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Bounded analytic functions on the polydisc

A dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Mathematics

by

David Scheinker

Committee in charge:

Professor Jim Agler, Chair
Professor Peter Ebenfelt
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2011

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Chair

University of California, San Diego

2011

DEDICATION

To Anna and Vladimir.

EPIGRAPH

Scientists too, as J. Robert Oppenheimer once remarked, “live always at the ‘edge of mystery’ - the boundary of the unknown.” But they transform the unknown into the known, haul it in like fisherman; artists get you out into that dark sea.

—Rebecca Solnit

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“This is terrible, anyone that reads this will think you are an idiot!”

“And what will Obama do then, convene his generals!?!”

“Are you purposely trying to confuse the reader? Have you ever read Euclid? You should read Eucild!”

Your creativity and enthusiasm for mathematics have been inspirational.

Thanks also to my mom, Anna, and my brother, Alex, for your support and encouragement. I specify your names not to eliminate ambiguity as to which mother and brother I am thanking, but because the affection I feel for you spills over into my writing. I enjoy typing your names and seeing them typed.

Thanks, of course, to my friends. Birgitte in particular. If not for the wonderful time I spent doing not-math with you I would have surely lacked the resolve to drink the hundreds of gallons of coffee that eventually led to the completion of this dissertation. Speaking of the coffee, thanks to the Pannikin, the wonderful people that work there and especially Amanda and Renee.

Thanks finally to my father, Vladimir, for his loving encouragement and questioning. Arguing with you has helped me understand what I want, and even more importantly what I do not want. If at the time of this reading I have a job it is in no small part due to your timely admonishment,

“David, a dissertation without publications is like love without sex.”

VITA

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PUBLICATIONS

Scheinker, D., A uniqueness theorem for bounded analytic functions on the polydisc. In submission. Available at <http://arxiv.org/abs/1012.3412>.

Scheinker, D., Kressin, K. *Math Problems and Solutions*. 307 pages. K Squared Publishing, San Diego, CA 2009.

Scheinker, D., The Riemann Hypothesis and a spectral problem for fractal strings. University of Pennsylvania Masters Thesis, Philadelphia, 2005.

ABSTRACT OF THE DISSERTATION

Bounded analytic functions on the polydisc

by

David Scheinker

Doctor of Philosophy in Mathematics

University of California, San Diego, 2011

Professor Jim Agler, Chair

In the paper ‘Distinguished Varieties’ Agler and McCarthy proved several connections between the theory of bounded analytic functions on the bidisc and 1-dimensional algebraic varieties that exit the bidisc through the distinguished boundary. In this paper we extend several of their results to the theory of bounded analytic functions on the polydisc. We give sufficient conditions for a rational inner function on the polydisc to be uniquely determined in the Schur class of the polydisc by it’s values on a finite set of points. This follows from giving sufficient conditions for a Pick problem on the polydisc to have a unique solution. We demonstrate that our results can be though of as a generalization to the polydisc of the Schwarz Lemma and Pick’s Theorem on the disc. We establish our results by studying the Pick problem with Hilbert function space techniques.

Chapter 1

Introduction

1.1 Overview and statement of main results

Let \mathbb{D} denote the unit disc centered at the origin in the complex plane, \mathbb{C} , and let \mathbb{T} denote the boundary of \mathbb{D} . The theory of bounded analytic functions on \mathbb{D} is elegant and well understood. However, much about the theory on \mathbb{D}^n remains mysterious. In this work, we study the **Schur class** of \mathbb{D}^n , $\mathcal{S}(\mathbb{D}^n)$, the set of analytic functions mapping \mathbb{D}^n to $\overline{\mathbb{D}}$. We seek to extend various results about $\mathcal{S}(\mathbb{D})$ to $\mathcal{S}(\mathbb{D}^n)$.

The classic Schwarz Lemma on \mathbb{D} provides a way to introduce our results.

Schwarz Lemma. *If $f : \mathbb{D} \rightarrow \mathbb{D}$ is analytic and $f(0) = 0$, then $|f(\lambda)| \leq |\lambda|$ for every $\lambda \in \mathbb{D}$. Furthermore, if $f(\lambda) = \lambda$ for some $\lambda \in \mathbb{D} \setminus 0$, then $f(z) = z$.*

The uniqueness part of the Schwarz Lemma says that the function $f(z) = z$ is uniquely determined in $\mathcal{S}(\mathbb{D})$ by its values on two points, 0 and any other point $\lambda \in \mathbb{D}$. That is, if $g \in \mathcal{S}(\mathbb{D})$ satisfies $g(0) = f(0)$ and $g(\lambda) = f(\lambda)$, then $g = f$ on \mathbb{D} . Our first result, Theorem **A** below, generalizes the Schwarz Lemma to $\mathcal{S}(\mathbb{D}^n)$. Theorem **A** states that each coordinate function $f(z_1, \dots, z_n) = z_k$ is uniquely determined in $\mathcal{S}(\mathbb{D}^n)$ by its values on two “well chosen” points. We use the notation (z, w) for a point in \mathbb{C}^n with $z = z_1$ and $w = (z_2, \dots, z_n)$. For the sake of simplicity, we state the theorem for $f(z, w) = z$ rather than for the general case of $f(z_1, \dots, z_n) = z_k$.

Theorem A. *If $f : \mathbb{D}^n \rightarrow \mathbb{D}$ is analytic and $f(0, w_0) = 0$ for some $w_0 \in \mathbb{D}^{n-1}$, then $|f(\lambda, w_0)| \leq |\lambda|$ for every $\lambda \in \mathbb{D}$. Furthermore, if $f(\lambda, w_0) = \lambda$ for some $\lambda \in \mathbb{D} \setminus 0$, then $f(z, w) = z$.*

Theorem A relates the degree of $f(z, w) = z$, $\deg(f) = 1$, to the number of points, $N = 2$, needed to uniquely determine f in $\mathcal{S}(\mathbb{D}^n)$. Our other results give various necessary and sufficient conditions for a function to be uniquely determined in $\mathcal{S}(\mathbb{D}^n)$ by its values on a finite set of points, with the number of points depending on the degree of the function. Our approach to proving these results is based on Pick's generalization of the Schwarz lemma in 1916. Pick's original setup for the disc is generalized to \mathbb{D}^n in the following definition.

Definition 1.1.1. *The **Pick problem on \mathbb{D}^n** is to determine, given N distinct points $\lambda_1, \dots, \lambda_N \in \mathbb{D}^n$ and N target points $\omega_1, \dots, \omega_N \in \mathbb{D}$, whether there exists a $f \in \mathcal{S}(\mathbb{D}^n)$ that satisfies $f(\lambda_i) = \omega_i$ for each $i = 1, \dots, N$.*

*A Pick problem is called **extremal** if a solution f satisfying $\|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)| = 1$ exists and no solution g satisfying $\|g\|_\infty < 1$ exists.*

*A rational function f is called **inner** if f is analytic on \mathbb{D}^n and $|f(\tau)| = 1$ for almost every $\tau \in \mathbb{T}^n$.*

In 1916 in [28], Pick gave necessary and sufficient conditions for a Pick problem on \mathbb{D} to have a solution and proved that every solvable problem is solvable by a rational inner function. Pick also showed that a Pick problem on \mathbb{D} is extremal if and only if it has a unique solution. The following is an immediate consequence of Pick's results.

Pick's Theorem. *(Pick, [28]) Fix $\lambda_1, \dots, \lambda_N \in \mathbb{D}$. If f is a rational inner function with less than N zeros on \mathbb{D} and $g \in \mathcal{S}(\mathbb{D})$ satisfies $g(\lambda_i) = f(\lambda_i)$ for $i = 1, \dots, N$, then $g = f$ on \mathbb{D} .*

In terms of the Pick problem on \mathbb{D} , Pick's Theorem states that for a rational inner function f with less than N zeros on \mathbb{D} , the problem with data $\lambda_1, \dots, \lambda_N$ and $f(\lambda_1), \dots, f(\lambda_N)$ has a unique solution.

In 1988 in [1], Agler gave necessary and sufficient conditions for a Pick problem on \mathbb{D}^2 to have a solution and proved that every solvable problem is solvable by a rational inner function. In [6], Agler and McCarthy showed that if a Pick problem on \mathbb{D}^2 is extremal then there exists an algebraic variety on which all solutions agree. The following is an example of an extremal problem and a variety on which all solutions agree. The example also shows that on \mathbb{D}^2 , unlike on \mathbb{D} , a Pick problem may be extremal and fail to have a unique solution.

Example 1.1.2 The Pick problem on \mathbb{D}^2 with data $(0, 0)$, $(\frac{1}{2}, \frac{1}{2})$ and $0, \frac{1}{2}$ is extremal and all solutions agree on the variety $V = \{(z, w) : z - w = 0\}$. That the problem is extremal follows from the fact that for all solutions agree on $V \cap \mathbb{D}^2$. To see that all solutions agree on V , let f be a solution and let $F(\lambda) = f(\lambda, \lambda)$. Since $F \in \mathcal{S}(\mathbb{D})$ satisfies $H(0) = 0$ and $F(\frac{1}{2}) = \frac{1}{2}$, the Schwarz Lemma implies that $f(\lambda) = \lambda$. Furthermore, $\|f\|_\infty \geq \|f|_V\|_\infty = \|F\|_{\mathbb{D}} = 1$ and the problem is extremal.

The following definition and theorem formalize that Example 1.1.2 is somewhat representative of extremal Pick problems on \mathbb{D}^2 .

Definition 1.1.3. For $n \geq 2$, an algebraic variety $V \subset \mathbb{C}^n$ is called *inner* if each of its irreducible components V_i meets \mathbb{D}^n and exits \mathbb{D}^n through the torus, i.e. $V_i \cap \mathbb{D}^n \neq \emptyset$ and $V_i \cap \partial(\mathbb{D}^n) \subset \mathbb{T}^n$.

Theorem 1.1.4. (Agler, McCarthy [6]) Given an extremal Pick problem on \mathbb{D}^2 , there exists a 1-dimensional inner variety $V \subset \mathbb{C}^2$ such that all solutions agree on $V \cap \mathbb{D}^2$.

The present work began as an investigation of Theorem 1.1.4. In the remainder of this section we describe our methods and state our main results.

