltonians* G^i *given by* $\varphi_H^t = \gamma_i^t \circ \varphi_{G^i}^t$ *and the loops* γ_i *satisfy the following conditions:*

(G1) *The point* x_0 *is a strict local maximum of* G_t^i *for* $t \in S^1$*.*

(G2) *There exist symplectic bases* Ξ^i *of* $T_{x_0}M$ *such that*

$$\left\| d^2(G_t^i)_{x_0} \right\|_{\Xi^i} \to 0 \text{ uniformly in } t \in S^1.$$

(G3) *The loop* $\gamma_i^{-1} \circ \gamma_j$ *has identity linearization at* x_0 *for all* i *and* j *(i.e. for all* $t \in S^1$ *we have* $d\big((\gamma_i^t)^{-1} \circ \gamma_j^t\big)_{x_0} = I$*), and is contractible to id in the class of such loops.*

A proof of this proposition and also of the fact that this description is equivalent to the definition of symplectically degenerate maxima can be found in [Gi2, GG3, GG2]. When the concept of a symplectically degenerate maximum was introduced in [Hi] by Hingston and also in the first formal definition in [Gi2], the geometric characterization was used as a definition of symplectically degenerate maxima. It is also shown in [GG3] that the conditions (G1) and (G2) already imply (G3) as a formal consequence.

For the proof of Theorem 2.2.6, we need an additional property of the loops γ_i for our tools to apply.

Remark 6.1.4 *The loops* $\eta_i^{-1} \circ \eta_j$ *are loops of Hamiltonian diffeomorphisms fixing* x_0*. The construction of these loops in [Gi2] shows that the loops* η_i *can be chosen such that* $\eta_i^{-1} \circ \eta_j$ *are supported in an arbitrarily small neighborhood of* x_0*.*

6.2 Symplectically degenerate minima

In this section we prove the geometric properties of symplectically degenerate minimaand show how its geometric properties compare to the properties of symplectically degenerate maxima. The existence of a symplectically degenerate minimum enters the proof of Theorem 2.2.6 only via those properties and was established in Chapter 5.

First, recall from Definition 2.2.4 in Section 2.2 that a symplectically degenerate minimum $\bar{x}$ satisfies

$$\Delta_H(\bar{x}) = 0 \text{ and } \mathrm{HF}_{-n}(H, \bar{x}) \neq 0.$$

Similarly to a symplectically degenerate maximum, also a symplectically degenerate minimum can be characterized by its geometric properties:

Proposition 6.2.1 ([GG3, GG2]) *Let* $\bar{x}$ *be a symplectically degenerate minimum of a Hamiltonian* H *and let* $x_0 = x(0) \in M$*. Then there exists a sequence of* *contractible loops η_i of Hamiltonian diffeomorphisms such that $x(t) = \eta_i^t(x_0)$, i.e each loop η_i sends x_0 to x. Furthermore, the Hamiltonians K^i given by $\varphi_H^t = \eta_i \circ \varphi_{K^i}^t$ and the loops η_i satisfy the following conditions:*

(K1) *the point x_0 is a strict local minimum of K_t^i for $t \in S^1$;*

(K2) *there exist symplectic bases Ξ^i of $T_{x_0}M$ such that*

$$\left\| d^2(K_t^i)_{x_0} \right\|_{\Xi^i} \to 0 \text{ uniformly in } t \in S^1;$$

(K3) *the loop $\eta_i^{-1} \circ \eta_j$ has identity linearization at x_0 for all i and j (i.e. for all $t \in S^1$ we have $d\big((\eta_i^t)^{-1} \circ \eta_j^t\big)_{x_0} = I$), and is contractible to id in the class of such loops.*

The proof of this proposition follows from the analog result of Proposition 6.1.3 for symplectically degenerate maxima. The main point is a modification of the Hamiltonian turning a symplectically degenerate minimum into a symplectically degenerate maximum and back to a minimum.

Proof To prove the proposition, we will consider the Hamiltonian $H^{inv} = -H_t \circ \varphi_H^t$ generating the inverse flow of φ_H^t. The (local) Floer homology of H^{inv} can be calculated from the Floer homology of H, since all one-periodic orbits of H give rise to one-periodic orbits of H^{inv} by reversing the orientation. The Conley-Zehnder index and the action of a periodic orbit of H^{inv} are the negatives of the Conley-Zehnder index and the action of the corresponding periodic orbit of H; and the Floer trajectories are the Floer trajectories of H with reversed orientation on S^1 and s replaced by $-s$.

Let $y(t) = x(-t)$ be the symplectically degenerate minimum traversed in opposite direction. Then y is a 1-periodic orbit of the Hamiltonian H_t^{inv}. As the only difference between the loops x and y is the orientation, we keep the same capping for both capped orbits. The capping is not relevant to the proof and thus is omitted in the notation. The properties of the symplectically degenerate minimum x imply that $\Delta_{H^{inv}}(y) = -\Delta_H(x) = 0$ and $\mathrm{HF}_n(H^{inv}, y) = \mathrm{HF}_{-n}(H, x) \neq 0$. Therefore, y is a symplectically degenerate maximum of H^{inv} and we can use the geometric characterization of symplectically degenerate maxima from [GG3, GG2] and Section 6.1 to construct Hamiltonians G_t^i and loops γ_i^t such that

(i) the point $x_0 = x(0) = y(0)$ is a strict local maximum of G^i_t for $t \in S^1$,

(ii) there exist symplectic bases Ξ^i of $T_{x_0}M$ such that

$$\left\| d^2(G^i_t)_{x_0} \right\|_{\Xi^i} \to 0 \text{ uniformly in } t \in S^1,$$

(iii) the loop $\gamma_i^{-1} \circ \gamma_j$ has identity linearization at x_0 for all i and j and is contractible to id in the class of such loops,

(iv) $\varphi_H^{-t} = \varphi^t_{G^i} \circ \gamma^t_i$.

