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UNIVERSITY OF CALIFORNIA, SAN DIEGO

The graded module category of a generalized Weyl algebra

A dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Mathematics

by

Robert Jeffrey Won

Committee in charge:

Professor Daniel Rogalski, Chair
Professor Dragos Oprea
Professor Alon Orlicsky
Professor Alireza Salehi Golsefidy
Professor Paul Siegel

2016

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The dissertation of Robert Jeffrey Won is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

Chair

University of California, San Diego

2016

DEDICATION

To my parents.

EPIGRAPH

‘I will take the Ring,’ he said, ‘though I do not know the way.’

—J.R.R. Tolkien, *The Fellowship of the Ring*

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ACKNOWLEDGEMENTS

First and foremost, I would like to thank my advisor, Dan Rogalski—for teaching me noncommutative ring theory and algebraic geometry, giving me interesting problems to think about and many very good ideas about how to solve them, answering many questions, and carefully reviewing this thesis. I could not have asked for a better or kinder advisor.

I am grateful to Alireza Salehi Golesefidy (from whom I first learned about algebraic groups), Dragos Oprea (from whom I first learned algebraic geometry), Alon Orlicsky, and Paul Siegel for serving on my committee. I am also grateful to Toby Stafford and Adam Bowers for writing letters, and to Lance Small, Sue Sierra, Paul Smith, Ellen Kirkman, James Zhang, Chelsea Walton, Jason Gaddis, and Manny Reyes for advice and helpful conversations.

My classmates made grad school unreasonably fun. Special thanks to my officemates: Mike Tait, for the laughs and the laps; Jay Cummings, for the friendly bets; Dan Hoff, for the empathy; and our honorary officemate Lexi Ambrogi, for the Jeo-parties. Shout-out to my algebras, Matt Grimm and Robert McGuigan, for making all those hours of qual studying fun. Thanks also to Frankie, Sinan, Josh, Fred, Corey, Longo, Shaunak, Naneh, Susan, Stephan, Bodnar, Kuang, Daniel, François, Pieter, Cal, Perry, Ryan, Daniel, Semko, Mark, and Craig.

The friendship of many others helped me survive grad school. I am thankful for new friends in the last five years: Chris, Maria, Gaby, Brian, Earl, ELiao, Sophia, Kat, Jenn, Lillian, Daniela, Gracia, Tim, Vincent, David, Scott, and Josh. Old friends scattered across the country remind me that friendship transcends distance: Kaitlyn, Nong, Toonie, Reep, Kursten, Emma, Barreto, Shim, Clam, Gordo, Mitch, James, Liz, Oreo, Eric, Jon, Geoff, Kathryn, Ron, Murph, Phil, Tiff, and Xiao.

I feel blessed to have had the opportunity to devote five years to the study of mathematics, and I know it would not have been possible without the help of my family. Thanks to Dan, Michelle, Madeleine, and Eric. Last but not least, thanks to my parents for their love and support. Mom and Dad, I couldn't have done it without you.

VITA

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ABSTRACT OF THE DISSERTATION

The graded module category of a generalized Weyl algebra

by

Robert Jeffrey Won

Doctor of Philosophy in Mathematics

University of California San Diego, 2016

Professor Daniel Rogalski, Chair

The first Weyl algebra, $A_1 = \mathbb{k}\langle x, y \rangle / (xy - yx - 1)$ is naturally \mathbb{Z} -graded by letting $\deg x = 1$ and $\deg y = -1$. In [Sie09], Sierra studied $\text{gr-}A_1$, the category of graded right A_1 -modules, computing its Picard group and classifying all rings graded equivalent to A_1 . In [Smi11], Smith showed that in fact $\text{gr-}A_1$ is equivalent to the category of quasicohherent sheaves on a certain quotient stack. He did this by constructing a commutative ring C , graded by finite subsets of the integers, and giving an equivalence of categories $\text{gr-}A_1 \cong \text{gr-}(C, \mathbb{Z}_{\text{fin}})$.

In this dissertation, we generalize results of Sierra and Smith by studying the graded module category of certain generalized Weyl algebras. We show that for a generalized Weyl algebra $A(f)$ with base ring $\mathbb{k}[z]$ defined by a quadratic polynomial f , the Picard group of $\text{gr-}A(f)$ is isomorphic to the Picard group of $\text{gr-}A_1$. For each $A(f)$, we also construct a commutative ring whose graded module category is equivalent to the quotient category $\text{qgr-}A(f)$, the category $\text{gr-}A(f)$ modulo its full subcategory of finite-dimensional modules.

Chapter 1

Introduction

Throughout this dissertation, fix an algebraically closed field \mathbb{k} of characteristic zero. All vector spaces and algebras are taken over \mathbb{k} and all categories and equivalences of categories are \mathbb{k} -linear.

1.1 Overview

Noncommutative rings are ubiquitous in mathematics. Given a commutative ring R and a nonabelian group G , the group ring $R[G]$ is a noncommutative ring. The $n \times n$ matrices with entries in \mathbb{C} or, more generally, endomorphism rings of modules are noncommutative rings. Noncommutative rings also arise as differential operators: the ring of differential operators on $\mathbb{k}[t]$ generated by multiplication by t and differentiation by t is isomorphic to the Weyl algebra,

$$A_1 := \mathbb{k}\langle x, y \rangle / (xy - yx - 1).$$

The Weyl algebra is a fundamental example in noncommutative ring theory. As A_1 is a differential polynomial ring, it can be viewed as an analogue of a commutative polynomial ring in two variables—in particular, it is a noetherian domain of Gelfand-Kirillov dimension 2. However, in contrast with any commutative ring, the Weyl algebra is a simple ring which is not a division algebra.

As noncommutative rings are a larger class of rings than commutative ones, not all of the same tools and techniques are available in the noncommutative set-

ting. For example, localization is only well-behaved at certain subsets of noncommutative rings called Ore sets. Care must also be taken when working with other ring-theoretic properties. The left ideals of a ring need not be (two-sided) ideals. One must be careful in distinguishing between left ideals and right ideals, as well as left and right noetherianness or artinianness.

In commutative algebra, there is a rich interplay between ring theory and algebraic geometry. One can associate to an \mathbb{N} -graded \mathbb{k} -algebra $R = \bigoplus_{i \in \mathbb{N}} R_i$ the projective scheme $\text{Proj } R$, the space of homogeneous prime ideals of R not containing the irrelevant ideal $R_{\geq 1}$. Localizing R at the homogeneous prime ideals, one can then construct the structure sheaf on $\text{Proj } R$. Algebraic properties of the ring R can be translated to geometric properties of the space $\text{Proj } R$ and vice versa.

In the noncommutative setting, there are obstructions to generalizing this idea. Therefore, in noncommutative projective geometry, rather than generalizing the space of homogeneous ideals of a ring, one approach has instead been to study the module category over the ring. This idea is due to a theorem of Serre: if a commutative graded \mathbb{k} -algebra R is generated over \mathbb{k} by elements of degree one then $\text{coh}(\text{Proj } R)$, the category of coherent sheaves on $\text{Proj } R$, is equivalent to $\text{qgr-}R$, the category of finitely generated graded R -modules modulo torsion.

Previous work in noncommutative projective geometry has largely focused on \mathbb{N} -graded rings. While the Weyl algebra A_1 is not \mathbb{N} -graded, it admits a natural \mathbb{Z} -grading given by letting $\deg x = 1$ and $\deg y = -1$. In [Sie09], Sierra studied $\text{gr-}A_1$, the category of finitely generated graded right A_1 -modules. She determined the group of autoequivalences of $\text{gr-}A_1$ and classified all rings graded equivalent to A_1 . The simple modules of $\text{gr-}A_1$ can be pictured as follows. For each $\lambda \in \mathbb{k} \setminus \mathbb{Z}$, there is a simple module $M_\lambda = A_1/(xy + \lambda)A_1$, while for each $n \in \mathbb{Z}$, the module $A_1/(xy + n)A_1$ is the nonsplit extension of two simple modules $X\langle n \rangle$ and $Y\langle n \rangle$. We can therefore represent the simple modules as the affine line with integer points doubled. Sierra gives the following picture:

Sierra studied the Picard group—the group of autoequivalences—of this category. She showed that there were many symmetries of this picture. There is an autoequivalence of $\text{gr-}A_1$ induced by a graded automorphism of A_1 which reflects

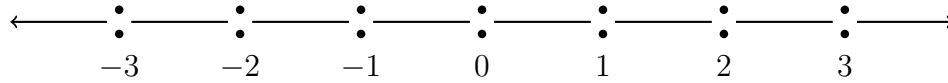


Figure 1.1.1: The simple modules of $\text{gr-}A_1$.

the picture, sending $A_1/(xy+n)A_1$ to $A_1/(xy-(n+1))A_1$ for all $n \in \mathbb{Z}$. The shift functor \mathcal{S}_{A_1} is an autoequivalence of the category which translates the picture, sending $A_1/(xy+n)A_1$ to $A_1/(xy+n+1)A_1$ for all $n \in \mathbb{Z}$. Hence, $\text{Pic}(\text{gr-}A_1)$ has a subgroup isomorphic to D_∞ . Further, Sierra constructed autoequivalences ι_n of $\text{gr-}A_1$, which permute $X\langle n \rangle$ and $Y\langle n \rangle$ and fix all other simple modules. Let \mathbb{Z}_{fin} denote the group of finite subsets of the integers, with operation exclusive or. The subgroup of $\text{Pic}(\text{gr-}A_1)$ generated by the ι_n is isomorphic to \mathbb{Z}_{fin} .

Theorem 1.1.1 (Sierra, [Sie09, Corollary 5.11]). *The group $\text{Pic}(\text{gr-}A_1)$ is isomorphic to $\mathbb{Z}_{\text{fin}} \rtimes D_\infty$.*

In [Smi11], Smith showed that in fact $\text{Gr-}A_1$, the category of graded right A_1 -modules, is equivalent to the category of quasicohherent sheaves on a certain quotient stack, χ . He constructed a commutative ring R graded by \mathbb{Z}_{fin} :

$$R := \mathbb{k}[z][\sqrt{z-n} \mid n \in \mathbb{Z}],$$

where $\deg \sqrt{z-n} = \{n\}$. Smith then proved:

Theorem 1.1.2 (Smith, [Smi11, Theorem 5.14 and Corollary 5.15]). *There is an equivalence of categories*

$$\text{Gr-}A_1 \equiv \text{Gr-}(R, \mathbb{Z}_{\text{fin}}) \equiv \text{Qcoh}(\chi).$$

In this dissertation, we generalize results of Sierra and Smith by studying the graded module category over certain generalized Weyl algebras (GWAs). We study GWAs $A(f)$ with base ring $\mathbb{k}[z]$ defined by a quadratic polynomial $f \in \mathbb{k}[z]$ and an automorphism $\sigma : \mathbb{k}[z] \rightarrow \mathbb{k}[z]$ mapping z to $z+1$. The GWA $A(f)$ has presentation

$$A(f) = \frac{\mathbb{k}[z]\langle x, y \rangle}{\begin{pmatrix} xz = \sigma(z)x & yz = \sigma^{-1}(z)y \\ xy = f & yx = \sigma^{-1}(f) \end{pmatrix}}.$$

When the defining polynomial f is clear from context, we use the notation A to denote $A(f)$. We assume, without loss of generality, that $f = z(z + \alpha)$ for some $\alpha \in \mathbb{k}$. The properties of $A(f)$ are determined by the roots of f . When $\alpha = 0$, we say that f has a multiple root, when $\alpha \in \mathbb{Z} \setminus \{0\}$, we say that f has congruent roots, and when $\alpha \in \mathbb{k} \setminus \mathbb{Z}$, we say that f has non-congruent roots.

In the non-congruent root case, the picture of $\text{gr-}A(f)$ can be thought of as a doubled version of the picture of $\text{gr-}A_1$.

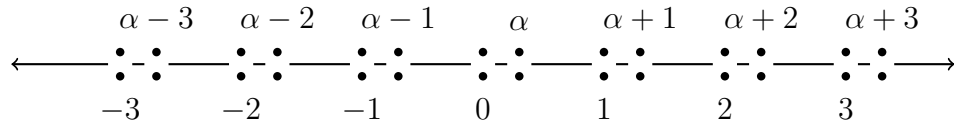


Figure 1.1.2: The simple modules of $\text{gr-}A(f)$ when f has non-congruent roots.

In this case, for each integer n , $A/(z + n)A$ is a nonsplit extension of two simple modules which we call $X_0^f\langle n \rangle$ and $Y_0^f\langle n \rangle$ and additionally, $A/(z + n + \alpha)$ is a nonsplit extension of two simple modules which we call $X_\alpha^f\langle n \rangle$ and $Y_\alpha^f\langle n \rangle$. Each pair of these simple modules behaves in the same way as the pair $X^f\langle n \rangle$ and $Y^f\langle n \rangle$ in $\text{gr-}A_1$.

In the multiple root case, the picture of $\text{gr-}A(f)$ is the same as for $\text{gr-}A_1$.

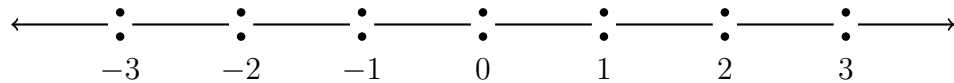


Figure 1.1.3: The simple modules of $\text{gr-}A(f)$ when f has a double root.

For every integer n , $A/(z + n)A$ is a nonsplit extension of two simple modules, $X^f\langle n \rangle$ and $Y^f\langle n \rangle$. However, although the picture is the same, the category is not—the simple modules $X^f\langle n \rangle$ and $Y^f\langle n \rangle$ also have nonsplit self-extensions.

Finally, in the congruent root case, there exist finite-dimensional simple modules: the shifts of a module we call Z^f . For each integer n , $A/(z + n)A$ has a composition series consisting of $X^f\langle n \rangle$, $Y^f\langle n \rangle$, and $Z^f\langle n \rangle$.

In this case, let $\text{fdim-}A(f)$ be the full subcategory of $\text{gr-}A(f)$ consisting of finite-dimensional modules. More concretely, the objects of $\text{fdim-}A(f)$ are given

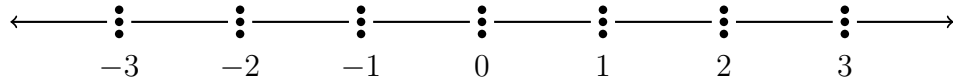


Figure 1.1.4: The simple modules of $\text{gr-}A(f)$ when f has congruent roots.

by finite direct sums of shifts of Z^f . We consider $\text{qgr-}A(f) = \text{gr-}A(f)/\text{fdim-}A(f)$, the quotient category of $\text{gr-}A(f)$ modulo its full subcategory generated by finite-dimensional modules and prove that $\text{qgr-}A(f)$ is equivalent to the category in the multiple root case.

Theorem 1.1.3 (Theorem 6.4.5). *Let $\alpha \in \mathbb{N}^+$. There is an equivalence of categories*

$$\text{qgr-}A(z(z + \alpha)) \cong \text{gr-}A(z^2).$$

In all three cases, we construct autoequivalences of $\text{gr-}A(f)$ which are analogous to Sierra's involutions ι_n . For all quadratic $f \in \mathbb{k}[z]$, we determine the Picard group of $A(f)$, showing that $\text{Pic}(\text{gr-}A(f)) \cong \text{Pic}(\text{gr-}A_1)$.

Theorem 1.1.4 (Theorem 5.2.4). *For any quadratic polynomial $f \in \mathbb{k}[z]$,*

$$\text{Pic}(\text{gr-}A(f)) \cong \mathbb{Z}_{\text{fin}} \rtimes D_\infty.$$

We also construct commutative rings similar to those in [Smi11] whose categories of graded modules are equivalent to $\text{qgr-}A(f)$. In the non-congruent root case, we define $\mathbb{Z}_{\text{fin}} \times \mathbb{Z}_{\text{fin}}$ graded-ring

$$C := \frac{\mathbb{k}[c_n, d_n \mid n \in \mathbb{Z}]}{(c_n^2 - n = c_m^2 - m, c_n^2 = d_n^2 - \alpha \mid m, n \in \mathbb{Z})}$$

where $\deg c_n = (n, \emptyset)$ and $\deg d_n = (\emptyset, n)$ and prove

Theorem 1.1.5 (Theorem 6.5.2). *Let $\alpha \in \mathbb{k} \setminus \mathbb{Z}$ and $f = z(z + \alpha)$. Then there is an equivalence of categories*

$$\text{gr-}A(f) \cong \text{gr-}(C, \mathbb{Z}_{\text{fin}} \times \mathbb{Z}_{\text{fin}}).$$

In the multiple and congruent root cases, we define the \mathbb{Z}_{fin} graded ring

$$B := \frac{\mathbb{k}[z][b_n \mid n \in \mathbb{Z}]}{(b_n^2 = (z + n)^2 \mid n \in \mathbb{Z})}$$

where $\deg b_n = \{n\}$ and prove

Theorem 1.1.6 (Theorem 6.4.5 and Corollary 6.4.6). *Let $\alpha \in \mathbb{N}^+$. There are equivalences of categories*

$$\text{gr-}A(z^2) \equiv \text{qgr-}A(z(z + \alpha)) \equiv \text{gr-}(B, \mathbb{Z}_{\text{fin}}).$$

We then prove some basic results about the ring B , characterizing its minimal prime ideals and showing that B is a non-noetherian, reduced ring of Krull dimension 1.

1.2 Structure of this dissertation

In the remainder of the introduction, we briefly summarize the contents of this dissertation. In Chapter 2, we establish notation and give background on graded rings, categories of graded modules, and generalized Weyl algebras. In Chapter 3 we study the category $\text{gr-}A(f)$. We study the finite length modules of $\text{gr-}A(f)$ in sections 3.1 and 3.3. In section 3.2, we introduce the important concept of the *structure constants* of a graded right A -submodule of $Q_{\text{gr}}(A(f))$ and in section 3.4, we investigate the rank one projective modules of $\text{gr-}A(f)$.

In Chapter 4, we develop some technical tools which allow us to define a functor on $\text{gr-}A$ by first defining it on the full subcategory of rank 1 projective modules. We use these tools in Chapter 5, where we construct autoequivalences of $\text{gr-}A(f)$ which are analogous to the ι_J of [Sie09] and compute the Picard group of $\text{gr-}A(f)$. Finally, in Chapter 6, we study the quotient category $\text{qgr-}A(f)$ and construct commutative rings whose graded module categories are equivalent to $\text{qgr-}A(f)$.

Chapter 2

Background

2.1 Graded rings and modules

We begin by fixing basic definitions, terminology, and notation. We follow the convention that 0 is a natural number so $\mathbb{N} = \mathbb{Z}_{\geq 0}$. We use the notation \mathbb{N}^+ to denote the positive natural numbers.

A \mathbb{k} -algebra R is a ring with a multiplicative identity which is a \mathbb{k} -vector space such that for all $r, s \in R$ and $\lambda \in \mathbb{k}$, $\lambda(rs) = (\lambda r)s = r(\lambda s)$. All rings in this dissertation are \mathbb{k} -algebras. If Γ is an abelian semigroup, then we say a \mathbb{k} -algebra R is Γ -graded if R has a \mathbb{k} -vector space decomposition

$$R = \bigoplus_{\gamma \in \Gamma} R_{\gamma}$$

such that $R_{\gamma} \cdot R_{\delta} \subseteq R_{\gamma+\delta}$ for all $\gamma, \delta \in \Gamma$. For a \mathbb{Z} -graded Ore domain R , we write $Q_{\text{gr}}(R)$ for the graded quotient ring of R , the localization of R at all nonzero homogeneous elements.

A *graded right R -module* is an R -module M with a \mathbb{k} -vector space decomposition $M = \bigoplus_{\gamma \in \Gamma} M_{\gamma}$ such that $M_{\gamma} \cdot R_{\delta} \subseteq M_{\gamma+\delta}$ for all $\gamma, \delta \in \Gamma$. If M and N are graded right R -modules and $\delta \in \Gamma$, then a Γ -graded right R -module homomorphism of degree δ is an R -module homomorphism φ such that $\varphi(M_{\gamma}) \subseteq N_{\gamma+\delta}$ for all $\gamma \in \Gamma$. If $\delta = 0$, then φ is referred to as a Γ -graded right R -module homomorphism. Define $\underline{\text{Hom}}_R(M, N)_{\delta}$ to be the set of all graded homomorphisms of degree δ from

M to N and define

$$\underline{\text{Hom}}_R(M, N) = \bigoplus_{\delta \in \Gamma} \underline{\text{Hom}}_R(M, N)_\delta.$$

The category of all Γ -graded right R -modules with Γ -graded right R -module homomorphisms is an abelian category denoted $\text{Gr-}(R, \Gamma)$. When $\Gamma = \mathbb{Z}$ we use the notation $\text{Gr-}R$ in place of $\text{Gr-}(R, \mathbb{Z})$. We follow the convention that if $\text{Xyz-}R$ is the name of a category then $\text{xyz-}R$ is the full subcategory of $\text{Xyz-}R$ consisting of noetherian objects. Hence, $\text{gr-}(R, \Gamma)$ is the category of all noetherian Γ -graded right R -modules. The category of all right R -modules with module homomorphisms is denoted $\text{Mod-}R$.

When we are working in the category $\text{gr-}R$, we denote the Γ -graded R -module homomorphisms by

$$\text{Hom}_{\text{gr-}R}(M, N) = \underline{\text{Hom}}_R(M, N)_0.$$

We write $\underline{\text{Ext}}_R$ and $\text{Ext}_{\text{gr-}R}$ for the derived functors of $\underline{\text{Hom}}_R$ and $\text{Hom}_{\text{gr-}R}$, respectively.

For a graded \mathbb{k} -algebra R , the *shift functor* on $\text{gr-}R$ sends a graded right module M to the new module $M\langle 1 \rangle = \bigoplus_{j \in \mathbb{Z}} M\langle 1 \rangle_j$, defined by $M\langle 1 \rangle_j = M_{j-1}$. We write this functor as $\mathcal{S}_R : M \mapsto M\langle 1 \rangle$. Similarly, $M\langle i \rangle_j = M_{j-i}$. This is in keeping with the convention of Sierra in [Sie09], although we warn that this is the opposite of the standard convention. For a graded \mathbb{k} -vector space V , we use the same notation to refer to the shift of grading on V : $V\langle i \rangle_j = V_{j-i}$. For right R -modules M and M' , note that as graded vector spaces,

$$\underline{\text{Hom}}_R(M\langle d \rangle, M'\langle d' \rangle) \cong \underline{\text{Hom}}_R(M, M')\langle d' - d \rangle.$$

Given two categories \mathcal{C} and \mathcal{D} , a covariant functor $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$ is called an *equivalence of categories* if there is a covariant functor $\mathcal{G} : \mathcal{D} \rightarrow \mathcal{C}$ such that $\mathcal{G} \circ \mathcal{F} \cong \text{Id}_{\mathcal{C}}$ and $\mathcal{F} \circ \mathcal{G} \cong \text{Id}_{\mathcal{D}}$. We say that \mathcal{C} and \mathcal{D} are *equivalent* and write $\mathcal{C} \equiv \mathcal{D}$. An equivalence of categories $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{C}$ is called an *autoequivalence of \mathcal{C}* . Given a category of Γ -graded modules, $\text{gr-}(R, \Gamma)$, let $\text{Aut}(\text{gr-}(R, \Gamma))$ be the group of autoequivalences of $\text{gr-}(R, \Gamma)$ with operation composition. Denote by \sim the

equivalence relation on $\text{Aut}(\text{gr-}(R, \Gamma))$ given by natural isomorphism. We define the *Picard group of gr- (R, Γ)*

$$\text{Pic}(\text{gr-}(R, \Gamma)) = \text{Aut}(\text{gr-}(R, \Gamma)) / \sim .$$

Following [Sie11b], we define a \mathbb{Z} -algebra R to be a \mathbb{k} -algebra without 1, with a $(\mathbb{Z} \times \mathbb{Z})$ -graded \mathbb{k} -vector space decomposition

$$R = \bigoplus_{i,j \in \mathbb{Z}} R_{i,j}$$

such that for any $i, j, k \in \mathbb{Z}$, $R_{i,j}R_{j,k} \subseteq R_{i,k}$ and if $j \neq j'$ then $R_{i,j}R_{j',k} = 0$. Additionally, we require that each of the subrings $R_{i,i}$ has a unit 1_i which acts as a left identity on all $R_{i,j}$ and right identity on all $R_{j,i}$. If R and S are \mathbb{Z} -algebras, we say that a \mathbb{k} -algebra homomorphism $\varphi : R \rightarrow S$ is *graded of degree d* if $\varphi(R_{i,j}) \subseteq S_{i+d,j+d}$. A \mathbb{Z} -algebra isomorphism is a degree 0 \mathbb{k} -algebra isomorphism $\varphi : R \rightarrow S$ such that $\varphi(1_i) = 1_i$ for all $i \in \mathbb{Z}$.

For a \mathbb{Z} -graded ring R , we define the \mathbb{Z} -algebra associated to R

$$\bar{R} = \bigoplus_{i,j \in \mathbb{Z}} \bar{R}_{i,j}$$

where $\bar{R}_{i,j} = R_{j,-i}$. A degree 0 automorphism of \bar{R} is called *inner* (see [Sie09, Theorem 3.10]) if for all $m, n \in \mathbb{Z}$, there exist $g_m \in \bar{R}_{m,m}$ and $h_n \in \bar{R}_{n,n}$ such that for all $w \in \bar{R}_{m,n}$,

$$\gamma(w) = g_m w h_n .$$

In [Sie09] and [Sie11b], Sierra studies the relationship between \bar{R} and $\text{gr-}R$. In particular, Sierra proves the following:

Theorem 2.1.1 (Sierra, [Sie11b, Theorem 3.6]). *Let R and S be \mathbb{Z} -graded \mathbb{k} -algebras. The following are equivalent:*

1. *The \mathbb{Z} -algebras \bar{R} and \bar{S} are isomorphic via a degree-preserving map.*
2. *There is an equivalence of categories $\Phi : \text{gr-}R \rightarrow \text{gr-}S$ such that for all $n \in \mathbb{Z}$, $\Phi(R\langle n \rangle) \cong S\langle n \rangle$.*

A functor Φ satisfying condition 2 in Theorem 2.1.1 above is called a *twist functor*. Sierra further gives the following result on twist functors.

Theorem 2.1.2 (Sierra, [Sie09, Corollary 3.11]). *Let R be a \mathbb{Z} -graded ring and suppose that all automorphisms of \overline{R} of degree 0 are inner. If $\Phi : \text{gr-}R \rightarrow \text{gr-}R$ is a twist functor, then Φ is naturally isomorphic to $\text{Id}_{\text{gr-}R}$.*

2.2 Category theoretic preliminaries

The category of modules over a ring is the prototypical example of an abelian category. In this section, we establish category theoretic definitions that we will need in chapters 4 and 6. If \mathbf{C} and \mathbf{D} are abelian categories, a functor $\mathcal{F} : \mathbf{C} \rightarrow \mathbf{D}$ is called *additive* if $\mathcal{F}(f + f') = \mathcal{F}f + \mathcal{F}f'$ for any morphisms $f, f' : X \rightarrow Y$. A *biproduct diagram* for objects $X, Y \in \mathbf{C}$ is a diagram

$$X \begin{array}{c} \xrightarrow{p_1} \\ \xleftarrow{i_1} \end{array} Z \begin{array}{c} \xrightarrow{p_2} \\ \xleftarrow{i_2} \end{array} Y$$

such that $p_1 i_1 = \text{Id}_X$, $p_2 i_2 = \text{Id}_Y$, and $i_1 p_1 + i_2 p_2 = \text{Id}_Z$. The object Z is a *biproduct of X and Y* , as the projections p_1 and p_2 make Z a categorical product while i_1 and i_2 make Z a categorical coproduct. A nice property of additive functors is that they preserve biproduct diagrams.

Proposition 2.2.1 (See [Mac78, Proposition VIII.2.4]). *A functor $\mathcal{F} : \mathbf{C} \rightarrow \mathbf{D}$ is additive if and only if \mathcal{F} carries each binary biproduct diagram in \mathbf{C} to a biproduct diagram in \mathbf{D} .*

We will also need the concept of the generator of a category. A set of objects $\{G_i \mid i \in I\}$ in \mathbf{C} is said to *generate* \mathbf{C} if for any two morphisms $f, g : X \rightarrow Y$ in \mathbf{C} such that $f \neq g$, there is some $i \in I$ and some morphism $h : G_i \rightarrow X$ such that $f \circ h \neq g \circ h$. If the set consists of a single object G , then G is called a *generator* of \mathbf{C} .

A subcategory \mathbf{D} of an abelian category \mathbf{C} is a *Serre subcategory* if for every short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

in \mathcal{C} , M is an object of \mathcal{D} if and only if M' and M'' are objects of \mathcal{D} —that is, \mathcal{D} is closed under subobjects, factor objects, and extensions.

The *quotient category* \mathcal{C}/\mathcal{D} of an abelian category by a Serre subcategory was first defined by Gabriel in his thesis [Gab62] as follows. The objects of \mathcal{C}/\mathcal{D} are the objects of \mathcal{C} . If M and N are objects of \mathcal{C} , then

$$\mathrm{Hom}_{\mathcal{C}/\mathcal{D}}(M, N) := \varinjlim \mathrm{Hom}_{\mathcal{C}}(M', N/N'),$$

where the direct limit is taken over all subobjects $M' \subseteq M$ and $N' \subseteq N$ where $M/M' \in \mathcal{D}$ and $N' \in \mathcal{D}$. Composition of morphisms in \mathcal{C}/\mathcal{D} is induced by composition in \mathcal{D} . There is also a canonical exact functor, the *quotient functor* $\pi : \mathcal{C} \rightarrow \mathcal{C}/\mathcal{D}$. For an object M , $\pi M = M$ and for a morphism f , πf is given by the image of f in the direct limit.

We state properties of morphisms in \mathcal{C}/\mathcal{D} as summarized in lecture notes of Paul Smith [Smi00].

Proposition 2.2.2 (See [Smi00, Proposition 13.7]). *Let $f \in \mathrm{Hom}_{\mathcal{C}}(M, N)$. Then*

1. *the kernel and cokernel of πf are $\pi(\ker f)$ and $\pi(\mathrm{coker} f)$ respectively;*
2. *πf is zero if and only if $\mathrm{im} f \in \mathcal{D}$;*
3. *πf is monic if and only if $\ker f \in \mathcal{D}$;*
4. *πf is epic if and only if $\mathrm{coker} f \in \mathcal{D}$;*
5. *πf is an isomorphism if and only if $\ker f \in \mathcal{D}$ and $\mathrm{coker} f \in \mathcal{D}$.*

2.3 Dimension in noncommutative rings

For a commutative ring S and an S -module M , the Krull dimension of M is defined to be the maximal length of a chain of prime ideals in $S/\mathrm{ann} M$. However, the relative scarcity of two-sided ideals makes this definition less useful for a noncommutative ring. Hence, there are several notions of dimension in the noncommutative setting. Of course, for a \mathbb{k} -algebra R , we can consider the dimension of R as a \mathbb{k} -vector space. We refer to this as the \mathbb{k} -dimension of R and denote it

$\dim_{\mathbb{k}}(R)$. We now discuss three different dimension functions for noncommutative rings.

The first notion of dimension we introduce is called the *Gelfand-Kirillov* dimension or GK dimension of R . A full treatment of GK dimension can be found in [KL00]. Let R be a finitely generated \mathbb{k} -algebra R and let V be a finite-dimensional \mathbb{k} -subspace of R containing 1 and generating R as a \mathbb{k} -algebra. We define the GK dimension of R to be

$$\text{GKdim } R = \limsup_{n \rightarrow \infty} \log_n \dim_{\mathbb{k}} V^n.$$

This definition is independent of the choice of generating subspace V .

Example 2.3.1. The first Weyl algebra $A_1 = \mathbb{k}\langle x, y \rangle / (xy - yx - 1)$ has GK dimension 2. We can choose the generating subspace $V = \{1, x, y\}$. Then for any monomial w in x and y of degree n , we can use the relation $xy - yx - 1$ to write w as a sum of monomials of the form $x^i y^j$ with $i + j \leq n$. Since there are exactly $n + 1$ such monomials of degree n , and V^n is spanned by the monomials of length at most n , we see that $\dim_{\mathbb{k}} V^n = (n + 1)(n + 2)/2$. Hence,

$$\text{GKdim } A_1 = \limsup_{n \rightarrow \infty} \log_n \frac{(n + 1)(n + 2)}{2} = 2.$$

We can also generalize the definition of Krull dimension to modules over noncommutative noetherian rings. A detailed discussion of Krull dimension can be found in chapter 6 of [MR87]. Let R be a noetherian ring and M a finitely generated right R -module. We define the *Krull dimension* of M , $\text{Kdim } M$, inductively as follows. If $M = 0$ then $\text{Kdim } M = -\infty$, otherwise if M is artinian, then $\text{Kdim } M = 0$. For $n \in \mathbb{N}$, we say that $\text{Kdim } M = n$ if (a) for each $m < n$, $\text{Kdim } M \neq m$ and (b) in any descending chain of submodules of M , all but finitely many factors have Krull dimension less than n .

The (*right*) *Krull dimension* of R is the Krull dimension of R as a right module over itself. This definition coincides with the usual Krull dimension in the case that R is commutative. For $n \in \mathbb{N}$, we say that a right R -module M is *n-critical* if $\text{Kdim } M = n$ and for each nonzero submodule $N \subseteq M$, $\text{Kdim } M/N < n$. Therefore, a 0-critical module is a simple module and any factor of a 1-critical module is artinian.

The last kind of dimension we consider here is *global dimension*. For a right R -module M , the *projective dimension* of M , denoted $\text{pd } M$, is defined to be the minimal length among all finite projective resolutions of M . If M does not admit a finite projective resolution, then $\text{pd } M = \infty$. The (*right*) *global dimension* of R , denoted $\text{gldim } R$, is defined to be

$$\text{gldim } R = \sup_{M \in \text{mod-}R} \{\text{pd } M\}.$$

2.4 Noncommutative projective schemes

In recent years, the field of *noncommutative projective algebraic geometry* has generalized the techniques of commutative algebraic geometry to the study of noncommutative rings. We begin by recalling some classical algebraic geometry. The necessary scheme-theoretic background is developed in detail in Hartshorne's text [Har77]. Throughout this section, let $R = \bigoplus_{i \in \mathbb{N}} R_i$ be a commutative \mathbb{N} -graded algebra, generated in degree 1 over $R_0 = \mathbb{k}$.

From the commutative ring R , we can construct the projective scheme $\text{Proj } R$. As a set, $\text{Proj } R$ consists of the homogeneous prime ideals of R not containing the irrelevant ideal $R_{\geq 1}$. We make $\text{Proj } R$ a topological space under the Zariski topology: the closed sets are of the form $V(\mathfrak{a}) = \{\mathfrak{p} \in \text{Proj } R \mid \mathfrak{p} \supseteq \mathfrak{a}\}$ for \mathfrak{a} a homogeneous ideal of R . We then construct the structure sheaf \mathcal{O} on $\text{Proj } R$ by localizing at homogeneous prime ideals. Specifically, let $R_{\mathfrak{p}}$ be the localization of R at the set of all homogeneous elements of R not in \mathfrak{p} . For an open set $U \subseteq \text{Proj } R$, define $\mathcal{O}(U)$ be the set of functions $f : U \rightarrow \prod_{\mathfrak{p} \in U} S_{\mathfrak{p}}$ such that $f(\mathfrak{p}) \in S_{\mathfrak{p}}$ and f is locally a quotient of elements in S .

As this construction of $\text{Proj } R$ shows, in classical algebraic geometry, the technique of localization and the abundance of prime ideals are important. A naïve generalization of this technique to noncommutative rings has at least two obstructions. First, in noncommutative rings, localization is well-behaved only at certain multiplicatively closed subsets called *Ore sets*. A more basic difficulty is the scarcity of two-sided ideals. Let $q \in \mathbb{k}^{\times}$ be a nonroot of 1. The quantum plane

$$\mathbb{k}_q[x, y] = \mathbb{k}\langle x, y \rangle / (xy - qyx)$$

is a noncommutative analogue of the projective line $\text{Proj } \mathbb{k}[x, y] = \mathbb{P}^1$. However, while $\mathbb{k}[x, y]$ has many homogeneous prime ideals, $\mathbb{k}_q[x, y]$ has only four: (0) , (x) , (y) and (x, y) .

Rather than attempting to generalize directly the construction of a projective scheme, one successful approach has instead been to generalize the study of sheaves over a scheme. This idea is motivated by a theorem of Serre. Let $\text{tors-}R$ be the full subcategory of $\text{gr-}R$ consisting of finite-dimensional modules. Let $\text{qgr-}R = \text{gr-}R / \text{tors-}R$.