Fix a non-constant rational inner function f on \mathbb{D}^n and a one dimensional inner variety $V \subset \mathbb{C}^n$. The restriction of f to $V \cap \mathbb{D}^n$ behaves somewhat like a function of one variable. Whereas the zero set of f is a $(n-1)$ -dimensional variety, the restriction of f to $V \cap \mathbb{D}^n$ has finitely many zeros. Let F denote the restriction of f to $V \cap \mathbb{D}^n$. In the special case that $V \cap \mathbb{D}^n$ can be parametrized with an analytic disc, it is possible to identify F with a function on \mathbb{D} via the parametrization. In

this case, the well understood theory of $\mathcal{S}(\mathbb{D})$ can be applied to F . However, it is not always possible to parameterize $V \cap \mathbb{D}^n$ with an analytic disc. Our approach is to use Hilbert function space theory to formalize that for a general 1-dimensional inner variety V , the function F behaves like a function of one variable. This approach allows us to generalize several results about $\mathcal{S}(\mathbb{D})$ to $\mathcal{S}(\mathbb{D}^n)$. One of the first results found using this approach was the following generalization of Pick's Theorem to \mathbb{D}^2 by Jim Agler. Agler's result was the inspiration for much of the research in this thesis and is of crucial importance to the results of this work. The theorem, as stated below, follows immediately from Lemma 3.0.12 in chapter 3.

Theorem 1.1.5. *(Agler, private communication) Fix an inner variety $V \subset \mathbb{D}^2$ and $\lambda_1, \dots, \lambda_N \in V \cap \mathbb{D}^2$ distinct. If f is a rational inner function on \mathbb{D}^2 with less than N zeros on $V \cap \mathbb{D}^2$ and $g \in \mathcal{S}(\mathbb{D}^2)$ satisfies $g(\lambda_i) = f(\lambda_i)$ for $i = 1, \dots, N$, then $\|g\|_\infty = 1$.*

In terms of the Pick problem on \mathbb{D}^2 , Theorem 1.1.5 states that for a function f with less than N zeros on $V \cap \mathbb{D}^n$, the problem on \mathbb{D}^2 with data $\lambda_1, \dots, \lambda_N$ and $f(\lambda_1), \dots, f(\lambda_N)$ is extremal.

The present work generalizes Agler's Theorem 1.1.5 to one dimensional inner varieties in \mathbb{C}^n and strengthens the conclusion.

Theorem B. *Fix a one dimensional inner variety $V \subset \mathbb{C}^n$ and distinct points $\lambda_1, \dots, \lambda_N \in V \cap \mathbb{D}^n$. If f is a rational inner function on \mathbb{D}^n with less than N zeros on $V \cap \mathbb{D}^n$ and $g \in \mathcal{S}(\mathbb{D}^n)$ satisfies $g(\lambda_i) = f(\lambda_i)$ for $i = 1, \dots, N$, then $g = f$ on $V \cap \mathbb{D}^n$.*

In terms of the Pick problem on \mathbb{D}^n , Theorem **B** states that for a function f with less than N zeros on $V \cap \mathbb{D}^n$, the problem on \mathbb{D}^n with data $\lambda_1, \dots, \lambda_N$ and $f(\lambda_1), \dots, f(\lambda_N)$ is extremal and all solutions agree on $V \cap \mathbb{D}^n$. The following example is a typical application of Theorem **B**.

Example 1.1.6 Let $\mathcal{N} \subset \mathbb{C}^2$ denote the Niel Parabola, $\{(z, w) : z^3 - w^2 = 0\}$, let $N = 6$ and fix $\lambda_1, \dots, \lambda_N \in \mathcal{N} \cap \mathbb{D}^2$. Rudin's Theorem 2.1.4 and Theorem **F** imply that for $a, b \neq 0$ with $|a| + |b| < 1$, the rational inner function defined by the following formula has 5 zeros on $\mathcal{N} \cap \mathbb{D}^2$.

$$f(z, w) = \frac{zw + az + bw}{1 + \bar{b}z + \bar{a}w}$$

Theorem **B** states that if $g \in \mathcal{S}(\mathbb{D}^2)$ satisfies $g(\lambda_i) = f(\lambda_i)$ for each $i = 1, \dots, N$, then $g = f$ on $\mathcal{N} \cap \mathbb{D}^2$. Equivalently, every solution to the Pick problem with data $\lambda_1, \dots, \lambda_N$ and $f(\lambda_1), \dots, f(\lambda_N)$ agrees with f on $\mathcal{N} \cap \mathbb{D}^2$.

We mention that since $\gamma(\lambda) = (\lambda^2, \lambda^3)$ defines a parameterization of $\mathcal{N} \cap \mathbb{D}^2$ as an analytic disc, we could have avoided invoking Theorems **B** and instead applied Pick's Theorem to $F(\lambda) = f(\gamma(\lambda))$. However, Pick's Theorem is only applicable to analytic discs whereas Theorem **B** can be applied to any one dimensional inner variety.

Our next result states that for each rational inner function f on \mathbb{D}^n there exists a finite set of points $\lambda_1, \dots, \lambda_{N^n}$ such that f is uniquely determined in $\mathcal{S}(\mathbb{D}^n)$ by its values on $\lambda_1, \dots, \lambda_{N^n}$.

Definition 1.1.7. For a rational inner function f define $\deg(f)$, the **degree** of f , by letting $f = \frac{q}{r}$ for $q, r \in \mathbb{C}[z_1, \dots, z_n]$ relatively prime and let $\deg(f) = \deg(q)$.

Theorem C. (Scheinker [31]) Fix $n, N \geq 1$. There exists a one dimensional inner variety $V \subset \mathbb{C}^n$ and $\lambda_1, \dots, \lambda_{N^n} \in V \cap \mathbb{D}^n$ with the following property. If f is a rational inner function of degree less than N and $g \in \mathcal{S}(\mathbb{D}^n)$ satisfies $g(\lambda_i) = f(\lambda_i)$ for each $i = 1, \dots, N^n$, then $g = f$ on \mathbb{D}^n .

Corollary 1.1.8. Fix $n, N \geq 1$. There exist points $\lambda_1, \dots, \lambda_{N^n} \in \mathbb{D}^n$ such that for each rational inner function f of degree less than N , the Pick problem with data $\lambda_1, \dots, \lambda_{N^n}$ and $f(\lambda_1), \dots, f(\lambda_{N^n})$ has a unique solution.

To the best of the author's knowledge Theorem **C** was the first result giving sufficient conditions for a general Pick problem on \mathbb{D}^n to have a unique solution.

To illustrate the significance of a function in $\mathcal{S}(\mathbb{D}^n)$ being uniquely determined by its values on a finite set of points, we mention that a function $f \in \mathcal{S}(\mathbb{D}^n)$ may not be uniquely determined in $\mathcal{S}(\mathbb{D}^n)$ by its values on a $(n - 1)$ -dimensional set. To see this, notice that for a generic $g \in \mathcal{S}(\mathbb{D}^n)$ the zero set of $f - g$ is an $(n - 1)$ -dimensional variety. The following is an example of this.

Example 1.1.9 Fix $n \geq 2$, let $z = z_1$ and as a departure from our usual notation let w denote the product of the other coordinate functions, $w = z_2 \cdots z_n$. Fix the $(n - 1)$ -dimensional inner variety $V = \{(z, w) : z - w = 0\} \subset \mathbb{C}^n$, fix $a, b \neq 0$ with $|a| + |b| < 1$ and consider the following functions.

$$f(z_1, \dots, z_n) = \frac{zw + az + bw}{1 + \bar{b}z + \bar{a}w} \text{ and } f_\epsilon(z_1, \dots, z_n) = \frac{zw + az + bw + \epsilon(z - w)}{1 + \bar{b}z + \bar{a}w + \epsilon(w - z)}$$

The function f is rational inner by Rudin's Theorem 2.1.4. The function f_ϵ satisfies $f_\epsilon = f$ on $V \cap \mathbb{D}^n$ and $f_\epsilon \neq f$ on \mathbb{D}^n . For ϵ sufficiently small, f_ϵ is also inner and thus in $\mathcal{S}(\mathbb{D}^n)$.

Our next result formalizes the phenomenon of Example 1.1.9 and can be thought of as a partial converse to Theorem **C**. The Theorem **D** states that if the degree of f is sufficiently high relative to the degree of a $(n - 1)$ -dimensional inner variety V , then f is not uniquely determined in $\mathcal{S}(\mathbb{D}^n)$ by its values on V .

Theorem D. *Fix a rational inner function f with no singularities on \mathbb{T}^n . If $V = Z_p$ is an inner variety given as the zero set of a polynomial p and the degree of p is less than or equal to the degree of f in each variable z_i , then there exists a rational inner function g on \mathbb{D}^n that satisfies $g = f$ on $V \cap \mathbb{D}^n$ and $g \neq f$ on \mathbb{D}^n .*

In terms of the Pick problem on \mathbb{D}^n , Theorem **D** gives sufficient conditions on a variety V and a function f such that for every positive integer M and any set of points $\lambda_1, \dots, \lambda_M \in V \cap \mathbb{D}^n$, there are many solutions to the problem on \mathbb{D}^n with data $\lambda_1, \dots, \lambda_M$ and $f(\lambda_1), \dots, f(\lambda_M)$. This is in striking contrast to the one dimensional case where Pick's Theorem guarantees that for each rational inner f , once enough points $\lambda_1, \dots, \lambda_M \in \mathbb{D}$ are chosen, the problem with data $\lambda_1, \dots, \lambda_M$ and $f(\lambda_1), \dots, f(\lambda_M)$ will have a unique solution.

Our next result is a generalization of Pick's Theorem to $\mathcal{S}(\mathbb{D}^n)$.

Theorem E. Fix $n, N > 1$ and $\lambda_1, \dots, \lambda_N \in \mathbb{D}$ distinct. Given a rational inner function of one variable $f(z, w) = F(z)$ of degree less than N , if $g \in \mathcal{S}(\mathbb{D}^n)$ satisfies $g(\lambda_i, w_0) = f(\lambda_i, w_0)$ for some $w_0 \in \mathbb{D}^{n-1}$ and $i = 1, \dots, N$, then $g = f$ on \mathbb{D}^n .

In terms of the Pick problem on \mathbb{D}^n , Theorem **E** gives sufficient conditions for the problem on \mathbb{D}^n with data $(\lambda_1, w_0), \dots, (\lambda_N, w_0)$ and $f(\lambda_1, w_0), \dots, f(\lambda_N, w_0)$ to have a unique solution.

Our next result establishes a formula for the number of zeros of a rational inner function f on an inner variety V , which we use to apply Theorems **B** and **C**.

Definition 1.1.10. Given a rational inner function f and an inner variety V , let $\deg_V(f)$ denote the number of zeros of f on $V \cap \mathbb{D}^n$ counted with multiplicity.

Agler and McCarthy discovered a formula for $\deg_V(f)$ in [6]. Although their original result was for \mathbb{D}^2 and inner functions with no singularities on \mathbb{T}^2 , their proof extends easily to general rational inner functions on \mathbb{T}^n . Before stating our generalization of their result we need some definitions.

