Title
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Publication Date
2022

Peer reviewed|Thesis/dissertation

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by

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A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Mathematics

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor Thomas Scanlon, Chair
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Spring 2022

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Abstract

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This thesis is motivated by the open question of whether there are transexponential o-minimal structures. As a candidate for a transexponential o-minimal structure, we suggest $\mathbb{R}_{\text{an,exp}}$ expanded with new symbols for a transexponential function, its derivatives, and their compositional inverses, which we call $\mathbb{R}_{\text{transexp}}$. The main result of the thesis is that the germs at $+\infty$ of $\mathbb{R}_{\text{transexp}}$-terms are ordered and thus form a Hardy field. If $\mathbb{R}_{\text{transexp}}$ is shown to have quantifier elimination in the future, then o-minimality would follow. Chapter 1 provides background on the open problem and more information on $\mathbb{R}_{\text{transexp}}$.

The work of the thesis is to adapt the construction of the field of logarithmic-exponential series in [8] to build an ordered differential field of sublogarithmic-transexponential series. The sublogarithmic-transexponential series field is constructed to embed the germs of $\mathbb{R}_{\text{transexp}}$-terms, showing that they are ordered. The challenge is to gradually build up the field of series so that we know how the partial structure should be ordered at each stage. We discuss how the ordering quickly becomes complicated even for relatively simple finite sums in Chapter 2.

The construction of the sublogarithmic-transexponential series can be divided into three parts. First, in Chapter 3, we prove that the group ring of certain kinds of finite sums is ordered by giving an algorithm to determine the sign on a sum and showing the algorithm terminates. Next, in Chapter 4, we adapt the construction of the logarithmic-exponential series in [8] to build a field of series closed under log, exp, and restricted analytic functions, starting from a field of coefficients and group of monomials satisfying certain assumption. Finally, in Chapter 5, we iterate the construction from Chapter 4 to build the full field of sublogarithmic-transexponential series. We also define a derivation that works like “differentiation with respect to the formal variable of the series.”

Dedicated to Josee Leitschuh

Contents

Acknowledgments

I am deeply grateful to my advisor Tom Scanlon for both his incisive observations and genuine kindness. Through our conversations, I learned so much about the process of refining mathematical ideas into a strategy.

Carol Wood, Pierre Simon, Sebastian Eterović, Ben Castle, Mariana Vicaría, Diego Bejarano Rayo, Scott Mutchnik, Jordan Brown, and Ronan O'Gorman made my time with the model theory group at Berkeley both enlightening and enjoyable. Mariana in particular has been incredibly generous, both as a friend and as a role model who I've looked to often over the past several years. Additional thanks to Pierre and Wes Holliday for serving on my dissertation committee.

There are many more of my fellow students who I would like to acknowledge. James Walsh, Reid, Mikayla, Patrick Lutz, and Guillaume Massas helped make Logic student events fun and frequent. My cohort was a great group of people—I would like to thank Lauren Heller, David Casey, and Kristina Nelson for many fun and interesting conversations; James Rowan, Yuan Yao, Xiaohan Yan, Yixuan Li, and Irit Huq-Kuruvilla for relaxing weekend teas; James Rowan (again) and James Dix for making me an honorary James; Jorge Garza Vargas and Roy Zhao for being such good friends to me; and Andrew Shi for his trust and for making me laugh.

Thanks to Emily Reinherz, my longest and dearest friend, for helping me keep perspective.

Finally, I am so thankful to my wonderful parents, Rosanne Fitko and Jim Padgett, and my brother, Carl Padgett, for all their love and support.

Chapter 1

Introduction

1.1 Background

The study of o-minimality began in the 1980s as a framework for generalizing the tools of semialgebraic geometry [5]. In 1991, Wilkie proved that the field of real numbers with exponentiation, $\mathbb{R}_{\text{exp}}$, is model complete [19]. This, along with Khovanskii's work on Fewnomials [13], proved that $\mathbb{R}_{\text{exp}}$ is o-minimal. Wilkie's result indicated that key qualities of semialgebraic geometry extend beyond the semialgebraic context to other important functions and that o-minimality is the right extension of semialgebraicity to consider.

The o-minimality of $\mathbb{R}_{\text{exp}}$ also spurred an investigation into the growth rates of functions defined in o-minimal fields: First, van den Dries, Macintyre, and Marker observed the following:

Lemma 1.1.1. *An expansion $\mathcal{R}$ of the ordered field of real numbers is o-minimal if and only if the germs at $+\infty$ of its definable functions form a Hardy field.*

They used this fact to greatly simplify the proof that the expansion of $\mathbb{R}_{\text{exp}}$ by restricted analytic functions, denoted $\mathbb{R}_{\text{an,exp}}$, is o-minimal [9]. Second, Miller proved that either $\mathcal{R}$ defines $x \mapsto e^x$, or all its definable functions are bounded in absolute value by a polynomial, i.e., $\mathcal{R}$ is either *exponential* or *polynomially bounded* [16].

It is a natural step to consider which o-minimal structures are *exponentially bounded*, meaning that every definable function is bounded in absolute value by $\exp^{\circ n}(x)$ for some $n \in \mathbb{N}$. Some o-minimal structures, including $\mathbb{R}_{\text{exp}}$ and $\mathbb{R}_{\text{an,exp}}$ [7], are known to be exponentially bounded, but currently none are known to be non-exponentially-bounded, i.e., *transexponential*. In fact, this is a longstanding open question:

Question 1.1.2. *Are there transexponential o-minimal structures?*

The first reference (known to the author) to the possible existence of transexponential o-minimal structures is a 1996 paper of van den Dries and Miller [10]. In [15], Marker and

Miller suggest that an expansion of $\mathbb{R}_{\text{exp}}$ by a new function symbol that represents a real analytic function $E : (a, +\infty) \to \mathbb{R}$ satisfying

$$E(x+1) = e^{E(x)}$$

could be o-minimal. Any function satisfying this difference equation must be transexponential, and in [3], Boshernitzan shows that there are solutions whose germs at $+\infty$ belong to a Hardy field [3]. Boshernitzan refers to an earlier paper of Kneser [14] for the existence of real analytic solutions to this difference equation.

Marker and Miller also discuss expanding $\mathbb{R}_{\text{exp}}$ by a "half-iterate"

$$e_{1/2}(x) = E\left(E^{-1}(x) + \frac{1}{2}\right)$$

of $\exp(x)$ whose germ at $+\infty$ also belongs to a Hardy field [3]. Since $(e_{1/2} \circ e_{1/2})(x) = \exp x$, the growth rate of $e_{1/2}$ is strictly between polynomial and exponential. It is also open whether such functions of intermediate growth could be defined in o-minimal structures.

Since Kneser's function E is transexponential and contained in a Hardy field, $\mathbb{R}_{\text{exp}}(E)$ is a natural candidate for a transexponential o-minimal structure. Thinking through a proof of o-minimality, however, suggests that $\mathbb{R}_{\text{an,exp}}(E)$ may be a more promising candidate. Van den Dries, Macintyre, and Marker's proof that $\mathbb{R}_{\text{an,exp}}$ is o-minimal is simpler than Wilkie's proof that $\mathbb{R}_{\text{exp}}$ is o-minimal, and it is also more straightforward to adapt. A key part of their proof is that the theory of $\mathbb{R}_{\text{an,exp}}(\log)$ has quantifier elimination and a universal axiomatization, which $\mathbb{R}_{\text{exp}}$ notably does not. With quantifier elimination, one need only consider germs of terms instead of germs of all definable functions when using Lemma 1.1.1. This gives a second version of Lemma 1.1.1, also from [9]:

Lemma 1.1.3. *If $\mathcal{R}$ is an expansion of the real field and $T = \text{Th}(\mathcal{R})$ has quantifier elimination, then T is o-minimal if and only if for each term $t(X)$ in a single variable with parameters in $\mathbb{R}$, there is $m \in \mathbb{R}$ such that either*

1. *$t(x) > 0$ for all $x > m$,*
2. *$t(x) = 0$ for all $x > m$, or*
3. *$t(x) < 0$ for all $x > m$.*

Lemma 1.1.3 gives the following general two-part strategy for proving an expansion $\mathcal{R}$ of the real field is o-minimal:

1. Find a language $\mathcal{L}$ in which $\text{Th}(\mathcal{R})$ has quantifier elimination.
2. Prove that the germs at $+\infty$ of $\mathcal{L}$-terms in a single variable with parameters in $\mathbb{R}$ are ordered.

To apply this strategy to $\mathbb{R}_{\text{an,exp}}(E)$, we first consider Part (1), i.e., what new symbols would be needed in order for $\text{Th}(\mathbb{R}_{\text{an,exp}}(E))$ to have quantifier elimination? Just as $\mathbb{R}_{\text{an,exp}}$ needs a symbol for log to have quantifier elimination, we include a symbol $L = E^{-1}$ for the functional inverse of E. Derivatives are definable with quantifiers using an order symbol, so we also add symbols $E', E'', E''', \ldots$, for the derivatives of E. (We will also use the notation $E^{(d)}$ to denote the dth derivative of E, for $d \in \mathbb{N}$.) These are new function symbols, so we must add symbols for their functional inverses $(E')^{-1}, (E'')^{-1}, (E''')^{-1}, \ldots$ too. We do not need new symbols for the derivatives of $L, (E')^{-1}, (E'')^{-1}, (E''')^{-1}, \ldots$ because we can express them in terms of the symbols we already have. For example,

$$L'(x) = \frac{1}{E'(L(x))}.$$

Thus, we suggest the following as a reasonable language in which to try to show $\text{Th}(\mathbb{R}_{\text{an,exp}}(E))$ has quantifier elimination:

Definition 1.1.4. *Let* $\mathcal{L}_{\text{transexp}} = \mathcal{L}_{\text{an}}(\exp, \log) \cup \left\{E^{(d)} : d \in \mathbb{N}\right\} \cup \left\{\left(E^{(d)}\right)^{-1} : d \in \mathbb{N}\right\}$.

