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TURBULENT DIFFUSION OF MAGNETIC FIELDS IN WEAKLY IONIZED GAS

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ABSTRACT

The diffusion of unidirectional magnetic fields by two-dimensional turbulent flows in a weakly ionized gas is studied. The fields here are orthogonal to the plane of fluid motion. This simple model arises in the context of the decay of the mean magnetic flux–to–mass ratio in the interstellar medium. When ions are strongly coupled to neutrals, the transport of a large-scale magnetic field is driven by both turbulent mixing and nonlinear, ambipolar drift. Using a standard homogeneous and Gaussian statistical model for turbulence, we show rigorously that a large-scale magnetic field can decay on turbulent mixing timescales when the field and neutral flow are strongly coupled. There is no enhancement of the decay rate by ambipolar diffusion. These results extend the Zeldovich theorem to encompass the regime of two-dimensional flows and orthogonal magnetic fields, recently considered by Zweibel. The limitation of the strong coupling approximation and its implications are discussed.

Subject headings: diffusion — ISM: magnetic fields — magnetic fields — turbulence

1. INTRODUCTION

In the interstellar medium, where the bulk of fluid consists of neutral gas, magnetic fields appear to be constantly lost from the fluid—a phenomenon often referred to as ambipolar diffusion (Spitzer 1978). This is simply because magnetic fields move with (or are tied to) ionized gases while there is slippage between the motion of ionized and neutral gases. For parameter values typical of interstellar clouds, however, the ambipolar drift appears to be too slow (roughly by 2 orders of magnitude) to explain the large dispersion in the correlation between density and magnetic field strength (Zweibel 2002). This naturally motivates us to explore other transport processes. In particular, the interaction of nonlinear, ambipolar diffusion and turbulent advective mixing is a question of obvious relevance. In this Letter, we seek to examine the interplay of these two processes and to determine the bound on the turbulent transport of magnetic fields.

While it is well known that turbulent mixing leads to a rapid diffusion of passive scalar fields, this is no longer the case for the diffusion of even a weak magnetic field (far below equipartition) in a fully ionized gas (Cattaneo & Vainshtein 1991; Gruzinov & Diamond 1994) because of the back-reaction of the Lorentz force. In a weakly ionized gas, such as in the interstellar medium, the problem becomes more complicated since turbulent mixing also depends on the collision frequency between ions and neutrals as well as on the strength of magnetic fields. This is because in a weakly ionized gas, ions undergo the frictional damping due to collisions with neutrals, which effectively reduces the effect of the Lorentz force (or Alfvén waves). In fact, Kim (1997) demonstrated that in two spatial dimensions, the diffusion is still reduced below its kinematic value but that the critical strength of a large-scale magnetic field above which the diffusion is reduced can be larger [by a factor of $(\nu_{i\text{-}n}\tau)^{1/2}$, where $\nu_{i\text{-}n}$ and τ are the ion-neutral collision frequency and the correlation time of neutrals, respectively] as compared with what happens in the case of a fully ionized gas.

For simplicity, in this Letter, we consider the mixing of unidirectional magnetic fields by two-dimensional flows that are perpendicular to these fields. Note that this configuration is different from traditional two-dimensional MHD, in which the fields and flows are coplanar. To maintain this geometry, it is necessary to make the strong coupling approximation (Spitzer 1978; Shu 1983; Zweibel 1988) by assuming that the drift between ions and neutrals is balanced by the Lorentz force on ions because of frequent ion-neutral collisions (frictional damping). Our work is the generalization of Zweibel (2002), who considered the diffusion of magnetic fields by highly idealized flows made up of an ensemble of hyperbolic stagnation points. Since it is not altogether clear how to relate these flows to realistic turbulence models, we take a statistical approach here and rigorously derive the diffusion rate by assuming a standard scenario of Gaussian and homogeneous turbulence. Note that it is possible that this simplified statistical model may fail when the nonlinear, ambipolar diffusion is dominant, in which case sharp frontlike structures are generated (Brandenburg & Zweibel 1994).

Under the strong coupling approximation, magnetic fields are advected passively by neutral flows and diffused by nonlinear ambipolar drift (in addition to the usual ohmic diffusion). Thus, in view of this nonlinear diffusion, one may naively expect that a large-scale magnetic field would decay at a rate that is significantly enhanced over the turbulent (kinematic) value. We show, however, that it is not the case because of strong fluctuations (see § 3). Specifically, in deriving a generalized Zeldovich theorem (which relates the macroscopic quantity [transport] to microscopic dissipation) for a weakly ionized gas in two-dimensional motion but containing magnetic fields orthogonal to the plane of motion, we show that the flux transport due to the advection by neutral flows has an upper bound given by a kinematic value while the nonlinear diffusion arising from ambipolar drift is insignificant. The remainder of the Letter is organized as follows. We present our model in § 2 and derive a diffusion rate in § 3. Section 4 contains the summary and discussion of the Letter.

