Title

Tribute to Krzysztof Wilmanski

Journal

Acta Mechanica, 225(8)

ISSN

0001-5970

Authors

Albers, Bettina

Bauer, Erich

Goddard, Joe D

et al.

Publication Date

2014-08-01

DOI

10.1007/s00707-014-1121-5

Peer reviewed

In "Continuous Media with Microstructure", B. Albers (ed.), Springer Verlag, Heidelberg, pp. 83-89, 2010.

A Note on the Representation of Cosserat Rotation

J.D. Goddard

Abstract. This brief article provides an independent derivation of a formula given by Kafadar and Eringen (1971) connecting two distinct Cosserat spins. The first of these, the *logarithmic spin* represents the time rate of change of the vector defining finite Cosserat rotation, whereas the second, the *instantaneous spin*, gives the local angular velocity representing the infinitesimal generator of that rotation. While the formula of Kadafar and Eringen has since been identified by Iserles et al. (2000) as the differential of the Lie-group exponential, the present work provides an independent derivation based on quaternions. As such, it serves to bring together certain scattered results on quaternionic algebra, which is currently employed as a computational tool for representing rigid-body rotation in various branches of physics, structural and robotic dynamics, and computer graphics.

1 Background: Cosserat Rotations

From the conventional continuum-mechanical viewpoint, a Cosserat continuum[1] is defined via a differentiable map assigning spatial position $\mathbf{x}(\mathbf{x}^\circ,t)$ and microstructural rotation $\mathbf{P}(\mathbf{x}^\circ,t)$ to each material particle, with $\mathbf{x}=\mathbf{x}^\circ$, $\mathbf{P}=\mathbf{I}$ in a given reference configuration, where $\mathbf{P}\in\mathbf{SO}(3)$ denotes a real, proper orthogonal tensor.

We can express the kinematics concisely in terms of the map $\mathbb{R}^3\to\mathbb{R}^3\times\mathbf{SO}(3)$ given by

$$\begin{aligned}&\mathbf{x}^\circ\to\{\mathbf{x},\theta\},\ \text{where}\ \boldsymbol{\theta}=-\frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{\varepsilon}:\boldsymbol{\Theta}\,,\ \text{and}\ \boldsymbol{\Theta}=-\boldsymbol{\varepsilon}\cdot\boldsymbol{\theta},\\ &\text{i.e.}\quad \theta_i=-\frac{1}{2}\varepsilon_{ijk}\Theta^{jk}\,,\ \text{and}\ \Theta_{ij}=-\varepsilon_{ijk}\theta^k,\end{aligned}\tag{1}$$

J.D. Goddard
Department of Mechanical and Aerospace Engineering 0411, University of California, San Diego, 9500 Gilman Dr., La Jolla, CA, 92093-0411, USA
e-mail: jgoddard@ucsd.edu

[1] As noted by numerous authors, e.g. [1, 2, 3, 4], this is a special case of Eringen's *microstretch* continuum [5], which involves both rotation and dilatation.

and the *Cayley-Gibbs-Rodrigues* relation [2, 6, 7, 8, 9, 10]

$$\mathbf{P} = \exp\boldsymbol{\Theta} = \mathbf{I} + \left(\frac{\sin\vartheta}{\vartheta}\right)\boldsymbol{\Theta} + \left(\frac{1-\cos\vartheta}{\vartheta^2}\right)\boldsymbol{\Theta}^2, \;\; \vartheta = \{-\mathrm{tr}(\boldsymbol{\Theta}^2)/2\}^{1/2} \qquad (2)$$

Here $\hat{\boldsymbol{\theta}} = \boldsymbol{\theta}/|\boldsymbol{\theta}|$ represents the axis of rotation and $\vartheta = |\boldsymbol{\theta}| = (\theta^i\theta_i)^{1/2}$ the angle of rotation about the axis.

If $\mathbf{P}$ is taken as primary variable, the skew-symmetric tensor $\boldsymbol{\Theta} = \log\mathbf{P} \in \mathfrak{so}(3)$ (Lie algebra) represents an inverse of the map $\mathfrak{so}(3) \to \mathbf{SO}(3)$ (Lie group), and it can be defined uniquely and computed by various methods [11]. Alternatively, and more conveniently, we may regard $\boldsymbol{\theta}$ as the primary variable, with (1) defining the associated map or *Cosserat placement* $\mathbb{R}^3 \to \mathbb{R}^6$.

