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Boundary Characterization of a Smooth Domain with Non-Compact Automorphism Group

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Boundary Characterization of a Smooth Domain with Non-Compact Automorphism Group

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by

Bradley Gray Thomas

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ABSTRACT OF THE DISSERTATION

Boundary Characterization of a Smooth Domain with Non-Compact Automorphism Group

by

Bradley Gray Thomas

Doctor of Philosophy, Graduate Program in Mathematics
University of California, Riverside, June 2012
Dr. Bun Wong , Chairperson

One of the most important problems in the field of several complex variables is the Greene-Krantz conjecture:

Conjecture *Let* $\Omega \subset \mathbb{C}^n$ *be a smoothly bounded domain with non-compact automorphism group. Then the* $\partial\Omega$ *is of finite type at any boundary orbit accumulation point.*

The purpose of this dissertation is to prove a result that suppports the truthfulness of this conjecture:

Theorem *Let* $\Omega \subset \mathbb{C}^n$ *be a smoothly bounded convex domain. Suppose there exists a point* $p \in \Omega$ *and a sequence* $\{\phi_j\} \subset Aut(\Omega)$ *such that* $\phi_j(p) \to q \in \partial\Omega$ *non-tangentially. Furthermore, suppose Condition LTW holds. Then* $\partial\Omega$ *is variety-free at* q.

Contents

Chapter 1

Introduction

In any category, it is natural to ask which objects in it are equivalent. This is no less true in the category consisting of bounded domanins in $\mathbb{C}^n$, with morphims being the holomorphic maps between them. In one dimension, the task of classification is already complete: Any simply connected bounded domain in the complex plane is biholomorphic to the unit disc. This incredible result is given by the Riemann Mapping Theorem. It is natural to ask, then, if such a result holds for bounded domains in $\mathbb{C}^n$. Unfortunately, the answer is no. Hence, if any classification is going to be obtained, the set of bounded domains in $\mathbb{C}^n$ must be restricted to a smaller collection satisfying some additional property. One such collection is the bounded domains in $\mathbb{C}^n$ with a non-compact automorphism group. Can any sort of classification be obtained in this case? One important tool needed in order classify such domains is the Greene-Krantz conjecture:

Conjecture 1.0.1 (Greene-Krantz) *Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded domain with non-compact automorphism group. Then $\partial\Omega$ is of finite type at any boundary orbit accumulation point.*

The purpose of this dissertation is to prove a result that supports the truthfulness of this conjecture, namely that the $\partial\Omega$ is variety-free at any boundary orbit accumulation point. This is a weaker conclusion, since finite type implies variety-free. Here is the exact statement of the result, with the assumption of the truth of Condition LTW:

Theorem 1.0.2 (Lee-Thomas-Wong) *Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded convex domain. Suppose there exists a point $p \in \Omega$ and a sequence $\{\phi_j\} \subset Aut(\Omega)$ such that $\phi_j(p) \to q \in \partial\Omega$ non-tangentially. Furthermore, assume Condition LTW holds. Then, S_q is trivial and hence $\partial\Omega$ is variety-free at q.*

Condition LTW is a techincal assumption. Please see page 7o for more information. Here is what follows:

In Chapter 2, a general study of the bounded domains in $\mathbb{C}^n$ with a non-compact automorphism group, with an emphasis on any known or potential results that provide a classification of such domains. The purpose of this chapter is to provide both a general framework and a common set of terminology to be used throughout the latter part of the document.

In Chapter 3, the definition of the invariant metrics and measures will be given, along with many of their important properties. These objects constitute one of the main tools that will be needed in the proof of the main theorem.

In Chapter 4, the main result will be proven. Leading up to it will be a sequence of definitions, lemmas, propositions, and corollaries that will be needed to prove this result. The basic idea behind the proof is Poincaré's Theorem, which states that the ball and the polydisc are not biholomorphic. In particular, assuming that the domain Ω is not variety-free at a boundary point q, the $\partial\Omega$ will be geometrically flat along this variety. Hence, near a strongly (pseudo)convex boundary point, the domain Ω is like a ball, whereas, near a flat boundary point, the domain Ω is like a polydisc. The non-compactness of the automorphism group allows one to mediate between these two types of boundary points, bringing forth a contradiction. The fact that the domain Ω near a strongly (pseudo)convex and flat boundary point is like a ball and polydisc, respectively, is codified precisely by the quotient of the Carathéodory and Kobayashi measures. A significant portion of this dissertation is a joint work with Dr. Lina Lee and Professor Bun Wong.

Chapter 2

Background

2.1 Domains with Non-Compact Automorphism Group

From now on, for brevity's sake, a bounded domain in $\mathbb{C}^n$ will be denoted by Ω and the automorphism group of Ω will be denoted by $\mathrm{Aut}(\Omega)$. Note that an element f of $\mathrm{Aut}(\Omega)$ is a biholomorphic map of Ω onto itself. As its name makes abundantly clear, $\mathrm{Aut}(\Omega)$ is indeed a group, the operation being function composition. In addition to being a group, $\mathrm{Aut}(\Omega)$ is also a topological group, the topology being given by the compact-open topology (since Ω, being a subset of $\mathbb{C}^n$, is equipped with a metric, the compact-open topology of $\mathrm{Aut}(\Omega)$ coincides with the topology of uniform convergence on compact sets). Furthermore, H. Cartan showed that $\mathrm{Aut}(\Omega)$ is a Lie Group. Before stating what is precisely meant by $\mathrm{Aut}(\Omega)$ being non-compact, a couple of definitions are required.

Definition 2.1.1 *Let G be a topological group and X a (Hausdorff) topological space. G acts upon X if there exists a continuous map $\varphi : G \times X \to X$, $\varphi(g,x) = gx$, such that $\varphi(e,x) = ex = x\ \forall x \in X$ and $\varphi(gg',x) = \varphi(g,\varphi(g',x))\ \forall\, g,g' \in G,\ x \in X$*

Definition 2.1.2 *Let G and X be as in the previous definition. The orbit of $x \in X$ under the action of G is $\{\varphi(g,x)\,|\,g \in G\}$.*

Definition 2.1.3 *A map $f : \Omega \longrightarrow \widetilde{\Omega}$, $\Omega \subset \mathbb{C}^n$, $\widetilde{\Omega} \subset \mathbb{C}^m$ is proper if, for any compact set $\widetilde{K} \subset \widetilde{\Omega}$, the set $f^{-1}(\widetilde{K})$ is compact in Ω.*

Note that this is equivalent to the following: For any sequence $\{z_j\} \subset \Omega$ which has no limit point in Ω, the sequence $\{f(z_j)\}$ has no limit point in $\widetilde{\Omega}$.

Definition 2.1.4 *If G and X are as in Definition 2.1.1 and are locally compact, then the action of G on X is proper if the map $G \times X \longrightarrow X \times X$, defined by $(g,x) \longmapsto (\varphi(g,x),x)$ is proper.*

Definition 2.1.5 *Aut(Ω) is non-compact if there exists a sequence $\{\phi_j\} \subset$ Aut(Ω) such that $\{\phi_j\}$ is divergent and $\{\phi_j\}$ has no convergent subsequence.*

Note that, since the action of Aut(Ω) on Ω is proper, for all $p \in \Omega$, $\phi_j(p) \longrightarrow q \in \partial\Omega$ as $j \longrightarrow \infty$, i.e. $\phi := \lim \phi_j$ maps Ω into $\partial\Omega$ (this can also be seen by looking at Montel's Theorem). Therefore, the orbit of any point $p \in \Omega$ is non-compact. The point q is called a *boundary orbit accumulation point* for the action of Aut(Ω) on Ω. More precisely, $q \in \partial\Omega$ is a boundary orbit accumulation point for the action of Aut(Ω) on Ω if there exists a point $p \in \Omega$ and a sequence $\{\phi_j\} \subset$ Aut(Ω) such that $\phi_j(p) \longrightarrow q$ as $j \longrightarrow \infty$.

Conversely, assume $\Omega \subset \mathbb{C}^n$ is bounded, with $q \in \partial\Omega$ a boundary orbit accumulation point.

Claim 2.1.6 *Aut(Ω) is non-compact.*

Proof. Assume that Aut(Ω) is compact. Then, for any sequence $\{\psi_j\} \subset \mathrm{Aut}(\Omega)$, there exists a subsequence $\{\psi_{j_\nu}\} \subset \{\psi_j\}$ such that $\psi_{j_\nu} \longrightarrow \psi \in \mathrm{Aut}(\Omega)$. Consider the sequence $\{\phi_j\} \subset \mathrm{Aut}(\Omega)$; by assumption, $\exists \{\phi_{j_\nu}\} \subset \{\phi_j\}$ such that $\phi_{j_\nu} \longrightarrow \phi \in \mathrm{Aut}(\Omega)$ as $\nu \longrightarrow \infty$. In particular, $\phi(p) = q \in \Omega \Rightarrow q \in \Omega \cap \partial\Omega$, which contradicts the fact that Ω is open. Therefore, Aut(Ω) is compact. ■

Therefore, there is no loss in assuming that Aut(Ω) non-compact means that at least one orbit of the action of Aut(Ω) on Ω is non-compact.

Now, here are some examples of some bounded domains with non-compact automorphism groups, along with an explanation why.

Example 2.1.7 *The unit disc in* $\mathbb{C}$

Let $\Delta := \{z \in \mathbb{C} \,|\, |z| < 1\}$ be the unit disc, where $|\cdot|$ denotes the Euclidean norm of z in $\mathbb{C}$. Then,

$$\mathrm{Aut}(\Delta) = \left\{ e^{i\theta}\frac{z-a}{1-\bar{a}z} \,\middle|\, a \in \Delta,\ \theta \in [0, 2\pi] \right\}.$$

Why is Aut(Δ) non-compact? To determine the answer to this question, a proposition is needed.

Proposition 2.1.8 *Let Ω be a bounded domain in* $\mathbb{C}^n$ *with a transitive automorphism group, i.e. Ω is homogeneous. The Aut(Ω) is non-compact.*

Proof. Let Ω be a bounded domain in $\mathbb{C}^n$ with a transitive automorphism group, i.e. given any two elements $a, b \in \Omega$, $\exists\, \phi \in \mathrm{Aut}(\Omega)$ such that $\phi(a) = b$. Let $z \in \Omega$. By the transitivity of Aut(Ω), the orbit of z is $\{w \in \Omega \,|\, w = \phi(z),\ \phi \in \mathrm{Aut}(\Omega)\} = \Omega$. Since Ω is open, it is not compact ⇒ the orbit of z is non-compact ⇒ Aut(Ω) is non-compact. ■

How, then, can this proposition be used in showing that $\mathrm{Aut}(\Delta)$ is non-compact? It can be invoked due to the fact that $\mathrm{Aut}(\Delta)$ is transitive: Given any two elements $a, b \in \Delta$, let

$$\phi_a(z) := \frac{z-a}{1-\bar{a}z}, \quad \phi_{-b}(z) := \frac{z+b}{1+\bar{b}z}.$$

Then, both ϕ_a and ϕ_{-b} are in $\mathrm{Aut}(\Delta)$. In fact, $\phi_{-b} \circ \phi_a(a) = \phi_{-b}(0) = b \Rightarrow \mathrm{Aut}(\Delta)$ is non-compact by the previous proposition.

Example 2.1.9 *The unit ball in $\mathbb{C}^n$.*

Let $\mathbb{B}_n := \left\{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n \mid \|z\| := \sum |z_k|^2 \ < 1 \right\}$ denote the unit ball. To write down the elements of $\mathrm{Aut}(\mathbb{B}_n)$ explicitly, recall that

$$\mathrm{U}(n) := \left\{ A \in M_n(\mathbb{C}) \mid A\overline{A}^t = \overline{A}^t A = I \right\}$$

is the (Lie) group (under matrix multiplication) of unitary matrices. Importantly, the elements of $\mathrm{U}(n)$ preserve the Euclidean norm: They correspond to complex rotations. Furthermore, consider the collection of maps $\{\phi_a\}$, where

$$\phi_a(z_1, \ldots, z_n) := \left(\frac{z_1 - a}{1 - \bar{a}z_1}, \frac{\sqrt{1-|a|^2}z_2}{1-\bar{a}z_1}, \ldots, \frac{\sqrt{1-|a|^2}z_n}{1-\bar{a}z_1} \right), |a| < 1$$

Note that $\phi(a, 0, \ldots, 0) = (0, \ldots, 0)$ and that ϕ_a is an automorphism of the ball. Therefore, $\mathrm{Aut}(\mathbb{B}_n)$ is the group generated by $\mathrm{U}(n)$ and $\{\phi_a\}$, i.e. every automorphism of the ball is a composition of elements from $\mathrm{U}(n)$ or $\{\phi_a\}$. Why is $\mathrm{Aut}(\mathbb{B}_n)$ noncompact?

Claim 2.1.10 *Aut($\mathbb{B}_n$) is transitive.*

Proof. Choose $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \in \mathbb{B}_n$. Then, $\exists\, \Phi_a \in \mathrm{U}(n)$ such that $\Phi_a(a) = (a_1, 0, \ldots, 0)$, i.e. Φ_a rotates a unto the z_1-axis. Choose $\Phi_{-b} \in \mathrm{U}(n)$ such that $\Phi_{-b}(b_1, 0, \ldots, 0) = b$ (Φ_{-b} is the inverse of Φ_b). Let ϕ_{a_1}, ϕ_{-b_1} be the automorphisms of the ball as described above. Note that $\phi_{-b_1} = (\phi_{b_1})^{-1}$ Therefore,

$(\Phi_{-b} \circ \phi_{-b} \circ \phi_a \circ \Phi_a)(a) = \Phi_{-b}(\phi_{-b}(\phi_a(\Phi_a(a)))) = \Phi_{-b}(\phi_{-b}(\phi_a(a_1, 0, \ldots, 0))) = \Phi_{-b}(\phi_{-b}(0)) =$

$\Phi_{-b}(b_1, 0, \ldots, 0) = b$, i.e. $(\Phi_{-b} \circ \phi_{-b} \circ \phi_a \circ \Phi_a)(a) = b \Rightarrow \mathrm{Aut}(\mathbb{B}_n)$ is transitive. ■

Therefore, by Proposition 2.1.8 , $\mathrm{Aut}(\mathbb{B}_n)$ in non-compact.

Example 2.1.11 *The unit polydisc in* $\mathbb{C}^n$.

Let $\Delta_n := \{z = (z_1, \ldots, z_n) \,|\, |z_k| < 1 \,\forall k\}$ denote the unit polydisc in $\mathbb{C}^n$. Notice that $\Delta_n = \Delta \times \ldots \times \Delta$, n times. Therefore,

$$\mathrm{Aut}(\Delta_n) = \left\{ \varphi(z) = \varphi(z_1, \ldots, z_n) := \left(e^{i\theta_1} \frac{z_{\sigma(1)} - a_1}{1 - \overline{a_1} z_{\sigma(1)}}, \ldots, e^{i\theta_n} \frac{z_{\sigma(n)} - a_n}{1 - \overline{a_n} z_{\sigma(n)}} \right) \right\},$$

where $a \in \Delta_n$, $0 \leq \theta_k \leq 2\pi$, and $\sigma \in S_n$, where S_n is the symmetric group on n letters. The fact that $\mathrm{Aut}(\Delta_n)$ is non-compact follows from the fact that $\mathrm{Aut}(\Delta_n)$ is transitive.

Claim 2.1.12 *Aut(Δ_n) is transitive.*

Proof. Let $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n) \in \Delta_n$. Consider the following automorphisms of Δ_n:

$$\varphi_a(z) = \left(\frac{z_1 - a_1}{1 - \overline{a_1} z_1}, \ldots, \frac{z_n - a_n}{1 - \overline{a_n} z_n} \right)$$

and

$$\varphi_{-b}(z) = \left(\frac{z_1 + b_1}{1 - \overline{b_1} z_1}, \ldots, \frac{z_n + b_n}{1 - \overline{b_n} z_n} \right).$$

Then, $(\varphi_{-b} \circ \varphi_a)(a) = \varphi_{-b}(0) = b \Rightarrow \mathrm{Aut}(\Delta_n)$ is transitive. ■

Therefore, it follows that $\mathrm{Aut}(\Delta_n)$ is non-compact.

Example 2.1.13 *The "egg" domain in* $\mathbb{C}^2$.

Let $E_m := \left\{(z_1, z_2) \in \mathbb{C}^2 \,|\, |z_1|^2 + |z_2|^{2m} < 1\right\}$ be the egg domain in $\mathbb{C}^2$, where $m \in \mathbb{Z}^+$. Then,

$$\mathrm{Aut}(E_m) = \left\{ (z_1, z_2) \longmapsto \left(\frac{z_1 + a}{1 + \bar{a} z_1},\, z_2 \left(\frac{\sqrt{1 - |a|^2}}{1 + \bar{a} z_1} \right)^{1/m} \right) \,\middle|\, |a| < 1 \right\}.$$

Why is $\mathrm{Aut}(E_m)$ non-compact? It is non-compact since it has a boundary orbit accumulation point.

Claim 2.1.14 *The point* $(1,0)$ *is a boundary orbit accumulation point for the action of Aut(*E_m*) on* E_m*.*

Proof. Choose a_j, $0 \leq a_j < 1$, such that $a_j \to 1$ as $j \to \infty$. Let $z = (z_1, z_2) \in E_m$ and $\phi_{a_j} \in \mathrm{Aut}(E_m)$. Then, $(1,0)$ is a boundary orbit accumulation point of the action of $\mathrm{Aut}(E_m)$ on E_m, since $\phi_{a_j}(z) \to (1,0)$ as $j \to \infty \Rightarrow \mathrm{Aut}(E_m)$ is non-compact by Claim 2.1.6. ■

With these examples in hand, it is natural to ask if any of these domains are biholomorphic. More generally, can the orginal desire for a higher-dimensional Riemann Mapping theorem be found for the set of bounded domains with non-compact automorphism group? As the following theorem of Poincaré demonstrates, without the imposition of additional conditons upon the domains under consideration, no such result holds. Before proving this theorem, a definition and a few facts are needed.

Definition 2.1.15 *Let* $\Omega \subset \mathbb{C}^n$ *be a bounded domain, where* $P \in \Omega$*. Then, the isotropy subgroup* $Aut(\Omega)_P \subset Aut(\Omega)$ *is the collection* $\{g \in Aut(\Omega) \,|\, g(P) = P\}$*.*

For the following proof, knowledge of the isotropy subgroups of Δ_n and $\mathbb{B}_n$ at the origin is needed.

1. $\mathrm{Aut}(\Delta_n)_0 = \left\{(e^{i\theta_1}, \ldots, e^{i\theta_n}) \,|\, 0 \leq \theta_j < 2\pi\right\} = S^1 \times \ldots \times S^1$, n times, which is *abelian.*

2. $\mathrm{Aut}(\mathbb{B}_n)_0 = \mathrm{U}(n)$, the unitary group, which is *not abelian.*

Theorem 2.1.16 (Poincaré) *There does not exist a biholomorphism between Δ_n and $\mathbb{B}_n$ for $n > 1$.*

Proof. Suppose $\exists$ a biholomorphism $g : \mathbb{B}_n \to \Delta_n$. Since $\mathbb{B}_n$ and Δ_n are homogeneous, without loss of generality, assume $g(0) = 0$. (If $g(0) = a \neq 0$, choose $f \in \mathrm{Aut}(\Delta_n)$ such that $f(a) = 0$. Then, $(f \circ g)(0) = 0$ and $f \circ g : \mathbb{B}_n \to \Delta_n$ is a biholomorphism.)

Claim 2.1.17 *g induces a group isomorphism between $Aut(\mathbb{B}_n)_0$ and $Aut(\Delta_n)_0$.*

Let $g_* : \mathrm{Aut}(\mathbb{B}_n)_0 \to \mathrm{Aut}(\Delta_n)_0$ be defined by $g_*(h) = g \circ h \circ g^{-1}$. Since $g(0) = 0$ and $h(0) = 0$, $g_*(h) \in \mathrm{Aut}(\Delta_n)_0$. Clearly, this map is well-defined and $g_*(h \circ j) = g \circ (h \circ j) \circ g^{-1}$

$= g \circ (h \circ 1_{\mathrm{Aut}(B_n)_0} \circ j) \circ g^{-1} = g \circ (h \circ (g^{-1} \circ g) \circ j) \circ g^{-1} = (g \circ h \circ g^{-1}) \circ (g \circ j \circ g^{-1}) =$ $g_*(h) \circ g_*(j) \Rightarrow g_*$ is a group homomorphism.

Let $g_*^{-1} : \mathrm{Aut}(\Delta_n)_0 \to \mathrm{Aut}(\mathbb{B}_n)_0$ be defined by $g_*^{-1}(h) = g^{-1} \circ h \circ g$, where $h \in \mathrm{Aut}(\Delta_n)_0$. By the exact same argument as in the previous paragraph, g_*^{-1} is a group homomorphism.

Subclaim: g_* and g_*^{-1} are inverses of each other.

$$g_*^{-1}(g_*(h)) = g^{-1} \circ (g \circ h \circ g^{-1}) \circ g = (g^{-1} \circ g) \circ h \circ (g^{-1} \circ g) = h$$

and

$$g_*(g_*^{-1}(j)) = g \circ (g^{-1} \circ j \circ g) \circ g^{-1} = (g \circ g^{-1}) \circ j \circ (g \circ g^{-1}) = j$$

where $h \in \mathrm{Aut}(\mathbb{B}_n)_0$ and $j \in \mathrm{Aut}(\Delta_n)_0$. Therefore, the subclaim is proved $\Rightarrow$ the claim is proved. But, this imples that $S^1 \times \ldots \times S^1 = \mathrm{Aut}(\Delta_n)_0 \cong \mathrm{Aut}(B_n)_0 = \mathrm{U}_n$, i.e $S^1 \times \ldots \times S^1 \cong \mathrm{U}_n$, which is a contradiction, since $S^1 \times \ldots \times S^1$ is abelian and U_n is non-abelian. Therefore, there does not exist a biholomorphism between $\mathbb{B}_n$ and Δ_n. ■

With this result, the next question one might ask is whether a classification result holds for a subset of the bounded domains in $\mathbb{C}^n$ with non-compact automorphism group, i.e. if some additional constraint is placed upon the domains under consideration, can any sort classification be obtained? The answer to this question is yes. In what follows, a discussion concerning the notion of *pseudoconvexity*, together with a known classification result, will ensue.

2.2 Pseudoconvexity and the Ball Characterization Theorem

Before getting to the Ball Characterization Theorem, the definition of pseudonconvexity will be given, along with the statement of several important properties of pseudoconvex domains. Examples will be discussed too. Good references for this material are contained in the books by Steven G. Krantz [K] and R.C. Gunning [Gu].