Definition 1.1.11. For a rational inner function f , define $n\text{-deg}(f)$, the ***n-degree*** of f , by letting $f = \frac{q}{r}$ for $q, r \in \mathbb{C}[z_1, \dots, z_n]$ relatively prime and let $n\text{-deg}(f)$ be the n -tuple of the degree of q in each coordinate.

Definition 1.1.12. For a rational inner function f and an inner variety V , define $\deg_V(f)$, the ***degree of f on V*** as the number of zeros of f on $V \cap \mathbb{D}^n$.

Definition 1.1.13. For a one dimensional variety $V \subset \mathbb{C}^n$, define the ***rank*** of V to be the n -tuple of the number of points in V over each fixed coordinate at a regular point of V .

Theorem F. If V is a one dimensional inner variety with rank $m = (m_1, \dots, m_n)$ and f is a rational inner function with n -degree $d = (d_1, \dots, d_n)$, then

$$\deg_V(f) \leq d \cdot m = d_1 m_1 + \dots + d_n m_n.$$

Furthermore, equality holds whenever f has no singular points on $V \cap \mathbb{T}^n$.

We close our overview of this thesis with an application of Theorem **B**. Given a variety $V \subset \mathbb{C}^n$, we say that a function F is analytic on $V \cap \mathbb{D}^n$ if for every point $\lambda \in V \cap \mathbb{D}^n$ there is an open ball θ in \mathbb{D}^n containing λ and an analytic function f of two variables on θ , such that $f|_{V \cap \theta} = F|_{V \cap \theta}$. H. Cartan's celebrated theorem in [13] implies that every analytic function F on $V \cap \mathbb{D}^n$ can be extended to an analytic function f on \mathbb{D}^n . If the function F is bounded, then one can ask if there exists an extension f that is also bounded. If an extension f satisfying $\sup_{\mathbb{D}^n} |f| = \sup_V |F|$ exists, then we call f a **norm preserving extension** of F .

If $V \cap \mathbb{D}^n$ is the graph of an analytic function $m : \mathbb{D}^k \rightarrow \mathbb{D}^{n-k}$, i.e. has, after a permutation of the coordinate functions, the form $V \cap \mathbb{D}^n = \{(\lambda, m(\lambda)) : \lambda \in \mathbb{D}^k\}$, then every bounded analytic function F on $V \cap \mathbb{D}^n$ has a norm preserving extension to \mathbb{D}^n given by

$$f(z_1, \dots, z_n) = F(z_1, \dots, z_k, m(z_1, \dots, z_k)).$$

If $V \cap \mathbb{D}^n$ is not the graph of an analytic function, then one can ask how to differentiate between those bounded functions on V that do and do not have norm preserving extensions to \mathbb{D}^n , e.g. see Agler and McCarthy's [5]. Theorem 1.1.14 states that determining whether a rational inner function on a 1-dimensional inner variety V has a norm preserving extension to \mathbb{D}^n is equivalent to solving a Pick problem on \mathbb{D}^n . Here, we call an analytic function F on $V \cap \mathbb{D}^n$ rational if it has finitely many zeros and inner if it satisfies $|F(\tau)| = 1$ for almost every $\tau \in V \cap \mathbb{T}^n$.

Theorem 1.1.14. *Let $V \subset \mathbb{C}^n$ be a 1-dimensional irreducible inner variety and F a rational inner function on $V \cap \mathbb{D}^n$. The following are equivalent.*

- a.** *There exists a $f \in \mathcal{S}(\mathbb{D}^n)$ such that $f|_V = F$.*
- b.** *There exists a N greater than the number of zeros of F on V and distinct $\lambda_1, \dots, \lambda_N \in V$ such that the Pick problem with data $\lambda_1, \dots, \lambda_N$ and $F(\lambda_1), \dots, F(\lambda_N)$ is solvable on \mathbb{D}^n .*

The implication **a** \rightarrow **b** is trivial. That **b** \rightarrow **a** follows from Theorem 4.1.1, a stronger version of Theorem **B** proved in chapter 4 section 1. We mention that a key fact used in the proof of Theorem 4.1.1 comes from a theorem in [25] by Jury, Knese and McCullough. I am thankful to the authors of that paper for the

discussion in which they explained their result and the subtlety associated with it. The role of the result from [25] is discussed in more detail in the proof of lemma 3.0.11.

1.2 Organization of the Paper

This work is organized as follows. In chapter 2 we give various background results on the Pick problems on \mathbb{D} and \mathbb{D}^2 , and on reproducing kernel Hilbert function spaces. In chapter 3 we prove several results about reproducing kernel Hilbert function spaces on inner varieties. In chapter 4 we prove Theorems **A-F** minus **C** using the results of chapter 3. In chapter 5 we give the proof of Theorem **C** from [31].

Chapter 2

Background

2.1 Rudin's theorem for rational inner functions

In [29], Rudin gave a formula for a general rational inner function on \mathbb{D}^n . We state this result with multi-index notation, using $f(z) = z^d$ to denote

$$f(z_1, \dots, z_n) = z_1^{d_1} \cdots z_n^{d_n}.$$

Definition 2.1.1. For a rational inner function f , let $f = \frac{q}{r}$ for $q, r \in \mathbb{C}[z_1, \dots, z_n]$ relatively prime. Define $\deg(f)$, the **degree** of f , to be the degree of q and define $n\text{-deg}(f)$, the **n-degree** of f , to be the n -tuple of the degree of q in each coordinate. Given a polynomial $q(z_1, \dots, z_n)$ with $n\text{-deg}(q) = (d_1, \dots, d_n)$ define \tilde{q} , the **reflection of q through \mathbb{T}^n** by the formula

$$\tilde{q}(z) = \tilde{q}(z_1, \dots, z_n) = z_1^{d_1} \cdots z_n^{d_n} \overline{q\left(\frac{1}{z_1}, \dots, \frac{1}{z_n}\right)} = z^d \overline{q\left(\frac{1}{z}\right)} \quad (2.1.2)$$

Theorem 2.1.3. (Rudin [29]) Given a polynomial q that does not vanish on \mathbb{D}^n the rational function $\frac{\tilde{q}}{q}$ is inner. Furthermore, every rational inner function on \mathbb{D}^n can be written as

$$f(z) = \tau z^m \frac{\tilde{q}}{q}, \quad \text{with } \tau \in \mathbb{T} \quad (2.1.4)$$

for some polynomial q that does not vanish on \mathbb{D}^n and $m = (m_1, \dots, m_n)$ an n -tuple of positive integers.

2.2 The Pick Problem on \mathbb{D}

Pick's study of the problem that now bears his name was one of the first major advancements in the theory of rational inner functions on \mathbb{D} . Pick gave a criteria for the solvability of a Pick problem with data $\lambda_1, \dots, \lambda_n$ and $\omega_1, \dots, \omega_N$ in terms of what is now called the Pick matrix, which we denote P . The Pick matrix P is a Hermitian N -by- N matrix defined as follows

$$P = \left(\frac{1 - \omega_i \overline{\omega_j}}{1 - \lambda_i \overline{\lambda_j}} \right).$$

The following theorem summarizes Pick's results.

Theorem 2.2.1. (Pick, [28]) *On \mathbb{D} , the following are equivalent.*

- a. The problem with data $\lambda_1, \dots, \lambda_n$ and $\omega_1, \dots, \omega_N$ has a solution.*
- b. The Pick matrix $P = \left(\frac{1 - \omega_i \overline{\omega_j}}{1 - \lambda_i \overline{\lambda_j}} \right)$ is positive semi-definite.*
- c. The problem with data $\lambda_1, \dots, \lambda_n$ and $\omega_1, \dots, \omega_N$ has a solution that is a rational inner function f with $\deg(f) = \text{rank} \left(\frac{1 - \omega_i \overline{\omega_j}}{1 - \lambda_i \overline{\lambda_j}} \right)$.*

A Pick problem is called **extremal** if there exists a solution f with $\|f\|_\infty = 1$ and no solution g of norm less than one exists. One can see that a Pick problem cannot have a unique solution if it is not extremal as follows. If there exists a solution g of norm strictly less than one, then adding a small multiple of a polynomial vanishing at the nodes will result in a second solution, $G = g + \epsilon p$. The converse follows from Pick's Theorem 2.2.1; if a Pick problem is extremal, then the problem has a unique solution. The following theorem summarizes those consequences of Pick's Theorem 2.2.1 that we will use in this work.

Theorem 2.2.2. *The following are equivalent.*

- a. The Pick problem with data $\lambda_1, \dots, \lambda_n$ and $\omega_1, \dots, \omega_N$ has a unique solution.*
- b. The Pick problem with data $\lambda_1, \dots, \lambda_n$ and $\omega_1, \dots, \omega_N$ is extremal.*
- c. The Pick matrix $P = \left(\frac{1 - \omega_i \overline{\omega_j}}{1 - \lambda_i \overline{\lambda_j}} \right)$ is positive semi-definite and singular.*
- d. The Pick problem with data $\lambda_1, \dots, \lambda_n$ and $\omega_1, \dots, \omega_N$ has a solution that is a rational inner solution f with $\deg(f) < N$.*

Pick's Theorem in section 1 of chapter 1 is equivalent to the implication $\mathbf{d} \rightarrow \mathbf{a}$ in Theorem 2.2.2.

2.3 The Pick problem on \mathbb{D}^2

In [1], Agler generalized Pick's theorem to the bidisc. Agler gave criteria for the solvability of a Pick problem on \mathbb{D}^2 in terms of the existence of two hermitian, positive semi-definite matrices. Agler also showed that every Pick problem on \mathbb{D}^2 that has a solutions has a solution that is a rational inner function. Agler's original paper [1] is difficult to find and we recommend the exposition in Agler and McCarthy's book [4]. We introduce some notation before stating Agler's theorem.

Given a Pick problem with data $\lambda_1, \dots, \lambda_n$ and $\omega_1, \dots, \omega_N$, let W , Λ^1 and Λ^2 denote the following N -by- N matrices.

$$W = (1 - w_i \bar{w}_j) \quad \Lambda^1 = \left(1 - \lambda_i^1 \bar{\lambda}_j^1\right) \quad \Lambda^2 = \left(1 - \lambda_i^2 \bar{\lambda}_j^2\right)$$

An **admissible kernel** K is an N -by- N positive definite matrix, with all the diagonal entries 1, such that

$$\Lambda^1 \cdot K = [(1 - \lambda_i^1 \bar{\lambda}_j^1) K_{ij}] \geq 0 \text{ and } \Lambda^2 \cdot K = [(1 - \lambda_i^2 \bar{\lambda}_j^2) K_{ij}] \geq 0.$$

Here \cdot denotes the Schur entrywise product: $(A \cdot B)_{ij} = A_{ij} B_{ij}$.