To connect the vector of the *logarithmic spin* $\boldsymbol{\Omega} = d\boldsymbol{\Theta}/dt$ to that of the *instantaneous spin* $\mathbf{N} = (d\mathbf{P}/dt)\mathbf{P}^{\mathrm{T}}$, where

$$\frac{d}{dt} := \left(\frac{\partial}{\partial t}\right)_{\mathbf{x}^\circ},$$

we recall the rather remarkable result of Kafadar and Eringen [5](Eqs.(2)-(9)), which can be expressed in the present notation as:

$$\boldsymbol{\nu} := -\frac{1}{2}\boldsymbol{\varepsilon} : \mathbf{N} = \boldsymbol{\Lambda}\boldsymbol{\omega}, \;\; \text{where} \;\; \boldsymbol{\omega} = -\frac{1}{2}\boldsymbol{\varepsilon} : \boldsymbol{\Omega},$$

with

$$\boldsymbol{\Lambda} = \mathbf{I} + \left(\frac{1-\cos\vartheta}{\vartheta^2}\right)\boldsymbol{\Theta} + \left(\frac{\vartheta-\sin\vartheta}{\vartheta^3}\right)\boldsymbol{\Theta}^2, \qquad (3)$$

and

$$\boldsymbol{\Lambda}^{-1} = \mathbf{I} - \frac{1}{2}\boldsymbol{\Theta} + \frac{1}{\vartheta^2}\left(1 - \frac{\vartheta}{2}\cot\frac{\vartheta}{2}\right)\boldsymbol{\Theta}^2$$

Either definition of spin is acceptable, and this relation makes it easy to relate their conjugate stresses.

Kafadar and Eringen [5] derive a formulas equivalent to (3), and Iserle et al. [8](Eqs. B.10-B.11) later have given them as differentials of the Lie-group exponential, cf. [7](Eqs. 17-19). The purpose of the present article is to give an independent derivation by means of the quaternionic representation of the Lie-algebra/Lie-group connection $\mathbf{SO}(3) = \exp\{\mathfrak{so}(3)\}$. It is hoped that this derivation will clarify certain relations between quaternions, matrix algebra and Lie groups.

2 Quaternions as Tensors

With no claim to originality, the object here is to provide a concise summary of scattered results from numerous treatises on quaternions, many of which are presented under a different guise elsewhere, e.g. in the much more comprehensive journal article [12]. The knowledgeable reader can skip to the following section for the derivation of the main result.

Hamilton's [13] quaternions represent a special case of the non-commutative hypercomplex (Clifford) algebras, which are known to be isomorphic to matrix algebras [14][2]. With this in mind, it is convenient for the present purposes to adopt the tensorial representation of quaternions:

$$\begin{aligned}\mathbb{Z} = z^i\mathbb{E}_i = z^0\mathbb{E}_0 + z^1\mathbb{E}_1 + z^2\mathbb{E}_2 + z^3\mathbb{E}_3 \;&\doteq\; \begin{pmatrix} z^0 & -z^1 & -z^2 & -z^3 \\ z^1 & z^0 & z^3 & -z^2 \\ z^2 & -z^3 & z^0 & z^1 \\ z^3 & z^2 & -z^1 & z^0 \end{pmatrix} \\ &= \begin{pmatrix} z^0\mathbb{I}+z^1\mathbb{J} & -z^2\mathbb{I}+z^3\mathbb{J} \\ z^2\mathbb{I}+z^3\mathbb{J} & z^0\mathbb{I}-z^1\mathbb{J} \end{pmatrix}, \qquad (4)\end{aligned}$$

$$\text{with} \quad \mathbb{I} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \;\; \mathbb{J} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \;\; \text{and} \;\; \mathbb{J}^2 = -\mathbb{I},$$