Definition 2.2.1 *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with a C^2 boundary (so, the defining function ρ for the boundary is C^2). Then, $\partial\Omega$ is pseudoconvex at p if*

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(p) w_j \overline{w}_k \geq 0 \quad \forall w \in T_p^{1,0}(\partial\Omega),$$

where

$$T_p^{1,0}(\partial\Omega) := \left\{ w \in \mathbb{C}^n \;\middle|\; \sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(p) w_j = 0 \right\}.$$

$T_p^{1,0}$ is called the *complex tangent space* to the $\partial\Omega$ at p. If the inequality above is strict, p is called a point of *strong pseudoconvexity.* So, a point $p \in \partial\Omega$ is a point of pseudoconvexity (resp. strong pseudoconvexity) if the complex Hessian (also known as the Levi form) is positive semi-definite (resp. positive defintie) at p on the complex tangent space. If every point $p \in \partial\Omega$ is a point of pseudoconvexity (resp. strong pseudoconvexity), then the domain Ω is said the be *pseudoconvex* (resp. *strongly pseudoconvex*). From now on, $T_p(\partial\Omega) := T_p^{1,0}(\partial\Omega)$.

Now, here are some important properties of pseudoconvex domains.

(1) Pseudoconvexity is independent of the defining function chosen.

Proof. Let $\tilde{\rho}$ be another defining function of $\partial\Omega$ in a neighborhood U of p, $p \in \partial\Omega$. Then, $\exists$ a C^1 function h defined in U such that $\tilde{\rho} = h\rho$, where $h(z) > 0\,\forall z \in U$. So,

$$\begin{aligned}
\frac{\partial^2 \tilde{\rho}}{\partial z_j \partial \overline{z}_k}(p) &= \frac{\partial^2 (\rho h)}{\partial z_j \partial \overline{z}_k}(p) = \frac{\partial}{\partial z_j}\left(\frac{\partial(\rho h)}{\partial \overline{z}_k}(p)\right) \\
&= \frac{\partial}{\partial z_j}\left(\frac{\partial \rho}{\partial \overline{z}_k}(p)\cdot h(p) + \rho(p)\cdot \frac{\partial h}{\partial \overline{z}_k}\right) \\
&= \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(p)\cdot h(p) + \frac{\partial \rho}{\partial \overline{z}_k}(p)\cdot \frac{\partial h}{\partial z_j}(p) \\
&\quad + \frac{\partial \rho}{\partial z_j}(p)\cdot \frac{\partial h}{\partial \overline{z}_k}(p) + \rho(p)\cdot \frac{\partial^2 h}{\partial z_j \partial \overline{z}_k}(p) \\
&= \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(p)\cdot h(p) + \frac{\partial \rho}{\partial \overline{z}_k}(p)\cdot \frac{\partial h}{\partial z_j}(p) + \frac{\partial \rho}{\partial z_j}(p)\cdot \frac{\partial h}{\partial \overline{z}_k}(p),
\end{aligned}$$

where the last equality follows from the fact that $\rho(p) = 0$. Therefore,

$$\begin{aligned}
\sum_{j,k=1}^{n} \frac{\partial^2 \tilde{\rho}}{\partial z_j \partial \overline{z}_k}(p) w_j \overline{w}_k &= h(p) \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(p) w_j \overline{w}_k \\
&\quad + \sum_{j,k=1}^{n}\left(\frac{\partial \rho}{\partial \overline{z}_k}(p)\cdot \frac{\partial h}{\partial z_j}(p) + \frac{\partial \rho}{\partial z_j}(p)\cdot \frac{\partial h}{\partial \overline{z}_k}(p)\right) w_j \overline{w}_k \\
&= h(p) \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(p) w_j \overline{w}_k \\
&\quad + 2\mathrm{Re} \sum_{j,k=1}^{n}\left(\frac{\partial \rho}{\partial z_j}(p)\cdot \frac{\partial h}{\partial \overline{z}_k}(p) w_j \overline{w}_k\right) \\
&= h(p) \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(p) w_j \overline{w}_k \;\; \text{if } w \in T_p(\partial\Omega).
\end{aligned}$$

Therefore, since $h(p) > 0$, $p \in \partial\Omega$ is a point of pseudoconvexity with respect to $\rho \Leftrightarrow$ it is a point of pseudoconvexity with respect to $\tilde{\rho}$ Therefore, the pseudoconvexity of a boundary point is irrespective of the defining function chosen. ■

(2) Pseudoconvexity is preserved under biholomorphic maps.

Proof. Let $\Phi : \Omega \longrightarrow \mathbb{C}^n$ be biholomorphic onto its image, where $\Omega' := \Phi(\Omega)$. So, $\Phi(z) = \Phi(z_1, \ldots, z_n) = (\Phi_1(z), \ldots, \Phi_n(z)) = (z'_1, \ldots, z'_n)$. Let $\rho : U \longrightarrow \mathbb{R}$ be a defining function for Ω, where $\overline{\Omega} \subset U$, U an open set. Then, $\tilde{\rho} := \rho \circ \Phi^{-1}$ is a defining function for Ω'. Choose $p \in \partial\Omega$ and $w \in T_p(\partial\Omega)$. Then, $\Phi(p) \in \partial\Omega'$ and $w' \in T_{\Phi(p)}(\partial\Omega')$, where

$$w' = \begin{pmatrix} w'_1 \\ \vdots \\ w'_n \end{pmatrix} = \begin{pmatrix} \frac{\partial \Phi_1}{\partial z_1}(p) & \ldots & \frac{\partial \Phi_1}{\partial z_n}(p) \\ \vdots & & \vdots \\ \frac{\partial \Phi_n}{\partial z_1}(p) & \ldots & \frac{\partial \Phi_n}{\partial z_n}(p) \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} \sum \frac{\partial \Phi_1}{\partial z_j}(p) w_j \\ \vdots \\ \sum \frac{\partial \Phi_n}{\partial z_j}(p) w_j \end{pmatrix}.$$

Now, since $\tilde{\rho} := \rho \circ \Phi^{-1}$, $\rho = \tilde{\rho} \circ \Phi$ implies

$$\begin{aligned} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(p) = \frac{\partial^2 (\tilde{\rho} \circ \Phi)}{\partial z_j \partial \overline{z}_k}(p) &= \frac{\partial}{\partial z_j} \left(\frac{\partial (\tilde{\rho} \circ \Phi)}{\partial \overline{z}_k}(p) \right) \\ &= \sum_{l,m=1}^{n} \frac{\partial^2 \tilde{\rho}}{\partial z'_m \partial \overline{z}'_l}(\Phi(p)) \cdot \frac{\partial \Phi_m}{\partial z_j}(p) \cdot \frac{\partial \overline{\Phi}_l}{\partial \overline{z}_k}(p) \end{aligned}$$

by the chain rule. Hence,

$$\begin{aligned} \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(p) w_j \overline{w}_k &= \sum_{j,k=1}^{n} \left(\sum_{l,m=1}^{n} \frac{\partial^2 \tilde{\rho}}{\partial z'_m \partial \overline{z}'_l}(\Phi(p)) \cdot \frac{\partial \Phi_m}{\partial z_j}(p) \cdot \frac{\partial \overline{\Phi}_l}{\partial \overline{z}_k}(p) \right) w_j \overline{w}_k \\ &= \sum_{l,m=1}^{n} \left(\sum_{j,k=1}^{n} \frac{\partial^2 \tilde{\rho}}{\partial z'_m \partial \overline{z}'_l}(\Phi(p)) \cdot \frac{\partial \Phi_m}{\partial z_j}(p) w_j \cdot \frac{\partial \overline{\Phi}_l}{\partial \overline{z}_k}(p) \overline{w}_k \right) \\ &= \sum_{l,m=1}^{n} \frac{\partial^2 \tilde{\rho}}{\partial z'_m \partial \overline{z}'_l}(\Phi(p)) w'_m \overline{w}'_l, \end{aligned}$$

i.e.

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(p) w_j \overline{w}_k = \sum_{l,m=1}^{n} \frac{\partial^2 \tilde{\rho}}{\partial z'_m \partial \overline{z}'_l}(\Phi(p)) w'_m \overline{w}'_l,$$

which imples that the Levi from is preserved under biholomorphic maps. In other words, pseudoconvexity is preserved under biholomorphic maps. ■

(3) If $p \in \partial\Omega$ is a point of strong pseudoconvexity, then there exists a neighborhood U of p such that for all $q \in \partial\Omega \cap U$, q is strongly pseudoconvex.

To prove this result, a techincal lemma is needed.(A proof can be found in [K] Chapter 3.)

Lemma 2.2.2 *If Ω is strongly pseudoconvex, then Ω has a defining function $\tilde{\rho}$ such that*

$$\sum_{j,k=1}^{n} \frac{\partial^2 \tilde{\rho}}{\partial z_j \partial \overline{z}_k}(p) w_j \overline{w}_k \geq C|w|^2$$

for all $p \in \partial\Omega$, $w \in \mathbb{C}^n$, where $C \in \mathbb{R}^+$.

Now, here is a proof.

Proof. By Lemma 2.2.2, there exists a defining function $\tilde{\rho}$ for Ω such that

$$\sum_{j,k=1}^{n} \frac{\partial^2 \tilde{\rho}}{\partial z_j \partial \overline{z}_k}(p) w_j \overline{w}_k \geq C|w|^2 \quad \forall w \in \mathbb{C}^n.$$

In particular,

$$\sum_{j,k=1}^{n} \frac{\partial^2 \tilde{\rho}}{\partial z_j \partial \overline{z}_k}(p) w_j \overline{w}_k > 0 \quad \forall w \neq 0,\, w \in \mathbb{C}^n.$$

Since $\tilde{\rho}$ is C^2, the function

$$q \stackrel{\Phi}{\to} \sum_{j,k=1}^{n} \frac{\partial^2 \tilde{\rho}}{\partial z_j \partial \overline{z}_k}(q) w_j \overline{w}_k$$

is continuous in a neighborhood U of $p \Rightarrow \forall q \in U \cap \partial\Omega$,

$$\sum_{j,k=1}^{n} \frac{\partial^2 \tilde{\rho}}{\partial z_j \partial \overline{z}_k}(q) w_j \overline{w}_k > 0 \quad \forall w \neq 0,\, w \in \mathbb{C}^n$$

by the continuity of $\Phi \Rightarrow q \in U \cap \partial\Omega$ is strongly pseudoconvex. This completes the proof. ∎

Note: The analogous result for pseudoconvex boundary points is false (see the examples below for details).

(4) Every domain in $\mathbb{C}$ (with a C^2 boundary) is vacuously pseudoconvex.

Proof. Let Ω be a domain in $\mathbb{C}$ with a C^2 boundary. So, the defining function ρ is C^2. Now, for all $p \in \partial\Omega$,

$$\nabla\rho(p) = \frac{d\rho}{dz}(p) \neq 0,$$

which implies $w \in T_p(\partial\Omega) \Leftrightarrow w = 0$. This implies that $T_p(\partial\Omega) = \{0\}$, giving that Ω is pseudoconvex, since the condition for pseudoconvexity in one dimension

$$\frac{d^2\rho}{dzd\overline{z}}(p)w^2 \geq 0$$

is always satsified. ■

To help elucidate this important notion, some examples are in order. To simplify calculations, only domains in $\mathbb{C}^2$ will be considered.

Example 2.2.3 *The unit ball* $\mathbb{B}_2$.

Recall that $\mathbb{B}_2 = \{z = (z_1, z_2) \in \mathbb{C}^2 \mid \rho(z) := |z_1|^2 + |z_2|^2 - 1 < 0\}$. The complex hessian for ρ is the matrix

$$\left(\frac{\partial^2\rho}{\partial z_j \partial\overline{z}_k}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which implies that

$$\sum_{j,k=1}^{2} \frac{\partial^2\rho}{\partial z_j \partial\overline{z}_k}(p) w_j \overline{w}_k = |w_1|^2 + |w_2|^2 > 0 \quad \forall w \in \mathbb{C}^2,\, w \neq 0$$

Since this is true for every $p \in \partial\Omega$, $\mathbb{B}_2$ is strongly pseudoconvex.

Example 2.2.4 *The egg domain* E_m

Recall that $E_m := \{z = (z_1, z_2) \mid \rho(z) := |z_1|^2 + |z_2|^{2m} - 1 < 0\}$. The complex hessian for ρ is the matrix

$$\left(\frac{\partial^2\rho}{\partial z_j \partial\overline{z}_k}\right) = \begin{pmatrix} 1 & 0 \\ 0 & m^2 z_2^{m-1}\overline{z}_2^{m-1} \end{pmatrix},$$

which implies that for any $p \in \partial\Omega$,

$$\sum_{j,k=1}^{2} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(p) w_j \overline{w}_k = \begin{pmatrix} \overline{w}_1 & \overline{w}_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & m^2 p_2^{m-1} \overline{p}_2^{m-1} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$
$$= |w_1|^2 + m^2 |p_2|^{2m-2} |w_2|^2,$$

which is greater than 0 only if $|p_2|^{2m-2} \neq 0$, i.e. any point $p = (p_1, p_2) \in \partial\Omega$ is a point of strong pseudoconvexity if $p_2 \neq 0$. Therefore, points of pseudoconvexity are of the form $(e^{i\theta}, 0)$ (the complex hessian is positive semi-definite at these points; take $w = (w_1, w_2) \in \mathbb{C}^2$, $w_1 = 0$).

Going back to the question posed prior to this discussion concerning pseudoconvexity, if additional restrictions are placed upon the domains under consideration – say strong pseudoconvexity and some sort of boundary regularity – can a meaningful classification be obtained? The answer is *yes*! The primary and most important one is known as the Ball Characterization Theorem, proved originally by Bun Wong [W1] and later refined by J.P. Rosay [R], using the same method introduced in [W1].

Theorem 2.2.5 (Bun Wong) *Let Ω be smoothly bounded strongly pseudoconvex domain in $\mathbb{C}^n$ with non-compact automorphism group. Then Ω is biholomorphic to the unit ball $\mathbb{B}_n$.*

With this result, the problem of classifying smoothly bounded strongly pseudoconvex domains with non-compact automorphism group is finished. The crucial hypothesis is that of strong pseudoconvexity. What if, though, the hypothesis of stong pseudoconvexity is weakend to just pseudoconvexity? Can any classification be obtained in this case? Again, the answer is yes, *but* additional conditions must be imposed upon the domain, which is illustrated by the following well-known result (see [BP]):

Theorem 2.2.6 *Let Ω be a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$ of finite type with non-compact automorphism group such that the Levi form of $\partial\Omega$ has no more than one zero eigenvalue at any point. Then Ω is biholomorphic to the ellipsoid E_m, where $m \in \mathbb{Z}^+$.*

This result classifies all smoothly bounded pseudoconvex domains of finite type. What is finite type? What follows, then, is a discussion of the notion of finite type in two dimensions.

2.3 Finite Type in $\mathbb{C}^2$

Because the notion of finite type is more complicated in dimensions greater than two, the forthcoming discussion concerning the type of a smoothly bounded domain will be restricted to $\mathbb{C}^2$. The way in which this will occur will follow the view promulgated in [K], Chapter 11.5.

Definition 2.3.1 *Let $\Omega := \{z \,|\, \rho(z) < 0\}$ be a smoothly bounded domain in $\mathbb{C}^2$, where $p \in \partial\Omega$. Then the analytic disc $\phi : \Delta \to \mathbb{C}^2$ is called a nonsingular disc tangent to $\partial\Omega$ at p if $\phi(0) = p$, $\phi'(0) \neq 0$, and $(\rho \circ \phi)'(0) = 0$.*

Definition 2.3.2 *Let $\Omega := \{z \,|\, \rho(z) < 0\}$ be a smoothly bounded domain with $p \in \partial\Omega$. Then $\partial\Omega$ is of finite (geometric) type m at p if the following condition holds: There exists a nonsingular disc tangent to $\partial\Omega$ at p such that*

$$|\rho \circ \phi(\zeta)| \leq C|\zeta|^m$$

for $|\zeta|$ small. But, there does not exist a nonsingular disc ψ tangent to $\partial\Omega$ at p such that

$$|\rho \circ \psi(\zeta)| \leq C|\zeta|^{m+1}$$

for $|\zeta|$ small. (C is some constant.) In this case, p is called a point of finite (geometric) type.

The idea behind the notion of type is that it measures the maximum order of contact of an analytic disc with a given boundary point. In two dimensions, there is actually a notion of *analytic* type, but this paper will focus on the former; there is no loss in doing this, since both notions of type are the same in two dimensions.

Before going through some examples to help clarify the idea of the type of a smooth domain at a boundary point, some important properties will be given.

(1) The definition of type is independent of the defining function chosen.

Proof. Let $\Omega \subset \mathbb{C}^2$ be smooth, with defining function ρ. Let $p \in \partial\Omega$ and $\tilde{\rho}$ be another defining function for Ω. Then, there exists a function h nonvanishing in a neighborhood of $\partial\Omega$ such that $\tilde{\rho} = h\rho$. Therefore, $\tilde{\rho} = h\rho \Rightarrow \rho = \frac{1}{h}\tilde{\rho}$. Therefore, for any nonsingular analytic disc ϕ tangent to $\partial\Omega$ at p,

$$|\rho(\phi(\zeta))| = \left|\left(\frac{\tilde{\rho}}{h}\right)(\phi(\zeta))\right| = \left|\frac{\tilde{\rho}(\phi(\zeta))}{h(\phi(\zeta))}\right|.$$

Let $p \in \partial\Omega$ be a point of finite type m with respect to ρ. So, there exists a nonsingular disc ϕ tangent to $\partial\Omega$ at p such that for $|\zeta|$ small,

$$|\rho \circ \phi(\zeta)| \leq C|\zeta|^m.$$

This implies that, by the above calculation, for $|\zeta|$ small,

$$\left|\frac{\tilde{\rho}(\phi(\zeta))}{h(\phi(\zeta))}\right| \leq C|\zeta|^m$$

i.e.

$$|\tilde{\rho}(\phi(\zeta))| \leq C|h(\phi(\zeta))||\zeta|^m \leq CM|\zeta|^m$$

for $|\zeta|$ small, where

$$M := \sup_{|\zeta| \,\text{small}} |h(\phi(\zeta))|.$$

Hence, for $|\zeta|$ small,

$$|\tilde{\rho}(\phi(\zeta))| \leq C_1|\zeta|^m.$$

Suppose there exists a nonsingular analytic disc ψ tangent to $\partial\Omega$ at p such that $|\tilde{\rho}(\psi(\zeta))| \leq C|\zeta|^{m+1}$ for $|\zeta|$ small. Then,

$$|\rho(\psi(\zeta))| \leq \frac{|\tilde{\rho}(\psi(\zeta))|}{|h(\psi(\zeta))|} \leq \frac{C|\zeta|^{m+1}}{|h(\psi(\zeta))|} \leq \frac{C}{M}|\zeta|^{m+1},$$

where

$$M := \inf_{|\zeta| \,\text{small}} |h(\psi(\zeta))|.$$

Hence,

$$|\rho(\psi(\zeta))| \leq C_1|\zeta|^{m+1}$$

for $|\zeta|$ small. This contradicts the fact that p is a point of finite type m with respect to ρ. Therefore, p is a point of finite type m with respect to $\tilde{\rho}$, which completes the proof.

■

(2) The condition of finite type is preserved under biholomorphic mappings.

Example 2.3.3 *The unit ball* $\mathbb{B}_2 = \left\{z \in \mathbb{C}^2 \,|\, \rho(z) = |z_1|^2 + |z_2|^2 - 1 < 0\right\}$.

Consider the boundary point $p = (1, 0)$. Is p a point of finite type? Now,

$$\nabla\rho = \begin{pmatrix} \overline{z}_1 \\ \\ \overline{z}_2 \end{pmatrix} \Longrightarrow \nabla\rho(p) = \begin{pmatrix} 1 \\ \\ 0 \end{pmatrix},$$

which implies that any curve tangent to $\partial\mathbb{B}_2$ at p must be of the form

$$\phi(\zeta) = (1 + O(\zeta^2), \zeta + O(\zeta^2)),$$

after a reparametrization (look at the Taylor expansion).

Consider the disc $\phi(\zeta) = (1, \zeta)$; it has order of contact 2 with $\partial\mathbb{B}_2$ at p, since

$$\rho(\phi(\zeta)) = \rho(1, \zeta) = |\zeta|^2.$$

Now, what happens in general, i.e. what is the maximum order of contact when ϕ is of the form

$$\phi(\zeta) = (1 + O(\zeta^2), \zeta + O(\zeta^2))?$$

Here's the computation:

$$\begin{aligned}\rho(\phi(\zeta)) &= |1 + O(\zeta^2)|^2 + |\zeta + O(\zeta^2)|^2 - 1 \\ &= |1 + O(\zeta^2)|^2 + |\zeta|^2 \cdot |1 + O(\zeta)|^2 - 1 \\ &\leq C|\zeta|^2\end{aligned}$$

for $|\zeta|$ small, since

$$|1 + O(\zeta^2)|^2 \to 1 \text{ as } |\zeta| \longrightarrow 0$$

and

$$|1 + O(\zeta)|^2 \to 1 \text{ as } |\zeta| \longrightarrow 0.$$

Therefore, $p = (1, 0) \in \partial B_2$ is a point of finite type 2.

Example 2.3.4 *The ellipsoid* $E_m = \left\{z \in \mathbb{C}^2 \mid \rho(z) = |z_1|^2 + |z_2|^{2m} - 1 < 0\right\}$

Consider the boundary point $p = (1, 0)$. To calculate the type at p, note that,

$$\nabla\rho = \begin{pmatrix} \overline{z}_1 \\ \\ m z_2^{m-1} \overline{z}_2^{m-1} \end{pmatrix} \Longrightarrow \nabla\rho(p) = \begin{pmatrix} 1 \\ \\ 0 \end{pmatrix},$$

which implies, after a reparametrization, a nonsingular analytic disc ϕ that intersects ∂E_m at p is of the form

$$\phi(\zeta) = (1 + O(\zeta^2), \zeta + O(\zeta^2)).$$

What is the maximum order of contact of such a curve with the boundary? First, consider the simple case where $\phi(\zeta) = (1, \zeta)$; this curve as order of contact $2m$ at the boundary point p, since

$$\rho(\phi(\zeta)) = |\zeta|^{2m}.$$

Can the order of contact improve? For an arbitrary curve ϕ as described above,

$$\begin{aligned} \rho(\phi(\zeta)) &= |1 + O(\zeta^2)|^2 + |\zeta + O(\zeta^2)|^{2m} - 1 \\ &= |1 + O(\zeta^2)|^2 + |\zeta|^{2m} \cdot |1 + O(\zeta)|^{2m} - 1 \\ &\leq C|\zeta|^{2m} \end{aligned}$$

for $|\zeta|$ small, since

$$|1+O(\zeta^2)|^2 \to 1 \text{ as } |\zeta| \longrightarrow 0$$

and

$$|1+O(\zeta)|^{2m} \to 1 \text{ as } |\zeta| \longrightarrow 0.$$

Therefore, the maximum order of contact of any nonsingular analytic disc tangent to ∂E_m at $p=(1,0)$ is $2m$, which implies that p is a point of finite type $2m$.

Example 2.3.5 *The domain* $E_\infty := \left\{ z \in \mathbb{C}^2 \,|\, \rho(z) := |z_1|^2 + 2e^{-1/|z_2|^2} - 1 < 0 \right\}$.