Theorem 2.4.1 (Serre, see [Har77, Ex 5.9]). *There is an equivalence of categories $\text{coh}(\text{Proj } R) \cong \text{qgr-}R$.*

Therefore, studying the coherent sheaves $\text{Proj } R$ is the same thing as studying the category of R -modules. Noncommutative rings with few two-sided ideals can have rich (right) module categories, so this approach has been fruitful. In [AZ94], Michael Artin and James Zhang defined the noncommutative projective scheme of a noncommutative ring purely module-theoretically.

Definition 2.4.2 (Artin-Zhang, [AZ94]). Let A be a right noetherian graded \mathbb{k} -algebra. The *noncommutative projective scheme of A* is the triple

$$(\text{qgr-}A, \pi A, \mathcal{S})$$

where \mathcal{S} denotes the shift functor on $\text{qgr-}A$.

Much of the literature on noncommutative projective schemes has focused on *connected graded \mathbb{k} -algebras*— \mathbb{N} -graded \mathbb{k} -algebras with $A_0 = \mathbb{k}$. Since the commutative polynomial ring under its usual grading satisfies these hypotheses, connected graded \mathbb{k} -algebras are analogues of (quotients of) commutative polynomial rings. For a connected graded \mathbb{k} -algebra A of GK dimension 2, we call $\text{qgr-}A$ a *noncommutative projective curve*. If A has GK dimension 3, we call $\text{qgr-}A$ a *noncommutative projective surface*.

The noncommutative projective curves were classified by Artin and Stafford, who showed that every noncommutative curve was equivalent to the category

of coherent sheaves on a commutative curve. The classification of noncommutative projective surfaces is an active area of current research. The noncommutative analogues of $\mathbb{k}[x, y, z]$ (so-called *Artin-Schelter regular algebras* of dimension 3) are well-understood (see [AS87, ATV91, Ste96, Ste97]) and many other examples of noncommutative projective surfaces have been studied (see [Van01, Van11, Rog04, Sie11a]).

2.5 Generalized Weyl algebras

Fix $f \in \mathbb{k}[z]$, let $\sigma : \mathbb{k}[z] \rightarrow \mathbb{k}[z]$ be the automorphism given by $\sigma(z) = z + 1$, and let

$$A(f) = \frac{\mathbb{k}[z]\langle x, y \rangle}{\begin{pmatrix} xz = \sigma(z)x & yz = \sigma^{-1}(z)y \\ xy = f & yx = \sigma^{-1}(f) \end{pmatrix}}.$$

Then $A(f) = \mathbb{k}[z](\sigma, f)$ is a *generalized Weyl algebra of degree 1* with base ring $\mathbb{k}[z]$, defining element f and defining automorphism σ . Generalized Weyl algebras were introduced by Vladimir Bavula, who studied rings of the form $A(f)$ for f of arbitrary degree [Bav93, BJ01]. Timothy Hodges studied the same rings under the name noncommutative deformations of type-A Kleinian singularities [Hod93]. By results in [Bav93], for all f , $A(f)$ is a noncommutative noetherian domain of Krull dimension 1.

By a theorem of Bavula and Jordan, $A(f) \cong A(g)$ if and only if $f(z) = \eta g(\tau \pm z)$ for some $\eta, \tau \in \mathbb{k}$ with $\eta \neq 0$ [BJ01, Theorem 3.28]. Hence, by adjusting η we may assume f is a monic polynomial and by adjusting τ we may assume that 0 is a root of f . We may also assume that 0 is the largest integer root of f . Results of Bavula and Hodges show that the properties of $A(f)$ are determined by the distance between the roots of f . In general we say that two distinct roots, λ and μ are *congruent* if $\lambda - \mu \in \mathbb{Z}$. The global dimension of $A(f)$ depends only on whether f has multiple or congruent roots, as follows.

Theorem 2.5.1 (Bavula and Hodges, [Bav93, Theorem 5] and [Hod93, Theorem

4.4]). *The global dimension of A is equal to*

$$\text{gldim } A = \begin{cases} \infty, & \text{if } f \text{ has at least one multiple root} \\ 2, & \text{if } f \text{ has no multiple roots but has congruent roots;} \\ 1, & \text{if } f \text{ has neither multiple nor congruent roots.} \end{cases}$$

In this dissertation, we study generalized Weyl algebras $A(f)$ for quadratic polynomials f . Without loss of generality, $f = z(z + \alpha)$. When $\alpha = 0$, since f has a multiple root, we say we are in the *multiple root case*. When $\alpha \in \mathbb{N}^+$, we say we are in the *congruent root case*. Finally, when $\alpha \in \mathbb{k} \setminus \mathbb{Z}$, we say that f has distinct non-congruent roots and refer to this case as the *non-congruent root case*.

Like the first Weyl algebra, the rings $A(f)$ are naturally \mathbb{Z} -graded by letting $\deg x = 1$, $\deg y = -1$, $\deg z = 0$. Note that every graded right $A(f)$ -module is actually a graded $(\mathbb{k}[z], A(f))$ -bimodule; for any $\varphi \in \mathbb{k}[z]$, we define its left action on a right $A(f)$ -module M by

$$\varphi \cdot m = m\sigma^{-i}(\varphi)$$

for any $m \in M_i$. This gives M a bimodule structure by the relations $xz = \sigma(z)x$ and $yz = \sigma^{-1}(z)y$.

Bavula and Jordan [BJ01] call a polynomial $g(z) \in \mathbb{k}[z]$ *reflective* if there exists some $\beta \in \mathbb{k}$ such that $g(\beta - z) = g(z)$. They observe that every quadratic polynomial is reflective. Indeed, if f is quadratic, there exists an outer automorphism ω of $A(f)$ such that $\omega(x) = y$, $\omega(y) = x$, and $\omega(z) = 1 - \alpha - z$ which reverses the grading on A . More specifically, there is a group automorphism of \mathbb{Z} , and the \mathbb{k} -algebra automorphism ω respects this automorphism. The group automorphism, which we denote $\bar{\omega}$, is given by negation and $\omega(A_n) = A_{\bar{\omega}(n)}$. Together, we call $(\omega, \bar{\omega})$ a *$\bar{\omega}$ -twisted graded ring automorphism* of A .

For any group automorphism $\bar{\theta}$ of \mathbb{Z} , any $\bar{\theta}$ -twisted graded ring automorphism $(\theta, \bar{\theta})$ of A induces an autoequivalence of $\text{gr } A$, denoted θ_* . Given a module $M \in \text{gr } A$, θ_*M is defined to be M with the grading $(\theta_*M)_n = M_{\bar{\theta}(n)}$. We write θ_*m to regard the element $m \in M$ as an element of θ_*M . The action of an element $a \in A$ on an element $\theta_*m \in \theta_*M$ is given by

$$\theta_*m \cdot a = \theta_*(m\theta(a)).$$

We suppress the asterisk in the subscript and simply refer to the autoequivalence induced by $(\omega, \bar{\omega})$ as ω . We also define ω on a graded \mathbb{k} -vector space as the functor that reverses grading, i.e. $(\omega V)_n = V_{-n}$. Then, as Sierra notes in [Sie09, (4.2)], if M and N are right A -modules, then ω gives isomorphisms of graded \mathbb{k} -vector spaces

$$\begin{aligned} \underline{\mathrm{Hom}}_A(\omega M, \omega N) &\cong \omega \underline{\mathrm{Hom}}_A(M, N) \\ \underline{\mathrm{Ext}}_A^1(\omega M, \omega N) &\cong \omega \underline{\mathrm{Ext}}_A^1(M, N). \end{aligned} \tag{2.1}$$

Chapter 3

The graded module category $\text{gr-}A$

In this chapter, we carefully study the graded right modules over generalized Weyl algebras $A(f)$ defined by quadratic polynomials f . We determine the simple modules and classify some of the finite length indecomposable modules. We then turn our attention to the rank one projective right modules and the morphisms between them.

3.1 Simple modules

We first describe the simple modules of $\text{gr-}A(f)$.

Lemma 3.1.1. *Let $f = z(z + \alpha)$.*

1. *If $\alpha = 0$, then up to graded isomorphism the graded simple $A(f)$ -modules are:*

- $X^f = A/(x, z)A$ and its shifts $X^f\langle n \rangle$ for each $n \in \mathbb{Z}$;
- $Y^f = (A/(y, z - 1)A)\langle 1 \rangle$ and its shifts $Y^f\langle n \rangle$ for each $n \in \mathbb{Z}$;
- $M_\lambda^f = A/(z + \lambda)A$ for each $\lambda \in \mathbb{k} \setminus \mathbb{Z}$.

2. *If $\alpha \in \mathbb{N}^+$, then up to graded isomorphism the graded simple $A(f)$ -modules are:*

- $X^f = (A/(x, z + \alpha)A)\langle -\alpha \rangle$ and its shifts $X^f\langle n \rangle$ for each $n \in \mathbb{Z}$;

- $Y^f = (A/(y, z - 1)A) \langle 1 \rangle$ and its shifts $Y^f \langle n \rangle$ for each $n \in \mathbb{Z}$;
- $Z^f = A/(y^\alpha, x, z)A$ and its shift $Z^f \langle n \rangle$ for each $n \in \mathbb{Z}$;
- $M_\lambda^f = A/(z + \lambda)A$ for each $\lambda \in \mathbb{k} \setminus \mathbb{Z}$.

3. If $\alpha \in \mathbb{k} \setminus \mathbb{Z}$, then up to graded isomorphism the graded simple $A(f)$ -modules are:

- $X_0^f = A/(x, z)A$ and its shifts $X_0^f \langle n \rangle$ for each $n \in \mathbb{Z}$;
- $Y_0^f = (A/(y, z - 1)A) \langle 1 \rangle$ and its shifts $Y_0^f \langle n \rangle$ for each $n \in \mathbb{Z}$;
- $X_\alpha^f = A/(x, z + \alpha)A$ and its shifts $X_\alpha^f \langle n \rangle$ for each $n \in \mathbb{Z}$;
- $Y_\alpha^f = (A/(y, z + \alpha - 1)A) \langle 1 \rangle$ and its shifts $Y_\alpha^f \langle n \rangle$ for each $n \in \mathbb{Z}$;
- $M_\lambda^f = A/(z + \lambda)A$ for each $\lambda \in \mathbb{k} \setminus (\mathbb{Z} \cup \mathbb{Z} + \alpha)$.

Proof. We do, as an example, the case that $\alpha = 0$. The other cases follow similarly from [Bav93, §3]. In [Bav93, §3], Bavula studies the simple $\mathbb{k}[z]$ -torsion A -modules, that is, modules for which $\text{tor}(M) := \{m \in M \mid m \cdot g = 0 \text{ for some } 0 \neq g \in \mathbb{k}[z]\}$ is equal to M . Every graded simple right A -module is isomorphic to A/I for some homogeneous right ideal I of A . Further, every homogeneous element of A can be written as gx^i or gy^i for some $g \in \mathbb{k}[z]$ and $i \in \mathbb{N}$. Hence, for every element a of A/I , there exists $h \in \mathbb{k}[z]$ such that $a \cdot h = 0$, so A/I is $\mathbb{k}[z]$ -torsion. By [Bav93, Theorem 3.2], up to ungraded isomorphism, the simple $\mathbb{k}[z]$ -torsion A -modules are

- $A/(x, z)A$
- $A/(y, z - 1)A$
- One module $A/(z + \lambda)A$ for each coset of \mathbb{k}/\mathbb{Z} .

For each $\lambda \in \mathbb{k} \setminus \mathbb{Z}$, observe that $M_{\lambda+1}^f \cong M_\lambda^f \langle 1 \rangle$ via the isomorphism mapping $\bar{1}$ to \bar{y} . Further, by Bavula's theorem, if $\lambda - \mu \notin \mathbb{Z}$, then M_λ^f and M_μ^f are not even ungraded isomorphic. Hence, if $\lambda \neq \mu$ then $M_\lambda^f \not\cong M_\mu^f$ in $\text{gr-}A$. Finally, we see that there are no other graded isomorphisms between any shift of X^f , Y^f , or M_λ^f simply by looking at the degrees in which these modules are nonzero (see Remark 3.1.2 below). Hence, we conclude that the graded isomorphism classes of

graded simples correspond to the shifts $X^f\langle n \rangle$, $Y^f\langle n \rangle$, and one module M_λ^f for each element of $\mathbb{k} \setminus \mathbb{Z}$. \square

When there is no danger of confusion, we will make two changes in notation for convenience. When it is clear which case we are in (multiple, congruent, or distinct roots), we will suppress the superscript on graded simple modules and refer to them as X , Y , Z , and M_λ . Also, for a right ideal I , we will often refer to the element $a + I \in A/I$ simply as a .

Remark 3.1.2. We also remark that for each integer n and each simple module S , $\dim_{\mathbb{k}} S \leq 1$. We explicitly give the degrees in which each simple module is nonzero. Additionally, by using the explicit description of the simple modules as quotients of A , we can also determine the action of the autoequivalence ω on the graded simple modules. In all cases, for $\lambda \in \mathbb{k} \setminus (\mathbb{Z} \cup \mathbb{Z} + \alpha)$, $\dim_{\mathbb{k}}(M_\lambda)_n = 1$ for all n and $\omega(M_\lambda) = M_\mu$ for some $\mu \in \mathbb{k} \setminus (\mathbb{Z} \cup \mathbb{Z} + \alpha)$.

1. If $\alpha = 0$, then

- $\dim_{\mathbb{k}} X_n = 1$ if and only if $n \leq 0$,
- $\dim_{\mathbb{k}} Y_n = 1$ if and only if $n > 0$,
- $\omega(X\langle n \rangle) \cong Y\langle -n - 1 \rangle$ and $\omega(Y\langle n \rangle) \cong X\langle -n - 1 \rangle$.

2. If $\alpha \in \mathbb{N}^+$, then

- $\dim_{\mathbb{k}} X_n = 1$ if and only if $n \leq -\alpha$,
- $\dim_{\mathbb{k}} Y_n = 1$ if and only if $n > 0$,
- $\dim_{\mathbb{k}} Z_n = 1$ if and only if $-\alpha < n \leq 0$,
- $\omega(X\langle n \rangle) \cong Y\langle \alpha - n - 1 \rangle$, $\omega(Y\langle n \rangle) \cong Y\langle \alpha - n - 1 \rangle$, and $\omega(Z\langle n \rangle) \cong Z\langle \alpha - n - 1 \rangle$.

3. If $\alpha \in \mathbb{k} \setminus \mathbb{Z}$, then

- $\dim_{\mathbb{k}}(X_0)_n = \dim_{\mathbb{k}}(X_\alpha)_n = 1$ if and only if $n \leq 0$,
- $\dim_{\mathbb{k}}(Y_0)_n = \dim_{\mathbb{k}}(Y_\alpha)_n = 1$ if and only if $n > 0$,

- $\omega(X_0\langle n \rangle) \cong Y_\alpha\langle -n - 1 \rangle$, $\omega(Y_0\langle n \rangle) \cong Y_\alpha\langle -n - 1 \rangle$,
- $\omega(X_\alpha\langle n \rangle) \cong Y_0\langle -n - 1 \rangle$, and $\omega(Y_\alpha\langle n \rangle) \cong Y_0\langle -n - 1 \rangle$.

Lemma 3.1.3. *Let $\alpha \in \mathbb{N}$ and let $n \in \mathbb{Z}$. Then $(z+n)(X\langle n \rangle) = (z+n)(Y\langle n \rangle) = 0$. If $\alpha \neq 0$, then $(z+n)(Z\langle n \rangle) = 0$. As graded left $\mathbb{k}[z]$ -modules, we have*

$$(A/zA)\langle n \rangle \cong (A/(z-1)A)\langle n+1 \rangle \cong \bigoplus_{j \in \mathbb{Z}} \frac{\mathbb{k}[z]}{(z+n)}.$$

Proof. Recall that there is a left action of $\mathbb{k}[z]$ on an A -module. If $\deg m = i$ then for $p \in \mathbb{k}[z]$, $p \cdot m = m\sigma^{-i}(p)$. This result follows from the fact that in $X\langle n \rangle = (A/(x, z + \alpha)A)\langle -\alpha + n \rangle$, $\deg 1 = n - \alpha$ and

$$(z+n) \cdot 1 = 1\sigma^{\alpha-n}(z+n) = z + \alpha = 0.$$

Similarly, in $Y\langle n \rangle$, $\deg 1 = n + 1$ so

$$(z+n) \cdot 1 = 1\sigma^{-n-1}(z+n) = z - 1 = 0,$$

and in $Z\langle n \rangle$, $\deg 1 = 0$ and

$$(z+n) \cdot 1 = z = 0.$$

For each $j \geq n$, $((A/zA)\langle n \rangle)_j$ is generated as a left $\mathbb{k}[z]$ module by x^{j-n} . Further, for all $g \in \mathbb{k}[z]$, $g \cdot x^{j-n} = x^{j-n}\sigma^{n-j}(g) = \sigma^n(g)x^{j-n}$. Hence, as a left $\mathbb{k}[z]$ -module the annihilator of $((A/zA)\langle n \rangle)_j$ is given exactly by the ideal $(z+n)$. For $j < n$, $((A/zA)\langle n \rangle)_j$ is generated as a left $\mathbb{k}[z]$ module by y^{n-j} . By a similar argument, the annihilator is given exactly by the ideal $(z+n)$ so

$$(A/zA)\langle n \rangle \cong \bigoplus_{j \in \mathbb{Z}} \frac{\mathbb{k}[z]}{(z+n)}.$$

The same proof shows that

$$(A/(z-1)A)\langle n+1 \rangle \cong \bigoplus_{j \in \mathbb{Z}} \frac{\mathbb{k}[z]}{(z+n)}.$$

□

Lemma 3.1.4. *Let $\alpha \in \mathbb{k} \setminus \mathbb{Z}$ and let $n \in \mathbb{Z}$. Then $(z+n)(X_0\langle n \rangle) = 0 = (z+n)(Y_0\langle n \rangle)$ and $(z+\alpha+n)(X_\alpha\langle n \rangle) = 0 = (z+\alpha+n)(Y_\alpha\langle n \rangle)$. As graded left $\mathbb{k}[z]$ -modules, we have*

$$(A/zA)\langle n \rangle \cong (A/(z-1)A)\langle n+1 \rangle \cong \bigoplus_{j \in \mathbb{Z}} \frac{\mathbb{k}[z]}{(z+n)} \text{ and}$$

$$(A/(z+\alpha)A)\langle n \rangle \cong (A/(z+\alpha-1)A)\langle n+1 \rangle \cong \bigoplus_{j \in \mathbb{Z}} \frac{\mathbb{k}[z]}{(z+\alpha+n)}.$$

Proof. This follows from the same proof as that of Lemma 3.1.3. □

Since $M_\lambda \cong \bigoplus_{j \in \mathbb{Z}} \mathbb{k}[z]/(z+\lambda)$ as a left $\mathbb{k}[z]$ -module, when combined with the previous lemmas, any finite length graded A -module, when considered as a left $\mathbb{k}[z]$ -module, is supported at finitely many \mathbb{k} -points of $\text{Spec } \mathbb{k}[z]$. We restate [Sie09, Definition 4.9].

Definition 3.1.5. If M is a graded A -module of finite length, define the *support* of M , $\text{Supp } M$, to be the support of M as a left $\mathbb{k}[z]$ -module. We are interested in the cases when $\text{Supp } M \subset \mathbb{Z}$ or $\text{Supp } M \subset \mathbb{Z} - \alpha$. When $\text{Supp } M \subset \mathbb{Z}$, we say that M is *integrally supported*. We are also interested in cases when $\text{Supp } M = \{n\}$ or $\text{Supp } M = \{n - \alpha\}$ for some $n \in \mathbb{Z}$ (we say M is *simply supported at n or $n - \alpha$*).

Lemma 3.1.3 shows that in the case that $\alpha \in \mathbb{N}$, $X\langle n \rangle$ and $Y\langle n \rangle$ are the unique simples supported at $-n$. In the case that $\alpha \notin \mathbb{Z}$, Lemma 3.1.4 shows that $X_0\langle n \rangle$ and $Y_0\langle n \rangle$ are the unique simples supported at $-n$ and $X_\alpha\langle n \rangle$ and $Y_\alpha\langle n \rangle$ are the unique simples supported at $-(n+\alpha)$. For $\lambda \in \mathbb{k} \setminus (\mathbb{Z} \cup \mathbb{Z} + \alpha)$, the simple module M_λ is the unique simple supported at $-\lambda$.

3.2 The structure constants of a graded submodule of the graded quotient ring of A

We seek to understand the rank one projective modules of $\text{gr-}A(f)$. Since $A(f)$ is noetherian, $A(f)$ is an Ore domain, so we can construct $Q_{\text{gr}}(A(f))$, the graded quotient ring of $A(f)$. Every homogeneous element of $A(f)$ can be written

as $x^i g(z)$ or $y^i g(z)$ for some $i \geq 0$ and some $g(z) \in \mathbb{k}[z]$. Since in the graded quotient ring $y = x^{-1}f$, we see that $Q_{\text{gr}}(A(f))$ embeds in the skew Laurent polynomial ring $\mathbb{k}(z)[x, x^{-1}; \sigma]$. Finally, since every element of $\mathbb{k}(z)[x, x^{-1}; \sigma]$ can be written as a quotient of elements of $A(f)$, therefore $Q_{\text{gr}}(A(f)) = \mathbb{k}(z)[x, x^{-1}; \sigma]$.

To understand the rank one projective modules, we begin by considering submodules of $Q_{\text{gr}}(A(f))$. Let I be a finitely generated graded right A -submodule of $Q_{\text{gr}}(A(f))$. Recall that $\sigma(z) = z + 1$ and every graded right A -module is a graded left $\mathbb{k}[z]$ -module with

$$\varphi \cdot m = m \cdot \sigma^{-i}(\varphi)$$

for $\varphi \in \mathbb{k}[z]$ and $\deg m = i$. We will examine I as a graded left $\mathbb{k}[z]$ -submodule of $Q_{\text{gr}}(A)$. As a left $\mathbb{k}[z]$ -module,

$$Q_{\text{gr}}(A) \cong \bigoplus_{i \in \mathbb{Z}} \mathbb{k}(z)x^i.$$

Suppose I is generated as an A -module by the homogeneous generators m_1, \dots, m_r , with $\deg m_i = d_i$. Then for each $n \in \mathbb{Z}$, $I_n = \sum m_i A_{n-d_i}$ where I_n is the degree n graded component of I . Since each graded component of A is finitely generated as a left $\mathbb{k}[z]$ -module, so is I_n . If we clear denominators and use the fact that $\mathbb{k}[z]$ is a PID, we deduce that I_n is generated as a left $\mathbb{k}[z]$ -module by a single element $a_n x^n$ where $a_n \in \mathbb{k}(z)$. Denote by (a_n) the left $\mathbb{k}[z]$ -submodule of $\mathbb{k}(z)$ generated by a_n . Then

$$I = \bigoplus_{i \in \mathbb{Z}} (a_i)x^i.$$

Because I is a right A -submodule, we have for each $i \in \mathbb{Z}$

$$\begin{aligned} (a_i)x^i \cdot x &\subseteq (a_{i+1})x^{i+1} && \text{and} \\ (a_{i+1})x^{i+1} \cdot y &= (a_{i+1})\sigma^i(f)x^i \subseteq (a_i)x^i. \end{aligned}$$

Therefore, for each $i \in \mathbb{Z}$, we have $(a_i) \subseteq (a_{i+1})$ and $(a_{i+1}\sigma^i(f)) \subseteq (a_i)$. Define $c_i = a_i a_{i+1}^{-1}$. We then have $1 \mid c_i$ and $c_i \mid \sigma^i(f)$, so $c_i \in \mathbb{k}[z]$. By multiplying by an appropriate element of \mathbb{k} , we assume that c_i is monic so

$$c_i \in \{1, \sigma^i(z), \sigma^i(z + \alpha), \sigma^i(f)\}.$$

Definition 3.2.1. We call the elements of this sequence $\{c_i\}_{i \in \mathbb{Z}}$ the *structure constants* of I . The lemma below shows that a finitely generated graded right A -submodule of $Q_{\text{gr}}(A)$ is determined up to graded isomorphism by its structure constants.

As an example, we compute the structure constants of the ring A .

Example 3.2.2. For $n \in \mathbb{Z}$, A_n is generated as a left $\mathbb{k}[z]$ -module by x^n when $n \geq 0$ and y^{-n} when $n < 0$. Also, for $n > 0$, $y^n x^n = \sigma^{-1}(f) \cdots \sigma^{-n}(f)$, so as a graded left $\mathbb{k}[z]$ -module,

$$A = \bigoplus_{i \in \mathbb{Z}} (a_i) x^i \text{ with } a_i = \begin{cases} 1, & i \geq 0, \\ \sigma^{-1}(f) \cdots \sigma^i(f), & i < 0. \end{cases}$$

The structure constants $\{c_i\}$ of A are therefore given by

$$c_i = \begin{cases} 1 & i \geq 0, \\ \sigma^i(f) & i < 0. \end{cases}$$

Lemma 3.2.3. Let I and J be finitely generated graded submodules of $Q_{\text{gr}}(A)$ with structure constants $\{c_i\}$ and $\{d_i\}$, respectively. Then $I \cong J$ as graded right A -modules if and only if $c_i = d_i$ for all $i \in \mathbb{Z}$.

Proof. As argued above there exist $\{a_i\}, \{b_i\} \subseteq \mathbb{k}(z)$ such that

$$I = \bigoplus_{i \in \mathbb{Z}} (a_i) x^i \quad \text{and} \quad J = \bigoplus_{i \in \mathbb{Z}} (b_i) x^i.$$

Then by definition, for each $i \in \mathbb{Z}$, $c_i = a_i a_{i+1}^{-1}$ and $d_i = b_i b_{i+1}^{-1}$. Let $g = a_0^{-1} b_0 \in \mathbb{k}(z)$. If $c_i = d_i$ for all $i \in \mathbb{Z}$, then $(a_i g) = (b_i)$ for all i . Hence, $I \cong J$ via left multiplication by g .

Conversely, suppose $\varphi : I \rightarrow J$ is a graded isomorphism of A -modules. Since, for each $i \in \mathbb{Z}$, $\varphi : I_i \rightarrow J_i$ is an isomorphism, we must have, up to a scalar in \mathbb{k}^\times , $\varphi(a_i x^i) = b_i x^i$. Then, up to a scalar,

$$\begin{aligned} b_i x^{i+1} &= \varphi(a_i x^i) x = \varphi(a_i x^{i+1}) = \varphi(a_{i+1} c_i x^{i+1}) = \varphi(a_{i+1} x^{i+1} \sigma^{-(i+1)}(c_i)) \\ &= \varphi(a_{i+1} x^{i+1}) \sigma^{-(i+1)}(c_i) = b_{i+1} x^{i+1} \sigma^{-(i+1)}(c_i) = b_{i+1} c_i x^{i+1}, \end{aligned}$$

so up to a scalar $d_i = b_i b_{i+1}^{-1} = c_i$ for all i . Since we assumed that structure constants were monic, $c_i = d_i$ for all $i \in \mathbb{Z}$. \square

Lemma 3.2.4. *Suppose $I = \bigoplus_{i \in \mathbb{Z}} (b_i) x^i$ is a finitely generated graded right A -submodule of $Q_{\text{gr}}(A)$ with structure constants $\{c_i\}$. Then for $n \gg 0$, $c_n = 1$ and $c_{-n} = \sigma^{-n}(f)$.*

Proof. Since I is finitely generated as an A -module, if $n \in \mathbb{Z}$ is greater than the highest degree of all generators, we have $I_n \cdot x = I_{n+1}$. Then $(b_n) = (b_{n+1})$ so $c_n = 1$. On the other hand, if n is less than the least degree of all generators, then $I_n \cdot y = I_{n-1}$. That is,

$$(b_{n-1})x^{n-1} = (b_n)x^n y = (b_n)x^{n-1} f = (b_n)\sigma^{n-1}(f)x^{n-1},$$

so $c_{n-1} = \sigma^{n-1}(f)$. Hence, for $n \gg 0$, $c_n = 1$ and $c_{-n} = \sigma^{-n}(f)$. \square

We remark that for any choice $\{c_i\}_{i \in \mathbb{Z}}$ satisfying (i) for each integer n , $c_n \in \{1, \sigma^n(z), \sigma^n(z + \alpha), \sigma^n(f)\}$ and (ii) $c_n = 1$ and $c_{-n} = \sigma^{-n}(f)$ for $n \gg 0$, we can construct a module with structure constants $\{c_i\}$. Let $b_0 = 1$ and for all integers i , define b_i such that $b_i b_{i+1}^{-1} = c_i$. Let $I = \bigoplus_{i \in \mathbb{Z}} (b_i) x^i$. Since $b_i x^i \cdot x \in (b_{i+1}) x^{i+1}$ and $b_i x^i \cdot y \in (b_{i-1}) x^{i-1}$, therefore I is a graded submodule of $Q_{\text{gr}}(A)$. Further, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $c_n = 1$ and $c_{-n} = \sigma^{-n}(f)$. Thus, the elements $\{b_i x^i \mid -N \leq i \leq N\}$ generate I as an A -module, so I is a finitely generated graded right A -submodule of $Q_{\text{gr}}(A)$.

Hence, isomorphism classes of finitely generated graded right A -submodules of $Q_{\text{gr}}(A)$ are in bijection with such sequences of structure constants, $\{c_i\}$. One reason taking this point of view is useful is that we can now state properties of a graded submodule $I \subseteq Q_{\text{gr}}(A)$ in terms of its structure constants and vice versa. First, we show that the simple factors of I are determined by its structure constants. We have the two following lemmas, one in the case that f has congruent or multiple roots ($\alpha \in \mathbb{N}$) and one in the case that the roots of f are distinct ($\alpha \in \mathbb{k} \setminus \mathbb{Z}$).

Lemma 3.2.5. *Let $\alpha \in \mathbb{N}$. Let $I = \bigoplus_{i \in \mathbb{Z}} (a_i) x^i$ be a finitely generated graded right A -submodule of $Q_{\text{gr}}(A)$ with structure constants $\{c_i\}$. Then*

1. I surjects onto $X\langle n \rangle$ if and only if $c_{n-\alpha} \in \{1, \sigma^{n-\alpha}(z)\}$,
2. I surjects onto $Y\langle n \rangle$ if and only if $c_n \in \{\sigma^n(z), \sigma^n(f)\}$,
3. If $\alpha > 0$, I surjects onto $Z\langle n \rangle$ if and only if $c_{n-\alpha} \in \{\sigma^{n-\alpha}(z + \alpha), \sigma^{n-\alpha}(f)\}$ and $c_n \in \{1, \sigma^n(z + \alpha)\}$.

Proof. Suppose I surjects onto $Z\langle n \rangle$. Then there exists a graded submodule $J = \bigoplus_{i \in \mathbb{Z}} (b_i)x^i$ of I such that

$$0 \longrightarrow J \longrightarrow I \longrightarrow Z\langle n \rangle \longrightarrow 0$$

is a short exact sequence. Let $\{d_i\}$ be the structure constants of J .

Because $Z\langle n \rangle_i = 0$ for all $i > n$ and all $i \leq n - \alpha$, we must have $b_i = a_i$ for all $i > n$ and all $i \leq n - \alpha$. Now by Lemma 3.1.3, as a left $\mathbb{k}[z]$ -module, for all $n - \alpha < i \leq n$, $Z\langle n \rangle_i \cong \mathbb{k}[z]/(z + n)$. Hence, $b_i = (z + n)a_i$ for all $n - \alpha < i \leq n$. Therefore, for all $i \neq n, n - \alpha$, $c_i = d_i$. However, $(z + n)d_{n-\alpha} = c_{n-\alpha}$ and $d_n = (z + n)c_n$. Since we know $d_{n-\alpha} \in \{1, \sigma^{n-\alpha}(z), \sigma^{n-\alpha}(z + \alpha), \sigma^{n-\alpha}(f)\}$, this forces $c_{n-\alpha} \in \{\sigma^{n-\alpha}(z + \alpha), \sigma^{n-\alpha}(f)\}$. Similarly, $c_n \in \{1, \sigma^n(z + \alpha)\}$.

Conversely, if $c_{n-\alpha} \in \{\sigma^{n-\alpha}(z + \alpha), \sigma^{n-\alpha}(f)\}$ and $c_n \in \{1, \sigma^n(z + \alpha)\}$, then we can construct $J = \bigoplus_{i \in \mathbb{Z}} (b_i)\mathbb{k}[z] \subseteq I$ by setting $b_i = a_i$ for all $i > n$ and $i \leq n - \alpha$ and $b_i = (z + n)a_i$ for all $n - \alpha < i \leq 0$. If we define $d_i = b_i b_{i+1}^{-1}$ to be monic (by multiplying by the appropriate scalar), then because $c_{n-\alpha} \in \{\sigma^{n-\alpha}(z + \alpha), \sigma^{n-\alpha}(f)\}$ and $c_n \in \{1, \sigma^n(z + \alpha)\}$, therefore $d_{n-\alpha} \in \{1, \sigma^{n-\alpha}(z + \alpha)\}$ and $d_n \in \{\sigma^n(z), \sigma^n(f)\}$. For all other integers i , $d_i = c_i$. Therefore, the $\{d_i\}$ are the structure constants of a finitely generated graded submodule of $Q_{\text{gr}}(A)$ which is isomorphic to J . By [Bav93, Theorem 2.1], A is 1-critical. Hence, the factor module I/J has finite length. Also, I/J is simply supported at $-n$, and is nonzero only in degrees $n - \alpha < i \leq 0$. Hence, $I/J \cong Z\langle n \rangle$ so I surjects onto $Z\langle n \rangle$.

The cases of I surjecting onto $X\langle n \rangle$ or $Y\langle n \rangle$ are similar but give a condition on just one structure constant each. \square

Lemma 3.2.6. *Let $\alpha \in \mathbb{k} \setminus \mathbb{Z}$. Let $I = \bigoplus_{i \in \mathbb{Z}} (a_i)x^i$ be a finitely generated graded right A -submodule of $Q_{\text{gr}}(A)$ with structure constants $\{c_i\}$. Then*

1. I surjects onto $X^0\langle n \rangle$ if and only if $c_n \in \{1, \sigma^n(z + \alpha)\}$,
2. I surjects onto $X^\alpha\langle n \rangle$ if and only if $c_n \in \{1, \sigma^n(z)\}$,
3. I surjects onto $Y^0\langle n \rangle$ if and only if $c_n \in \{\sigma^n(z), \sigma^n(f)\}$,
4. I surjects onto $Y^\alpha\langle n \rangle$ if and only if $c_n \in \{\sigma^n(z + \alpha), \sigma^n(f)\}$.

Proof. This follows from essentially the same proof as Lemma 3.2.5. □

Observe that in the cases of distinct roots or a multiple root, whether a simple module is a factor of a finitely generated graded right A -submodule of $Q_{\text{gr}}(A)$ or not depends on only a single structure constant. In the case of congruent roots, each simple factor of the form $Z\langle n \rangle$ is determined by two different structure constants. Additionally, as a consequence of the constructions in Lemmas 3.2.5 and 3.2.6, we obtain the following two corollaries, the latter of which is an analogue of [Sie09, Lemma 4.11].

Corollary 3.2.7. *Let I be a finitely generated graded right A -submodule of $Q_{\text{gr}}(A)$ and let S be a simple graded A -module. Then for each $n \in \mathbb{Z}$*

$$\dim_{\mathbb{k}} \underline{\text{Hom}}_A(I, S)_n \leq 1.$$

Proof. We claim that there is a unique kernel $J \subseteq I$ for any surjection $I \rightarrow S$. If S is supported at $\mathbb{Z} \cup (\mathbb{Z} - \alpha)$ then this follows from the construction in Lemmas 3.2.5 and 3.2.6.

If $S = M_\lambda$ for some $\lambda \in \mathbb{k}$, $\lambda \notin \mathbb{Z} \cup \mathbb{Z} + \alpha$, then notice that we can construct $J = (z + \lambda)I$ such that $I/J \cong M_\lambda$. Further, since M_λ is simply supported at $-\lambda$ with degree 1 in each graded component, J is the unique kernel for any surjection $I \rightarrow M_\lambda$.