The main result of this thesis is the following:

Theorem 1.1.5. *The germs at* $+\infty$ *of* $\mathcal{L}_{\text{transexp}}$*-terms in a single variable with parameters in* $\mathbb{R}$ *are ordered.*

This solves Part (2) for the language $\mathcal{L}_{\text{transexp}}$.

1.2 Thesis work

To prove Theorem 1.1.5, we will build an ordered differential field of sublogarithmic-transexponential series, based on van den Dries, Macintyre, and Marker's construction of the field of logarithmic-exponential transseries in [8]. The sublogarithmic-transexponential series field will be constructed to embed the germs at $+\infty$ of $\mathcal{L}_{\text{transexp}}$-terms as a substructure. The order on the sublogarithmic-transexponential series then induces an order on the germs.

The sublogarithmic-transexponential series satisfy a theory T_{transexp}, defined as follows. Let $T_{\text{an}}(\exp, \log)$ be the universal axiomatization for $R_{\text{an}}(\exp)$ given in [9] in the language $\mathcal{L}_{\text{an}}(\exp, \log)$. Let T_{transexp} be the following set of universal axioms:

1. Universal axioms for $T_{\text{an}}(\exp, \log)$;
2. Axioms identifying the restrictions of each of the new function symbols to $[0, 1]$ with the corresponding restricted analytic function symbols, i.e.,
   a) $0 \le x \le 1 \to E^{(d)}(x) = \widetilde{E^{(d)}}(x)$, where $\widetilde{E^{(d)}}$ is the restricted analytic function symbol corresponding to the function $E^{(d)}$;
3. For all $x \ge 0$, $E(x+1) = \exp E(x)$;

4. For all $x < 0$, $E^{(d)}(x) = 0$, for $d \in \mathbb{N}$;

5. Axioms stating that the symbols for the inverse functions are indeed inverses, i.e.,

   a) $(x \geq 1 \rightarrow E(L(x)) = x) \wedge (x < 1 \rightarrow L(x) = 0)$ and

   b) $\left(x \geq 1 \rightarrow E^{(d)}\left((E^{(d)})^{-1}(x)\right) = x\right) \wedge \left(x < 1 \rightarrow (E^{(d)})^{-1}(x) = 0\right)$;

6. An axiom for each $d \in \mathbb{N}$ stating that $E^{(d+1)}$ is indeed equal to the derivative of $E^{(d)}$, i.e.,

$$\forall x > 0, \forall \epsilon > 0, \forall y > 0 \left(0 < |x - y| < \min\left(1, \frac{\epsilon}{E(x+1)}\right) \rightarrow \left| \frac{E^{(d)}(y) - E^{(d)}(x)}{y - x} - E^{(d+1)}(x) \right| < \epsilon \right)$$

   Each axiom in this schema is just the usual $\epsilon - \delta$ statement of the derivative, with $\delta := \min\left(1, \frac{\epsilon}{E(x+1)}\right)$.

Given $F \models T_{\text{transexp}}$, we will build a field M_F of sublogarithmic-transexponential transseries with coefficients in F that satisfies T_{transexp}. We also define a derivation on M_F that works like differentiation with respect to the variable of the transseries field. The difficulty is to build M_F so that a unique ordering is maintained at each stage of the construction.

In Chapter 2, we present some basic computations involving E and its derivatives that we will use without reference later on. We also introduce the main challenge in defining an ordering on M_F, which arises from expressions involving E and its derivatives that grow at equivalent rates.

The construction of M_F is divided into Chapters 3-5. In Chapter 3, we prove that certain *finite* sums of monomials built from E and its derivatives are uniquely ordered, if we assume some simple partial ordering relations.

In Chapter 4, we adapt the construction of the logarithmic-exponential series in [8] to start with monomials built from E and its derivatives. The resulting structure is a model of $T_{\text{an,exp}}(\log)$. The key change from the original construction is to allow only *finite* sums of certain kinds of monomials at a time. This restriction allows us to use the result from Chapter 3 to order the structure we build.

In Chapter 5, we build M_F by iterating the construction in Chapter 4 to close off under E, its derivatives, and the inverse function symbols. We also define the derivation on M_F.

1.3 Fields of series and transseries

Transseries can be understood as a generalization of Laurent series that represent (often divergent) asymptotic expansions of real functions at $+\infty$. They were initially studied independently by Dahn and Göring [4] in work on Tarski's problem on the real exponential field and Ecalle in work on the Dulac problem [11]. We refer to van den Dries, Macintyre, and Marker's construction of the logarithmic-exponential (log-exp) series in [8], which closely follows Dahn and Göring's original construction. Elements of the field of log-exp series are infinite sums of "log-exp monomials" arranged in decreasing order. For example,

$$2e^{e^x} - \frac{1}{2}xe^{2x} - x^{1/3} + 3\log x + 1 + x^{-1} + x^{-2} + x^{-3} + \cdots + xe^{-x}$$

is a log-exp series. Chapter 4 of this thesis is an adaptation of van den Dries, Macintyre, and Marker's presentation of the log-exp series construction to "transexponential monomials."

Both the construction of the log-exp series and the work in Chapter 4 make frequent use of Hahn series. We will present an overview of Hahn series in the next section and conclude the chapter with a brief discussion of transexponential transseries constructions.

Hahn series and Mal'cev-Neumann series

The definitions and results in this section can be found in [17], though we follow the notation of [8].

Let $\boldsymbol{k}$ be a ring and G a multiplicative ordered group. The *Mal'cev-Neumann ring* $\boldsymbol{k}((G))$ with monomials in G and coefficients in $\boldsymbol{k}$ consists of formal sums

$$s = \sum_{g \in G} c_g g$$

with $c_g \in \boldsymbol{k}$ such that $\mathrm{Supp}(s) := \{g \in G : c_g \neq 0\}$ is *reverse well-ordered* in G.

We must check that $\boldsymbol{k}((G))$ is indeed a ring. Addition is defined in the obvious way, and it is clear than $\boldsymbol{k}((G))$ is closed under addition and scalar multiplication. Multiplication of series is also defined in the usual way: If $\sum_{g\in G} a_g g, \sum_{g\in G} b_g g \in \boldsymbol{k}((G))$, define

$$\left(\sum_{g\in G} a_g g\right)\left(\sum_{g\in G} b_g g\right) := \sum_{g\in G}\sum_{g_1 g_2 = g} a_{g_1} b_{g_2} g.$$

However, it is not obvious that the series on the right is an element of $\boldsymbol{k}((G))$. The following lemma [17, 3.2 and 3.21] tells us that the inner sum on the right is always finite:

Lemma 1.3.1. *Let $A, B \subset G$ be reverse well-ordered. Then AB is also reverse well-ordered. Furthermore, for any fixed $u \in AB$, there are only finitely many pairs $(a, b) \in A \times B$ such that $u = ab$.*

We will also need to know that for any $\epsilon \in \boldsymbol{k}((G))$ with $\mathrm{Supp}(\epsilon) < 1 \in G$ and any sequence $c_0, c_1, c_2, \cdots \in \boldsymbol{k}$, the sum

$$\sum_{n=0}^{\infty} c_n \epsilon^n$$

is an element of $\boldsymbol{k}((G))$. It follows from Lemma 1.3.1 that for any well-ordered set $A \subset G_{<1}$, A^n is reverse well-ordered, and thus $\bigcup_{n=1}^{k} A^n$ is reverse well-ordered. However, it takes significant work to show that $\bigcup_{n=1}^{\infty} A^n$ is reverse well-ordered and to show that any element of $\bigcup_{n=1}^{\infty} A^n$ appears in only finitely many sets in this union. This result is usually called Neumann's Lemma [17, 3.4 and 3.5]:

Lemma 1.3.2 (Neumann's Lemma). *Let $A \subset G^{<1}$ be reverse well-ordered. Then $\bigcup_{n=1}^{\infty} A^n$ is reverse well-ordered, and any element of $\bigcup_{n=1}^{\infty} A^n$ lies in A^n for only finitely many $n \in \mathbb{N}$.*

By Neumann's Lemma, $\boldsymbol{k}((G))$ is closed under taking power series with coefficients in $\boldsymbol{k}$ of infinitesimal elements.

We now introduce some notation. For an element $0 \neq s = \sum c_g g \in \boldsymbol{k}((G))$, define

1. $\mathrm{Lm}(s) = \max(\mathrm{Supp}(s))$ to be the leading monomial of s,
2. $\mathrm{Lc}(s) = c_{\mathrm{Lm}(s)}$ to be the leading coefficient of s, and
3. $\mathrm{Lt}(s) = \mathrm{Lc}(s)\mathrm{Lm}(s)$ to be the leading term of s.

Define $\mathrm{Lm}(0) = \mathrm{Lc}(0) = \mathrm{Lt}(0) = 0$.