2. MODEL

In a weakly ionized gas with $\rho \sim \rho_n \gg \rho_i$, $\nu_{i\text{-}n}/\nu_{n\text{-}i}$ is large; i.e., $\nu_{i\text{-}n}/\nu_{n\text{-}i} = \rho_n/\rho_i \gg 1$. Here ρ, ρ_n, and ρ_i are the density of the bulk of fluid, neutrals, and ions, respectively, and $\nu_{i\text{-}n}$ and $\nu_{n\text{-}i}$ are ion-neutral and neutral-ion collision frequencies, respectively. Infrequent neutral-ion collisions permit us to prescribe the motion of neutral gas, provided that

$$\frac{B^2}{4\pi\rho_i}\frac{\nu_{i\text{-}n}}{\nu_{n\text{-}i}}(\simeq v_A^2) \ll v^2, \quad \tau\nu_{n\text{-}i} \ll 1, \qquad (1)$$

where $v \sim v_n$ and τ are the neutral velocity and correlation time, respectively, and $v_A = B/(4\pi\rho)^{1/2}$ is the Alfvén speed with respect to the bulk of the fluid (see Kim 1997). Given a neutral velocity, the ion velocity follows simply from the strong coupling approximation as $\boldsymbol{v}_i = \boldsymbol{v}_n + [(\boldsymbol{\nabla}\times\boldsymbol{B})\times\boldsymbol{B}]/4\pi\rho_i\nu_{i\text{-}n}$. Note here that the strong coupling approximation is valid when

$$\nu_{i\text{-}n} > \tilde{\nu}_A, \qquad (2)$$

where $\tilde{\nu}_A = kB/(4\pi\rho_i)^{1/2} = kv_A(\nu_{i\text{-}n}/\nu_{n\text{-}i})^{1/2}$ is the Alfvén frequency defined by using the density of the ion (cf. eq. [1]), with k being the wavenumber. When this condition is violated, as is likely to be case on small scales because of the high frequency of Alfvén waves, the drift between ion and neutral motions is no longer balanced by the Lorentz force, thereby requiring a self-consistent treatment of ion dynamics (Kim 1997).

We consider the mixing of unidirectional magnetic fields [say, $\boldsymbol{B} = B(x, y)\hat{z}$] by incompressible neutral flows $\boldsymbol{v}(x, y)$ in the x-y plane perpendicular to these fields. That is, we treat the two-dimensional motions of three-dimensional fields. Note that our problem is somewhat similar to that in the Goldreich-Sridhar model (Goldreich & Sridhar 1997), which also considers the perpendicular mixing of anisotropic structures. The evolution equation for the strength of the magnetic field in this geometry can be written in the following form (see also Zweibel 2002):

$$(\partial_t + \boldsymbol{v}\cdot\boldsymbol{\nabla})Q = \alpha\nabla\cdot(Q^2\nabla Q). \qquad (3)$$

Here $Q \equiv B/\rho$, $\alpha \equiv \rho/4\pi\nu_{n\text{-}i}$, and the (small) ohmic diffusivity has been ignored. The second term on the left-hand side of equation (2) represents the advection by neutral flow, while the term on the right-hand side is the nonlinear diffusion by ambipolar drift. The ratio of the effects of these two can be measured by the ambipolar Reynolds number $R_{AD} = vl/\lambda_T = (v^2/v_A^2)(\tau\nu_{n\text{-}i})$. Here $vl = \eta_k$ is the kinematic diffusion rate, and $\lambda_T = v_A^2/\nu_{n\text{-}i}$ is ambipolar drift due to total magnetic fields. Note that R_{AD} can be larger or smaller than unity while still being consistent with equation (1). The question is then what the total diffusion rate is in the presence of these two (advection and ambipolar drift) effects. Would they act together to significantly enhance the diffusion rate over the kinematic value η_k? To answer this question, we assume a Gaussian, homogeneous turbulence and evaluate the diffusion rate by using a quasi-linear closure (for instance, see Moffatt 1978) in the next section. The observant reader is no doubt puzzled by the fact that equation (3) is two-dimensional. The motivation for this simplification is that turbulence with $\boldsymbol{k}\cdot\boldsymbol{B}_0 \neq 0$ will bend magnetic field lines, resulting in its conversion to Alfvén waves and radiation along $\boldsymbol{B}_0$. Such fluctuations are intrinsically less effective at transporting Q since much of their energy is expended on bending. However, flutelike eddies, with $\boldsymbol{k}\cdot\boldsymbol{B}_0 = 0$, are energetically favored for transport and also remain correlated with the flux tube being transported for a longer time. Thus, the incorporation of eddies with a finite wavenumber along $\boldsymbol{B}_0$ (i.e., $\boldsymbol{k}\cdot\boldsymbol{B}_0 \neq 0$) will reduce the transport of magnetic fields.