Theorem 2.3.1. (Agler, [1]) *The following are equivalent.*

- a.** *The Pick problem with data $\lambda_1, \dots, \lambda_n$ and $\omega_1, \dots, \omega_N$ has a solution.*
- b.** *The Pick problem with data $\lambda_1, \dots, \lambda_n$ and $\omega_1, \dots, \omega_N$ has a solution that is a rational inner function.*
- c.** *There exists a pair of N -by- N positive semi-definite matrices, Γ, Δ , such that*

$$W = \Lambda^1 \cdot \Gamma + \Lambda^2 \cdot \Delta \quad \text{i.e. } (1 - \omega_i \bar{\omega}_j) = (1 - \lambda_i^1 \bar{\lambda}_j^1) \Gamma_{ij} + (1 - \lambda_i^2 \bar{\lambda}_j^2) \Delta_{ij}.$$

- d.** *For every admissible kernel K , the Pick matrix associated with K , $W \cdot K = [(1 - w_i \bar{w}_j) K_{ij}]$, is positive semi-definite.*

A Pick problem on \mathbb{D}^2 cannot have a unique solution if it is not extremal. To see this, notice that if a Pick problem has a solution f of norm less than one then adding a small multiple of a polynomial vanishing at the nodes will result in another solution, $F = f + \epsilon p$. Agler and McCarthy gave criteria for a Pick problem to be extremal in terms of the singularity of a matrix associated to the problem.

Theorem 2.3.2. (Agler, McCarthy, [6]) *The following are equivalent.*

- a. The Pick problem data $\lambda_1, \dots, \lambda_n$ and $\omega_1, \dots, \omega_N$ is extremal.*
- b. There exists an admissible kernel K such that the Pick matrix given by, $W \cdot K = ((1 - \omega_i \overline{\omega_j})K_{ij})$, is singular.*

2.4 Analytic Hilbert function spaces

In [6], Agler and McCarthy study inner varieties using Hilbert function spaces and reproducing kernels. In this thesis, we use their methods and generalize several of their results. In this section, we present a very brief introduction to Hilbert function spaces. For a more extensive discussion see [4].

Given a set X , a **Hilbert function space on X** is a Hilbert space $H(X)$, consisting of functions on X , such that evaluation at each point $\lambda \in X$ is a non-zero continuous linear functional on $H(X)$. The element of $H(X)$ that induces the linear functional of evaluation at λ , the existence of which is guaranteed by the Riesz Representation Theorem, is called the **reproducing kernel at λ** and is denoted k_λ . That is, for each $g \in H$ the equality $\langle g, k_\lambda \rangle = g(\lambda)$ holds. In this work, the set X will be \mathbb{D}^n or $V \cap \mathbb{D}^n$ for a 1-dimensional inner variety $V \subset \mathbb{C}^n$, and $H(X)$ will consist of functions that are analytic on X . The following definition formalizes two of the properties of such spaces.

Definition 2.4.1. *For a Hilbert function space $H(X)$, we call $H(X)$ **free** if there do not exist distinct points $\lambda_1, \dots, \lambda_N \in X$ and non-zero scalars $a_1, \dots, a_N \in \mathbb{C}$ such that every $f \in H(X)$ satisfies*

$$a_1 f(\lambda_1) + \dots + a_N f(\lambda_N) = 0. \quad (2.4.2)$$

If X equals \mathbb{D}^n or $V \cap \mathbb{D}^n$ for a 1-dimensional inner variety $V \subset \mathbb{C}^n$ and $H(X)$ contains the polynomials, then $H(X)$ is free.

Given a Hilbert function space $H(X)$, we define a **kernel function** $K : X \times X \rightarrow \mathbb{C}$ with the formula

$$K(\lambda, \xi) = k_\xi(\lambda) = \langle k_\xi, k_\lambda \rangle \quad \text{for } \lambda, \xi \in X.$$

For each distinct set of points $\lambda_1, \dots, \lambda_N \in X$ the N by N self-adjoint matrix (K_{ij}) , denoted (k_{ij}) and defined by the following formula, is positive semi-definite.

$$(K_{ij}) = (k_{ij}) \text{ where } k_{ij} = K(\lambda_i, \lambda_j) = \langle k_{\lambda_j}, k_{\lambda_i} \rangle$$

A kernel is called **strictly positive** if for every choice of points $\lambda_1, \dots, \lambda_N \in X$, the matrix (K_{ij}) associated to the points is strictly positive. If the space $H(X)$ is free, then the kernel K is strictly positive and can be normalized to have all diagonal entries equal to 1, that is $K(\lambda_i, \lambda_i) = 1$ for every $\lambda_i \in X$. A kernel K can be thought of as an infinite positive semi-definite matrix where each entry is associated to a pair of points in X and the matrix (K_{ij}) can be thought of as the restriction of K to $\lambda_1, \dots, \lambda_N$. When there is no danger of confusion, we will use K to denote the matrix (K_{ij}) .

The **multiplier algebra** of a Hilbert function space $H(X)$, $Mult(H(X))$, is the algebra of functions ϕ on X that satisfy $\phi f \in H$ for every $f \in H$. Addition and multiplication in $Mult(H(X))$ are defined in the obvious way. If $\phi \in Mult(H(X))$, then it follows from the Closed Graph Theorem that multiplication by ϕ , denoted M_ϕ , is a bounded linear operator on $H(X)$. We will make use of the fact that for a point $\lambda \in X$ and a multiplication operator M_ϕ , the evaluation functional k_λ is an eigenvector of M_ϕ^* , the adjoint of M_ϕ , with eigenvalue $\overline{\phi(\lambda)}$. One can see this as follows. Fix $g \in H(X)$.

$$\langle g, M_\phi^* k_\lambda \rangle = \langle M_\phi g, k_\lambda \rangle = \langle \phi g, k_\lambda \rangle = \phi(\lambda) g(\lambda) = \langle g, \overline{\phi(\lambda)} k_\lambda \rangle.$$

A particular Hilbert function space that we are interested in is $H^2(\mathbb{D}^n)$, the **Hardy space on \mathbb{D}^n** . $H^2(\mathbb{D}^n)$ is defined as the closure of the polynomials in $L^2(\mathbb{T}^n, d\tau)$, where $d\tau$ is normalized Lebesgue measure on \mathbb{T}^n . Each $f \in H^2(\mathbb{D}^n)$ is analytic on \mathbb{D}^n and can be continuously extended to almost every point on \mathbb{T}^n . Furthermore, the extension of f to \mathbb{T}^n is in $L^2(\mathbb{T}^n)$. We summarize this as follows.

$$H^2(\mathbb{D}^n) = \{f : f \text{ is analytic and } \|f\|^2 = \int_{\mathbb{T}^n} |f(\tau)|^2 d\tau < \infty\}.$$

The space $H^2(\mathbb{D}^n)$ is free since it contains the polynomials. In $H^2(\mathbb{D}^n)$, the reproducing kernel for a point $\lambda \in \mathbb{D}^n$ is called the Szego kernel and is given by the

following formula.

$$k_\lambda(z) = \prod_{i=1}^N \frac{1}{(1 - \bar{\lambda}^i z_i)}$$

For $f \in H^2(\mathbb{D}^n)$, the reproducing property of k_λ is expressed as follows.

$$f(\lambda) = \langle f, k_\lambda \rangle = \int_{\mathbb{T}^n} f(\tau) \overline{k_\lambda(\tau)} d\tau = \int_{\mathbb{T}^n} \frac{f(\tau)}{(1 - \lambda^1 \bar{\tau}^1) \dots (1 - \lambda^n \bar{\tau}^n)} d\tau \quad (2.4.3)$$

Formula 2.4.3 follows from the Cauchy integral formula for \mathbb{D}^n .

The connection between functional analysis on $H^2(\mathbb{D}^n)$ and function theory on \mathbb{D}^n arises from the fact that $Mult(H^2(\mathbb{D}^n))$ equals $H^\infty(\mathbb{D}^n)$, the Banach algebra of bounded analytic functions on \mathbb{D}^n endowed with the sup norm. Furthermore, for each $\phi \in Mult(H^2(\mathbb{D}^n))$ the operator $\|M_\phi\|$ satisfies $\|M_\phi\| = \|\phi\|_\infty$. We summarize this as follows.

$$Mult(H^2(\mathbb{D}^n)) = H^\infty(\mathbb{D}^n) = \{f : f \text{ analytic on } \mathbb{D}^n \text{ and } \|f\|_\infty < \infty\}.$$

The algebra $H^\infty(\mathbb{D}^n)$ is in turn connected to the Pick problem on \mathbb{D}^n since a function f is in the unit ball of $H^\infty(\mathbb{D}^n)$ if and only if f is in $\mathcal{S}(\mathbb{D}^n)$, the Schur class of \mathbb{D}^n . Another example of the connection is that for a Pick problem on \mathbb{D} with data $\lambda_1, \dots, \lambda_N$ and $\omega_1, \dots, \omega_N$ and two N by N matrices W and K given by

$$W = (1 - \omega_i \bar{\omega}_j) \text{ and } K = (\langle k_{\lambda_j}, k_{\lambda_i} \rangle) = \left(\frac{1}{1 - \lambda_i \bar{\lambda}_j} \right)$$

the classical Pick matrix P equals $W \cdot K$. Here \cdot denotes the Schur entrywise product: $(A \cdot B)_{ij} = A_{ij} B_{ij}$.

For a Banach space B we use B_1 to denote the unit ball of B . In particular,

$$Mult_1(H^2(\mathbb{D}^n)) = Ball(Mult(H^2(\mathbb{D}^n))) \text{ and } H_1^\infty(V) = Ball(H^\infty(\mathbb{D}^n)).$$

Chapter 3

Hilbert Function Space on V

The generalized Pick problem on a Hilbert function space $H(X)$ with kernel K is to determine, given distinct nodes $\lambda_1, \dots, \lambda_N \in X$ and $\omega_1, \dots, \omega_N \in \mathbb{D}$, whether there exists a function f in the unit ball of $Mult(H)$, $Mult_1(H)$, that satisfies $f(\lambda_i) = \omega_i$. Given a Pick problem, we let W and K denote the following N by N matrices

$$W = (1 - \omega_i \bar{\omega}_j) \text{ and } K = (k_{ij}) \text{ where } k_{ij} = K(\lambda_i, \lambda_j) = \langle k_{\lambda_j}, k_{\lambda_i} \rangle .$$

We use \cdot denotes the Schur entrywise product and call $W \cdot K = ((1 - \omega_i \bar{\omega}_j)k_{ij})$ the Pick matrix associated to the problem.