With additional assumptions on $\boldsymbol{k}$, we can say more about $\boldsymbol{k}((G))$. First, if $\boldsymbol{k}$ is ordered, then we can order $\boldsymbol{k}((G))$ by taking $s > 0$ if and only if $\mathrm{Lc}(s) > 0$. Second, if $\mathrm{Lc}(s) \in \boldsymbol{k}^{\times}$, then we can compute the multiplicative inverse of s in $\boldsymbol{k}((G))$ using the usual trick for computing the inverse of an infinite series. So if $\boldsymbol{k}$ is a field, then so is $\boldsymbol{k}((G))$. Since Hahn was the first to study the structure $\boldsymbol{k}((G))$ when $\boldsymbol{k}$ is a field [12], in this case $\boldsymbol{k}((G))$ is called the *Hahn series field* with coefficients in $\boldsymbol{k}$ and monomials in G.

Previous transexponential transseries constructions

In this subsection, we discuss existing work on transseries fields built using a transexponential function. Schmeling, in his 2001 PhD thesis [18], presents a variety of interesting results on transseries fields containing transfinite iterates $\exp_\alpha$ and $\log_\alpha$ of exp and log for $\alpha < \omega^\omega$. For Schmeling, the ωth iterate of exp satisfies the same difference equation

$$f(x+1) = \exp f(x)$$

that we take E to satisfy in this thesis. Schmeling also gives a formal definition of what it means for a generalized power series field $C[[\mathfrak{M}]]$ to be a *transseries* field and develops a theory of derivations and compositions for transseries fields and their transfinite exponential expansions. However, Schmeling does not construct a derivation on the field of transseries containing the transfinite iterates of exp and log.

Van den Dries, van der Hoeven, and Kaplan extend Schmeling's thesis work in [6] to build a field $\mathbb{L}$ of *logarithmic hyperseries* that contains *all* transfinite iterates of log. They also construct natural differentiation, integration, and composition operations on $\mathbb{L}$.

In [2], Bagayoko, van der Hoeven, and Kaplan generalize Schmeling's methods and exploit the properties of $\mathbb{L}$ to build a *hyperserial field* $\mathbb{H}$ that contains all transfinite iterates of log *and* exp, though $\mathbb{H}$ is not yet shown to be closed under differentiation. Bagayoko's upcoming thesis also contains promising results in related areas [1].

The hyperserial field $\mathbb{H}$ differs from the sublogarithmic-transexponential series constructed in this thesis in two ways: First, in addition to being closed under all ordinal iterates of exp and log, $\mathbb{H}$ also allows many kinds of sums that are not in the field of sublogarithmic-transexponential series. The notion of summability in the sublogarithmic-transexponential series is quite restrictive. In order to use the ordering result of Chapter 3 and to define a derivation in a natural way, certain kinds of finiteness are built in through the whole construction.

Second, $\mathbb{H}$ is not yet known to be a differential field, while a derivation on the sublogarithmic-transexponential series is defined in Section 5.3 and respects exp and E. It is necessary to the proof of Theorem 1.1.5 for the sublogarithmic-transexponential series to be a differential field.

Chapter 2

Basic properties of E and its derivatives

In [14], Kneser constructs a real analytic "half-iterate" of e^x, i.e., a function h such that $h(h(x)) = e^x$. Finding half-iterates of a function ψ reduces to finding solutions to the Abel functional equation, which in its full generality is the following:

$$g(\psi(x)) = g(x) + c.$$

After constructing a real analytic solution g to the Abel equation with $c = 1$ and $\psi = \exp$, Kneser defines $h(x) = g^{-1}\left(g(x) + \frac{1}{2}\right)$, so that

$$\begin{aligned} h(h(x)) &= g^{-1}\left(g\left(g^{-1}\left(g(x) + \frac{1}{2}\right)\right) + \frac{1}{2}\right) \\ &= g^{-1}\left(g(x) + 1\right) \\ &= g^{-1}(g(\exp x)) \\ &= \exp x. \end{aligned}$$

However, we are concerned not with partial iterates of exp, but with Kneser's solution g to the Abel equation with $c = 1$ and $\psi = \exp$. If g satisfies $g(\exp(x)) = g(x) + 1$, then g^{-1} satisfies

$$g^{-1}(x + 1) = \exp(g^{-1}(x)).$$

Any such g^{-1} must be transexponential. Kneser's construction gives the following:

Theorem 2.0.1 (Kneser)**.** *The functional equation $g(\exp x) = g(x) + 1$ has a real analytic solution on $x > 0$.*

This functional equation actually has infinitely many real-analytic solutions, but we will use Kneser's, which may be taken to satisfy $g(1) = 0$. We will call the solution given by Kneser's construction L. We will call its inverse E. We will also use E and L to refer to the corresponding operators on germs of functions at $+\infty$ and as formal symbols.

We now give several simple identities and relations derived from the functional equation E satisfies.

Remark 2.0.2. Whenever we write a relation, it should be understood that it holds for all large enough x.

1. $E(x+1) > E(x)^a$ for any $a \in \mathbb{R}$, and $\lim\limits_{x\to\infty} \dfrac{E(x)^a}{E(x+1)} = 0$.

2. $E'(x+1) = e^{E(x)}E'(x) = E(x+1)E'(x)$ by the chain rule, so that $E'(x) > E(x)$. Again $\lim\limits_{x\to\infty} \dfrac{E(x)}{E'(x)} = 0$. This also shows that $E^{(d_1)}(x) > E^{(d_2)}(x)$ if $d_1 > d_2$.

3. Repeatedly differentiating both sides of $E(x+1) = \exp E(x)$ gives algebraic-difference-differential equations of all orders, which are of the form

$$E^{(d)}(x+1) = E(x+1)\sum_{k=1}^{d}\sum_{\bar{j}} \frac{d!}{j_1!\cdots j_{d-k+1}!}\left(\frac{E'(x)}{1!}\right)^{j_1}\cdots\left(\frac{E^{(d-k+1)}(x)}{(d-k+1)!}\right)^{j_{d-k+1}}$$

where the second sum is taken over all sequences $j_1, \ldots, j_{d-k+1}$ in $\mathbb{Z}_+$ satisfying

$$\begin{aligned} j_1 + j_2 + \cdots + j_{d-k+1} &= k \\ j_1 + 2j_2 + \cdots + (d-k+1)j_{d-k+1} &= d. \end{aligned}$$

The double sum in the above equation is the formula for the dth complete Bell polynomial with arguments $E'(x), \ldots, E^{(d)}(x)$, which we will denote by $B_d(x)$, i.e.,

$$B_d(x) := \sum_{k=1}^{d}\sum_{\bar{j}} \frac{d!}{j_1!\cdots j_{d-k+1}!}\left(\frac{E'(x)}{1!}\right)^{j_1}\cdots\left(\frac{E^{(d-k+1)}(x)}{(d-k+1)!}\right)^{j_{d-k+1}}.$$

Bell polynomials are used in the study of set partitions, though they arise here as an instance of Faà di Bruno's formula, which generalizes the chain rule by computing higher derivatives of a composition of functions.

The following lemma comes from Boshernitzan in [3] and will help us derive more identities involving E and its derivatives.

Lemma 2.0.3. *Let $g(x), h(x) > 0$ be continuous, $\lim\limits_{x\to\infty} h(x) = +\infty$, and assume that*

$$\begin{aligned} h(x+1) &> 2h(x) \\ |g(x+1) - g(x)| &\le h(x+1) \end{aligned}$$

for all large enough x. Then $|g(x)| < 3h(x)$ for all large enough x.

Proof. Using the second inequality n times, we have

$$\begin{aligned} g(x+n) &\leq g(x+(n-1)) + h(x+n) \\ &\leq [g(x+(n-2)) + h(x+(n-1))] + h(x+n) \\ &\vdots \\ &\leq g(x) + \sum_{k=0}^{n-1} h(x+(n-k)). \end{aligned}$$

Now using the first inequality n times, $h(x+n) > 2h(x+(n-1)) > \cdots > 2^n h(x)$, so $\frac{h(x)}{h(x+n)} < \frac{1}{2^n}$. So we have

$$\begin{aligned} \lim_{n\to\infty} \frac{g(x+n)}{h(x+n)} &\leq \lim_{n\to\infty} \frac{g(x) + \sum_{k=0}^{n-1} h(x+(n-k))}{h(x+n)} \\ &\leq \lim_{n\to\infty} \left(\sum_{k=0}^{n-1} \frac{1}{2^k} + \frac{g(x)}{h(x+n)} \right) \\ &< 3. \end{aligned}$$

□

If we take $g(x) = \log(E'(x))$ and $h(x) = \log(E(x))$ then we can apply Lemma 2.0.3 because:

$$\begin{aligned} h(x+1) &= \log(E(x+1)) = \log(e^{E(x)}) = E(x) = e^{E(x-1)} \\ &> 2E(x-1) = 2h(x) \\ |g(x+1) - g(x)| &= \log(E'(x+1)) - \log(E'(x)) \\ &= \log\left(\frac{E'(x+1)}{E'(x)}\right) \\ &= \log(E(x+1)) = h(x+1). \end{aligned}$$

Lemma 2.0.3 gives that $\log(E'(x)) < 3\log(E(x))$, and exponentiating, we get $E'(x) < E(x)^3$. From here, we can make a sequence of comparisons.