Now we define the loops η_i and the Hamiltonians K^i by inverting the loops γ_i and the flows of the Hamiltonians G^i, i.e., $\eta^t_i = \gamma_i^{-t}$ and $K^i_t = -G^i_t \circ \varphi^t_{G^i}$. Then we have $\varphi^t_H = \eta_i \circ \varphi^t_{K^i}$, as required in the proposition.

The properties (K1), (K2) and (K3) of the loops η_i and the Hamiltonians K^i follow directly from the properties (G1), (G2) and (G3) of γ_i and G^i with the same coordinate systems Ξ^i of $T_{x_0}M$. □

Remark 6.2.2 *The equation $\varphi_H^{-t} = \varphi^t_{G^i} \circ \gamma^t_i$ in (iv) is a modification of the geometric characterization of symplectically degenerate maxima in [GG3, GG2]. In those papers and in the definition of a symplectically degenerate maximum in [Gi2], the requirement takes the form $\varphi_H^{-t} = \gamma^t_i \circ \varphi^t_{G^i}$. But in the construction of the loops and the Hamiltonians, the order of composition is not crucial, see [Gi2] for details. In the proof of Theorem 2.2.6, we need the order of composition to be $\eta_i \circ \varphi^t_{K^i}$ to ensure that a composition $\eta^t_i \circ \varphi^t_F$ for an autonomous Hamiltonian F is generated by a one-periodic Hamiltonian. This would not necessarily be the case if the order of composition is changed.*

Remark 6.2.3 *The loops $\eta_i^{-1} \circ \eta_j$ are loops of Hamiltonian diffeomorphisms fixing x_0. The construction of the loops γ_i in [Gi2] for the case of a symplectically degenerate maximum shows that the loops γ_i can be chosen such that $\gamma_i^{-1} \circ \gamma_j$ are supported in an arbitrarily small neighborhood of x_0. Hence also the loops $\eta_i = \gamma_i^{-1}$ can be chosen to be supported near x_0. This will be important to apply the direct sum decomposition from Proposition 7.1.1 to a neighborhood of a symplectically degenerate minimum in order to prove the theorems.*

Chapter 7

Direct sum decomposition in filtered Floer homology

In this chapter, we prove the existence of a direct sum decomposition of filtered Floer homology for short action intervals and Hamiltonians of a certain form. This decomposition enables us to localize the problem and to restrict our attention one of the summands. This summand only depends on the behavior of the Hamiltonian on a fixed open set and we can apply methods from the symplectically aspherical case on manifolds in our setting. The construction of this direct sum decomposition is done originally in [He1].

7.1 The direct sum decomposition

In this section, we construct the direct sum decomposition for certain non-degenerate Hamiltonians and discuss in which situations it can be extended to the degenerate case.

To construct this direct sum decomposition, we need the Hamiltonian to be of a certain form and to choose particular open sets. Let K be a non-degenerate Hamiltonian on M. Consider two open sets U and V such that $U \subset V$ and both sets are bounded by level sets of K. On the shell $\bar{V} \setminus U$, assume that the Hamiltonian K is autonomous and does not have one-periodic orbits. In particular, this implies that U and V are homotopy equivalent. At this point, we also fix an almost complex structure J on M, which is compatible with ω.

Recall that the Floer chain groups are spanned by capped one-periodic orbits, i.e. all one-periodic orbits enter the chain groups with multiple equivalence classes of cappings. Consider the splitting of Floer chain groups into the direct sum

$$\mathrm{CF}_*^{(a,\,b)}(K) = \mathrm{CF}_*^{(a,\,b)}(K, U) \oplus \mathrm{CF}_*^{(a,\,b)}(K; M, U), \tag{7.1.1}$$

where the first summand is generated by the one-periodic orbits in U with capping equivalent to a capping contained in U. The second summand is spanned by all the remaining capped orbits.

Proposition 7.1.1 *Let the Hamiltonian K and the open sets U and V be as above. There exists an $\epsilon > 0$, depending only on J, the open sets U and V and on $K|_{V\setminus U}$ such that* (7.1.1) *gives rise to a direct sum decomposition of homology*

$$\mathrm{HF}_*^{(a,\,b)}(K) = \mathrm{HF}_*^{(a,\,b)}(K, U) \oplus \mathrm{HF}_*^{(a,\,b)}(K; M, U) \tag{7.1.2}$$

whenever the action interval $(a,\, b)$ is chosen such that $b - a < \epsilon$.

In the case of a symplectically rational manifold, an analogous direct sum decomposition was proven in [GG2] if K is constant on $V \setminus U$. The proof in the rational case relies on energy bounds for J-holomorphic curves using the rationality constant. In the irrational case in this thesis, we use more general bounds on the energy of Floer trajectories and relax the conditions on the Hamiltonian. To prove Theorem 2.2.6 we are going to apply Proposition 7.1.1 to the functions $H_\pm$ and F.