Consider the point $p=(1,0) \in \partial\Omega$. Then,

$$\nabla\rho = \begin{pmatrix} \overline{z}_1 \\ \\ \frac{2e^{-1/|z_2|^2}}{z_2^2 \overline{z}_2} \end{pmatrix} \Longrightarrow \nabla\rho(p) = \begin{pmatrix} 1 \\ \\ 0 \end{pmatrix}.$$

Consider the curve $\phi(\zeta) = (1,\zeta)$; it is tangent to $\partial\Omega$ at p. Now,

$$\rho(\phi(\zeta)) = 2e^{-1/|\zeta|^2},$$

which implies that

$$\frac{|\rho(\phi(\zeta))|}{|\zeta|^m} = \frac{2e^{-1/|\zeta|^2}}{|\zeta|^m} \longrightarrow 0 \text{ as } \zeta \to 0$$

by L'Hopital's rule, since

$$\frac{d^k}{d\zeta^k}(2e^{-1/|\zeta|^2})\bigg|_{\zeta=0} = 0 \ \forall k \in \mathbb{Z}^+.$$

Since this is true for any $m \in \mathbb{Z}^+$,

$$|\rho(\phi(\zeta))| \le C|\zeta|^m$$

as $|\zeta| \to 0 \ \forall m \in \mathbb{Z}^+$, which implies that $p=(1,0)$ is a point of *infinite type*.

Example 2.3.6 *The unit polydisc* $\Delta_2 = \big\{ z \in \mathbb{C}^2 \,|\, |z_j| < 1,\, j = 1, 2 \big\}$.

Choose $p = (1, 0) \in \partial\Delta_2$. In a neighborhood U_p of p, let $\rho(z) := |z_1| - 1$ be a local defining function for the boundary defined in $U_p \cap \partial\Delta_2$. Since

$$\nabla\rho = \begin{pmatrix} \overline{z}_1 \\ \\ 0 \end{pmatrix} \Longrightarrow \nabla\rho(p) = \begin{pmatrix} 1 \\ \\ 0 \end{pmatrix},$$

the nonsingular analytic disc $\phi(\zeta) := (1, \zeta)$ is tangent to $\partial\Delta_2$ at p. So

$$\rho(\phi(\zeta)) = \rho(1, \zeta) = |1| - 1 = 0 \;\forall \zeta \in D,$$

which implies that

$$|\rho(\phi(\zeta))| \leq C|\zeta|^m$$

as $|\zeta| \to 0 \;\forall m \in \mathbb{Z}^+$. Therefore, $p = (1, 0) \in \partial\Omega$ is a point of infinite type.

As these examples illustrate, the greater the type at a boundary point, the flatter the boundary is in a neighborhood of that point.

2.4 Importance of the Greene-Krantz Conjecture

In the above well-known result (Theorem 2.2.7) finite type is assumed on the whole boundary, because it is not known where the boundary orbit accumulation points are located. But, the crucial fact that is needed for this result to be true is the *finiteness of type at the boundary orbit accumulation points*. Consider, now, the Greene-Krantz Conjecture, named after Robert E. Greene and Steven G. Krantz:

Conjecture 2.4.1 (Greene-Krantz) *Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded domain with non-compact automorphism group. Then $\partial\Omega$ is of finite type at any boundary orbit accumulation point.*

If this result is true, then any smoothly bounded pseudoconvex domain with non-compact automorphism group in $\mathbb{C}^2$ is biholomorphic to E_m by Theorem 2.2.7 stated above. Why? The Greene-Krantz conjecture would give that the boundary orbit accumulation points are of finite type and, in two dimensions, the restriction on the Levi form concerning its eigenvalues coincides with the notion of pseudoconvexity already assumed. This consequence, together with many more, show why the truthfulness of the Greene-Krantz conjecture is so important concerning the classification of smoothly bounded domains with non-compact automorphism group.

In the last example of the previous section (Example 2.3.6), the analytic disc ϕ was actually contained in the boundary of the bidisc Δ_2, passing through the point $p = (1, 0)$. Whenever this happens, i.e. whenever there exists a positive dimensional complex analytic variety on $\partial\Omega$, passing through some point $p \in \partial\Omega$, the domain will be of infinite type at p. Here is statement of this fact, along with its proof.

Lemma 2.4.2 *In $\mathbb{C}^2$, finite type at $p \in \partial\Omega \Longrightarrow$ variety free at p, i.e. there is no positive dimensional complex analytic variety on $\partial\Omega$, passing through the point p.*

Proof. Suppose $\partial\Omega$ is not variety free at p. So, there exists a complex variety $V \subset \partial\Omega$ such that $p \in V$ and $\dim_{\mathbb{C}} V > 0$. So, $\dim_{\mathbb{C}} V = 1 \Longrightarrow V$ is nonsingular at p (see Lemma 3.2 in [FW]). Let $\phi : \Delta \longrightarrow V$ be local parameterization of V in a neighborhood of p. Hence, $\phi(0) = p$ and $\phi'(0) \neq 0$. But, $V \subset \partial\Omega \Longrightarrow |\rho(\phi(\zeta))| = 0 \Longrightarrow |\rho(\phi(\zeta))| = 0 \leq C|\zeta|^m \ \forall m \in \mathbb{Z}^+ \Longrightarrow \partial\Omega$ is of infinite type. ■

The concept of a domain being variety free at a boundary point will be very important in the sequal, since the main result to be proven will be that if a domain satisfies certain conditions, it will be variety free at its boundary orbit accumulation points. In other words, the main result of this document will be to prove a result that supports the truthfulness of the Greene-Krantz conjecture. A very important tool needed to prove the main result will be the invariant metrics and measures. A discussion of them and their important properties will take place in the next chapter.

Chapter 3

Invariant Metrics and Measures

3.1 Invariant Metrics

Definition 3.1.1 *Let Ω be a domain in $\mathbb{C}^n$, $p \in \Omega$, and $\xi \in T_p\Omega$. Then the (infinitesimal) Kobayashi metric on Ω at p in the direction of ξ is defined as*

$$F_K^{\Omega}(p,\xi) = \inf \left\{ \frac{1}{\alpha} \,\middle|\, \exists \phi \in Hol(\Delta, \Omega) \text{ such that } \phi(0) = p,\ \phi'(0) = \alpha\xi \right\}.$$

The Carathéodory metric on Ω at p in the direction of ξ is defined as

$$F_C^{\Omega}(p,\xi) = \sup \left\{ |\phi_*(p)\xi| \,|\, \exists \phi \in Hol(\Omega, \Delta) \text{ such that } \phi(p) = 0 \right\}.$$

Note that $T_p\Omega = \mathbb{C}^n$ and that $\mathrm{Hol}(\Omega_1, \Omega_2)$ is the set of all holomorphic mappings from Ω_1 to Ω_2. A very important property that these two metrics satisfy is the following decreasing property.

Lemma 3.1.2 *If $f \in Hol(\Omega_1, \Omega_2)$, $p \in \Omega_1$, $\xi \in \mathbb{C}^n$, then*

$$F_K^{\Omega_1}(p,\xi) \geq F_K^{\Omega_2}(f(p), f_*(p)\xi)$$

and

$$F_C^{\Omega_1}(p,\xi) \geq F_C^{\Omega_2}(f(p), f_*(p)\xi).$$

Proof. First, the Kobayashi case. Let $\phi \in \text{Hol}(\Delta, \Omega_1)$ such that $\phi(0) = p$, $\phi'(0) = \alpha\xi$. Consider $f \circ \phi \in \text{Hol}(\Delta, \Omega_2)$. $(f \circ \phi)(0) = f(p)$ and $(f \circ \phi)'(0) = f_*(\phi(0))\phi'(0) = f_*(p)\alpha\xi = \alpha f_*(p)\xi$. Hence,

$$F_K^{\Omega_2}(f(p), f_*(p)\xi) \leq \frac{1}{\alpha}.$$

Taking the infimum over all ϕ with the desired property implies

$$F_K^{\Omega_1}(p, \xi) \geq F_K^{\Omega_2}(f(p), f_*(p)\xi).$$

Now, the Carathéodory case. Let $\phi \in \text{Hol}(\Omega_2, \Delta)$ such that $\phi(f(p)) = 0$. Consider $\phi \circ f \in \text{Hol}(\Omega_1, \Delta)$. $(\phi \circ f)(p) = \phi(f(p)) = 0$. Hence, $F_C^{\Omega_1}(p, \xi) \geq |(\phi \circ f)_*(p)\xi| = |\phi_*(f(p))(f_*(p)\xi)|$; taking the supremum over all ϕ with the desired property gives that

$$F_C^{\Omega_1}(p, \xi) \geq F_C^{\Omega_2}(f(p), f_*(p)\xi).$$

■

Corollary 3.1.3 *If $f : \Omega_1 \longrightarrow \Omega_2$ is a biholomorphism, $p \in \Omega_1$, $\xi \in \mathbb{C}^n$, then*

$$F_K^{\Omega_1}(p, \xi) = F_K^{\Omega_2}(f(p), f_*(p)\xi)$$

and

$$F_C^{\Omega_1}(p, \xi) = F_C^{\Omega_2}(f(p), f_*(p)\xi).$$

Proof. First, the Kobayshi case. By the previous lemma, $F_K^{\Omega_1}(p, \xi) \geq F_K^{\Omega_2}(f(p), f_*(p)\xi)$. Now, apply that lemma to the mapping $f^{-1} : \Omega_2 \longrightarrow \Omega_1$:

$$F_K^{\Omega_2}(f(p), f_*(p)\xi) \geq F_K^{\Omega_1}(f^{-1}(f(p)), f_*^{-1}(f(p))(f_*(p)\xi)).$$

But, $f^{-1}(f(p)) = p$ and $f_*^{-1}(f(p))(f_*(p)\xi) = (f^{-1} \circ f)_*(p)\xi = (1_{\Omega_1})_*(p)\xi = \xi$, where 1_{Ω_1} is the identity mapping on Ω_1. This implies that

$$F_K^{\Omega_1}(p, \xi) \geq F_K^{\Omega_2}(f(p), f_*(p)\xi) \geq F_K^{\Omega_1}(p, \xi)$$

$\Longrightarrow$

$$F_K^{\Omega_1}(p,\xi) = F_K^{\Omega_2}(f(p), f_*(p)\xi).$$

The exact same argument holds for the Carathéodory metric. ■

Lemma 3.1.4 $F_K^{\Omega}(p,c\xi) = |c|F_K^{\Omega}(p,\xi)$ *for any* $c \in \mathbb{C}$*.*

Proof. Let $f \in \mathrm{Hol}(\Delta,\Omega)$ such that $f(0) = p$ and $f'(0) = \alpha\xi$. Let $g(z) := f\left(\frac{cz}{|c|}\right)$. Then, $g(0) = f(0) = p$ and $g'(0) = \frac{c}{|c|}(\alpha\xi) = \frac{\alpha}{|c|}(c\xi)$. Therefore,

$$F_K^{\Omega}(p,c\xi) \leq \frac{|c|}{\alpha} = |c| \cdot \frac{1}{\alpha}.$$

Taking the infimum over $\frac{1}{\alpha}$ implies that

$$F_K^{\Omega}(p,c\xi) \leq |c|F_K^{\Omega}(p,\xi).$$

To show the reverse inequality, let $f \in \mathrm{Hol}(\Delta,\Omega)$ such that $f(0) = p$ and $f'(0) = \alpha\,(c\xi)$. Let $g(z) := f\left(\frac{|c|z}{c}\right)$. Then $g(0) = f(0) = p$ and $g'(0) = \frac{|c|}{c}\alpha c\xi = |c|\alpha\xi = (|c|\alpha)\,\xi$. Therefore,

$$F_K^{\Omega}(p,\xi) \leq \frac{1}{|c|\alpha} = \frac{1}{|c|} \cdot \frac{1}{\alpha}.$$

Taking the infimum over $\frac{1}{\alpha}$ implies that

$$F_K^{\Omega}(p,\xi) \leq \frac{1}{|c|}F_K^{\Omega}(p,c\xi)$$

$\Longrightarrow$

$$|c|F_K^{\Omega}(p,\xi) \leq F_K^{\Omega}(p,c\xi)$$

$\Longrightarrow$

$$|c|F_K^{\Omega}(p,\xi) = F_K^{\Omega}(p,c\xi).$$

■

Note that the same property holds for the Carathéodory metric. That is,

$$F_C^\Omega(p, c\xi) = |c| F_C^\Omega(p, \xi)$$

for any $c \in \mathbb{C}$.

Now, the defintions of the Carathéodory and Kobayashi distance on a bounded domain $\Omega \subset \mathbb{C}^n$. Like the metrics, they also have a decreasing property, which will be proved as a lemma.

Definition 3.1.5 *Given two points $z, w \in \Omega \subset \mathbb{C}^n$, the Carathéodory distance between z and w is defined as*

$$d_C^\Omega(z, w) := \sup \{\rho(\phi(z), \phi(w)) \,|\, \phi \in Hol(\Omega, \Delta)\},$$

where ρ is the Poincaré distance on Δ.

Definition 3.1.6 *Given two points $z, w \in \Omega \subset \mathbb{C}^n$, the (integrated) Kobayashi distance between z and w is defined as*

$$d_K^\Omega(z, w) := \inf_\gamma \left\{ \int_0^1 F_K^\Omega(\gamma(t), \gamma'(t)) dt \right\},$$

where $\gamma : [0, 1] \to \Omega$ is a piecewise C^1 curve joining z and w.

Lemma 3.1.7 *For $f \in Hol(\Omega_1, \Omega_2)$ and $z, w \in \Omega_1$,*

$$d_C^{\Omega_1}(z, w) \geq d_C^{\Omega_2}(f(z), f(w))$$

and

$$d_K^{\Omega_1}(z, w) \geq d_K^{\Omega_2}(f(z), f(w)).$$

Proof. First, the Carathéodory case. Let $\phi \in \mathrm{Hol}(\Omega_2, \Delta)$. Then, $\phi \circ f \in \mathrm{Hol}(\Omega_1, \Delta)$. Hence, $\rho((\phi \circ f)(z), (\phi \circ f)(w)) \leq d_C^{\Omega_1}(z, w) \Rightarrow \rho(\phi(f(z)), \phi(f(w))) \leq d_C^{\Omega_1}(z, w)$. Taking the supremum over all ϕ implies that $d_C^{\Omega_1}(z,w) \geq d_C^{\Omega_2}(f(z), f(w))$.

Now, the Kobayashi case. Let $\gamma : [0,1] \to \Omega_1$ be a C^1 curve joining z to w (without loss of generality; if γ is piecewise C^1, repeat the argument for each C^1 piece of γ (and $f \circ \gamma$)). Then, $f \circ \gamma : [0,1] \to \Omega_2$ is a C^1 curve joining $f(z)$ to $f(w)$. Hence,

$$F_K^{\Omega_1}(\gamma(t), \gamma'(t)) \geq F_K^{\Omega_2}(f(\gamma(t)), f_*(\gamma(t))\gamma'(t))$$

$\Longrightarrow$

$$\begin{aligned}\int_0^1 F_K^{\Omega_1}(\gamma(t), \gamma'(t))dt &\geq \int_0^1 F_K^{\Omega_2}(f(\gamma(t)), f_*(\gamma(t))\gamma'(t))dt \\ &\geq d_K^{\Omega_2}(f(z), f(w));\end{aligned}$$

taking the infimum over all C^1 curves γ implies

$$d_K^{\Omega_1}(z,w) \geq d_K^{\Omega_2}(f(z), f(w)).$$

■

Corollary 3.1.8 *If $f : \Omega_1 \longrightarrow \Omega_2$ is a biholomorphism, then*

$$d_C^{\Omega_1}(z,w) = d_C^{\Omega_2}(f(z), f(w))$$

and

$$d_K^{\Omega_1}(z,w) = d_K^{\Omega_2}(f(z), f(w)).$$

Proof. Apply the previous lemma to both f and f^{-1}. ■

The orginal definition of the Kobayashi (pseudo)distance is not the one above. Here is what it is (see [Ko], p.45):

Definition 3.1.9 *Let M be a complex manifold. Given two points z and w, choose points $z = z_0, z_1, \ldots, z_{k-1}, z_k = w$ of M, points $a_1, \ldots, a_k, b_1, \ldots, b_k$ of Δ, and holomorphic mappings $f_1, \ldots, f_k$ of Δ into M such that $f_j(a_j) = z_{j-1}$ and $f_j(b_j) = z_j$ for*

$j = 1, \ldots, k$. *For each choice of points and mappings thus made, consider the number*

$$\rho(a_1, b_1) + \ldots + \rho(a_k, b_k).$$

$d_K^M(z, w)$ *is defined as the infimum of the numbers obtained in this manner for all possible choices.*

Royden in [R] showed that Kobayashi's orginal definition is equivalent to the integrated form given above.

For a complex manifold, d_K^M in general is only a pseudodistance. For any complex manifold M, whenever d_K^M is actually a distance, the manifold is called *hyperbolic.* If d_K^M happens to be complete, M is called *complete hyperbolic.* M is complete with respect to d_K^M if, for each point $z \in M$ and any $r \in \mathbb{R}^+$, $\overline{\beta_K(z; r)}$ (the closed ball, centered at z, with radius r measured with respect to the Kobayashi distance) is a compact subset of M. For the purposes of this paper, the following facts will be needed:

1. Every bounded domain $\Omega \subset \mathbb{C}^n$ is hyperbolic.
2. Every smoothly bounded convex domain $\Omega \subset \mathbb{C}^n$ is complete hyperbolic.

Now, more important properties that will be needed in the sequal will be given, along with their proofs.

Lemma 3.1.10 *Let* $\Omega \subset \mathbb{C}^n$ *be bounded, with* $p \in \Omega$ *and* $\xi \in \mathbb{C}^n$. *Then,*

$$F_K^\Omega(p, \xi) \leq \frac{|\xi|}{r},$$

where $r = dist(p, \partial\Omega)$.

Proof. Let $r := \text{dist}(p, \partial\Omega)$; consider $f : \Delta \longrightarrow \Omega$ given by $f(z) = p + \frac{r\xi}{|\xi|}z$. f is holomorphic, $f(0) = p$, and $f'(0) = \frac{r\xi}{|\xi|}$. Therefore, by definition, $F_K^\Omega(p, \xi) \leq \frac{|\xi|}{r}$. ∎

Lemma 3.1.11 *Let $\Omega \subset \mathbb{C}^n$ be a domain with a C^2 boundary. Let $q \in \partial\Omega$ be strongly pseudoconvex and ν the real normal vector to $\partial\Omega$ at q . Let $z, w \in \ell_q$, where*

$$\ell_q = \{q - t\nu\}\,,\, t \in \mathbb{R}$$

(i.e. ℓ_q is the real normal line to the $\partial\Omega$ at q). Let $\gamma : [0,1] \longrightarrow \Omega$ be defined by $\gamma(t) = (1-t)w + tz$. Then,

$$d_K^\Omega(z, w) = \int_0^1 F_K^\Omega(\gamma(t), \gamma'(t))dt.$$

Proof. For this proof, the following well-known result will be needed (see [A], [G]):

Let $\Omega \subset \mathbb{C}^n$ be a domain with a C^2 boundary. Let $q \in \partial\Omega$ be strongly pseudoconvex. For any $p \in \Omega$, $\xi \in T_p\Omega$,

$$F_K^\Omega(p, \xi) \approx \frac{1}{\delta(p)}\xi_N + \frac{1}{\sqrt{\delta(p)}}\xi_T,$$

where ξ_T and ξ_N are the tangential and normal components of ξ at q and $\delta(p) := \mathrm{dist}(p, \partial\Omega)$.

Note that

$$F_K^\Omega(p, \xi) \approx \frac{1}{\delta(p)}\xi_N + \frac{1}{\sqrt{\delta(p)}}\xi_T$$

means that there are constants c_1 and c_2 such that

$$c_1\left(\frac{1}{\delta(p)}|\xi_N| + \frac{1}{\sqrt{\delta(p)}}|\xi_T|\right) \le F_K^\Omega(p, \xi) \le c_2\left(\frac{1}{\delta(p)}|\xi_N| + \frac{1}{\sqrt{\delta(p)}}|\xi_T|\right).$$

Let ν be in the direction of the $\mathrm{Re}\, z_n$-axis (choose local coordinates). Let γ be as above. For all $t \in [0,1]$, $\gamma'(t)$ is normal to $\partial\Omega$ at q. Let σ be any other piecewise C^1 curve joining z to w. Then, $\sigma(t) = \sigma_N(t) + \sigma_T(t) = \gamma(t) + \sigma_T(t)$, where $\sigma_N(t) = \gamma(t)$ is the straight line joining z to w and $\sigma_T(t)$ is piecewise C^1 for which $\sigma_T'(t) \in T_q\partial\Omega$ for at least one $t \in (0,1)$. By definition,

$$d_K^\Omega(z, w) \le \int_0^1 F_K^\Omega(\gamma(t), \gamma'(t))dt.$$

But,

$$\int_0^1 F_K^\Omega(\gamma(t),\gamma'(t))dt \le c_2 \int_0^1 \frac{|\gamma'(t)|}{\delta(\gamma(t))}dt \le \int_0^1 \left(\frac{|\gamma'(t)|}{\delta(\gamma(t))} + \frac{|\sigma_T'(t)|}{\sqrt{\delta(\sigma(t))}}\right) dt$$
$$\le c_2 \int_0^1 \left(\frac{|\sigma_N'(t)|}{\delta(\sigma(t))} + \frac{|\sigma_T'(t)|}{\sqrt{\delta(\sigma(t))}}\right) dt$$
$$\le \frac{c_2}{c_1} \int_0^1 F_K^\Omega(\sigma(t),\sigma'(t))dt = \int_0^1 F_K^\Omega\left(\sigma(t), \frac{c_2}{c_1}\sigma'(t)\right) dt.$$

Let $u(t) := \sigma\left(\frac{c_2}{c_1}t\right)$, $u : [0, c_1/c_2] \longrightarrow \Omega$. Then,

$$\int_0^1 F_K^\Omega\left(\sigma(t), \frac{c_2}{c_1}\sigma'(t)\right) dt = \int_0^{\frac{c_1}{c_2}} F_K^\Omega\left(\sigma\left(\frac{c_2}{c_1}t\right), \frac{c_2}{c_1}\sigma'\left(\frac{c_2}{c_1}t\right)\right) dt$$
$$= \int_0^{\frac{c_1}{c_2}} F_K^\Omega\left(u(t), u'(t)\right) dt.$$

Therefore,

$$d_K^\Omega(z,w) \le \int_0^1 F_K^\Omega(\gamma(t),\gamma'(t))dt \le \int_0^{\frac{c_1}{c_2}} F_K^\Omega\left(u(t), u'(t)\right) dt;$$

taking the infimum over all piecewise C^1 curves u joining z to w implies

$$d_K^\Omega(z,w) \le \int_0^1 F_K^\Omega(\gamma(t),\gamma'(t))dt \le d_K^\Omega(z,w)$$

$\Longrightarrow$

$$d_K^\Omega(z,w) = \int_0^1 F_K^\Omega(\gamma(t),\gamma'(t))dt.$$

■

Lemma 3.1.12 *Let $\Omega \subset \mathbb{C}^n$ be a domain with a C^2 boundary. Let $q \in \partial\Omega$ be strongly pseudoconvex and ν the real normal vector to $\partial\Omega$ at q . Suppose $z \in \ell_q$, $w \in \Omega$, and $\tilde{w}$ is the projection of w onto ℓ_q. Then, $d_K^\Omega(z,\tilde{w}) \le d_K^\Omega(z,w)$.*

Proof. Choose coordinates so that ν is in the $\mathrm{Re}\, z_n$ direction and q is the origin. Then, $\tilde{w} = (0,0,\ldots,0,\mathrm{Re} w_n)$. Let $\gamma(t) := (1-t)z + t\tilde{w}$, $t \in [0,1]$. Let σ be any piecewise C^1 curve joining z to w. So, $\sigma(t) = \gamma(t) + \sigma_T(t)$, where σ_T is piecewise C^1 such that

$\sigma_T(0) = 0$ and $\sigma_T(1) = (w_1, w_2, \ldots, w_{n-1}, \operatorname{Im} w_n)$. Therefore,

$$\begin{aligned}
d_K^\Omega(z, \tilde{w}) &= \int_0^1 F_K^\Omega(\gamma(t), \gamma'(t))dt \leq c_2 \int_0^1 \frac{|\gamma'(t)|}{\delta(\gamma(t))} dt \\
&\leq c_2 \int_0^1 \left(\frac{|\gamma'(t)|}{\delta(\gamma(t))} + \frac{|\sigma_T'(t)|}{\sqrt{\delta(\sigma(t))}} \right) dt \\
&\leq c_2 \int_0^1 \left(\frac{|\sigma_N'(t)|}{\delta(\sigma(t))} + \frac{|\sigma_T'(t)|}{\sqrt{\delta(\sigma(t))}} \right) dt \\
&\leq \frac{c_2}{c_1} \int_0^1 F_K^\Omega(\sigma(t), \sigma'(t))dt = \int_0^1 F_K^\Omega\left(\sigma(t), \frac{c_2}{c_1}\sigma'(t)\right) dt.
\end{aligned}$$

The first equality follows from the previous lemma. Let $u(t) := \sigma\left(\frac{c_2}{c_1}t\right)$, $u : [0, c_1/c_2] \longrightarrow \Omega$. Then,

$$\begin{aligned}
\int_0^1 F_K^\Omega\left(\sigma(t), \frac{c_2}{c_1}\sigma'(t)\right) dt &= \int_0^{\frac{c_1}{c_2}} F_K^\Omega\left(\sigma\left(\frac{c_2}{c_1}t\right), \frac{c_2}{c_1}\sigma'\left(\frac{c_2}{c_1}t\right)\right) dt. \\
&= \int_0^{\frac{c_1}{c_2}} F_K^\Omega\left(u(t), u'(t)\right) dt.
\end{aligned}$$

Therefore,

$$d_K^\Omega(z, \tilde{w}) \leq \int_0^{\frac{c_1}{c_2}} F_K^\Omega\left(u(t), u'(t)\right) dt;$$

taking the infimum over all piecewiese C^1 curves joing z to w

$\Longrightarrow$

$$d_K^\Omega(z, \tilde{w}) \leq d_K^\Omega(z, w).$$

■

For the next lemma, let H be the upper half-plane in $\mathbb{C}$ and $\Gamma_\theta(q)$ a cone in H with vertex $q = \mathbf{0}$ and angle θ between the imaginary axis and the arms of the cone. The method of proof will follow Theorem 5.1 in [W2].