Lemma 2.0.4. *For all large enough x, we have*

1. $E^{(d)}(x) < E(x)E(x-1)^{3d}$
2. $E^{(d)}(x) < E(x)^2$
3. $E^{(d)}(x-1)^a < E(x)$ *for all* $a \in \mathbb{R}$
4. $E^{(d)}(x-r)^a < E(x)$ *for all* $a \in \mathbb{R}$ *and* $r > 0$

Proof. The first part is proven in [3], using Lemma 2.0.3. The second part follows from the first part and Lemma 2.0.3:

$$\begin{aligned}\lim_{x\to\infty}\frac{E^{(d)}(x+1)}{E(x+1)^2} &= \lim_{x\to\infty}\frac{E(x+1)E(x)^{3d}}{E(x+1)^2}\\ &\leq \lim_{x\to\infty}\frac{E(x)^{3d}}{E(x+1)}\\ &= \lim_{x\to\infty}\frac{E(x)^{3d}}{e^{E(x)}}\\ &= 0.\end{aligned}$$

The third part uses Lemma 2.0.3 and the fact that $\lim_{x\to\infty}\frac{E(x)^a}{E(x+1)} = 0$ for any $a \in \mathbb{R}$:

$$\begin{aligned}\lim_{x\to\infty}\frac{E^{(d)}(x-1)^a}{E(x)} &\leq \lim_{x\to\infty}\frac{(E(x-1)E(x-2)^{3d})^a}{E(x)}\\ &\leq \lim_{x\to\infty}\frac{E(x-1)^{a+1}}{E(x)}\\ &= 0.\end{aligned}$$

For the fourth part, we need only show that $\lim_{x\to\infty}\frac{E(x-r)^m}{E(x)} = 0$ for any $m \in \mathbb{N}$, since the second part tells us that $\lim_{x\to\infty}\frac{E^{(d)}(x-r)^a}{(E(x-r)^2)^a} = 0$. The proof is the same as for the third part, using the fact that partial iterates of the exponential also have faster-than-polynomial growth. First assume r is rational. Write $e_{1/q}$ to denote a $1/q$th iterate of exp (meaning $\exp = \overbrace{e_{1/q} \circ \cdots \circ e_{1/q}}^{q \text{ times}}$) and write $e_{p/q}$ to denote p iterates of $e_{1/q}$, i.e., $e_{p/q} = \overbrace{e_{1/q} \circ \cdots \circ e_{1/q}}^{p \text{ times}}$. Then

$$\begin{aligned}\lim_{x\to\infty}\frac{E(x-r)^m}{E(x)} &= \lim_{x\to\infty}\frac{e_{1-r}(E(x-1))^m}{e^{E(x-1)}}\\ &= \lim_{y\to\infty}\frac{y^m}{e_r(y)}\\ &\leq \lim_{y\to\infty}\frac{y^m}{y^a} && \text{for any } a\\ &= 0 && \text{for } a > m.\end{aligned}$$

If r is irrational, then there is a rational r' such that $0 < r' < r$, so

$$\lim_{x\to\infty}\frac{E(x-r)^m}{E(x)} \leq \lim_{x\to\infty}\frac{E(x-r')^m}{E(x)} = 0.$$

□

2.1 Monomials with equivalent growth rates and logarithmic derivatives

Note that what is actually proven in each part of Lemma 2.0.4 is that the limit of the quotient of the smaller expression by the larger expression is 0. This means that in each of these comparisons, the larger expression is not only larger, but also has a faster growth rate as $x \to \infty$.

However, it is possible for simple expressions involving E and its derivatives to have equivalent growth rates. For example,

$$\begin{aligned}\lim_{x\to\infty} \frac{E(x)E''(x)}{E'(x)^2} &= \lim_{x\to\infty} \frac{E(x)^2\big(E'(x-1)^2 + E''(x-1)\big)}{E(x)^2E'(x-1)^2} \\ &= \lim_{x\to\infty} \left(1 + \frac{E''(x-1)}{E'(x-1)^2}\right) \\ &= 1.\end{aligned}$$

by Lemma 2.0.4, so $E(x)E''(x)$ and $E'(x)^2$ grow at the same rate as x approaches infinity.

Even though $E(x)E''(x)$ and $E'(x)^2$ have the same growth rate, it is easy to determine that $E(x)E''(x) > E'(x)^2$:

$$\begin{aligned}E(x)E''(x) - E'(x)^2 &= E(x)^2\big(E'(x-1)^2 + E''(x-1)\big) - E(x)^2E'(x-1)^2 \\ &= E(x)^2E''(x-1) \\ &> 0.\end{aligned}$$

However, it can be quite complicated to compute the sign of an expression involving many terms with equivalent growth rates. To remedy this, it will be useful to be able to rewrite expressions involving E and its derivatives in terms of different functions that all have distinct growth rates. To this end, we follow Boshernitzan in [3] and introduce a useful sequence of functions.

Definition 2.1.1. *Let $E_0(x) = E(x)$ and let $E_{d+1}(x) = \frac{E_d'(x)}{E_d(x)}$ for $d \in \mathbb{N}$.*

Each function in the sequence is the logarithmic derivative of the previous one.

Lemma 2.1.2. *For $d \geq 2$, we have $E_d(x) = E'(x-d) + R_d(x)$ where $R_d(x)$ is such that $\lim_{x\to\infty} \frac{R_d(x)}{E'(x-d-1)} = 1$.*

Proof. We know $E_0(x) = E(x)$ and $E_1(x) = \frac{E'(x)}{E(x)} = E'(x-1)$. Then,

$$E_2(x) = \frac{E''(x-1)}{E'(x-1)} = \frac{E'(x-2)^2 + E''(x-2)}{E'(x-2)} = E'(x-2) + \frac{E''(x-2)}{E'(x-2)}.$$

Let $R_2(x) := \frac{E''(x-2)}{E'(x-2)}$, which is real analytic on $x > 0$. The same calculation as above, shifted down by 1, shows that $R_2(x) = E'(x-3) + \frac{E''(x-3)}{E'(x-3)}$. So $\lim_{x\to\infty} \frac{R_2(x)}{E'(x-3)} = 1$.

We proceed by induction. Suppose

1. $R_d(x)$ has been defined so that $\lim_{x\to\infty}\frac{R_d(x)}{E'(x-d-1)}=1$ and R_d is real analytic at all large enough values of x.

2. $E_d(x)=E'(x-d)+R_d(x)$.

Then

$$\begin{aligned}E_{d+1}(x)=&\frac{E''(x-d)+R_d'(x)}{E'(x-d)+R_d(x)}\\=&\frac{E(x-d)\big(E'(x-d-1)^2+E''(x-d-1)\big)+R_d'(x)}{E'(x-d)+R_d(x)}\\&+\frac{R_d(x)E'(x-d-1)-R_d(x)E'(x-d-1)}{E'(x-d)+R_d(x)}\\=&\frac{E(x-d)E'(x-d-1)^2+R_d(x)E'(x-d-1)}{E(x-d)E'(x-d-1)+R_d(x)}\\&+\frac{E(x-d)E''(x-d-1)+R_d'(x)-R_d(x)E'(x-d-1)}{E'(x-d)+R_d(x)}\\=&E'(x-d-1)+\frac{E(x-d)E''(x-d-1)-R_d(x)E'(x-d-1)+R_d'(x)}{E'(x-d)+R_d(x)}.\end{aligned}$$

Now we would like to define $R_{d+1}(x)$ to be

$$\frac{E(x-d)E''(x-d-1)-R_d(x)E'(x-d-1)+R_d'(x)}{E'(x-d)+R_d(x)}$$

so we must show this expression satisfies the induction hypothesis. The expression is real analytic at large x because the denominator is large when x is large. So it remains to show the expression has the same growth rate as $E'(x-d-2)$. We will show $\frac{E(x-d)E''(x-d-1)}{E'(x-d)+R_d(x)}$ has the same growth rate at $E'(x-d-2)$, and $\frac{-R_d(x)E'(x-d-1)+R_d'(x)}{E'(x-d)+R_d(x)}$ approaches 0 as $x\to\infty$.

By induction, we know $\lim\limits_{x\to\infty}\dfrac{R_d(x)}{E'(x-d-1)}=1$. Since R_d is differentiable at large x, $\lim\limits_{x\to\infty}\dfrac{R_d'(x)}{E''(x-d-1)}=1$. So

$$\lim_{x\to\infty}\frac{-R_d(x)E'(x-d-1)+R_d'(x)}{E'(x-d)+R_d(x)}=\lim_{x\to\infty}\frac{-R_d(x)E'(x-d-1)+R_d'(x)}{E(x-d)\left(E'(x-d-1)+\frac{R_d(x)}{E(x-d)}\right)}=0$$

since $E'(x-d-1)^n<E(x-d)$ for all $n\in\mathbb{N}$ by Lemma 2.0.4.

Now we can rewrite

$$\begin{aligned}\frac{E(x-d)E''(x-d-1)}{E(x-d)E'(x-d-1)+R_d(x)}&=\frac{E(x-d-1)\big(E'(x-d-2)^2+E''(x-d-2)}{E(x-d-1)E'(x-d-2)+\frac{R_d(x)}{E(x-d)}}\\&=\frac{E'(x-d-2)^2+E''(x-d-2)}{E'(x-d-2)+\frac{R_d(x)}{E(x-d)E(x-d-1)}}.\end{aligned}$$

So again by Lemma 2.0.4

$$\lim_{x\to\infty} \frac{\frac{E'(x-d-2)^2+E''(x-d-2)}{E'(x-d-2)+\frac{R_d(x)}{E(x-d)E(x-d-1)}}}{E'(x-d-2)} = \lim_{x\to\infty} \frac{E'(x-d-2)^2+E''(x-d-2)}{E'(x-d-2)^2} = 1. \qquad \square$$

Corollary 2.1.3. *For all large enough x and all $d, n \in \mathbb{N}$, we have $E_d(x) > E_{d+1}(x)^n$.*

This corollary follows immediately from Lemmas 2.0.4 and 2.1.2, and it is also proved in [3] via different computations.