Remark 7.1.2 *The direct sum decomposition in* (7.1.2) *generalizes the concept of local Floer homology. Indeed, if $\bar{x}$ is an isolated capped periodic orbit with action c, one can find neighborhoods U and V of $\bar{x}$ containing no other periodic orbits. Then for a sufficiently small $\epsilon > 0$, the summand $\mathrm{HF}_*^{(c-\epsilon,\,c+\epsilon)}(K, U)$ is exactly the local Floer homology $\mathrm{HF}_*(K, \bar{x})$ of the Hamiltonian K at $\bar{x}$. Indeed, for sufficiently small non-degenerate perturbations $\tilde{K}$ of K, the periodic orbits of $\tilde{K}$ near $\bar{x}$ have action in the interval $(c - \epsilon,\, c + \epsilon)$ and therefore are in this summand of the Floer homology.*

Remark 7.1.3 *The definition of local Floer homology in Remark 7.1.2 using the direct sum decomposition is fundamentally different from the localization in the original definition of local Floer homology. Here we only fix the Hamiltonian on a shell $V \setminus U$*

between two bounded open sets U and V. Then we use the small action interval, and thus small energy of Floer trajectories, to ensure that the trajectories do not leave V using the energy bounds from Section 7.2.

In the construction of local Floer homology we do not directly restrict the action interval but fix the Hamiltonian outside an open set U that only contains one isolated one-periodic orbit $\bar{x}$. Then we take a small non-degenerate perturbation of the Hamiltonian on U to split this one-periodic orbits up into non-degenerate periodic orbits. The action of those is close to the action of $\bar{x}$ and thus the energy of Floer trajectories connecting them is small. As the Hamiltonian is fixed outside U, this ensures that Floer trajectories between orbits in U stay in U. Then the local Floer homology is defined by restricting the definition of Floer homology to U.

To prove Proposition 7.1.1, we need to show that for such Hamiltonians K no Floer trajectory can connect orbits from different summands, if the action interval is sufficiently small. The key to that is proving the following proposition which provides a lower bound on the energy for those Floer trajectories.

Proposition 7.1.4 *Let K be a non-degenerate Hamiltonian and let U and V be open sets that are both bounded by level sets of K. Assume furthermore that K does not have one-periodic orbits in $\bar{V} \setminus U$ and is autonomous on this shell. Let $u\colon S^1 \times \mathbb{R} \to M$ be a Floer trajectory that intersects ∂U and ∂V. Then there is a constant $\epsilon > 0$, only depending on the open sets U and V, the restriction of the Hamiltonian K and the almost complex structure J to $\bar{V} \setminus U$, such that $E(u) > \epsilon$.*

This bound on the energy of Floer trajectories is sufficient to prove the direct sum decomposition for small action intervals.

Proof [Proof of Proposition 7.1.1] Let $\bar{x}$ and $\bar{y}$ be two capped orbits in $\mathrm{HF}_*^{(a,b)}(K)$. Assume that $\bar{x}$ and $\bar{y}$ are connected by a Floer trajectory u, and let $\bar{x}$ be in $\mathrm{HF}_*^{(a,b)}(K, U)$. We need to show that y is contained in U and the capping of $\bar{y}$ is equivalent to a capping in U.

By construction, V is homotopy equivalent to U and the capping of $\bar{y}$ is equivalent to $u\#$(the capping of $\bar{x}$). Thus it suffices to show that the Floer trajectory u is contained in V. If u did leave V, it would have to intersect both boundary components of $V \setminus U$, as u is converging to the orbit x, which is contained in U.

By Proposition 7.1.4, such a trajectory would have to have energy $E(u) > \epsilon$ for some constant $\epsilon > 0$. Thus, if we choose the action interval (a, b) such that $b - a$ is smaller than the lower bound ϵ in Proposition 7.1.4, the Floer trajectory u has to be contained in V and we have the desired direct sum in homology. □

Remark 7.1.5 *In general, the direct sum decomposition from Proposition 7.1.1 need not be compatible with monotone homotopy maps. In some important cases, however, this is the case. For example, consider two Hamiltonians K^1 and K^2 that agree on $V \setminus U$ up to an additive constant and assume $K^1 \geq K^2$. Then the above direct sum decomposition is compatible with the monotone homotopy map $\mathrm{HF}_*^{(a,\,b)}(K^1) \to \mathrm{HF}_*^{(a,\,b)}(K^2)$. Indeed, the monotone homotopy map is defined using a version of the Floer equation. If the two Hamiltonians agree up to a constant, their Hamiltonian vector fields agree and this equation is exactly the standard Floer equation. Thus Proposition 7.1.4 the above proof of Proposition 7.1.1 also apply in this setting and show that the monotone homotopy map is compatible with the direct sum decomposition for sufficiently small action intervals.*

Corollary 7.1.6 *Let K be any Hamiltonian, which we do not necessarily assume to be non-degenerate. Assume that the open sets U and V are bounded by level sets of K as above. If the Hamiltonian K is autonomous on $V \setminus U$ and does not have periodic orbits in $\bar{V} \setminus U$, then for sufficiently small action interval (a, b) the direct sum decomposition* (7.1.2) *holds.*

Proof It suffices to construct a cofinal set of non-degenerate perturbations of K, such that the direct sum decomposition (7.1.2) holds for all of those Hamiltonians and is compatible with the monotone homotopy maps.