3. DIFFUSION RATE

We employ the decomposition of Q into large-scale $\langle Q\rangle$ and small-scale Q' components and assume that flows are on small scales ($\langle\boldsymbol{v}\rangle = 0$). The equations for $\langle Q\rangle$ and Q' are then easily obtained as follows:

$$\partial_t\langle Q\rangle + \boldsymbol{\nabla}\cdot\langle\boldsymbol{v}Q'\rangle = \alpha\nabla\cdot\langle Q^2\nabla Q\rangle, \qquad (4)$$

$$\partial_t Q' + \boldsymbol{v}\cdot\boldsymbol{\nabla}\langle Q\rangle = \alpha\boldsymbol{\nabla}\cdot\boldsymbol{F}, \qquad (5)$$

where

$$\boldsymbol{F} \equiv (Q'^2 - \langle Q'^2\rangle)\nabla\langle Q\rangle + 2Q'\langle Q\rangle\nabla\langle Q\rangle + \langle Q\rangle^2\nabla Q' + Q'^2\nabla Q' - \langle Q'^2\nabla Q'\rangle + 2\langle Q\rangle(Q'\nabla Q' - \langle Q'\nabla Q'\rangle).$$

As can be seen from equation (4), the determination of the diffusion rate requires the computation of the flux $\Gamma_i = \langle v_i Q'\rangle$ and the cubic nonlinear term in Q. To compute the cubic nonlinear term, as well as other nonlinear terms that appear in the following analysis, we assume that the statistics of fluctuations are Gaussian and that the turbulence is homogeneous. Then

$$\langle Q^2\nabla Q\rangle = (\langle Q\rangle^2 + \langle Q'^2\rangle)\nabla\langle Q\rangle. \qquad (6)$$

On the other hand, the flux Γ_i is evaluated by assuming stationary turbulence. We first multiply equation (5) by v_i and then take the average to obtain

$$\Gamma_i = -\frac{\eta_k}{1 + \alpha\tau k_{\rm eff}^2\langle Q^2\rangle}\partial_i\langle Q\rangle = -\frac{\eta_k}{1 + R_{AD}^{-1}}\partial_i\langle Q\rangle. \qquad (7)$$

Here $\eta_k = \tau\langle v^2\rangle/2 \simeq vl$ is the kinematic diffusion rate; $k_{\rm eff} \simeq 1/l$ is the inverse of the characteristic scale of fluctuating magnetic fields; $R_{AD} = \eta_k/\lambda_T$, where $\lambda_T = (\langle v_A\rangle^2 + \langle v_A'^2\rangle)/\nu_{n\text{-}i}$; and τ is the correlation time of fluctuating magnetic fields, which is assumed to be comparable to that of neutral velocity. Note that λ_T and R_{AD}, now defined in terms of averaged quantities, include both mean and fluctuating components. In deriving equation (7), we used $\langle vQ'^2\rangle = 0$ and $\langle v_i v_j\rangle = \delta_{ij}\langle v^2\rangle/2$ by assuming an isotropic turbulence. Equation (7) states how much flux is transported from large to small scales, thereby leading to the decay of $\langle Q\rangle$. Interestingly, the diffusion rate by advection, $-\Gamma_i/\partial_i\langle Q\rangle$, has an upper bound given by kinematic diffusion η_k and becomes smaller as R_{AD} decreases. This is because that ambipolar drift "renormalizes" the correlation time so as to reduce the transport. Thus, it is clear that the kinematic turbulent flux is an *upper bound* on Γ_i.