It is well known that if the Pick problem with data $\lambda_1, \dots, \lambda_N \in X$ and $\omega_1, \dots, \omega_N \in \mathbb{D}$ has a solution, then the associated Pick matrix $W \cdot K$ is positive semi-definite, see [4]. In the following theorem we show that if a Hilbert function space $H(X)$ satisfies certain mild assumptions, then a kind of converse holds. If the Pick matrix $W \cdot K$ has a non-trivial kernel, then the associated Pick problem on $H(X)$ has a unique solution. The proof of the following theorem is a generalization of a proof in [6].

Theorem 3.0.4. *Let $H(X)$ be free Hilbert function space with kernel K and the property that if a $f \in H(X)$ vanishes on an open subset of X , then $f = 0$ in $H(X)$. Fix a generalized Pick problem with data $\lambda_1, \dots, \lambda_N \in X$ and $\omega_1, \dots, \omega_N \in \mathbb{D}$ that has a solution. If the Pick matrix $W \cdot K = ((1 - \omega_i \bar{\omega}_j)K_{ij})$ is singular, then the solution is unique.*

PROOF: Consider a generalized Pick problem with data $\lambda_1, \dots, \lambda_N \in X$ and $\omega_1, \dots, \omega_N \in \mathbb{D}$ and assume that the problem has a solution. Since $H(X)$ is free the kernel associated to K is strictly positive and we can normalize the kernel K so that the diagonal entries of K equal 1. Let $W \cdot K$ be the Pick matrix associated to the problem and let γ be a non-zero vector in the null space of $W \cdot K$. Notice that not all of the points ω_i equal 0, for if they did, then we would have $W \cdot K = K$ which is strictly positive.

Consider the case where all of the entries of γ are non-zero. Let λ_{N+1} be any point in X that is not one of the original nodes. Let w_{N+1} be a possible value that a solution to the original problem can take at λ_{N+1} . Since the Pick problem with data $\lambda_1, \dots, \lambda_{N+1}$ and $\omega_1, \dots, \omega_{N+1}$ is solvable, the matrix $[(1 - w_i \bar{w}_j) K_{ij}]_1^{N+1}$ is positive semi-definite and for each $z \in \mathbb{C}$

$$\langle [(1 - w_i \bar{w}_j) K_{ij}]_1^{N+1} \begin{pmatrix} \gamma \\ z \end{pmatrix}, \begin{pmatrix} \gamma \\ z \end{pmatrix} \rangle \geq 0. \quad (3.0.5)$$

Since γ is in the null-space of $[(1 - w_i \bar{w}_j) K_{ij}]_1^N$ and each $K_{ii} = 1$, inequality (3.0.5) reduces to

$$2\Re[\bar{z} \sum_{j=1}^N (1 - \bar{w}_j w_{N+1}) K_{N+1,j} \gamma_j] + |z|^2 (1 - |w_{N+1}|^2) \geq 0. \quad (3.0.6)$$

Since equation 3.0.6 holds for all z , it follows that $\sum_{j=1}^N (1 - \bar{w}_j w_{N+1}) K_{N+1,j} \gamma_j = 0$ and the following gives an implicit formula for w_{N+1} ,

$$w_{N+1} \left(\sum_{j=1}^N \bar{w}_j K_{N+1,j} \gamma_j \right) = \sum_{j=1}^N K_{N+1,j} \gamma_j. \quad (3.0.7)$$

Claim: There exists an open set of points λ_{N+1} on which both sides of formula 3.0.7 do not reduce to 0.

Proof of claim: Let $\bar{a}_j = \bar{w}_j \gamma_j$ and define $G \in H(X)$ with the formula

$$G(x) = \sum_{j=1}^N \bar{a}_j k_{\lambda_j}(x).$$

Thus defined, $G(\lambda_{N+1})$ equals the left side of formula 3.0.7,

$$G(\lambda_{N+1}) = \sum_{j=1}^N \bar{a}_j k_{\lambda_j}(\lambda_{N+1}) = \sum_{j=1}^N \bar{w}_j K_{N+1,j} \gamma_j.$$

Towards a contradiction, suppose that the left side of formula 3.0.7 equals zero, i.e. $G = 0$ in $H(X)$. Take the inner product of G and an arbitrary $f \in H(X)$,

$$0 = \langle f, G \rangle = \langle f, \sum_{j=1}^N \bar{a}_j k_{\lambda_j} \rangle = \sum_{j=1}^N a_j \langle f, k_{\lambda_j} \rangle = \sum_{j=1}^N a_j f(\lambda_j) \quad (3.0.8)$$

Since the entries of γ are by assumption non-zero, the scalars a_i are non-zero and formula 3.0.8 contradicts the assumption that $H(X)$ is free. Thus, there exists an open set on which formula 3.0.7 does not vanish and on this open set w_{N+1} is given by the formula

$$w_{N+1} = \left(\sum_{j=1}^N K_{N+1,j} \gamma_j \right) / \left(\sum_{j=1}^N \bar{w}_j K_{N+1,j} \gamma_j \right). \quad (3.0.9)$$

Since $H(X)$ has the property that if a $f \in H(X)$ vanishes on an open subset of X , then $f = 0$ in $H(X)$, the existence of a formula for w_{N+1} on an open set uniquely determines w_{N+1} .

In the case where γ has exactly $M < N$ non-zero entries, the corresponding M point subproblem of the original problem has a unique solution. To see this, proceed as follows. Without loss of generality, assume that the first M entries of γ are non-zero and consider the M point subproblem of the original Pick problem with data $\lambda_1, \dots, \lambda_M$ and $\omega_1, \dots, \omega_M$. Let $(W \cdot K)_M$ be the M by M Pick matrix associated to the new problem. Since the last $N - M$ entries of γ are zero,

$$\langle (W \cdot K)_M \gamma_M, \gamma_M \rangle = \langle W \cdot K \gamma, \gamma \rangle = 0.$$

Thus, γ_M is in the kernel of $(W \cdot K)_M$. Since γ_M has only non-zero entries, one can apply the original argument of the proof to $(W \cdot K)_M$ and γ_M to conclude that the solution to the subproblem $\lambda_{i_1}, \dots, \lambda_{i_M}$ and $\omega_{i_1}, \dots, \omega_{i_M}$ is unique. Since the solution to the original problem solves the subproblem, it is also unique. \square

Given a 1-dimensional inner variety $V \subset \mathbb{C}^n$, we will use the previous theorem to prove Theorem 4.1.1, that a certain type of Pick problem on $V \cap \mathbb{D}^n$ has a unique solution. To prove Theorem 4.1.1, we first prove Lemma 3.0.11 and Lemma 3.0.12 below.

Lemma 3.0.11 states that for each 1-dimensional inner variety $V \subset \mathbb{C}^n$ there exists a Hilbert function space of analytic functions on $V \cap \mathbb{D}^n$ that satisfies the hypotheses of the previous theorem. In [6], Agler and McCarthy established the existence of the desired Hilbert function space structure for each 1-dimensional inner variety in \mathbb{C}^2 . A slight modification of their proof extends the result to a 1-dimensional inner variety in \mathbb{C}^n . Before stating and proving the result we give some notation and a well-known lemma, also from [6].

We say that a function F is analytic on $V \cap \mathbb{D}^n$ if for every point $\lambda \in V \cap \mathbb{D}^n$ there exists an open ball Ω in \mathbb{D}^n containing λ and an analytic function f on Ω , such that $f|_{V \cap \Omega} = F|_{V \cap \Omega}$. We use ∂V to denote the topological boundary of $V \cap \mathbb{D}^n$ as a subset of V , $\partial V = V \cap \mathbb{T}^n$. An analytic function F on $V \cap \mathbb{D}^n$ is called **inner** if for almost every $\tau \in \partial V$, F continues continuously to τ and satisfies $|F(\tau)| = 1$. An analytic function F on $V \cap \mathbb{D}^n$ is called **rational** if F has finitely many zeros, that is $\deg_V(F) < \infty$. If f is a rational inner function on \mathbb{D}^n and F is the restriction of f to $V \cap \mathbb{D}^n$, then F is a rational inner function on $V \cap \mathbb{D}^n$ and $\deg_V(F) = \deg_V(f)$. We use $H_1^\infty(V)$ to denote the unit ball of the Banach algebra of bounded analytic functions on $V \cap \mathbb{D}^n$. Given a measure μ , we use $H^2(\mu)$ to denote the closure of the polynomials in $L^2(\mu)$. For S a Riemann surface, Ω an open subset of S and ν a finite measure on $\overline{\Omega}$, we let $A^2(\nu)$ denote the closure in $L^2(\nu)$ of $A(\Omega)$, the functions that are analytic on Ω and continuous on $\overline{\Omega}$.

Lemma 3.0.10. *Let S be a compact Riemann surface. Let $\Omega \subseteq S$ be a domain whose boundary is a finite union of piecewise smooth Jordan curves. There exists a finite measure ν on $\partial\Omega$ such that evaluation at every λ in Ω is a bounded linear functional on $A^2(\nu)$ and such that the linear span of the corresponding evaluation kernels is dense in $A^2(\nu)$.*

Lemma 3.0.11. *Let $V \subset \mathbb{C}^n$ be a 1-dimensional irreducible inner variety. There exists a finite measure μ on $V \cap \mathbb{T}^n$, such that $H^2(\mu)$ has the following properties.*

- i.* $H^2(\mu)$ is a free Hilbert function space.
- ii.* If a $f \in H^2(\mu)$ vanishes on an open subset of $V \cap \mathbb{D}^n$, then $f = 0$ in $H^2(\mu)$.
- iii.* $H_1^\infty(V) \subset \text{Mult}_1(H^2(\mu))$.

The existence of μ satisfying conditions **i** and **ii** of 3.0.11 follows easily by generalizing arguments from [6]. Establishing $H_1^\infty(V) \subset Mult_1(V)$ in condition **iii**, however, requires far more delicacy. The fact that $H_1^\infty(V)$ is a subset of the closure of the polynomials is a theorem in [25] by Jury, Knese and McCullough. The author would like to thank the authors of that paper for a valuable discussion clarifying the subtlety of this condition.