Hence for any simple module S , if there is a surjection $I \rightarrow S$, we have

$$\underline{\text{Hom}}_A(I, S) \cong \underline{\text{Hom}}_A(I/J, S) \cong \underline{\text{Hom}}_A(S, S).$$

Since all graded simple modules have \mathbb{k} -dimension 0 or 1 in each graded component, we conclude that for each $n \in \mathbb{Z}$, $\dim_{\mathbb{k}} \underline{\text{Hom}}_A(I, S)_n \leq 1$. □

Corollary 3.2.8. *Let I be a finitely generated graded right A -submodule of $Q_{\text{gr}}(A)$. If $\alpha \in \mathbb{N}$, then for all $n \gg 0$,*

$$\text{Hom}_{\text{gr-}A}(I, X\langle n \rangle) = \text{Hom}_{\text{gr-}A}(I, Y\langle -n \rangle) = \mathbb{k}.$$

If $\alpha \in \mathbb{k} \setminus \mathbb{Z}$, then for all $n \gg 0$,

$$\text{Hom}_{\text{gr-}A}(I, X_0\langle n \rangle) = \text{Hom}_{\text{gr-}A}(I, Y_0\langle -n \rangle) = \mathbb{k} \text{ and}$$

$$\text{Hom}_{\text{gr-}A}(I, X_\alpha\langle n \rangle) = \text{Hom}_{\text{gr-}A}(I, Y_\alpha\langle -n \rangle) = \mathbb{k}.$$

Proof. The result follows from Lemmas 3.2.4, 3.2.5, 3.2.6, and Corollary 3.2.7. \square

3.3 Finite length modules

We now seek to understand the finite length modules of $\text{gr-}A(f)$. We will see that the extensions between graded simple modules are few: if M and M' are graded simple modules, then $\underline{\text{Ext}}_A^1(M, M')$ is either 0 or \mathbb{k} . For clarity, we consider the three cases (congruent, multiple, and non-congruent roots) separately.

3.3.1 Multiple root

Let $\alpha = 0$ so that $f = z^2$ and we are in the multiple root case. We record the Ext groups between simple modules.

Lemma 3.3.1. *1. As graded vector spaces, $\underline{\text{Ext}}_A^1(X, Y) = \underline{\text{Ext}}_A^1(Y, X) = \mathbb{k}$, concentrated in degree 0.*

2. As graded vector spaces, $\underline{\text{Ext}}_A^1(X, X) = \underline{\text{Ext}}_A^1(Y, Y) = \mathbb{k}$, concentrated in degree 0.

3. Let $\lambda, \mu \in \mathbb{k} \setminus \mathbb{Z}$. If $\lambda \neq \mu$, then $\text{Ext}_{\text{gr-}A}^1(M_\lambda, M_\mu) = 0$, but $\text{Ext}_{\text{gr-}A}^1(M_\lambda, M_\lambda) = \mathbb{k}$.

4. Let $\lambda \in \mathbb{k} \setminus \mathbb{Z}$. Let $S \in \{X, Y\}$. Then $\underline{\text{Ext}}_A^1(M_\lambda, S) = \underline{\text{Ext}}_A^1(S, M_\lambda) = 0$.

Proof. Recall that $X = A/(x, z)A$ and $Y = A/(y, z - 1)A\langle 1 \rangle$. Let I_X and I_Y be the ideals of A defining X and Y respectively, that is, $I_X = (x, z)A$ and $I_Y = (y, z - 1)A$. Now for $S \in \{X, Y\}$, we have the exact sequence

$$0 \rightarrow I_S\langle d \rangle \rightarrow A\langle d \rangle \rightarrow S \rightarrow 0 \quad (3.1)$$

where if $S = X$ then $d = 0$ and if $S = Y$ then $d = 1$. For any graded simple module S' , we then apply the functor $\underline{\text{Hom}}_A(-, S')$ to yield

$$\begin{aligned} 0 \rightarrow \underline{\text{Hom}}_A(S, S') \rightarrow \underline{\text{Hom}}_A(A, S')\langle -d \rangle \rightarrow \underline{\text{Hom}}_A(I_S, S')\langle -d \rangle \\ \rightarrow \underline{\text{Ext}}_A^1(S, S') \rightarrow 0. \end{aligned} \quad (3.2)$$

We know that $\underline{\text{Hom}}_A(S, S') = \mathbb{k}$, concentrated in degree 0 if and only if $S = S'$, otherwise $\underline{\text{Hom}}_A(S, S') = 0$. We also know that as a graded \mathbb{k} -vector space, $\underline{\text{Hom}}_A(A, S')\langle -d \rangle \cong S'\langle -d \rangle$. Additionally, based on Lemma 3.2.5 and Corollary 3.2.7, we can compute $\underline{\text{Hom}}_A(I_S, S')\langle -d \rangle$. Hence, we will be able to deduce $\underline{\text{Ext}}_A^1(S, S')$.

1. Let $S = X$, $S' = Y$, and $d = 0$ in the exact sequence (3.2). Notice that by the construction in Lemma 3.2.5, A and I_X have the same structure constants except in degree 0, where A has structure constant 1 and I_X has structure constant z . Hence, $\underline{\text{Hom}}_A(A, Y)$ and $\underline{\text{Hom}}_A(I_X, Y)$ differ only in degree 0 where $\text{Hom}_{\text{gr-}A}(A, Y) = 0$, $\text{Hom}_{\text{gr-}A}(I_X, Y) = \mathbb{k}$ so $\underline{\text{Ext}}_A^1(X, Y) = \mathbb{k}$, concentrated in degree 0. By applying the autoequivalence ω as in equation (2.1), we deduce $\underline{\text{Ext}}_A^1(Y, X) = \mathbb{k}$.
2. Let $S = S' = X$ and $d = 0$ in the exact sequence (3.2). In this case, $\underline{\text{Hom}}_A(X, X) = \mathbb{k}$. Again, by the construction in Lemma 3.2.5, A and I_X have the same structure constants except in degree 0, where A has structure constant 1 and I_X has structure constant z . Notice then that $\underline{\text{Hom}}_A(A, X)_n = \underline{\text{Hom}}_A(I_X, X)_n$ for every degree $n \in \mathbb{Z}$. By considering the exact sequence in each degree, we conclude $\underline{\text{Ext}}_A^1(X, X) = \mathbb{k}$, concentrated in degree 0. Applying ω yields the result $\underline{\text{Ext}}_A^1(Y, Y) = \mathbb{k}$.

3. Apply $\text{Hom}_{\text{gr-}A}(-, M_\mu)$ to the short exact sequence

$$0 \rightarrow (z + \lambda)A \rightarrow A \rightarrow M_\lambda \rightarrow 0 \quad (3.3)$$

to yield

$$\begin{aligned} 0 &\rightarrow \text{Hom}_{\text{gr-}A}(M_\lambda, M_\mu) \rightarrow \text{Hom}_{\text{gr-}A}(A, M_\mu) \rightarrow \text{Hom}_{\text{gr-}A}((z + \lambda)A, M_\mu) \\ &\rightarrow \text{Ext}_{\text{gr-}A}^1(M_\lambda, M_\mu) \rightarrow 0. \end{aligned}$$

Since $\dim_{\mathbb{k}}(M_\mu)_0 = 1$, $\text{Hom}_{\text{gr-}A}(A, M_\mu) \cong \text{Hom}_{\text{gr-}A}((z + \lambda)A, M_\mu) = \mathbb{k}$. Hence, it follows that $\text{Ext}_{\text{gr-}A}^1(M_\lambda, M_\mu) \cong \text{Hom}_{\text{gr-}A}(M_\lambda, M_\mu)$ which is \mathbb{k} when $\lambda = \mu$ and 0 otherwise.

4. Let $\lambda \in \mathbb{k} \setminus \mathbb{Z}$ and apply $\underline{\text{Hom}}_A(-, M_\lambda)$ to the short exact sequence (3.1). This yields

$$0 \rightarrow \underline{\text{Hom}}_A(A, M_\lambda)\langle -d \rangle \rightarrow \underline{\text{Hom}}_A(I_S, M_\lambda)\langle -d \rangle \rightarrow \underline{\text{Ext}}_A^1(S, M_\lambda) \rightarrow 0.$$

Now by Corollary 3.2.7, in every graded component

$$\underline{\text{Hom}}_A(A, M_\lambda) \cong \underline{\text{Hom}}_A(I_S, M_\lambda)$$

so $\underline{\text{Ext}}_A^1(S, M_\lambda) = 0$.

Now let $S \in \{X, Y\}$ and apply $\underline{\text{Hom}}_A(-, S)$ to the short exact sequence (3.3) yielding

$$0 \rightarrow \underline{\text{Hom}}_A(A, S) \rightarrow \underline{\text{Hom}}_A((z + \lambda)A, S) \rightarrow \underline{\text{Ext}}_A^1(M_\lambda, S) \rightarrow 0.$$

But since $z + \lambda$ has degree 0, each $\underline{\text{Hom}}_A((z + \lambda)A, S) \cong \underline{\text{Hom}}_A(A, S)$ so $\underline{\text{Ext}}_A^1(M_\lambda, S) = 0$. \square

This lemma allows us to characterize all length two indecomposables in $\text{gr-}A$. Since $\underline{\text{Ext}}_A^1(X, Y) = \mathbb{k}$, there is a unique (up to isomorphism) nonsplit extension of X by Y . We denote this module $E_{X,Y}$ and similarly we denote by $E_{Y,X}$, $E_{X,X}$ and $E_{Y,Y}$ the extensions of Y by X , X by X , and Y by Y , respectively. We record these modules explicitly:

Lemma 3.3.2. *Let $\alpha = 0$. The length two indecomposable modules of $\text{gr-}A$ whose simple factors are X or Y are precisely the modules*

1. $E_{X,Y} \cong A/zA$;
2. $E_{Y,X} \cong A/(z-1)A\langle 1 \rangle$;
3. $E_{X,X} \cong A/xA$;
4. $E_{Y,Y} \cong A/yA\langle 1 \rangle$.

Proof. We record the nonsplit exact sequences. To show that A/zA is an extension of X by Y , we consider the natural quotient map $A/zA \rightarrow A/(x,z)A = X$ whose kernel is the submodule $(xA + zA)/zA$. We can construct an isomorphism $Y \rightarrow (xA + zA)/zA$ mapping 1 to x . This map is clearly surjective, and is injective since $yA + (z-1)A$ is the right annihilator of x in A/zA . Similar arguments apply in the other cases. All four nonsplit exact sequences are recorded below.

$$\begin{aligned}
0 &\longrightarrow Y \xrightarrow{x} A/zA \longrightarrow X \longrightarrow 0 \\
0 &\longrightarrow X \xrightarrow{y} (A/(z-1)A)\langle 1 \rangle \longrightarrow Y \longrightarrow 0 \\
0 &\longrightarrow X \xrightarrow{z} A/xA \longrightarrow X \longrightarrow 0 \\
0 &\longrightarrow Y \xrightarrow{(z-1)} (A/yA)\langle 1 \rangle \longrightarrow Y \longrightarrow 0
\end{aligned}$$

These short exact sequences do not split. As an example, in the first case, any nonzero element of non-positive degree generates A/zA so there does not exist a nonzero map $X \rightarrow A/zA$. \square

3.3.2 Congruent roots

Let $\alpha \in \mathbb{N}^+$ so that we are in the congruent root case.

Lemma 3.3.3. 1. *As graded vector spaces,*

$$\underline{\text{Ext}}_A^1(X, Z) = \underline{\text{Ext}}_A^1(Z, X) = \underline{\text{Ext}}_A^1(Y, Z) = \underline{\text{Ext}}_A^1(Z, Y) = \mathbb{k},$$

concentrated in degree 0.

2. *Let $S \in \{X, Y, Z\}$. Then $\underline{\text{Ext}}_A^1(S, S) = 0$.*
3. $\underline{\text{Ext}}_A^1(X, Y) = \underline{\text{Ext}}_A^1(Y, X) = 0$.

4. If $\lambda \neq \mu$, then $\text{Ext}_{\text{gr-}A}^1(M_\lambda, M_\mu) = 0$, but $\text{Ext}_{\text{gr-}A}^1(M_\lambda, M_\lambda) = \mathbb{k}$.
5. Let $\lambda \in \mathbb{k} \setminus \mathbb{Z}$. Let $S \in \{X, Y, Z\}$. Then $\underline{\text{Ext}}_A^1(M_\lambda, S) = \underline{\text{Ext}}_A^1(S, M_\lambda) = 0$.
6. $\underline{\text{Ext}}_A^1(Z, A) = 0$.

Proof. We will use similar arguments as in the proof of Lemma 3.3.1. Let $I_X = (x, z + \alpha)A$, $I_Y = (y, z - 1)A$, and $I_Z = (y^\alpha, x, z)A$ be the right ideals of A defining X , Y , and Z , respectively. Now for $S \in \{X, Y, Z\}$, we have the exact sequence

$$0 \rightarrow I_S \langle d \rangle \rightarrow A \langle d \rangle \rightarrow S \rightarrow 0 \quad (3.4)$$

where if $S = X$ then $d = -\alpha$, if $S = Y$ then $d = 1$, and if $S = Z$ then $d = 0$. For any graded simple module S' , we then apply the functor $\underline{\text{Hom}}_A(-, S')$ to yield

$$\begin{aligned} 0 \rightarrow \underline{\text{Hom}}_A(S, S') \rightarrow \underline{\text{Hom}}_A(A, S') \langle -d \rangle \rightarrow \underline{\text{Hom}}_A(I_S, S') \langle -d \rangle \\ \rightarrow \underline{\text{Ext}}_A^1(S, S') \rightarrow 0. \end{aligned} \quad (3.5)$$

We will use Lemma 3.2.5 and Corollary 3.2.7 to compute $\underline{\text{Hom}}_A(I_S, S') \langle -d \rangle$ from which we will deduce $\underline{\text{Ext}}_A^1(S, S')$.

1. Let $S = X$ and $S' = Z$ in the exact sequence (3.5). Since $\underline{\text{Hom}}_A(X, Z) = 0$, we need only determine in which degrees $\underline{\text{Hom}}_A(A, Z) \langle \alpha \rangle$ differs from $\underline{\text{Hom}}_A(I_X, Z) \langle \alpha \rangle$. By the construction in Lemma 3.2.5, observe that A and I_X have the same structure constants except in degree 0 where A has structure constant 1 and I_X has structure constant $z + \alpha$. Again, by Lemma 3.2.5, I_X surjects onto exactly those shifts of Z that A does except I_X additionally surjects onto $Z \langle \alpha \rangle$. That is for each $n \in \mathbb{Z}$,

$$\underline{\text{Hom}}_A(A, Z) \langle \alpha \rangle_n = \underline{\text{Hom}}_A(I_X, Z) \langle \alpha \rangle_n,$$

except when $n = 0$. In degree 0, we have that $\text{Hom}_{\text{gr-}A}(A, Z) \langle \alpha \rangle = 0$ and $\text{Hom}_{\text{gr-}A}(I_X, Z) \langle \alpha \rangle = \mathbb{k}$. Hence, $\underline{\text{Ext}}_A^1(X, Z) = \mathbb{k}$, concentrated in degree 0. Applying the autoequivalence ω implies that $\underline{\text{Ext}}_A^1(Y, Z) = \mathbb{k}$.

Now let $S = Z$ and $S' = X$ in the exact sequence (3.5). By Lemma 3.2.5, A and I_Z have the same structure constants except in degrees 0 and $-\alpha$.

In degree 0, A has structure constant 1 and I_Z has structure constant z whereas in degree $-\alpha$, A has structure constant $\sigma^{-\alpha}(f)$ while I_Z has structure constant $\sigma^{-\alpha}(z)$. Again, by Lemma 3.2.5, I_Z surjects onto exactly those shifts of X that A does except I_X additionally surjects onto X . Hence, $\underline{\text{Ext}}_A^1(Z, X) = \mathbb{k}$ and by applying the autoequivalence ω , we also see that $\underline{\text{Ext}}_A^1(Z, Y) = \mathbb{k}$.

2. This follows by a similar argument.
3. This follows by a similar argument.
4. This follows by the same argument as in Lemma 3.3.1.
5. This follows by the same argument as in Lemma 3.3.1.
6. Suppose for contradiction that for some $n \in \mathbb{Z}$, there exists a nonsplit extension of $Z\langle n \rangle$ by A . That is, there exists a graded right A -module I such that

$$0 \rightarrow A \rightarrow I \rightarrow Z\langle n \rangle \rightarrow 0$$

is a nonsplit exact sequence.

We claim that I is isomorphic to a submodule of the graded quotient ring of A , so is isomorphic to a right ideal of A . We first show that $A \subseteq I$ is an essential extension of modules. Suppose for contradiction that $A \subseteq I$ is not essential so there exists $0 \subsetneq J \subseteq I$ such that $J \cap A = 0$. Observe that we have the inclusion

$$\frac{J+A}{A} \subseteq \frac{I}{A} \cong Z\langle n \rangle$$

so $J+A/A$ is a submodule of $Z\langle n \rangle$. Since $Z\langle n \rangle$ is simple, and since $J \cap A = 0$, this implies that $J+A/A \cong Z\langle n \rangle$ so we have the isomorphisms

$$J \cong \frac{J}{J \cap A} \cong \frac{J+A}{A} \cong \frac{I}{A} \cong Z\langle n \rangle.$$

Tracing these canonical isomorphisms gives us a splitting $I/A \rightarrow I$, which is a contradiction since I is a nonsplit extension of A by $Z\langle n \rangle$.

Now since $A \rightarrow I$ is an essential extension, we can embed in the injective hull of A , which is $Q_{\text{gr}}(A)$, proving our claim. Let $\{d_i\}$ be the structure constants of A . By the construction in Lemma 3.2.5, since A is the kernel of the surjection $I \rightarrow Z\langle n \rangle$, therefore $d_{n-\alpha} \in \{1, \sigma^{n-\alpha}(z)\}$ and $d_n \in \{\sigma^n(z), \sigma^n(f)\}$. But by the computation in Example 3.2.2, this is impossible. Hence, there are no nonsplit extensions of $Z\langle n \rangle$ by A and $\underline{\text{Ext}}_A^1(Z, A) = 0$. \square

The preceding lemma allows us to characterize all length two indecomposables in $\text{gr-}A$. There is a unique (up to isomorphism) nonsplit extension of X by Z . We denote this module $E_{X,Z}$ and similarly we denote by $E_{Z,X}$, $E_{Y,Z}$ and $E_{Z,Y}$ the extensions of Z by X , Y by Z , and Z by Y , respectively. We record these modules explicitly:

Lemma 3.3.4. *Let $\alpha \in \mathbb{N}^+$. The length two indecomposable modules of $\text{gr-}A$ whose simple factors are X , Y , or Z are precisely the modules*

1. $E_{Z,X} \cong A/(x, z)A$;
2. $E_{X,Z} \cong A/(x^{\alpha+1}, z + \alpha)A\langle -\alpha \rangle$;
3. $E_{Z,Y} \cong A/(y, z + \alpha - 1)\langle 1 - \alpha \rangle$;
4. $E_{Y,Z} \cong A/(y^{\alpha+1}, z - 1)A\langle 1 \rangle$.

Proof. We will show that $E_{Z,X}$ is a nonsplit extension of Z by X . There is a natural projection

$$E_{Z,X} = \frac{A}{(x, z)A} \longrightarrow \frac{A}{(x, y^\alpha, z)A} = Z \quad (3.6)$$

whose kernel is given by

$$\frac{(x, y^\alpha, z)A}{(x, z)A} \cong \frac{y^\alpha A}{y^\alpha A \cap (x, z)A}.$$

It is easy to check that there is a well-defined homomorphism

$$X = \frac{A}{(x, z + \alpha)A}\langle -\alpha \rangle \longrightarrow \frac{y^\alpha A}{y^\alpha A \cap (x, z)A}$$

mapping 1 to y^α . Since X is simple and this map is a surjection, therefore the kernel of (3.6) is isomorphic to X .

To see that $E_{Z,X}$ is a nonsplit extension of Z by X , we will show that $\text{Hom}_{\text{gr-}A}(Z, E_{Z,X}) = 0$. If φ is a homomorphism $Z \rightarrow E_{Z,X}$, then $\varphi(1) = g$ for some $g \in \mathbb{k}[z]$. Then $\varphi(y^{\alpha+1}) = gy^{\alpha+1}$. But since $y^{\alpha+1} = 0$ in Z , therefore $g = 0$ in $E_{Z,X}$ so φ is the zero map. The remaining computations are done analogously. By Lemma 3.3.3, these modules represent all length two indecomposable modules of $\text{gr-}A$. \square

We will also need some understanding of indecomposable modules of length 3. While we will not characterize all such modules, we will show, for various lists of simple modules, that there is a unique indecomposable module with those ordered Jordan-Hölder quotients. This will allow us to name these modules by specifying their Jordan-Hölder quotients, in order.

Lemma 3.3.5. *Let $\alpha \in \mathbb{N}^+$.*

1. $E_{X,Z,X} := \frac{A}{(x^{\alpha+1}, (z + \alpha)x, (z + \alpha)^2)A} \langle -\alpha \rangle$ is the unique indecomposable module which is a nonsplit extension of $E_{X,Z}$ by X .
2. $E_{Y,Z,Y} := \frac{A}{(y^{\alpha+1}, (z - 1)y, (z - 1)^2)A} \langle 1 \rangle$ is the unique indecomposable module which is a nonsplit extension of $E_{Y,Z}$ by Y .
3. $E_{Z,Y,X} := A/zA$ is the unique indecomposable module which is a nonsplit extension of $E_{Z,Y}$ by X and a nonsplit extension of $E_{Z,X}$ by Y .

Proof. We will prove the first claim. The other two follow from similar arguments. There is a natural surjection

$$E_{X,Z,X} = \frac{A}{(x^{\alpha+1}, (z + \alpha)x, (z + \alpha)^2)A} \langle -\alpha \rangle \longrightarrow \frac{A}{(x^{\alpha+1}, z + \alpha)A} \langle -\alpha \rangle = E_{X,Z}$$

whose kernel is given by

$$\frac{(x^{\alpha+1}, z + \alpha)A}{(x^{\alpha+1}, (z + \alpha)x, (z + \alpha)^2)A} \langle -\alpha \rangle \cong \frac{(z + \alpha)A}{(x^{\alpha+1}, (z + \alpha)x, (z + \alpha)^2)A} \langle -\alpha \rangle.$$

This kernel is isomorphic to X via the homomorphism mapping 1 to $z + \alpha$. Therefore, $E_{X,Z,X}$ is an extension of $E_{X,Z}$ by X .

To see that this is a nonsplit extension, suppose $\varphi \in \text{Hom}_{\text{gr-}A}(E_{X,Z}, E_{X,Z,X})$. Then $\varphi(1) = g$ for some $g \in \mathbb{k}[z]$. To ensure that φ is well-defined, $\varphi(z + \alpha) = 0$, so $z + \alpha \mid g$. But then $\varphi(x) = gx = 0$ so φ is not an isomorphism onto its image.

Finally, we show that $E_{X,Z,X}$ is the unique module that is an extension of $E_{X,Z}$ by X . Apply $\underline{\text{Hom}}_A(-, X)$ to the exact sequence

$$0 \rightarrow Z \rightarrow E_{X,Z} \rightarrow X \rightarrow 0$$

to obtain the sequence

$$\cdots \rightarrow \underline{\text{Ext}}_A^1(X, X) \rightarrow \underline{\text{Ext}}_A^1(E_{X,Z}, X) \rightarrow \underline{\text{Ext}}_A^1(Z, X) \rightarrow \underline{\text{Ext}}_A^2(X, X) \rightarrow \cdots .$$

Now by [Bav93, Theorem 5], X has projective dimension 1 so $\underline{\text{Ext}}_A^2(X, X) = 0$. Then by Lemma 3.3.3, $\underline{\text{Ext}}_A^1(E_{X,Z}, X) = \mathbb{k}$, concentrated in degree 0. So there is a unique extension of $E_{X,Z}$ by X , namely $E_{X,Z,X}$. \square

3.3.3 Non-congruent roots

Let $\alpha \in \mathbb{k} \setminus \mathbb{Z}$ so that we are in the non-congruent root case. We again begin by careful analysis of the extensions between graded simple $A(f)$ -modules.

Lemma 3.3.6. *1. As graded vector spaces, $\underline{\text{Ext}}_A^1(X_0, Y_0) = \underline{\text{Ext}}_A^1(Y_0, X_0) = \underline{\text{Ext}}_A^1(X_\alpha, Y_\alpha) = \underline{\text{Ext}}_A^1(Y_\alpha, X_\alpha) = \mathbb{k}$, concentrated in degree 0 .*

2. Let $S \in \{X_0, Y_0, X_\alpha, Y_\alpha\}$. Then $\underline{\text{Ext}}_A^1(S, S) = 0$.

3. Let $S_0 \in \{X_0, Y_0\}$ and $S_\alpha \in \{X_\alpha, Y_\alpha\}$. Then $\underline{\text{Ext}}_A^1(S_0, S_\alpha) = \underline{\text{Ext}}_A^1(S_\alpha, S_0) = 0$.

4. If $\lambda \neq \alpha$, then $\text{Ext}_{\text{gr-}A}^1(M_\lambda, M_\alpha) = 0$, but $\text{Ext}_{\text{gr-}A}^1(M_\lambda, M_\lambda) = \mathbb{k}$.

5. Let $\lambda \in \mathbb{k} \setminus (\mathbb{Z} \cup \mathbb{Z} + \alpha)$. Let $S \in \{X_0, Y_0, X_\alpha, Y_\alpha\}$. Then $\underline{\text{Ext}}_A^1(M_\lambda, S) = \underline{\text{Ext}}_A^1(S, M_\lambda) = 0$.

Proof. We use the same techniques as in our proofs of Lemmas 3.3.1 and 3.3.3. Let $I_{X_0} = (x, z)A$, $I_{Y_0} = (y, z-1)A$, $I_{X_\alpha} = (x, z+\alpha)A$, and $I_{Y_\alpha} = (y, z+\alpha-1)A$ be the

ideals of A defining X_0, Y_0, X_α and Y_α respectively. Now for $S \in \{X_0, Y_0, X_\alpha, Y_\alpha\}$, we have the exact sequence

$$0 \rightarrow I_S\langle d \rangle \rightarrow A\langle d \rangle \rightarrow S \rightarrow 0 \quad (3.7)$$

where if $S = X_0$ or X_α then $d = 0$ and if $S = Y_0$ or Y_α then $d = 1$. For any graded simple module S' , we then apply the functor $\underline{\text{Hom}}_A(-, S')$ to yield

$$\begin{aligned} 0 \rightarrow \underline{\text{Hom}}_A(S, S') \rightarrow \underline{\text{Hom}}_A(A, S')\langle -d \rangle \rightarrow \underline{\text{Hom}}_A(I_S, S')\langle -d \rangle \\ \rightarrow \underline{\text{Ext}}_A^1(S, S') \rightarrow 0. \end{aligned} \quad (3.8)$$

Based on Lemma 3.2.6 and Corollary 3.2.7, we can compute $\underline{\text{Hom}}_A(I_S, S')\langle -d \rangle$. Hence, we will be able to deduce $\underline{\text{Ext}}_A^1(S, S')$.

1. Let $S = X_0, S' = Y_0$, and $d = 0$ in the exact sequence (3.8). Now $\underline{\text{Hom}}_A(X_0, Y_0) = 0$ and $\underline{\text{Hom}}_A(A, Y_0) \cong Y_0$ as graded \mathbb{k} -vector spaces, so to compute $\underline{\text{Ext}}_A^1(X_0, Y_0)$ we need to compute in which degrees $\underline{\text{Hom}}_A(I_{X_0}, Y_0)$ is nonzero. By the construction in Lemma 3.2.6, A and I_{X_0} have the same structure constants except in degree 0, where A has the structure constant 1 and I_{X_0} has the structure constant z . Hence, by Lemma 3.2.6 and Corollary 3.2.7, A and I_{X_0} surject onto exactly the same shifts of Y_0 except that I_{X_0} surjects onto Y_0 while A does not. Therefore, $\underline{\text{Ext}}_A^1(X_0, Y_0) = \mathbb{k}$, concentrated in degree 0. Applying the autoequivalence ω shows that $\underline{\text{Ext}}_A^1(Y_\alpha, X_\alpha) = \mathbb{k}$. An analogous argument shows that $\underline{\text{Ext}}_A^1(Y_0, X_0) = \underline{\text{Ext}}_A^1(X_\alpha, Y_\alpha) = \mathbb{k}$, concentrated in degree 0.
2. This follows by a similar argument.
3. This follows by a similar argument.
4. This follows by the same argument as in Lemma 3.3.1.
5. This follows by the same argument as in Lemma 3.3.1. □

We now know that there is a unique extension of X_0 by Y_0 , Y_0 by X_0 , X_α by Y_α and Y_α by X_α . We classify these extensions explicitly. We leave the proof to the reader, since it is similar to the other cases.

Corollary 3.3.7. *Let $\alpha \in \mathbb{k} \setminus \mathbb{Z}$. The following nonsplit exact sequences in $\text{gr-}A$ represent all length two indecomposables of $\text{gr-}A$ whose simple factors are supported at $\mathbb{Z} \cup \mathbb{Z} - \alpha$:*

$$\begin{aligned} 0 \rightarrow Y_0 \xrightarrow{x} A/zA \rightarrow X_0 \rightarrow 0, \\ 0 \rightarrow X_0 \xrightarrow{y} (A/(z-1)A)\langle 1 \rangle \rightarrow Y_0 \rightarrow 0, \\ 0 \rightarrow Y_\alpha \xrightarrow{x} A/(z+\alpha)A \rightarrow X_\alpha \rightarrow 0, \\ 0 \rightarrow X_\alpha \xrightarrow{y} (A/(z+\alpha-1)A)\langle 1 \rangle \rightarrow Y_\alpha \rightarrow 0. \end{aligned}$$

□

3.4 Rank one projective modules

3.4.1 Structure constants and projectivity

We now seek to understand the finitely generated graded right projective A -modules of rank one. Since every rank one graded projective A -module embeds in $Q_{\text{gr}}(A)$, we may use the results of section 3.2. For a finitely generated rank one graded projective A -module P , if we have a graded embedding $P \subseteq Q_{\text{gr}}(A)$, then there exists some $g \in \mathbb{k}[z]$, such that $gP \subseteq A$. Since $gP \cong P$, we can view P as a graded right ideal of A . We will see that the projectivity of a finitely generated right A -submodule of $Q_{\text{gr}}(A)$ can be determined by its structure constants.

By Theorem 2.5.1, the global dimension of $A(f)$ depends on the roots of f . In the non-congruent root case ($\alpha \in \mathbb{k} \setminus \mathbb{Z}$), $\text{gldim } A(f) = 1$, i.e. $A(f)$ is hereditary. Hence, the isomorphism classes of right ideals of A are the same as the isomorphism classes of rank one projective right A -modules. However, in the multiple root case ($\alpha = 0$) $\text{gldim } A(f) = \infty$ and in the congruent root case ($\alpha \in \mathbb{N}^+$), $\text{gldim } A(f) = 2$. In these cases, it takes some work to determine whether a finitely generated graded right A -submodule of $Q_{\text{gr}}(A)$ is projective. Eventually we will give conditions on

structure constants (or equivalently, conditions on simple factors) that determine the projectivity of a right ideal. We begin by stating two standard results on projective dimension. For a right R -module M , let $\text{pd}(M)$ denote the projective dimension of M .

Proposition 3.4.1. *[Rot08, Proposition 8.6] Let R be any ring. For a right R -module M , $\text{pd}(M) \leq n$ if and only if $\underline{\text{Ext}}_R^k(M, N) = 0$ for all right A -modules N and all $k \geq n + 1$.*

Corollary 3.4.2. *If $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ is an exact sequence of right A -modules then*

1. $\text{pd}(L) \leq \max\{\text{pd}(M), \text{pd}(N) - 1\}$.
2. $\text{pd}(M) \leq \max\{\text{pd}(L), \text{pd}(N)\}$.
3. $\text{pd}(N) \leq \max\{\text{pd}(L) + 1, \text{pd}(M)\}$.

Proof. This follows from the long exact Ext sequence induced by $\underline{\text{Hom}}_A(-, P)$ over all A -modules P and Proposition 3.4.1. \square

In the congruent root case, by [Bav93, Theorem 5], the finite-dimensional simple module Z is the only simple module with projective dimension 2. We begin our study of the projectivity of a right ideal by seeing that an ideal P is projective if and only if there is no extension of Z by P .

Lemma 3.4.3. *Let $\alpha \in \mathbb{N}^+$ and let P be a right ideal of A . Then the following are equivalent:*

- (i) P is projective;
- (ii) $\underline{\text{Ext}}_A^1(Z, P) = 0$;
- (iii) $\underline{\text{Hom}}_A(Z, A/P) = 0$;
- (iv) A/P has projective dimension at most 1.

Proof. By Corollary 3.4.2, (i) and (iv) are equivalent. Now apply the functor $\underline{\text{Hom}}_A(Z, -)$ to the exact sequence

$$0 \longrightarrow P \longrightarrow A \longrightarrow A/P \longrightarrow 0 \quad (3.9)$$

to obtain

$$0 \longrightarrow \underline{\text{Hom}}_A(Z, A/P) \longrightarrow \underline{\text{Ext}}_A^1(Z, P) \longrightarrow 0,$$

because by Lemma 3.3.3, $\underline{\text{Ext}}_A^1(Z, A) = 0$. Hence, (ii) and (iii) are equivalent.

We now prove that (i) implies (iii). Suppose for contradiction that Z is a submodule of A/P for a projective right ideal P . Then we have the exact sequence

$$0 \longrightarrow Z \longrightarrow A/P \longrightarrow C \longrightarrow 0,$$

where C is the cokernel of the morphism $Z \longrightarrow A/P$. Consider any right A -module N . By applying $\underline{\text{Hom}}_A(-, N)$ we have the long exact Ext sequence

$$\cdots \longrightarrow \underline{\text{Ext}}_A^2(A/P, N) \longrightarrow \underline{\text{Ext}}_A^2(Z, N) \longrightarrow 0,$$

since A has global dimension 2. By the equivalence of (i) and (iv), A/P has projective dimension 1, so by Proposition 3.4.1, $\underline{\text{Ext}}_A^2(A/P, N) = 0$ and hence $\underline{\text{Ext}}_A^2(Z, N) = 0$. Since this holds for all N , again by Proposition 3.4.1, Z has projective dimension at most 1. But this is a contradiction, for in [Bav93, Theorem 5], Bavula shows that Z has projective dimension 2.

Finally, we show that (iii) implies (iv), completing the proof. We prove the statement by induction on the length of A/P . For simple modules, [Bav93, Theorem 5] shows that all simples except Z have projective dimension 1. Now suppose for all modules M of length at most n , $\underline{\text{Hom}}_A(Z, M) = 0$ implies M has projective dimension 1. Suppose A/P has length $n + 1$ and $\underline{\text{Hom}}_A(Z, A/P) = 0$. Then A/P has a submodule S isomorphic to either $X\langle n \rangle$, $Y\langle n \rangle$ or M_λ for some $\lambda \in \mathbb{k} \setminus \mathbb{Z}$. The quotient $(A/P)/S$ has length n , so by the induction hypothesis, as long as $\underline{\text{Hom}}_A(Z, (A/P)/S) = 0$, both S and $(A/P)/S$ have projective dimension 1 and thus A/P will have projective dimension 1.

So suppose that $\underline{\text{Hom}}_A(Z, (A/P)/S) \neq 0$, that is, that $(A/P)/S$ has a submodule isomorphic to $Z\langle n \rangle$ for some n . Since this $Z\langle n \rangle$ is not a submodule of

A/P , in fact A/P must have a submodule that is a nontrivial extension of $Z\langle n \rangle$ by S . By Lemma 3.3.4, the only such extensions are $E_{Z,X}\langle n \rangle$ or $E_{Z,Y}\langle n \rangle$. Call this submodule T . By [Bav93, Theorem 5], T has projective dimension 1, so as long as $\underline{\text{Hom}}_A(Z, (A/P)/T) = 0$, we are done by the induction hypothesis.

Suppose for contradiction that $\underline{\text{Hom}}_A(Z, (A/P)/T) \neq 0$. Arguing as above, A/P must now have a submodule that is a nontrivial extension of $Z\langle m \rangle$ by T for some integer m . Now the exact sequence

$$0 \longrightarrow X \longrightarrow E_{Z,X} \longrightarrow Z \longrightarrow 0$$

induces the long exact Ext sequence

$$0 \longrightarrow \underline{\text{Hom}}_A(Z, Z) \longrightarrow \underline{\text{Ext}}_A^1(Z, X) \longrightarrow \underline{\text{Ext}}_A^1(Z, E_{Z,X}) \longrightarrow 0,$$

since $\underline{\text{Ext}}_A^1(Z, Z) = 0$ by Lemma 3.3.3. Again, by Lemma 3.3.3, $\underline{\text{Ext}}_A^1(Z, X) = \mathbb{k}$ and since $\underline{\text{Hom}}_A(Z, Z) = \mathbb{k}$, we conclude that $\underline{\text{Ext}}_A^1(Z, E_{Z,X}) = 0$. Hence, there is no nontrivial extension of $Z\langle m \rangle$ by $E_{Z,X}\langle n \rangle$, which is a contradiction. A similar contradiction arises if T is a nontrivial extension of Z by $E_{Z,Y}\langle n \rangle$. \square

Corollary 3.4.4. *Let $\alpha \in \mathbb{N}^+$ and let P be a graded right ideal of A with structure constants $\{c_i\}$. Then P is projective if and only if for each $n \in \mathbb{Z}$, it is not the case that both $c_{n-\alpha} \in \{1, \sigma^{n-\alpha}(z)\}$ and $c_n \in \{\sigma^n(z), \sigma^n(f)\}$.*

Proof. Suppose $P = \bigoplus_{i \in \mathbb{Z}} (a_i)x^i$ is a projective right ideal of A and suppose for contradiction that for some $n \in \mathbb{Z}$, both $c_{n-\alpha} \in \{1, \sigma^{n-\alpha}(z)\}$ and $c_n \in \{\sigma^n(z), \sigma^n(f)\}$.