Lemma 3.1.13 *Let $z = (0, r)$ be a point on the imaginary axis. Then,*

$$d_K^H(z, \partial\Gamma_\theta(q)) = \ln(\tan\theta + \sec\theta).$$

Proof. Let $w = x + iy$ be coordinates for $\mathbb{C}$. The Kobayashi metric for H is $\frac{\sqrt{dx^2+dy^2}}{y}$ (this can be seen by pulling back the Poincaré metric $\frac{\sqrt{dx^2+dy^2}}{1-x^2-y^2}$ on Δ via the mapping

$\omega = \frac{i-w}{i+w}$). Let z' and z'' be the points of intersection of $\Gamma_\theta(q)$ and the geodesic of $\frac{\sqrt{dx^2+dy^2}}{y}$ through z (which is a circle of radius r with center q). Let $\gamma : [0, \theta] \longrightarrow H$ be the arc of the circle joining z to z'. Then, the length of the arc of the circle joining z to z' is

$$\int_\gamma \frac{\sqrt{dx^2 + dy^2}}{y}.$$

Now, $\gamma(\phi) = r(\sin\phi + i\cos\phi)$ implies

$$\begin{aligned}
\int_\gamma \frac{\sqrt{dx^2 + dy^2}}{y} &= \int_0^\theta \frac{r\, d\phi}{r\cos\phi} = \int_0^\theta \sec\phi\, d\phi \\
&= \int_0^\theta \sec\phi \cdot \frac{\sec\phi + \tan\phi}{\sec\phi + \tan\phi} d\phi \\
&= \int_0^\theta \frac{\sec^2\phi + \sec\phi\tan\phi}{\sec\phi + \tan\phi} d\phi.
\end{aligned}$$

Letting $u = \sec\phi + \tan\phi$ gives that

$$d_K^H(z, z') = \int_\gamma \frac{\sqrt{dx^2 + dy^2}}{y} = \ln(\tan\theta + \sec\theta).$$

It remains to show that this is the minimum distance between z and any point on $\partial\Gamma_\theta(q)$. The following well-known theorem from Riemannian geometry is needed.

Theorem 3.1.14 *Let M be a simply connected complete Riemannian manifold of negative sectional curvature. For any two points in M, there exists one and only one minimizing geodesic joining them.*

Note that H, equipped with $\frac{\sqrt{dx^2+dy^2}}{y}$, satisfies the hypothesis of this theorem. Let w' be a point on $\partial\Gamma_\theta(q)$ that achieves $d_K^H(z, \partial\Gamma_\theta(q))$. By the symmetric properties of $\Gamma_\theta(q)$ and $\frac{\sqrt{dx^2+dy^2}}{y}$, there exists a point $w'' \in \partial\Gamma_\theta(q)$ such that the line segments $\overline{qw'}$ and $\overline{qw''}$ have the same euclidean length. Let w be the point of intersection of the imaginary axis and the circle of radius $\overline{qw'}$ centered at q. Since the arc of the circle joining w to w' is a minimizing geodesic, then by the theorem it is the *minimizing* geodesic, and hence $d_K^H(w, w') \leq d_K^H(z, w')$. Note that, by the above computation,

$$d_K^H(w, w') = d_K^H(z, z') = \ln(\tan\theta + \sec\theta).$$

Therefore,

$$d_K^H(z, \partial\Gamma_\theta(q)) \leq d_K^H(z, z') \leq d_K^H(z, w') = d_K^H(z, \partial\Gamma_\theta(q)).$$

$\Longrightarrow$

$$d_K^H(z, z') = d_K^H(z, \partial\Gamma_\theta(q)) = \ln(\tan\theta + \sec\theta).$$

■

3.2 Invariant Measures

Definition 3.2.1 *Let $\Omega \subset \mathbb{C}^n$ be a domain, with $p \in \Omega$. Let $\{\xi_k\}_{k=1}^m \subset T_p\Omega$ be a set of linearly independent vectors. Let M be a (m, m) form at p, with $1 \le m \le n$, such that $M(\xi_1, \ldots, \xi_m, \overline{\xi}_1, \ldots, \overline{\xi}_m) = 1$. Put*

$$\mu_m = \prod_{j=1}^{m} \frac{i}{2} dz_j \wedge d\overline{z}_j.$$

Let $U := \mathbb{B}_{m-j} \times \Delta_j$, where $0 \le j \le m$. Then, define the Eisenman-Kobayashi m-measure on Ω at p with M by

$$K_U^{\Omega}(p; \xi_1, \ldots, \xi_m) = \inf \left\{ \frac{1}{\alpha} \,\middle|\, \exists \Phi \in Hol(U, \Omega),\ \Phi(0) = p,\ \Phi^*(M)_0 = \alpha\mu_m,\ \alpha > 0 \right\}.$$

Similarly, define the Carathéodory m-measure on Ω at p with M by

$$C_U^{\Omega}(p; \xi_1, \ldots, \xi_m) = \sup \left\{ \beta \,\middle|\, \exists \Phi \in Hol(\Omega, U),\ \Phi(p) = 0,\ \Phi^*(\mu_m)_p = \beta M,\ \beta > 0 \right\}.$$

Now, properties of these measures will be given as a sequence of lemmas.

Lemma 3.2.2 (Decreasing Properties) *Suppose $\phi : \Omega_1 \longrightarrow \Omega_2$ is holomorphic, where $p \in \Omega_1$, $\Omega_1 \subset \mathbb{C}^n$, and $\Omega_2 \subset \mathbb{C}^{n'}$. Let U be as above, where $0 \le j \le m \le$ $min\{n, n'\}$. Then, if $\{\xi_k\}_{k=1}^m$ and $\{\phi_*(p)\xi_k\}_{k=1}^m$ are linearly independent,*

$$K_U^{\Omega_1}(p; \xi_1, \ldots, \xi_m) \ge K_U^{\Omega_2}(\phi(p); \phi_*(p)\xi_1, \ldots, \phi_*(p)\xi_m)$$

and

$$C_U^{\Omega_1}(p; \xi_1, \ldots, \xi_m) \ge C_U^{\Omega_2}(\phi(p); \phi_*(p)\xi_1, \ldots, \phi_*(p)\xi_m).$$

Proof. First, show $K_U^{\Omega_1}(p; \xi_1, \ldots, \xi_m) \ge K_U^{\Omega_2}(\phi(p); \phi_*(p)\xi_1, \ldots, \phi_*(p)\xi_m)$. Let p and ϕ be as above, and M an (m, m) form on Ω_1 such that

$M(\xi_1, \ldots, \xi_m, \overline{\xi}_1, \ldots, \overline{\xi}_m) = 1$. Suppose $\Phi \in \mathrm{Hol}(U, \Omega_1)$ such that $\Phi(0) = p$ and $\Phi^*(M)_0 = \alpha\mu_m$. Let M' be an (m, m) form on Ω_2 such that $(\phi^* M')_p = M$. (Note that this means that $M'(\phi_*(p)\xi_1, \ldots, \phi_*(p)\xi_m, \phi_*(p)\overline{\xi}_1, \ldots, \phi_*(p)\overline{\xi}_m) = 1$.) Consider the mapping $\phi \circ \Phi : U \longrightarrow \Omega_2$; it is holomorphic mapping that satisfies $(\phi \circ \Phi)(0) = \phi(p)$ and $(\phi \circ \Phi)^* M' = \Phi^* (\phi^* M') = \alpha \mu_m$. Therefore,

$$\frac{1}{\alpha} \geq K_U^{\Omega_2}(\phi(p); \phi_*(p)\xi_1, \ldots, \phi_*(p)\xi_m);$$

taking the inf over all $\Phi \in \mathrm{Hol}(U, \Omega_1)$ with the desired property

$\Rightarrow$

$$K_U^{\Omega_1}(p; \xi_1, \ldots, \xi_m) \geq K_U^{\Omega_2}(\phi(p); \phi_*(p)\xi_1, \ldots, \phi_*(p)\xi_m).$$

For the Carathéodory case, let $\Phi \in \mathrm{Hol}(\Omega_2, U)$ such that $\Phi(\phi(p)) = 0$ and $\Phi^*(\mu_m)_{\phi(p)} = \beta M'$ where M' is an (m, m) form on Ω_2 satisfying $M'(\phi_*(p)\xi_1, \ldots, \phi_*(p)\xi_m, \phi_*(p)\overline{\xi}_1, \ldots, \phi_*(p)\overline{\xi}_m) = 1$. Consider the mapping $\Phi \circ \phi : \Omega_1 \longrightarrow U$; it is holomorphic and satisfies $(\Phi \circ \phi)(p) = 0$ and $(\Phi \circ \phi)^*(\mu_m)_p = \phi^*(\Phi^*(\mu_m)_{\phi(p)}) = \phi^*(\beta M')_p = \beta \phi^*(M')_p$. (Note that $(\phi^* M')_p(\xi_1, \ldots, \xi_m, \overline{\xi}_1, \ldots, \overline{\xi}_m) = M'(\phi_*(p)\xi_1, \ldots, \phi_*(p)\xi_m, \phi_*(p)\overline{\xi}_1, \ldots, \phi_*(p)\overline{\xi}_m) = 1$.) Hence

$$C_U^{\Omega_1}(p; \xi_1, \ldots, \xi_m) \geq \beta;$$

taking the sup over all $\Phi \in \mathrm{Hol}(\Omega_2, U)$ with the desired property yields

$$C_U^{\Omega_1}(p; \xi_1, \ldots, \xi_m) \geq C_U^{\Omega_2}(\phi(p); \phi_*(p)\xi_1, \ldots, \phi_*(p)\xi_m).$$

■

Corollary 3.2.3 *Suppose $\Omega_1, \Omega_2 \subset \mathbb{C}^n$ are domains and U is as above, where $0 \leq j \leq m \leq n$. Let $p \in \Omega_1$ and $\{\xi_k\}_{k=1}^m \subset T_p\Omega_1$ be linearly independent. If $\phi : \Omega_1 \longrightarrow \Omega_2$ is a biholomorphism, then*

$$K_U^{\Omega_1}(p; \xi_1, \ldots, \xi_m) = K_U^{\Omega_2}(\phi(p); \phi_*(p)\xi_1, \ldots, \phi_*(p)\xi_m)$$

and

$$C_U^{\Omega_1}(p;\xi_1,\ldots,\xi_m) = C_U^{\Omega_2}(\phi(p);\phi_*(p)\xi_1,\ldots,\phi_*(p)\xi_m).$$

Proof. First, the result will be shown for the Eisenman-Kobayashi case. By Lemma 3.2.2,

$$K_U^{\Omega_1}(p;\xi_1,\ldots,\xi_m) \geq K_U^{\Omega_2}(\phi(p);\phi_*(p)\xi_1,\ldots,\phi_*(p)\xi_m).$$

Now, apply that same lemma to the mapping ϕ^{-1} to obtain

$$K_U^{\Omega_2}(\phi(p);\phi_*(p)\xi_1,\ldots,\phi_*(p)\xi_m) \geq K_U^{\Omega_1}(p;\phi_*^{-1}(\phi(p))(\phi_*(p)\xi_1),\ldots,\phi_*^{-1}(\phi(p))(\phi_*(p)\xi_m)).$$

But, $\phi_*^{-1}(\phi(p))(\phi_*(p)\xi_k) = (\phi^{-1}\circ\phi)_*(p)\xi_k = (1_{\Omega_1})_*(p)\xi_k = \xi_k$, where 1_{Ω_1} is the identity mapping on Ω_1. Therefore,

$$K_U^{\Omega_2}(\phi(p);\phi_*(p)\xi_1,\ldots,\phi_*(p)\xi_m) \geq K_U^{\Omega_1}(p;\xi_1,\ldots,\xi_m)$$

$\Rightarrow$

$$K_U^{\Omega_1}(p;\xi_1,\ldots,\xi_m) = K_U^{\Omega_2}(\phi(p);\phi_*(p)\xi_1,\ldots,\phi_*(p)\xi_m).$$

The same exact argument proves the result for the Carathéodory case. ■

Lemma 3.2.4 *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain, with $p \in \Omega$. Let $\{\xi_k\}_{k=1}^m \subset T_p\Omega$ be linearly independent and let M be a (m,m) form at p such that $M(\xi_1,\ldots,\xi_m,\overline{\xi}_1,\ldots,\overline{\xi}_m) = 1$, i.e.*

$$M = \frac{1}{V}\mu_m,$$

where V = volume of $\xi_1,\ldots,\xi_m$. Then,

$$K_U^{\Omega}(p;\xi_1,\ldots,\xi_m) = V\cdot K_U^{\Omega}(p;e_1,\ldots,e_m)$$

and

$$C_U^{\Omega}(p;\xi_1,\ldots,\xi_m) = V\cdot C_U^{\Omega}(p;e_1,\ldots,e_m),$$

where $\{e_k\}_{k=1}^m$ *is an orthonormal basis for span*$(\xi_1, \ldots, \xi_m)$ *(subspace of* $T_p\Omega$ *spanned by* $\xi_1, \ldots, \xi_m$*).*

Proof. Recall that

$$K_U^\Omega(p; \xi_1, \ldots, \xi_m) = \inf\left\{\frac{1}{\alpha}\,\middle|\,\exists\Phi \in \text{Hol}(U, \Omega),\ \Phi(0) = p,\ \Phi^*(M)_0 = \alpha\mu_m,\ \alpha > 0\right\}.$$

Let $\Phi \in \text{Hol}(U, \Omega)$ such that $\Phi(0) = p$ and $\Phi^*(M)_0 = \alpha\mu_m$. Now $\Phi^*(M)_0 = \alpha\mu_m \Rightarrow \frac{1}{V}\Phi^*(\mu_m)_0 = \alpha\mu_m \Rightarrow \Phi^*(\mu_m)_0 = (V\alpha)\,\mu_m$. Therefore,

$$K_U^\Omega(p; e_1, \ldots, e_m) \leq \frac{1}{V\alpha} = \frac{1}{V} \cdot \frac{1}{\alpha}.$$

Taking the infimum over $\frac{1}{\alpha}$ implies that

$$K_U^\Omega(p; e_1, \ldots, e_m) \leq \frac{1}{V} \cdot K_U^\Omega(p; \xi_1, \ldots, \xi_m)$$

$\implies$

$$V \cdot K_U^\Omega(p; e_1, \ldots, e_m) \leq K_U^\Omega(p; \xi_1, \ldots, \xi_m).$$

To show the reverse inequality, let $\Phi \in \text{Hol}(U, \Omega)$ such that $\Phi(0) = p$ and $\Phi^*(\mu_m)_0 = \alpha\mu_m$. Now, $\Phi^*(\mu_m)_0 = \alpha\mu_m \Rightarrow \Phi^*(\frac{\mu_m}{V})_0 = \frac{\alpha}{V}\mu_m \Rightarrow \Phi^*(M)_0 = \frac{\alpha}{V}\mu_m$, which implies that

$$K_U^\Omega(p; \xi_1, \ldots, \xi_m) \leq \frac{V}{\alpha} = V \cdot \frac{1}{\alpha}.$$

Taking the infimum over $\frac{1}{\alpha}$ gives that

$$K_U^\Omega(p; \xi_1, \ldots, \xi_m) \leq V \cdot K_U^\Omega(p; e_1, \ldots, e_m)$$

$\implies$

$$K_U^\Omega(p; \xi_1, \ldots, \xi_m) = V \cdot K_U^\Omega(p; e_1, \ldots, e_m).$$

Now, for the Carathéodory case. Recall that

$$C_U^\Omega(p; \xi_1, \ldots, \xi_m) = \sup\left\{\beta\,\middle|\,\exists\Phi \in \text{Hol}(\Omega, U),\ \Phi(p) = 0,\ \Phi^*(\mu_m)_p = \beta M,\ \beta > 0\right\}.$$

Let $\Phi \in \text{Hol}(\Omega, U)$ such that $\Phi(p) = 0$ and $\Phi^*(\mu_m)_p = \beta M$. Note that this means that $\Phi^*(\mu_m)_p = \frac{\beta}{V}\mu_m$. Hence,

$$C_U^\Omega(p; e_1, \ldots, e_m) \geq \frac{\beta}{V}$$

$\Longrightarrow$

$$V \cdot C_U^\Omega(p; e_1, \ldots, e_m) \geq \beta;$$

taking the supremum over all β implies that

$$V \cdot C_U^\Omega(p; e_1, \ldots, e_m) \geq C_U^\Omega(p; \xi_1, \ldots, \xi_m).$$

For the reverse inequlity, let $\Phi \in \text{Hol}(\Omega, U)$ such that $\Phi(p) = 0$ and $\Phi^*(\mu_m)_p = \beta\mu_m$. This impies that $\Phi^*(\mu_m)_p = (\beta V)\frac{\mu_m}{V} = (\beta V)\, M$. Hence,

$$C_U^\Omega(p; \xi_1, \ldots, \xi_m) \geq \beta V.$$

Taking the supremum over all β implies that

$$C_U^\Omega(p; \xi_1, \ldots, \xi_m) \geq V \cdot C_U^\Omega(p; e_1, \ldots, e_m)$$

$\Longrightarrow$

$$C_U^\Omega(p; \xi_1, \ldots, \xi_m) = V \cdot C_U^\Omega(p; e_1, \ldots, e_m).$$

■

The following theorem, known as the Carathéodory-Cartan-Kaup-Wu Theorem, will be needed for some of the subsequent lemmas. This statement is taken from [K], page 444.

Theorem 3.2.5 (Carathéodory-Cartan-Kaup-Wu Theorem) *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain. Let $f \in Hol(\Omega, \Omega)$ and $p \in \Omega$. Assume that $f(p) = p$. Then:*

1. *The eigenvalues of $df(p)$ all have modulus not exceeding 1;*

2. $|\det df(p)| \leq 1$*;*

3. If $|\det df(p)| = 1$*, then* $f \in Aut(\Omega)$*;*

4. If $df(p) = I_n$*, then* $f = 1_\Omega$*.*

Lemma 3.2.6 *Let* $\Omega \subset \mathbb{C}^n$ *be a domain and let* U *be as above. Suppose* $p \in \Omega$ *and* $\{\xi_k\}_{k=1}^m \subset T_p\Omega$ *be a linearly independent set of vectors. Then,*

$$\frac{C_U^\Omega(p; \xi_1, \ldots, \xi_m)}{K_U^\Omega(p; \xi_1, \ldots, \xi_m)} \leq 1$$

Proof. Let $\Phi \in \mathrm{Hol}(U, \Omega)$ such that $\Phi(0) = p$ and $\Phi^*(M)_0 = \alpha\mu_m$; $\Psi \in \mathrm{Hol}(\Omega, U)$ such that $\Psi(p) = 0$ and $\Psi^*(\mu_m)_p = \beta M$, where M is an (m, m) form on Ω such that $M(\xi_1, \ldots, \xi_m, \overline{\xi}_1, \ldots, \overline{\xi}_m) = 1$.