Consider the perturbations that differ from K on $V \setminus U$ only by a constant. These form a cofinal set, since for every perturbation $H \geq K$ we can find a smaller one with that additional property. We can choose these perturbations to be non-degenerate, as K does not have periodic orbits in $\bar{V} \setminus U$ and there are no restrictions on the perturbation outside $\bar{V} \setminus U$. The connecting maps between the Floer homologies of the perturbations are monotone homotopy maps and respect the direct sum decomposition. Thus we also have a direct sum in the limit. □

7.2 Proof of the direct sum decomposition

To prove the proposition, we need to find a lower bound for the energy of Floer trajectories crossing the shell $V \setminus \bar{U}$. The following lemma can be used to bound the time-integral in the expression for the energy away from zero for the part of a Floer trajectory in a compact set not containing one-periodic orbits.

Lemma 7.2.1 *Let W be a bounded open set with smooth boundary and at least two boundary components. Further, let K be an autonomous Hamiltonian on $\bar{W}$. Assume that K is constant on each boundary component and does not have one-periodic orbits in $\bar{W}$.*

Then there exists a constant $C_1 > 0$, depending only on the almost complex structure J, the open set W and K such that:

(i) For $T \leq 1$, any path $\gamma\colon [0,T] \to \bar{W}$, which connects two distinct boundary components of W, satisfies

$$\int_0^T \|\dot{\gamma}(t) - X_K(\gamma(t))\|^2 \; dt > C_1.$$

(ii) Any loop $\gamma\colon S^1 \to \bar{W}$ satisfies

$$\int_{S^1} \|\dot{\gamma}(t) - X_K(\gamma(t))\|^2 \; dt > C_1.$$

This lemma is a generalization of lemmas in [Us2], but the existence of similar lower bounds goes back to [Sa]. The proof given in Section 7.3, however, differs from the proofs in [Sa, Us2]. With $W = V \setminus \bar{U}$, this lemma implies Proposition 7.1.4 if the area of $u^{-1}(W)$ is small. If this area is not small, we need the following lemma to relate the area of the domain and the energy for certain parts of a Floer trajectory.

Lemma 7.2.2 (Usher's lemma) *Let W be a bounded open set with smooth boundary and at least two boundary components and let K be an autonomous Hamiltonian on $\bar{W}$. Let S be a connected subset of the cylinder $S^1 \times \mathbb{R}$ and let $u\colon S \to \bar{W}$ satisfy the Floer equation* (4.1.1) *with Hamiltonian K. Assume that $u(\partial S) \subseteq \partial W$. If $u(S)$ intersects two distinct boundary components of W, then there exists a constant C_2,* *depending only on the domain W, the Hamiltonian K and the complex structure J on W, such that*

$$Area(S) + E(u) \geq C_2.$$

This lemma is a generalization of Lemma 2.3 in [Us2] and also the proof follows the same lines the proof in [Us2]. We include the proof of this lemma in Section 7.4 and continue here with the proof of Proposition 7.1.4. Similarly to the special case in [Us2], both lemmas are applied in the case when W is a shell between two open sets to bound the energy of certain Floer trajectories away from zero.

To prove Proposition 7.1.4, we choose two more open sets U' and V' bounded by level sets of K such that

$$U \subset U' \subset V' \subset V.$$

Denote the loop $t \mapsto u(t, s)$ for fixed s by $\gamma_s(t)$ and consider the set

$$Z = \left\{ s \in \mathbb{R} \mid \gamma_s \text{ intersects } V' \setminus U' \right\}.$$

Then for every $s \in Z$, we either have $\gamma_s \subseteq V \setminus U$ or γ_s intersects one of the boundary components of $V \setminus U$. In the first case, we can apply Lemma 7.2.1 (ii) to the Hamiltonian K and $W = V \setminus \bar{U}$. In the second case, the path γ_s also intersects one of the boundary components of $V' \setminus U'$ and we can apply Lemma 7.2.1 (i) with W taken to be one of the shells $V \setminus V'$ or $U' \setminus U$. Denote by C the minimum of the constants C_1 from Lemma 7.2.1 for the shells $V \setminus U$, $V \setminus V'$ and $U' \setminus U$ and our fixed Hamiltonian K and almost complex structure J.

Then we have the following estimate for the energy of u:

$$\begin{aligned} E(u) &= \int_{\mathbb{R}} \int_{S^1} \|\partial_s u\|^2 \; dt \; ds \\ &\geq \int_Z \int_{S^1} \|\partial_t u(s, t) - X_K(u(s, t))\|^2 \; dt \; ds \\ &\geq \int_Z C \; ds = C \; m_{Leb}(Z). \end{aligned}$$

If $m_{Leb}(Z) \geq C_2/2$, where C_2 is the constant from Lemma 7.2.2 for the shell $W = V' \setminus U'$, we have found a lower bound $CC_2/2$ for the energy of u.