Let

$$P' = (z+n)P = \bigoplus_{i \in \mathbb{Z}} ((z+n)a_i)\mathbb{k}[z]$$

and note that P' is graded isomorphic to P . We will construct an ideal I of A which is an extension of $Z\langle n \rangle$ by P' .

Let $I = \bigoplus_{i \in \mathbb{Z}} (b_i)x^i$ where

$$b_i = \begin{cases} (z+n)a_i, & i \leq n-\alpha \\ a_i, & n-\alpha+1 \leq i \leq n \\ (z+n)a_i, & i \geq n+1. \end{cases}$$

This is a well-defined ideal of A as long as $d_i = b_i b_{i+1}^{-1} \in \{1, \sigma^i(z), \sigma^i(z + \alpha), \sigma^i(f)\}$ for all $i \in \mathbb{Z}$. But for all $i \neq n, n - \alpha$, by construction $d_i = c_i$. Further, since $c_{n-\alpha} \in \{1, \sigma^{n-\alpha}(z)\}$ therefore $d_{n-\alpha} = (z + n)c_{n-\alpha} \in \{z + n, \sigma^{n-\alpha}(f)\}$. And finally $c_n \in \{\sigma^n(z), \sigma^n(f)\}$ implies that $d_n = (z + n)^{-1}c_n \in \{1, \sigma^n(z + \alpha)\}$.

Now note that $P' \subseteq I$ and I/P' has finite length, is simply supported at $-n$, and is nonzero precisely in degrees between $n - \alpha + 1$ and n , inclusive. So $I/P' \cong Z\langle n \rangle$ and I is an extension of $Z\langle n \rangle$ by P' . Clearly, such an extension is nontrivial, since $Z\langle n \rangle$ is not a submodule of A . But since P' is projective, this contradicts Lemma 3.4.3.

Conversely, suppose P is not projective. By Lemma 3.4.3, there exists a nontrivial extension

$$0 \longrightarrow P \longrightarrow I \longrightarrow Z\langle n \rangle \longrightarrow 0.$$

By an argument identical to the one in part 6 of Lemma 3.3.3, I is isomorphic to a submodule of $Q_{\text{gr}}(A)$. By clearing denominators we may assume that I is a right ideal of A , say $I = \bigoplus_{i \in \mathbb{Z}} (b_i)x^i$. But now as in the proof of Lemma 3.2.5, we can construct P as the kernel of the morphism $I \rightarrow Z\langle n \rangle$, and so conclude that $c_{n-\alpha} \in \{1, \sigma^{n-\alpha}(z)\}$ and $c_n \in \{\sigma^n(z), \sigma^n(f)\}$. \square

In the multiple root case ($\alpha = 0$), we get an analogue of Corollary 3.4.4, though we use a different technique as there is no analogue of Lemma 3.4.3. Indeed, the following lemma is the same result as if we let $\alpha = 0$ in Corollary 3.4.4.

Lemma 3.4.5. *Let $\alpha = 0$. Let P be a graded right ideal of A with structure constants $\{c_i\}$. Then P is projective if and only if for each $n \in \mathbb{Z}$, $c_n \neq \sigma^n(z)$.*

Proof. First we show that if, for all $n \in \mathbb{Z}$, $c_n \neq \sigma^n(z)$, then P is projective. Note that both xA and yA are projective, so $E_{X,X} = A/xA$ and $E_{Y,Y} = A/yA\langle 1 \rangle$ both have projective dimension 1. Hence, if Q is any rank one projective that surjects onto $E_{X,X}$ or $E_{Y,Y}$, the kernel of this surjection will also be projective. Since P is finitely generated, by Lemma 3.2.4, there exist $N_1, N_2 \in \mathbb{Z}$ such that for all $n \geq N_1$, $c_n = 1$ and for all $n \leq N_2$, $c_n = \sigma^n(f)$.

We will construct P by constructing a finite sequence of projective modules $\{P_i\}$ until we arrive at a module with the same structure constants as P . Let

$P_0 = A\langle N_1 \rangle$. Clearly P_0 is projective. If the structure constants of P_0 are $\{d_i\}$ then since P_0 is a shift of A , we know that $d_n = 1$ for all $n \geq N_1$ and $d_n = \sigma^n(f)$ for all $n < N_1$. Let i_0 be the largest index where d_{i_0} differs from c_{i_0} . Since $d_{i_0} = \sigma^{i_0}(f)$, then $c_{i_0} = 1$ (since we are assuming for all n that $c_n \neq \sigma^n(z)$). We construct P_1 by letting $(P_1)_n = (P_0)_n$ for all $n \leq i_0$ and letting $(P_1)_n = (z + i_0)^2(P_0)_n$ for all $n > i_0$. Then P_1 is a submodule of $Q_{\text{gr}}(A)$ with structure constants equal to those of P_0 except in degree i_0 , where P_1 has structure constant 1. Further, the quotient P_0/P_1 is supported at i_0 and has \mathbb{k} -dimension two in each graded component of degree greater than i_0 . Hence, P_0/P_1 is isomorphic to $E_{Y,Y}\langle i_0 \rangle$ or $Y\langle i_0 \rangle \oplus Y\langle i_0 \rangle$. There is a unique submodule of P_0 containing P_1 (the module where P_0 is multiplied by $z + i_0$ in all degrees greater than i_0), and therefore $P_0/P_1 \cong E_{Y,Y}\langle i_0 \rangle$. Therefore, P_1 is projective.

We continue this process for the finitely many indices where c_i differs from d_i until we reach a module P_N such that P_0/P_N has composition series consisting of only distinct shifts of $E_{Y,Y}$. By Corollary 3.4.2, P_0/P_N has projective dimension 1 so P_N is projective. Further, since they have the same structure constants, $P_N \cong P$, so P is projective.

Conversely, suppose there exist some indices n such that $c_n = \sigma^n(z)$. Construct a module Q with structure constants $\{d_i\}$ such that Q has the same structure constants as P except if $c_n = \sigma^n(z)$ then $d_n = 1$. By the above argument, Q is projective. Now we construct a finite sequence $\{Q_i\}$ of submodules of Q such that $Q_N \cong P$, and we can show Q_N is not projective. Let $Q_0 = Q$. Let i_0 be the largest index where Q_0 differs from P . Construct Q_1 by letting $(Q_1)_n = (Q_0)_n$ for all $n \leq i_0$ and $(Q_1)_n = (z + i_0)(Q_0)_n$ for all $n > i_0$. Then Q_1 is a submodule of $Q_{\text{gr}}(A)$ (with structure constants equal to those of Q_0 except in degree i_0 , where Q_1 has structure constant 1). The quotient Q_0/Q_1 is supported at i_0 and has \mathbb{k} -dimension one in each graded component of degree greater than or equal to i_0 . Hence, Q_0/Q_1 is isomorphic to $Y\langle i_0 \rangle$.

Continue this process for the finitely many indices where Q differs from P to reach a Q_N such that Q_0/Q_N has composition series consisting only of distinct shifts of Y . Since there are no extensions between any distinct shifts of Y , Q_0/Q_N

must be a direct sum of shifts of Y , and since Y has infinite projective dimension, so does Q_0/Q_N . But since Q_0 was projective, this proves that Q_N is not projective. Further, $Q_N \cong P$, proving that P is not projective. \square

As a corollary, we see that a graded rank one projective module has a unique simple factor supported at n for each $n \in \mathbb{Z}$. As a further corollary, we see that a rank one projective is determined by its integrally supported simple factors.

Corollary 3.4.6. *Let P be a rank one graded projective A -module. Let $n \in \mathbb{Z}$.*

- *If $\alpha = 0$, then P surjects onto exactly one of $X\langle n \rangle$ and $Y\langle n \rangle$.*
- *If $\alpha \in \mathbb{N}^+$, then P surjects onto exactly one of $X\langle n \rangle$, $Y\langle n \rangle$, and $Z\langle n \rangle$.*
- *If $\alpha \in \mathbb{k} \setminus \mathbb{Z}$, then P surjects onto exactly one of $X_0\langle n \rangle$ and $Y_0\langle n \rangle$. Likewise, P surjects onto exactly one of $X_\alpha\langle n \rangle$ and $Y_\alpha\langle n \rangle$.*

Proof. If $\alpha = 0$ or $\alpha \in \mathbb{k} \setminus \mathbb{Z}$, then this result follows immediately from Corollary 3.4.5, Lemma 3.2.6, and Lemma 3.2.5.

If $\alpha \in \mathbb{N}^+$, then let $\{c_i\}$ be the structure constants of P . First, by Lemma 3.2.5 if P surjects onto $X\langle n \rangle$ or $Y\langle n \rangle$, then $c_n \in \{\sigma^n(z), \sigma^n(f)\}$ or $c_{n-\alpha} \in \{1, \sigma^{n-\alpha}(z)\}$, so P cannot surject onto $Z\langle n \rangle$. Similarly, if P surjects onto $Z\langle n \rangle$, it cannot surject onto either $X\langle n \rangle$ or $Y\langle n \rangle$. And by Corollary 3.4.4, it is not possible that P surjects onto both $X\langle n \rangle$ and $Y\langle n \rangle$. Hence, P surjects onto at most one of $X\langle n \rangle$, $Y\langle n \rangle$, or $Z\langle n \rangle$.

Now we show that P surjects onto exactly one of $X\langle n \rangle$, $Y\langle n \rangle$, or $Z\langle n \rangle$. If P does not surject onto $Y\langle n \rangle$, $c_n \in \{1, \sigma^n(z + \alpha)\}$. If, additionally, P does not surject onto $X\langle n \rangle$ then $c_{n-\alpha} \in \{\sigma^{n-\alpha}(z + \alpha), \sigma^{n-\alpha}(f)\}$. In either case, by Lemma 3.2.5, P then surjects onto $Z\langle n \rangle$. \square

Definition 3.4.7. For a rank one projective P we have shown that for each $j \in \mathbb{Z}$, P has a unique simple factor supported at $-j$. We use the notation $F_j(P)$ to refer to this simple factor. Similarly, if $\alpha \notin \mathbb{Z}$, for every $j \in \mathbb{Z}$, P has a unique simple factor supported at $-j - \alpha$. We use the notation $F_j^\alpha(P)$ to refer to this simple factor.

Corollary 3.4.8. *A rank one graded projective A -module is determined up to isomorphism by its simple factors which are supported at $\mathbb{Z} \cup \mathbb{Z} - \alpha$.*

Proof. Let P be a rank one graded projective A -module with structure constants $\{c_i\}$. In all cases, the integrally supported simple factors determine the $\{c_i\}$ and therefore determine P . In the non-congruent root case, the result follows from Lemma 3.2.6. In the multiple root case, this follows from Lemma 3.2.5 and Corollary 3.4.6. In particular, if P has a factor of $X\langle n \rangle$ then $c_n = 1$ and if P has a factor of $Y\langle n \rangle$ then $c_n = \sigma^n(f)$.

In the congruent root case, note that by Corollary 3.4.6, for each $n \in \mathbb{Z}$, P surjects onto exactly one of $X\langle n \rangle$, $Y\langle n \rangle$ or $Z\langle n \rangle$. Now, for each $j \in \mathbb{Z}$, we can determine c_j by using Lemma 3.2.5. In particular, whether or not $X\langle j + \alpha \rangle$ and $Y\langle j \rangle$ are factors of P completely determine c_j . If both $X\langle j + \alpha \rangle$ and $Y\langle j \rangle$ are factors, then $c_j = \sigma^j(z)$. If $X\langle j + \alpha \rangle$ is a factor but $Y\langle j \rangle$ is not then $c_j = 1$. If $X\langle j + \alpha \rangle$ is not a factor but $Y\langle j \rangle$ is, then $c_j = \sigma^j(f)$. And if neither $X\langle j + \alpha \rangle$ nor $Y\langle j \rangle$ are factors of P , then $c_j = \sigma^j(z + \alpha)$. We make this explicit in Table 3.4.1, below. \square

Table 3.4.1: The structure constants of a rank one projective module.

	$F_j(P) = X\langle j \rangle$	$F_j(P) = Y\langle j \rangle$	$F_j(P) = Z\langle j \rangle$
$F_{j+\alpha}(P) = X\langle j + \alpha \rangle$	$c_j = 1$	$c_j = \sigma^j(z)$	$c_j = 1$
$F_{j+\alpha}(P) = Y\langle j + \alpha \rangle$	$c_j = \sigma^j(z + \alpha)$	$c_j = \sigma^j(f)$	$c_j = \sigma^j(z + \alpha)$
$F_{j+\alpha}(P) = Z\langle j + \alpha \rangle$	$c_j = \sigma^j(z + \alpha)$	$c_j = \sigma^j(f)$	$c_j = \sigma^j(z + \alpha)$

Lemma 3.4.9. 1. *Let $\alpha \in \mathbb{N}$. For each $n \in \mathbb{Z}$ choose $S_n \in \{X\langle n \rangle, Y\langle n \rangle, Z\langle n \rangle\}$ such that for $n \gg 0$, $S_n = X\langle n \rangle$ and $S_{-n} = Y\langle -n \rangle$. Then there exists a rank one graded projective P such that $F_n(P) = S_n$ for all n .*

2. *Let $\alpha = 0$. For each $n \in \mathbb{Z}$ choose $S_n \in \{X\langle n \rangle, Y\langle n \rangle\}$ such that for $n \gg 0$, $S_n = X\langle n \rangle$ and $S_{-n} = Y\langle -n \rangle$. Then there exists a rank one graded projective P such that $F_n(P) = S_n$ for all n .*

Proof. First, let $\alpha \in \mathbb{N}$. We can construct a module $P \subseteq Q_{\text{gr}}(A)$ that surjects onto S_n for all n . We do this by specifying the structure constants $\{c_i\}$ of P : for each n , c_n is determined by S_n and $S_{n+\alpha}$ as described in Table 3.4.1. By the remark following Lemma 3.2.4, P is a finitely generated graded right submodule of $Q_{\text{gr}}(A)$.

By Corollary 3.4.4, P will be projective as long as, for each $n \in \mathbb{Z}$, it is not the case that both $c_n \in \{1, \sigma^n(z)\}$ and $c_{n+\alpha} \in \{\sigma^{n+\alpha}(z), \sigma^{n+\alpha}(f)\}$. But if $c_n \in \{1, \sigma^n(z)\}$, then by the first row of the table, P surjects onto $X\langle j + \alpha \rangle$. If P surjects onto $X\langle j + \alpha \rangle$, then by the first column of the table, we see that $c_{n+\alpha} \in \{1, \sigma^{n+\alpha}(z + \alpha)\}$. Therefore, in fact P is projective. By construction, $F_n(P) = S_n$ for all $n \in \mathbb{Z}$.

The case $\alpha = 0$ is analogous. In this case, if $S_n = X\langle n \rangle$, then let $c_n = 1$ and if $S_n = Y\langle n \rangle$ then let $c_n = \sigma^{-n}(f)$. \square

3.4.2 Morphisms between rank one projectives

In this section, we describe the morphisms between finitely generated rank one projective A -modules. We again make use of the fact that every right A -module is also a left $\mathbb{k}[z]$ -module. The fact that $\mathbb{k}[z]$ is a PID leads to a nice characterization of the morphisms between rank one projectives. We first note that if P and Q are finitely generated rank one projectives, then any morphism $f \in \text{Hom}_{\text{gr-}A}(P, Q)$ is either 0 or else an injection. This is because A is 1-critical and therefore $P/\ker f$ is a finite length module, but Q has no nonzero finite length submodules. Hence, either $\ker f = P$ or else $\ker f = 0$. We define a maximal embedding between right A -modules.

Definition 3.4.10. Let P and Q be finitely generated right A -modules. An A -module homomorphism $f : P \rightarrow Q$ is called a *maximal embedding* if there does not exist an A -module homomorphism $g : P \rightarrow Q$ such that $f(P) \subsetneq g(P)$.

We will prove that in fact, if P and Q are finitely generated rank one projectives, there exists a unique (up to scalar) maximal embedding $P \rightarrow Q$.

Proposition 3.4.11. *Let P and Q be finitely generated graded rank one projective A -modules embedded in $Q_{\text{gr}}(A)$. Then every homomorphism $P \rightarrow Q$ is given by left*

multiplication by some element of $\mathbb{k}(z)$ and as a left $\mathbb{k}[z]$ -module, $\text{Hom}_{\text{gr-}A}(P, Q)$ is free of rank one.

Proof. Since P and Q are finitely generated graded submodules of $Q_{\text{gr}}(A)$, we may multiply by some element of $\mathbb{k}[z]$ and assume P and Q are right ideals of A . Let $f \in \text{Hom}_{\text{gr-}A}(P, Q)$. Now since $P_0, Q_0 \subseteq A_0 = \mathbb{k}[z]$, P_0 and Q_0 are actually left $\mathbb{k}[z]$ -submodules of $\mathbb{k}[z]$ and hence ideals of $\mathbb{k}[z]$. Because $\mathbb{k}[z]$ is a PID, we can write $P_0 = (p_0)$ and $Q_0 = (q_0)$ for some $p_0, q_0 \in \mathbb{k}[z]$. Let f_0 be the restriction of f to P_0 . Since f is a right A -module homomorphism, it is also a left $\mathbb{k}[z]$ -module homomorphism. Therefore, f_0 is given by left multiplication by some $\varphi \in \mathbb{k}(z)$ such that $(\varphi p_0) \subseteq (q_0)$ in $\mathbb{k}[z]$.

We claim that f_0 determines f . Define $m_n = x^n$ for $n \geq 0$ and $m_n = y^n$ for $n < 0$. Now in each graded component, we have that $P_n = (p_n)m_n$ and $Q_n = (q_n)m_n$ where $p_n, q_n \in \mathbb{k}[z]$. Let f_n be the restriction of f to P_n . Note that $(p_0)m_n$ is a nonzero submodule of $(p_n)m_n$, and

$$f(p_0 m_n) = f(p_0) m_n = \varphi p_0 m_n,$$

so on the submodule $(p_0)m_n \subseteq P_n$, f is given by left multiplication by φ . Viewing f_n as a left $\mathbb{k}[z]$ -module homomorphism, we conclude that f_n is given by left multiplication by φ for each n , so f is simply left multiplication by φ . Conversely, left multiplication by $\varphi \in \mathbb{k}(z)$ will be an element of $\text{Hom}_{\text{gr-}A}(P, Q)$ if and only if $(\varphi p_n) \subseteq (q_n)$ for all n .

Now $\text{Hom}_{\text{gr-}A}(P, Q)$ is a left $\mathbb{k}[z]$ -submodule of $\text{Hom}_{\text{gr-}\mathbb{k}[z]}(P, Q)$ which the above shows is isomorphic to some left $\mathbb{k}[z]$ -submodule of $\mathbb{k}(z)$. Since in addition multiplication by φ is a homomorphism only if $\varphi \in \mathbb{k}[z]q_0/p_0$, we can clear denominators so that $(p_0) \text{Hom}_{\text{gr-}A}(P, Q)$ is actually a left $\mathbb{k}[z]$ -submodule of $\mathbb{k}[z]$. Since $\mathbb{k}[z]$ is a PID, $(p_0) \text{Hom}_{\text{gr-}A}(P, Q) = (g)$ for some $g \in \mathbb{k}[z]$. Finally, we see that $\text{Hom}_{\text{gr-}A}(P, Q)$ is generated as a left $\mathbb{k}[z]$ -module by $g/p_0 \in \mathbb{k}(z)$. Let $\theta = g/p_0$. Then all homomorphisms $\text{Hom}_{\text{gr-}A}(P, Q)$ are given by left multiplication by some $\mathbb{k}[z]$ -multiple of θ . \square

The above proposition tells us that a Hom set between rank one projective A -modules is generated as a left $\mathbb{k}[z]$ -module by a unique (up to scalar) maxi-

mal embedding. Note, however, the element $\theta \in \mathbb{k}(z)$ as in the previous lemma depends on a fixed embedding of P and Q in A . As an example, $zA \cong A$, but $\text{Hom}_{\text{gr-}A}(zA, A)$ is generated as $\mathbb{k}[z]$ -module by left multiplication by z^{-1} while $\text{Hom}_{\text{gr-}A}(A, A)$ is generated by left multiplication by 1.

Given rank one projectives P and Q we would like to have canonical representations of P and Q as submodules of the graded quotient ring $Q_{\text{gr}}(A)$. This would also give us a canonical representation of the maximal embedding $P \rightarrow Q$. We will do this graded component by graded component. Suppose P has structure constants $\{c_i\}$. We will give a canonical sequence $\{p_i\}$ such that

$$P \cong \bigoplus_{i \in \mathbb{Z}} (p_i)x^i \subseteq Q_{\text{gr}}(A).$$

By Lemma 3.2.4, there exists an integer N such that for all $n > N$, $c_n = 1$. Now recalling that $p_i = c_i p_{i+1}$, for any $n \in \mathbb{Z}$, let $p_n = \prod_{j \geq n} c_j$. Observe that $p_n \in \mathbb{k}[z]$ since all but finitely many of the factors are 1.

Definition 3.4.12. Given a graded rank one projective module P , the representation of $\bigoplus (p_i)x^i \subseteq Q_{\text{gr}}(A)$ given above is called *canonical representation of P* . We call $\bigoplus (p_i)x^i$ a *canonical rank one projective module*.

Now given canonical graded rank one projective modules $P = \bigoplus (p_i)x^i$ and $Q = \bigoplus (q_i)x^i$, we can compute the maximal embedding of P into Q , which is unique if we require that θ be monic.

Lemma 3.4.13. *Let P and Q be rank one graded projective A -modules with structure constants $\{c_i\}$ and $\{d_i\}$, respectively. As above, write*

$$P = \bigoplus_{i \in \mathbb{Z}} (p_i)x^i = \bigoplus_{i \in \mathbb{Z}} \left(\prod_{j \geq i} c_j \right) x^i \text{ and } Q = \bigoplus_{i \in \mathbb{Z}} (q_i)x^i = \bigoplus_{i \in \mathbb{Z}} \left(\prod_{j \geq i} d_j \right) x^i.$$

Then the maximal embedding $P \rightarrow Q$ is given by multiplication by

$$\theta_{P,Q} = \text{lcm}_{i \in \mathbb{Z}} \left(\frac{q_i}{\text{gcd}(p_i, q_i)} \right) = \text{lcm}_{i \in \mathbb{Z}} \left(\frac{\prod_{j \geq i} d_j}{\text{gcd} \left(\prod_{j \geq i} c_j, \prod_{j \geq i} d_j \right)} \right)$$

where by lcm we mean the unique monic least common multiple $\theta_{P,Q} \in \mathbb{k}[z]$.

Proof. As we saw in Proposition 3.4.11, the maximal embedding $P \rightarrow Q$ is given by multiplication by some $\theta_{P,Q} \in \mathbb{k}(z)$ such that $(\theta_{P,Q}p_i) \subseteq (q_i)$ for all $i \in \mathbb{Z}$. Because, for large enough i , $p_i = q_i = 1$, we see that $\theta_{P,Q} \in \mathbb{k}[z]$. In fact, that $(\theta_{P,Q}p_i) \subseteq (q_i)$ implies that $\theta_{P,Q}$ must be a $\mathbb{k}[z]$ -multiple of $q_i/\gcd(p_i, q_i)$ for all i . The minimal such $\theta_{P,Q}$ is given by

$$\theta_{P,Q} = \operatorname{lcm}_{i \in \mathbb{Z}} \left(\frac{q_i}{\gcd(p_i, q_i)} \right)$$

Since the structure constants of a graded submodule of $Q_{\text{gr}}(A)$ were defined to be monic, $\theta_{P,Q}$ is monic. Because there exists some $N \in \mathbb{N}$ such that for all $n \geq N$, $c_n = d_n$ and $c_{-n} = d_{-n}$, therefore $q_i/\gcd(p_i, q_i) = q_j/\gcd(p_j, q_j)$ for all $i, j \geq N$ and all $i, j \leq -N$, so the least common multiple can be taken over finitely many indices. \square

Corollary 3.4.14. *Suppose $\alpha \in \mathbb{k} \setminus \mathbb{Z}$ or $\alpha = 0$. Assume the hypotheses of Lemma 3.4.13. Then there exists an $N \in \mathbb{Z}$ such that*

$$\theta_{P,Q} = \frac{q_N}{\gcd(p_N, q_N)} = \frac{\prod_{j \geq N} d_j}{\gcd\left(\prod_{j \geq N} c_j, \prod_{j \geq N} d_j\right)} = \frac{\prod_{j \geq N} d_j}{\prod_{j \geq N} \gcd(c_j, d_j)}.$$

Proof. If $\alpha \in \mathbb{k} \setminus \mathbb{Z}$ or $\alpha = 0$, we can compute the maximal embedding $P \rightarrow Q$ without taking a least common multiple, as follows. The irreducible factors of $\theta_{P,Q}$ are all $\sigma^i(z)$ or $\sigma^i(z + \alpha)$ for some $i \in \mathbb{Z}$. If $\alpha \in \mathbb{k} \setminus \mathbb{Z}$ or $\alpha = 0$, then $\sigma^i(z)$ and $\sigma^i(z + \alpha)$ can only appear as factors of the structure constant in degree i . Thus, if $i \neq j$, then q_i shares no irreducible factors with p_j or q_j . Hence, if $n < m$, then $q_m/\gcd(p_m, q_m)$ divides $q_n/\gcd(p_n, q_n)$. By Lemma 3.2.4, there exists an $N \in \mathbb{Z}$ such that for all $i \leq N$, $p_i = q_i$. For this N , we can write

$$\theta_{P,Q} = \frac{q_N}{\gcd(p_N, q_N)} = \frac{\prod_{j \geq N} d_j}{\gcd\left(\prod_{j \geq N} c_j, \prod_{j \geq N} d_j\right)} = \frac{\prod_{j \geq N} d_j}{\prod_{j \geq N} \gcd(c_j, d_j)}. \quad \square$$

We now show that the cokernel of a maximal embedding has a special structure.

Lemma 3.4.15. *Let P and Q be graded rank one projective A -modules and let $f : P \rightarrow Q$ be a maximal embedding. Then the module $Q/f(P)$ is supported at $\mathbb{Z} \cup \mathbb{Z} - \alpha$.*

Proof. Let $N = Q/f(P)$. As A has Krull dimension 1, N has finite length. By Lemma 3.1.1, N has a finite composition series whose factors are either supported at $\mathbb{Z} \cup \mathbb{Z} - \alpha$ or isomorphic to some M_λ with $\lambda \notin \mathbb{Z} \cup \mathbb{Z} - \alpha$. Suppose for contradiction that M_λ is a subfactor of N . Then we have $f(P) \subseteq Q_1 \subseteq Q_2 \subseteq Q$ with $Q_2/Q_1 \cong M_\lambda$ and so $Q_1 = (z + \lambda)Q_2$. But now $(z + \lambda)^{-1}f(P) \subseteq (z + \lambda)^{-1}Q_1 = Q_2 \subseteq Q$, contradicting the maximality of f . \square

Finally, we give necessary and sufficient conditions for a collection of projective objects in $\text{gr-}A$ to generate the category.

Proposition 3.4.16. *A set of rank one graded projective A -modules $\mathcal{P} = \{P_i\}_{i \in I}$ generates $\text{gr-}A$ if and only if for every graded simple module M which is supported at $\mathbb{Z} \cup \mathbb{Z} - \alpha$, there exists a surjection to M from a direct sum of modules in \mathcal{P} .*

Proof. One direction is clear. Now suppose \mathcal{P} is a set of graded projective A -modules that generates all graded simple modules supported at $\mathbb{Z} \cup \mathbb{Z} - \alpha$. We will show that every shift of A is the image of a surjection from a direct sum of modules in \mathcal{P} , and so \mathcal{P} generates $\text{gr-}A$. Let $P \in \mathcal{P}$ and choose a maximal embedding $\varphi : P \rightarrow A\langle n \rangle$. It suffices to construct a surjection $\psi : \bigoplus_{j \in J} P_j \rightarrow A\langle n \rangle/P$ for some $J \subseteq I$. This is because, by the projectivity of the P_j , there exists a lift $\bar{\psi} : \bigoplus_{j \in J} P_j \rightarrow A\langle n \rangle$ and because $\text{im } \varphi + \text{im } \bar{\psi} = A\langle n \rangle$.

Since A has Krull dimension 1, the quotient $A\langle n \rangle/P$ has finite length. By Lemma 3.4.15, it is supported at $\mathbb{Z} \cup \mathbb{Z} - \alpha$. We induct on the length of $A\langle n \rangle/P$. Now there exists some integrally supported simple module M_0 which fits into the exact sequence

$$0 \longrightarrow K_0 \longrightarrow A/P \longrightarrow M_0 \longrightarrow 0.$$

Again, it suffices to give surjections onto M_0 and K_0 . By hypothesis, \mathcal{P} generates M_0 . By induction, \mathcal{P} generates K_0 , completing the proof. \square

Chapter 4

Functors defined on subcategories of projectives

The results of section 3.4 suggest that much information about the category $\text{gr-}A$ is contained in its rank one projective modules and the morphisms between them. We will therefore develop the machinery necessary to define a functor on $\text{gr-}A$ by first defining it on the full subcategory of direct sums of rank one projective modules. We remark that a dual statement to Lemma 4.0.1 is stated in [Van01, Proposition 3.1.1 (3)] (in terms of injective objects), but no detailed proof is given.

Lemma 4.0.1. *Let \mathcal{C} be an abelian category with enough projectives. Let \mathcal{P} be a full subcategory consisting of projective objects in \mathcal{C} such that every object in \mathcal{C} has a projective resolution by objects in \mathcal{P} . Given an additive functor $\mathcal{F} : \mathcal{P} \rightarrow \mathcal{C}$, there is a unique (up to natural isomorphism of functors) extension of \mathcal{F} to an additive functor $\tilde{\mathcal{F}} : \mathcal{C} \rightarrow \mathcal{C}$ which is right exact.*

Proof. We begin by defining $\tilde{\mathcal{F}}$ on objects. For each $M \in \mathcal{C}$, fix a partial projective resolution

$$P_1 \xrightarrow{d_0} P_0 \longrightarrow M \longrightarrow 0$$

where P_0, P_1 are objects in \mathcal{P} . Then apply \mathcal{F} to yield

$$\mathcal{F}(P_1) \xrightarrow{\mathcal{F}(d_0)} \mathcal{F}(P_0)$$

and let $\tilde{\mathcal{F}}(M) = \text{coker } \mathcal{F}(d_0)$ so there is a right exact sequence

$$\mathcal{F}(P_1) \xrightarrow{\mathcal{F}(d_0)} \mathcal{F}(P_0) \longrightarrow \tilde{\mathcal{F}}(M) \longrightarrow 0.$$

For $P \in \mathcal{P}$, we always fix the resolution

$$0 \longrightarrow P \longrightarrow P \longrightarrow 0$$

so that $\tilde{\mathcal{F}}(P) = \mathcal{F}(P)$.

Next we define $\tilde{\mathcal{F}}$ on morphisms. Let $g : M \rightarrow N$ be a morphism and let $P_1 \rightarrow P_0 \rightarrow M$ and $Q_1 \rightarrow Q_0 \rightarrow N$ be the partial projective resolutions of M and N , respectively. Because P_0 and P_1 are projective, there exist lifts of g , homomorphisms h_0 and h_1 such that

$$\begin{array}{ccccccc} P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\ h_1 \downarrow & & h_0 \downarrow & & g \downarrow & & \\ Q_1 & \longrightarrow & Q_0 & \longrightarrow & N & \longrightarrow & 0 \end{array} \quad (4.1)$$

commutes. Although h_0 and h_1 are not necessarily unique, they do induce unique maps on homology (i.e. any choices for h_0, h_1 are chain homotopic). Then applying \mathcal{F} to this commutative diagram yields

$$\begin{array}{ccccccc} \mathcal{F}(P_1) & \longrightarrow & \mathcal{F}(P_0) & \longrightarrow & \tilde{\mathcal{F}}(M) & \longrightarrow & 0 \\ \mathcal{F}(h_1) \downarrow & & \mathcal{F}(h_0) \downarrow & & & & \\ \mathcal{F}(Q_1) & \longrightarrow & \mathcal{F}(Q_0) & \longrightarrow & \tilde{\mathcal{F}}(N) & \longrightarrow & 0 \end{array}$$

which induces a unique map $\tilde{\mathcal{F}}(g) : \tilde{\mathcal{F}}(M) \rightarrow \tilde{\mathcal{F}}(N)$ such that

$$\begin{array}{ccccccc} \mathcal{F}(P_1) & \longrightarrow & \mathcal{F}(P_0) & \longrightarrow & \tilde{\mathcal{F}}(M) & \longrightarrow & 0 \\ \mathcal{F}(h_1) \downarrow & & \mathcal{F}(h_0) \downarrow & & \tilde{\mathcal{F}}(g) \downarrow & & \\ \mathcal{F}(Q_1) & \longrightarrow & \mathcal{F}(Q_0) & \longrightarrow & \tilde{\mathcal{F}}(N) & \longrightarrow & 0 \end{array}$$

commutes. Further, since homotopic maps stay homotopic after applying a functor, and since any choices of h_0, h_1 were homotopic, therefore $\mathcal{F}(h_0)$ and $\mathcal{F}(h_1)$ are also unique up to homotopy. Hence, the induced map on zeroth homology, $\tilde{\mathcal{F}}(g)$, is well-defined. Note also that clearly $\tilde{\mathcal{F}}$ extends \mathcal{F} and since \mathcal{F} is additive, $\tilde{\mathcal{F}}$ is additive.

Again, since any choice of lifts h_0, h_1 induce the same map on homology, in particular, we get unique maps on zeroth homology. Thus, if $g : M \rightarrow M$ is the identity, we may choose h_0, h_1 to be the identity also. Then clearly $\tilde{\mathcal{F}}(g) : \tilde{\mathcal{F}}(M) \rightarrow \tilde{\mathcal{F}}(M)$ is the identity. Given $g' : L \rightarrow M$ and $g : M \rightarrow N$, the uniqueness of the induced maps on homology gives $\tilde{\mathcal{F}}(g \circ g') = \tilde{\mathcal{F}}(g) \circ \tilde{\mathcal{F}}(g')$ so $\tilde{\mathcal{F}}$ is functorial.

We now show that $\tilde{\mathcal{F}}$ is independent of the choice of projective resolution. Suppose that we had fixed a different choice of projective resolutions leading to a functor $\tilde{\mathcal{F}}'$. Let $P_1 \xrightarrow{d_0} P_0 \rightarrow M$ and $P'_1 \xrightarrow{d'_0} P'_0 \rightarrow M$ be the choices of partial projective resolutions of M in defining $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}'$, respectively. Choosing lifts of the identity $M \rightarrow M$ as before, we obtain the commutative diagram

$$\begin{array}{ccccccc}
 P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\
 j_1 \downarrow & & j_0 \downarrow & & \text{Id}_M \downarrow & & \\
 P'_1 & \longrightarrow & P'_0 & \longrightarrow & M & \longrightarrow & 0 \\
 k_1 \downarrow & & k_0 \downarrow & & \text{Id}_M \downarrow & & \\
 P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0
 \end{array} \tag{4.2}$$

and since $k_0 \circ j_0$ is the identity on $P_0/\text{im } d_0$, therefore j_0 gives an isomorphism $P_0/\text{im } d_0 \rightarrow P'_0/\text{im } d'_0$. Applying \mathcal{F} to this diagram yields

$$\begin{array}{ccccccc}
 \mathcal{F}(P_1) & \longrightarrow & \mathcal{F}(P_0) & \longrightarrow & \tilde{\mathcal{F}}(M) & \longrightarrow & 0 \\
 \mathcal{F}(j_1) \downarrow & & \mathcal{F}(j_0) \downarrow & & & & \\
 \mathcal{F}(P'_1) & \longrightarrow & \mathcal{F}(P'_0) & \longrightarrow & \tilde{\mathcal{F}}'(M) & \longrightarrow & 0
 \end{array}$$

from which we get a map $\tilde{\mathcal{F}}(M) \rightarrow \tilde{\mathcal{F}}'(M)$ induced by $\mathcal{F}(j_0)$. Since j_0 was an isomorphism on homology, therefore $\mathcal{F}(j_0)$ induces an isomorphism $\tilde{\mathcal{F}}(M) \rightarrow \tilde{\mathcal{F}}'(M)$.