For each $d \in \mathbb{N}$, we can rewrite $E^{(d)}$ as a polynomial with integer coefficients in $E_0, \dots, E_d$. For example,

$$E' = E_0 \cdot E_1$$
$$E'' = E_0 \cdot (E_1)^2 + E_0 \cdot E_1 \cdot E_2.$$

We get similar expressions for all derivatives of E because

$$E'_d = E_d E_{d+1}$$

for all $d \in \mathbb{N}$.

The importance of the above corollary is that it allows us to rewrite expressions involving the derivatives of E in terms of $E_0, E_1, E_2, \dots$ to identify a dominant term. For example, we showed earlier that $E(x)E''(x)$ and $E'(x)^2$ grow at the same rate as $x \to \infty$. This is easy to see when we rewrite these expressions in terms of $E_0, E_1, E_2, \dots$

$$E(x)E''(x) = E_0(x)^2 E_1(x)^2 + E_0(x)^2 E_1(x) E_2(x)$$
$$E'(x)^2 = E_0(x)^2 E_1(x)^2.$$

We can also rewrite $E_d(x)$ for $d > 0$ in terms of $E_0(x-1), \dots, E_d(x-1)$. For example,

$$\begin{aligned}
E_1(x) &= E'(x-1) = E_0(x-1)E_1(x-1) \\
E_2(x) &= \frac{E'_1(x)}{E_1(x)} \\
&= \frac{E'_0(x-1)E_1(x-1) + E_0(x-1)E'_1(x-1)}{E_0(x-1)E_1(x-1)} \\
&= \frac{E_0(x-1)E_1(x-1)^2 + E_0(x-1)E_1(x-1)E_2(x-1)}{E_0(x-1)E_1(x-1)} \\
&= E_1(x-1)\left(1 + \frac{E_2(x-1)}{E_1(x-1)}\right).
\end{aligned}$$

The computation for $d = 3$ below illustrates the general pattern:

$$\begin{aligned}
E_3(x) &= \frac{E_2'(x)}{E_2(x)} \\
&= \frac{E_1'(x-1) + E_2'(x-1)}{E_1(x-1) + E_2(x-1)} \\
&= \frac{E_1(x-1)E_2(x-1) + E_2(x-1)E_3(x-1)}{E_1(x-1) + E_2(x-1)} \\
&= E_2(x-1)\left(1 + \frac{-E_2(x-1) + E_3(x-1)}{E_1(x-1) + E_2(x-1)}\right).
\end{aligned}$$

The following lemma shows how roughly this same computation can be used for all $d \geq 3$.

Lemma 2.1.4. *Let* $\epsilon_2(x) = \frac{E_2(x-1)}{E_1(x-1)}$*. For all* $d \geq 2$*,* $E_{d+1}(x) = E_d(x-1)(1 + \epsilon_{d+1}(x))$*, where* $\epsilon_{d+1}(x) = \frac{\epsilon_d'(x)}{E_d(x-1)} \cdot \frac{1}{1+\epsilon_d(x)}$ *and* $\epsilon_{d+1}(x)$ *is expressed in terms of* $E_1(x-1), \ldots, E_{d+1}(x-1)$*.*

Proof. We will use induction. In the base case,

$$\begin{aligned}
\epsilon_3(x) &= \frac{\left(\frac{E_2(x-1)}{E_1(x-1)}\right)'}{E_2(x-1)\left(1 + \frac{E_2(x-1)}{E_1(x-1)}\right)} \\
&= \frac{\frac{E_1(x-1)E_2(x-1)E_3(x-1) - E_1(x-1)E_2(x-1)^2}{E_1(x-1)^2}}{E_2(x-1)\left(1 + \frac{E_2(x-1)}{E_1(x-1)}\right)} \\
&= \frac{E_3(x-1) - E_2(x-1)}{E_1(x-1) + E_2(x-1)}
\end{aligned}$$

and our earlier computation shows $E_3(x) = E_2(x)(1 + \epsilon_3(x))$.

Suppose the hypotheses hold for $n = 2, \ldots, d$. Then

$$\begin{aligned}
E_{d+1} &= \frac{E_d'(x)}{E_d(x)} \\
&= \frac{E_{d-1}'(x-1)(1 + \epsilon_d(x)) + E_{d-1}(x-1)\epsilon_d'(x)}{E_{d-1}(x-1)(1 + \epsilon_d(x))} \\
&= \frac{E_{d-1}'(x-1)}{E_{d-1}(x-1)} + \frac{\epsilon_d'(x)}{1 + \epsilon_d(x)} \\
&= E_d(x-1)\left(1 + \frac{\epsilon_d'(x)}{E_d(x-1)} \cdot \frac{1}{1 + \epsilon_d(x)}\right) \\
&= E_d(x-1)(1 + \epsilon_{d+1}(x)).
\end{aligned}$$

By induction, $\epsilon_d(x)$ can be expressed in terms of $E_1(x-1), \ldots, E_d(x-1)$. Since for all $n \in \mathbb{N}$ we have

$$E_n'(x-1) = E_n(x-1)E_{n+1}(x-1)$$

we can express $\epsilon'_d(x)$ in terms of $E_1(x-1), \ldots, E_{d+1}(x-1)$. □

Corollary 2.1.5. *For all* $d \geq 1$, $\lim_{x\to\infty} \frac{\left(\frac{E_{d-1}(x-1)}{E_d(x-1)}\right)}{E(x-d)} = 1$.

Proof. By Lemma 2.1.2

$$\begin{aligned}\frac{E_{d-1}(x-1)}{E_d(x-1)} &= \frac{E'(x-d) + R_{d-1}(x-1)}{E'(x-d-1) + R_d(x-1)} \\ &= \frac{E'(x-d) + R_{d-1}(x-1)}{E'(x-d-1)} \left(\frac{1}{1 + \frac{R_d(x-1)}{E'(x-d-1)}}\right).\end{aligned}$$

Also by Lemma 2.1.2, we know

$$\lim_{x\to\infty} \frac{R_{d-1}(x-1)}{E'(x-d-1)} = 1$$
$$\lim_{x\to\infty} \frac{R_d(x-1)}{E'(x-d-1)} = 0.$$

So $\lim_{x\to\infty} \frac{E_{d-1}(x-1)}{E_d(x-1)} = E(x-d) + 1$ and the result follows. □

We will use these results, in which $E_0, E_1, \ldots$ are functions, to give intuition for how to order purely formal sums of monomials in the remaining chapters.

Chapter 3

Ordering the E-sums

In the previous section, we used the difference equation for E and the difference-differential equations for its derivatives to derive some simple relations among these functions. We also introduced the sequence of logarithmic derivatives $E_0, E_1, \ldots$ as a different "basis" for expressions involving E and its derivatives, which allows us to identify a dominant monomial. In this chapter, we introduce the formal notions of E-sums and E_*-sums and impose relations upon them to match the properties derived in Chapter 2. We also show that under some simple hypotheses, the E-sums can be uniquely ordered.

In the following definitions, we use the same symbols we used for functions in the previous chapter (e.g., $E^{(d)}(x), E_d(x)$, etc.) to represent *purely formal* objects. We do this to highlight the connection to results of the previous chapter and for consistency of notation.

Definition 3.0.1 (E, m-sums)**.** *Let X be a set of variables, and let $\boldsymbol{k}$ be a field. Fix $m \in \mathbb{Z}$.*

1. *First, let G_m be the multiplicative abelian group generated by expressions of the form $E^{(d)}(x-m)^a$ for $d \in \mathbb{N}$, $x \in X$, and $a \in \boldsymbol{k}$, where we identify expressions of the form $E^{(d)}(x-m)^a E^{(d)}(x-m)^b$ with $E^{(d)}(x-m)^{a+b}$.*

2. *Second, let Λ_m be the multiplicative abelian group generated by expressions of the form $\big(\log E'(x-m)\big)^a$ for $x \in X$ and $a \in \boldsymbol{k}$, where we identify expressions of the form $\big(\log E'(x-m)\big)^a \big(\log E'(x-m)\big)^b$ with $\big(\log E'(x-m)\big)^{a+b}$.*

We will call the group ring $\boldsymbol{k}[G_m]$ the E, m-sums. We will call the group ring $\boldsymbol{k}[G_m \Lambda_m]$ the log-E, m-sums.

We will write $\log E'(x-m)^a$ instead of $\big(\log E'(x-m)\big)^a$ to avoid using too many parentheses. The parentheses should be implicitly understood as surrounding $\log E'(x-m)$. If we intend an exponent to apply only to $E'(x-m)$, we will instead write $\log\big(E'(x-m)^a\big)$.