If $m_{Leb}(Z) < C_2/2$, we want to use Lemma 7.2.2. To do so, we choose S as one connected component of $u^{-1}(V' \setminus U')$, such that $u(S)$ intersects both boundary components. Since u intersects both ∂U and ∂V, such a set S exists and $S \subseteq Z \times S^1$. Then we have $Area(S) \leq m_{Leb}(Z) \leq C_2/2$ and $u(\partial S) \subseteq \partial(V' \setminus U')$. Now Lemma 7.2.2 applies with $W = V' \setminus \bar{U}'$ and we find that

$$E(u) \geq E(u|_S) \geq C_2 - Area(S) \geq C_2/2.$$

Thus with $\epsilon = \min\{CC_2/2, C_2/2\}$ we have found a lower bound for the energy in either case.

7.3 Proof of Lemma 7.2.1

To find lower bounds for the integrals in question, we first use the Schwarz inequality to get

$$\int_0^T \|\dot{\gamma}(t) - X_K(\gamma(t))\|^2 \, dt \quad \geq \quad \left(\int_0^T \|\dot{\gamma}(t) - X_K(\gamma(t))\| \, dt\right)^2$$

for $T \leq 1$ and it suffices to find a lower bound for the L^1-norm.

To that end, for a path $\gamma(t)$ in $\bar{W}$, we define the path $\eta(t) = \varphi_K^{-t}(\gamma(t))$. By the chain rule we have

$$\begin{aligned}\dot{\gamma}(t) &= d\varphi_K^t(\eta(t))\dot{\eta}(t) + \left(\frac{d}{dt}\varphi_K^t\right)(\eta(t)) \\ &= d\varphi_K^t(\eta(t))\dot{\eta}(t) + X_K(\gamma(t)).\end{aligned}$$

Recall for part (i) that we assume K to be autonomous and constant on the boundary components of W. The two boundary components of W are thus preserved under the flow. Since γ connects two distinct boundary components of W, the same is true for η. Denote the distance of these boundary components with respect to the metric given by ω and J by δ. Then we find the desired lower bound by the following calculation:

$$\begin{aligned}\int_0^T \|\dot{\gamma}(t) - X_K(\gamma(t))\| \, dt &= \int_0^T \|d\varphi_K^t(\eta(t))\dot{\eta}(t)\| \, dt \\ &> c \cdot \int_0^T \|\dot{\eta}(t)\| \, dt \\ &\geq c \cdot d(\eta(0), \eta(T)) \\ &\geq c \cdot \delta.\end{aligned}$$

The constant c is positive since K is smooth and both t and $\eta(t)$ are varying in compact sets and K has no critical points in $\bar{W}$.

Similarly, we find for part (ii)

$$\begin{aligned}\int_{S^1} \|\dot{\gamma}(t) - X_K(\gamma(t))\| \ dt &= \int_0^1 \|d\varphi_K^t(\eta(t))\dot{\eta}(t)\| \ dt \\ &\geq c \cdot d(\eta(0), \eta(1)) \\ &= c \cdot d\big(\gamma(0), \varphi_K^{-1}(\gamma(0))\big).\end{aligned}$$

As $\bar{W}$ is compact and φ_K is continuous with no one-periodic orbits in $\bar{W}$, this distance is bounded away from zero.

Thus in both parts we have found a lower bound and we set C_1 to be the minimum of those bounds.

7.4 Proof of Usher's lemma

Recall that we want to prove the existence of a lower bound for the sum of the energy and the area of the domain of a Floer trajectory u, i.e., for $Area(S)+E(u)$, which is independent of u. For simplicity of notation, we assume that W has exactly two boundary components. Since u is a solution of the Floer equation and S is a subset of the cylinder $S^1 \times \mathbb{R}$, the graph $\tilde{u}\colon S \to S^1 \times \mathbb{R} \times \bar{W}$ is a $\tilde{J}$-holomorphic curve for a certain almost complex structure $\tilde{J}$ which is tamed by $\tilde{\omega} = ds \wedge dt - dt \wedge dK + \omega$. (For the precise definition of $\tilde{J}$, see the proof of Lemma 2.3 in [Us2]). This almost complex structure $\tilde{J}$ depends only on the almost complex structure J on $\bar{W}$ and on the Hamiltonian K on $\bar{W}$.

For any subset $S' \subset S$, this definition of $\tilde{\omega}$ gives

$$\int_{S'} \tilde{u}^* \tilde{\omega} = \int_{S'} ds \wedge dt + \int_{S'} \left|\frac{\partial u}{\partial s}\right|_{J_t}^2 \ ds\ dt = Area(S') + E(u|_{S'}).$$

Let Σ be a closed hypersurface in W which separates the two boundary components. By assumption, there is a $z_0 \in S$ such that $u_0 = u(z_0) \in \Sigma$. Let D be a disk centered at z_0 with radius independent of all other choices, say, e.g., 1/3. Also choose a ball B centered at u_0 such that $B \subseteq W$.

Now we consider a ball $\tilde{B}$ centered at (z_0, u_0) and contained in $D \times B$. Since the radius of D is fixed, the radius of this ball $\tilde{B}$ depends only on the radius of B and thus only on the open set W. Then we define

$$\tilde{S} = \{z \in S \mid \tilde{u}(z) \in \tilde{B}\}.$$

By definition, the boundary of $\tilde{S}$ is mapped to the boundary of $\tilde{B}$. Indeed, $\tilde{u}(\partial S)$ is contained in $(\partial S) \times (\partial W)$ and therefore not in $\tilde{B} \subseteq D \times B$.