In the case of stationary turbulence, the flux transport is balanced by dissipation on small scales as

$$\Gamma_i\partial_i\langle Q\rangle = -\alpha(\langle Q\rangle^2 + \langle Q'^2\rangle)\langle(\partial_i Q')^2\rangle = -\lambda_T\langle(\partial_i Q')^2\rangle. \qquad (8)$$

This was obtained by multiplying equation (5) by Q' and then taking the average. Equation (8), together with equation (7), establishes the relation between small- and large-scale fields as

$$\frac{\eta_k}{1 + R_{AD}^{-1}}(\partial_i\langle Q\rangle)^2 = \lambda_T\langle(\partial_i Q')^2\rangle. \qquad (9)$$

Equation (9) is a generalized Zeldovich theorem for a weakly ionized, strongly coupled gas. It gives the relation between the mean field and its gradient and relates the macroscopic quantity (i.e., flux transport) to microscopic dissipation. Note that the original Zeldovich theorem in a fully ionized gas can be recovered by replacing the ambipolar drift by ohmic diffusion η ($\lambda_T \to \eta$) and by taking $R_{\rm AD}^{-1} = 0$, which gives $\langle(\partial_i Q')^2\rangle/(\partial_i\langle Q\rangle)^2 = \eta_k/\eta = R_m$ (R_m is the magnetic Reynolds number). Of course, the situation discussed here is three-dimensional, with fields orthogonal to the plane of two-dimensional motion.

Finally, the (total) diffusion rate of $\langle Q\rangle$ follows from equations (5)–(7) and equation (9) as

$$\eta_T = \frac{\eta_k}{1 + R_{\rm AD}^{-1}}\left[1 + \frac{|\nabla\langle Q\rangle|^2}{\langle(\nabla Q')^2\rangle}\right]. \qquad (10)$$

Given in this form, η_T illustrates the two complementary effects of ambipolar drift on diffusion—the first is the reduction of η_T by the renormalization of τ (the term $R_{\rm AD}^{-1}$), and the second is the enhancement of η_T by nonlinear diffusion (the second term in the square brackets). Due to the second effect, it is, in principle, possible that $\eta_T \gg \eta_k$. This, however, turns out to be very unlikely. To see this, we first estimate η_T in an interesting and more relevant case in which $R_{\rm AD} > 1$.

When $R_{\rm AD} > 1$, equation (9) leads to

$$\frac{\langle(\nabla Q')^2\rangle}{|\nabla\langle Q\rangle|^2} \sim R_{\rm AD}(>1), \qquad (11)$$

suggesting strong fluctuations. Equation (10) then becomes

$$\eta_T \sim \eta_k. \qquad (12)$$

Thus, η_T approaches the kinematic value η_k. That is, the diffusion cannot exceed the kinematic rate because of strong fluctuations. Note that in this limit, the field is transported as an effectively passive scalar!

We now look at a less interesting limit, $R_{\rm AD} < 1$, in which the effect of turbulence does not play an important role. Note that in this limit, the strong coupling approximation can easily break down. This can be seen by rewriting the validity condition for the strong coupling approximation (eq. [2]) as $R_{\rm AD} > 1/(\tau\nu_{i\text{-}n})$. That is, the smaller $R_{\rm AD}$ (the stronger magnetic field), the easier it is to violate the strong coupling approximation. Thus, the results obtained in the limit $R_{\rm AD} \ll 1$ may not be consistent with this approximation. With this in mind, we reduce equation (9) to

$$\frac{\langle(\nabla Q')^2\rangle}{|\nabla\langle Q\rangle|^2} \sim R_{\rm AD}^2(<1), \qquad (13)$$

for $R_{\rm AD} < 1$. Equation (13) indicates that to satisfy the stationarity condition (eq. [9]), $\langle Q\rangle^2 > \langle Q'^2\rangle$. This follows because the dissipation due to ambipolar drift on small scales is too large to be balanced by flux transport. However, as noted previously, the strong coupling approximation (which leads to the nonlinear diffusion) is likely to be invalid on small scales, especially when magnetic fields are strong. What should happen on small scales is the propagation of Alfvén waves rather than nonlinear diffusion. Alternatively put, when the full dynamics of ions is taken into account, stationary turbulence may still be possible even when $\langle Q\rangle^2 > \langle Q'^2\rangle$. Now, the diffusion rate in this case ($R_{\rm AD} < 1$) follows from equations (10) and (13) as

$$\eta_T \sim \eta_k/R_{\rm AD} = \lambda_T \simeq \langle v_{\rm A}\rangle^2/\nu_{n\text{-}i}. \qquad (14)$$

This is a somewhat expected result in the sense that for small $R_{\rm AD}$, the diffusion rate is set by the ambipolar drift. Our nontrivial result is the observation that in this case, the ambipolar drift due to fluctuating magnetic fields must be negligible in order to maintain stationarity.