Definition 3.0.2 (E_*, m-sums)**.** *Let X be an ordered set of variables, and let $\boldsymbol{k}$ be an ordered field. Fix $m \in \mathbb{Z}$. Let H_m be the multiplicative abelian group generated by expressions of the form $E_d(x-m)^a$ and $E_0(x-m+k)^a$ for $d, k \in \mathbb{N}$, $x \in X$, and $a \in \boldsymbol{k}$, where we identify* *expressions of the form* $E_d(x-m)^a E_d(x-m)^b$ *with* $E_d(x-m)^{a+b}$ *and* $E_0(x-m+k)^a E_0(x-m+k)^b$ *with* $E_0(x-m+k)^{a+b}$.

Define an order on H_m *as follows: We can write any* $h \in H_m$ *as*

$$h = \prod_{j=1}^{p} E_0(x_j - m + k)^{\alpha_{j,k}} \cdots E_0(x_j - m)^{\alpha_{j,0}} E_1(x_j - m)^{\beta_j} E_2(x_j - m)^{a_{j,2}} \cdots E_d(x_j - m)^{a_{j,d}}$$

for some $p, k, d \in \mathbb{N}$ *and* $x_1 > \cdots > x_p$. *Let*

1. $\overline{\alpha_l} = (\alpha_{1,l}, \alpha_{2,l}, \ldots, \alpha_{p,l})$ *for* $l = 0, \ldots, k$
2. $\overline{\beta} = (\beta_1, \beta_2, \ldots, \beta_p)$
3. $\overline{a_l} = (a_{1,l}, a_{2,l}, \ldots, a_{p,l})$ *for* $l = 2, \ldots, d$

Define $h > 1$ *if and only if the first nonzero element of the following sequence is positive:*

$$\overline{\alpha_k}, \overline{\alpha_{k-1}}, \ldots, \overline{\alpha_0}, \overline{\beta}, \overline{a_2}, \ldots, \overline{a_d}.$$

Let the E_*, m*-sums be the Hahn series field* $\boldsymbol{k}((H_m))$.

At this point, it is not yet clear how we should define a derivation on $\boldsymbol{k}((H_m))$, but we can define a derivation on a subfield generated by monomials with only integer exponents.

Definition 3.0.3. *Let* X *be an ordered set of variables, and let* $\boldsymbol{k}$ *be an ordered differential field. Let* $H'_m \subset H_m$ *be the subgroup generated by expressions of the form* $E_d(x-m)^n$ *for* $d \in \mathbb{N}$, $x \in X$, $n \in \mathbb{Z}$. *Then* $\boldsymbol{k}((H'_m))$ *is a subfield of* $\boldsymbol{k}((H_m))$. *Define a derivation on* $\boldsymbol{k}((H'_m))$ *by*

$$\partial_m(E_d(x-m)) = E_d(x-m)E_{d+1}(x-m)$$

which comes from the definition $E_{d+1} = \frac{E'_d}{E_d}$ *of the sequence of logarithmic derivatives from section 2.1.*

In this section, we will only discuss $\boldsymbol{k}[G_m]$ and $\boldsymbol{k}[G_m\Lambda_m]$ with $m = 0$, and $\boldsymbol{k}((H_m))$ with $m = 0, 1$, though the proofs work the same for any consecutive pair $m = n, n+1 \in \mathbb{Z}$.

Remark 3.0.4. The goal of this section is to show that the log-E-sums $\boldsymbol{k}[G_0\Lambda_0]$ are ordered. We will first show that $\boldsymbol{k}[G_0]$ is ordered by defining an injective ring homomorphism

$$\sigma_0 : \boldsymbol{k}[G_0] \to \boldsymbol{k}((H_0)).$$

Since $\boldsymbol{k}((H_0))$ is an ordered Hahn series field, σ_0 will induce an order on $\boldsymbol{k}[G_0]$. We will then define an order-preserving embedding $\nu_0 : \sigma_0(\boldsymbol{k}[G_0]) \to \boldsymbol{k}((H_1))$ and use it to define an "approximation map"

$$\rho_0 : \boldsymbol{k}[G_0\Lambda_0] \to \boldsymbol{k}((H_1))$$

so-called because it approximates log-E-sums by elements of $\boldsymbol{k}((H_1))$ well enough to induce a unique ordering of $\boldsymbol{k}[G_0\Lambda_0]$ compatible with the order induced on $\boldsymbol{k}[G_0]$.

3.1 The order on $\boldsymbol{k}((H_0))$ induces an order on $\boldsymbol{k}[G_0]$

We will define a homomorphism $\sigma_0 : \boldsymbol{k}[G_0] \to \boldsymbol{k}((H_0))$. Intuitively, we want $\sigma_0(s)$ to be $s \in \boldsymbol{k}[G_0]$ "rewritten" using the expressions for the derivatives of E as polynomials in $E_0, E_1, \ldots$ from Chapter 2. We can begin by defining $\sigma_0(E^{(d)}(x))$ exactly according to this intuition:

$$\sigma_0(E^{(d)}(x)) := E_0(x)E_1(x)^d + \cdots + E_0(x)\cdots E_d(x)$$

is the polynomial expression for $E^{(d)}(x)$ in terms of the logarithmic derivative sequence $E_0(x), \ldots, E_d(x)$.

Next, we must extend the definition to generators of G_0 of the form $E^{(d)}(x)^a$. If $a = n \in \mathbb{N}$, then we can define $\sigma_0(E^{(d)}(x)^n) := \sigma_0(E^{(d)}(x))^n$. If $a \notin \mathbb{N}$, then we cannot define $\sigma_0(E^{(d)}(x)^a)$ to be $\sigma_0(E^{(d)}(x))^a$ because $\sigma_0(E^{(d)}(x))^a$ is not formally an element of $\boldsymbol{k}((H_0))$ (unless $d = 1$). To represent $\sigma_0(E^{(d)}(x))^a$ as an element of $\boldsymbol{k}((H_0))$, we reason as follows:

$$\begin{aligned}
\sigma_0(E^{(d)}(x))^a &= \left(E_0(x)E_1(x)^d + \cdots + E_0(x)\cdots E_d(x)\right)^a \\
&= E_0(x)^a E_1(x)^{da}\left(1 + \frac{E_1(x)^{d-1}E_2(x) + \cdots + E_1(x)\cdots E_d(x)}{E_1(x)^d}\right)^a \\
&= E_0(x)^a E_1(x)^{da}\sum_{k=0}^{\infty}\binom{a}{k}\left(\frac{E_1(x)^{d-1}E_2(x) + \cdots + E_1(x)\cdots E_d(x)}{E_1(x)^d}\right)^k
\end{aligned}$$

where the infinite sum in the final line comes from the Taylor series for $(1 + (\cdot))^a$. The sum in the final line expands to a valid element of $\boldsymbol{k}((H_0))$, so we take this to be the definition of $\sigma_0(E^{(d)}(x)^a)$. Note that this definition extends the definition of $\sigma_0(E^{(d)}(x))^n$ for $n \in \mathbb{N}$ above.

To summarize, we define $\sigma_0 : \boldsymbol{k}[G_0] \to \boldsymbol{k}((H_0))$ by

1. $\sigma_0(E(x)^a) = E_0(x)^a$
2. $\sigma_0(E'(x)^a) = E_0(x)^a E_1(x)^a$
3. $\sigma_0(E^{(d)}(x)^a) = E_0(x)^a E_1(x)^{da} \sum_{k=0}^{\infty} \binom{a}{k} \left(\frac{E_1(x)^{d-1}E_2(x)+\cdots+E_1(x)\cdots E_d(x)}{E_1(x)^d}\right)^k$ for $d > 1$
4. Extend σ_0 to products and sums so that it is a $\boldsymbol{k}$-algebra homomorphism, i.e., for $g_1, \ldots, g_n \in G_0$ and $c_1, \ldots, c_n \in \boldsymbol{k}$, define
   a) $\sigma_0(g_1 \cdots g_n) = \sigma_0(g_1) \cdots \sigma_0(g_n)$
   b) $\sigma_0(c_1 g_1 + \cdots + c_n g_n) = c_1\sigma_0(g_1) + \cdots c_n\sigma_0(g_n)$

We must check that σ_0 is well defined, i.e., that

1. for each generator g of G_0, $\sigma_0(g)$ and g satisfy the same relations, and
2. for each $s \in \boldsymbol{k}[G_0]$, $\sigma_0(s)$ is a valid sum in the Hahn series field $\boldsymbol{k}((H_0))$.

First, the only relations among the generators of G_0 are

$$E^{(d)}(x)^{a+b} = E^{(d)}(x)^a E^{(d)}(x)^b.$$

For $d > 1$, we can compute that

$$\begin{aligned}\sigma_0\left(E^{(d)}(x)^{a+b}\right) &= E_0(x)^{a+b}E_1(x)^{d(a+b)}\sum_{k=0}^{\infty}\binom{a+b}{k}\left(\frac{\sigma_0(E^{(d)}(x))}{E_0(x)E_1(x)^d}-1\right)^k \\ &= E_0(x)^a E_1(x)^{da}\sum_{k=0}^{\infty}\binom{a}{k}\left(\frac{\sigma_0(E^{(d)}(x))}{E_0(x)E_1(x)^d}-1\right)^k \\ &\cdot E_0(x)^b E_1(x)^{db}\sum_{k=0}^{\infty}\binom{b}{k}\left(\frac{\sigma_0(E^{(d)}(x))}{E_0(x)E_1(x)^d}-1\right)^k \\ &= \sigma_0\left(E^{(d)}(x)^a\right)\sigma_0\left(E^{(d)}(x)^b\right).\end{aligned}$$

Also, we immediately have

$$\begin{aligned}\sigma_0\left(E(x)^{a+b}\right) &= \sigma_0\left(E(x)^a\right)\sigma_0\left(E(x)^b\right) \\ \sigma_0\left(E'(x)^{a+b}\right) &= \sigma_0\left(E'(x)^a\right)\sigma_0\left(E'(x)^b\right).\end{aligned}$$

Second, since $\sigma_0(g)$ is a valid sum in $\boldsymbol{k}((H_0))$ for each generator g of G_0, and only finitely many generators of G_0 appear in any $s \in \boldsymbol{k}[G_0]$, $\sigma_0(s)$ is a valid sum in $\boldsymbol{k}((H_0))$.