Consider $\Psi \circ \Phi \in \mathrm{Hol}(U, U)$: $(\Psi \circ \Phi)(0) = 0$ and $(\Psi \circ \Phi)^*(\mu_m)_0 = \Phi^*((\Psi^*\mu_m)_p)_0 = \Phi^*(\beta M)_0 = \beta\Phi^*(M)_0 = (\beta\alpha)\mu_m \Rightarrow |\det d(\Psi \circ \Phi)(0)|^2 = \beta\alpha$; since $|\det d(\Psi \circ \Phi)(0)|^2 \leq 1$ by the Carathéodory-Cartan-Kaup-Wu Theorem, $\beta\alpha \leq 1 \Rightarrow \beta \leq \frac{1}{\alpha}$. Hence,

$$C_U^\Omega(p; \xi_1, \ldots, \xi_m) \leq K_U^\Omega(p; \xi_1, \ldots, \xi_m)$$

after taking the infimum over all Φ and the supremum over all Ψ. The result follows. ∎

Lemma 3.2.7 *Let* $\Omega \subset \mathbb{C}^n$*,* $p \in \Omega$*, and* $U = \mathbb{B}_{n-j} \times \Delta_j$ *(i.e.* $U \subset \mathbb{C}^n$*). Let* M *be an* (n, n) *form on* Ω *defined as*

$$M = \frac{1}{V} \prod_{j=1}^{n} \frac{i}{2} dz_j \wedge d\overline{z}_j,$$

where V *= volume of* $\xi_1, \ldots, \xi_n$ *(i.e.* $M(\xi_1, \ldots, \xi_n, \overline{\xi}_1, \ldots, \overline{\xi}_n) = 1$*. Then,*

$$K_U^\Omega(p; \xi_1, \ldots, \xi_m) = V \cdot \left| M_\Omega^E(p) \right|$$

and

$$C_U^\Omega(p; \xi_1, \ldots, \xi_m) = V \cdot \left| M_\Omega^C(p) \right|,$$

where

$$\left|M_\Omega^E(p)\right| = \inf\left\{\frac{1}{|\det d\Phi(0)|^2}\,\middle|\,\exists\Phi\in Hol(U,\Omega),\ \Phi(0)=p\right\}$$

and

$$\left|M_\Omega^C(p)\right| = \sup\left\{|\det d\Phi(p)|^2\,\middle|\,\exists\Phi\in Hol(\Omega,U),\ \Phi(p)=0\right\}$$

Proof. First, the Eisenman/Kobayashi case. Let $\Phi\in\mathrm{Hol}(U,\Omega)$ such that $\Phi(0)=p$ and $\Phi^*(M)_0=\alpha\mu_n$. Now,

$$M=\frac{1}{V}\prod_{j=1}^{n}\frac{i}{2}dz_j\wedge d\bar{z}_j=\frac{1}{V}\mu_n$$

$\Rightarrow \Phi^*(M)_0=\frac{1}{V}\cdot|\det d\Phi(0)|^2\mu_n \Rightarrow \frac{1}{V}\cdot|\det d\Phi(0)|^2\mu_n=\alpha\mu_n \Rightarrow \frac{1}{V}\cdot|\det d\Phi(0)|^2=\alpha.$

So,

$$\frac{1}{\alpha}=\frac{V}{|\det d\Phi(0)|^2}.$$

Therefore,

$$\begin{aligned}K_U^\Omega(p;\xi_1,\ldots,\xi_m)&=\inf\left\{\frac{V}{|\det d\Phi(0)|^2}\,\middle|\,\exists\Phi\in\mathrm{Hol}(U,\Omega),\ \Phi(0)=p\right\}\\&=V\cdot\inf\left\{\frac{1}{|\det d\Phi(0)|^2}\,\middle|\,\exists\Phi\in\mathrm{Hol}(U,\Omega),\ \Phi(0)=p\right\}\\&=V\cdot\left|M_\Omega^E(p)\right|.\end{aligned}$$

Now, the Carathéodory case. Let $\Phi\in\mathrm{Hol}(\Omega,U)$ such that $\Phi(p)=0$ and $\Phi^*(\mu_n)_p=\beta M$. Now, $\Phi^*(\mu_n)_p=|\det d\Phi(p)|^2\mu_n\Rightarrow|\det d\Phi(p)|^2\mu_n=\beta M\Rightarrow|\det d\Phi(p)|^2\mu_n=\frac{\beta}{V}\mu_n\Rightarrow|\det d\Phi(p)|^2=\frac{\beta}{V}$. Hence,

$$V\cdot|\det d\Phi(p)|^2=\beta.$$

Therefore, using the same reasoning as before,

$$C_U^\Omega(p;\xi_1,\ldots,\xi_m)=V\cdot\left|M_\Omega^C(p)\right|.$$

■

Lemma 3.2.8 *If $U = \mathbb{B}_{n-j} \times \Delta_j$, then Aut(U) is transitive.*

Proof. Let $\mathbf{a} = (a_1, \ldots, a_{n-j}, a_{n-j+1}, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_{n-j}, b_{n-j+1}, \ldots, b_n) \in U$. Let $\phi \in \text{Aut}(\mathbb{B}_{n-j}), \varphi \in \text{Aut}(\Delta_j)$ be such that $\phi(a_1, \ldots, a_{n-j}) = (b_1, \ldots, b_{n-j})$ and $\varphi(a_{n-j+1}, \ldots, a_n) = (b_{n-j+1}, \ldots, b_n)$ (we can find ϕ and φ since $\text{Aut}(\mathbb{B}_{n-j})$ and $\text{Aut}(\Delta_j)$ are both transitive). Let $\Phi \in \text{Aut}(U)$ be defined as $\Phi(z_1, \ldots, z_{n-j}, z_{n-j+1}, \ldots, z_n) = (\phi(z_1, \ldots, z_{n-j}), \varphi(z_{n-j+1}, \ldots, z_n))$. Then, $\Phi(\mathbf{a}) = \mathbf{b} \Rightarrow \text{Aut}(U)$ is transitive. ■

Lemma 3.2.9 *Let $\Omega \subset \mathbb{C}^n$, $p \in \Omega$, and $\{\xi_k\}_{k=1}^m \subset T_p\Omega$ be linearly independent. Then*

$$\frac{C_U^\Omega(p; \xi_1, \ldots, \xi_m)}{K_U^\Omega(p; \xi_1, \ldots, \xi_m)} = 1$$

if and only if Ω is biholomorphic to U.

Proof. ($\Leftarrow$) Suppose $\Omega \cong U$. Then, U must be a domain in $\mathbb{C}^n$, so $U = \mathbb{B}_{n-j} \times \Delta_j$. Therefore, to show that

$$\frac{C_U^\Omega(p; \xi_1, \ldots, \xi_m)}{K_U^\Omega(p; \xi_1, \ldots, \xi_m)} = 1,$$

it suffices to show that

$$\frac{\left|M_\Omega^C(p)\right|}{\left|M_\Omega^E(p)\right|} = 1$$

by Lemma 3.2.7.

Denote the biholomorphism between Ω and U by ϕ, $\phi : \Omega \longrightarrow U$. Let $\varphi \in \text{Aut}(U)$ be the automorphism that maps $\phi(p) \mapsto 0$ (recall that $\text{Aut}(U)$ is transitive by the previous lemma). Consider the mapping $\Phi := \varphi \circ \phi$; Φ is a biholomorphic mapping between Ω and U and $\Phi(p) = 0$. Therefore, by Corollary 3.2.3,

$$\left|M_\Omega^C(p)\right| = |\det d\Phi(p)|^2 \cdot \left|M_U^C(0)\right|.$$

Taking the supremum over all $\Phi \in \text{Hol}(\Omega, U)$ such that $\Phi(p) = 0$

$\Rightarrow$

$$\left|M_\Omega^C(p)\right| \leq \left|M_\Omega^C(p)\right| \cdot \left|M_U^C(0)\right|$$

$\Rightarrow$

$$1 \leq \left| M_U^C(0) \right| .$$

Now, consider the inverse mapping $\phi^{-1} : U \longrightarrow \Omega$. Let $\psi \in \mathrm{Aut}(\Omega)$ be the automorphism that maps $\phi^{-1}(0) \mapsto p$. Therefore, let $\Psi := \psi \circ \phi^{-1}$; Ψ is a biholomorphism, with $\Psi(0) = p$. Hence,

$$\left| M_U^E(0) \right| = \left| \det d\Psi(0) \right|^2 \cdot \left| M_\Omega^E(p) \right|$$

$\Rightarrow$

$$\frac{1}{\left| \det d\Psi(0) \right|^2} = \frac{\left| M_\Omega^E(p) \right|}{\left| M_U^E(0) \right|};$$

taking the infimum over all $\Psi \in \mathrm{Hol}(U, \Omega)$ with the property that $\Psi(0) = p \Rightarrow$

$$\left| M_\Omega^E(p) \right| \leq \frac{\left| M_\Omega^E(p) \right|}{\left| M_U^E(0) \right|}$$

$\Rightarrow$

$$\left| M_U^E(0) \right| \leq 1.$$

Hence,

$$\left| M_U^E(0) \right| \leq 1 \leq \left| M_U^C(0) \right| .$$

By Lemma 3.2.6,

$$\left| M_U^C(0) \right| \leq \left| M_U^E(0) \right|$$

$\Rightarrow$

$$\left| M_U^C(0) \right| = \left| M_U^E(0) \right| .$$

Hence,

$$\frac{\left| M_\Omega^C(p) \right|}{\left| M_\Omega^E(p) \right|} = \frac{\left| \det d\Phi(p) \right|^2 \cdot \left| M_U^C(0) \right|}{\left| \det d\Phi(p) \right|^2 \cdot \left| M_U^E(0) \right|} = \frac{\left| M_U^C(0) \right|}{\left| M_U^E(0) \right|} = 1.$$

($\Rightarrow$) This implication is due to Bun Wong (see Theorem E in [W1]). Since two domains of different dimension are not even homeomorphic, U must be a domain in $\mathbb{C}^n$, i.e.

$U = \mathbb{B}_{n-j} \times \Delta_j$. Therefore the assumption that

$$\frac{C_U^\Omega(p;\xi_1,\ldots,\xi_m)}{K_U^\Omega(p;\xi_1,\ldots,\xi_m)} = 1$$

is equivalent to

$$\frac{\left|M_\Omega^C(p)\right|}{\left|M_\Omega^E(p)\right|} = 1$$

by Lemma 3.2.7.

Let $\{f_j\} \subset \mathrm{Hol}(\Omega, U)$ be such that $f_j(p) = 0$ and $|\det df_j(p)|^2 \longrightarrow \left|M_\Omega^C(p)\right|$. Note that $\{F \in \mathrm{Hol}(\Omega, U)\,,\, F(p) = 0\}$ is a compact subset of $\mathrm{Hol}(\Omega, U)$, so there exists a subsequence $\{f_{j_k}\} \subset \{f_j\}$ such that $f_{j_k}(p) = 0$ and $f_{j_k} \longrightarrow f \in \mathrm{Hol}(\Omega, U)$ (in the CO topology, which is equivalent to uniform convergence on compact sets) . Hence, $|\det df_j(p)|^2 \longrightarrow |\det df(p)|^2$, which implies that $|\det df(p)|^2 = \left|M_\Omega^C(p)\right|$.

Let $\{g_j\} \subset \mathrm{Hol}(U, \Omega)$ such that $g_j(0) = p$ and

$$\frac{1}{|\det dg_j(0)|^2} \longrightarrow \left|M_\Omega^E(p)\right|.$$

Since Ω is a bounded subset of $\mathbb{C}^n$, $\{G \in \mathrm{Hol}(U, \Omega)\,,\, G(0) = p\}$ is a compact subset of $\mathrm{Hol}(U, \Omega)$, so there exists a subsequence $\{g_{j_k}\} \subset \{g_j\}$ such that $g_{j_k}(0) = p$ and $g_{j_k} \longrightarrow g \in \mathrm{Hol}(U, \Omega)$ (uniformly on compact sets)

$\Longrightarrow$

$$\frac{1}{|\det dg_{j_k}(0)|^2} \longrightarrow \frac{1}{|\det dg(0)|^2}$$

$\Longrightarrow$

$$\frac{1}{|\det dg(0)|^2} = \left|M_\Omega^E(p)\right|.$$

Consider $f \circ g \in \mathrm{Hol}(U, U)$. $(f \circ g)(0) = 0$ and

$$\begin{aligned}|\det d(f \circ g)(0)|^2 &= |\det df(p)|^2 \cdot |\det dg(0)|^2 \\ &= \frac{\left|M_\Omega^C(p)\right|}{\left|M_\Omega^E(p)\right|} = 1\end{aligned}$$

$\Longrightarrow |\det d(f \circ g)(0)| = 1 \Longrightarrow f \circ g \in \mathrm{Aut}(U)$ by the Carathéodory-Cartan-Kaup-Wu theorem $\Longrightarrow g$ is injective.

Consider $g \circ f \in \mathrm{Hol}(\Omega, \Omega)$. $(g \circ f)(p) = p$ and

$$\begin{aligned} |\det d(g \circ f)(p)|^2 &= |\det dg(0)|^2 \cdot |\det df(p)|^2 \\ &= \frac{\left|M_\Omega^C(p)\right|}{\left|M_\Omega^E(p)\right|} = 1 \end{aligned}$$

$\Longrightarrow |\det d(g \circ f)(p)| = 1 \Longrightarrow g \circ f \in \mathrm{Aut}(\Omega)$ by the Carathéodory-Cartan-Kaup-Wu theorem $\Longrightarrow g$ is surjective $\Longrightarrow g : \Omega \longrightarrow U$ is biholomorphic. ■

Lemma 3.2.10 *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with a C^2 boundary and $q \in \partial\Omega$ strongly pseudoconvex. Let W be a neighborhood of q in $\mathbb{C}^n$. Then,*

$$\frac{K_U^\Omega(p; \xi_1, \ldots, \xi_m)}{K_U^{\Omega \cap W}(p; \xi_1, \ldots, \xi_m)} \longrightarrow 1$$

and

$$\frac{C_U^\Omega(p; \xi_1, \ldots, \xi_m)}{C_U^{\Omega \cap W}(p; \xi_1, \ldots, \xi_m)} \longrightarrow 1$$

as $p \longrightarrow q$.

Proof. Should follow using similar arguments found in [G] or Chapter 11.3 in [K]. ■

The following two lemmas deal with the invariant measures defined for a specific domain $W_\beta := \Gamma_\beta^r(q) \times \mathbb{B}_{n-1}^\epsilon$, where

1. $\Gamma_\beta^r(q)$ is a cone in $\mathbb{C}$ with vertex at $q \in \mathbb{C}$, angle $\beta \in [0, \pi]$, and radius r.

2. $\mathbb{B}_{n-1}^\epsilon$ is $n-1$ dimensional ball of radius ϵ.

In particular, the invariant measures for W_β will be defined with respect to the product domain $U = \Delta \times \mathbb{B}_{n-1}$. In other words, let

$$\left|M_{W_\beta}^E(z)\right| = \inf\left\{ \frac{1}{|\det df(0)|^2} \;\middle|\; f \in \mathrm{Hol}(U, W_\beta), f(0) = z \right\}$$

and

$$\left| M^C_{W_\beta}(z) \right| = \sup \left\{ |\det df(z)|^2 \;\middle|\; f \in \mathrm{Hol}(W_\beta, U), f(z) = 0 \right\}.$$

Lemma 3.2.11 *For the wedge domain W_β described above,*

$$\left| M^C_{W_\beta}(z,w) \right| \geq \left| M^C_{\Gamma^r_\beta(q)}(z) \right| \cdot \left| M^C_{\mathbb{B}^\epsilon_{n-1}}(w) \right|,$$

where $z \in \Gamma^r_\beta(q)$ and $w \in \mathbb{B}^\epsilon_{n-1}$ (the Carathéodory measure for $\Gamma^r_\beta(q)$ and $\mathbb{B}^\epsilon_{n-1}$ are defined with respect to Δ and $\mathbb{B}_{n-1}$, respectively).

Proof. Let $g \in \mathrm{Hol}\left(\Gamma^r_\beta(q), \Delta\right)$, $h \in \mathrm{Hol}(\mathbb{B}^\epsilon_{n-1}, \mathbb{B}_{n-1})$, where $g(z) = h(w) = 0$. Hence, the mapping $f(z,w) := (g(z), h(w))$ is a holomorphic mapping from W_β to $\Delta \times \mathbb{B}_{n-1}$ with the property that $f(z,w) = (0,0)$. Let $(z,w) = (z, w_1, \ldots, w_{n-1})$ be the coordinates for $\mathbb{C}^n$. Hence, the mapping f takes $(z, w_1, \ldots, w_{n-1})$ to $(g(z), h_1(w_1, \ldots, w_{n-1}), \ldots, h_{n-1}(w_1, \ldots, w_{n-1}))$. Therefore,

$$df = \begin{pmatrix} \frac{\partial g}{\partial z} & \frac{\partial g}{\partial w_1} & \cdots & \frac{\partial g}{\partial w_{n-1}} \\ \frac{\partial h_1}{\partial z} & \frac{\partial h_1}{\partial w_1} & \cdots & \frac{\partial h_1}{\partial w_{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial h_{n-1}}{\partial z} & \frac{\partial h_{n-1}}{\partial w_1} & \cdots & \frac{\partial h_{n-1}}{\partial w_{n-1}} \end{pmatrix} = \begin{pmatrix} \frac{\partial g}{\partial z} & 0 & \cdots & 0 \\ 0 & \frac{\partial h_1}{\partial w_1} & \cdots & \frac{\partial h_1}{\partial w_{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{\partial h_{n-1}}{\partial w_1} & \cdots & \frac{\partial h_{n-1}}{\partial w_{n-1}} \end{pmatrix}$$

$\Rightarrow$

$$\det df(z,w) = \det dg(z) \cdot \det dh(w)$$

$\Rightarrow$

$$|\det df(z,w)|^2 = |\det dg(z)|^2 \cdot |\det dh(w)|^2.$$

Now, taking the supremum over all $f \in \mathrm{Hol}\,(W_\beta, \Delta \times \mathbb{B}_{n-1})$ implies that

$$\left| M^C_{W_\beta}(z,w) \right| \geq |\det dg(z)|^2 \cdot |\det dh(w)|^2$$

$\Rightarrow$

$$\left|M^C_{W_\beta}(z,w)\right| \geq \left|M^C_{\Gamma^r_\beta(q)}(z)\right| \cdot \left|M^C_{\mathbb{B}^\epsilon_{n-1}}(w)\right|$$

after taking the sup over all g and h. ■

Lemma 3.2.12 *For the wedge domain W_β described above,*

$$\left|M^E_{W_\beta}(z,w)\right| \leq \left|M^E_{\Gamma^r_\beta(q)}(z)\right| \cdot \left|M^E_{\mathbb{B}^\epsilon_{n-1}}(w)\right|.$$

Proof. The proof is almost the same as the previous one. Let $g \in \mathrm{Hol}\left(\Delta, \Gamma^r_\beta(q)\right)$, $h \in \mathrm{Hol}(\mathbb{B}_{n-1}, \mathbb{B}^\epsilon_{n-1})$, where $g(0) = z$ and $h(0) = w$. Hence the mapping $f : \Delta \times \mathbb{B}_{n-1} \longrightarrow W_\beta$, $(z,w) = (z, w_1, \ldots, w_{n-1}) \longmapsto (g(z), h(w))$, is holomorphic and $f(0,0) = (z,w)$. Like in the case of the previous lemma,

$$|\det df(0,0)|^2 = |\det dg(0)|^2 \cdot |\det dh(0)|^2$$

$\Rightarrow$

$$\frac{1}{|\det df(0,0)|^2} = \frac{1}{|\det dg(0)|^2} \cdot \frac{1}{|\det dh(0)|^2}.$$

Taking the inf over all $f \in \mathrm{Hol}\,(\Delta \times \mathbb{B}_{n-1}, W_\beta)$ yields

$$\left|M^E_{W_\beta}(z,w)\right| \leq \frac{1}{|\det dg(0)|^2} \cdot \frac{1}{|\det dh(0)|^2}$$

$\Rightarrow$

$$\left|M^E_{W_\beta}(z,w)\right| \leq \left|M^E_{\Gamma^{\beta,r}_q}(z)\right| \cdot \left|M^E_{\mathbb{B}^\epsilon_{n-1}}(w)\right|$$

after taking the inf over all g and h. ■

For the next lemma, when considering $\Gamma^r_\beta(q)$, assume q is the origin and β is the angle between $\mathrm{Re}\,z$-axis and the arms of the cone. The reason for this will become clear later. This lemma is actually Lemma 2.6 in [FW] (Fu and Wong's paper).

Lemma 3.2.13 *Let θ and α be two numbers such that $0 < \theta < \alpha < \pi$. Let $\{z_j\}$ be a sequence in $\Gamma_\theta^r(q)$ and let ν_j be the angle between the $\mathrm{Re}\, z$-axis and the vector joining q to z_j. Suppose $z_j \to 0$ and $\nu_j \to \nu$. Then, for any $r > 0$,*

$$\lim_{j\to\infty} \frac{\left|M_{\Gamma_\alpha^r(q)}(z_j)\right|}{\left|M_{\Gamma_\theta^r(q)}(z_j)\right|} = \left(\frac{\theta \cos\left(\frac{\pi\nu}{2\theta}\right)}{\alpha \cos\left(\frac{\pi\nu}{2\alpha}\right)}\right)^2,$$

where M is either of the invariant measures.

Proof. Let $w = u + iv$ and $z = x + iy$. Then, $\left|M_H^C(w)\right| = \left|M_H^E(w)\right| = \frac{1}{v^2}$, where H is the upper half-plane (this can be obtained by pulling back the Poincaré metric on Δ via the biholomorphic mapping $z = \frac{i-w}{i+w}$ between H and Δ). Let

$$f(w) := \left(e^{-\pi i/2} w\right)^{2\beta/\pi} = e^{-i\beta} w^{2\beta/\pi};$$

f is a biholomorphic mapping between H and $\Gamma_\beta(q)$, $0 < \beta < \pi$ (r is infinitely large). Then,

$$f^{-1}(z) = e^{\pi i/2} z^{\pi/2\beta}.$$

Let $w_j := f^{-1}(z_j)$, with z_j as in the statement of the lemma. Then, $w_j \to 0$ as $z_j \to 0$.

So, $|M_H(w_j)| = |f'(w_j)|^2 \cdot \left|M_{\Gamma_\beta}(z_j)\right|$ (write Γ_β for $\Gamma_\beta(q)$ for brevity of notation).