Let us now view the graph of u as a map $\tilde{u}\colon \tilde{S} \to \tilde{B}$. As B is contained in W, where K is fixed, the almost complex structure $\tilde{J}$ on $D \times B$ depends only on $K|_W$ and the complex structure J on B. The almost complex structure J depends on the ball B and therefore on the point u_0.

By definition of $\tilde{S}$, the center $(z_0, u_0) = \tilde{u}(z_0)$ of the ball $\tilde{B}$ is contained in the image of $\tilde{u}$. Now $\tilde{u}$ is considered to be a $\tilde{J}$-holomorphic curve in $\tilde{B}$, which is passing through the center $\tilde{u}(z_0)$ and has no boundary in the interior of $\tilde{B}$. For such $\tilde{u}$, Proposition 4.3.1(ii) in [Si] applies and we have $Area(\tilde{S}) + E(u|_{\tilde{S}}) \geq C_2(u_0)$.

This constant $C_2(u_0)$ still depends on u, since the choice of u_0 depends on u. To achieve a constant that does not depend on u, we take the infimum over all $u_0 \in \Sigma$ and define

$$C_2 = \inf\{C_2(u_0) \mid u_0 \in \Sigma\}.$$

Since the hypersurface Σ is compact, this constant C_2 is positive and independent of u. With this choice of C_2, we have the desired result that $Area(S) + E(u) \geq Area(\tilde{S}) + E(u|_{\tilde{S}}) \geq C_2 > 0$.

Chapter 8

Proof of Theorem 2.2.6

The proof of this theorem relies on the original proof of the Conley conjecture in the case of a symplectically aspherical manifold in [Gi2]. Since we do not restrict the properties of the symplectic form, the computations in the proof cannot be done explicitly in our settings. To apply the computation from the aspherical case in our setting, we use the direct sum decomposition from Chapter 7 to localize the problem. Then Darboux theorem removes the dependence on the ambient manifold and we can apply tools from more restrictive cases. For the direct sum decomposition to hold, we need to change some details in the construction of auxiliary Hamiltonians. Furthermore, we use a symplectically degenerate minimum in contrast to previous work using a symplectically degenerate maximum. This change also requires some modifications in the proof.

8.1 Outline of the proof

The keys to proving Theorem 2.2.6 are the geometrical description of symplectically degenerate minima given in Proposition 6.2.1 and the direct sum decomposition from Chapter 7. In particular, we can assume the symplectically degenerate minimum to be a constant orbit x_0 with trivial capping and omit the capping in the notation from now on. Furthermore, we can assume that x_0 is a strict local minimum of H and that H has arbitrarily small Hessian at x_0.

Then we use the squeezing method from [BPS, Gi2, GG1] and construct Hamiltonians H_+ and H_- such that $H_- < H < H_+$. It suffices to show that a linear homotopy from H_+ to H_- induces a non-zero map between the filtered Floer homology groups of $H_\pm$ for the action interval in question. This map is independent of the choice of the monotone decreasing homotopy. Since $H_- < H < H_+$, the homotopy can be chosen to pass through the Hamiltonian H and thus also the homotopy map factors through the filtered Floer homology group of H, which can therefore not be trivial.

Similarly to the construction in [Gi2, GG2], the functions $H_\pm$ are radially symmetric and centered at the symplectically degenerate minimum; see Section 8.2 for details. The only difference to the construction in [Gi2, GG2] is that we need these functions to satisfy the requirements for the direct sum decomposition from Proposition 7.1.1 to apply. For the cotangent bundle case, we also choose the functions to satisfy the growth condition at infinity.

To prove that the monotone homotopy map is non-zero, we use the direct sum decomposition proved in Chapter 7. It suffices to show that the restriction to one of the summands is an isomorphism. The considered summand $\mathrm{HF}_*(H_\pm, U)$ is defined using a small neighborhood U of the symplectically degenerate minimum x_0. This summand depends only on the restriction of the functions $H_\pm$ to U and the symplectic structure in U and is independent of the ambient manifold. By Darboux theorem for symplectic manifolds, we can view U as an open set in any symplectic manifold of dimension $2n$ and the theorem follows as in the closed, symplectically aspherical case in [Gi2].

8.2 The functions H_+ and H_-

By the geometric characterization of the symplectically degenerate minimum in Proposition 6.2.1 above, it suffices to prove the theorem for the function K^1 and the constant orbit x_0 as symplectically degenerate minimum. We keep the notation H for K^1. Fix a neighborhood W of x_0 such that x_0 is a strict global minimum of H on W and that there exist Darboux coordinates for M in W. On the cotangent bundle, we choose W such that $\|p\| \leq C$ in W for some possibly very large constant $C > 0$. We also fix now an almost complex structure J on M that is compatible with ω.

Let U and V be balls centered at x_0 and contained in W. We then construct the function H_- and an auxiliary function F to be of the form shown in Figure 8.2. The construction differs slightly in the two different settings. In the case of a closed manifold, we can choose both functions to be constant outside a neighborhood of the symplectically degenerate minimum. In the case of a cotangent bundle, we need these functions to be in our class of Hamiltonian and satisfy the growth condition at infinity.