PROOF: Let p_1, \dots, p_r be a set of irreducible polynomials such that V is the intersection of Z_{p_1}, \dots, Z_{p_r} . Let C be the projective closure of $Z_{p_1} \cap \dots \cap Z_{p_r}$ in $\mathbb{C}\mathbb{P}^n$ and identify $V \cap \mathbb{D}^n$ with a subset of C . Let S be the desingularization of C . This means S is a compact Riemann surface (not connected if C is not irreducible) and there is a holomorphic function $\phi : S \rightarrow C$ that is biholomorphic from S' onto C' and finite-to-one from $S \setminus S'$ onto $C \setminus C'$. Here C' is the set of non-singular points in C , and S' is the preimage of C' . See *e.g.* [18] or [22] for details of the desingularization.

Let $\Omega = \phi^{-1}(V \cap \mathbb{D}^n)$. Then $\partial\Omega$ is a finite union of disjoint curves, each of which is analytic except possibly at a finite number of cusps. Let ν be the measure from Lemma 3.0.10 (or the sum of these if Ω is not connected).

The desired measure μ is the push-forward of ν by ϕ , normalized to have mass 1 on $\partial V := \partial(V \cap \mathbb{D}^n) = V \cap \mathbb{T}^n$. In particular, μ is defined by

$$\mu(E) = \nu(\phi^{-1}(E)).$$

If $\lambda \in V \cap \mathbb{D}^n$ is a regular point of V , then there exists a unique ζ such that $\phi(\zeta) = \lambda$ and k_ζ is the reproducing kernel associated to ζ in $A^2(\Omega)$. The function $k_\lambda = k_\zeta \circ \phi^{-1}$ is defined μ almost every where on ∂V and for each $f \in H^2(\mu)$ satisfies

$$\langle f, k_\lambda \rangle_{H^2(\mu)} = \int_{\partial V} f \cdot \overline{k_\zeta \circ \phi^{-1}} d\mu = \int_{\partial\Omega} f \circ \phi \cdot \overline{k_\zeta} d\nu = \langle f \circ \phi, k_\zeta \rangle_{A^2(\Omega)} = f(\lambda).$$

If $\lambda \in V \cap \mathbb{D}^n$ is a singular point of V , then there exist finitely many ζ_1, \dots, ζ_s such that $\phi(\zeta_i) = \lambda$ and the function

$$k_\lambda = \frac{1}{s} \sum k_{\zeta_i} \circ \phi^{-1}$$

is the corresponding reproducing kernel function for λ .

That $H^2(\mu)$ satisfies conditions **i** and **ii** follows from that $H^2(\mu)$ is the closure of the polynomials. That $H_1^\infty(V) \subset \text{Mult}_1(H^2(\mu))$ follows from a theorem in [25] stating that each $\phi \in H_1^\infty(V)$ is uniformly approximatable by polynomials with sup-norm 1. \square

The second lemma that we need to prove Theorem 4.1.1 is Lemma 3.0.12 below. The lemma states that for a 1-dimensional inner variety $V \subset \mathbb{C}^n$ and a rational inner function F on $V \cap \mathbb{D}^n$, the rank of the Pick matrix associated to a Pick problem with solution F is less than or equal to the degree of F . This implies that if an N point Pick problem has a solution of degree less than N , then the Pick matrix associated to the problem is singular. This lemma is of crucial importance to this work and was the inspiration for much of the research in this thesis. It was discovered by Jim Agler and privately communicated to the author.

Lemma 3.0.12. *Let $V \subset \mathbb{C}^n$ be an irreducible 1-dimensional inner variety, fix $\lambda_1, \dots, \lambda_N \in V$ distinct and let F be a rational inner function on $V \cap \mathbb{D}^n$. Let $H^2(\mu)$ be a Hilbert function space on $V \cap \mathbb{D}^n$, the existence of which is guaranteed by Lemma 3.0.11 and let K be the kernel associated with $H^2(\mu)$. Let $\omega_i = F(\lambda_i)$. The rank of the N by N matrix $W \cdot K = ((1 - \omega_i \overline{\omega_j})K_{ij})$ is less than or equal to the number of zeros of F on V counted with multiplicity, i.e. $\text{rank}(W \cdot K) \leq \deg_V(F)$.*

PROOF: Fix a measure μ on $V \cap \mathbb{T}^n$, the existence of which is guaranteed by lemma 3.0.11, and let $H = H^2(\mu)$. Consider multiplication by F , denoted M_F , as a bounded linear operator on H . Since F is inner, M_F is an isometry and the operator $1 - M_F M_F^*$ is a projection onto

$$P = H \ominus FH = \{g \in H : \text{for each } f \in H, \langle g, Ff \rangle = 0\} \quad (3.0.13)$$

The rank of the grammian of a set of vectors is less than or equal to the dimension of the span of the vectors. Since the dimension of P equals $\deg_V(F)$, the rank of the grammian of a set of vectors contained in P is less than or equal to $\deg_V(F)$. We claim the Pick matrix associated to the problem, $W \cdot K$, equals the grammian of the vectors $\{(1 - M_F M_F^*)k_{\lambda_i}\}_1^N \subset P$, i.e.

$$W \cdot K = ((1 - \omega_i \overline{\omega_j})K_{ij}) = (\langle (1 - M_F M_F^*)k_{\lambda_i}, (1 - M_F M_F^*)k_{\lambda_j} \rangle). \quad (3.0.14)$$

That $W \cdot K$ equals the above grammian implies that

$$\text{rank}(W \cdot K) \leq \dim(\text{span}\{(I - M_F M_F^*)k_{\lambda_i}\}) \leq \dim(\text{image}(I - M_F M_F^*)) = \deg_V(F).$$

The following calculation establishes equation 3.0.14.

$$\begin{aligned} & \langle (I - M_F M_F^*)k_{\lambda_j}, (I - M_F M_F^*)k_{\lambda_i} \rangle = \langle (I - M_F M_F^*)k_{\lambda_j}, k_{\lambda_i} \rangle = \\ & \langle k_{\lambda_j}, k_{\lambda_i} \rangle - \langle M_F^* k_{\lambda_j}, M_F^* k_{\lambda_i} \rangle = \langle k_{\lambda_j}, k_{\lambda_i} \rangle - \langle \overline{F(\lambda_j)} k_{\lambda_j}, \overline{F(\lambda_i)} k_{\lambda_i} \rangle = \\ & \langle k_{\lambda_j}, k_{\lambda_i} \rangle - \omega_i \overline{\omega_j} \langle k_{\lambda_j}, k_{\lambda_i} \rangle = (1 - \omega_i \overline{\omega_j}) K_{i,j} \end{aligned}$$

The first equality comes from the fact that $I - M_F M_F^*$ is a projection. The equalities in the second line come from that each evaluation functional k_λ is the eigenvector of the multiplication operator M_F with eigenvalue $\overline{F(\lambda)}$, that is $M_F^* k_{\lambda_i} = \overline{F(\lambda_i)} k_{\lambda_i} = \overline{\omega_i} k_{\lambda_i}$. \square

Chapter 4

Proof of main results

4.1 Proof of Theorem B

The following theorem implies Theorem **B** from the introduction. Recall that $H_1^\infty(V)$ denotes the unit ball of $H^\infty(V)$.

Theorem 4.1.1. *Let $n, N > 0$, let $V \subset \mathbb{C}^n$ be an irreducible 1-dimensional inner variety and let $\lambda_1, \dots, \lambda_N \in V \cap \mathbb{D}^n$ be distinct. If F is a rational inner function on $V \cap \mathbb{D}^n$ with $\deg_V(F) < N$ and $G \in H_1^\infty(V)$ satisfies $G(\lambda_i) = F(\lambda_i)$ for each $i = 1, \dots, N$, then $G = F$ on $V \cap \mathbb{D}^n$.*

PROOF: Fix $\lambda_1, \dots, \lambda_N \in V \cap \mathbb{D}^n$, a rational inner function F on $V \cap \mathbb{D}^n$ with $\deg_V(F) < N$ and let $\omega_i = F(\lambda_i)$. Let $H = H^2(\mu)$ be a Hilbert function space on $V \cap \mathbb{D}^n$, the existence of which is guaranteed by lemma 3.0.11. By Lemma 3.0.12 the N by N Pick matrix corresponding to the problem, $W \cdot K = ((1 - w_i \overline{w_j})K_{ij})$, has rank less than or equal to $\deg_V(F)$. Since, by assumption, $\deg_V(F) < N$ it follows that $W \cdot K$ is singular and Theorem 3.0.4 implies that F is the unique solution in $Mult_1(H)$ to the Pick problem with data $\lambda_1, \dots, \lambda_N$ and $\omega_1, \dots, \omega_N$. Since Lemma 3.0.11 implies that $H_1^\infty(V) \subset Mult_1(H)$, if $G \in H_1^\infty(V)$ satisfies $G(\lambda_i) = F(\lambda_i)$ for each $i = 1, \dots, N$, then G is also a solution and thus, $G = F$ on $V \cap \mathbb{D}^n$. \square

Theorem **B** follows from Theorem 4.1.1, since if F is the restriction of a rational inner f on D^n to $V \cap \mathbb{D}^n$, then F is rational, inner and $\deg_V(F) = \deg_V(f)$.

4.2 Proof of Theorem D

We use the following technical result in the proof of Theorem D. For a polynomial q , \tilde{q} denotes the reflection of q through \mathbb{T}^n given by formula 2.1.2.

Lemma 4.2.1. *(Agler, McCarthy and Stankus, [7]) If $V \subset \mathbb{C}^n$ is an inner variety, then there exists a polynomial p such that $V = Z_p$ and $\tilde{p} = p$.*

The idea of Theorem D is that for a rational inner function $f = \frac{\tilde{q}}{q}$, if the degree of a polynomial p is less than the degree of q in each variable, then for ϵ small the function $f_\epsilon = \frac{\widetilde{q+\epsilon p}}{q+\epsilon p}$ equals f on the zero set of p and is inner.

PROOF: Fix a rational inner function f and assume that it equals $z^d \frac{\tilde{q}}{q}$. Fix $V = Z_p$ an inner variety such that the degree of p is less than or equal to the degree of f in each variable. Since V is inner, we may assume that $\tilde{p} = p$.

Claim: for ϵ sufficiently small, the following function is inner and equals f on V .

$$f_\epsilon = z^d \frac{\widetilde{(q + \epsilon p)}}{q + \epsilon p}$$

Proof of claim: If $q + \epsilon p$ does not vanish on \mathbb{D}^n , then f_ϵ is inner. If f has no singular points on \mathbb{T}^n , then q does not vanish on $\overline{\mathbb{D}^n}$ and $\delta = \inf_{\overline{\mathbb{D}^n}} |q| > 0$. Choosing $\epsilon < \frac{\|p\|_\infty}{\delta}$ guarantees that $q + \epsilon p \neq 0$ on \mathbb{D}^n . If $\widetilde{(q + \epsilon p)}$ equals \tilde{q} on the zero set of p , and this condition is satisfied precisely when the degree of q is greater than or equal to the degree of p in each variable, then f_ϵ equals f on V .