Hence,

$$\left|M_{\Gamma_\beta}(z_j)\right| = \frac{1}{(\mathrm{Im}\ w_j)^2} \cdot \frac{1}{|f'(w_j)|^2}.$$

Suppose $z_j = r_j e^{i\nu_j}$. Then,

$$\begin{aligned} w_j = e^{\pi i/2}\left(r_j e^{i\nu_j}\right)^{\pi/2\beta} &= r_j^{\pi/2\beta} e^{i(\pi/2 + \pi i \nu_j/2\beta)} \\ &= r_j^{\pi/2\beta}\left(\cos\left(\frac{\pi}{2} + \frac{\pi\nu_j}{2\beta}\right) + i \sin\left(\frac{\pi}{2} + \frac{\pi\nu_j}{2\beta}\right)\right). \end{aligned}$$

So,

$$\begin{aligned} \mathrm{Im}\ w_j &= r_j^{\pi/2\beta} \sin\left(\frac{\pi}{2} + \frac{\pi\nu_j}{2\beta}\right) \\ &= r_j^{\pi/2\beta} \cos\left(\frac{\pi\nu_j}{2\beta}\right). \end{aligned}$$

Since

$$f'(w) = e^{-i\beta}\left(\frac{2\beta}{\pi}w^{2\beta/\pi-1}\right),$$

$$\left|f'(w_j)\right| = \frac{2\beta}{\pi}\left|w_j\right|^{2\beta/-1}$$

$\Rightarrow$

$$\begin{aligned}\left|M_{\Gamma_\beta}(z_j)\right| &= \frac{1}{r_j^{\pi/\beta}\cos^2\left(\frac{\pi\nu_j}{2\beta}\right)}\cdot\frac{1}{\frac{4\beta^2}{\pi^2}\left|w_j\right|^{4\beta/\pi-2}} \\ &= \frac{1}{r_j^2\cos^2\left(\frac{\pi\nu_j}{2\beta}\right)\frac{4\beta^2}{\pi^2}},\end{aligned}$$

since

$$r_j^{\pi/\beta}\left|w_j\right|^{4\beta/\pi-2} = r_j^{\pi/\beta}\left(r_j^{\pi/2\beta}\right)^{4\beta/\pi-2} = r_j^2.$$

Therefore,

$$\frac{\left|M_{\Gamma_\alpha}(z_j)\right|}{\left|M_{\Gamma_\theta}(z_j)\right|} = \frac{\frac{1}{r_j^2\cos^2\left(\frac{\pi\nu_j}{2\alpha}\right)\frac{4\alpha^2}{\pi^2}}}{\frac{1}{r_j^2\cos^2\left(\frac{\pi\nu_j}{2\theta}\right)\frac{4\theta^2}{\pi^2}}} = \left(\frac{\theta\cos\left(\frac{\pi\nu_j}{2\theta}\right)}{\alpha\cos\left(\frac{\pi\nu_j}{2\alpha}\right)}\right)^2.$$

Since

$$\lim_{j\to\infty}\frac{\left|M_{\Gamma_\beta^r}(z_j)\right|}{\left|M_{\Gamma_\beta}(z_j)\right|} = 1,$$

it follows immediately that

$$\lim_{j\to\infty}\frac{\left|M_{\Gamma_\alpha^r(q)}(z_j)\right|}{\left|M_{\Gamma_\theta^r(q)}(z_j)\right|} = \left(\frac{\theta\cos\left(\frac{\pi\nu}{2\theta}\right)}{\alpha\cos\left(\frac{\pi\nu}{2\alpha}\right)}\right)^2.$$

■

The next lemma is a generalization of Lemma 2.4 in [FW], which will be needed in the sequel. In it, $U = \Delta \times \mathbb{B}_m$ and $\{e_k\}_{k=1}^{m+1} \subset \mathbb{C}^n$ is a set of orthonormal vectors. A sketch of the proof will be given.

Lemma 3.2.14 *Let $\Omega \subset \mathbb{C}^n$ be a bounded convex domain, where $\partial\Omega$ is C^2 smooth and strongly convex near $q \in \partial\Omega$. Let $\widetilde{\Omega} = \Omega \cap B(q;r)$, where $B(q;r) \subset \mathbb{C}^n$. Then,*

$$\overline{\lim_{x\to q}}\frac{C_U^{\widetilde{\Omega}}(x;e_1,\ldots,e_{m+1})}{K_U^{\widetilde{\Omega}}(x;e_1,\ldots,e_{m+1})} < 1.$$

Sketch of Proof. The idea is to approximate $\widetilde{\Omega}$ by analytic elipsoids, which are biholomorphic to balls. The inequivalence of a ball with the product domain U will give the desired conclusion.

Lemma 3.2.15 *Let $\Omega \subset \mathbb{C}^n$ be bounded, $p \in \Omega$, and $\{\xi_k\}_{k=1}^m \subset \mathbb{C}^n$ be linearly independent. Suppose $T : span(\xi_1, \ldots, \xi_m) \longrightarrow span(\xi'_1, \ldots, \xi'_m)$ is an isomorphism, where $T(\xi_k) = \xi'_k$. Then,*

$$C_U^{\Omega}(p; \xi_1, \ldots, \xi_m) \cdot |\det T|^2 = C_U^{\Omega}(p; \xi'_1, \ldots, \xi'_m)$$

and

$$K_U^{\Omega}(p; \xi_1, \ldots, \xi_m) \cdot |\det T|^2 = K_U^{\Omega}(p; \xi'_1, \ldots, \xi'_m).$$

Chapter 4

Boundary Accumulation Points of a Convex Domain in $\mathbb{C}^n$

4.1 Some Geometry of Convex Domains

Definition 4.1.1 *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with a C^1 boundary. Then $\{q_j\} \subset \Omega$ converges to $q \in \partial\Omega$ non-tangentially if there exists $\alpha > 1$ such that, for j large enough,*

$$q_j \in \Gamma_q(\alpha) := \{z \in \Omega \,|\, |z - q| < \alpha dist(z, \partial\Omega)\}.$$

Let ℓ_q be the real normal line to $\partial\Omega$ through q. Then $q_j \longrightarrow q$ normally if $\{q_j\} \subset \ell_q$ for j large enough.

A stronger form of non-tanagential approach is if the sequence $q_j \longrightarrow q \in \partial\Omega$ within the region $\widetilde{\Gamma}_q(\alpha)$, where

$$\widetilde{\Gamma}_q(\alpha) := \left\{z \in \Omega \,|\, 0 \leq \angle zq\tilde{z} < \cos^{-1}(1/\alpha)\right\}, \quad \tilde{z} \in \ell_q.$$

In general, $\widetilde{\Gamma}_q(\alpha) \subset \Gamma_q(\alpha)$; the other inclusion holds when the domain Ω is convex, as is demonstrated in the next lemma.

Lemma 4.1.2 *Let $\Omega \subset \mathbb{C}^n$ be a bounded convex domain with a C^1 boundary. Suppose $q \in \partial\Omega$. Then,*

$$\Gamma_q(\alpha) \subset \widetilde{\Gamma}_q(\alpha).$$

Proof. Define local coordinates so that $q = \mathbf{0}$ and the outward unit normal vector to $\partial\Omega$, ν, is in the Rez_n direction (so, $\nu = (0, \ldots, 0, 1)$). Since Ω is convex, $\Omega \subset \{w \,|\, \mathrm{Re} w_n \leq 0\}$; denote this region by H. Now, $\mathrm{dist}(z, \partial\Omega) \leq \mathrm{dist}(z, \partial H) = |\mathrm{Re} z_n|$.

Let $\pi : \Omega \longrightarrow \ell_q \cap \Omega$, $z \mapsto \tilde{z}$, be the projection of Ω onto ℓ_q. Note that $|\tilde{z}| = |\tilde{z} - q| = |\mathrm{Re} z_n|$. Therefore, if $z \in \Gamma_q(\alpha)$, then

$$|z - q| < \alpha \mathrm{dist}(z, \partial\Omega)$$

$\Longrightarrow$

$$|z - q| < |\tilde{z} - q|\alpha$$

$\Longrightarrow$

$$\frac{1}{\alpha} < \frac{|\tilde{z} - q|}{|z - q|}$$

$\Longrightarrow$

$$\angle zq\tilde{z} = \cos^{-1}\left(\frac{|\tilde{z} - q|}{|z - q|}\right) \leq \cos^{-1}\left(\frac{1}{\alpha}\right)$$

$\Longrightarrow$

$$z \in \widetilde{\Gamma}_q(\alpha).$$

■

Lemma 4.1.3 *Let $\Omega \subset \mathbb{C}^n$ be a bounded convex domain with a C^1 boundary. Suppose there exists $p \in \Omega$ and $\{\phi_j\} \subset Aut(\Omega)$ such that $\phi_j(p) \longrightarrow q \in \partial\Omega$ non-tangentially. Then, there exists $\{p_j\} \subset \Omega$ such that $\phi_j(p_j) \longrightarrow q$ normally and that $d_K^\Omega(p, p_j) \leq r$ for some $r > 0$.*

Proof. Let $q_j := \phi_j(p)$. Since $q_j \longrightarrow q$ non-tangentially, there exists $\alpha > 1$ such that $q_j \in \Gamma_q(\alpha)$ for all j large enough. Let $\pi : \Omega \longrightarrow \Omega \cap \ell_q$ be the projection mapping of Ω onto the real normal line ℓ_q; let $\tilde{q}_j = \pi(q_j)$. Then, $|q_j - \tilde{q}_j| \le |q_j - q| < \alpha \mathrm{dist}(q_j, \partial\Omega) \longrightarrow 0$ as $j \longrightarrow \infty$. Therefore, let $p_j := \phi_j^{-1}(\tilde{q}_j)$. Then, $\tilde{q}_j \longrightarrow q$ normally for all j large enough.

Now by Lemma 4.1.2, since Ω is convex, $q_j \in \widetilde{\Gamma}_q(\alpha)$, so $0 \le \angle q_j q \tilde{q}_j < \cos^{-1}(1/\alpha)$. Hence,

$$\cos\left(\angle q_j q \tilde{q}_j\right) = \frac{|\tilde{q}_j - q|}{|q_j - q|} > \frac{1}{\alpha}.$$

Consider the straight line $\gamma(t) = (1-t)q_j + t\tilde{q}_j$. Then,

$$\begin{aligned} d_K^\Omega(p, p_j) = d_K^\Omega(q_j, \tilde{q}_j) &\le \int_0^1 F_K^\Omega(\gamma(t), \gamma'(t))\, dt \\ &\le \int_0^1 \frac{|\gamma'(t)|}{\mathrm{dist}(\gamma(t), \partial\Omega)}\, dt \le \int_0^1 \frac{\alpha|\gamma'(t)|}{|\gamma'(t) - q|}\, dt \\ &\le \frac{\alpha|\tilde{q}_j - q_j|}{|\tilde{q}_j - q|} \le \frac{\alpha|q_j - q|}{|\tilde{q}_j - q|} < \alpha^2. \end{aligned}$$

Note that the second inequality comes from Lemma 3.1.10. Therefore, letting $r = \alpha^2$, it follows that $d_K^\Omega(p, p_j) < r$ for j large enough. ■

Lemma 4.1.4 *Let $\Omega \subset \mathbb{C}^n$ be a bounded, complete hyperbolic domain with a C^2 boundary. Suppose $p \in \partial\Omega$ is strongly convex. Then, for any fixed $r > 0$, the euclidean diameter of $\beta_K^\Omega(z; r) \longrightarrow 0$ as $z \longrightarrow p$.*

Proof. Let ρ be a local defining function for Ω in a neighborhood of p, where p is strongly convex. Let $z' \in \partial\Omega$ such that $|z - z'| = \mathrm{dist}(z, \partial\Omega)$. Now, since p is strongly convex, for z close to p, the $\partial\Omega$ is strongly convex in a neighborhood of z', and so the diameter of the set

$$\left\{ w \in \Omega \,\middle|\, \mathrm{Re} f(w) > -\sqrt{|z - z'|} \right\}$$

converges to 0 as $z \longrightarrow p$, where

$$f(w) = \frac{1}{|\nabla\rho(z')|} \sum_{j=1}^n \frac{\partial\rho}{\partial z_j}(z')(w_j - z_j').$$

Therefore, the result will follow if it can be shown that

$$\beta_K^\Omega(z;r) \subset \left\{ w \in \Omega \,\middle|\, \mathrm{Re} f(w) > \sqrt{|z - z'|} \right\}.$$

To simplify notation, choose coordinates so that $z' = \mathbf{0}$ and that the outward normal vector to $\partial\Omega$ at z' is in the $\mathrm{Re} z_n$-direction. Then, with respect to these coordinates, $z = (0, \ldots, 0, -x)$ for some $x > 0$. Hence,

$$\begin{aligned} f(w) &= \frac{1}{|\nabla\rho(z')|} \sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(z')(w_j - z_j') \\ &= \frac{1}{|\nabla\rho(\mathbf{0})|} \left(0(w_1 - 0) + \ldots + 0(w_{n-1} - 0) + |\nabla\rho(\mathbf{0})|(w_n - 0)\right) \\ &= w_n \end{aligned}$$

$\Longrightarrow$

$$\mathrm{Re} f(w) = \mathrm{Re} w_n.$$

Therefore, the result will follow if

$$\beta_K^\Omega(z;r) \subset \left\{ w \in \Omega \,\middle|\, \mathrm{Re} w_n > -\sqrt{x} \right\}.$$

Suppose this isn't true, i.e. there exists $w \in \beta_K^\Omega(z;r)$ for which $\mathrm{Re} w_n \leq -\sqrt{x}$. Let $\tilde{w} = (0, \ldots, 0, \mathrm{Re} w_n)$ and $\gamma(t) = (1-t)\tilde{w} + tz$. Therefore,

$$r \geq d_K^\Omega(z, w) \geq d_K^\Omega(z, \tilde{w}) = \int_0^1 F_K^\Omega(\gamma(t), \gamma'(t))\, dt.$$

The first inequality holds because the domain is complete hyperbolic and the second by Lemma 3.1.12; the equality holds by Lemma 3.1.11. Since $\gamma(t)$ is near the stongly pseudoconvex boundary point z', it follows from from a well-known result (see [A] and [G]; an explicit statement is contained in Lemma 3.1.11) that

$$\int_0^1 F_K^\Omega(\gamma(t), \gamma'(t))\, dt \gtrsim \int_0^1 \frac{|\gamma_N'(t)|}{\delta(\gamma(t))}\, dt = \int_0^1 \frac{|-\mathrm{Re} w_n - x|}{|(1-t)\mathrm{Re} w_n - tx|}\, dt.$$

Now, for x close to 0, $x < \sqrt{x} \Rightarrow -x > -\sqrt{x} \geq \mathrm{Re}w_n$. Therefore, $-x \geq \mathrm{Re}w_n \Rightarrow -\mathrm{Re}w_n - x \geq -\mathrm{Re}w_n + \mathrm{Re}w_n = 0$. Hence,

$$\begin{aligned}\int_0^1 \frac{|-\mathrm{Re}w_n - x|}{|(1-t)\mathrm{Re}w_n - tx|}\,dt &= \int_0^1 \frac{(-\mathrm{Re}w_n - x)}{-((1-t)\mathrm{Re}w_n - tx)}\,dt \\ &= -\int_0^1 \frac{(-\mathrm{Re}w_n - x)}{(1-t)\mathrm{Re}w_n - tx}\,dt.\end{aligned}$$

Let $u = (1-t)\mathrm{Re}w_n - tx$. Then, $du = (-\mathrm{Re}w_n - x)dt$, which implies that

$$-\int_0^1 \frac{(-\mathrm{Re}w_n - x)}{(1-t)\mathrm{Re}w_n - tx}\,dt = -\int_{\mathrm{Re}w_n}^{-x} \frac{du}{u} = \ln\left(\frac{|\mathrm{Re}w_n|}{x}\right) \geq \ln\left(\frac{1}{\sqrt{x}}\right).$$

Therefore,

$$\infty > r \geq \ln\left(\frac{1}{\sqrt{x}}\right) \longrightarrow \infty$$

as $x \longrightarrow 0$, which is a contradiction. ■

4.2 Maximal Chain of Analytic Disks

Definition 4.2.1 *Let $\Omega \subset \mathbb{C}^n$ and S a connnected subset of $\partial\Omega$. Then $\partial\Omega$ is geometrically flat along S if $\partial\Omega$ is C^2 in a neighborhood of S and the outward normal direction to $\partial\Omega$ is constant on S.*

Proposition 4.2.2 *Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded convex domain. If $\phi : \Delta \longrightarrow \partial\Omega$ is a holomorphic mapping, the $\partial\Omega$ is geometrically flat along $\phi(\Delta)$.*

Proof. Suppose $\phi : \Delta \longrightarrow \partial\Omega$ is holomorphic and $\phi(0) = q$. Choose holomorphic coordinates $(z_1, \ldots, z_n)$ so that $q = \mathbf{0}$ and the $\mathrm{Re} z_n$ axis is the outward normal direction to $\partial\Omega$ at q. Since Ω is convex, the set $\{w \,|\, \mathrm{Re} w_n = 0\}$ is the real supporting hyperplane to $\partial\Omega$ at q and $\overline{\Omega} \subset \{w \,|\, \mathrm{Re} w_n \leq 0\}$. Let $\mathbf{x} : \mathbb{C}^n \longrightarrow \mathbb{C}$ be the function $(z_1, \ldots, z_n) \mapsto z_n$. Note the $\mathbf{x}$ is holomorphic, and Re $\mathbf{x} = \mathrm{Re} z_n$ is harmonic. Consider the composition $\mathbf{x} \circ \phi$; it is holomorphic and $\mathrm{Re}(\mathbf{x} \circ \phi)(w)$ is harmonic. Now, $\mathrm{Re}(\mathbf{x} \circ \phi)(0) = \mathrm{Re}(\mathbf{x}(q)) = 0$ and $\mathrm{Re}(\mathbf{x} \circ \phi)(w) \leq 0 \;\forall w \in \Delta$ by convexity $\Rightarrow \mathrm{Re}(\mathbf{x} \circ \phi)(w) \equiv 0$ on Δ by the Maximum Principle $\Rightarrow \mathrm{Im}(\mathbf{x} \circ \phi)(w) \equiv 0$ on Δ by the Cauchy-Riemann equations. Therefore, $(\mathbf{x} \circ \phi)(w) = \mathbf{x}|_{\phi(\Delta)} \equiv 0 \Rightarrow \phi(\Delta) \subset \{w \,|\, w_n = 0\} \Rightarrow \partial\Omega$ is geometrically flat along $\phi(\Delta)$.
■

Note that, as a consequence of this proof, $\phi(\Delta)$ is contained in the maximal complex subspace of the real suppoting hyperplane to the boundary at q, i.e. $\phi(\Delta) \subset T_q\partial\Omega$.

Definition 4.2.3 *Let $H \subset \mathbb{C}^n$ and $q \in H$. Then the maximal chain of analytic discs on H through q, denoted Δ_q^H, is*

$$\Delta_q^H = \{z \in H \,|\, \textit{there exists a finite chain of analytic discs joining } z \textit{ to } q\}.$$

In other words, $z \in \Delta_q^H$ if

1. *There exists holomorphic mappings* $\phi_j : \Delta \longrightarrow H$, $1 \leq j \leq k$;

2. *Points* $z_j \in H$, $a_j, b_j \in \Delta$, $1 \leq j \leq k$, *such that* $\phi_j(a_j) = z_{j-1}$ *and* $\phi_j(b_j) = z_j$, *where* $z_0 = z$ *and* $z_k = q$.

From now on, whenever it is clear, the notation $\{\phi_1, \ldots, \phi_k\}$ will be used to represent that z can be joined to q by a finte chain of analytic discs. Also, for the sake of brevity, let $S_q := \Delta_q^H$.

Now, some important properties of S_q, which will be given as a sequence of corollaries.

Corollary 4.2.4 *If* $z \in S_q$, *then* $S_q = S_z$

Proof. ($\subseteq$) Suppose $w \in S_q$. Let $\{\phi_1, \ldots, \phi_k\}$ be a chain of analytic discs joining z to q, and $\{\psi_1, \ldots, \psi_l\}$ a chain of analytic discs joining q to w. Then $\{\phi_1, \ldots, \phi_k, \psi_1, \ldots, \psi_l\}$ is a chain of analytic discs joining z to $w \Rightarrow w \in S_z \Rightarrow S_q \subseteq S_z$.

($\supseteq$) Suppose $w \in S_z$; let $\{\phi_1, \ldots, \phi_k\}$ be a chain of analytic discs joining w to z. Since $z \in S_q$, there exists a chain of analytic discs $\{\psi_1, \ldots, \psi_l\}$ joining z to $q \Rightarrow$ $\{\phi_1, \ldots, \phi_k, \psi_1, \ldots, \psi_l\}$ is a chain joining w to $q \Rightarrow S_z \subseteq S_q \Rightarrow S_z = S_q$. ■

Corollary 4.2.5 *If* $V \subset H$ *is a complex variety through* q, *then* $V \subset S_q$.

Proof. Since any two points of a complex variety can be joined by a chain of analytic disks, it follows immediately that $V \subset S_q$ (see [Ko], p.97). ■

Corollary 4.2.6 *If* $\Omega \subset \mathbb{C}^n$ *is a smoothly bounded convex domain, then* $\partial\Omega$ *is geometrically flat along* S_q *for all* $q \in \partial\Omega$.

Proof. Let $z \in S_q$. Then, there exists a finite chain of analytic discs $\{\phi_1, \ldots, \phi_k\}$ joining z to q. Hence, $\partial\Omega$ is geometrically flat long $\phi_j(\Delta)$, $j = 1, \ldots, k$, by Proposition

4.2.2. Since z was chosen arbitrarily, S_q consists of a union of (potentially arbitrarily many) geometrically flat faces. Since Ω is smooth, then $\partial\Omega$ is geometrically flat along S_q. ■

Theorem 4.2.7 *Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded convex domain. Then S_q is geometrically convex for all $q \in \partial\Omega$, i.e. if $z, w \in S_q$, then the line segment $tz + (1-t)w \in S_q$ for all $t \in [0, 1]$.*

Proof. First, it will be shown that if $z, w \in S_q$, then the line segment $tz + (1-t)w \in \partial\Omega$ for all $t \in [0, 1]$. Since Ω is convex, choose coordinates $(z_1, \ldots, z_n)$ such that $\overline{\Omega} \subset \{z \,|\, \text{Re} z_n \leq 0\}$ and $S_q \subset \{z \,|\, \text{Re} z_n = 0\}$ (since S_q geometrically flat by the previous corollary). Because Ω is convex, $tz + (1-t)w \in \overline{\Omega}$ for all $t \in [0, 1]$, and $\text{Re}(tz + (1-t)w)_n = t\text{Re}z_n + (1-t)\text{Re}w_n = 0$ for all $t \in [0, 1] \Rightarrow tz + (1-t)w \in \partial\Omega$ for all $t \in [0, 1]$.

Now, in order to show that $tz + (1-t)w \in S_q$ for all $t \in [0, 1]$, induction on the length of the chain joining z to w will be used.

Base Case: Let $z, w \in S_q$. Suppose there exists an analytic disc $\phi : \Delta \longrightarrow \partial\Omega$ such that $\phi(a) = z$, $\phi(b) = w$. Consider the mapping

$$(t, \zeta) \longmapsto t\phi(\zeta) + (1-t)\phi(b).$$

For any ζ, $\phi(\zeta)$, $\phi(b) \in S_q \Rightarrow t\phi(\zeta) + (1-t)\phi(b) \in \partial\Omega\ \forall t$ (by above). Now, for any fixed t,

$$t\phi(\zeta) + (1-t)\phi(b) : \Delta \longrightarrow \partial\Omega$$

is an analytic disc. Let $\zeta = b$. Then, $t\phi(b) + (1-t)\phi(b) = \phi(b) = w \Rightarrow$ the image of $t\phi(\zeta) + (1-t)\phi(b)$ contains a point in S_q (it is $w = \phi(b)$) $\Rightarrow t\phi(\zeta) + (1-t)\phi(b) \in S_q$

$\forall\zeta$; since t was arbitrary, $t\phi(\zeta) + (1-t)\phi(b) \in S_q$ $\forall\zeta$ $\forall t$. Therefore, let $\zeta = a$. Then, $t\phi(a) + (1-t)\phi(b) = tz + (1-t)w \in S_q$ $\forall t$.

Induction Hypothesis: Assume the result is true if $z, w \in S_q$ and they can be joined by at most n analytic discs.