Suppose that $Q_1 \xrightarrow{e_0} Q_0 \rightarrow N$ and $Q'_1 \xrightarrow{e'_0} Q'_0 \rightarrow N$ are the choices of partial projective resolutions of N in defining $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}'$ respectively. Let $g : M \rightarrow N$ be given. Construct h_0 and h'_0 lifting g as in diagram (4.1) and construct maps $j_0 : P_0 \rightarrow P'_0$ and $\ell_0 : Q_0 \rightarrow Q'_0$ as in diagram (4.2) to obtain the diagram

$$\begin{array}{ccc}
 P_0 & \xrightarrow{j_0} & P'_0 \\
 h_0 \downarrow & & h'_0 \downarrow \\
 Q_0 & \xrightarrow{\ell_0} & Q'_0
 \end{array} \tag{4.3}$$

A diagram chase shows that this square commutes on zeroth homology. That is,

$$\begin{array}{ccc} P_0/\operatorname{im} d_0 & \xrightarrow{j_0} & P'_0/\operatorname{im} d'_0 \\ h_0 \downarrow & & h'_0 \downarrow \\ Q_0/\operatorname{im} e_0 & \xrightarrow{\ell_0} & Q'_0/\operatorname{im} e'_0 \end{array}$$

commutes (the maps are well-defined on homology). Applying \mathcal{F} to (4.3) yields

$$\begin{array}{ccc} \mathcal{F}(P_0) & \xrightarrow{\mathcal{F}(j_0)} & \mathcal{F}(P'_0) \\ \mathcal{F}(h_0) \downarrow & & \mathcal{F}(h'_0) \downarrow \\ \mathcal{F}(Q_0) & \xrightarrow{\mathcal{F}(\ell_0)} & \mathcal{F}(Q'_0) \end{array}$$

which again commutes on zeroth homology. Therefore, we conclude that the isomorphisms $\eta_M : \tilde{\mathcal{F}}(M) \rightarrow \tilde{\mathcal{F}}'(M)$ and $\eta_N : \tilde{\mathcal{F}}(N) \rightarrow \tilde{\mathcal{F}}'(N)$ induced by $\mathcal{F}(j_0)$ and $\mathcal{F}(\ell_0)$ give a commuting square

$$\begin{array}{ccc} \tilde{\mathcal{F}}(M) & \xrightarrow{\eta_M} & \tilde{\mathcal{F}}'(M) \\ \tilde{\mathcal{F}}(g) \downarrow & & \tilde{\mathcal{F}}'(g) \downarrow \\ \tilde{\mathcal{F}}(N) & \xrightarrow{\eta_N} & \tilde{\mathcal{F}}'(N) \end{array}$$

and so $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}'$ are naturally isomorphic via η .

Finally, to check that $\tilde{\mathcal{F}}$ is right exact, let

$$0 \longrightarrow L \xrightarrow{g} M \xrightarrow{f} N \longrightarrow 0$$

be an exact sequence in \mathbf{C} . Let $P_1 \rightarrow P_0 \rightarrow L$ and $Q_1 \rightarrow Q_0 \rightarrow N$ be partial projective resolutions for L and N respectively. Now the Horseshoe Lemma gives the commutative diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & P_1 & \longrightarrow & P_0 & \longrightarrow & L \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow g \\ P_1 \oplus Q_1 & \longrightarrow & P_0 \oplus Q_0 & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow f & & \downarrow \\ & & Q_1 & \longrightarrow & Q_0 & \longrightarrow & N \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

in which the columns are exact. Since \mathcal{F} is additive, by Proposition 2.2.1 it preserves finite direct sums so

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
& & \mathcal{F}(P_1) & \longrightarrow & \mathcal{F}(P_0) & \longrightarrow & \tilde{\mathcal{F}}(L) \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \tilde{\mathcal{F}}(g) \downarrow \\
\mathcal{F}(P_1) \oplus \mathcal{F}(Q_1) & \longrightarrow & \mathcal{F}(P_0) \oplus \mathcal{F}(Q_0) & \longrightarrow & \tilde{\mathcal{F}}(M) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \tilde{\mathcal{F}}(f) \downarrow \\
\mathcal{F}(Q_1) & \longrightarrow & \mathcal{F}(Q_0) & \longrightarrow & \tilde{\mathcal{F}}(N) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \\
& & 0 & & 0 & &
\end{array}$$

commutes. A diagram chase then shows the right column is exact at $\tilde{\mathcal{F}}(M)$ and that $\tilde{\mathcal{F}}(M) \xrightarrow{\tilde{\mathcal{F}}(f)} \tilde{\mathcal{F}}(N)$ is a surjection. Hence, $\tilde{\mathcal{F}}$ is right exact. \square

As a corollary of the above lemma, we prove that we can check if two categories are equivalent by checking if they are equivalent on a full subcategory of projective objects.

Corollary 4.0.2. *Let \mathcal{C} and \mathcal{C}' be abelian categories with enough projectives. Let \mathcal{P} (respectively \mathcal{P}') be a full subcategory of projective objects such that every object of \mathcal{C} (respectively \mathcal{C}') has a projective resolution by objects of \mathcal{P} (respectively \mathcal{P}'). Then if $\mathcal{F} : \mathcal{P} \rightarrow \mathcal{P}'$ is an equivalence of categories, then there exists an equivalence of categories $\tilde{\mathcal{F}} : \mathcal{C} \rightarrow \mathcal{C}'$ which extends \mathcal{F} .*

Proof. Suppose $\mathcal{F} : \mathcal{P} \rightarrow \mathcal{P}'$ is an equivalence of categories with quasi-inverse $\mathcal{G} : \mathcal{P}' \rightarrow \mathcal{P}$. We may regard \mathcal{F} as a functor $\mathcal{F} : \mathcal{P} \rightarrow \mathcal{C}'$ and \mathcal{G} as a functor $\mathcal{G} : \mathcal{P}' \rightarrow \mathcal{C}$. Now by Lemma 4.0.1, there exist functors $\tilde{\mathcal{F}} : \mathcal{C} \rightarrow \mathcal{C}'$ and $\tilde{\mathcal{G}} : \mathcal{C}' \rightarrow \mathcal{C}$ which extend the functors \mathcal{F} and \mathcal{G} .

Now $\mathcal{G} \circ \mathcal{F} : \mathcal{P} \rightarrow \mathcal{P}$ is naturally isomorphic to the identity functor $\text{Id}_{\mathcal{P}}$. Consider the composition $\tilde{\mathcal{G}} \circ \tilde{\mathcal{F}} : \mathcal{C} \rightarrow \mathcal{C}$. Then $\tilde{\mathcal{G}} \circ \tilde{\mathcal{F}}$ is an extension of $\mathcal{G} \circ \mathcal{F}$ to the category \mathcal{C} so by Lemma 4.0.1 is the unique such extension up to natural

isomorphism. Since $\text{Id}_{\mathcal{C}}$ is an extension of $\text{Id}_{\mathcal{P}}$, we conclude that $\tilde{\mathcal{G}} \circ \tilde{\mathcal{F}} \cong \text{Id}_{\mathcal{C}}$. Similarly, $\tilde{\mathcal{G}} \circ \tilde{\mathcal{F}} \cong \text{Id}_{\mathcal{C}'}$ and hence, $\mathcal{C} \equiv \mathcal{C}'$. \square

We have shown that in order to define a functor on a category, it suffices to construct a functor on “enough” of the projective objects in that category. In the next chapter, we will construct autoequivalences of $\text{gr-}A$ in this way. We now show that, in fact, the subcategory consisting of only canonical rank one projective modules is “big enough”, as additive functors defined on these objects extend uniquely to direct sums.

Lemma 4.0.3. *Let \mathcal{C} be an abelian category. Let \mathcal{R} be the full subcategory of $\text{gr-}A$ consisting of the canonical rank one projective modules and let \mathcal{P} be the full subcategory of $\text{gr-}A$ consisting of all finite direct sums of canonical rank one projective modules. If $\mathcal{F} : \mathcal{R} \rightarrow \mathcal{C}$ is an additive functor, then \mathcal{F} extends to an additive functor $\tilde{\mathcal{F}} : \mathcal{P} \rightarrow \mathcal{C}$.*

Proof. We begin by defining $\tilde{\mathcal{F}}$ on objects. For $P \in \mathcal{P}$, choose canonical rank one projective modules P_i and write $P = \bigoplus_{i=1}^n P_i$. Define $\tilde{\mathcal{F}}(P) = \bigoplus_{i=1}^n \mathcal{F}(P_i)$.

We now define $\tilde{\mathcal{F}}$ on morphisms. Let $P, Q, R \in \mathcal{P}$ and write $P = \bigoplus_{i=1}^n P_i$, $Q = \bigoplus_{j=1}^m Q_j$, and $R = \bigoplus_{k=1}^r R_k$. First we observe that

$$\text{Hom}_{\text{gr-}A}(P, Q) = \text{Hom}_{\text{gr-}A} \left(\bigoplus_{i=1}^n P_i, \bigoplus_{j=1}^m Q_j \right) \cong \bigoplus_{i=1}^n \bigoplus_{j=1}^m \text{Hom}_{\text{gr-}A}(P_i, Q_j)$$

so we can represent $\varphi \in \text{Hom}_{\text{gr-}A}(P, Q)$ as a sum $\varphi = \sum_{i=1}^n \sum_{j=1}^m \varphi_{i,j}$ where $\varphi_{i,j} \in \text{Hom}_{\text{gr-}A}(P_i, Q_j)$. Define

$$\tilde{\mathcal{F}}(\varphi) = \sum_{i=1}^n \sum_{j=1}^m \mathcal{F}(\varphi_{i,j}).$$

Now we can write $\text{Id}_P = \sum_{i=1}^n \text{Id}_{P_i}$ and use the fact that \mathcal{F} is a functor to deduce that

$$\tilde{\mathcal{F}}(\text{Id}_P) = \sum_{i=1}^n \mathcal{F}(\text{Id}_{P_i}) = \sum_{i=1}^n \text{Id}_{\mathcal{F}(P_i)} = \text{Id}_{\tilde{\mathcal{F}}(P)}.$$

We now check that $\tilde{\mathcal{F}}$ preserves compositions. If $\psi \in \text{Hom}_{\text{gr-}A}(Q, R)$, then we can write

$$\psi \circ \varphi = \sum_{i=1}^n \sum_{k=1}^r (\psi \circ \varphi)_{i,k} = \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^r \psi_{j,k} \circ \varphi_{i,j}$$

so, since \mathcal{F} is an additive functor,

$$\begin{aligned}
\tilde{\mathcal{F}}(\psi \circ \varphi) &= \sum_{i=1}^n \sum_{k=1}^r \mathcal{F}((\psi \circ \varphi)_{i,k}) = \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^r \mathcal{F}(\psi_{j,k} \circ \varphi_{i,j}) \\
&= \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^r \mathcal{F}(\psi_{j,k}) \circ \mathcal{F}(\varphi_{i,j}) \\
&= \sum_{j=1}^m \sum_{k=1}^r \mathcal{F}(\psi_{j,k}) \circ \sum_{i=1}^n \sum_{j=1}^m \mathcal{F}(\varphi_{i,j}) = \tilde{\mathcal{F}}(\psi) \circ \tilde{\mathcal{F}}(\varphi).
\end{aligned}$$

Hence, $\tilde{\mathcal{F}}$ preserves compositions of morphisms and so $\tilde{\mathcal{F}} : \mathbf{P} \rightarrow \mathbf{C}$ is a functor extending \mathcal{F} . \square

The remaining results in this section give the technical tools used to construct autoequivalences of $\text{gr-}A$ switching X and Y .

Lemma 4.0.4. *Let \mathbf{C} be an abelian category with enough projectives, let \mathbf{P} be a full subcategory of \mathbf{C} consisting of projective objects, and let $\mathbf{D} \subseteq \mathbf{C}$ be a full subcategory which is closed under subobjects. Suppose that for every $M \in \mathbf{P}$, there exists a unique smallest subobject $N \subseteq M$ such that $M/N \in \mathbf{D}$. Write $N = \mathcal{F}(M)$. Then there is an additive functor $\mathcal{F} : \mathbf{P} \rightarrow \mathbf{C}$ where for each projective P , $\mathcal{F}(P)$ is as defined above, with the action of \mathcal{F} on morphisms being restriction.*

Proof. Let P and Q be projective objects and $f \in \text{Hom}_{\mathbf{C}}(P, Q)$. If we can show that $f(\mathcal{F}(P)) \subseteq \mathcal{F}(Q)$ then $f|_{\mathcal{F}(P)} : \mathcal{F}(P) \rightarrow \mathcal{F}(Q)$ is a well-defined restriction, and it is easy to see the functor $\mathcal{F} : \mathbf{P} \rightarrow \mathbf{C}$ defined as above is additive.

Now $Q/\mathcal{F}(Q) \in \mathbf{D}$ by the definition of $\mathcal{F}(Q)$ and $P/f^{-1}(\mathcal{F}(Q))$ embeds into $Q/\mathcal{F}(Q)$. Since \mathbf{D} is closed under subobjects, $P/f^{-1}(\mathcal{F}(Q)) \in \mathbf{D}$. Then $\mathcal{F}(P) \subseteq f^{-1}(\mathcal{F}(Q))$ because $\mathcal{F}(P)$ was defined to be the unique smallest $N \subseteq P$ such that $P/N \in \mathbf{D}$. Hence, $f(\mathcal{F}(P)) \subseteq \mathcal{F}(Q)$, as desired. \square

The preceding lemma is the main tool we will use to construct our autoequivalence. We will construct a full subcategory \mathbf{D} of $\text{gr-}A$ and a functor ι_0 that maps a rank one projective module to the smallest kernel of morphisms to elements of \mathbf{D} .

Proposition 4.0.5. *Let \mathcal{C} be an abelian \mathbb{k} -linear category. Let $\mathcal{I} = \{I_1, \dots, I_n\}$ be a finite set of indecomposable objects in \mathcal{C} and let $\mathcal{D} \subseteq \mathcal{C}$ be the full subcategory consisting of all finite direct sums of elements of \mathcal{I} , possibly with repeats. Suppose further for every $C \in \mathcal{C}$ and every $D \in \mathcal{D}$ that $\text{Hom}_{\mathcal{C}}(C, D)$ is finite-dimensional and that \mathcal{D} is closed under subobjects. Then for each $M \in \mathcal{C}$, there exists a unique smallest $N \subseteq M$ such that $M/N \in \mathcal{D}$.*

Proof. Let M be an object in \mathcal{C} . By hypothesis, for each $1 \leq i \leq n$, $\text{Hom}_{\mathcal{C}}(M, I_i)$ is finite-dimensional, say spanned by $\varphi_{i_1}, \dots, \varphi_{i_d}$. Note that for $1 \leq j \leq d$, $M/\ker \varphi_{i_j} \subseteq I_i$ is an object in \mathcal{D} , because \mathcal{D} is closed under subobjects.

Define $J_i = \bigcap_{j=1}^d \ker \varphi_{i_j}$. First, since $M/J_i \subseteq \bigoplus_{j=1}^d M/\ker \varphi_{i_j}$ and \mathcal{D} is closed under direct sums and subobjects, $M/J_i \in \mathcal{D}$. Further, for any $\psi \in \text{Hom}_{\mathcal{C}}(M, I_i)$, $J_i \subseteq \ker \psi$.

Now consider the intersection $N = \bigcap_{i=1}^n J_i \subseteq M$. Again, since each $M/J_i \in \mathcal{D}$, we have $M/N \in \mathcal{D}$. To show that N is the unique smallest such object, let $L = \bigoplus_{j=1}^r I_{\alpha_j}$ be an object in \mathcal{D} and let $\psi \in \text{Hom}_{\mathcal{C}}(M, L)$. Because \mathcal{C} is an abelian category, $\text{Hom}_{\mathcal{C}}(M, L) = \text{Hom}_{\mathcal{C}}(M, \bigoplus_{j=1}^r I_{\alpha_j}) = \bigoplus_{j=1}^r \text{Hom}_{\mathcal{C}}(M, I_{\alpha_j})$. But as $N = \bigcap_{i=1}^n \bigcap_{j=1}^d \ker \varphi_{i_j}$, therefore $N \subseteq \ker \psi$. \square

Chapter 5

The Picard group of $\text{gr-}A$

In this chapter, we determine the Picard group of $\text{gr-}A$. Let \mathbb{Z}_{fin} be the group of finite subsets of \mathbb{Z} with operation \oplus given by exclusive or. For the first Weyl algebra, A_1 , in [Sie09, Corollary 5.11] Sierra computed that

$$\text{Pic}(\text{gr-}A_1) \cong \mathbb{Z}_{\text{fin}} \rtimes D_\infty.$$

The subgroup of $\text{Pic}(\text{gr-}A_1)$ isomorphic to D_∞ is generated by the shift functor \mathcal{S}_{A_1} and the autoequivalence reversing the graded structure of A_1 . For quadratic polynomials f , $\text{gr-}A(f)$ is still equipped with both a shift functor $\mathcal{S}_{A(f)}$ as well as the grading-reversing autoequivalence ω . We therefore expect D_∞ to appear as a subgroup of $\text{Pic}(\text{gr-}A(f))$.

The subgroup of $\text{Pic}(\text{gr-}A_1)$ isomorphic to \mathbb{Z}_{fin} is generated by autoequivalences that Sierra calls *involutions* of $\text{gr-}A_1$. In section 5.2, we will construct analogous involutions of $\text{gr-}A(f)$, and so we will show that for any quadratic polynomial $f \in \mathbb{k}[z]$,

$$\text{Pic}(\text{gr-}A(f)) \cong \mathbb{Z}_{\text{fin}} \rtimes D_\infty \cong \text{Pic}(\text{gr-}A_1).$$

5.1 The rigidity of $\text{gr-}A$

In this section, we will show that $\text{gr-}A$ exhibits the same sort of rigidity as $\text{gr-}A_1$. The general structure of our arguments parallel [Sie09, §5]. In particular,

we will first prove that an autoequivalence of $\text{gr-}A$ is determined by its action on only the simple modules supported at $\mathbb{Z} \cup \mathbb{Z} - \alpha$. We will then prove that any autoequivalence of $\text{gr-}A$ must permute the simple modules in a rigid way.

We begin by proving an analogue of [Sie09, Lemma 5.1]. Recall from section 2.1, that for a \mathbb{Z} -graded ring R , we defined the \mathbb{Z} -algebra associated to R

$$\bar{R} = \bigoplus_{i,j \in \mathbb{Z}} \bar{R}_{i,j}$$

where $\bar{R}_{i,j} = R_{j-i}$. Recall also that an automorphism γ of \bar{R} is called *inner* if for all $m, n \in \mathbb{Z}$, there exist $g_m \in \bar{R}_{m,m}$ and $h_n \in \bar{R}_{n,n}$ such that for all $w \in \bar{R}_{m,n}$, $\gamma(w) = g_m w h_n$.

We will first study the automorphisms of \bar{A} . Following the notation of Sierra, define $m_{ij} \in \bar{A}_{ij} = A_{j-i}$ to be the canonical $\mathbb{k}[z]$ -module generator of A_{j-i} ; that is, m_{ij} is x^{j-i} if $j \geq i$ and y^{i-j} if $i > j$.

Lemma 5.1.1. *Every automorphism of \bar{A} of degree 0 is inner.*

Proof. Let γ be an automorphism of \bar{A} . Since γ is an automorphism, for all $i, j \in \mathbb{Z}$ there is a unit $\zeta_{ij} \in \mathbb{k}[z]$ such that $\gamma(m_{ij}) = \zeta_{ij} m_{ij}$. Thus, $\zeta_{ij} \in \mathbb{k}^*$. Further, for all $n \in \mathbb{Z}$, we have $\zeta_{nn} = 1$. Denote by γ_n the restriction of γ to $\bar{A}_{nn} = \mathbb{k}[z]$. Since γ_n is an automorphism of $\mathbb{k}[z]$, we know it is of the form $z \mapsto a_n z + b_n$ for some $a_n, b_n \in \mathbb{k}$.

Applying γ to the identity

$$m_{n,n+1} m_{n+1,n} = z(z + \alpha) \cdot 1_n$$

yields

$$\zeta_{n,n+1} \zeta_{n+1,n} m_{n,n+1} m_{n+1,n} = [a_n^2 z^2 + a_n(2b_n + \alpha)z + b_n(b_n + \alpha)] \cdot 1_n.$$

Since, in addition

$$\zeta_{n,n+1} \zeta_{n+1,n} m_{n,n+1} m_{n+1,n} = (\zeta_{n,n+1} \zeta_{n+1,n} z^2 + \zeta_{n,n+1} \zeta_{n+1,n} \alpha z) \cdot 1_n,$$

by comparing coefficients, either $b_n = 0$ or $b_n = -\alpha$. If $b_n = 0$, then $a_n = \zeta_{n,n+1} \zeta_{n+1,n} = a_n^2$. Since $\zeta_{n,n+1} \zeta_{n+1,n}$ is a unit, therefore $a_n = \zeta_{n,n+1} \zeta_{n+1,n} = 1$. On

the other hand, if $b_n = -\alpha$, then $-a_n = \zeta_{n,n+1}\zeta_{n+1,n} = a_n^2$. In this case, $a_n = -1$. In either case, $\zeta_{n,n+1}\zeta_{n+1,n} = 1$.

Now, apply γ to the identity

$$m_{n,n+1}m_{n+1,n} - m_{n,n-1}m_{n-1,n} = (2z + \alpha - 1) \cdot 1_n \quad (5.1)$$

to obtain

$$m_{n,n+1}m_{n+1,n} - \zeta_{n,n-1}\zeta_{n-1,n}m_{n,n-1}m_{n-1,n} = (2a_n z + 2b_n + \alpha - 1) \cdot 1_n. \quad (5.2)$$

Subtracting these equations yields

$$(\zeta_{n,n-1}\zeta_{n-1,n} - 1)m_{n,n-1}m_{n-1,n} = (2z - 2a_n z - 2b_n) \cdot 1_n$$

and so

$$(\zeta_{n,n-1}\zeta_{n-1,n} - 1)(z - 1)(z + \alpha - 1) \cdot 1_n = (2z - 2a_n z - 2b_n) \cdot 1_n.$$

If $\zeta_{n,n-1}\zeta_{n-1,n} - 1 \neq 0$, then the left-hand side is quadratic in z , but the right-hand side is linear z . Hence, $\zeta_{n,n-1}\zeta_{n-1,n} = 1$. But now, comparing equations (5.1) and (5.2) means

$$\gamma_n(z \cdot 1_n) = z \cdot 1_n.$$

So for all n , γ_n is the identity on \bar{A}_{nn} . Hence, for any $g \in \mathbb{k}[z] = \bar{A}_{ii}$ and $i, j \in \mathbb{Z}$, we have

$$\gamma(g \cdot m_{ij}) = \zeta_{ij}g \cdot m_{ij}.$$

Now, for all $i, j, l \in \mathbb{Z}$, $m_{ij}m_{jl} = hm_{il}$ for some $h \in \mathbb{k}[z] = \bar{A}_{ii}$. By applying γ , we conclude $\zeta_{ij}\zeta_{jl} = \zeta_{il}$. So if $v \in \bar{A}_{ij}$, we have $\gamma(v) = \zeta_{ij}v = \zeta_{i0}v\zeta_{0j}$, so γ is inner by [Sie09, Theorem 3.10]. \square

This technical result allows us to prove an analogue of [Sie09, Corollary 5.6]. As in the case of $\text{gr-}A_1$, we can check if two autoequivalences of $\text{gr-}A$ are naturally isomorphic by checking on a relatively small set of simple modules.

Lemma 5.1.2. *Let \mathcal{F} and \mathcal{F}' be autoequivalences of $\text{gr-}A$. Then $\mathcal{F} \cong \mathcal{F}'$ if and only if $\mathcal{F}(S) \cong \mathcal{F}'(S)$ for all simple modules S supported at $\mathbb{Z} \cup \mathbb{Z} - \alpha$.*

Proof. Suppose $\mathcal{F}(S) \cong \mathcal{F}'(S)$ for all simple modules S supported at $\mathbb{Z} \cup \mathbb{Z} - \alpha$. By Corollary 3.4.8, since any rank one projective is determined by its simple factors supported at $\mathbb{Z} \cup \mathbb{Z} - \alpha$, we have that $\mathcal{F}(P) \cong \mathcal{F}'(P)$ for all rank one projectives P . In particular, for all integers n , $(\mathcal{F}')^{-1}\mathcal{F}(A\langle n \rangle) \cong A\langle n \rangle$. Hence, $(\mathcal{F}')^{-1}\mathcal{F}$ is a twist functor. Finally, by Lemma 5.1.1 and Theorem 2.1.2, $(\mathcal{F}')^{-1}\mathcal{F} \cong \text{Id}_{\text{gr-}A}$ and so $\mathcal{F} \cong \mathcal{F}'$. \square

The proof of the preceding lemma also demonstrates what we noticed in section 3.4—that the structure of $\text{gr-}A$ is largely determined by its subcategory of rank one projective modules. We now prove analogues of [Sie09, Theorem 5.3].

Theorem 5.1.3. *Let $\alpha \in \mathbb{N}$ and let \mathcal{F} be an autoequivalence of $\text{gr-}A$. Then there exist unique integers $a = \pm 1$ and b such that for all $n \in \mathbb{Z}$*

$$\{\mathcal{F}(X\langle n \rangle), \mathcal{F}(Y\langle n \rangle)\} \cong \{X\langle an + b \rangle, Y\langle an + b \rangle\},$$

and for all $\lambda \in \mathbb{k} \setminus \mathbb{Z}$,

$$\mathcal{F}(M_\lambda) \cong M_{a\lambda+b}.$$

If $\alpha > 0$, then additionally

$$\mathcal{F}(Z\langle n \rangle) \cong Z\langle an + b \rangle.$$

Proof. The proof is essentially the same as Sierra's. First, let $\alpha = 0$. Since \mathcal{F} is an autoequivalence, for any integer n there exists an integer n' such that \mathcal{F} maps the pair $\{X\langle n \rangle, Y\langle n \rangle\}$ to the pair $\{X\langle n' \rangle, Y\langle n' \rangle\}$ since by Lemmas 3.3.1, these are the only pairs of simple modules with both nonsplit self-extensions as well as nonsplit extensions by each other. Now let $\alpha > 0$. In this case, for any integer n there exists an integer n' such $\mathcal{F}(Z\langle n \rangle) \cong Z\langle n' \rangle$ since the shifts of Z are the only simple modules that have projective dimension 2. Then we also see that \mathcal{F} maps the pair $\{X\langle n \rangle, Y\langle n \rangle\}$ to the pair $\{X\langle n' \rangle, Y\langle n' \rangle\}$ since these are the unique simple modules which have nonsplit extensions with $Z\langle n' \rangle$.

For any $\alpha \in \mathbb{Z}$, for any $\lambda \in \mathbb{k} \setminus \mathbb{Z}$, there exists a $\mu \in \mathbb{k} \setminus \mathbb{Z}$ such that $\mathcal{F}(M_\lambda) \cong M_\mu$ since these are the only simple modules whose only nonsplit extensions are self-extensions. Altogether then, there exists a bijective function $g : \mathbb{k} \rightarrow \mathbb{k}$ such that

- (i) If $\lambda \in \mathbb{Z}$, then $g(\lambda) \in \mathbb{Z}$ and $\mathcal{F}(\{X\langle\lambda\rangle, Y\langle\lambda\rangle\}) \cong \{X\langle g(\lambda)\rangle, Y\langle g(\lambda)\rangle\}$ (and if $\alpha > 0$, $\mathcal{F}(Z\langle\lambda\rangle) \cong Z\langle g(\lambda)\rangle$).
- (ii) If $\lambda \notin \mathbb{Z}$, then $g(\lambda) \notin \mathbb{Z}$ and $\mathcal{F}(M_\lambda) \cong M_{g(\lambda)}$.

Now consider the functor $\mathcal{F}_0 = \mathcal{F}(- \otimes_{\mathbb{k}[z]} A)_0 : \text{mod-}\mathbb{k}[z] \rightarrow \text{mod-}\mathbb{k}[z]$. Notice that $\mathcal{F}_0(\mathbb{k}[z]) \cong \mathbb{k}[z]$. Further, for all $\lambda \in \mathbb{k}$, we have that $\mathcal{F}_0(\mathbb{k}[z]/(z + \lambda)) \cong \mathbb{k}[z]/(z + g(\lambda))$. If $\lambda \in \mathbb{Z}$, this follows from Lemma 3.1.3, otherwise it follows from the definition of M_λ and $M_{g(\lambda)}$. The functor \mathcal{F}_0 gives a \mathbb{k} -algebra homomorphism $\varphi : \text{Hom}_{\mathbb{k}[z]}(\mathbb{k}[z], \mathbb{k}[z]) \rightarrow \text{Hom}_{\mathbb{k}[z]}(\mathcal{F}_0\mathbb{k}[z], \mathcal{F}_0\mathbb{k}[z])$. Identify $\mathbb{k}[z]$ with $\text{Hom}_{\mathbb{k}[z]}(\mathbb{k}[z], \mathbb{k}[z])$, where $h \in \mathbb{k}[z]$ corresponds to left multiplication by h .

The functor \mathcal{F}_0 takes the short exact sequence

$$0 \rightarrow \mathbb{k}[z] \xrightarrow{(z+\lambda)\cdot} \mathbb{k}[z] \rightarrow \mathbb{k}[z]/(z + \lambda) \rightarrow 0$$

to

$$0 \rightarrow \mathcal{F}_0\mathbb{k}[z] \rightarrow \mathcal{F}_0\mathbb{k}[z] \rightarrow \mathbb{k}[z]/(z + g(\lambda)) \rightarrow 0,$$

so φ maps multiplication by $z + \lambda$ to multiplication by $c(z + g(\lambda))$ for some $c \in \mathbb{k}^*$. Therefore, $\varphi(z)$ must be linear in z , i.e. $\varphi(z) = \gamma z + \delta$ for some $\gamma, \delta \in \mathbb{k}$. Then

$$\gamma z + \delta + \lambda = \varphi(z + \lambda) = c(z + g(\lambda))$$

and so $g(\lambda) = (\lambda + \delta)/\gamma$. Since g maps \mathbb{Z} bijectively to \mathbb{Z} , we conclude that $\gamma = \pm 1$ and $\delta \in \mathbb{Z}$. Take $a = \gamma$ and $b = a\delta$. \square

Theorem 5.1.4. *Let $\alpha \in \mathbb{k} \setminus \mathbb{Z}$, and $\alpha \notin \mathbb{Z} + 1/2$. Let \mathcal{F} be an autoequivalence of $\text{gr-}A$. Then exactly one of the following is true:*

1. *There exists a unique integer b such that for all $n \in \mathbb{Z}$*

$$\{\mathcal{F}(X_0\langle n \rangle), \mathcal{F}(Y_0\langle n \rangle)\} \cong \{X_0\langle n + b \rangle, Y_0\langle n + b \rangle\},$$

$$\{\mathcal{F}(X_\alpha\langle n \rangle), \mathcal{F}(Y_\alpha\langle n \rangle)\} \cong \{X_\alpha\langle n + b \rangle, Y_\alpha\langle n + b \rangle\},$$

and for all $\lambda \in \mathbb{k} \setminus (\mathbb{Z} \cup \mathbb{Z} + \alpha)$

$$\mathcal{F}(M_\lambda) \cong M_{\lambda+b}.$$

2. There exists a unique integer b such that for all $n \in \mathbb{Z}$

$$\{\mathcal{F}(X_0\langle n \rangle), \mathcal{F}(Y_0\langle n \rangle)\} \cong \{X_\alpha\langle -n + b \rangle, Y_\alpha\langle -n + b \rangle\},$$

$$\{\mathcal{F}(X_\alpha\langle n \rangle), \mathcal{F}(Y_\alpha\langle n \rangle)\} \cong \{X_0\langle -n + b \rangle, Y_0\langle -n + b \rangle\},$$

and for all $\lambda \in \mathbb{k} \setminus (\mathbb{Z} \cup \mathbb{Z} + \alpha)$

$$\mathcal{F}(M_\lambda) \cong M_{-\lambda + \alpha + b}.$$

Proof. This proof is quite similar to the previous one, although the details are slightly messier.

From Lemma 3.1.1 and Lemma 3.3.6, for any $\lambda \in \mathbb{k} \setminus (\mathbb{Z} \cup \mathbb{Z} + \alpha)$, there exists a $\mu \in \mathbb{k} \setminus (\mathbb{Z} \cup \mathbb{Z} + \alpha)$ such that $\mathcal{F}(M_\lambda) \cong M_\mu$, since these are the only simple modules with nonsplit self extensions. Further, by Lemma 3.3.6, for all $n \in \mathbb{Z}$ \mathcal{F} must map the pair $\{X_0\langle n \rangle, Y_0\langle n \rangle\}$ to either a pair $\{X_0\langle n' \rangle, Y_0\langle n' \rangle\}$ or a pair $\{X_\alpha\langle n' \rangle, Y_\alpha\langle n' \rangle\}$ for some integer n' , since these pairs form the only nonsplit extensions of two nonisomorphic simples. Likewise for the pair $\{X_\alpha\langle n \rangle, Y_\alpha\langle n \rangle\}$. That is, there is a bijection $g : \mathbb{k} \rightarrow \mathbb{k}$ such that

(1) If $\lambda \in \mathbb{Z}$, then either

- $g(\lambda) \in \mathbb{Z}$ and $\mathcal{F}(\{X_0\langle \lambda \rangle, Y_0\langle \lambda \rangle\}) \cong \{X_0\langle g(\lambda) \rangle, Y_0\langle g(\lambda) \rangle\}$ or else
- $g(\lambda) \in \mathbb{Z} + \alpha$ and $\mathcal{F}(\{X_0\langle \lambda \rangle, Y_0\langle \lambda \rangle\}) \cong \{X_\alpha\langle g(\lambda) - \alpha \rangle, Y_\alpha\langle g(\lambda) - \alpha \rangle\}$.

(2) If $\lambda \in \mathbb{Z} + \alpha$, then either

- $g(\lambda) \in \mathbb{Z} + \alpha$ and $\mathcal{F}(\{X_\alpha\langle \lambda - \alpha \rangle, Y_\alpha\langle \lambda - \alpha \rangle\}) \cong \{X_\alpha\langle g(\lambda) - \alpha \rangle, Y_\alpha\langle g(\lambda) - \alpha \rangle\}$ or else
- $g(\lambda) \in \mathbb{Z}$ and $\mathcal{F}(\{X_\alpha\langle \lambda - \alpha \rangle, Y_\alpha\langle \lambda - \alpha \rangle\}) \cong \{X_0\langle g(\lambda) \rangle, Y_0\langle g(\lambda) \rangle\}$.

(3) If $\lambda \notin (\mathbb{Z} \cup \mathbb{Z} + \alpha)$, then $g(\lambda) \notin (\mathbb{Z} \cup \mathbb{Z} + \alpha)$ and $\mathcal{F}(M_\lambda) \cong M_{g(\lambda)}$.

As in the proof of Lemma 5.1.3, consider the functor $\mathcal{F}_0 = \mathcal{F}(- \otimes_{\mathbb{k}[z]} A)_0 : \text{mod-}\mathbb{k}[z] \rightarrow \text{mod-}\mathbb{k}[z]$. Just as in the previous proof, there exist $\delta, \gamma \in \mathbb{k}$ such that $g(\lambda) = (\lambda + \delta)/\gamma$. Since g maps $\mathbb{Z} \cup \mathbb{Z} + \alpha$ bijectively to itself, we conclude that

$\gamma = \pm 1$. Since we assumed $\alpha \notin \mathbb{Z} + 1/2$, if $\gamma = 1$ then $\delta \in \mathbb{Z}$, and if $\gamma = -1$ then $\delta \in \mathbb{Z} - \alpha$.