Assume that $f = \frac{q}{\tilde{q}}$ and that $n\text{-deg}(q) = (d_1, \dots, d_n) = n\text{-deg}(\tilde{q})$. By Rudin's Theorem 2.1.4 and the fact that $\tilde{p} = p$, it follows that

$$f_\epsilon = \frac{\widetilde{(q + \epsilon p)}}{q + \epsilon p} = \frac{\tilde{q} + \epsilon p}{q + \epsilon p}.$$

In the general case let $f = z^d \frac{\tilde{q}}{q}$, let $s = n\text{-deg}(q)$, let $m = n\text{-deg}(p)$ and let

$$f_\epsilon = z^{d+s} \frac{\overline{q(\frac{1}{z})} + \epsilon \overline{p(\frac{1}{z})}}{q + \epsilon p} = z^d \frac{\tilde{q} + \epsilon z^{s-m} \tilde{p}}{q + \epsilon p} = z^d \frac{\tilde{q} + \epsilon z^{s-m} p}{q + \epsilon p}.$$

It follows that $f_\epsilon = f$ on V and Rudin's Theorem 2.1.4 implies that f_ϵ is inner. \square

4.3 Proof of Theorems E and A

Fix $n, N > 1$, $\lambda_1, \dots, \lambda_N \in \mathbb{D}$ distinct and $\gamma \in \mathbb{D}^{n-1}$. Suppose that $f(z_1, \dots, z_n) = F(z_1)$ is a rational inner function of degree less than N .

We want to show that the Pick problem with data $(\lambda_1, \gamma), \dots, (\lambda_N, \gamma)$ and $f(\lambda_1, \gamma), \dots, f(\lambda_N, \gamma)$ has a unique solution on \mathbb{D}^n . We consider the problem as a generalized Pick problem on $H^2(\mathbb{D}^n)$. Recall that the kernel function for $H^2(\mathbb{D}^n)$ associated with each $\lambda \in \mathbb{D}^2$ is the Szego kernel

$$k_{(\lambda, \gamma)}(z, w) = \frac{1}{(1 - \bar{\lambda}^1 z) \prod_{i=2}^N (1 - \bar{\gamma}_i z_i)}.$$

Let $\omega_i = f(\lambda_i, \gamma)$. The Pick matrix associated the the problem, $W \cdot K$, is given by the following formula

$$W \cdot K = \left(\frac{1 - \overline{f(\lambda_i, \gamma)} f(\lambda_j, \gamma)}{(1 - \bar{\lambda}_i \lambda_j) \prod_{i=2}^N (1 - \bar{\gamma}_i \gamma_j)} \right)$$

By applying the appropriate automorphism to \mathbb{D}^n in the variables z_2, \dots, z_n , we may assume that $\gamma = (0, \dots, 0)$. This implies that the $W \cdot K$ is given by the following formula

$$W \cdot K = \left(\frac{1 - \overline{F(\lambda_i)} F(\lambda_j)}{(1 - \bar{\lambda}_i \lambda_j)} \right).$$

Since F is a rational inner function on \mathbb{D} of degree strictly less than N , the implication $\mathbf{d} \rightarrow \mathbf{c}$ in Pick's Theorem 2.2.2 implies that $W \cdot K$ is singular. Since $W \cdot K$ is singular, Theorem 3.0.4 implies that the solution to the problem is unique. \square

Theorem **A** is an immediate corollary.

4.4 Proof of Theorem F

Our proof is a slight modification of an argument from [6].

PROOF: Let V be a 1-dimensional inner variety with rank m and f a rational inner function with $n\text{-deg}(f) = (d_1, \dots, d_n)$. Since V is 1-dimensional, we can assume that no point of V has more than one coordinate equal to 0 and that each point of V that has exactly one 0 coordinate is regular.

Consider the case when $f(z_1, \dots, z_n) = z_1^{d_1} \cdots z_n^{d_n}$. At each of the m_k points in V with k th coordinate equal to 0, f has a zero of multiplicity d_k . Thus, $\deg_V(z_1^{d_1} \cdots z_n^{d_n}) = d \cdot m = d_1 m_1 + \dots + d_n m_n$.

For a general inner $f(z) = \tau z^{d+s} \frac{q(\frac{1}{z})}{q(z)} = \tau z^d \frac{\tilde{q}}{q}$, where $s = n\text{-deg}(q)$, normalize q so that $q(0) = 1$ and let

$$q(z) = 1 + Q(z) \text{ and } q_r(z) = q(rz) = 1 + Q(rz).$$

Since $Q(0, \dots, 0) = 0$, one can factor a coordinate function out of each term of Q

$$q_r(z_1, \dots, z_n) = 1 + r^n z_1 \cdots z_n R(rz_1, \dots, rz_n) \text{ and } f_r(z) = \frac{z^d + \tilde{R}(z)}{1 + r^n z R(rz)}.$$

As r increases from 0 to 1, the function f_r changes continuously from z^d to $f(z)$. The zeros of f_r are bounded away from $V \cap \mathbb{T}^n$ if and only if the zeros of \tilde{q} are. Since \tilde{q} is the reflection of q through \mathbb{T}^n , for a $\rho \in \mathbb{T}^n$ we have that $\widetilde{q(\rho)} = 0$ if and only if $q(\rho) = 0$. This implies that the zeros of f_r are bounded away from $V \cap \mathbb{T}^n$ if and only if the zeros of q , the singular points of f , are bounded away from $V \cap \mathbb{T}^n$. Thus, the number of zeros of f equals $d \cdot m$ whenever f does not have a singular point on $V \cap \mathbb{T}^n$. This establishes the case of equality.

Suppose f has a singular point $\rho \in V \cap \mathbb{T}^n$, i.e. $q(\rho) = 0 = \tilde{q}(\rho)$. If $s(r)$ is a zero of f_r on $V \cap \mathbb{D}^n$ that approaches ρ as r goes to 1, then $1/\overline{s(r)}$ is a singular point of f_r on V that approaches ρ from outside of \mathbb{D}^n . That $1/\overline{s(r)} \in V$ follows from that V is symmetric with respect to \mathbb{T}^n , Lemma 4.2.1. Since $V \cap \mathbb{D}^n$ is 1-dimensional, the restriction of f to $V \cap \mathbb{D}^n$, denoted F , is locally a function of one variable. Since F is bounded in a neighborhood of ρ , F has a removable singularity at ρ and the number of zeros of F on $V \cap \mathbb{D}^n$ is less than $d \cdot m$ by the multiplicity of the cancellation of the removable singularity of F at ρ . \square

Chapter 5

Theorem C

The following is a refinement of Theorem C. The proof is from [31].

Theorem 5.0.1. (Scheinker [31]) Fix $n, N \geq 1$. There exists a one dimensional inner variety $V \subset \mathbb{C}^n$ and $\lambda_1, \dots, \lambda_{N^n} \in V \cap \mathbb{D}^n$ with the following property. If f is a rational inner function of degree less than N and $g \in \mathcal{S}(\mathbb{D}^n)$ satisfies $g(\lambda_i) = f(\lambda_i)$ for each $i = 1, \dots, N^n$, then $g = f$ on \mathbb{D}^n . In particular, V can be taken to be the union of any $M = N^{n-1}$ analytic discs of the following form. For $r = 2, \dots, n$ let $\tau_1^r, \dots, \tau_N^r \in \mathbb{T}$ be distinct and let D_1, \dots, D_M be distinct analytic discs given by

$$D_k : \mathbb{D} \rightarrow \mathbb{D}^n \text{ with } D_k(z) = (z, \tau_{i_{2,k}}^2 z, \dots, \tau_{i_{n,k}}^n z). \quad (5.0.2)$$

Before proving Theorem 5.0.1, we introduce some definitions and a technical lemma.

Definition 5.0.3. For each $n, N \geq 1$ let $\mathcal{I}_N(\mathbb{D}^n)$ denote the set of rational inner functions on \mathbb{D}^n of degree strictly less than N .

Definition 5.0.4. For $n > m \geq 1$, we call an analytic function $E : \mathbb{D}^m \rightarrow \mathbb{D}^n$ an **analytic m -disc**. We use $E(\mathbb{D}^m)$ to denote the range of E .

Definition 5.0.5. Let $f \in \mathcal{S}(\mathbb{D}^n)$, $\tau \in \mathbb{T}$ and E be an analytic $(n-1)$ -disc

$$\text{given by } E : \mathbb{D}^{n-1} \rightarrow \mathbb{D}^n \text{ with } E(z_1, \dots, z_{n-1}) = (z_1, \dots, z_{n-1}, \tau z_1) \quad (5.0.6)$$

We define f_E as follows, $f_E(z_1, \dots, z_{n-1}) = f(E(z_1, \dots, z_{n-1}))$.

The function f_E is in $\mathcal{S}(\mathbb{D}^{n-1})$ and parametrizes the restriction of f to E .

Lemma 5.0.7. *For $n \geq m > 1$ and $\tau \in \mathbb{T}$, if $f \in \mathcal{I}_N(\mathbb{D}^n)$ and $E(z_1, \dots, z_{n-1}) = (z_1, \dots, z_{n-1}, \tau z_1)$, then $f_E \in \mathcal{I}_N(\mathbb{D}^{n-1})$.*

PROOF: Since f is inner, the denominator of f has a non-zero constant term and Rudin's Theorem 2.1.4 implies that f can be written as follows.

$$f(z_1, \dots, z_n) = \tau \frac{z^{d_1} \cdots z^{d_n} + r_0(z_1, \dots, z_n)}{1 + q_0(z_1, \dots, z_n)} \text{ for some } \tau \in \mathbb{T} \quad (5.0.8)$$

where the degree of f equals $d_1 + \dots + d_n$ and each term of r_0 has degree less than or equal to d_i in each z_i and less than d_i in at least one z_i . The corollary follows from substituting $f(z_1, \dots, z_{n-1}, \tau z_1)$ into equation 5.0.8. \square

5.1 A result on \mathbb{D}^2

We will use the case $n = 2$ of Lemma 5.1.2 in the proof of Theorem 5.0.1.

We will use the following technical lemma to prove the case $n = 2$ of Lemma 5.1.2. We use $B_\epsilon(z)$ to denote the ball of radius ϵ around z and we use $m_{t,a}(z)$ to denote the automorphism of \mathbb{D} given by $t \frac{z-a}{1-\bar{a}z}$.