Induction Step: Suppose $z, w \in S_q$ and z and w can be joined by $n+1$ analytic discs, i.e. there exists $\phi_1, \ldots, \phi_{n+1} : \Delta \longrightarrow \partial\Omega$, $a_j, b_j \in \Delta$, $z_j \in \partial\Omega$, $j = 1, \ldots, n+1$, such that $\phi_j(a_j) = z_{j-1}$, $\phi_j(b_j) = z_j$, $\phi_1(a_1) = z_0 = z$, and $\phi_{n+1}(b_{n+1}) = z_{n+1} = w$. Now, consider the mapping

$$(t, \zeta) \longmapsto t\phi_1(\zeta) + (1-t)\phi_{n+1}(b_{n+1}).$$

For any $\zeta \in \Delta$, $\phi_1(\zeta)$, $\phi_{n+1}(b_{n+1}) \in S_q \Rightarrow$

$$t\phi_1(\zeta) + (1-t)\phi_{n+1}(b_{n+1}) \in \partial\Omega \, \forall t, \forall\zeta,$$

since ζ was chosen arbitrarily. Therefore, it remains to show that

$$t\phi_1(\zeta) + (1-t)\phi_{n+1}(b_{n+1}) \in S_q \, \forall t, \forall\zeta.$$

Now, for any t, the mapping

$$\zeta \longmapsto t\phi_1(\zeta) + (1-t)\phi_{n+1}(b_{n+1})$$

is an analytic disc. But, when $\zeta = b_1$,

$$t\phi_1(b_1) + (1-t)\phi_{n+1}(b_{n+1}) = t\phi_2(a_2) + (1-t)\phi_{n+1}(b_{n+1}) = tz_1 + (1-t)z_{n+1} \in S_q,$$

since z_1 and $z_{n+1} = w$ can be joined by n analytic discs. Therefore, $t\phi_1(\zeta) + (1-t)\phi_{n+1}(b_{n+1}) : \Delta \longrightarrow \partial\Omega$ is an analytic disc joining $tz_1 + (1-t)z_{n+1} \in S_q$ to $t\phi_1(\zeta) +$

$(1-t)\phi_{n+1}(b_{n+1})$ (i.e. the image of the analytic disc $t\phi_1(\zeta)+(1-t)\phi_{n+1}(b_{n+1})$ contains a point in S_q) $\Rightarrow$

$$t\phi_1(\zeta) + (1-t)\phi_{n+1}(b_{n+1}) \in S_q \, \forall \zeta;$$

since t was chosen arbitrarily,

$$t\phi_1(\zeta) + (1-t)\phi_{n+1}(b_{n+1}) \in S_q \, \forall \zeta, \, \forall t.$$

Therefore, letting $\zeta = a_1$, it follows that $tz + (1-t)w \in S_q$ $\forall t$. ∎

4.3 Normal Convergence and Boundary Orbit Accumulation Points

Proposition 4.3.1 *Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded convex domain. Suppose there exists $p \in \Omega$ and $\{\phi_j\} \subset Aut(\Omega)$ such that $\phi_j(p) \longrightarrow q \in \partial\Omega$ non-tangentially. Then, there exists a non-constant holomorphic onto mapping $\phi : \Omega \longrightarrow S_q$ such that $\phi_j \longrightarrow \phi$, passing to a subsequence if necessary.*

Proof. Recall the Theorem of Montel (see [Wu], p. 199): A uniformly bounded family of holomorphic mappings from a complex manifold M into $\mathbb{C}^n$ is equicontinuous and hence relatively compact in $\mathrm{Hol}(M, \mathbb{C}^n)$.
Therefore, since $\{\phi_j\} \subset \mathrm{Aut}(\Omega)$ is uniformly bounded, $\phi_j \longrightarrow \phi \in \mathrm{Hol}(\Omega, \mathbb{C}^n)$, passing to a subsequence if necessary. Since $\phi \notin \mathrm{Aut}(\Omega)$, $\phi \in \mathrm{Hol}(\Omega, \partial\Omega)$ (for more details, see Cartan's Theorem, Chapter 5 in [N]).

Therefore, it remains to show that $\phi(\Omega) = S_q$. It is clear that $\phi(\Omega) \subseteq S_q$. To show the other inculsion, let $q' \in S_q$. Therefore, a point $p' \in \Omega$ needs to be found such that $\phi(p') = q'$. Now, by Corollary 4.2.6, $\partial\Omega$ is geometrically flat along S_q; let ν by the unit outward normal vector to S_q. By Lemma 4.1.3, there exists a sequence $\{p_j\} \subset \beta_K^{\Omega}(p; r)$, $0 < r < \infty$, such that $\phi_j(p_j) := q_j \longrightarrow q \in \partial\Omega$ normally. Let δ_j be the number defined by

$$q_j = q - \delta_j \nu.$$

Consider the point $q'_j := q' - \delta_j \nu$. Now, for any j,

$$d_K^{\Omega}(q_j, q'_j) < r' < \infty.$$

Let $p'_j := \phi_j^{-1}(q'_j)$. Therefore,

$$\begin{aligned} d_K^\Omega(p, p'_j) &\le d_K^\Omega(p, p_j) + d_K^\Omega(p_j, p'_j) \\ &< r + d_K^\Omega(q_j, q'_j) < r + r' < \infty, \forall j. \end{aligned}$$

Now,

$$\overline{\beta_K^\Omega(p; r + r')} \subset \Omega,$$

because of complete hyperbolicity. Hence,

$$p'_j \longrightarrow p' \in \overline{\beta_K^\Omega(p; r + r')} \subset \Omega,$$

passing to a subsequence if necessary. Therefore,

$$\phi(p') = \lim_{j\to\infty} \phi_j(p'_j) = \lim_{j\to\infty} q'_j = q'$$

$\Longrightarrow S_q \subseteq \phi(\Omega) \Longrightarrow \phi : \Omega \longrightarrow S_q$ is surjective. ■

Theorem 4.3.2 *Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded convex domain. Suppose there exists a point $p \in \Omega$ and a sequence $\{\phi_j\} \subset Aut(\Omega)$ such that $\phi_j(p) \longrightarrow q \in \partial\Omega$ non-tangentially. If S_q is not trivial, then S_q is an open convex set contained in a complex m-dimensional plane, where* $\dim_{\mathbb{C}} S_q = m$.

Proof. By Theorem 4.2.7, S_q is a convex set, and hence is contained in a complex m-dimensional plane, where $\dim_{\mathbb{C}} S_q = m$. Therefore, all that remains to be shown is that S_q is open. Suppose not, i.e. there exists a point $w \in \partial S_q$. Now, by Proposition 4.2.1, $\phi := \lim_{j\to\infty} \phi_j$ is a surjective holomorphic mapping, which implies that there exists $z \in \Omega$ such that $\phi(z) = w$. Choose an open set U containing z such that $\dim_{\mathbb{C}} \phi(U) = m$.

Let H be the complex $m - 1$ dimensional subspace of the real supporting hyperplane to ∂S_q at $w = \phi(z)$, chosen as follows: Choose coordinates $(z_1, \ldots, z_m)$ such that w is the origin and $S_q \subset \{z \,|\, \operatorname{Re} z_m \le 0\}$. Then, $L := \{z \,|\, \operatorname{Re} z_m = 0\}$ is a real supporting hyperplane to ∂S_q at w and $H \subset L$, where $H = \{z \mid z_m = 0\}$. (Note that there is no assumption of any boundary regularity at w; convexity guarantees the existence of a real supporting hyperplane.)

Let $h : \mathbb{C}^m \longrightarrow \mathbb{C}$ be defined by $h(a) = h(a_1, \ldots, a_m) = a_m$; h is a holomorphic mapping. Now, consider the compostion $H := h \circ \phi$, defined on U. H is holomorphic, so the function $\tilde{H}(a) := \operatorname{Re} H(a)$ is harmonic, so the Maximum Principle holds for $\tilde{H}$. Now, $\tilde{H}(z) = \operatorname{Re} h(w) = \operatorname{Re} w_m = 0$ and $\tilde{H}|_U \leq 0$ (because of convexity), which implies that

$$\tilde{H}|_U \equiv 0$$

by the Maximum Principle. Therefore,

$$\tilde{H}|_U = \operatorname{Re}(h \circ \phi)|_U = \operatorname{Re} h|_{\phi(U)} \equiv 0$$

$\Longrightarrow$

$$\operatorname{Im} h|_{\phi(U)} \equiv 0,$$

because h is holomorphic (look at the Cauchy-Riemann equations for h) $\Rightarrow h \equiv 0$ on $\phi(U) \Rightarrow \phi(U) \subset H$. But,

$$\dim_{\mathbb{C}} \phi(U) = m > m - 1 = \dim_{\mathbb{C}} H,$$

which is a contradiction. Therefore, $\partial S_q = \emptyset \Rightarrow S_q$ is open. ■

For the next theorem, the idea of a holomorphic support function is needed. First, a defintition (see Section 3.2.1 in [K]).

Definition 4.3.3 *Let $\Omega \subset \mathbb{C}^n$ be a domain and $p \in \partial\Omega$. Then p possesses a holomorphic support function for the domain Ω provided that there is a neighborhood U_p of p and a holomorphic function $f_p : U_p \longrightarrow \mathbb{C}$ such that*

$$\{z \in U_p \mid f_p(z) = 0\} \cap \overline{\Omega} = \{p\}.$$

For $p \in \partial\Omega$ that is strongly pseudoconvex, a holomorphic support function can always be found. To see this, choose a local defining fucntion ρ for Ω such that

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(p) w_j \overline{w}_k \geq C|w|^2, \ \forall w \in \mathbb{C}^n.$$

Note that such a ρ can be found for $p \in \partial\Omega$ that is strongly pseudoconvex. Then, the *Levi polynomial*, defined by the equation

$$h(z) = \sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(p)(z_j - p_j) + \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial z_k}(p)(z_j - p_j)(z_k - p_k)$$

is a holomorphic support funtion for Ω at p. For a proof of this fact, see [K], p.139-140.

Theorem 4.3.4 *Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded domain. Let $q \in \partial\Omega$, and suppose S_q is not trivial. Furthermore, suppose $\phi : \Omega \longrightarrow S_q$ is a surjective holomorphic mapping. Then, there exists a sequence $\{p_j\} \subset \Omega$ such that $p_j \longrightarrow p \in \partial\Omega$, p is strongly pseudoconvex, and $\{\phi(p_j)\} \subset S_q$ converges to a point in S_q.*

Proof. Let Ω and ϕ be as above. Then, since Ω is smooth, there exists a strongly pseudoconvex boundary point $p \in \partial\Omega$. Let h be a holomorphic support function for Ω at p, and let

$$H := \{z \in \mathbb{C}^n \mid h(z) = 0\}.$$

Note that $\dim_{\mathbb{C}} H = n - 1$. Let ν be the unit outward normal vector to $\partial\Omega$ at p, and define the set H_n by

$$H_n = \left\{ z - \nu \cdot \frac{1}{n} \,\middle|\, z \in H \right\}$$

(i.e. translate H in the direction of $-\nu$ by a length of $\frac{1}{n}$). Since $\partial\Omega$ is strongly pseudoconvex at p, there exists a neighborhood U of p such that $\partial\Omega \cap U$ is strongly pseudoconvex. Choose $N \in \mathbb{Z}^+$ such that $H_n \cap \Omega \subset \Omega \cap U \ \forall n \geq N$.

Suppose $\dim_{\mathbb{C}} S_q = m$; Let $V_n \subset H_n$ be an m dimensional analytic subset such that $\phi|_{V_n \cap \Omega}$ has rank m almost everywhere - perturb V_n if needed - and that $\partial(V_n \cap \Omega) \subset \partial\Omega$. (Note that the Jacobian of ϕ restricted to V_n, $J\phi|_{V_n}$, will not have full rank (i.e. rank m) if V_n is contained in the common zero set of the component functions of one of its rows.) Let $V_n' := V_n \cap \Omega$.

Let $\phi_n := \phi|_{V_n'}$. Suppose there exists an $n \geq N$ such that ϕ_n isn't proper. Hence, for some subset $K \subset S_q$ that is compact, $\phi_n^{-1}(K)$ will not be compact in V_n'. Hence, there exists $\{p_j\} \subset \phi_n^{-1}(K) \subset V_n'$ such that $p_j \longrightarrow p' \in \partial V_n' \subset \partial\Omega$, p' is strongly pseudoconvex, and $\phi(p_j) = \phi_n(p_j) \longrightarrow q' \in S_q$.

Suppose ϕ_n is proper $\forall n \geq N$. Since ϕ_n has rank m almost everywhere, it must be surjective by the Proper Mapping Theorem of Remmert. Now, for each $n \geq N$, choose $K_n \subset S_q$ that is compact. Hence, $\phi_n^{-1}(K_n)$ is compact in V_n'; choose $p_n \in \phi_n^{-1}(K_n) \subset V_n'$. Hence, $p_n \longrightarrow p$ as $n \longrightarrow \infty$ and $\phi(p_n) = \phi_n(p_n) \in K_n \subset S_q$ $\forall n \geq N$. ∎

Definition 4.3.5 *Let $\Omega \subset \mathbb{C}^n$ be a bounded convex domain with a C^2 boundary. Suppose $\phi_j(p) \to q$, where $p \in \Omega$ and $\{\phi_j\} \subset Aut(\Omega)$ Let $\pi : \Omega \longrightarrow \Omega$ be the projection onto the complex plane normal to $\partial\Omega$ at q. Then, $\phi_j(p) \to q$ non-tangentially in the complex normal direction if*

$$\pi(\phi_j(p)) \in \Gamma_q(\alpha)$$

for some $\alpha > 1$ and j large enough.

Lemma 4.3.6 *Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded convex domain. Suppose, for some $q \in \partial\Omega$, S_q is not trivial. Furthermore, assume there exists $p \in \Omega$ and a sequence $\{\phi_j\} \subset Aut(\Omega)$ such that $\phi_j(p) \longrightarrow q$ non-tangentially. Then, for any $a \in \Omega$, $\phi_j(a) \longrightarrow b \in S_q$ non-tangentially in the complex normal direction for some $b \in S_q$.*

Proof. Choose coordinates $(z_1, \ldots, z_n)$ such that $q = \mathbf{0}$ and the Rez_n direction is the outward normal direction to $\partial\Omega$ at q. Since Ω is convex, $\overline{\Omega} \subset \{z \in \mathbb{C}^n \,|\, \text{Re}z_n \leq 0\}$. Since $\phi_j(p) \longrightarrow q$ non-tangentially, there exists a sequence $\{p_j\} \subset \Omega$ such that $\phi_j(p_j) =: q_j \longrightarrow q$ normally and $d_K^\Omega(p, p_j) < r < \infty \ \forall j$ by Lemma 4.1.3. Note that $d_K^\Omega(p, a) = s < \infty$, since Ω is complete hyperbolic.

Let $\phi_j(a) = a_j$ and a_j' the projection of a_j onto the z_n-plane. Note that, under this projection, $b \longmapsto q$, since S_q is contained in a complex m-dimensional subspace of the complex tangent space, where $m \leq n-1$, by Theorem 4.3.2. Therefore, $a_j' = (0, \ldots, 0, t_j)$, $t_j = A_j e^{i\alpha_j}$, and $q_j = (0, \ldots, 0, x_j)$, $x_j < 0$. Hence, $\theta_j := \pi - \alpha_j$ is the angle between the line segment $\overline{a_j' q}$ and the $-$Rez_n-axis (without loss of generality; assume $a_j' \longrightarrow q$ within the second quadrant of the z_n-plane). Therefore,

$$\begin{aligned} \infty > s + r &> d_K^\Omega(p, a) + d_K^\Omega(p, p_j) \\ &> d_K^\Omega(p_j, a) \\ &= d_K^\Omega(q_j, a_j) \\ &\geq d_K^\Omega(q_j, a_j'), \end{aligned}$$

since the mapping $(z_1, \ldots, z_n) \longmapsto (0, \ldots, 0, z_n)$ is well-defined, since Ω is convex. But,

$$\begin{aligned} d_K^\Omega(q_j, a_j') &\geq d_K^H(x_j, t_j) \\ &= \ln\left(\sec\theta_j + \tan\theta_j\right). \end{aligned}$$

The inequality holds because the projection onto the z_n-plane is a holomorphic mapping, where $H = \{z \in \mathbb{C} \,|\, \text{Re}z \leq 0\}$; the equality holds by Lemma 3.1.13. But,

$$\ln\left(\sec\theta_j + \tan\theta_j\right) \longrightarrow \infty \text{ if } \theta_j \longrightarrow \frac{\pi}{2}.$$

Putting all of this together,

$$\infty > d_K^H(x_j, t_j) \longrightarrow \infty \text{ if } t_j \longrightarrow q \text{ tangentially,}$$

which is a contradiction. Hence $\phi_j(a) \longrightarrow b$ non-tangentially in the complex normal direction. ■

An additional hypothesis that will be needed in the main theorem is called Condition LTW. Here is its statement:

Condition 4.3.7 (Condition LTW) *Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded convex domain. Suppose, for some $q \in \partial\Omega$, S_q is not trivial. Furthermore, assume there exists $p \in \Omega$ and a sequence $\{\phi_j\} \subset Aut(\Omega)$ such that $\phi_j(p) \longrightarrow q$ non-tangentially. Then, there exists a sequence $\{z_i\} \subset \Omega$ converging to a strongly (pseudo)convex boundary point p' such that $\phi(z_i) \to \tilde{q} \in S_q$. Also, for each z_i, there exists a sequence $\left\{z_i^j\right\} \subset \Omega$ such that $z_i^j \in \overline{\beta_K^\Omega(z_i;r)}$ and $\phi_j(z_i^j)$ is contained in the complex planes normal to S_q for j large enough. Furthermore, for any $\epsilon > 0$, i can be chosen large enough so that $\overline{\beta_K^\Omega(z_i;r)} \subset B(p';\epsilon) \cap \Omega$*

This condition is a technical detail that is needed for the proof of the main theorem. In this condition, $\beta_K^\Omega(z_i;r)$ is the ball, centered at z_i, with radius r measured with respect to the Kobayashi distance. Also, the statement that $\phi_j(z_i^j)$ is contained in the complex planes normal to S_q for j large enough means that $\phi_j(z_i^j)$ is contained in the product $\Pi \times S_q$, where Π is the complex plane normal to S_q (recall that $\partial\Omega$ is geometrically flat along S_q). Note that, if it can be shown that for any $a \in \Omega$, $\phi_j(a) \to S_q$ non-tangentially, which is very reasonable to expect to be true, the condition is a corollary to previous results. Here is how the proof would follow:

Proof. Let $\epsilon > 0$.

1. By Proposition 4.3.1, $\phi_j \to \phi : \Omega \longrightarrow S_q$, where ϕ is a surjective holomorphic mapping.

2. Since $\Omega \subset \mathbb{C}^n$ is smoothly bounded and convex, there exists a strongly convex boundary point p' (which is also strongly pseudoconvex), so by Theorem 4.3.4, a sequence z_i can be chosen so that $\phi(z_i) \to \tilde{q} \in S_q$.

3. Now, $\phi(z_i) = \tilde{q}_i \in S_q \Rightarrow \lim_{j\to\infty} \phi_j(z_i) = \tilde{q}_i \in S_q$, so $\phi_j(z) \longrightarrow \tilde{q}_i$ non-tangentially (assuming it can be shown).

4. Therefore, by Lemma 4.1.3, there exists a sequence $\left\{z_i^j\right\} \subset \Omega$ such that $\phi_j(z_i^j) \longrightarrow \tilde{q}_i$ normally and $d_K^\Omega(z_i, z_i^j) < r < \infty$ for some $r > 0$.

5. Since $\operatorname{diam} \beta_K^\Omega(z_i; r) \longrightarrow 0$ as $z_i \longrightarrow p'$ by Lemma 4.1.4, choose z_i close enough to p' so that $\overline{\beta_K^\Omega(z_i; r)} \subset B(p'; \epsilon)$; hence $z_i^j \in B(p'; \epsilon) \cap \Omega$.

This concludes the argument. ■

Condition LTW is sufficient for the main theorem to hold. In order to simplify the proof, the following Corollary to Condition LTW will be used.

Corollary 4.3.8 *Assume Condition LTW. Then, for each z_i, there exists a sequence $\left\{\tilde{p}_i^j\right\} \subset \Omega$ such that $\phi_j(\tilde{p}_i^j) \to \tilde{q}_i \in S_q$ in the complex plane normal to S_q through $\tilde{q}_i$, non-tangentially, and $\tilde{p}_i^j \in \overline{\beta_K^\Omega(z_i; r')} \subset B(p'; \epsilon) \cap \Omega$, for z_i chosen sufficiently close to p'*

Proof. Suppose $\phi_j(z_i^j) \to \tilde{q}_i$ as $j \to \infty$, where $\phi_j(z_i^j)$ is as in Condition LTW and $\tilde{q}_i \in S_q$. For simpler notation, let $V := S_q$. Let π denote the projection of Ω onto the complex plane normal to V through $\tilde{q}_i$. Since Ω is convex, this mapping is well-defined and holomorphic. Let $p_i^j := \pi(\phi_j(z_i^j))$. Then, by Lemma 4.3.6, $p_i^j \to \tilde{q}_i$ non-tangentially.

Let V_j be the translation of V to p_i^j:

$$V_j := \left\{ z + (p_i^j - \tilde{q}_i) \,\middle|\, z \in V \right\}.$$

Then, for j large enough, $V_j \subset \Omega$. Let $\tilde{p}_i^j := \phi_j^{-1}\left(p_i^j\right)$. Then,

$$\begin{aligned} d_K^\Omega(\tilde{p}_i^j, z_i) &\leq d_K^\Omega(\tilde{p}_i^j, z_i^j) + d_K^\Omega(z_i^j, z_i) \\ &\leq d_K^\Omega(p_i^j, \phi_j(z_i^j)) + r \\ &\leq d_K^{V_j}(p_i^j, \phi_j(z_i^j)) + r < s + r < \infty \end{aligned}$$

for j large enough. Let $r' := r + s$. Hence, $\tilde{p}_i^j \in \overline{\beta_K^\Omega(z_i; r')} \subset B(p'; \epsilon) \cap \Omega$, for z_i chosen sufficiently close to p', and $\phi_j(\tilde{p}_i^j) \to \tilde{q}_i$ in the complex plane normal to V through $\tilde{q}_i$. ■

4.4 Principal Result

Proposition 4.4.1 *Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded convex domain. Suppose there exists a point $p \in \Omega$ and a sequence $\{\phi_j\} \subset Aut(\Omega)$ such that $\phi_j(p) \to q \in \partial\Omega$ non-tangentially. Furthermore, suppose S_q is not trivial and that Condition LTW holds. Then, S_q is biholomorphic to $\mathbb{B}_m$, where $m = \dim_{\mathbb{C}} S_q$.*

Proof. By the corollary of Condition LTW, for any $\epsilon > 0$ and some strongly convex boundary point p', there exists a sequence $\{p_j\} \subset \Omega$ such that $p_j \in \overline{\beta_K^{\Omega}(z_i; r')} \subset B(p'; \epsilon) \cap \Omega$ and $q_j := \phi_j(p_j) \to q' \in S_q$ in the complex plane normal to $\partial\Omega$ through q'. (Note that p_j is $\tilde{p}_i^j$ and q' is $\tilde{q}_i$ in the corollary of Condition LTW; notation is being changed for simplicity.) By sequential compactness, $p_j \to \tilde{p} \in \overline{\beta_K^{\Omega}(z_i; r')}$, passing to a subsequence if necessary. Hence, $\phi(\tilde{p}) = q'$ non-tangentially. Furthermore, by choosing p' and ϵ so that

$$B(p'; \epsilon) \cap Z_j = \emptyset,\ j \in \{1, \ldots, m\},$$

where Z_j is the common zero set of the component functions in the jth row of $d\phi$, $\operatorname{rank} d\phi(\tilde{p}) = m$.