4. SUMMARY AND DISCUSSION

The problem of the transport of magnetic fields in the interstellar medium is studied by incorporating the effect of turbulence. Specifically, we consider the diffusion of unidirectional magnetic fields in the presence of two-dimensional, incompressible, turbulent (neutral) flows perpendicular to these magnetic fields, embedded in a weakly ionized gas. By assuming that the strong coupling approximation is valid on all scales, we compute the total diffusion rate of a large-scale magnetic field through a quasi-linear analysis. When the turbulence is homogeneous, stationary, and Gaussian, the diffusion rate η_T is found to depend on $R_{\rm AD}$ and the level of fluctuations (see eq. [10]), with ambipolar drift playing two complementary roles (see § 3). In particular, when $R_{\rm AD} > 1$, η_T is shown to be at most of the order of the turbulent rate $\eta_k = vl$. In this case, the field is effectively a passive scalar. Interestingly, this suggests that even in the strong coupling regime, it is unlikely that magnetic fields will diffuse at a rate faster than the simple kinematic value, in spite of the nonlinear diffusion operator. In the opposite case ($R_{\rm AD} < 1$), we demonstrated that $\eta_T \simeq \lambda_T$ as long as fluctuations are negligible compared with the mean field. Note, however, that this limit may not be consistent with the strong coupling approximation. Therefore, our result not only confirms the main point of Zweibel (2002) but also puts it on a simple, rigorous foundation.

The results of this Letter are applicable to any system with a neutral population and a weak magnetic field, such that the strong coupling approximation ($\boldsymbol{v}_i = \boldsymbol{v}_n + [(\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B}]/4\pi\rho_i\nu_{i\text{-}n}$) is valid. However, because of the assumed incompressibility of neutral flows, a more uniform loading of magnetic fields discussed in the Letter is basically due to the diffusion of magnetic fields in a constant density background. As the magnetic field diffuses while the density remains constant, the system progresses toward a state of more uniformly loaded magnetic fields. In the turbulent case with $R_{\rm AD} > 1$, this mass-loading uniformization occurs by the turbulent cascade of magnetic energy (by diffusive mixing) to small scale, where it is eliminated by ohmic dissipation. This uniformization occurs in one large eddy turnover time, as one expects, since $\tau_{\rm un} = \sum_n \tau_n = \sum_n l_n/v_n = \sum_n (l_n/\epsilon^{1/3} l_n^{1/3}) = (l_0^{2/3}/\epsilon^{1/3})\,[1/(1-\alpha^{2/3})] \sim \tau_0$. Here ϵ is the energy dissipation rate, and $l_n = \alpha l_{n-1}$ with $\alpha \sim \frac{1}{2}$ was used. It is important to realize that the breakdown of flux freezing on small scales due to ohmic diffusion is critical to the uniformization of loading. Strictly speaking, our results cannot be directly applied to star-forming regions where the compressibility of flows and gravity are crucial. Nevertheless, our results imply that the ambipolar drift in the turbulent medium can make magnetically subcritical clouds supercritical and also that the turbulent mixing can uniformize the loading of magnetic field lines on a large eddy turnover timescale.

In the interstellar medium, the diffusion rate due to ambipolar drift alone is too small (by a few orders of magnitude) to explain the observations. Given that R_{AD} based on a large-scale magnetic field seems to be larger than unity, turbulent mixing perhaps provides a mechanism by which the uniformity of the density and the strength of magnetic fields is achieved on the eddy turnover timescale. However, ions and neutrals are unlikely to be strongly coupled on small scales because of the high frequencies of Alfvén waves, thereby invalidating the strong coupling approximation. Therefore, the complete answer to the problem ultimately requires the self-consistent treatment of ion dynamics. Furthermore, for a better estimate on R_{AD}, some information on the strength of fluctuating magnetic fields is needed. For instance, when fluctuations are much stronger than mean fields, R_{AD} based on fluctuations may be smaller than unity, and thus ambipolar drift alone may lead to a fast diffusion. Of course, even in this case, the validity of the nonlinear (ambipolar) diffusion may become questionable because of the breakdown of the strong coupling approximation. In either cases, the relaxation of the strong coupling approximation is expected to bring in the reduction of the diffusion since magnetic fields are no longer passively advected/distorted (Kim 1997). Of course, other effects, such as gravity, the detailed microscale mechanism for dissipating magnetic energy, and turbulence intermittency, must be considered as well. A detailed study of this issue will be addressed in a future work.

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