Remark 3.1.1. Let $s \in \boldsymbol{k}[G_0]$. Because σ_0 is defined using the Taylor expansion of $(1+(\cdot))^a$, any monomial in $\mathrm{Supp}(\sigma_0(s))$ must be of the form

$$\prod_{j=1}^{p} E_0(x_j)^{\alpha_j}\cdot E_1(x_j)^{\beta_j}E_2(x_j)^{n_{j,2}}\cdots E_d(x_j)^{n_{j,d}}$$

with $\alpha_j, \beta_j \in \boldsymbol{k}$ and $n_{j,2},\ldots,n_{j,d} \in \mathbb{N}$ for all $j = 1,\ldots,p$, for some $d, k \in \mathbb{N}$. The key observation is that the exponents of $E_2,\ldots,E_d$ generators are not just any elements of $\boldsymbol{k}$—they must be *natural numbers.*

Remark 3.1.2 (Ordering and separation assumptions)**.** We will now give assumptions from which we can prove that $\boldsymbol{k}[G_0]$ is totally ordered:

1. $\boldsymbol{k}$ is an ordered field.

2. X is a subset of an ordered field L.

3. We have the following partial order: For all $m \in \mathbb{N}$, all $x, y \in X$ with $x > y$, and all $a \in \boldsymbol{k}$

    a) $E(x - m) > \boldsymbol{k}$

b) $E(x-m) > E(y-m)^a$.

4. There is a map $r : X \times X \to \mathbb{Q} \cap (0,1)$ such that for all $x, y \in X$ with $x > y$, we have $x - y < r(x, y)$.

Notice that if we identify
$$E_0(x-m) = E(x-m)$$
then the ordering already defined on each $\boldsymbol{k}((H_m))$ extends the partial order generated by these assumptions.

Lemma 3.1.3. *Suppose X and $\boldsymbol{k}$ satisfy the ordering and separation assumptions in Remark 3.1.2. Then σ_0 is injective.*

Proof. Let $0 \neq s = \sum_{i=1}^{n} c_i g_i \in \boldsymbol{k}[G_0]$ with all $c_i \neq 0$. Then $\sigma_0(s)$ is a possibly infinite sum in $\boldsymbol{k}((H_0))$. To show that $\sigma_0(s) \neq 0$, we will find a monomial of $\sigma_0(s)$ with nonzero coefficient.

Enumerate all the variables that appear in $g_1, \ldots, g_n$ as $x_1 > \cdots > x_p$. Split each $g_i = \prod_{j=1}^{p} g_i(x_j)$ into blocks for each of the x_j's. Let d be largest such that for some j, $E^{(d)}(x_j)$ appears in some g_i. By definition of σ_0, we can express

$$\begin{aligned}\sigma_0(g_i(x_j)) &= \sigma_0(E(x_j)^{a_{i,j,0}})\sigma_0(E'(x_j)^{a_{i,j,1}})\sigma_0(E''(x_j)^{a_{i,j,2}})\cdots\sigma_0(E^{(d)}(x_j)^{a_{i,j,d}}) \\ &= E_0(x_j)^{a_{i,j,0}+\cdots+a_{i,j,d}}E_1(x_j)^{a_{i,j,1}+2a_{i,j,2}+\cdots+da_{i,j,d}}\cdot \\ &\left(\sum_{k_2=0}^{\infty}\binom{a_{i,j,2}}{k_2}\left(\frac{E_2(x_j)}{E_1(x_j)}\right)^{k_2}\right)\cdots\left(\sum_{k_d=0}^{\infty}\binom{a_{i,j,d}}{k_d}\left(\cdots+\frac{E_2(x_j)\cdots E_d(x_j)}{E_1(x_j)^{d-1}}\right)^{k_d}\right)\end{aligned}$$

with $a_{i,j,0}, \ldots, a_{i,j,d} \in \boldsymbol{k}$. When we fully expand out this product of sums, we have that for each $k_d \in \mathbb{Z}_+$, the coefficient of the term

$$E_0(x_j)^{a_{i,j,0}+\cdots+a_{i,j,d}}E_1(x_j)^{a_{i,j,1}+2a_{i,j,2}+\cdots+da_{i,j,d}}\left(\frac{E_2(x_j)\cdots E_d(x_j)}{E_1(x_j)^{d-1}}\right)^{k_d}$$

is $\binom{a_{i,j,d}}{k_d}$. This is because the only times $E_d(x_j)$ appears anywhere in $\sigma_0(g_i(x_j))$ come from instances of $\frac{E_2(x_j)\cdots E_d(x_j)}{E_1(x_j)^{d-1}}$. So the coefficient of the term

$$\left(\frac{E_2(x_j)\cdots E_d(x_j)}{E_1(x_j)^{d-1}}\right)^{k_d}\prod_{l=1}^{p}E_0(x_l)^{a_{i,l,0}+\cdots+a_{i,l,d}}E_1(x_l)^{a_{i,l,1}+2a_{i,l,2}+\cdots+da_{i,l,d}}$$

in $\sigma_0(g_i) = \sigma_0(g_i(x_1))\cdots\sigma_0(g_i(x_p))$ is still $\binom{a_{i,j,d}}{k_d}$, since $E_d(x_j)$ does not appear in $\sigma_0(g_i(x_l))$ for $l \neq j$.

Let $I \subset \{1, \ldots, n\}$ be maximal such that for all $j = 1, \ldots, p$ and all $i_1, i_2 \in I$, we have

$$\begin{aligned}a_{i_1,j,0}+\cdots+a_{i_1,j,d} &= a_{i_2,j,0}+\cdots+a_{i_2,j,d} \\ a_{i_1,j,1}+2a_{i_1,j,2}+\cdots+da_{i_1,j,d} &= a_{i_2,j,1}+2a_{i_2,j,2}+\cdots+da_{i_2,j,d}.\end{aligned}$$

For the remainder of the argument, fix some $j \in \{1, \ldots, p\}$. Then the coefficient of

$$\left(\frac{E_2(x_j)\cdots E_d(x_j)}{E_1(x_j)^{d-1}}\right)^{k_d} \prod_{l=1}^{p} E_0(x_l)^{a_{i,l,0}+\cdots+a_{i,l,d}} E_1(x_l)^{a_{i,l,1}+\cdots+da_{i,l,d}}$$

in $\sigma_0(\sum_{i=1}^{n} c_i g_i)$ is

$$\sum_{i\in I} c_i \binom{a_{i,j,d}}{k_d}.$$

Suppose $a_{i,j,d}$ for $i \in I$ are distinct. If $\sum_{i\in I} c_i \binom{a_{i,j,d}}{k_d} = 0$ for $k_d = 0, \ldots, |I|-1$, then we would have $c_1 = \cdots = c_{|I|} = 0$, a contradiction. If $\sum_{i\in I} c_i \binom{a_{i,j,d}}{k_d} \neq 0$ for some $k_d < |I|$, then we are done because this shows $\sigma_0(s) \neq 0$.

So assume $a_{i,j,d}$ for $i \in I$ are not all distinct. Let $1 < q < |I|$ be the number of distinct values among $a_{i,j,d}$ for $i \in I$. Let $(P_1, \ldots, P_q)$ partition I so that for any $i_1, i_2 \in P_l$, $\alpha_{l,j} := a_{i_1,j,d} = a_{i_2,j,d}$, and $\alpha_{l_1,j} \neq \alpha_{l_2,j}$ for $l_1 \neq l_2$. Consider the sums

$$\sum_{i\in P_l} c_i \frac{g_i}{E^{(d)}(x_j)^{\alpha_{l,j}}}$$

for $l = 1, \ldots, q$. Each of these sums has strictly fewer terms than s, and its monomials have one less multiplicand, since $E^{(d)}(x_j)$ does not appear.

We will proceed by induction with the hypothesis that for each $l = 1, \ldots, q$, we can find a leading term $b_l t_l$ of $\sigma_0\left(\sum_{i\in P_l} c_i \frac{g_i}{E^{(d)}(x_j)^{\alpha_{l,j}}}\right)$, where $0 \neq b_l \in \boldsymbol{k}$ and $t_l \in G_0$. Since $E^{(d)}(x_j)$ does not appear in $\sum_{i\in P_l} c_i \frac{g_i}{E^{(d)}(x_j)^{\alpha_{l,j}}}$, $E_d(x_j)$ does not appear in t_l. Now let $I_0 \subset \{1, \ldots, q\}$ be the set of indices l at which

$$E_0(x_j)^{\alpha_{l,j}} E_1(x_j)^{d\alpha_{l,j}} t_l$$

is maximized in H_0.