Let ν be the constant unit outward normal vector to $\partial\Omega$ along S_q. For simpler notation, let $V := S_q$ and let V_j be the translation of V to q_j, i.e.

$$V_j = \left\{ z + (q_j - q'), \,|\, z \in V \right\}.$$

Then, $V_j \subset \Omega$ for j large enough. Let $\{\xi_k\}_{k=1}^m \subset T_{q'}V$ be linearly independent. Since $\partial\Omega$ is geometrically flat along V, $\{\xi_k\}_{k=1}^m \subset T_{q_j}V_j$ is linearly independent (for j large enough) and ν will be normal to V_j. Since $V_j \subset \Omega$ for j large, this set is linearly independent in $T_{q_j}\Omega$. Since $d\phi_j^{-1}(q_j)$ is an isomorphism of vector spaces,

$$\left\{ d\phi_j^{-1}(q_j)\xi_1, \ldots, d\phi_j^{-1}(q_j)\xi_m \right\}$$

is linearly independent in $T_{p_j}\Omega$. For brevity, let

$$\xi_k' := d\phi_j^{-1}(q_j)\xi_k \text{ and } \tilde{\xi}_k := \lim_{j\to\infty} d\phi_j^{-1}(q_j)\xi_k.$$

Claim: $\left\{\tilde{\xi}_1, \ldots, \tilde{\xi}_m\right\} \subset T_{\tilde{p}}\Omega$ is linearly independent.

Now, for any k, the set $\left\{d\phi_j^{-1}(q_j)\xi_k\right\}_j$ is a sequence of points in $\mathbb{C}^n$. Since $\mathbb{C}^n$ is a sequentially compact metric space, $\left\{d\phi_j^{-1}(q_j)\xi_k\right\}_j$ has a convergent subsequence. Therefore, $\tilde{\xi}_k := \lim_{j\to\infty} d\phi_j^{-1}(q_j)\xi_k$ exists, passing to a subsequence if necessary.

Now, for any $k \in \{1, \ldots, m\}$,

$$\begin{aligned} d\phi(\tilde{p})(\tilde{\xi}_k) &= d\phi(\tilde{p})\left(\lim_{j\to\infty} d\phi_j^{-1}(q_j)\xi_k\right) = \lim_{j\to\infty} d\phi_j(p_j)\left(\lim_{j\to\infty} d\phi_j^{-1}(q_j)\xi_k\right) \\ &= \lim_{j\to\infty} d\phi_j(p_j)\left(d\phi_j^{-1}(q_j)\xi_k\right) = \lim_{j\to\infty} d\left(\phi_j \circ \phi_j^{-1}\right)(q_j)\xi_k \\ &= \lim_{j\to\infty} d1_\Omega(q_j)\xi_k = \xi_k. \end{aligned}$$

Therefore, $\xi_1, \ldots, \xi_m$ linearly independent and $d\phi(\tilde{p})$ linear implies that $\tilde{\xi}_1, \ldots, \tilde{\xi}_m$ linearly indepedent. This proves the claim.

Let $U = \mathbb{B}_m$. Now, by the well-known properties of the invariant measures,

$$\begin{aligned} C_U^\Omega(p_j; \xi_1', \ldots, \xi_m') &= C_U^\Omega(q_j; \xi_1, \ldots, \xi_m) \\ &\leq C_U^{V_j}(q_j; \xi_1, \ldots, \xi_m) \\ &= C_U^V(q'; \xi_1, \ldots, \xi_m), \end{aligned}$$

and

$$K_U^\Omega(\tilde{p}; \tilde{\xi}_1, \ldots, \tilde{\xi}_m) \geq K_U^V(q'; \xi_1, \ldots, \xi_m).$$

Therefore,

$$1 \geq \frac{C_U^V(q'; \xi_1, \ldots, \xi_m)}{K_U^V(q'; \xi_1, \ldots, \xi_m)} \geq \frac{C_U^\Omega(p_j; \xi_1', \ldots, \xi_m')}{K_U^\Omega(\tilde{p}; \tilde{\xi}_1, \ldots, \tilde{\xi}_m)}$$

for any j large. Hence, letting $j \to \infty$,

$$1 \geq \frac{C_U^V(q'; \xi_1, \ldots, \xi_m)}{K_U^V(q'; \xi_1, \ldots, \xi_m)} \geq \frac{C_U^\Omega(\tilde{p}; \tilde{\xi}_1, \ldots, \tilde{\xi}_m)}{K_U^\Omega(\tilde{p}; \tilde{\xi}_1, \ldots, \tilde{\xi}_m)}.$$

Since

$$\frac{C_U^{\Omega}(\tilde{p};\tilde{\xi}_1,\ldots,\tilde{\xi}_m)}{K_U^{\Omega}(\tilde{p};\tilde{\xi}_1,\ldots,\tilde{\xi}_m)} \longrightarrow 1$$

as $\tilde{p} \to$ a strongly (pseudo)convex boundary point ($\tilde{p}$ can be chosen arbitrarily close to a strongly (pseudo)convex boundary point by construction), it follows that

$$\frac{C_U^{V}(q';\xi_1,\ldots,\xi_m)}{K_U^{V}(q';\xi_1,\ldots,\xi_m)} = 1$$

$\Longrightarrow$

$$V \cong \mathbb{B}_m.$$

■

Theorem 4.4.2 (Lee-Thomas-Wong) *Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded convex domain. Suppose there exists a point $p \in \Omega$ and a sequence $\{\phi_j\} \subset Aut(\Omega)$ such that $\phi_j(p) \to q \in \partial\Omega$ non-tangentially. Furthermore, suppose that Condition LTW holds. Then, S_q is trivial and hence $\partial\Omega$ is variety-free at q.*

Proof. Suppose S_q is not trivial. For notational simplicity, let $V := S_q$. Then, V is a convex open subset lying in a m-dimensional subspace of $T_q\partial\Omega$, where $m = \dim_{\mathbb{C}} V$ by Theorem 4.3.2. By a linear change of coordinates, assume $\nu = (1, 0, \ldots, 0)$ is the constant outward normal vector to $\partial\Omega$ along V (recall that the $\partial\Omega$ is geometrically flat along V) and V lies in the $z_2 \cdots z_{m+1}$ plane, where q is the origin. Let $\pi : \Omega \longrightarrow \{z \in \mathbb{C}^n \,|\, z_{m+2} = \cdots = z_n = 0\}$ be the projection mapping (i.e. π is the projection onto the $z_1 \cdots z_{m+1}$ plane).

Now, by the corollary of Condition LTW, there exists a strongly convex boundary point $p' \in \partial\Omega$ such that, for any $\epsilon > 0$, there exists a sequence $\{p_j\} \subset \Omega$ such that $q_j := \phi_j(p_j) \to q' \in S_q$ in the complex plane normal to $\partial\Omega$ through q' non-tangentially and $p_j \in B(p';\epsilon) \cap \Omega$ $\forall j$. Also, $p_j \in \overline{\beta_K^{\Omega}(z_i;r')} \subset B(p';\epsilon) \cap \Omega$, so $p_j \to \tilde{p} \in \overline{\beta_K^{\Omega}(z_i;r')}$, passing to a subsequence if necessary. (Recall that, in the corollary, p_j is $\tilde{p}_i^j$ and q' is $\tilde{q}_i$; notation has been changed for simplicity.) Let $\{\Omega_\ell\}$ be a sequence of relatively compact open subsets of Ω such that $\Omega_\ell \subset\subset \Omega_{\ell+1}, \bigcup_{\ell=1}^{\infty} \Omega_\ell = \Omega$, and, choosing ℓ large enough, $p_j, \tilde{p} \in \Omega_\ell \; \forall j$.

Let V_j be the translation of V to q_j, i.e.

$$V_j = \left\{ z + (q_j - q') \,|\, z \in V \right\}.$$

Then, $V_j \subset \Omega$ for j large enough. Let $\{\xi_k\}_{k=1}^m \subset T_{q'}V$ be linearly independent. Since $\partial\Omega$ is geometrically flat along V, $\{\xi_k\}_{k=1}^m \subset T_{q_j}V_j$ is linearly independent (for j large enough) and ν will be normal to V_j. Hence, the set $\{\xi_1, \ldots, \xi_m, \nu\} \subset T_{q_j}\Omega$ is linearly independent. Since $d\phi_j^{-1}(q_j)$ is an isomorphism of vector spaces, the set

$$\left\{ d\phi_j^{-1}(q_j)\xi_1, \ldots, d\phi_j^{-1}(q_j)\nu \right\} \subset T_{p_j}\Omega$$

is linearly independent. For brevity, let

$$\xi_k' := d\phi_j^{-1}(q_j)\xi_k, \quad \nu' := d\phi_j^{-1}(q_j)\nu.$$

Now, let $U = \Delta \times \mathbb{B}_m$ and let $f : V \longrightarrow \mathbb{B}_m$ be the biholomorphism from the previous proposition. Let $\{V_\delta\}$ be a sequence of subsets of V such that $V_\delta \nearrow V$ as $\delta \to 0$. Consider the wedge domain

$$\Gamma_\alpha^r\left(q'\right) \times V_\delta,$$

where $\Gamma_\alpha^r\left(q'\right)$ is a cone contained in the complex plane (z_1-plane) normal to $\partial\Omega$ at q', with angle between its arms and $-\mathrm{Re}\, z_1$-axis $\alpha \in \left(\frac{\pi}{2}, \pi\right)$ and arms of length r. Let $\theta \in \left(0, \frac{\pi}{2}\right)$ be chosen so that $q_j \in \Gamma_\theta^r(q')$. Then,

$$\Gamma_\theta^r\left(q'\right) \times V_\delta \subset \Omega$$

for r small enough, since $\partial\Omega$ is geometrically flat along V. For j sufficiently large,

$$\pi\left(\phi_j\left(\Omega_\ell\right)\right) \subset \Gamma_\alpha^r\left(q'\right) \times V_\delta.$$

Hence, for j large enough, by using the decreasing properties of the invariant measures, the following inequalities hold:

$$\begin{aligned}\frac{C_U^{\Omega_\ell}(p_j;\xi_1',\ldots,\xi_m',\nu')}{K_U^{\Omega}(p_j;\xi_1',\ldots,\xi_m',\nu')} &\geq \frac{C_U^{\phi_j(\Omega_\ell)}(q_j;\xi_1,\ldots,\xi_m,\nu)}{K_U^{\Omega}(q_j;\xi_1,\ldots,\xi_m,\nu)} \\ &\geq \frac{C_U^{\pi(\phi_j(\Omega_\ell))}(q_j;\xi_1,\ldots,\xi_m,\nu)}{K_U^{\Omega}(q_j;\xi_1,\ldots,\xi_m,\nu)} \\ &\geq \frac{C_U^{\Gamma_\alpha^r(q')\times V_\delta}(q_j;\xi_1,\ldots,\xi_m,\nu)}{K_U^{\Gamma_\theta^r(q')\times V_\delta}(q_j;\xi_1,\ldots,\xi_m,\nu)}\end{aligned}.$$

Now, for any j, the volume of $\xi_1',\ldots,\xi_m',\nu'$ is non-zero – denote it by μ_j – so

$$\begin{aligned}\frac{C_U^{\Omega_\ell}(p_j;\xi_1',\ldots,\xi_m',\nu')}{K_U^{\Omega}(p_j;\xi_1',\ldots,\xi_m',\nu')} &= \frac{\mu_j\cdot C_U^{\Omega_\ell}(p_j;e_1,\ldots,e_{m+1})}{\mu_j\cdot K_U^{\Omega}(p_j;e_1,\ldots,e_{m+1})} \\ &= \frac{C_U^{\Omega_\ell}(p_j;e_1,\ldots,e_{m+1})}{K_U^{\Omega}(p_j;e_1,\ldots,e_{m+1})},\end{aligned}$$

where $\{e_k\}_{k=1}^{m+1}$ is an orthonormal basis for $\operatorname{span}(\xi_1',\ldots,\xi_m')$ by Lemma 3.2.4. Hence,

$$\frac{C_U^{\Omega_\ell}(p_j;e_1,\ldots,e_{m+1})}{K_U^{\Omega}(p_j;e_1,\ldots,e_{m+1})} \geq \frac{C_U^{\Gamma_\alpha^r(q')\times V_\delta}(q_j;\xi_1,\ldots,\xi_m,\nu)}{K_U^{\Gamma_\theta^r(q')\times V_\delta}(q_j;\xi_1,\ldots,\xi_m,\nu)}$$

(call this inequality $*$). Since U, $\Gamma_\alpha^r(q')\times V_\delta$, $\Gamma_\theta^r(q')\times V_\delta$ are domains in $\mathbb{C}^{m+1}$, by Lemma 3.2.7,

$$\frac{C_U^{\Gamma_\alpha^r(q')\times V_\delta}(q_j;\xi_1,\ldots,\xi_m,\nu)}{K_U^{\Gamma_\theta^r(q')\times V_\delta}(q_j;\xi_1,\ldots,\xi_m,\nu)} = \frac{\left|M^C_{\Gamma_\alpha^r(q')\times V_\delta}(q_j)\right|}{\left|M^E_{\Gamma_\theta^r(q')\times V_\delta}(q_j)\right|}.$$

Let $q_j = (q_{j\,1},\ldots,q_{j\,m+1}) = (q_{j\,1},w) := (q_j,w)$, where q_j is in the complex plane normal to $\partial\Omega$ through q', with $\operatorname{Re} q_j < 0$, and $w \in V_\delta$. Therefore, by Lemma 3.2.11, 3.2.12,

$$\frac{\left|M^C_{\Gamma_\alpha^r(q')\times V_\delta}(q_j)\right|}{\left|M^E_{\Gamma_\theta^r(q')\times V_\delta}(q_j)\right|} \geq \frac{\left|M^C_{\Gamma_\alpha^r(q')}(q_j)\right|\cdot\left|M^C_{V_\delta}(w)\right|}{\left|M^E_{\Gamma_\theta^r(q')}(q_j)\right|\cdot\left|M^E_{V_\delta}(w)\right|}.$$

Letting $j\to\infty$, $\delta\to 0$, and so the right-hand side of the above inequality becomes

$$\lim_{j\to\infty}\frac{\left|M^C_{\Gamma_\alpha^r(q')}(q_j)\right|}{\left|M^E_{\Gamma_\theta^r(q')}(q_j)\right|}\cdot\frac{\left|M^C_V(w)\right|}{\left|M^E_V(w)\right|}.$$

Since V is biholomorphic to the ball $\mathbb{B}_m$,

$$\lim_{j\to\infty} \frac{\left|M^C_{\Gamma^r_\alpha(q')}(q_j)\right|}{\left|M^E_{\Gamma^r_\theta(q')}(q_j)\right|} \cdot \frac{\left|M^C_V(w)\right|}{\left|M^E_V(w)\right|} = \lim_{j\to\infty} \frac{\left|M^C_{\Gamma^r_\alpha(q')}(q_j)\right|}{\left|M^E_{\Gamma^r_\theta(q')}(q_j)\right|}.$$

Let ω_j be the angle between the line segment joining q_j to q' and the $-\operatorname{Re} z_1$-axis. Suppose $\omega_j \to \omega$, where $\omega \in (0, \theta)$. Therefore, by Lemma 3.2.13,

$$\lim_{j\to\infty} \frac{\left|M^C_{\Gamma^r_\alpha(q')}(q_j)\right|}{\left|M^E_{\Gamma^r_\theta(q')}(q_j)\right|} = \frac{\theta \cos\left(\frac{\pi}{2\theta}\cdot\omega\right)}{\alpha\cos\left(\frac{\pi}{2\alpha}\cdot\omega\right)} \longrightarrow 1,$$

letting $r \to 0$, since that allows $\theta \to \frac{\pi}{2}^-$ and $\alpha \to \frac{\pi}{2}^+$. What does this all mean? It means that, letting $j \to \infty$ in inequaltiy $*$ that

$$\lim_{j\to\infty} \frac{C^{\Omega_\ell}_U(p_j; e_1, \ldots, e_{m+1})}{K^\Omega_U(p_j; e_1, \ldots, e_{m+1})} \geq 1,$$

and so

$$\frac{C^{\Omega_\ell}_U(\tilde{p}; e_1, \ldots, e_{m+1})}{K^\Omega_U(\tilde{p}; e_1, \ldots, e_{m+1})} \geq 1.$$

Letting $\ell \to \infty$, it follows that

$$\frac{C^{\Omega}_U(\tilde{p}; e_1, \ldots, e_{m+1})}{K^\Omega_U(\tilde{p}; e_1, \ldots, e_{m+1})} \geq 1.$$

Let $\widetilde{\Omega} = \Omega \cap B(p'; \epsilon)$. Using the localization properties of the invariant measures (Lemma 3.2.10),

$$\overline{\lim_{\tilde{p}\to p'}} \frac{C^{\widetilde{\Omega}}_U(\tilde{p}; e_1, \ldots, e_{m+1})}{K^{\widetilde{\Omega}}_U(\tilde{p}; e_1, \ldots, e_{m+1})} = \overline{\lim_{\tilde{p}\to p'}} \frac{C^{\Omega}_U(\tilde{p}; e_1, \ldots, e_{m+1})}{K^{\Omega}_U(\tilde{p}; e_1, \ldots, e_{m+1})} \geq 1.$$

But, by Lemma 3.2.14,

$$\overline{\lim_{\tilde{p}\to p'}} \frac{C^{\widetilde{\Omega}}_U(\tilde{p}; e_1, \ldots, e_{m+1})}{K^{\widetilde{\Omega}}_U(\tilde{p}; e_1, \ldots, e_{m+1})} < 1,$$

which implies that

$$1 > \overline{\lim_{\tilde{p}\to p'}} \frac{C^{\widetilde{\Omega}}_U(\tilde{p}; e_1, \ldots, e_{m+1})}{K^{\widetilde{\Omega}}_U(\tilde{p}; e_1, \ldots, e_{m+1})} \geq 1,$$

which is a contradiction. Therefore, the assumption was false and so the $\partial\Omega$ must be variety-free at q. ■

Theorem 4.4.3 *Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded convex domain. Suppose there exists a point $p \in \Omega$ and a sequence $\{\phi_j\} \subset Aut(\Omega)$ such that $\phi_j(p) \to q \in \partial\Omega$ non-tangentially. Then,* $\dim_{\mathbb{C}} S_q < n - 1$.

Proof. Assume the conclusion isn't true, i.e. $\dim_{\mathbb{C}} S_q = n-1$. The idea is to show that Condition LTW holds, and then invoke the same argument as in the main theorem to derive a contradiction. Let $V := S_q$.

By Proposition 4.3.1, $\phi_j \to \phi : \Omega \longrightarrow V$, where ϕ is a surjective holomorphic mapping. Since $\Omega \subset \mathbb{C}^n$ is smoothly bounded and convex, there exists a strongly (pseudo)convex boundary point p' , so by Theorem 4.3.4, a sequence $\{z_i\}$ can be chosen so that $\phi(z_i) \to \tilde{q} \in V$. Now, $\phi(z_i) \in V$ implies that $\lim_{j\to\infty} \phi_j(z_i) \in V$, so $\phi_j(z_i) \to q' \in V$ non-tangentially in the complex normal direction by Lemma 4.3.6. Let π be the projection of Ω onto the complex plane normal to V through q'. Since Ω is convex, this mapping is well-defined and hence holomorphic. Let $z_i^j := \pi(\phi_j(z_i))$, and let V_j be the translation of V to z_i^j, i.e.

$$V_j := \left\{ z + (z_i^j - q') \,\middle|\, z \in V \right\}.$$

For j large enough, $V_j \subset \Omega$. Let $\tilde{z}_i^j := \phi_j^{-1}(z_i^j)$. Then,

$$\begin{aligned} d_K^{\Omega}(z_i, \tilde{z}_i^j) &= d_K^{\Omega}(\phi_j(z_i), z_i^j) \\ &\le d_K^{V_j}(\phi_j(z_i), z_i^j) < s < \infty \end{aligned}$$

for j large enough. So,

$$\tilde{z}_i^j \in \overline{\beta_K^{\Omega}(z_i; s)},$$

and, for any $\epsilon > 0$, by choosing z_i sufficiently close to p',

$$\overline{\beta_K^{\Omega}(z_i; s)} \subset B(p'; \epsilon) \cap \Omega,$$

since diam $\beta_K^{\Omega}(z_i; s) \to 0$ as $z_i \to p'$ by Lemma 4.1.4.

At this point, repeat the argument in the main theorem. In fact, letting $p_j := \tilde{z}_i^j$ and $q_j := \phi_j(p_j)$, one can apply that argument verbatim to derive the contradiction. ■

Chapter 5

Conclusions

This result supports the truthfulness of the Greene-Krantz conjecture and provides a partial generalization of the result of Fu and Wong [FW] (it is only partial, since in their work, the domain considered in $\mathbb{C}^2$ was pseudoconvex, whereas the domain considered here in $\mathbb{C}^n$ was assumed to be convex, which is a stronger assumption; also, there is the assumption of Condition LTW).

There is much work to be done. The result of this dissertation is far from the Greene-Krantz conjecture in its full generality: The hypothesis is much stronger and the conclusion is weaker. As a start towards proving the conjecture, here are some problems the author is interested in pursuing:

Problem 5.0.4 *Prove the main result without the assumption of Condition LTW.*

The author believes that the main result will hold without the assumption of this condition, since the condition can be easily proven if it can be shown that $\phi_j(a) \to S_q$, $a \in \Omega$, non-tangentially, which is very reasonable to think to be true.

Problem 5.0.5 *Suppose* $\Omega \subset \mathbb{C}^2$ *is smoothly bounded and pseudoconvex. Furthermore, suppose there exists* $p \in \Omega$ *and* $\{\phi_j\} \subset Aut(\Omega)$ *such that* $\phi_j(p) \to q \in \partial\Omega$ *non-tangentially. Then* $\partial\Omega$ *is of finite type at* q*.*

Proving this would be a generalization of Fu and Wong's result in that a stronger conclusion would be obtained.

Problem 5.0.6 *Suppose* $\Omega \subset \mathbb{C}^2$ *is smoothly bounded and pseudoconvex. Furthermore, suppose there exists* $p \in \Omega$ *and* $\{\phi_j\} \subset Aut(\Omega)$ *such that* $\phi_j(p) \to q \in \partial\Omega$*. Then* $\partial\Omega$ *is of finite type (or variety-free) at* q*.*

In this problem, the assumption of non-tangential convergence is removed, i.e. the possibility of $\phi_j(p) \to q$ *tangentially* is allowed. It is unnatural to have the assumption of non-tangential convergence, so proving a result allowing tangential convergence would be good progress. Of course, one can not forget the Greene-Krantz conjecture itself:

Problem 5.0.7 *Prove the Greene-Krantz conjecture.*

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