Lemma 5.1.1. *Let $\tau_1, \dots, \tau_N \in \mathbb{T}$ be distinct and E_1, \dots, E_M be analytic discs*

$$\text{given by } E_i : \mathbb{D} \rightarrow \mathbb{D}^2 \text{ with } E_i(z) = (z, \tau_i z).$$

There exist $\tau \in \mathbb{T}$ and $\epsilon > 0$ such that for every $t \in B_\epsilon(\tau) \cap \mathbb{T}$ and $a \in B_\epsilon(0)$, the image of the analytic disc $C_{m_{t,a}}$ given by

$$C_{m_{t,a}} : \mathbb{D} \rightarrow \mathbb{D}^2 \text{ with } C_{m_{t,a}}(z) = (z, m_{t,a}(z))$$

intersects each $E_i(\mathbb{D})$ at a distinct point $(r_i, \tau_i r_i)$.

Furthermore, C , defined as the union of every $C_{m_{t,a}}(\mathbb{D})$ over $t \in B_\epsilon(\tau) \cap \mathbb{T}$ and $a \in B_\epsilon(0)$ is a set of uniqueness for analytic functions on \mathbb{D}^2 .

PROOF: Fix $\tau \in \mathbb{T}$ such that $\tau \neq \tau_i$ for each i and let $\epsilon_1 > 0$ be small enough so that for each i , $\tau_i \notin B_{\epsilon_1}(\tau)$. Let $C_m = C_{m_{\tau,a}}$, with a to be specified later. The sets $C_m(\mathbb{D})$ and $E_i(\mathbb{D})$ intersect if and only if one of the roots of the equation $\tau_i z = m_{t,a}(z)$ lies in \mathbb{D} . Let r_i and s_i denote the roots. If $a = 0$ then $r_i = 0$ and $s_i = \infty$ for each i . For sufficiently small $\epsilon_1 > \epsilon > 0$, if a is perturbed away from zero and remains in $B_\epsilon(0)$, then each of the roots r_i becomes non-zero and stays in \mathbb{D} . That the roots r_1, \dots, r_M are distinct follows from the fact that they are non-zero and that $\tau_i \neq \tau_j$.

To see that C is a set of uniqueness let f be analytic on \mathbb{D}^2 and suppose that $f|_C = 0$. Fix $x \in \mathbb{D}$, $a \in B_\epsilon(0)$ and let

$$A_x = \{(x, m_{t,a}(x)) \in \mathbb{D}^2 : t \in B_\epsilon(\tau) \cap \mathbb{T}\} \subset C.$$

Since $f(x, z)$ is an analytic function in the single variable z and vanishes on the arc A_x , $f = 0$. Since $f(x, \cdot) = 0$ for each $x \in \mathbb{D}$, $f = 0$ on \mathbb{D}^2 . \square

If Theorem 5.0.1 holds for n then the following lemma immediately follows for n . We prove the following lemma for $n = 2$.

Lemma 5.1.2. *Fix N , let $f \in \mathcal{I}_N(\mathbb{D}^n)$, let $\tau_1, \dots, \tau_N \in \mathbb{T}$ be distinct and let E_1, \dots, E_N be analytic $(n-1)$ -discs given by*

$$E_k : \mathbb{D}^{n-1} \rightarrow \mathbb{D}^n \text{ with } E_k(z_1, \dots, z_{n-1}) = (z_1, \dots, z_{n-1}, \tau_k z_1)$$

If $g \in \mathcal{S}(\mathbb{D}^n)$ satisfies $g = f$ on each $E_k(\mathbb{D}^{n-1})$, then $g = f$ on \mathbb{D}^n .

PROOF OF LEMMA 5.1.2(case $n=2$): By Lemma 5.1.1 there exists an analytic disc $C_m(\mathbb{D})$ that intersects each of $E_1(\mathbb{D}), \dots, E_N(\mathbb{D})$ at a distinct point $R_i = (r_i, \tau_i r_i)$. Fix $f \in \mathcal{I}_N(\mathbb{D}^2)$ and assume that $g \in \mathcal{S}(\mathbb{D}^2)$ satisfies $g = f$ on each $E_k(\mathbb{D}^{n-1})$. Let $f_m = f_{C_m}$ and $g_m = g_{C_m}$. Notice that $g_m \in \mathcal{S}(\mathbb{D})$ and by Lemma 5.0.7, $f_m \in \mathcal{I}_N(\mathbb{D})$. It follows that for $i = 1, \dots, N$,

$$g_m(r_i) = g(D_i(r_i)) = g(r_i, \tau_i r_i) = f(r_i, \tau_i r_i) = f(D_i(r_i)) = f_m(r_i)$$

Since $g_m(r_i) = f_m(r_i)$ for $i = 1, \dots, N$, Pick's Theorem implies that $g_m = f_m$ on \mathbb{D} and thus, $g = f$ on each $C_m(\mathbb{D})$. By Lemma 5.1.1, the discs $C_m(\mathbb{D})$ sweep out a set of uniqueness and thus, $g = f$ on \mathbb{D}^2 . \square

5.2 Proof of Theorem 5.0.2

In this section we use induction to prove Theorem 5.0.1. The case $n = 1$ is Pick's Theorem. Fix $n \geq 2$ and suppose that Theorem 5.0.1 holds for each $m < n$. We show that Theorem 5.0.1 holds for n in 3 steps.

In the first step we fix N , fix a set of analytic $(n-1)$ -discs E_1, \dots, E_N , and fix a set of N^{n-1} points $\{\lambda_{j_s}\} \subset \mathbb{D}^{n-1}$ to which we will imply the induction hypothesis. We lift the set $\{\lambda_{j_s}\}$ to the set of N^n points $\{\lambda_{k_{j_s}}\}$ in \mathbb{D}^n by letting $\lambda_{k_{j_s}} = E_k(\lambda_{j_s})$.

In the second step we apply the induction hypothesis to show that for each $f \in \mathcal{I}_N(\mathbb{D}^n)$, if $g \in \mathcal{S}(\mathbb{D}^n)$ satisfies $g(\lambda_{k_{j_s}}) = f(\lambda_{k_{j_s}})$ for k, j, s , then $g = f$ on E_1, \dots, E_N .

In the third step we use Lemma 5.1.2 (which holds for $n-1$ by the induction hypothesis) to show that since g equals f on E_1, \dots, E_N , $g = f$ on \mathbb{D}^n .

STEP 1: Fix N and let $\tau_1, \dots, \tau_N \in \mathbb{T}$ be distinct and E_1, \dots, E_N be analytic $(n-1)$ -discs given by

$$E_k : \mathbb{D}^{n-1} \rightarrow \mathbb{D}^n \text{ with } E_k = (z_1, \dots, z_m, \tau_i z_1).$$

Let $M = N^{n-2}$. For each $r = 2, \dots, n-1$ let $\tau_1^r, \dots, \tau_N^r \in \mathbb{T}$ be distinct. Let $D_1, \dots, D_{N^{n-2}}$ be the N^{n-2} analytic discs given by

$$D_j : \mathbb{D} \rightarrow \mathbb{D}^{n-1} \text{ with } D_j(z) = (z, \tau_{i_2,j}^2 z, \dots, \tau_{i_{n-1},j}^{n-1} z).$$

For each j , let $\lambda_{j_1}, \dots, \lambda_{j_N} \in D_j(\mathbb{D}) \subset \mathbb{D}^{n-1}$ be distinct and lift each point λ_{j_s} to \mathbb{D}^n , N times, by letting $\lambda_{k_{j_s}} = E_k(\lambda_{j_s})$.

STEP 2: Fix $f \in \mathcal{I}_N(\mathbb{D}^n)$ and suppose $g \in \mathcal{S}(\mathbb{D}^n)$ satisfies $g(\lambda_{k_{j_s}}) = f(\lambda_{k_{j_s}})$ for each k, j, s . For each k , let $f_k = f_{E_k}$ and $g_k = g_{E_k}$. Notice that $g_k \in \mathcal{S}(\mathbb{D}^{n-1})$ and by Lemma 5.0.7, $f_k \in \mathcal{I}_N(\mathbb{D}^{n-1})$. It follows that for $k = 1, \dots, N$, $j = 1, \dots, N^{n-2}$ and $s = 1, \dots, N$,

$$g_k(\lambda_{j,s}) = g(E_k(\lambda_{j,s})) = g(\lambda_{k,j,s}) = f(\lambda_{k,j,s}) = f(E_k(\lambda_{l,s})) = f_k(\lambda_{j,s}).$$

Since for each k , $g_k(\lambda_{j_s}) = f_k(\lambda_{j_s})$ for each j and s , the induction hypothesis implies that $g_k = f_k$ on \mathbb{D}^{n-1} . Thus, $g = f$ on each E_k .

STEP 3: If $n = 2$, then case $n = 2$ of Lemma 5.1.2 implies that $g = f$ on \mathbb{D}^2 .
Suppose $n \geq 3$.

For $\rho \in \mathbb{T}$ let C_ρ be the analytic (n-1)-disc given by

$$C_\rho : \mathbb{D}^{n-1} \rightarrow \mathbb{D}^n \text{ with } C_\rho(z_1, \dots, z_{n-1}) = (z_1, \dots, z_{n-2}, z_{n-1}, \bar{\rho}z_{n-1}).$$

For each ρ , let $f_\rho = f_{C_\rho}$, $g_\rho = g_{C_\rho}$. Let $I_{\rho,k} : \mathbb{D}^{n-2} \rightarrow \mathbb{D}^n$ and $H_{\rho,k} : \mathbb{D}^{n-2} \rightarrow \mathbb{D}^{n-1}$ be analytic (n-2)-discs such that

$$I_{\rho,k}(\mathbb{D}^{n-2}) = C_\rho(\mathbb{D}^{n-1}) \cap E_k(\mathbb{D}^{n-1}) \text{ and } H_{\rho,k}(\mathbb{D}^{n-2}) = C_\rho^{-1}(I_{\rho,k}(\mathbb{D}^{n-2})).$$

Since $g = f$ on $I_{\rho,1}(\mathbb{D}^{n-2}), \dots, I_{\rho,N}(\mathbb{D}^{n-2})$ it follows that $g_\rho = f_\rho$ on $H_{\rho,1}(\mathbb{D}^{n-2}), \dots, H_{\rho,N}(\mathbb{D}^{n-2})$ and Lemma 5.1.2 (which holds for $n-1$ by the induction hypothesis) implies that $g_\rho = f_\rho$. Thus, $g = f$ on C_ρ and since $\mathbb{D}^n = \bigcup_{\rho \in \mathbb{T}} C_\rho$, it follows that $g = f$ on \mathbb{D}^n . \square

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