If $\gamma = 1$, let $b = \delta \in \mathbb{Z}$. In this case,

$$\begin{aligned} \{\mathcal{F}(X_0\langle n \rangle), \mathcal{F}(Y_0\langle n \rangle)\} &\cong \{X_0\langle n + b \rangle, Y_0\langle n + b \rangle\} \text{ and} \\ \{\mathcal{F}(X_\alpha\langle n \rangle), \mathcal{F}(Y_\alpha\langle n \rangle)\} &\cong \{X_\alpha\langle n + b \rangle, Y_\alpha\langle n + b \rangle\}. \end{aligned}$$

If $\gamma = -1$, let $b = -\alpha - \delta$. In this case,

$$\begin{aligned} \{\mathcal{F}(X_0\langle n \rangle), \mathcal{F}(Y_0\langle n \rangle)\} &\cong \{X_\alpha\langle -n + b \rangle, Y_\alpha\langle -n + b \rangle\} \text{ and} \\ \{\mathcal{F}(X_\alpha\langle n \rangle), \mathcal{F}(Y_\alpha\langle n \rangle)\} &\cong \{X_0\langle -n + b \rangle, Y_0\langle -n + b \rangle\}. \end{aligned}$$

□

Definition 5.1.5 ([Sie09, Definition 5.4]). If \mathcal{F} is an autoequivalence of $\text{gr-}A$, we call the integer b above the *rank* of \mathcal{F} . The integer a above is called the *sign* of \mathcal{F} . If $a = 1$, then we say \mathcal{F} is *even* and if $a = -1$, we say \mathcal{F} is *odd*. If \mathcal{F} is even and has rank 0, we say that \mathcal{F} is *numerically trivial*.

Example 5.1.6. As was the case for autoequivalences of $\text{gr-}A_1$, \mathcal{S}_A^n is an even autoequivalence of rank n and ω is an odd autoequivalence of rank -1 .

Notice that if $\alpha \in \mathbb{Z} + 1/2$, then there are potentially extra symmetries of $\text{gr-}A$. In the proof of Theorem 5.1.4, the function $g : \mathbb{k} \rightarrow \mathbb{k}$ could be of the form $g(\lambda) = \lambda + 1/2$, as this maps $\mathbb{Z} + \alpha$ bijectively to itself if $\alpha \in \mathbb{Z} + 1/2$. We show that in fact, in this case, there is an autoequivalence which acts in this way.

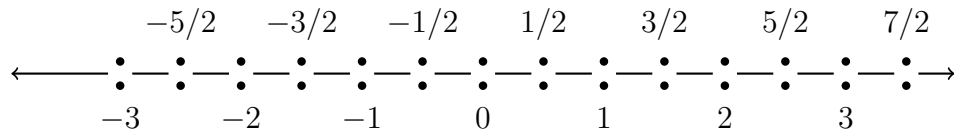


Figure 5.1.1: The simple modules of $\text{gr-}A(f)$ when $\alpha = 1/2$.

Proposition 5.1.7. *Let $\alpha \in \mathbb{Z} + 1/2$. There exists an autoequivalence \mathcal{F} of $\text{gr-}A$ such that for all $n \in \mathbb{Z}$,*

$$\begin{aligned} \mathcal{F}(X_0\langle n \rangle) &\cong X_\alpha\langle n + 1/2 - \alpha \rangle, & \mathcal{F}(Y_0\langle n \rangle) &\cong Y_\alpha\langle n + 1/2 - \alpha \rangle, \\ \mathcal{F}(X_\alpha\langle n \rangle) &\cong X_0\langle n + 1/2 + \alpha \rangle, & \mathcal{F}(Y_\alpha\langle n \rangle) &\cong Y_0\langle n + 1/2 + \alpha \rangle, \end{aligned}$$

and for all $\lambda \in \mathbb{k} \setminus (\mathbb{Z} \cup \mathbb{Z} + 1/2)$

$$\mathcal{F}(M_\lambda) \cong M_{\lambda+1/2}.$$

Proof. We will construct an autoequivalence \mathcal{F} that translates Figure 5.1.1 by $1/2$. By Lemmas 4.0.3 and 4.0.1, it suffices to define \mathcal{F} on the full subcategory \mathbf{R} of $\text{gr-}A$ consisting of the canonical rank one projective right A -modules. Let P be a rank one canonical projective module with structure constants $\{c_i\}$. By the work in section 3.2, we know we can write each structure constant as $c_i = a_i b_i$ where $a_i \in \{1, \sigma^i(z)\}$ and $b_i \in \{1, \sigma^i(z + \alpha)\}$. Let $\sigma^{1/2}$ be the automorphism of $\mathbb{k}[z]$ with $\sigma^{1/2}(z) = z + 1/2$.

Define $\mathcal{F}(P)$ as the canonical rank one projective module whose structure constants $\{c'_i\}$ are defined as follows. If $a_{n+\alpha-1/2} = \sigma^{n+\alpha-1/2}(z)$, then c'_n has a factor of $\sigma^n(z + \alpha)$. If $b_{n-\alpha-1/2} = \sigma^{n-\alpha-1/2}(z + \alpha)$, then c'_n has a factor of $\sigma^n(z)$. Overall, the irreducible factors of the $\{c'_i\}$ are given by $\{\sigma^{1/2}(a_i)\}$ and $\{\sigma^{1/2}(b_i)\}$, where the factors appear in the structure constant of the appropriate degree. Since, for all n , $c'_n \in \{1, \sigma^n(z), \sigma^n(z + 1/2), \sigma^n(f)\}$, and for $n \gg 0$, $c'_n = 1$ and $c'_{-n} = \sigma^{-n}(f)$, these structure constants define a canonical rank one projective module.

Let P and Q be canonical rank one projective modules with structure constants $\{c_i\}$ and $\{d_i\}$, respectively. Let the structure constants of $\mathcal{F}(P)$ and $\mathcal{F}(Q)$ be $\{c'_i\}$ and $\{d'_i\}$ respectively. By Lemma 3.4.13 and Corollary 3.4.14, there is an $N \in \mathbb{Z}$ such that $\underline{\text{Hom}}_A(P, Q)$ and $\underline{\text{Hom}}_A(\mathcal{F}(P), \mathcal{F}(Q))$ are generated as a $\mathbb{k}[z]$ -module by multiplication by

$$\theta_{P,Q} = \frac{\prod_{j \geq N} d_j}{\prod_{j \geq N} \gcd(c_j, d_j)} \quad \text{and}$$

$$\theta_{\mathcal{F}(P), \mathcal{F}(Q)} = \frac{\prod_{j \geq N} d'_j}{\prod_{j \geq N} \gcd(c'_j, d'_j)},$$

respectively. Notice that by the way we defined the structure constants of $\mathcal{F}(P)$ and $\mathcal{F}(Q)$,

$$\theta_{\mathcal{F}(P), \mathcal{F}(Q)} = \sigma^{1/2}(\theta_{P,Q}).$$

Let $g \in \underline{\text{Hom}}_A(P, Q)$, so $g = \varphi \theta_{P,Q}$ for some $\varphi \in \mathbb{k}[z]$. Define $\mathcal{F}(g)$ to be left multiplication by $\sigma^{1/2}(\varphi \theta_{P,Q}) = \sigma^{1/2}(\varphi) \theta_{\mathcal{F}(P), \mathcal{F}(Q)}$.

Since every morphism in \mathbf{R} is given by left multiplication by an element of $\mathbb{k}[z]$ and \mathcal{F} acts on morphisms by applying $\sigma^{1/2}$ to this element, clearly \mathcal{F} is functorial. Since the identity morphism is just multiplication by 1, we have $\mathcal{F}(\text{Id}_P) = \text{Id}_{\mathcal{F}(P)}$. Hence, \mathcal{F} is a functor. It is also easy to see that \mathcal{F} is essentially surjective on canonical rank one projective modules. Given a rank one projective module P' , we reverse the structure constant construction to construct a canonical rank one projective P such that $\mathcal{F}(P) \cong P'$. Since $\sigma^{1/2}$ is an automorphism of $\mathbb{k}[z]$, it gives an isomorphism

$$\underline{\text{Hom}}_A(P, Q) = \theta_{P, Q} \mathbb{k}[z] \cong \sigma^{1/2}(\theta_{P, Q} \mathbb{k}[z]) = \underline{\text{Hom}}_A(\mathcal{F}(P), \mathcal{F}(Q)).$$

Hence, \mathcal{F} is full and faithful and so is an autoequivalence of \mathbf{R} , which extends uniquely to an autoequivalence of $\text{gr-}A$ by Corollary 4.0.2 and Lemma 4.0.3.

We need only show that \mathcal{F} has the claimed action on simple modules. For each $\lambda \in \mathbb{k}$, \mathcal{F} maps the exact sequence

$$0 \longrightarrow A \xrightarrow{(z+\lambda)\cdot} A \longrightarrow A/(z+\lambda)A \longrightarrow 0$$

to the exact sequence

$$0 \longrightarrow \mathcal{F}(A) \xrightarrow{(z+\lambda+1/2)\cdot} \mathcal{F}(A) \longrightarrow \mathcal{F}(A/(z+\lambda)A) \longrightarrow 0.$$

Hence, $\mathcal{F}(A/(z+\lambda)A)$ is supported at $-(\lambda+1/2)$ so if $\lambda \notin \mathbb{Z} + 1/2$, $\mathcal{F}(M_\lambda) \cong M_{\lambda+1/2}$. The pair of simple modules $\{X_0\langle n \rangle, Y_0\langle n \rangle\}$ which are supported at $-n$ must map to the pair $\{X_\alpha\langle n+1/2-\alpha \rangle, Y_\alpha\langle n+1/2-\alpha \rangle\}$, as these are the simples supported at $-(n+1/2)$. Similarly, the pair $\{X_\alpha\langle n \rangle, Y_\alpha\langle n \rangle\}$ must map to the pair $\{X_0\langle n-1/2+\alpha \rangle, Y_0\langle n-1/2+\alpha \rangle\}$.

Now consider the short exact sequence

$$0 \longrightarrow (xA+zA)\langle n \rangle \xrightarrow{1\cdot} A\langle n \rangle \longrightarrow X_0\langle n \rangle \longrightarrow 0.$$

Using the construction in Lemmas 3.2.5 and 3.2.6, we can explicitly compute the structure constants of $xA+zA$. Since in each graded component of degree $i \leq 0$, $(xA+zA)_i = z(A)_i$ and for $i > 0$, $(xA+zA)_i = A_i$, therefore $xA+zA$ has structure constants which are the same as that of A , except in degree 0 where $xA+zA$ has a

structure constant of z . Multiplying structure constants to compute the canonical representation of $xA + zA$, we observe that $xA + zA$ is itself a canonical rank one projective module. Since maximal embeddings of canonical rank one projectives are given by multiplication by elements in $\mathbb{k}[z]$, the inclusion $xA + zA \rightarrow A$ in (5.1) is a maximal embedding.

Since \mathcal{F} of multiplication by 1 is again given by multiplication by 1, \mathcal{F} maps (5.1) to the exact sequence

$$0 \longrightarrow \mathcal{F}((xA + zA)\langle n \rangle) \xrightarrow{1} \mathcal{F}(A\langle n \rangle) \longrightarrow \mathcal{F}(X_0\langle n \rangle) \longrightarrow 0.$$

Hence, $\mathcal{F}(X_0\langle n \rangle)$ is zero in sufficiently large degree, so $\mathcal{F}(X_0\langle n \rangle) \cong X_\alpha\langle n+1/2-\alpha \rangle$. This then implies $\mathcal{F}(Y_0\langle n \rangle) \cong Y_\alpha\langle n+1/2-\alpha \rangle$. A similar computation for $X_\alpha\langle n \rangle$ completes the proof. \square

Remark 5.1.8. The autoequivalence constructed in Proposition 5.1.7 translates the picture of the simple modules by $1/2$. Since the square of this autoequivalence is isomorphic to \mathcal{S}_A , we call it $\mathcal{S}^{1/2}$.

For completeness, we can extend Definition 5.1.5 to the case $\alpha \in \mathbb{Z} + 1/2$ in a natural way. If \mathcal{F} is an autoequivalence of $\text{gr-}A(f)$, and for all $\beta \in \mathbb{k}$, \mathcal{F} maps the simples supported at β to simples supported at $\beta + n/2$ for some $n \in \mathbb{Z}$, we say that \mathcal{F} is *even* and has *rank* $n/2$. If \mathcal{F} maps the simples supported at β to simples supported at $-\beta + n/2$ for some $n \in \mathbb{Z}$, we say that \mathcal{F} is *odd* and has *rank* $n/2$. If \mathcal{F} is even and has rank 0 then we say \mathcal{F} is *numerically trivial*. The autoequivalence $\mathcal{S}^{1/2}$ is even and has rank $1/2$.

Corollary 5.1.9. *Let $\text{Pic}_0(\text{gr-}A)$ be the subgroup of $\text{Pic}(\text{gr-}A)$ of numerically trivial autoequivalences. Then $\text{Pic}(\text{gr-}A) \cong \text{Pic}_0(\text{gr-}A) \rtimes D_\infty$.*

Proof. By Lemma 5.1.2, each autoequivalence in $\text{Pic}(\text{gr-}A)$ is determined by its action on the simple modules supported at $\mathbb{Z} \cup \mathbb{Z} - \alpha$. Let $\alpha \notin \mathbb{Z} + 1/2$. By checking their action on the simple modules supported at $\mathbb{Z} \cup \mathbb{Z} - \alpha$, we observe that $\omega\mathcal{S}_A \cong \mathcal{S}_A^{-1}\omega$ and so the subgroup

$$\langle \omega, \mathcal{S}_A \rangle \subseteq \text{Pic}(\text{gr-}A)$$

is isomorphic to D_∞ .

Again, by considering the action of numerically trivial autoequivalences on the simple modules, we observe that $\text{Pic}_0(\text{gr-}A)$ is a normal subgroup of $\text{Pic}(\text{gr-}A)$. If \mathcal{F} is a numerically trivial autoequivalence in $\langle \omega, \mathcal{S}_A \rangle$, then we can write $\mathcal{F} = \mathcal{S}_A^i \omega^j$ for some $i, j \in \mathbb{Z}$. Since \mathcal{F} is numerically trivial and $\omega^2 = \text{Id}_{\text{gr-}A}$, in fact $\mathcal{F} = \mathcal{S}_A^i$ which then implies $i = 0$ so $\mathcal{F} = \text{Id}_{\text{gr-}A}$. Therefore,

$$\text{Pic}_0(\text{gr-}A) \cap \langle \omega, \mathcal{S}_A \rangle = \{\text{Id}_{\text{gr-}A}\}.$$

By Theorems 5.1.3 and 5.1.4, any autoequivalence can be written as the product of an autoequivalence in $\langle \omega, \mathcal{S}_A \rangle$ and a numerically trivial autoequivalence. Therefore, $\text{Pic}(\text{gr-}A) \cong \text{Pic}_0(\text{gr-}A) \rtimes D_\infty$, as desired.

In the case $\alpha \in \mathbb{Z} + 1/2$, $\text{Pic}(\text{gr-}A)$ contains a subgroup isomorphic to D_∞ generated by the autoequivalences $\mathcal{S}_A^{1/2}$ and ω . This is a finer copy of D_∞ , containing $\langle \omega, \mathcal{S}_A \rangle$ as a subgroup. The remainder of the proof is identical to the previous case. \square

5.2 Involutions of gr- A

Having shown that $\text{gr-}A(f)$ has the same rigidity as $\text{gr-}A_1$, we now show that $\text{Pic}_0(\text{gr-}A(f))$ is also isomorphic to $\text{Pic}_0(\text{gr-}A_1)$. We construct autoequivalences which are analogous to the involutions ι_j of Sierra.

Proposition 5.2.1. *Let $\alpha \in \mathbb{N}$. Then for any $j \in \mathbb{Z}$, there is a numerically trivial autoequivalence ι_j of $\text{gr-}A$ such that $\iota_j(X\langle j \rangle) \cong Y\langle j \rangle$, $\iota_j(Y\langle j \rangle) \cong X\langle j \rangle$, and $\iota_j(S) \cong S$ for all other simple modules S . For any $i, j \in \mathbb{Z}$, $\mathcal{S}_A^i \iota_j \cong \iota_{i+j} \mathcal{S}_A^i$, and $\iota_j^2 \cong \text{Id}_{\text{gr-}A}$.*

Proof. First, suppose $\alpha \in \mathbb{N}^+$. It will suffice to construct ι_0 , for if ι_0 exists then we may define $\iota_j = \mathcal{S}_A^j \iota_0 \mathcal{S}_A^{-j}$. Let \mathbf{P} be the full subcategory of $\text{gr-}A$ whose objects are direct sums of canonical rank one projective modules and let \mathbf{R} be the full subcategory of $\text{gr-}A$ whose objects are the canonical rank one projective modules. Note that since every finitely generated graded right A -module has a projective

resolution by objects in \mathbf{P} , by Lemma 4.0.1, we can construct ι_0 by defining it on \mathbf{P} , then extend to $\text{gr-}A$. By Lemma 4.0.3, it suffices to define ι_0 only on \mathbf{R} .

Let

$$S = \{0, X, Y, Z, E_{Z,X}, E_{Z,Y}, E_{X,Z}, E_{Y,Z}, E_{Z,Y,X}, E_{X,Z,X}, E_{Y,Z,Y}\}$$

and let \mathbf{D} be the full subcategory of $\text{gr-}A$ whose objects are exactly the elements of S . It is clear that \mathbf{D} is closed under subobjects. Let P be a graded rank one projective module. By Corollary 3.4.6, P surjects onto exactly one of X , Y , and Z , a module that we call $F_0(P)$. Suppose $F_0(P) = X$. Since P is projective, the surjection $f_0 : P \rightarrow X$ lifts to a morphism $f_1 : P \rightarrow E_{X,Z}$. Since f_1 is a lift of a surjection to X , f_1 is surjective, as $E_{X,Z}$ has no subobject isomorphic to X . Again, since P is projective, f_1 lifts to a surjection $f_2 : P \rightarrow E_{X,Z,X}$, as $E_{X,Z,X}$ has no subobject isomorphic to $E_{X,Z}$.

Given the structure constants $\{c_i\}$ of P , we can in fact construct the submodule of P that is the kernel of f_2 . We construct this submodule in three steps. First, we construct K_0 , the kernel of f_0 , which is unique by Corollary 3.2.7. Following the construction in Lemma 3.2.5, K_0 is the submodule of P that has structure constants equal to c_i for all $i \in \mathbb{Z}$ except when $i = -\alpha$, where K_0 has structure constant $zc_{-\alpha}$. We then construct $\ker f_1 = K_1$ as a submodule of K_0 . Note that since $(P/K_1)/(K_0/K_1) \cong P/K_0$, we must have that K_1 is the unique submodule of K_0 which is the kernel of the surjection $K_0 \rightarrow Z$. Again, by the construction in Lemma 3.2.5, K_1 has structure constants c_i for all $i \in \mathbb{Z}$ except when or $i = 0$ where K_1 has structure constant zc_0 . Finally, we can construct $\ker f_2 = K_2$ as a submodule of K_1 , by constructing the unique submodule such that $K_1/K_2 \cong X$. Observe that K_2 has structure constants c_i for all $i \in \mathbb{Z}$, except when $i = -\alpha$, where K_2 has structure constant $zc_{-\alpha}$ and when $i = 0$, where K_2 has structure constant zc_0 .

Similarly, if $F_0(P) = Y$, there is a unique submodule of P which is the kernel of a surjection $P \rightarrow E_{Y,Z,Y}$. If $F_0(P) = Z$, then $\text{Hom}_{\text{gr-}A}(P, E_{Z,Y,X}) = \mathbb{k}$ so there is a unique submodule which is the kernel of a surjection $P \rightarrow E_{Z,Y,X}$. In any case, there exists a unique smallest submodule N of P such that $P/N \in \mathbf{D}$. Define $\iota_0 P = N$. By Lemma 4.0.4, ι_0 gives an additive functor $\mathbf{R} \rightarrow \text{gr-}A$ such that ι_0

acts on morphisms by restriction. By using Lemma 4.0.3 and Proposition 4.0.1, we extend ι_0 to a functor $\iota_0 : \text{gr-}A \rightarrow \text{gr-}A$.

We now show that ι_0 has the claimed properties. Suppose P has structure constants $\{c_i\}$. Above, we computed the structure constants, $\{d_i\}$ of $\iota_0 P$. By Lemma 3.2.5 and Corollary 3.4.4 if $F_0(P) = X$, then $c_{-\alpha} \in \{1, \sigma^{-\alpha}(z)\}$ and $c_0 \in \{1, z + \alpha\}$. We showed that $d_{-\alpha} = zc_{-\alpha}$ and $d_0 = zc_0 \in \{z, f\}$. For all $i \neq 0, -\alpha$, we saw that $c_i = d_i$. Hence, we can find $\iota_0(P)$ explicitly as a submodule of P as follows:

$$(\iota_0 P)_i = \begin{cases} z^2 P_i & \text{if } i \leq -\alpha \\ z P_i & \text{if } -\alpha < i \leq 0 \\ P_i & \text{if } i > 0. \end{cases}$$

Similarly, if $F_0(P) = Y$, then $c_0 \in \{z, f\}$ and $c_{-\alpha} \in \{\sigma^{-\alpha}(z + \alpha), \sigma^{-\alpha}(f)\}$ and $d_0 = z^{-1}c_0$ and $d_{-\alpha} = z^{-1}c_{-\alpha}$. We can explicitly construct $\iota_0 P$ as follows:

$$(\iota_0 P)_i = \begin{cases} P_i & \text{if } i \leq -\alpha \\ z P_i & \text{if } -\alpha < i \leq 0 \\ z^2 P_i & \text{if } i > 0. \end{cases}$$

Finally, if $F_0(P) = Z$, then P surjects onto $E_{Z,Y,X} = A/zA$, and $(\iota_0 P)_i = zP_i$ for all $i \in \mathbb{Z}$, so $c_i = d_i$ for all $i \in \mathbb{Z}$.

We can describe the action of ι_0 on P purely in terms of its structure constants, as follows. If both c_0 and $c_{-\alpha}$ can be multiplied by z (i.e. for $i = 0, -\alpha$, $zc_i \in \{1, \sigma^i(z), \sigma^i(z + \alpha), \sigma^i(f)\}$) then $\iota_0 P$ has $d_0 = zc_0$ and $d_{-\alpha} = zc_{-\alpha}$. Likewise, if both c_0 and $c_{-\alpha}$ can be divided by z , then $\iota_0 P$ has $d_0 = z^{-1}c_0$ and $d_{-\alpha} = z^{-1}c_{-\alpha}$. Otherwise, $d_0 = c_0$ and $d_{-\alpha} = c_{-\alpha}$. Observe that by Lemma 3.2.5, if $F_0(P) = X$ then $F_0(\iota_0 P) = Y$ and vice versa. Hence, by repeating the above process once by taking the kernel to $E_{X,Z,X}$ and next the kernel to $E_{Y,Z,Y}$ we compute that $\iota_0^2 P = z^2 P$.

So for any rank one projective, P , $\iota_0^2 P = z^2 P \cong P$. Additionally, if P' is another rank one projective, Proposition 3.4.11 tells us that $\text{Hom}_{\text{gr-}A}(z^2 P, z^2 P') = \text{Hom}_{\text{gr-}A}(P, P')$ is given by left multiplication by a $\mathbb{k}[z]$ -multiple of some $\theta \in \mathbb{k}(z)$. Since ι_0 is defined on morphisms to be restriction, this shows that ι_0^2 also gives

an isomorphism $\text{Hom}_{\text{gr-}A}(P, P') \cong \text{Hom}_{\text{gr-}A}(\iota_0^2 P, \iota_0^2 P')$. Since ι_0 is additive, it preserves finite direct sums, so for a direct sum of rank one projectives, ι_0^2 is given by multiplication by z^2 in each component. Hence, ι_0^2 is naturally isomorphic to the identity functor on the full subcategory of finite direct sums of rank one projectives. Extending to all of $\text{gr-}A$, this shows that ι_0 is an autoequivalence of $\text{gr-}A$ (with quasi-inverse ι_0).

Now, because for any rank one graded projective module P , the structure constants of $\iota_0 P$ differ from those of P only in degrees 0 and $-\alpha$, $\iota_0 P$ has the same integrally supported simple factors as P except possibly X , $X\langle\alpha\rangle$, Y , $Y\langle-\alpha\rangle$, Z , and $Z\langle\pm\alpha\rangle$. But if $F_\alpha(P) = X\langle\alpha\rangle$, then by Table 3.4.1, $c_0 \in \{1, z\}$. By the construction of $\iota_0 P$ (multiplying c_0 by 1, z or z^{-1} to compute d_0), this means $d_0 \in \{1, z\}$ as well, so $F_\alpha(\iota_0 P) = X\langle\alpha\rangle$. Similarly, if $F_{-\alpha}(P) = Y\langle-\alpha\rangle$ then $F_{-\alpha}(\iota_0 P) = Y\langle-\alpha\rangle$ as well. If $F_0(P) = Z$, then we observed above that $\iota P \cong P$ so $F_0(\iota_0 P) = Z$. If $F_\alpha(P) = Z\langle\alpha\rangle$ then $c_0 \in \{z + \alpha, f\}$. Again, since $\iota_0 P$ acts on structure constants only multiplying by 1, z , or z^{-1} , this means $d_0 \in \{z + \alpha, f\}$. Since $\iota_0 P$ does not affect the structure constant in degree α , by Lemma 3.2.5, $F_\alpha(\iota_0 P) = Z\langle\alpha\rangle$. Similarly, if $F_{-\alpha}(P) = Z\langle-\alpha\rangle$, then $F_{-\alpha}(\iota_0 P) = Z\langle-\alpha\rangle$. Altogether, all of the integrally supported simple factors of $\iota_0 P$ are the same as those of P , except that if $F_0(P) = X$ then $F_0(\iota_0 P) = Y$ and vice versa.

We will now show that ι_0 fixes all simples modules other than X and Y . Let $S \notin \{X, Y\}$ be an integrally supported simple module. Suppose for contradiction that $\iota_0 S \not\cong S$. By Lemma 3.4.9, we can construct a canonical rank one projective module P such that S is a factor of P but $\iota_0 S$ is not. Note that $\iota_0 S$ is a factor of $\iota_0 P$. But by the discussion above, P and $\iota_0 P$ have the same integrally supported simple factors except possibly X and Y , so S is also a factor of $\iota_0 P$. Applying ι_0 again, we conclude that both $\iota_0 S$ and $\iota_0^2 S$ are factors of $\iota_0^2 P$, but since $\iota_0^2 \cong \text{Id}_{\text{gr-}A}$, this is a contradiction, as $\iota_0 S$ is not a factor of P . Hence, ι_0 fixes all integrally supported simple modules other than X and Y .

Since ι_0 fixes all integrally supported simple modules other than X and Y , ι_0 is numerically trivial. By Theorem 5.1.3, $\iota_0 M_\lambda \cong M_\lambda$ for all $\lambda \in \mathbb{k} \setminus \mathbb{Z}$. Further, for a projective module P , if $F_0(P) = X$ then $F_0(\iota_0 P) = Y$. Therefore, $\iota_0 X \cong Y$.

For the case that $\alpha = 0$, we use the same argument, letting \mathbf{D} be the full subcategory of $\text{gr-}A$ whose objects are in the set $S = \{0, X, Y, E_{X,X}, E_{Y,Y}\}$. In this case, for a rank one projective P , we can again explicitly construct $\iota_0 P \subseteq P$. If $F_0(P) = X$, then

$$(\iota_0 P)_i = \begin{cases} z^2 P_i & \text{if } i \leq 0 \\ P_i & \text{if } i > 0 \end{cases}$$

and if $F_0(P) = Y$, then

$$(\iota_0 P)_i = \begin{cases} P_i & \text{if } i \leq 0 \\ z^2 P_i & \text{if } i > 0. \end{cases}$$

Let $\{d_i\}$ be the structure constants of $\iota_0 P$. In this case, if $c_0 = 1$ then $d_0 = z^2$. If $c_0 = z^2$ then $d_0 = 1$. The remainder of the proof is analogous to the previous case. \square

In the case that $\alpha \in \mathbb{N}$, we have thus constructed analogues of Sierra's autoequivalences ι_j [Sie09, Proposition 5.7]. These autoequivalences, which we also call ι_j share many of the same properties as Sierra's. First notice that since we defined $\iota_j = \mathcal{S}_A^j \iota_0 \mathcal{S}_A^{-j}$, we construct ι_j by shifting all the modules of S by j and repeating the same construction. This also that means that for a rank one graded projective module P , $\iota_j^2 P = (z + j)^2 P$, by the same argument as in the previous proof. Also, reviewing the construction above, it is clear that for any integers i and j , $\iota_i \iota_j = \iota_j \iota_i$ and so the subgroup of $\text{Pic}(\text{gr-}A)$ generated by the $\{\iota_j\}$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{(\mathbb{Z})}$.

We identify $(\mathbb{Z}/2\mathbb{Z})^{(\mathbb{Z})}$ with finite subsets of the integers, \mathbb{Z}_{fin} with operation given by exclusive or. We often denote the singleton set $\{n\} \in \mathbb{Z}_{\text{fin}}$ as simply n . For each $J \in \mathbb{Z}_{\text{fin}}$, we define the autoequivalence

$$\iota_J = \prod_{j \in J} \iota_j.$$

As we noted in the previous proof, ι_j has quasi-inverse ι_j so ι_J has quasi-inverse ι_J . For completeness, define $\iota_\emptyset = \text{Id}_{\text{gr-}A}$.

Having constructed involutions in the case $\alpha \in \mathbb{N}$, we now turn our attention to the case $\alpha \in \mathbb{k} \setminus \mathbb{Z}$. In this case we will also be able to construct generalizations of Sierra's autoequivalences.

Proposition 5.2.2. *Let $\alpha \in \mathbb{k} \setminus \mathbb{Z}$. Then for any $j \in \mathbb{Z}$, there is a numerically trivial autoequivalence $\iota_{(j,\emptyset)}$ of $\text{gr-}A$ such that $\iota_{(j,\emptyset)}(X_0\langle j \rangle) \cong Y_0\langle j \rangle$, $\iota_{(j,\emptyset)}(Y_0\langle j \rangle) \cong X_0\langle j \rangle$, and $\iota_{(j,\emptyset)}(S) \cong S$ for all other simple modules S . For any $i, j \in \mathbb{Z}$, $\mathcal{S}_A^i \iota_{(j,\emptyset)} \cong \iota_{(i+j,\emptyset)} \mathcal{S}_A^i$, and $(\iota_{(j,\emptyset)})^2 \cong \text{Id}_{\text{gr-}A}$.*

Similarly, for any $j \in \mathbb{Z}$, there is a numerically trivial autoequivalence $\iota_{(\emptyset,j)}$ of $\text{gr-}A$ such that $\iota_{(\emptyset,j)}(X_\alpha\langle j \rangle) \cong Y_\alpha\langle j \rangle$, $\iota_{(\emptyset,j)}(Y_\alpha\langle j \rangle) \cong X_\alpha\langle j \rangle$, and $\iota_{(\emptyset,j)}(S) \cong S$ for all other simple modules S . For any $i, j \in \mathbb{Z}$, $\mathcal{S}_A^i \iota_{(\emptyset,j)} \cong \iota_{(\emptyset,i+j)} \mathcal{S}_A^i$, and $(\iota_{(\emptyset,j)})^2 \cong \text{Id}_{\text{gr-}A}$.

Proof. The construction is similar to that in the proof of Proposition 5.2.1. We construct $\iota_{(\emptyset,0)}$ and define $\iota_{(\emptyset,j)} = \mathcal{S}_A^j \iota_{(\emptyset,0)} \mathcal{S}_A^{-j}$. Let \mathbf{R} be the full subcategory of $\text{gr-}A$ whose objects are the canonical rank one projectives. We define $\iota_{(\emptyset,0)}$ on \mathbf{R} , then use Lemmas 4.0.3 and 4.0.1 to extend to a functor defined on all of $\text{gr-}A$. The construction of $\iota_{(0,\emptyset)}$ is completely analogous.

Let $S = \{0, X_\alpha, Y_\alpha\}$ and let \mathbf{D} be the full subcategory of $\text{gr-}A$ whose objects are the elements of S . Clearly \mathbf{D} is closed under subobjects. Let P be a graded rank one projective module. By Corollary 3.4.6, P surjects onto exactly one of X_α and Y_α , a module that we called $F_0^\alpha(P)$. Hence, there exists a unique smallest submodule $N \subseteq P$ such that $P/N \in S$. Let $N = \iota_{(\emptyset,0)}P$. By Lemma 4.0.4, $\iota_{(\emptyset,0)}$ gives an additive functor $\mathbf{R} \rightarrow \text{gr-}A$ whose action on morphisms is given by restriction.

Focusing now on structure constants, let $\{c_i\}$ and $\{d_i\}$ be the structure constants for P and $\iota_{(\emptyset,0)}P$, respectively. We can compute d_0 by constructing the unique kernel to the surjection $P \rightarrow F_0^\alpha(P)$. If $F_0^\alpha(P) = X_\alpha$, then

$$(\iota_{(\emptyset,0)}P)_i = \begin{cases} (z + \alpha)P_i & \text{if } i \leq 0 \\ P_i & \text{if } i > 0 \end{cases}$$

and if $F_0^\alpha(P) = Y$, then

$$(\iota_{(\emptyset,0)}P)_i = \begin{cases} P_i & \text{if } i \leq 0 \\ (z + \alpha)P_i & \text{if } i > 0. \end{cases}$$

In particular, If $c_0 \in \{1, z\}$ then $d_0 = (z + \alpha)c_0$ and if $c_0 \in \{z + \alpha, f\}$, then $d_0 = (z + \alpha)^{-1}c_0$. By repeating this construction, we see that $\iota_{(\emptyset,0)}^2 P = (z + \alpha)P$ and so $\text{gr-}A$ is an autoequivalence of $\text{gr-}A$ with quasi-inverse $\iota_{(\emptyset,0)}$.

Because for any rank one graded projective module P the structure constants for $\iota_{(\emptyset,0)}P$ differ from those of P only in degree 0, where they differ only by a factor of $z + \alpha$, by Lemma 3.2.6, $\iota_{(\emptyset,0)}P$ has the same integrally supported simple factors as P except if $F_0^\alpha(P) = X_\alpha$ then $F_0^\alpha(\iota_{(\emptyset,0)}P) = Y_\alpha$ and vice versa. Again, examining the action of $\iota_{(\emptyset,0)}$ over all rank one projectives P , we deduce that $\iota_{(\emptyset,0)}P$ has the claimed action on simple modules, fixing all but X_α and Y_α . \square

We have therefore also constructed analogues of Sierra's ι_j in the case that $\alpha \in \mathbb{k} \setminus \mathbb{Z}$. The subscript on the ι keeps track of which of the simples modules is being permuted: the first coordinate corresponds to the shifts of X_0 and Y_0 while the second coordinate corresponds to the shifts of X_α and Y_α . Observe that in this case the subgroup of $\text{Pic}(\text{gr-}A)$ generated by the $\{\iota_{(j,\emptyset)}, \iota_{(\emptyset,j)}\}$ is isomorphic to the direct product $\mathbb{Z}_{\text{fin}} \times \mathbb{Z}_{\text{fin}}$. For every $J, J' \in \mathbb{Z}_{\text{fin}} \times \mathbb{Z}_{\text{fin}}$ we define

$$\iota_{(J,J')} = \prod_{j \in J} \iota_{(j,\emptyset)} \prod_{j \in J'} \iota_{(\emptyset,j)}.$$

For completeness, define $\iota_{(\emptyset,\emptyset)} = \text{Id}_{\text{gr-}A}$.

Finally, we are able to determine $\text{Pic}_0(\text{gr-}A)$ and therefore $\text{Pic}(\text{gr-}A)$.

Lemma 5.2.3. *Let $\alpha \in \mathbb{N}$. Then the map*

$$\begin{aligned} \Phi : \mathbb{Z}_{\text{fin}} &\rightarrow \text{Pic}_0(\text{gr-}A(f)) \\ J &\mapsto \iota_J \end{aligned}$$

is a group isomorphism.

Let $\alpha \in \mathbb{k} \setminus \mathbb{Z}$. Then the map

$$\begin{aligned} \Psi : \mathbb{Z}_{\text{fin}} \times \mathbb{Z}_{\text{fin}} &\rightarrow \text{Pic}_0(\text{gr-}A(f)) \\ (J, J') &\mapsto \iota_{(J,J')} \end{aligned}$$

is a group isomorphism.

Proof. First let $\alpha \in \mathbb{N}$. For any $i \in \mathbb{Z}$, we saw that $\iota_i^2 \cong \text{Id}_{\text{gr-}A}$. Hence, $\iota_J \iota_{J'} \cong \iota_{J \oplus J'}$ so Φ is a group homomorphism. It is clear that Φ is injective. To show surjectivity, suppose $\mathcal{F} \in \text{Pic}_0(\text{gr-}A)$. Since \mathcal{F} is numerically trivial, by Theorem 5.1.3, \mathcal{F} fixes all shifts of Z .