which represents a 4-dimensional subspace $\mathfrak{Q}$ of the 7-dimensional vector space consisting of the sum of skew-symmetric and isotropic 4-tensors. The z^i are real, and $\doteq$ indicates the matrix of tensor components relative to given basis $\mathbf{e}_i,\ i = 0\ldots,3$. Superscripts on z^i allow for curvilinear tensors, and the Einstein summation convention is observed here and in the following[3]. The 2×2 *identity* matrix $\mathbb{I}$ and *symplectic* matrix $\mathbb{J}$ are to be interpreted as belonging to the appropriate blocks of the 4×4 matrix, and the replacement $\mathbb{I}\to 1$, $\mathbb{J}\to\imath$ establishes an isomorphism with the algebra of 2×2 complex matrices[4]. The basis elements $\mathbb{E}_i$, which are obtained from (4) by taking $z^j = \delta_i^j$, satisfy

$$\mathbb{E}_i^2 = \begin{cases} \mathbb{E}_0, \ i = 0, \\ -\mathbb{E}_0, \ i \neq 0 \end{cases} \quad \text{and} \quad \mathbb{E}_i\mathbb{E}_j = \begin{cases} \varepsilon_{ij}{}^k\mathbb{E}_k, \ \text{for } i,j,k \neq 0, \\ \mathbb{E}_i, \ \text{for } j = 0, \\ \mathbb{E}_j, \ \text{for } i = 0, \end{cases} \qquad (5)$$

which yields the well-known product rules for general quaternions. The *deviator* (or "vector part") $\mathbb{Z}'$, *conjugate* $\mathbb{Z}^*$, *modulus* $|\mathbb{Z}|$, and *inverse* of a general quaternion $\mathbb{Z}$ are defined respectively by

$$\begin{aligned} &\mathbb{Z}' := \mathrm{Dev}(\mathbb{Z}) = \mathbb{Z}|_{z^0=0}, \ \text{with } \mathbb{Z} = z^0\mathbb{E}_0 + \mathbb{Z}', \ \mathbb{Z}^* = z^0\mathbb{E}_0 - \mathbb{Z}', \\ &|\mathbb{Z}| = \frac{1}{2}\left[\mathrm{tr}(\mathbb{Z}\mathbb{Z}^*)\right]^{1/2}, \ \text{and } \mathbb{Z}^{-1} = \mathbb{Z}^*/|\mathbb{Z}|^2, \end{aligned} \qquad (6)$$

As discussed below, *unitary* quaternions, defined by $\mathbb{Q}^{-1} = \mathbb{Q}^*$, represent spatial rotations.

[2] Given this fact, the pure mathematician might wish to be excused from further reading of the present work.

[3] However, the z^i in (4) do not obey the standard tensor-transformation rules unless interpreted in terms of 4-vectors.

[4] In that case, the $\mathbb{E}_i,\ i = 1,2,3$, are identical with the well-known *Pauli spin matrices* [12], up to permutation and multiplication by $\pm\imath$.

With matrix scalar product for matrices $\mathbb{A},\mathbb{B}$ defined by $\mathbb{A}:\mathbb{B}=\mathrm{tr}(\mathbb{A}^*\mathbb{B})$, it is clear from the preceding relations that the $\mathbb{E}_i$ represent an orthogonal basis, whose reciprocal basis is $\mathbb{E}^i=\frac{1}{4}\mathbb{E}_i$. Therefore, the transformation of $\mathfrak{Q}$ into the space of 4-vectors with basis $\mathbf{e}_i$ is given by

$$\boldsymbol{\gamma}=\mathbf{e}_i\otimes\mathbb{E}^i \text{ with } \mathbf{z}=\boldsymbol{\gamma}:\mathbb{Z}, \text{ i.e. } \gamma^j{}_{kl}=(\mathbb{E}^j)_{kl} \text{ with } z^j=\gamma^{jkl}(\mathbb{Z})_{kl}, \tag{7}$$

and the inverse $\mathbb{Z}=\boldsymbol{\gamma}^{-1}\cdot\mathbf{z}$ is obviously given by $(\boldsymbol{\gamma}^{-1})_{klj}=(\mathbb{E}_j)_{kl}$.