To construct the Hamiltonian H_- near the symplectically degenerate minimum x_0, we fix balls

$$B_{r_-} \subset B_{r_+} \subset B_r \subset U \subset V \subset B_R \subset B_{R_-} \subset B_{R_+} \Subset W$$

centered at x_0. In W, the function H_- takes the following form as a function of the distance from x_0:

- $H_- \geq H$ and $H_- \equiv c = H(x_0)$ on B_{r_-};
- on $B_{r_+} \setminus B_{r_-}$ the function H_- is monotone increasing;
- on $B_r \setminus B_{r_+}$ the function is constant;
- in the shell $B_R \setminus B_r$ the function is monotone decreasing, linear as a function of the square of the distance from x_0 with small slope α on $V \setminus U$ such that there are no one-periodic orbits in $V \setminus B_r$;
- the function H_- is again constant on $B_{R_-} \setminus B_R$ with a value less than c;
- it is monotone decreasing on $B_{R_+} \setminus B_{R_-}$;
- outside B_{R_+}, the function H_- is constant and equal to its minimum.

Outside W, we define the function H_- differently in our two different settings. For the case of a closed manifold, we take H_- to be constant outside W.

On the cotangent bundle, the function H_- also has to meet the requirements (H1) and (H2). This can be realized by defining H_- to be a positive, non-degenerate quadratic form for $\|p\| \geq C$ i.e. we choose H_- to satisfy (2.1.1) and thus conditions (H1) and (H2). The coefficients of the quadratic form are chosen such that we have $H_- \leq H$ on M. Thus H_- is of the general form mentioned in Example 2.1.2.

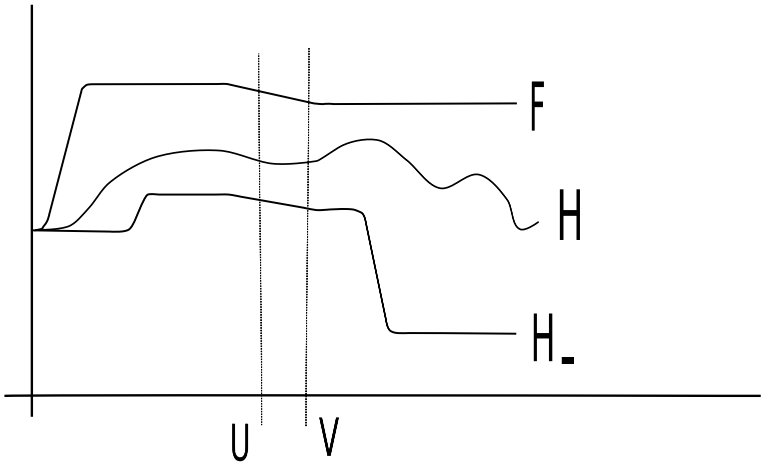

Figure 8.1: The functions H_- and F as functions of the distance from x_0.

This function is constructed very similarly to the ones used in [Gi2, GG2] near a symplectically degenerate maximum. More concretely, up to an additive constant (and the quadratic growth condition in the cotangent bundle case), this is the negative of the function H_+ used in [He2], since we are using a symplectically degenerate minimum here. See also Section 8.3 for details about the choices made in the construction of H_-.

Let us now turn to the construction of H_+, which is again constructed similarly to the function H_- in [Gi2] for the case of a symplectically degenerate maximum. This is the point where we use the existence of a symplectically degenerate minimum and its geometric characterization discussed in Section 6.2. The geometric characterization of symplectically degenerate minima in Proposition 6.2.1 and Remark 6.2.3 imply that there is

- a loop η^t of Hamiltonian diffeomorphisms fixing x_0, which is supported in U and
- a system of coordinates ξ on a neighborhood W of x_0

such that the Hamiltonian K generating the flow $\eta^{-t} \circ \varphi_H^t$ has a strict local minimum at x_0 and $\max_t \left\| d^2(K_t)_{x_0} \right\|_\xi$ is sufficiently small. The loop η is contractible in the class of loops having identity linearization at x_0, i.e. there exists a homotopy of loops of Hamiltonian diffeomorphisms with identity linearization at x_0 between η and a constant loop. Let G_s^t be a Hamiltonian generating such a homotopy η_s^t normalized by $G_s^t(x_0) \equiv 0$. We then normalize K by the additional requirement that $K_t(x_0) \equiv c$ (or equivalently that $H = K\#G^0$).

By the characterization of the symplectically degenerate minimum, there exists a function F, depending on the coordinate system ξ, such that

- $\left\|d^2 F_{x_0}\right\|_\xi$ is sufficiently small,
- $F \geq K$ and $F(x_0) = c = H(x_0)$ is the global minimum of F.

To be more precise, in B_r the function F is the negative of a bump function centered at x_0. Furthermore, F is chosen to differ from H_- only by a constant in $V \setminus U$. The last condition is needed to have the direct sum decomposition in Proposition 7.1.1 below for the filtered Floer homology groups of both H_- and F and to ensure that these decompositions are compatible with the monotone homotopy map for a linear homotopy from H_- to F; see Remark 7.1.5.

Outside B_R, we choose F similar to H_-. In the case of a closed manifold, we can define F to be constant outside B_R. For the cotangent bundle case, we define F to be constant outside B_R for $\|p\| \leq C$. For $\|p\| > C$, we define F to be of the form described in Example 2.1.2, i.e., a positive, non-degenerate quadratic form, to ensure that F satisfies the growth conditions (H1) and (H2). Relevant for the proof of Theorem 2.2.6 is mostly the shape of the functions H_- and F inside W and the inequality $H_- < H < H_+$ between the Hamiltonians, but not the concrete definition of $H_\pm$ outside W. The coefficients of the quadratic form are fixed now in such a way that all functions of the homotopy $F^s = G^s\#F$ satisfies $F^s \geq H_-$ for all s and $F^1 = F \geq K$.