By Lemma 3.4.9, we can construct a canonical rank one projective P such that $F_n(P) = X\langle n \rangle$ for all $n \geq 0$ and $F_n(P) = Y\langle n \rangle$ for all $n < 0$. Note that since \mathcal{F} is numerically trivial, for any $j \in \mathbb{Z}$, $\mathcal{F}(F_j(P)) \cong F_j(\mathcal{F}(P))$. Now by Corollary 3.2.8, $\mathcal{F}(F_j(P))$ can only differ from $F_j(\mathcal{F}(P))$ for finitely many j . Let J be precisely those indices at which they differ. By Lemma 5.1.2, $\iota_J \cong \mathcal{F}$, so Φ is surjective.

The case $\alpha \in \mathbb{k} \setminus \mathbb{Z}$ follows from the same argument, doubling the number of indices where necessary. \square

Theorem 5.2.4. *Let $f \in \mathbb{k}[z]$ be quadratic. Then*

$$\text{Pic}(\text{gr-}A(f)) \cong \mathbb{Z}_{\text{fin}} \rtimes D_\infty.$$

Proof. This follows from Corollary 5.1.9, Lemma 5.2.3, and the fact that

$$\mathbb{Z}_{\text{fin}} \times \mathbb{Z}_{\text{fin}} \cong \mathbb{Z}_{\text{fin}}. \quad \square$$

Finally, we are interested in when the collections of modules $\{\iota_J A \mid J \in \mathbb{Z}_{\text{fin}}\}$ (in the case $\alpha \in \mathbb{N}$) or $\{\iota_{(J,J')} \mid (J, J') \in \mathbb{Z}_{\text{fin}} \times \mathbb{Z}_{\text{fin}}\}$ (in the case $\alpha \in \mathbb{k} \setminus \mathbb{Z}$) generate $\text{gr-}A(f)$. In the cases of a multiple root or non-congruent roots, then just as in the case for the first Weyl algebra, these collections of modules generate $\text{gr-}A(f)$. However, in the case of a congruent root, we see that $\{\iota_J A \mid J \in \mathbb{Z}_{\text{fin}}\}$ does not.

Lemma 5.2.5. *Let $f = z(z + \alpha)$.*

1. *If $\alpha = 0$, then $\{\iota_J A \mid J \in \mathbb{Z}_{\text{fin}}\}$ generates $\text{gr-}A(f)$.*
2. *If $\alpha \in \mathbb{k} \setminus \mathbb{Z}$, then $\{\iota_{(J,J')} A \mid (J, J') \in \mathbb{Z}_{\text{fin}} \times \mathbb{Z}_{\text{fin}}\}$ generates $\text{gr-}A(f)$.*
3. *If $\alpha \in \mathbb{N}^+$, then $\{\iota_J A \mid J \in \mathbb{Z}_{\text{fin}}\}$ does not generate $\text{gr-}A(f)$.*

Proof. In the first two cases, this follows from Proposition 3.4.16, Proposition 5.2.1, and Proposition 5.2.2. In the case that $\alpha \in \mathbb{N}^+$, for all $J \in \mathbb{Z}_{\text{fin}}$, the shifts of Z that are factors $\iota_J(A)$ are exactly those that are factors of A . Namely, $\text{Hom}_{\text{gr-}A}(\iota_J(A), Z\langle n \rangle) = 0$ for $n < 0$ and $n \geq \alpha$. Hence, no $\iota_J(A)$ has a surjection to $Z\langle -1 \rangle$, and so $\{\iota_J(A) \mid J \in \mathbb{Z}_{\text{fin}}\}$ does not generate $\text{gr-}A(f)$. \square

Lemma 5.2.6. *Let $\alpha \in \mathbb{N}^+$. The action of $\text{Pic}_0(\text{gr-}A)$ on the set of graded rank one projective modules has infinitely many orbits, one for each finite subset of \mathbb{Z} .*

Proof. By Lemma 3.4.9, if for each n we choose $S_n \in \{X\langle n \rangle, Y\langle n \rangle, Z\langle n \rangle\}$ such that for $n \gg 0$, $S_n = X\langle n \rangle$ and $S_{-n} = Y\langle -n \rangle$, then there exists a rank one projective whose integrally supported simple factors are precisely the S_n . By Theorem 5.1.3, an autoequivalence $\mathcal{F} \in \text{Pic}_0(\text{gr-}A)$ will have $\mathcal{F}(Z\langle n \rangle) \cong Z\langle n \rangle$. Also, by Proposition 5.2.1, there exist numerically trivial autoequivalences permuting $X\langle n \rangle$ and $Y\langle n \rangle$ for any n . Hence, for each finite subset J of \mathbb{Z} there is an orbit consisting of all rank one graded projective modules that surject onto exactly $Z\langle j \rangle$ for each $j \in J$. \square

Remark 5.2.7. In the case $\alpha = 0$ or $\alpha \in \mathbb{k} \setminus \mathbb{Z}$, it is easily checked that the action of $\text{Pic}_0(\text{gr-}A)$ is transitive on the set of graded rank one projective modules.

Chapter 6

Constructing a homogeneous coordinate ring for $\text{gr-}A$

The main result in this chapter is that for a generalized Weyl algebra $A(f)$ defined by a quadratic polynomial f , there exists a commutative \mathbb{k} -algebra $B(f)$, graded by a group Γ such that $\text{qgr-}A(f) \cong \text{gr-}(B(f), \Gamma)$. We will construct the graded ring $B(f)$ explicitly in all cases. The main tool in proving this equivalence of categories is a theorem of Angel del R  o [del91, Theorem 7].

6.1 Notation

In order to use del R  o's theorem, we will use essentially the same notation as found in the discussion before [Smi11, Theorem 5.13]. Let R and S be \mathbb{k} -algebras graded by the abelian groups Γ and G , respectively. Following the definition of del R  o in [del91], we define a *bigraded R - S -bimodule* to be an R - S -bimodule P with a \mathbb{k} -vector space decomposition

$$P = \bigoplus_{(\gamma, g) \in \Gamma \times G} P_{(\gamma, g)}$$

that respects the graded structure of R on the left and S on the right. That is, for any $\gamma, \delta \in \Gamma$ and any $g, h \in G$,

$$R_\gamma \cdot P_{(\delta, h)} \cdot S_g \subseteq P_{(\gamma + \delta, g + h)}.$$

When we want to specify the degrees of an element $p \in P_{(\gamma,g)}$, we use the notation γp^g .

For any $\gamma \in \Gamma$ we have the G -graded right S -module

$$P_{(\gamma,*)} = \bigoplus_{g \in G} P_{(\gamma,g)}.$$

Note that if $r \in R_\delta$, then left multiplication by r is an S -module homomorphism $P_{(\gamma,*)} \rightarrow P_{(\gamma+\delta,*)}$ that preserves G -degree and hence we get a \mathbb{k} -linear map

$$\varphi : R_\delta \rightarrow \text{Hom}_{\text{gr}-(S,G)}(P_{(\gamma,*)}, P_{(\gamma+\delta,*)}). \quad (6.1)$$

We now define a functor $H_S(P, -) : \text{gr}-(S, G) \rightarrow \text{gr}-(R, \Gamma)$. if M is a G -graded right S -module, let

$$H_S(P, M) = \bigoplus_{\gamma \in \Gamma} \text{Hom}_{\text{gr}-(S,G)}(P_{(-\gamma,*)}, M).$$

First, note that $H_S(P, M)$ is Γ -graded. The right R -module structure is given as follows : given $h \in \text{Hom}_{\text{gr}-(S,G)}(P_{(-\gamma,*)}, M)$ and $r \in R_\delta$, recall that as in equation (6.1), multiplication by r gives a G -graded S -module homomorphism $\varphi(r) : P_{(-\gamma-\delta,*)} \rightarrow P_{(-\gamma,*)}$. Then

$$h \cdot r = h \circ \varphi(r) \in \text{Hom}_{\text{gr}-(S,G)}(P_{(-\gamma-\delta,*)}, M).$$

We obtain a functor

$$H_S(P, -) : \text{gr}-(S, G) \rightarrow \text{gr}-(R, \Gamma) \quad (6.2)$$

by defining $H_S(P, -)$ on a morphism $h : M \rightarrow N$ to be composition with h . In his discussion before [del91, Proposition 2], del R o notes that $H_S(P, -)$ is naturally isomorphic to the functor he denotes $(-)_*^P$.

In the subsequent sections we will attempt to construct a graded commutative ring B and a bigraded B - A -bimodule P , and then use del R o's theorem to prove that $H_{A(f)}(P, -)$ is an equivalence of categories. In the cases of a multiple root or distinct non-congruent roots, we will be able to construct such a ring and bimodule. In the case of congruent roots, we will pass to the quotient category $\text{qgr} - A(f)$ obtained by taking $\text{gr} - A(f)$ modulo its full subcategory

of finite-dimensional modules. We then show that $\text{qgr-}A(f) \cong \text{gr-}(B, \mathbb{Z}_{\text{fin}})$ for a commutative ring B . In the next section, we develop machinery which constructs Γ -graded rings R from autoequivalences in the Picard group of $\text{gr-}(S, G)$.

6.2 Defining a ring from autoequivalences

Suppose we have an abelian subgroup $\Gamma \subseteq \text{Pic}(\text{gr-}(S, G))$ and for each $\gamma \in \Gamma$, choose one autoequivalence $\mathcal{F}_\gamma \in \text{Aut}(\text{gr-}(S, G))$. Since $\text{Pic}(\text{gr-}(S, G))$ is the group $\text{Aut}(\text{gr-}(R, \Gamma))$ modulo natural isomorphism, we have, for all $\gamma, \delta \in \Gamma$

$$\mathcal{F}_\gamma \mathcal{F}_\delta \cong \mathcal{F}_\delta \mathcal{F}_\gamma \cong \mathcal{F}_{\gamma+\delta}.$$

Let $\Theta_{\gamma, \delta}$ be the G -graded S -module isomorphism $\mathcal{F}_\gamma \mathcal{F}_\delta S \rightarrow \mathcal{F}_{\gamma+\delta} S$ (i.e. $\Theta_{\gamma, \delta}$ is the natural isomorphism at S). Motivated by Paul Smith's construction in [Smi11, §10], we can define a Γ -graded ring R if the isomorphisms $\Theta_{\gamma, \delta}$ satisfy the following condition: for all $\gamma, \delta, \epsilon \in \Gamma$ and for all $\varphi \in \text{Hom}_{\text{gr-}(S, G)}(S, \iota_\gamma S)$

$$\Theta_{\epsilon, \gamma+\delta} \circ \mathcal{F}_\epsilon(\Theta_{\delta, \gamma}) \circ \mathcal{F}_\epsilon \mathcal{F}_\delta(\varphi) = \Theta_{\delta+\epsilon, \gamma} \circ \mathcal{F}_{\delta+\epsilon}(\varphi) \circ \Theta_{\epsilon, \delta}. \quad (6.3)$$

Morally, this says that the map $\text{Aut}(\text{gr-}(S, G)) \times \text{Aut}(\text{gr-}(S, G)) \rightarrow \text{Aut}(\text{gr-}(S, G))$ mapping $(\mathcal{F}_\gamma, \mathcal{F}_\delta)$ to $\mathcal{F}_{\gamma+\delta}$ is associative, although it is a weaker condition, since the isomorphisms $\Theta_{\gamma, \delta}$ are only at the module S_S .

Proposition 6.2.1. *Assume the setup and notation above. If the autoequivalences $\{\mathcal{F}_\gamma \mid \gamma \in \Gamma\}$ and isomorphisms $\{\Theta_{\gamma, \delta} \mid \gamma, \delta \in \Gamma\}$ satisfy condition (6.3) then*

$$R = \bigoplus_{\gamma \in \Gamma} \text{Hom}_{\text{gr-}(S, G)}(S, \mathcal{F}_\gamma S)$$

is an associative Γ -graded ring with multiplication defined as follows. For $\varphi \in R_\gamma$ and $\psi \in R_\delta$

$$\varphi \cdot \psi = \Theta_{\delta, \gamma} \circ \mathcal{F}_\delta(\varphi) \circ \psi.$$

Proof. We need to check that the multiplication defined above is associative. It suffices to check on homogeneous elements, so let $\varphi, \psi, \xi \in R$ be homogeneous elements of degree γ, δ , and ϵ , respectively. Then, by definition

$$(\varphi \cdot \psi) \cdot \xi = \Theta_{\epsilon, \gamma+\delta} \circ \mathcal{F}_\epsilon(\Theta_{\delta, \gamma}) \circ \mathcal{F}_\epsilon \mathcal{F}_\delta(\varphi) \circ \mathcal{F}_\epsilon(\psi) \circ \xi$$

and

$$\varphi \cdot (\psi \cdot \xi) = \Theta_{\delta+\epsilon, \gamma} \circ \mathcal{F}_{\delta+\epsilon}(\varphi) \circ \Theta_{\epsilon, \delta} \circ \mathcal{F}_\epsilon(\psi) \circ \xi.$$

Since we assumed the isomorphisms satisfied condition (6.3), R is associative. \square

If, for example, S is a \mathbb{Z} -graded ring and we take $\Gamma = \mathbb{Z}$ generated by the shift functor on $\text{gr-}S$, then $\mathcal{S}^n \mathcal{S}^m = \mathcal{S}^{n+m}$ so the isomorphisms $\Theta_{n,m}$ are trivial and condition (6.3) is automatic. By the construction in Proposition 6.2.1, we recover our original ring as $R \cong S$. Since we found that for a GWA A , $\text{gr-}A$ has many autoequivalences, we can choose a more interesting subgroup of autoequivalences to define such a ring.

Given rings R and S as in Proposition 6.2.1, condition (6.3) also allows us to define a bigraded R - S -bimodule P . Let

$$P = \bigoplus_{\gamma \in \Gamma} \mathcal{F}_\gamma S.$$

The G -graded right S -module structure on each $\mathcal{F}_\gamma S$ gives P a G -graded right S -module structure. Let $\varphi \in R_\gamma$ and $p \in P_{(\delta,*)} = \mathcal{F}_\delta S$. Then P has a Γ -graded left R -module structure given by

$$\varphi \cdot p = [\Theta_{\delta, \gamma} \circ \mathcal{F}_\delta(\varphi)](p) \in P_{(\gamma+\delta,*)}. \quad (6.4)$$

For $\varphi \in R_\gamma$, $\psi \in R_\delta$,

$$(\psi \cdot \varphi) \cdot p = [\Theta_{\epsilon, \gamma+\delta} \circ \mathcal{F}_\epsilon(\Theta_{\delta, \gamma}) \circ \mathcal{F}_\epsilon \mathcal{F}_\delta(\varphi) \circ \mathcal{F}_\epsilon(\psi)](p)$$

while

$$\psi \cdot (\varphi \cdot p) = [\Theta_{\delta+\epsilon, \gamma} \circ \mathcal{F}_{\delta+\epsilon}(\varphi) \circ \Theta_{\epsilon, \delta} \circ \mathcal{F}_\epsilon(\psi)](p).$$

and since we assumed condition (6.3), therefore equation (6.4) gives P a Γ -graded left R -module structure. Since \mathcal{F}_δ is an autoequivalence of $\text{gr-}(S, G)$ and $\Theta_{\delta, \gamma}$ is a G -graded S -module isomorphism, it is easily checked that this makes P a bigraded R - S -bimodule. This extra left R -module structure makes $H_S(P, P)$ a bigraded R - R -bimodule.

As a graded ring, R has its usual bigraded R - R -bimodule structure. We define the bigraded R - R -bimodule

$$\hat{R} = \bigoplus_{\gamma \in \Gamma} \bigoplus_{\delta \in \Gamma} R_{(\gamma, \delta)} = \bigoplus_{\gamma \in \Gamma} \bigoplus_{\delta \in \Gamma} R_{\gamma + \delta} = \bigoplus_{\gamma \in \Gamma} \bigoplus_{\delta \in \Gamma} \text{Hom}_{\text{gr}-(S, G)}(S, \mathcal{F}_{\gamma + \delta} S)$$

and note that there exists a canonical homomorphism of bigraded R - R -bimodules

$$\varrho_R^P : \hat{R} \longrightarrow H_S(P, P) = \bigoplus_{\gamma \in \Gamma} \text{Hom}_{\text{gr}-(S, G)} \left(\mathcal{F}_{-\gamma} S, \bigoplus_{\delta \in \Gamma} \mathcal{F}_{\delta} S \right)$$

where ϱ_R^P maps the element $\gamma\varphi^\delta$ to the homomorphism which maps $p \in P_{(\gamma, *)} = \mathcal{F}_{-\gamma} S$ to $\varphi \cdot p \in \mathcal{F}_{\delta} S$ where φ acts as in equation (6.4) and maps all other homogeneous elements to 0. This map ϱ_R^P is the same as the one del Río calls ϱ_A^P in [del91, Lemma 5], though del Río's bigraded bimodule's module structures are on opposite sides.

Proposition 6.2.2. *Assume the setup and notation above. If P is a generator of $\text{gr}-(S, G)$, then $H_S(P, -)$ gives an equivalence of categories*

$$\text{gr}-(R, \Gamma) \rightarrow \text{gr}-(S, G).$$

Proof. This follows immediately from [del91, Theorem 7(c)] as long as P is a projective generator of $\text{gr}-(S, G)$ and ϱ_R^P is an isomorphism. Since each \mathcal{F}_γ is an autoequivalence, $\mathcal{F}_\gamma S$ is automatically projective so if P is a generator then it is a projective generator. Hence, we need only show that ϱ_R^P is an isomorphism. Assuming the setup above, recall that for an element

$$\gamma\varphi^\delta \in {}_\gamma R_\delta = \text{Hom}_{\text{gr}-(S, G)}(S, \mathcal{F}_{\gamma + \delta} S)$$

$\varrho_R^P(\varphi)$ is given by the homomorphism $\Theta_{\delta, \gamma} \circ \mathcal{F}_\delta(\varphi) : \mathcal{F}_{-\gamma} S \rightarrow \mathcal{F}_\delta S$. Now since \mathcal{F}_γ is an autoequivalence it gives an isomorphism

$$\text{Hom}_{\text{gr}-(S, G)}(S, \mathcal{F}_{\gamma + \delta} S) \cong \text{Hom}_{\text{gr}-(S, G)}(\mathcal{F}_{-\gamma} S, \mathcal{F}_{-\gamma} \mathcal{F}_{\gamma + \delta} S)$$

and $\Theta_{-\gamma, \gamma + \delta}$ gives an isomorphism $\mathcal{F}_{-\gamma} \mathcal{F}_{\gamma + \delta} S \rightarrow S$. Hence, ϱ_R^P is an isomorphism and $H_S(P, -)$ is an equivalence of categories. \square

With this framework in place, we need only find subgroups of $\text{Pic}(\text{gr}-(S, G))$ such that the autoequivalences $\{\mathcal{F}_\gamma \mid \gamma \in \Gamma\}$ satisfy condition (6.3) and the $\mathcal{F}_\gamma S$ generate $\text{gr}-(S, G)$. Cranking the machinery yields a Γ -graded ring R such that $\text{gr}-(R, \Gamma) \cong \text{gr}-(S, G)$. For a generalized Weyl algebra $A(f)$, we will see that the autoequivalences $\{\iota_J \mid J \in \mathbb{Z}_{\text{fin}}\}$ constructed in Chapter 5 often satisfy these conditions.

6.3 Multiple root

Let $\alpha = 0$. We saw that in Chapter 5 the autoequivalences $\{\iota_n \mid n \in \mathbb{Z}\}$ formed a subgroup of $\text{Pic}(\text{gr}-A)$ isomorphic to \mathbb{Z}_{fin} . By Lemma 5.2.5, we know that the set $\{\iota_J A \mid J \in \mathbb{Z}_{\text{fin}}\}$ generates $\text{gr}-A$. We will show that these autoequivalences satisfy condition (6.3), and hence we can construct a \mathbb{Z}_{fin} -graded commutative ring B such that $\text{gr}-(B, \mathbb{Z}_{\text{fin}}) \cong \text{gr}-A$. For each $J \in \mathbb{Z}_{\text{fin}}$, we define the polynomial

$$h_J = \prod_{j \in J} (z + j)^2.$$

For completeness, define $h_\emptyset = 1$. Recall that in Proposition 5.2.1 we showed that for a projective module P , $\iota_n^2 P = (z + n)^2 P$ and so $\iota_n^2 \cong \text{Id}_{\text{gr}-A}$. We denote by σ_n the isomorphism $\iota_n^2 A \rightarrow A$. Since A is projective, σ_n is given by left multiplication by h_n^{-1} . Similarly, for $J \in \mathbb{Z}_{\text{fin}}$, we define

$$\sigma_J : \iota_J^2 A \rightarrow A$$

given by left multiplication by h_J^{-1} . Now, for $I, J \in \mathbb{Z}_{\text{fin}}$, we define

$$\Theta_{I,J} = \iota_{I \oplus J}(\sigma_{I \cap J}) = \Theta_{J,I}$$

$$\Theta_{I,J} : \iota_I \iota_J A = \iota_{I \oplus J} \iota_{I \cap J}^2 A \rightarrow \iota_{I \oplus J} A.$$

Lemma 6.3.1. *The isomorphisms $\{\Theta_{I,J} \mid I, J \in \mathbb{Z}_{\text{fin}}\}$ and the autoequivalences $\{\iota_K \mid K \in \mathbb{Z}_{\text{fin}}\}$ satisfy condition (6.3).*

Proof. We must show that for all $I, J, K \in \mathbb{Z}_{\text{fin}}$ and all $\varphi \in \text{Hom}_{\text{gr}-A}(A, \iota_I A)$,

$$\Theta_{K, I \oplus J} \circ \iota_K(\Theta_{J,I}) \circ \iota_K \iota_J(\varphi) = \Theta_{J+K, I} \circ \iota_{J+K}(\varphi) \circ \Theta_{K, J}$$

or equivalently that

$$\begin{aligned} & \iota_{I \oplus J \oplus K}(\sigma_{(I \oplus J) \cap K}) \circ \iota_K \iota_{I \oplus J}(\sigma_{I \cap J}) \circ \iota_K \iota_J(\varphi) \\ &= \iota_{I, J \oplus K}(\sigma_{I \cap (J \oplus K)}) \circ \iota_{J \oplus K}(\varphi) \circ \iota_{J \oplus K}(\sigma_{J \cap K}) \end{aligned}$$

Recall that by Proposition 3.4.11, the homomorphisms between rank one projective modules are all given by multiplication by an element in the commutative ring $\mathbb{k}(z)$. Since the autoequivalences ι_L act on morphisms by restriction, we need only check that multiplication by $h_{I \cap J}^{-1} h_{(I \oplus J) \cap K}^{-1}$ is the same as $h_{J \cap K}^{-1} h_{I \cap (J \oplus K)}^{-1}$. This is true since

$$(I \cap J) \cap (I \oplus J) \cap K = (J \cap K) \cap I \cap (J \oplus K) = \emptyset$$

and

$$(I \cap J) \cup (I \oplus J) \cap K = (I \cap J) \cup (I \cap K) \cup (J \cap K) = (J \cap K) \cup (I \cap (J \oplus K)).$$

Therefore, we conclude that the isomorphisms $\{\Theta_{I,J} \mid I, J \in \mathbb{Z}_{\text{fin}}\}$ and autoequivalences $\{\iota_K \mid K \in \mathbb{Z}_{\text{fin}}\}$ satisfy condition (6.3). \square

As in Proposition 6.2.1, we can define the \mathbb{Z}_{fin} -graded ring

$$B = \bigoplus_{J \in \mathbb{Z}_{\text{fin}}} B_J = \bigoplus_{J \in \mathbb{Z}_{\text{fin}}} \text{Hom}_{\text{gr-}A}(A, \iota_J A).$$

To be explicit, the multiplication in B is defined as follows. For $a \in B_I$ and $b \in B_J$, $a \cdot b \in \text{Hom}_{\text{gr-}A}(A, \iota_{I \oplus J} A)$ is defined by

$$a \cdot b = \iota_{I \oplus J}(\sigma_{I \cap J}) \circ \iota_J(a) \circ b.$$

Theorem 6.3.2. *There is an equivalence of categories*

$$\text{gr-}A \equiv \text{gr-}(B, \mathbb{Z}_{\text{fin}}).$$

Proof. This is an immediate corollary of Propositions 6.2.1 and 6.2.2 together with Lemma 6.3.1. \square

Our next results describe some properties of the ring B which will allow us to give a presentation for B . We first establish some notation. For $J \in \mathbb{Z}_{\text{fin}}$ let φ_J be the map

$$\varphi_J : (\iota_J A)_0 \rightarrow \text{Hom}_{\text{gr-}A}(A, \iota_J A)$$

which takes $m \in (\iota_J A)_0$ to the homomorphism defined by $\varphi_J(m)(a) = m \cdot a$. It is clear that φ_J is an isomorphism of $\mathbb{k}[z]$ -modules, and we will use φ_J to identify B_J with $(\iota_J A)_0$. For $J \in \mathbb{Z}_{\text{fin}}$, define $b_J := \varphi_J(h_J) \in B_J$ with $b_\emptyset := \varphi_\emptyset(1)$.

Lemma 6.3.3. *Let $\alpha = 0$ and let $I, J \in \mathbb{Z}_{\text{fin}}$.*

1. $(\iota_J A)_0 = h_J \mathbb{k}[z]$ so b_J freely generates B_J as a right B_\emptyset -module.
2. $b_I b_J = b_{I \cap J}^2 b_{I \oplus J}$ so $b_J = \prod_{j \in J} b_j$.
3. For all $n \in \mathbb{Z}$, $b_n^2 = \varphi_\emptyset(h_n)$.
4. B is a commutative \mathbb{k} -algebra generated by $\{b_n \mid n \in \mathbb{Z}\}$.

Proof. This is an analogue of [Smi11, Lemma 10.2]; we use similar arguments to Smith, altering them slightly when necessary. Though these results are similar, we will see later in this section that, interestingly, the ring B exhibits properties that are rather different those of from Smith's ring C .

1. Recall that in Proposition 5.2.1, for each $n \in \mathbb{Z}$ and for each rank one projective P , we constructed $\iota_n P$ as a submodule of P , in particular the kernel of a nonzero morphism $P \rightarrow E_{X,X}\langle n \rangle$ or $P \rightarrow E_{Y,Y}\langle n \rangle$. We also saw that $(\iota_n^2 P)_0 = (z+n)^2 P_0$. Therefore, $(z+n)^2 A_0 = (\iota_n^2 A)_0 \subseteq (\iota_n A)_0$. But since $\iota_n A$ is the kernel of the nonzero morphism to $E_{X,X}\langle n \rangle$ or $E_{Y,Y}\langle n \rangle$, and these modules have \mathbb{k} -dimension 2 in all graded components where they are nonzero, so $(\iota_n A)_0$ has \mathbb{k} -codimension 2 in A_0 . Hence,

$$(\iota_n A)_0 = (z+n)^2 A_0 = h_n \mathbb{k}[z].$$

Now, since the autoequivalences ι_n commute, $(\iota_J A)_0 \subseteq (\iota_j A)_0 = h_j \mathbb{k}[z]$ for each $j \in J$. Additionally, $(\iota_J A)_0$ has \mathbb{k} -codimension $2|J|$ in A_0 . Therefore,

$$(\iota_J A)_0 = h_J \mathbb{k}[z].$$

Identifying $(\iota_J A)_0$ with B_J via φ , we see that $b_J = \varphi_J(h_J)$ freely generates B_J as a $\mathbb{k}[z]$ -module. Now since $\iota_\emptyset = \text{Id}_{\text{gr-}A}$, multiplication $B_J \times B_\emptyset \rightarrow B_J$ sends (f, g) to $f \circ g$. Since $B_\emptyset = \mathbb{k}[z]$, the result follows.

2. This result follows from a proof identical to the proof of [Smi11, Lemma 10.2.(5)-(6)]. For convenience, we summarize it here. By the definition of multiplication in B ,

$$b_I b_J = \iota_{I \oplus J}(\sigma_{I \cap J}) \circ \iota_J(\varphi_I(h_I)) \circ \varphi_J(h_J).$$

Recalling that the involutions $\{\iota_I \mid I \in \mathbb{Z}_{\text{fin}}\}$ act on morphisms by restriction, we see that $b_I b_J : A \rightarrow \iota_{I \oplus J} A$ is given by left multiplication by $h_{I \cup J}$, and therefore

$$b_I b_J = \varphi_{I \oplus J}(h_{I \cup J}).$$

Now note that

$$\begin{aligned} b_{I \cap J}^2 b_{I \oplus J} &= b_{I \cap J}(b_{I \cap J} b_{I \oplus J}) = b_{I \cap J}(\varphi_{I \cup J}(h_{I \cup J})) = b_{I \cap J} b_{I \cup J} \\ &= \varphi_{(I \cap J) \oplus (I \cup J)}(h_{(I \cap J) \cup (I \cup J)}) \\ &= \varphi_{I \oplus J}(h_{I \cup J}) = b_I b_J. \end{aligned}$$

Induction on $|J|$ yields $b_J = \prod_{j \in J} b_j$.

3. For $a \in A$,

$$(b_n \cdot b_n)(a) = \sigma_n(h_n^2)(a) = (z + n)^{-2} h_n^2 a = h_n a.$$

Hence, b_n^2 is given by multiplication by $(z + n)^2$, that is,

$$b_n^2 = \varphi_\emptyset((z + n)^2).$$

4. Notice that

$$b_1^2 - b_0^2 = \varphi_\emptyset((z + 1)^2 - z^2) = \varphi_\emptyset(2z + 1)$$

so

$$\varphi_\emptyset(z) = \frac{1}{2} (b_1^2 - b_0^2 - \varphi_\emptyset(1)).$$

Hence, the b_n generate B_\emptyset as a \mathbb{k} -algebra, and combined with parts 1 and 2, the b_n generate B as a \mathbb{k} -algebra. By part 2, $b_n b_m = b_m b_n$ for all $n, m \in \mathbb{Z}$, and the result follows.

□

Proposition 6.3.4. *The \mathbb{Z}_{fin} -graded ring B has presentation*

$$B \cong \frac{\mathbb{k}[z][b_n \mid n \in \mathbb{Z}]}{(b_n^2 = (z+n)^2 \mid n \in \mathbb{Z})}$$

where $\deg z = \emptyset$ and $\deg b_n = n$.

Proof. By Lemma 6.3.3, the elements $\{b_n \mid n \in \mathbb{Z}\}$ generate B as a \mathbb{k} -algebra and satisfy the relations $b_n^2 = (z+n)^2$ for all $n \in \mathbb{Z}$. Hence, we need only show that the ideal generated by these relations contains all relations in B .

Let $r = 0$ be a relation in B . Since B is graded, we may assume that r is homogeneous of degree I . By Lemma 6.3.3, we can write

$$r = b_I \beta = 0$$

where β is a $\mathbb{k}[z]$ -linear combination of products of b_j^2 's for some integers j . By using the relations $b_j^2 = (z+j)^2$ for each j , we can rewrite β in B as a polynomial $g(z) \in \mathbb{k}[z]$. Hence

$$r = b_I g(z) = 0$$

but since B_I is freely generated as a right $B_0 = \mathbb{k}[z]$ -module by b_I , this implies that $g(z) = 0$, so the relation r was already in the ideal $(b_n^2 = (z+n)^2 \mid n \in \mathbb{Z})$, completing our proof. □

We use this presentation to prove some basic results about B . While the construction of B was analogous to that of Smith's ring C in [Smi11], the two rings are different enough to warrant closer examination. Smith proves that C is an ascending union of Dedekind domains. In contrast, since $b_n^2 - (z+n)^2 = (b_n + z+n)(b_n - z-n)$, B is not even a domain.

Lemma 6.3.5. *The minimal prime ideals of B are of the form*

$$(b_n + (-1)^{\epsilon_n}(z+n) \mid n \in \mathbb{Z})$$

for some choice of $\epsilon_n \in \{0, 1\}$ for each $n \in \mathbb{Z}$.

Proof. We work in the polynomial ring $S = \mathbb{k}[z][b_n \mid n \in \mathbb{Z}]$. The prime ideals of B correspond to prime ideals of S containing $(b_n^2 = (z+n)^2 \mid n \in \mathbb{Z})$. For each $n \in \mathbb{Z}$, choose $\epsilon_n \in \{0, 1\}$. Viewing S as a polynomial ring with coefficients in $\mathbb{k}[z]$, we see that

$$\mathfrak{p} = (b_n + (-1)^{\epsilon_n}(z+n) \mid n \in \mathbb{Z})$$

is the kernel of the map evaluating a polynomial $g(b_n \mid n \in \mathbb{Z})$ at the point $((-1)^{\epsilon_n}(z+n) \mid n \in \mathbb{Z})$ and so \mathfrak{p} is a prime ideal of S . To see that \mathfrak{p} corresponds to a minimal prime, we observe that for every $n \in \mathbb{Z}$, a prime ideal containing $b_n^2 - (z+n)^2$ must contain either $b_n + (z+n)$ or $b_n - (z+n)$. Hence \mathfrak{p} corresponds to a minimal prime. \square

For $J \in \mathbb{Z}_{\text{fin}}$ we write R_J for the \mathbb{k} -subalgebra of B generated by the elements $\{1, z\} \cup \{b_n \mid n \in J\}$. By the same argument as in Proposition 6.3.4, R_J has the presentation

$$R_J \cong \frac{\mathbb{k}[z][b_n \mid n \in J]}{(b_n^2 = (z+n)^2 \mid n \in J)}.$$

We will use the fact that B is an ascending union of the subrings R_J for any ascending, exhaustive chain of subsets $J \in \mathbb{Z}_{\text{fin}}$.

Proposition 6.3.6. *B is a non-noetherian, reduced ring of Krull dimension 1.*

Proof. We showed in Lemma 6.3.5 that B has infinitely many minimal prime ideals, so B is not noetherian. Further, the quotient of B by a minimal prime is isomorphic to $\mathbb{k}[z]$, and hence B has Krull dimension 1. We will show that the intersection of all minimal primes is (0) , so the nilradical $\mathfrak{n}(B) = (0)$. Let a be an element in the intersection of all minimal primes. We can write a as a sum of finitely many homogeneous terms, so a is an element of the subring R_J for some $J \in \mathbb{Z}_{\text{fin}}$. Suppose $j \in J$. Since

$$\begin{aligned} a &\in (b_j + (z+j)) + (b_n - (z+n) \mid n \in J \setminus \{j\}) \text{ and} \\ a &\in (b_j - (z+j)) + (b_n - (z+n) \mid n \in J \setminus \{j\}), \end{aligned}$$

we can write

$$a = (b_j + (z+j))r + s = (b_j - (z+j))r' + s', \tag{6.5}$$

for some $r, r' \in R_J$ and some $s, s' \in (b_n - (z + n) \mid n \in J)$. Setting $b_n = (z + n)$ for all $n \in J$, the right hand side of (6.5) is identically 0 and hence

$$r \in (b_n - (z + n) \mid n \in J)$$

so

$$a = (b_j + (z + j))r + s \in (b_j^2 - (z + j)^2) + (b_n - (z + n) \mid n \in J \setminus \{j\}).$$

Since $(b_j^2 - (z + j)^2) = (0)$ in B , therefore

$$a \in (b_n - (z + n) \mid n \in J \setminus \{j\}).$$

Inducting on the size of J , we conclude that $a = 0$ and since a was an arbitrary element in the intersection of all primes, we conclude that $\mathfrak{n}(B) = (0)$. \square

6.4 Congruent roots and the quotient category $\mathbf{qgr}\text{-}A$

Let $\alpha \in \mathbb{N}^+$ so that we are in the congruent root case. The fact that the set $\{\iota_J A \mid J \in \mathbb{Z}_{\text{fin}}\}$ does not generate $\text{gr}\text{-}A$ is a significant difference from the other cases. In particular, we are unable to use Proposition 6.2.2. One of the main obstructions is that there are infinitely many orbits of rank one graded projectives under the action of numerically trivial autoequivalences since numerically trivial autoequivalences fix the finite-dimensional simple modules, $Z\langle i \rangle$.

To deal with this obstruction, we will take the quotient category of $\text{gr}\text{-}A$ modulo its full subcategory of finite-dimensional modules. This is the same construction that Artin and Zhang [AZ94] use in their definition of the noncommutative projective scheme associated to an \mathbb{N} -graded ring R . However, as A is \mathbb{Z} -graded, the details are somewhat different. We will investigate this quotient category fairly explicitly.

Let $\text{fdim}\text{-}A$ denote the full subcategory of $\text{gr}\text{-}A$ consisting of all finite-dimensional modules. The only finite-dimensional simple modules are the shifts of Z so each object in $\text{fdim}\text{-}A$ has a composition series consisting entirely of shifts of

Z . By Lemma 3.3.3, therefore every object in $\text{fdim-}A$ is a direct sum of shifts of Z . Since $\text{fdim-}A$ is a Serre subcategory, we can define

$$\text{qgr-}A = \text{gr-}A / \text{fdim-}A.$$

The next lemma allows us to write down a Hom set in $\text{qgr-}A$ in a very concrete way.