The projection $\mathfrak{Q}\to\mathfrak{so}(3)$

$$\mathbf{S}'=\mathrm{dev}(\mathbf{S}), \text{ with } \mathbf{S}=\boldsymbol{\Pi}\mathbb{S}, \ \boldsymbol{\Pi}=\mathbb{E}_0-\mathbf{e}_0\otimes\mathbf{e}_0 \tag{8}$$

provides a connection between the Lie algebra $\mathfrak{so}(3)$ and the algebra of quaternions, as defined by the preceding matrix representations. Thus, given a 4-vector $\mathbf{x}$ with quaternion $\mathbb{X}=\boldsymbol{\gamma}^{-1}\cdot\mathbf{x}$, one obtains from it the 3-vector ("physical-space" vector) $\mathbf{x}'$:

$$\mathbf{x}'=\boldsymbol{\gamma}:\mathbb{X}', \text{with } \mathbf{x}=x^0\mathbf{e}_0+\mathbf{x}', \text{ and } \mathbf{x}^*=x^0\mathbf{e}_0-\mathbf{x}', \tag{9}$$

Then, the quaterionic product[5] is given in terms of 3-vector operations as:

$$\begin{aligned}\mathbf{xy}:=\boldsymbol{\gamma}:(\mathbb{X}\,\mathbb{Y})&=x^0\mathbf{y}'+y^0\mathbf{x}'+(x^0y^0-\mathbf{x}'\cdot\mathbf{y}')\mathbf{e}_0+\mathbf{x}'\times\mathbf{y}',\\ \text{with } \mathbf{x}'\times\mathbf{y}'&=\frac{1}{2}(\mathbf{x}'\mathbf{y}'-\mathbf{y}'\mathbf{x}'),\end{aligned} \tag{10}$$

which, with the proviso that the vector space be enlarged to 4-vectors, adds a new operation to the usual 3-vector operations.

Without loss of generality, we employ the usual quaterionic convention $\mathbf{e}_0\equiv 1$, and we make a distinction between a quaternion and its 4-vector only when necessary to clarify tensor-transformation formulae. Hence, letting lower-case bold Greek refer either to 3-vectors or 3rd-rank tensors, we have the well-known polar or exponential representation (cf. e.g. [12])

$$\begin{aligned}&\mathbf{z}=\rho e^{\boldsymbol{\phi}}=\rho(\cos\varphi+\hat{\boldsymbol{\phi}}\sin\varphi),\\ \text{where}\quad &\varphi=|\boldsymbol{\phi}|,\ \rho=|\mathbf{z}|,\ \boldsymbol{\phi}=\boldsymbol{\phi}',\ \hat{\boldsymbol{\phi}}=\boldsymbol{\phi}/\varphi=-\hat{\boldsymbol{\phi}}^*,\end{aligned} \tag{11}$$

which remains valid when the pair $\mathbf{z},\boldsymbol{\phi}$ is replaced by the corresponding quaternions $\mathbb{Z}=\boldsymbol{\gamma}^{-1}\cdot\mathbf{z}$ and $\mathbb{F}=\boldsymbol{\gamma}^{-1}\cdot\boldsymbol{\phi}$. The same formula serves to define the logarithm and its various branches.

3 Application to Cosserat Rotations

The important special case $\rho=1$ of (11) gives a unitary quaternion $\mathbf{q}=\boldsymbol{\gamma}:\mathbb{Q}$, which represents an orthogonal transformation $\mathbf{P}\in\mathbf{SO}(3)$ of space-vectors, according to

$$\mathbf{y}'=\boldsymbol{\gamma}:(\mathbb{Q}\mathbb{X}'\mathbb{Q}^*)=\mathbf{q}\mathbf{x}'\mathbf{q}^*=\mathbf{P}\mathbf{x}'=\mathbf{P}^{1/2}(\mathbf{x}'\mathbf{P}^{-1/2}) \tag{12}$$

[5] Not to be confused with the Gibbs dyadic notation for the tensor product $\mathbf{x}\otimes\mathbf{y}$.