This is an isospectral homotopy, since the homotopy G^s is normalized to be of the form discussed after Definition 4.2.1. We define the function H_+ by

$$H_+ := G^0\#F \geq G^0\#K = H.$$

We have chosen the loop η to be supported in U to ensure that the function G^0 is constant outside U. This implies that H_+ differs from F only by the constant value of G^0 on $\bar{V} \setminus U$. Thus H_+ also differs only by a constant from H_- on this set by the definition of F. Therefore we also have the direct sum decomposition from Proposition 7.1.1 for H_+. It is compatible with the homomorphism induced by the homotopy F^s and the monotone homotopy map for a homotopy from H_+ to H_-.

8.3 The Floer homology of $H_\pm$ and the monotone homotopy map

Before we go into detail on the Floer homology of the functions $H_\pm$, we need to specify the choices made in the construction of these functions. The order in which the constants are chosen is important to guarantee the conditions in all auxiliary propositions. To be more precise, we first choose some small constant $\alpha_0 > 0$ such that α_0/π is irrational. Then we fix the Hamiltonian H_- on B_r and pick $\epsilon > 0$ smaller than the energy bound from Proposition 7.1.1 for a Hamiltonian, which is linear in the radius with slope α_0 on $V \setminus U$. Using these choices, we take a sufficiently large order of iteration k as in [Gi2, GG2]. Furthermore, we now fix H_- outside B_r with slope $\alpha = \alpha_0/k$ on $V \setminus U$. We thus have the direct sum decomposition of filtered Floer homology by Proposition 7.1.1 for $H_-^{(k)}$. At this point we choose some $\delta_k \in (0, \epsilon/2)$, depending on the order of iteration k, to ensure that the action intervals $(kc + \delta_k,\, kc + \epsilon)$ and $(kc - \delta_k,\, kc + \delta_k)$ are small enough for the direct sum decomposition (7.1.2) to exist.

Thus Proposition 7.1.1 applies to the functions $H_\pm$ and F and we can restrict ourselves to the summand of the Floer homology containing the orbits in U. This direct sum decomposition is also constructed to be compatible with all monotone homotopy maps arising in the proof. Since U is contained in a Darboux neighborhood of x_0, this summand does only depend on the restriction of the function to U and is independent of the ambient manifold.

For the calculation of this part of the Floer homology groups, we refer the reader to [Gi2, GG3] as the functions and homotopies used here are very similar to the constructions used in these papers. Namely, the functions H_- and F in this thesis are autonomous and generate the inverses of the flows of the Hamiltonians H_+ and F used in [Gi2, He2], hence they are just the negatives of those functions. The Floer equation (4.1.1) is the same equation as for the functions in [Gi2] with the orientation in both coordinates of $S^1 \times \mathbb{R}$ reversed and the negative Hamiltonians. Thus the numbers of connecting Floer trajectories are equal.

Similarly, the homotopy from F to H_- here is the negative of the homotopy from H_+ to F in [Gi2] and we obtain an isomorphism of Floer homology groups

$$\mathbb{Z}_2 \cong \mathrm{HF}_{-n-1}^{(kc-\epsilon,\, kc-\delta_k)}(F^{(k)}, U) \to \mathrm{HF}_{-n-1}^{(kc-\epsilon,\, kc+\delta_k)}(H_-^{(k)}, U) \cong \mathbb{Z}_2$$

by the same argument.

Since the homotopy F^s is isospectral with $F^s \geq H_-$ and for this summand, the filtered Floer homology does not depend on the manifold, we can use (4.2.3) for iterations of the Hamiltonians F and $H_\pm$. As in the symplectically aspherical case in [Gi2], we have the commutative diagram

$$\begin{array}{ccc}
\mathrm{HF}_{-n-1}^{(kc-\epsilon,\, kc-\delta_k)}(F^{(k)}, U) & \xrightarrow{\cong} & \mathrm{HF}_{-n-1}^{(kc-\epsilon,\, kc-\delta_k)}(H_+^{(k)}, U) \\
& \searrow^{\cong} & \downarrow \Psi \\
& & \mathrm{HF}_{-n-1}^{(kc-\epsilon,\, kc-\delta_k)}(H_-^{(k)}, U)
\end{array}$$

for this summand, where the horizontal map is induced by the isospectral homotopy F^s and the other maps are monotone homotopy maps. The diagonal map is an isomorphism by the same argument as in [Gi2] using the long exact sequence (4.2.1) of filtered Floer homology to go over to the action interval $(kc - \delta_k, kc + \delta_k)$. By the commutativity of this diagram, the map Ψ is also an isomorphism of this summand if the filtered Floer homology groups. This implies that the monotone homotopy map

$$\mathrm{HF}_{-n-1}^{(kc-\epsilon,\, kc-\delta_k)}(H_+^{(k)}) \to \mathrm{HF}_{-n-1}^{(kc-\epsilon,\, kc-\delta_k)}(H_-^{(k)})$$

for the complete filtered Floer homology is non-zero. This map factors through the Floer homology group of H, which we want to show to be non-trivial. This proves Theorem 2.2.6 and thus also completes the proofs of Theorems 2.2.1 and 2.2.5.

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