- Lemma 6.4.1.** *1. For every finitely generated graded right A -module M , there exists a unique smallest submodule $\kappa(M) \subseteq M$ such that $M/\kappa(M) \in \text{fdim-}A$.*
- 2. For every finitely generated graded right A -module N , there exists a unique largest submodule $\tau(N) \subseteq N$ such that $\tau(N) \in \text{fdim-}A$.*

Proof. 1. Since M is finitely generated, $\underline{\text{Hom}}_A(M, Z)$ is finite-dimensional over \mathbb{k} and is nonzero in only finitely many degrees, say d_1, \dots, d_n . Letting $\mathcal{I} = \{Z\langle d_1 \rangle, \dots, Z\langle d_n \rangle\}$ and \mathbf{D} be the full subcategory of $\text{gr-}A$ consisting of all finite direct sums of elements from \mathcal{I} , by Proposition 4.0.5, there exists a unique smallest submodule M' such that $M/M' \in \mathbf{D}$. Let $\kappa(M) = M'$. Note that in fact $\kappa(M)$ is the unique smallest submodule such that $M/\kappa(M) \in \text{fdim-}A$ because $M/\kappa(M) \in \text{fdim-}A$ and any factor of M in $\text{fdim-}A$ is also in \mathbf{D} .

2. Since N is finitely generated and A is noetherian, N is noetherian. Since the sum of two objects in $\text{fdim-}A$ is again an object in $\text{fdim-}A$, there exists a unique largest submodule of N that is a direct sum of shifts of Z . Call this largest submodule $\tau(N)$. □

The preceding lemma allows us to describe $\text{Hom}_{\text{qgr-}A}(M, N)$ without any reference to a direct limit. In particular,

$$\text{Hom}_{\text{qgr-}A}(\pi M, \pi N) = \text{Hom}_{\text{gr-}A}(\kappa(M), N/\tau(N)).$$

Further, for a module M , Proposition 4.0.5 not only gives the existence of $\kappa(M)$, it also gives the construction. Indeed, we can construct $\kappa(M)$ by taking the intersection of all kernels of maps $M \rightarrow Z\langle i \rangle$.

Proposition 6.4.2. *Let P be a projective graded right A -module. Then πP is projective in $\text{qgr-}A$ if and only if $\kappa(P) = P$.*

Proof. Suppose P is projective in $\text{gr-}A$ and $\kappa(P) = P$. Let M and N be graded A -modules and let $g \in \text{Hom}_{\text{qgr-}A}(\pi M, \pi N)$ be an epimorphism. We show that for any morphism $h : \pi P \rightarrow \pi N$, there exists a lift $j : \pi P \rightarrow \pi M$ so that πP is projective.

To show projectivity of πP , we are only concerned with these morphisms in $\text{qgr-}A$, so we may replace M by $\kappa(M)$ since $\pi M = \pi\kappa(M)$ and similarly we may replace N by $N/\tau(N)$. Then g is represented by a morphism $\bar{g} \in \underline{\text{Hom}}_A(M, N)$ such that $\text{coker}(\bar{g}) \in \text{fdim-}A$. But since $\text{coker}(\bar{g}) \in \text{fdim-}A$, $\pi N = \pi \text{im } \bar{g}$, so we may replace N with $\text{im } \bar{g}$ and assume that \bar{g} is surjective.

Now note that since $\kappa(P) = P$, we have $\text{Hom}_{\text{qgr-}A}(\pi P, \pi N) = \underline{\text{Hom}}_A(P, N)$. So h is represented by a morphism $\bar{h} : P \rightarrow N$, which by the projectivity of P in $\text{gr-}A$ lifts to a morphism $\bar{j} : P \rightarrow M$. Let $j = \pi(\bar{j})$.

Conversely, suppose that $\kappa(P) \neq P$. Then $\kappa(P)$ fits into the exact sequence

$$0 \longrightarrow \kappa(P) \longrightarrow P \longrightarrow \bigoplus_{i \in I} Z\langle i \rangle \longrightarrow 0$$

for some finite set of integers I . Note that by Lemma 3.4.3 this exact sequence shows that $\kappa(P)$ is not projective. By Lemma 3.4.2, $\kappa(P)$ has projective dimension at most 1, and since $\kappa(P)$ is not projective, it has a projective resolution in $\text{gr-}A$ of length 1:

$$0 \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} \kappa(P) \rightarrow 0. \quad (6.6)$$

Suppose for contradiction that $\pi P = \pi\kappa(P)$ is projective in $\text{qgr-}A$. Since π is an exact functor, we have the following exact sequence in $\text{qgr-}A$:

$$0 \longrightarrow \pi P_1 \xrightarrow{\pi d_1} \pi P_0 \xrightarrow{\pi d_0} \pi\kappa(P) \rightarrow 0.$$

Since we assumed $\pi\kappa(P)$ is projective, there exists a splitting $h : \pi\kappa(P) \rightarrow \pi P_0$ such that $\pi d_0 \circ h = \text{Id}_{\pi\kappa(P)}$. The splitting h is represented by a morphism $\bar{h} \in \underline{\text{Hom}}_A(\kappa(P), P_0)$, since P_0 has no finite-dimensional submodules. Now \bar{h} gives a splitting of (6.6), since $\pi(\bar{h} \circ d_0) = \text{Id}_{\pi\kappa(P)}$ and since $\kappa(P)$ has no finite-dimensional

submodules, we know that $\text{Hom}_{\text{qgr-}A}(\pi\kappa(P), \pi\kappa(P)) = \underline{\text{Hom}}_A(\kappa(P), \kappa(P))$. Hence, $\bar{h} \circ d_0 = \text{Id}_{\kappa(P)}$. But a splitting of (6.6) shows that $\kappa(P)$ is projective, which is a contradiction. \square

Corollary 6.4.3. *If P is a rank one projective in $\text{gr-}A$ then $\pi(P)$ is projective in $\text{qgr-}A$ if and only if for each $n \in \mathbb{Z}$, P surjects onto exactly one of $X\langle n \rangle$ or $Y\langle n \rangle$.*

Proof. This follows immediately from Proposition 6.4.2 and Corollary 3.4.6. \square

Note that in particular, the previous corollary says that A is not projective in $\text{qgr-}A$.

Proposition 6.4.4. *Let $\mathcal{P} = \{P_i\}_{i \in I}$ be a set of projective modules in $\text{gr-}A$. If \mathcal{P} generates every shift of X and Y then $\pi\mathcal{P} = \{\pi P_i\}_{i \in I}$ generates $\text{qgr-}A$.*

Proof. Suppose \mathcal{P} is a set of projective modules in $\text{gr-}A$ which generates every shift of X and Y . Since the shifts of A generate $\text{gr-}A$, likewise the shifts of πA generate $\text{qgr-}A$. Since $\pi A = \pi\kappa(A)$, we will show that \mathcal{P} generates every shift of $\kappa(A)$ and hence $\pi\mathcal{P}$ generates every shift of $\pi\kappa(A)$.

Let $P \in \mathcal{P}$ and choose a maximal embedding $\varphi : P \rightarrow \kappa(A)\langle n \rangle$. It suffices to construct a surjection $\psi : \bigoplus_{j \in J} P_j \rightarrow \kappa(A)/P$ for some $J \subseteq I$. This is because, by the projectivity of the P_j , there exists a lift $\bar{\psi} : \bigoplus_{j \in J} P_j \rightarrow \kappa(A)\langle n \rangle$ and because $\text{im } \varphi + \text{im } \bar{\psi} = \kappa(A)\langle n \rangle$.

Since A has Krull dimension 1, the quotient $\kappa(A)\langle n \rangle/P$ has finite length. Further, since $\underline{\text{Hom}}_A(\kappa(A), Z) = 0$, so $\kappa(A)\langle n \rangle/P$ is a direct sum of indecomposables, none of which has a factor of Z . It thus suffices to show that \mathcal{P} generates every such indecomposable.

Without loss of generality, suppose $\kappa(A)\langle n \rangle/P$ is an indecomposable with a factor of $X\langle i \rangle$. We induct on the length of the indecomposable. By hypothesis, some $P_0 \in \mathcal{P}$ surjects onto $X\langle i \rangle$. By the projectivity of P_0 , this surjection then lifts to a map $g : P_0 \rightarrow \kappa(A)\langle n \rangle/P$. If this is a surjection, then we are done. Otherwise, P_0 surjects onto a proper submodule of $\kappa(A)\langle n \rangle/P$. Again, it suffices to give a surjection onto the cokernel of g . But now note that since $\kappa(A)\langle n \rangle$ surjects onto $(\kappa(A)\langle n \rangle/P)/\text{im}(g)$, then $(\kappa(A)\langle n \rangle/P)/\text{im}(g)$ again has no factor of Z and

has shorter length. By induction, \mathcal{P} generates $(\kappa(A)\langle n \rangle / P) / \text{im}(g)$ and thus \mathcal{P} generates $\kappa(A)\langle n \rangle$. \square

Theorem 6.4.5. *Let $\alpha \in \mathbb{N}^+$. There is an equivalence of categories*

$$\text{gr-}A(z^2) \cong \text{qgr-}A(z(z + \alpha)).$$

Proof. Let $f = z^2$ and $g = z(z + \alpha)$. Let \mathbf{P} be the full subcategory of $\text{gr-}A(f)$ consisting of direct sums of the canonical rank one projective A -modules (as described in Lemma 3.4.13). Let \mathbf{P}' be the full subcategory of $\text{qgr-}A(g)$ consisting of the images in $\text{qgr-}A(g)$ of direct sums of the canonical rank one projectives of $\text{gr-}A(g)$ that remain projective in $\text{qgr-}A(g)$. We will define an equivalence of categories $\mathcal{G} : \mathbf{P} \rightarrow \mathbf{P}'$. We will then use Lemma 4.0.2 to extend this to an equivalence $\text{gr-}A(f) \cong \text{qgr-}A(g)$.

First, we define \mathcal{G} on objects. Let P be a canonical rank one projective $A(f)$ -module. By Corollary 3.4.6, for each $n \in \mathbb{Z}$, P surjects onto exactly one of $X^f\langle n \rangle$ and $Y^f\langle n \rangle$. Define P' to be the canonical projective object of $\text{gr-}A(g)$ with simple factors corresponding to those of P , that is, for all $n \in \mathbb{Z}$, if $F_n(P) = X^f\langle n \rangle$ then $F_n(P') = X^g\langle n \rangle$ and if $F_n(P) = Y^f\langle n \rangle$ then $F_n(P') = Y^g\langle n \rangle$. Such a projective P' exists by Lemma 3.4.9. By Corollary 6.4.3, P' is a projective object of $\text{qgr-}A(g)$. Now define $\mathcal{G}(P) = \pi(P')$. By abuse of notation, we will also refer to the object of $\text{gr-}A(g)$ as $\mathcal{G}(P)$. For a direct sum of canonical rank one projectives, $P = \bigoplus_{i \in I} P_i$, define $\mathcal{G}(P) := \bigoplus_{i \in I} \mathcal{G}(P_i)$.

Suppose now that $P = \bigoplus_{i \in \mathbb{Z}} (p_i)x^i$ and $Q = \bigoplus_{i \in \mathbb{Z}} (q_i)x^i$ are canonical rank one projectives of $\text{gr-}A(f)$ with structure constants $\{c_i\}$ and $\{d_i\}$, respectively. By Lemma 3.4.13, $\text{Hom}_{\text{gr-}A(f)}(P, Q)$ is generated as a $\mathbb{k}[z]$ -module by left multiplication by $\theta_{P,Q} = \text{lcm}_{i \in \mathbb{Z}}(q_i / \text{gcd}(p_i, q_i))$. Since $\mathcal{G}(P)$ and $\mathcal{G}(Q)$ are the images under π of projectives P' and Q' in $\text{gr-}A(g)$, they have no finite-dimensional submodules, $\kappa(P') = P'$, and $\kappa(Q') = Q'$. Hence, $\text{Hom}_{\text{qgr-}A(g)}(\mathcal{G}(P), \mathcal{G}(Q)) \cong \text{Hom}_{\text{gr-}A(g)}(P', Q')$ is also generated as a $\mathbb{k}[z]$ -module by some maximal embedding. We will show that in fact this maximal embedding is given by multiplication by the same $\theta_{P,Q}$.

By Lemma 3.2.5 and Corollary 3.4.5, for every $i \in \mathbb{Z}$, $c_i = 1$ if and only if $F_i(P) = X^f\langle i \rangle$, and $c_i = (z + i)^2$ if and only if $F_i(P) = Y^f\langle i \rangle$. The same is true for

the structure constants $\{d_i\}$ of Q . Now let $I = \{i_1, \dots, i_r\} \subset \mathbb{Z}$ be precisely those indices i_j such that $F_{i_j}(P) = X^f \langle i_j \rangle$ and $F_{i_j}(Q) = Y^f \langle i_j \rangle$. These are the indices i_j such that $c_{i_j} = 1$ and $d_{i_j} = (z + i_j)^2$. Observe that for distinct integers n and m then c_n and d_n are relatively prime to c_m and d_m so that

$$\theta_{P,Q} = \text{lcm}_{i \in \mathbb{Z}} \frac{q_i}{\text{gcd}(p_i, q_i)} = \prod_{i_j \in I} (z + i_j)^2.$$

Let $\{c'_i\}$ and $\{d'_i\}$ be the structure constants for $P' = \mathcal{G}(P) = \bigoplus_{i \in \mathbb{Z}} (p'_i)x^i$ and $Q' = \mathcal{G}(Q) = \bigoplus_{i \in \mathbb{Z}} (q'_i)x^i$. As in Corollary 3.4.8, for each $n \in \mathbb{Z}$, we can calculate c'_n from the simple factors of P' . Specifically c'_n depends only on $F_n(P')$ and $F_{n+\alpha}(P')$. For any integer n , the polynomial $z + n$ is only possibly a factor of c_n or $c_{n-\alpha}$ and these structure constants depend only on the simple factors at n , $n + \alpha$ and $n - \alpha$. We use Table 3.4.1 from Corollary 3.4.8 and the fact that no shift of Z is a factor of P' to construct the following table:

Table 6.4.1: The structure constants of P' in terms of its simple factors.

	$F_n(P') = X^g \langle n \rangle$	$F_n(P') = Y^g \langle n \rangle$
$F_{n+\alpha}(P') = X^g \langle n + \alpha \rangle$	$c'_n = 1$	$c'_n = \sigma^n(z)$
$F_{n+\alpha}(P') = Y^g \langle n + \alpha \rangle$	$c'_n = \sigma^n(z + \alpha)$	$c'_n = \sigma^n(f)$
$F_{n-\alpha}(P') = X^g \langle n - \alpha \rangle$	$c'_{n-\alpha} = 1$	$c'_{n-\alpha} = \sigma^{n-\alpha}(z + \alpha)$
$F_{n-\alpha}(P') = Y^g \langle n - \alpha \rangle$	$c'_{n-\alpha} = \sigma^{n-\alpha}(z)$	$c'_{n-\alpha} = \sigma^{n-\alpha}(f)$

Observe from the table that either

- (i) $F_n(P') = Y^g \langle n \rangle$ in which case $z + n$ is a factor of both c'_n and $c'_{n-\alpha}$, or else
- (ii) $F_n(P') = X^g \langle n \rangle$ in which case $z + n$ is not a factor of either c'_n or $c'_{n-\alpha}$.

Now when we calculate the maximal embedding $P' \rightarrow Q'$

$$\theta_{P',Q'} = \text{lcm}_{i \in \mathbb{Z}} \frac{q'_i}{\text{gcd}(p'_i, q'_i)}$$

we see that for $i_j \in I$, $q_{i_j-\alpha}$ has a factor of $(z + i_j)^2$ while $p_{i_j-\alpha}$ has no factor of $(z + i_j)$. For all other integers n , the factor $(z + n)$ appears either only as a factor of p_n and $p_{n-\alpha}$ or else as a factor of p_n , $p_{n-\alpha}$, q_n , and $q_{n-\alpha}$. Hence,

$$\theta_{P',Q'} = \prod_{i_j \in I} (z + i_j)^2 = \theta_{P,Q}.$$

So the maximal embeddings $P \rightarrow Q$ and $\mathcal{G}(P) \rightarrow \mathcal{G}(Q)$ are both given by multiplication by the same element $\theta_{P,Q}$. If $f \in \text{Hom}_{\text{gr-}A(f)}(P, Q)$ is given by multiplication by $\theta_{P,Q}\beta$ then define $\mathcal{G}(f) \in \text{Hom}_{\text{qgr-}A(g)}(\mathcal{G}(P), \mathcal{G}(Q))$ to be multiplication by $\theta_{P,Q}\beta$ also. By our definition of \mathcal{G} on morphisms, it is clear that $\mathcal{G}(\text{Id})$ is the identity and \mathcal{G} respects composition.

Altogether, we have defined \mathcal{G} on the canonical rank one projectives of $\text{gr-}A(f)$. By Lemma 4.0.3, we can extend \mathcal{G} to a functor $\mathcal{G} : \mathbf{P} \rightarrow \mathbf{P}'$. It is easily seen that \mathcal{G} is an equivalence on these subcategories. To see that \mathcal{G} is full and faithful, notice that for canonical rank one projectives P and Q of $\text{gr-}A(f)$ our construction of \mathcal{G} gave an isomorphism

$$\text{Hom}_{\text{gr-}A}(P, Q) \cong \text{Hom}_{\text{qgr-}A(f)}(\mathcal{G}(P), \mathcal{G}(Q)).$$

Now given direct sums of canonical rank one projectives $\bigoplus_{i \in I} P_i$ and $\bigoplus_{j \in J} Q_j$, the construction in Lemma 4.0.3 gives an isomorphism

$$\begin{aligned} \text{Hom}_{\text{gr-}A} \left(\bigoplus_{i \in I} P_i, \bigoplus_{j \in J} Q_j \right) &\cong \bigoplus_{i \in I} \bigoplus_{j \in J} \text{Hom}_{\text{gr-}A}(P_i, Q_j) \\ &\cong \bigoplus_{i \in I} \bigoplus_{j \in J} \text{Hom}_{\text{qgr-}A(f)}(\mathcal{G}(P_i), \mathcal{G}(Q_j)) \cong \text{Hom}_{\text{qgr-}A(f)} \left(\mathcal{G} \bigoplus_{i \in I} P_i, \mathcal{G} \bigoplus_{j \in J} Q_j \right). \end{aligned}$$

To see that \mathcal{G} is essentially surjective, notice that given P' in \mathbf{P}' , we can construct a module P of $\text{gr-}A(f)$ such that $\mathcal{G}(P) \cong P'$ by constructing a direct sum of canonical rank one projectives in $\text{gr-}A(f)$ with corresponding simple factors.

By Proposition 3.4.16, every object of $\text{gr-}A(f)$ has a projective resolution by objects of \mathbf{P} . Similarly, by Proposition 6.4.4, every object of $\text{qgr-}A(g)$ has a projective resolution by objects of \mathbf{P}' . Hence, by Corollary 4.0.2, there is an equivalence $\text{gr-}A(f) \cong \text{qgr-}A(g)$. \square

Corollary 6.4.6. *Let $\alpha \in \mathbb{N}^+$. Then there is an equivalence of categories*

$$\text{qgr-}A(z(z + \alpha)) \cong \text{gr}(B, \mathbb{Z}_{\text{fin}}).$$

Proof. This follows immediately from Theorem 6.4.5 and Theorem 6.3.2. \square

6.5 Non-congruent roots

Let $\alpha \in \mathbb{k} \setminus \mathbb{Z}$, so we are in the distinct, non-congruent root case. Recall that this case bore similarities to the case of the first Weyl algebra, A_1 . We saw that the category $\text{gr-}A$ looked like a “doubled” version of $\text{gr-}A_1$. In particular, we indexed our autoequivalences $\iota_{(J,J')}$ by $\mathbb{Z}_{\text{fin}} \times \mathbb{Z}_{\text{fin}}$, and many of the properties of these autoequivalences were the same as Sierra’s ι_J , with indices doubled. For notational convenience, let $\Gamma = \mathbb{Z}_{\text{fin}} \times \mathbb{Z}_{\text{fin}}$.

This case also bears resemblance to the multiple root case. By Lemma 5.2.5, we know that the set $\{\iota_\gamma A \mid \gamma \in \Gamma\}$ generates $\text{gr-}A$. We will show that the isomorphisms between these autoequivalences satisfy condition 6.3, so we can use Proposition 6.2.1 to define a ring C with an equivalent graded module category. We will then show that C is commutative, and give a presentation for C .

For each $(J, J') \in \Gamma$, define the polynomial

$$h_{(J,J')} = \prod_{j \in J} (z + j) \prod_{j' \in J'} (z + j' + \alpha). \quad (6.7)$$

For a projective module P we showed $\iota_{(n,\emptyset)}^2 P = (z + n)P$ and $\iota_{(\emptyset,n)}^2 P = (z + n + \alpha)P$ and so $\iota_{(n,\emptyset)}^2 \cong \text{Id}_{\text{gr-}A} \cong \iota_{(\emptyset,n)}^2$. We denote by $\sigma_{(n,\emptyset)}$ the isomorphism $\iota_{(n,\emptyset)}^2 A \rightarrow A$ and by $\sigma_{(\emptyset,n)}$ the isomorphism $\iota_{(\emptyset,n)}^2 A \rightarrow A$. Since A is projective, $\sigma_{(n,\emptyset)}$ is given by left multiplication by $h_{(n,\emptyset)}^{-1}$ and $\sigma_{(\emptyset,n)}$ is given by left multiplication by $h_{(\emptyset,n)}^{-1}$. Similarly, for $(J, J') \in \Gamma$, we define

$$\sigma_{(J,J')} = \prod_{j \in J} \sigma_{(j,\emptyset)} \prod_{j' \in J'} \sigma_{(\emptyset,j')} : \iota_{(J,J')}^2 A \rightarrow A$$

and note that $\sigma_{(J,J')}$ is also given by left multiplication by $h_{(J,J')}^{-1}$.

Now, for $(I, I'), (J, J') \in \Gamma$, we define

$$\Theta_{(I,I'),(J,J')} = \iota_{(I \oplus J, I' \oplus J')}(\sigma_{(I \cap J, I' \cap J')}) = \Theta_{(J,J'),(I,I')}$$

$$\Theta_{(I,I'),(J,J')} : \iota_{(I,I')}\iota_{(J,J')}A = \iota_{(I \oplus J, I' \oplus J')}\iota_{(I \cap J, I' \cap J')}^2 A \rightarrow \iota_{(I \oplus J, I' \oplus J')}A.$$

Lemma 6.5.1. *The isomorphisms $\{\Theta_{\gamma,\delta} \mid \gamma, \delta \in \Gamma\}$ and the autoequivalences $\{\iota_\gamma \mid \gamma \in \Gamma\}$ satisfy condition (6.3).*

Proof. This result follows from the same proof as Lemma 6.3.1 with doubled indices. \square

Having checked that our autoequivalences satisfy condition (6.3), we use Proposition 6.2.1, to define the Γ -graded ring

$$C = \bigoplus_{\gamma \in \Gamma} C_\gamma = \bigoplus_{\gamma \in \Gamma} \text{Hom}_{\text{gr-}A}(A, \iota_\gamma A) = \bigoplus_{(J, J') \in \mathbb{Z}_{\text{fin}} \times \mathbb{Z}_{\text{fin}}} \text{Hom}_{\text{gr-}A}(A, \iota_{(J, J')} A).$$

Theorem 6.5.2. *There is an equivalence of categories*

$$\text{gr-}A \cong \text{gr-}(C, \Gamma).$$

Proof. This is an immediate corollary of Propositions 6.2.1 and 6.2.2 together with Lemma 6.5.1. \square

Finally, we describe properties of and give a presentation for the ring C . For $\gamma \in \Gamma$ let φ_γ be the map

$$\varphi_\gamma : (\iota_\gamma A)_0 \rightarrow \text{Hom}_{\text{gr-}A}(A, \iota_\gamma A), \quad \varphi_\gamma(m)(a) = m \cdot a.$$

We use the $\mathbb{k}[z]$ -module isomorphism φ_γ to identify C_γ with $(\iota_\gamma A)_0$. For $\gamma = (J, J') \in \Gamma$ we also define

$$\begin{aligned} c_J &= \varphi_{(J, \emptyset)}(h_{(J, \emptyset)}) \in C_{(J, \emptyset)} \text{ and} \\ d_{J'} &= \varphi_{(\emptyset, J')} (h_{(\emptyset, J')}) \in C_{(\emptyset, J')}. \end{aligned}$$

For completeness, define $c_\emptyset = d_\emptyset = 1$.

Lemma 6.5.3. *Let $\alpha \in \mathbb{k} \setminus \mathbb{Z}$ and let $\gamma = (I, I') \in \Gamma$.*

1. $(\iota_\gamma A)_0 = h_\gamma \mathbb{k}[z]$.
2. $c_I d_{I'} = d_{I'} c_I$.
3. *The element $c_I d_{I'}$ freely generates C_γ as a $C_{(\emptyset, \emptyset)}$ -module.*
4. $c_I c_J = c_{I \cap J}^2 c_{I \oplus J}$ and $d_I d_J = d_{I \cap J}^2 d_{I \oplus J}$ and so $c_I d_{I'} = \prod_{i \in I} c_i \prod_{i' \in I'} d_{i'}$.

5. For all $n, m \in \mathbb{Z}$

$$c_n^2 - n = c_m^2 - m$$

$$d_n^2 - n = d_m^2 - m$$

$$c_n^2 = d_n^2 - \alpha.$$

6. C is a commutative \mathbb{k} -algebra generated by $\{c_n, d_n \mid n \in \mathbb{Z}\}$.

Proof. The arguments in this proof are similar to those found in Lemma 6.3.3 and therefore [Smi11, Lemma 10.2].

1. For each $n \in \mathbb{Z}$, we constructed $\iota_{(n,\emptyset)}A$ by taking the kernel of the nonzero morphism to $X_0\langle n \rangle$ or $Y_0\langle n \rangle$. Since

$$(z+n)A_0 = (\iota_{(n,\emptyset)}^2 A)_0 \subseteq (\iota_{(n,\emptyset)} A)_0$$

and $(\iota_{(n,\emptyset)} A)_0$ has \mathbb{k} -codimension 1 in A_0 , therefore

$$(\iota_{(n,\emptyset)} A)_0 = (z+n)A_0 = (z+n)\mathbb{k}[z].$$

By an analogous argument, $(\iota_{(\emptyset,n)} A)_0 = (z+n+\alpha)\mathbb{k}[z]$. And since the autoequivalences commute

$$(\iota_\gamma A)_0 \subseteq (\iota_{(i,\emptyset)} A)_0 = (z+i)\mathbb{k}[z] \text{ and } (\iota_\gamma A)_0 \subseteq (\iota_{(\emptyset,i')} A)_0 = (z+i+\alpha)\mathbb{k}[z]$$

for each $i \in I$ and $i' \in I'$. Since $(\iota_\gamma A)_0$ has \mathbb{k} -codimension $|I| + |I'|$ in A_0 . Therefore,

$$(\iota_\gamma A)_0 = h_\gamma \mathbb{k}[z].$$

2. This follows since the ι_γ act on morphisms by restriction. Both $c_J d_{J'}$ and $d_{J'} c_J$ are given by multiplication by $h_{(J,\emptyset)} h_{(\emptyset,J')} = h_{(\emptyset,J')} h_{(J,\emptyset)}$.

3. Since $\iota_{(\emptyset,\emptyset)} = \text{Id}_{\text{gr-}A}$, multiplication $C_\gamma \times C_{(\emptyset,\emptyset)} \rightarrow C_\gamma$ sends (g, h) to $g \circ h$. By part 1, $C_\gamma = \varphi(h_\gamma \mathbb{k}[z])$. Since $C_{(\emptyset,\emptyset)} = \mathbb{k}[z]$ it follows that C_γ is generated as a right $C_{(\emptyset,\emptyset)}$ -module by $\varphi_\gamma(h_\gamma)$, or left multiplication by h_γ . Now by the definition of multiplication in C ,

$$c_I d_{I'} = \iota_{(\emptyset,I')} (\varphi(h_{(I,\emptyset)})) \varphi(h_{(\emptyset,I')}).$$

and since $\iota_{(I,\emptyset)}$ acts on morphisms by restriction, $c_I d_{I'}$ is given by multiplication by $h_{(I,\emptyset)} h_{(\emptyset,I')} = h_\gamma$.

4. This has the same proof as part 2 of Lemma 6.3.3.

5. For $a \in A$,

$$(c_n.c_n)(a) = \sigma_{(n,\emptyset)}((z+n)^2)(a) = (z+n)^{-1}(z+n)^2a = (z+n)a.$$

Hence, c_n^2 is given by multiplication by $z+n$, that is,

$$c_n^2 = \varphi_{(\emptyset,\emptyset)}(z+n).$$

Similarly,

$$d_n^2 = \varphi_{(\emptyset,\emptyset)}(z+n+\alpha),$$

from which the claim follows.

6. This follows from parts 2, 3, and 4. □

Proposition 6.5.4. *The Γ -graded ring C has presentation*

$$C \cong \frac{\mathbb{k}[c_n, d_n \mid n \in \mathbb{Z}]}{(c_n^2 - n = c_m^2 - m, c_n^2 = d_n^2 - \alpha \mid m, n \in \mathbb{Z})}$$

where $\deg c_n = (n, \emptyset)$ and $\deg d_n = (\emptyset, n)$.

Proof. By Lemma 6.5.3, the elements $\{c_n\}$ and $\{d_n\}$ generate C as a \mathbb{k} -algebra and satisfy the relations $c_n^2 - n = c_m^2 - m$ and $c_n^2 = d_n^2 - \alpha$ for all $n, m \in \mathbb{Z}$. We need to show that the ideal generated by these relations contains all relations in C .

Let $r = 0$ be a relation in C . We may assume that r is homogeneous of degree (I, I') . By Lemma 6.5.3, we can write

$$r = c_I d_{I'} \beta = 0$$

where β is a $\mathbb{k}[z]$ -linear combination of products of c_j^2 's and d_j^2 's for some integers j . By using the relations $c_j^2 = (z+j)$ and $d_j^2 = c_j^2 + \alpha$ for each j , we can rewrite β in C as a polynomial $g(z) \in \mathbb{k}[z]$. Hence

$$r = c_I d_{I'} g(z) = 0$$

but since $B_{(I,I')}$ is freely generated as a right $B_{(\emptyset,\emptyset)} = \mathbb{k}[z]$ -module by $c_I d_{I'}$, this implies that $g(z) = 0$, so the relation r was already in the ideal

$$(c_n^2 - n = c_m^2 - m, c_n^2 = d_n^2 - \alpha \mid m, n \in \mathbb{Z}),$$

completing our proof. □

Chapter 7

Future questions

Noncommutative algebraic geometry has made much progress in understanding the geometry of connected \mathbb{N} -graded \mathbb{k} -algebras. These rings are natural to consider, as they are analogues of quotients of commutative polynomial rings. The work of Sierra ([Sie09]) and Smith ([Smi11]) on the first Weyl algebra suggested that there might be interesting geometry hiding within \mathbb{Z} -graded \mathbb{k} -algebras which are not necessarily connected. In this dissertation, we have generalized some of these results to certain generalized Weyl algebras. Like the Weyl algebra, the GWAs $A(f)$ studied here are \mathbb{Z} -graded domains of GK dimension 2 with $(A(f))_0 \cong \mathbb{k}[z]$.

These results are especially interesting in light of Artin and Stafford's classification of noncommutative projective curves in [AS95].

Theorem 7.0.1 (Artin-Stafford, [AS95]). *Let A be a connected \mathbb{N} -graded domain, generated in degree 1 with GK dimension 2. Then there exists a projective curve X such that $\text{qgr-}A \equiv \text{coh}(X)$.*

Might there be an analogue to Theorem 7.0.1 for \mathbb{Z} -graded domains of GK dimension 2? In this final chapter, we propose directions for future work in understanding noncommutative \mathbb{Z} -graded rings.

The rings B and C in Theorems 6.3.2 and 6.5.2 would be interesting objects for future study. One would like to know the properties of B and C and of the schemes $\text{Spec } B$ and $\text{Spec } C$. Their algebro-geometric properties are important for

the following reason. For a Γ -graded commutative ring R , there is an action of the affine algebraic group $\text{Spec } \mathbb{k}\Gamma$ on $\text{Spec } R$ corresponding to the grading on R . Therefore, we can form the quotient stack $[\text{Spec } R / \text{Spec } \mathbb{k}\Gamma]$.

Question 7.0.2. *What are the properties of the stacks $[\text{Spec } B / \text{Spec } \mathbb{k}\mathbb{Z}_{\text{fin}}]$ and $[\text{Spec } C / \text{Spec } \mathbb{k}(\mathbb{Z}_{\text{fin}} \times \mathbb{Z}_{\text{fin}})]$?*

Question 7.0.3. *For a \mathbb{Z} -graded domain R of GK dimension 2, under what conditions is there a stack χ such that $\text{qgr-}R \equiv \text{coh}(\chi)$?*

One generalization of the work in this dissertation might be to begin with the study of other GWAs of GK dimension 2: those defined by non-quadratic polynomials or different base rings. Many interesting rings are GWAs. The quantum Weyl algebra $A_q = \mathbb{k}\langle x, y \rangle / (xy - qyx - 1)$ and, more generally, ambiskew polynomial rings are GWAs (see [Jor00]). The universal enveloping algebra of $\mathfrak{sl}(2)$ and similar algebras studied by Smith in [Smi90] can also be constructed as GWAs.

One could also consider \mathbb{Z} -graded rings of higher dimension. In [BR16], Bell and Rogalski classified certain simple \mathbb{Z} -graded rings of arbitrary dimension. They proved that in GK dimension 2, all of these rings are graded Morita equivalent to GWAs. It would be interesting to study the higher-dimensional simple \mathbb{Z} -graded rings studied by Bell and Rogalski.

For all of these \mathbb{Z} -graded rings R , it would be interesting to study $\text{gr-}R$. It is particularly important to understand the Picard group $\text{Pic}(\text{gr-}R)$.

Question 7.0.4. *For the \mathbb{Z} -graded rings R above, what is $\text{Pic}(\text{gr-}R)$?*

Understanding $\text{Pic}(\text{gr-}R)$ is an important first step in constructing a commutative ring whose graded modules are equivalent to $\text{gr-}R$. Recall the construction used in section 6.2. For an abelian subgroup $\Gamma \subset \text{Pic}(\text{gr-}R)$, we choose autoequivalences $\{\mathcal{F}_\gamma \mid \gamma \in \Gamma\}$. If these autoequivalences satisfy condition (6.3), then we can define an associative Γ -graded ring

$$S = \bigoplus_{\gamma \in \Gamma} \text{Hom}_{\text{gr-}R}(R, \mathcal{F}_\gamma R) \quad (7.1)$$

The category $\text{gr-}(S, \Gamma)$ is equivalent to $\text{gr-}R$ if $\bigoplus_{\gamma \in \Gamma} \mathcal{F}_\gamma R$ is a projective generator of $\text{gr-}R$.

In some sense, we would like to find “small” subgroups Γ of $\text{Pic}(\text{gr-}R)$. If, for example, $\Gamma = \mathbb{Z}$ is the subgroup of $\text{Pic}(\text{gr-}R)$ generated by the shift functor, then we recover the ring $R \cong S$. Hence, if Γ contains \mathcal{S} , then S contains R as a subring and so S is noncommutative. However, Γ must be large enough that $\mathcal{F}_\gamma R$ generates $\text{gr-}R$.

Question 7.0.5. *For a noncommutative \mathbb{Z} -graded ring R , how can we choose subgroups $\Gamma \subseteq \text{Pic}(\text{gr-}R)$ and autoequivalences $\{\mathcal{F}_\gamma \mid \gamma \in \Gamma\}$ satisfying condition (6.3)? When does $\bigoplus_{\gamma \in \Gamma} \mathcal{F}_\gamma R$ generate $\text{gr-}R$? When is S commutative?*

We can also take an opposite view of (7.1). That is, given a stack χ , a quasicoherent sheaf \mathcal{O} on χ , and an autoequivalence \mathcal{S} of $\text{coh}(\chi)$ of infinite order, we can consider the \mathbb{Z} -graded ring

$$\bigoplus_{n \in \mathbb{Z}} \text{Hom}_{\text{Qcoh}(\chi)}(\mathcal{O}, \mathcal{S}^n \mathcal{O}). \quad (7.2)$$

Question 7.0.6. *Under what conditions on the stack χ and the sheaf \mathcal{O} does (7.2) give a \mathbb{Z} -graded domain of GK dimension 2?*

The study of the geometry of connected \mathbb{N} -graded rings has led to a deeper understanding of these rings. In exploring the questions posed in this chapter, we hope that a more complete theory of the geometry of \mathbb{Z} -graded rings is developed.

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