It is easy to show [9] by (10)-(12) that

$$\begin{aligned} &\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}},\ |\boldsymbol{\theta}| = 2|\boldsymbol{\phi}|\ (\mathrm{mod}\, 2\pi),\ \text{hence } \mathbf{p} = \mathbf{q}^2 \text{ or } \mathbf{q} = \pm\mathbf{p}^{1/2}, \\ &\text{where} \quad \mathbf{q} = e^{\boldsymbol{\phi}} \text{ and } \mathbf{p} = e^{\boldsymbol{\theta}}, \end{aligned} \tag{13}$$

which involves the map between skew-Hermitian quaternions $\mathbb{Z} = -\mathbb{Z}^*$ and rotations:

$$\begin{aligned} e^{\boldsymbol{\zeta}} &= \cos\zeta + \sin\zeta\ \hat{\boldsymbol{\zeta}} = \boldsymbol{\gamma} : e^{\mathbf{Z}}, \\ \text{with} \quad e^{\mathbf{Z}} &= \mathbf{I} + \sin\zeta\ \hat{\mathbf{Z}} + (1-\cos\zeta)\ \hat{\mathbf{Z}}^2,\ \mathbf{Z} = \boldsymbol{\Pi}\mathbb{Z}, \end{aligned} \tag{14}$$

where the unit skew-Hermitian vector $\hat{\boldsymbol{\zeta}} = -\hat{\boldsymbol{\zeta}}^*$ represents the axis of rotation, and ζ the angle of rotation about the axis.

The relations (13)-(14) allow for an easy verification of certain results of Kafadar and Eringen [5] involving the derivatives of the one-parameter Lie group represented by orthogonal transformations $\mathbf{P}(t) = \exp\boldsymbol{\Theta}(t)$. In particular, given the relations,

$$\boldsymbol{\nu} = -\frac{1}{2}\boldsymbol{\varepsilon} : \mathbf{N},\ \mathbf{N} = \frac{d\mathbf{P}}{dt}\mathbf{P}^{\mathrm{T}},\ \text{with } \boldsymbol{\nu} = \frac{d\mathbf{p}}{dt}\mathbf{p}^* = 2\frac{d\mathbf{q}}{dt}\mathbf{q}^* \tag{15}$$

we can now derive the desired relation between $\mathbf{N}$ or $\boldsymbol{\nu}$ and $\boldsymbol{\omega} = -\frac{1}{2}\boldsymbol{\varepsilon} : \boldsymbol{\Omega}$, where $\boldsymbol{\Omega} := d\boldsymbol{\Theta}/dt$.

Letting $\dot{(\ )} = d(\ )/dt$ and employing the representation $\mathbf{q} = \exp\boldsymbol{\phi} = \cos\varphi + \hat{\boldsymbol{\phi}}\sin\varphi$, one finds readily by (13) and the relation $\hat{\boldsymbol{\theta}}^2 = -1$ that

$$\boldsymbol{\nu} = 2\dot{\mathbf{q}}\mathbf{q}^* = \dot{\vartheta}\hat{\boldsymbol{\theta}} + \sin\vartheta\dot{\hat{\boldsymbol{\theta}}} + (1-\cos\vartheta)\hat{\boldsymbol{\theta}}\dot{\hat{\boldsymbol{\theta}}}, \tag{16}$$

which corresponds to Eq. (7) of [5]. However, in view of the relation $\boldsymbol{\theta}^2 = -\vartheta^2$, it follows that

$$\dot{\vartheta} = -\frac{1}{2}\left(\hat{\boldsymbol{\theta}}\dot{\boldsymbol{\theta}} + \dot{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}}\right) \text{ and } \hat{\boldsymbol{\theta}}\dot{\vartheta} = \frac{1}{2}\left(\dot{\boldsymbol{\theta}} - \hat{\boldsymbol{\theta}}\dot{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}}\right) \tag{17}$$

Substitution of these expressions into (16) and application of (10) gives after some algebra

$$\begin{aligned} \boldsymbol{\nu} &= \dot{\boldsymbol{\theta}} + \left(\frac{1-\cos\vartheta}{\vartheta^2}\right)\boldsymbol{\theta}\times\dot{\boldsymbol{\theta}} + \left(\frac{\vartheta-\sin\vartheta}{\vartheta^3}\right)\boldsymbol{\theta}\times(\boldsymbol{\theta}\times\dot{\boldsymbol{\theta}}) \\ &\equiv \left\{\mathbf{I} + \left(\frac{1-\cos\vartheta}{\vartheta^2}\right)\boldsymbol{\Theta} + \left(\frac{\vartheta-\sin\vartheta}{\vartheta^3}\right)\boldsymbol{\Theta}^2\right\}\boldsymbol{\omega} \end{aligned} \tag{18}$$

representing Eq. (8) of [5], as cited above in (3).

4 Conclusions

The connection between Cosserat spins given by Kafadar and Eringen [5] is rather easily established by means of the quaternionic representation of rotations. While

the present work is intended mainly to establish that fact, Section 2 provides tensor representations that may be useful in a broader range of applications. In particular, the formula (11) with $\rho \neq 1$ describes superposed Cosserat rotation and dilatation, suggesting a convenient of representation of Eringen's microstretch continuum. In particular, one sees that a complex quaternion of the form

$$\mathbb{Z}(\mathbf{x}^\circ) = \mathbf{x} + \imath \log \mathbf{z}$$

where $\mathbf{z}$ is defined by (11), defines a more general *microstretch placement*, a subject to be considered in a future work.

Acknowledgement

This is offered as a token of friendship to Professor Krzysztof Wilmanski, on the occasion of his seventieth birthday, and to acknowledge his early recognition of the relevance of structured-continuum models to the mechanics of heterogeneous materials [15]. I also thank Professor Reuven Segev for clarifying certain algebraic details in the derivation of Kafadar and Eringen [5].

References

1. Ericksen, J., Truesdell, C.: Exact theory of stress and strain in rods and shells. Arch. Rat. Mech. Anal. 1(1), 295–323 (1957)
2. Truesdell, C., Toupin, R.A.: Principles of classical mechanics and field theory. In: Flügge, S. (ed.) Encyclopedia of Physics (Handbuch der Physik), vol. 3/1. Springer, Berlin (1960)
3. Mindlin, R.D.: Micro-structure in linear elasticity. Arch. Ratl. Mech. Anal. 16(1), 51–78 (1964)
4. Cowin, S.C.: Stress functions for cosserat elasticity. Int. J. Solids Struct. 6(4), 389–398 (1970)
5. Kafadar, C.B., Eringen, A.C.: Micropolar media. i. the classical theory. Int. J. Eng. Sci. 9(3), 271–305 (1971)
6. Mehrabadi, M.M., Cowin, S.C., Jaric, J.: Six-dimensional orthogonal tensor representation of the rotation about an axis in three dimensions. Int. J. Solids Struct. 32(3-4), 439–449 (1995)
7. Park, J., Chung, W.K.: Geometric integration on euclidean group with application to articulated multibody systems. IEEE Trans. Robotics 21(5), 850–863 (2005)
8. Iserles, A., Munthe-Kaas, H.Z., Nørsett, S.P., Zanna, A.: Lie-group methods. Acta numerica 9(1), 215–365 (2000)
9. Spring, K.W.: Euler parameters and the use of quaternion algebra in the manipulation of finite rotations: A review. Mech. Mach. Theo. 21(5), 365–373 (1986)
10. Bauchau, O.A., Choi, J.Y.: The vector parameterization of motion. Nonlin. Dyn. 33(2), 165–188 (2003)
11. Gallier, J., Xu, D.: Computing exponentials of skew symmetric matrices and logarithms of orthogonal matrices. Int. J. Robot. Autom. 17, 1–11 (2002)

12. Lüdkovsky, S.V., van Oystaeyen, F.: Differentiable functions of quaternion variables. Bull. Sci. math. 127(9), 755–796 (2003)
13. Hamilton, W.R.S.: Elements of quaternions. Longmans, Green & Co., London (1866)
14. Morris, A.O.: On a generalized clifford algebra (ii). Q J. Math 19(1), 289–299 (1968)
15. Wilmanski, K.: Dynamics of bodies with microstructure. Arch. Mech. Stos. 20(6), 705–744 (1968)