Consider the sum

$$\sum_{l\in I_0} \sigma_0(E^{(d)}(x_j)^{\alpha_{l,j}}) b_l t_l = \sum_{l\in I_0} E_0(x_j)^{\alpha_{l,j}} E_1(x_j)^{d\alpha_{l,j}} b_l t_l \sum_{k_d=0}^{\infty} \binom{\alpha_{l,j}}{k_d} \left(\cdots + \frac{E_2(x_j)\cdots E_d(x_j)}{E_1(x_j)^{d-1}}\right)^{k_d}.$$

For each k_d, the following is the largest monomial of the sum with k_d as the exponent of $E_d(x_j)$:

$$E_0(x_j)^{\alpha_{l,j}} E_1(x_j)^{d\alpha_{l,j}} t_l \left(\frac{E_2(x_j)\cdots E_d(x_j)}{E_1(x_j)^{d-1}}\right)^{k_d}.$$

Its coefficient is

$$\sum_{l\in I_0} \binom{\alpha_{l,j}}{k_d} b_l$$

since $E_d(x_j)$ only appears in instances of $\frac{E_2(x_j)\cdots E_d(x_j)}{E_1(x_j)^{d-1}}$. If the coefficient is 0 for all $k_d = 0, \ldots, |I_0|-1$, then that forces $b_1 = \cdots = b_q = 0$, since the $\alpha_{l,j}$'s are all distinct. But this contradicts the induction hypothesis. So some $\sum_{l=1}^{q} \binom{\alpha_{l,j}}{k_d} b_l$ must be nonzero for $k_d < |I_0|$, which means we have found a monomial of $\sigma_0(s)$ with nonzero coefficient. □

Since σ_0 is injective, the ordering of $\boldsymbol{k}((H_0))$ induces an order on $\boldsymbol{k}[G_0]$ by defining $s > 0$ if and only if $\sigma_0(s) > 0$.

3.2 Defining the order preserving embedding ν_0

In Lemma 2.1.4, we showed that for the functions E_d, E_{d+1}, and ϵ_d with $d \geq 2$, we have

$$E_d(x) = E_{d-1}(x-1)(1+\epsilon_d(x)).$$

The proof of this lemma does not actually use any properties of E_d, E_{d+1}, and ϵ_d as functions. Only the identities $E_0 = E$, $E_{d+1} = \frac{E_d'}{E_d}$ for $d \in \mathbb{N}$, and the difference-differential identities for the derivatives of E are needed. Thus, the conclusion of Lemma 2.1.4 still holds in any purely formal context in which the necessary identities have been imposed.

Our goal is to formally build an ordered field of series that embeds the germs at $+\infty$ of $\mathcal{L}_{\text{transexp}}$-terms as an ordered differential field. Thus, we must impose the conclusion of Lemma 2.1.4 if we hope to build something consistent with the identities used to prove this Lemma.

In this section, we will first show how to represent $\epsilon_d(x)$ as an element of $\boldsymbol{k}((H_1))$ for each $d \in \mathbb{N}$. We will also prove a lemma about the form this representation takes, which will be necessary in the next section. Then we will define an order preserving embedding

$$\nu_0 : \sigma_0(\boldsymbol{k}[G_0]) \to \boldsymbol{k}((H_1)).$$

It is important that we restrict the domain of ν_0 to $\sigma_0(\boldsymbol{k}[G_0]) \subsetneq \boldsymbol{k}((H_0))$. Sums in the image of σ_0 can be infinite, but they are limited in two key ways:

1. Only finitely many elements of X appear in any sum.

2. For each sum, there is some $d \in \mathbb{N}$ such that only $E_0, \ldots, E_d$ generators may appear.

Arbitrary elements of $\boldsymbol{k}((H_0))$ do not have these two properties, and this causes problems when trying to extend the definition of ν_0 to all of $\boldsymbol{k}((H_0))$ in the natural way. Fortunately, we will only ever need to embed elements of $\sigma_0(\boldsymbol{k}[G_0])$ into $\boldsymbol{k}((H_1))$, so this is not a problem for us.

Fix $x \in X$. We will represent each $\epsilon_d(x)$ by an element of $\boldsymbol{k}((H_1')) \subset \boldsymbol{k}((H_1))$, by induction. In the base case, we can immediately represent $\epsilon_2(x)$ by $\frac{E_2(x-1)}{E_1(x-1)} \in \boldsymbol{k}((H_1))$. We can also immediately identify the nth derivative of ϵ_2 with $(\partial_1)^{\circ n}\left(\frac{E_2(x-1)}{E_1(x-1)}\right) \in \boldsymbol{k}((H_1'))$ for all $n \in \mathbb{N}$.

Now suppose $\epsilon_d(x)$ is represented by $s_d \in \boldsymbol{k}((H_1'))$. We will use the formula

$$\epsilon_{d+1}(x) = \frac{\epsilon_d'(x)}{E_d(x-1)} \cdot \frac{1}{1+\epsilon_d(x)}$$

from Lemma 2.1.4, replacing $\epsilon_d(x)$ by s_d, $\epsilon_d'(x)$ by $\partial_1(s_d)$, and $\frac{1}{1+\epsilon_d(x)}$ by an infinite sum. Following the formula, we represent $\epsilon_{d+1}(x)$ by

$$\frac{\partial_1(s_d)}{E_d(x-1)} \sum_{j=0}^{\infty} (-s_d)^j \in \boldsymbol{k}((H_1')).$$

We can then represent the nth derivative of ϵ_{d+1} by

$$(\partial_1)^{\circ n} \left(\frac{\partial_1(s_d)}{E_d(x-1)} \sum_{j=0}^{\infty} (-s_d)^j \right) \in \boldsymbol{k}((H_1'))$$

for all $n \in \mathbb{N}$. In what follows, we will write $\epsilon_d(x)$ instead of writing out the sums as above in order to highlight the connection with the computations of Section 2.1.

Lemma 3.2.1. *Let $x \in X$ and $d \in \mathbb{N}$. Then we have the following:*

1. *The exponent of $E_d(x-1)$ in any monomial of $\epsilon_d(x)$ is either 0 or 1.*
2. *$E_{d+k}(x-1)$ does not appear in $\epsilon_d(x)$ for any $k \geq 1$.*
3. *The sum of the exponents of generators in any monomial of $\epsilon_d(x)$ is 0.*

Proof. All three claims are immediate for $d = 2$. Suppose the results hold for $\epsilon_d(x)$. We will show they hold for $\epsilon_{d+1}(x)$.

Since $\epsilon_{d+1}(x) = \frac{\epsilon_d'(x)}{E_d(x-1)} \sum_{j=1}^{\infty} (-\epsilon_d(x))^j$, every monomial of $\epsilon_{d+1}(x)$ is of the form

$$\frac{M_1}{E_d(x-1)} M_2$$

where M_1 is a monomial of $\epsilon_d'(x)$ and M_2 is a monomial of $\sum_{j=1}^{\infty} (-\epsilon_d(x))^j$.

Since $E_{d+k}(x-1)$ does not appear in $\epsilon_d(x)$, it does not appear in $\sum_{j=1}^{\infty} (-\epsilon_d(x))^j$ either. Also, since the sum of exponents in every monomial of $\epsilon_d(x)$ is 0, the same is true for $\sum_{j=1}^{\infty} (-\epsilon_d(x))^j$. So we can ignore M_2 when proving (1), (2), and (3).

Now we will consider the possible forms M_1 can take. Any monomial of $\epsilon_d'(x)$ arises as a monomial of $\partial_1(M)$ for some monomial M of $\epsilon_d(x)$. Let $E_1(x-1)^{n_1} \cdots E_d(x-1)^{n_d}$ be a monomial of $\epsilon_d(x)$, with $n_1, \ldots, n_{d-1} \in \mathbb{Z}$ and $n_d \in \{0, 1\}$. Then

$$\partial_1 \big(E_1(x-1)^{n_1} \cdots E_d(x-1)^{n_d} \big) = \sum_{l=1}^{d} n_l \big(E_1(x-1)^{n_1} \cdots E_d(x-1)^{n_d} \big) \cdot E_{l+1}(x-1).$$

In this sum, $E_{d+1}(x-1)$ can only appear with exponent 0 or 1, which finishes (1). No $E_{d+k}(x-1)$ with $k \geq 2$ appears, which finishes (2). For (3), observe that each monomial in the sum has one new generator $E_{l+1}(x-1)$ with exponent 1. So the sum of exponents in any M_1 must be 1, which means the total sum of exponents in $\frac{M_1}{E_d(x-1)}M_2$ is 0. □

We now define an order-preserving field embedding $\nu_0 : \sigma_0(\boldsymbol{k}[G_0]) \to \boldsymbol{k}((H_1))$ with the intuition of rewriting elements of $\sigma_0(\boldsymbol{k}[G_0]) \subset \boldsymbol{k}((H_0))$ using the difference equations

$$E_d(x) = E_{d-1}(x-1)(1+\epsilon_d(x))$$

from Lemma 2.1.4. Define $\nu_0 : \sigma_0(\boldsymbol{k}[G_0]) \to \boldsymbol{k}((H_1))$ as follows for all $a \in \boldsymbol{k}$ and $n \in \mathbb{N}$:

1. $\nu_0(E_0(x)^a) = E_0(x)^a$
2. $\nu_0(E_1(x)^a) = E_0(x-1)^a E_1(x-1)^a$
3. $\nu_0(E_d(x)^n) = E_{d-1}(x-1)^n(1+\epsilon_d(x))^n$ for $d \geq 2$.
4. Extend ν_0 to products and sums so that it is a $\boldsymbol{k}$-algebra homomorphism. For generators $g_1, \ldots, g_l$ of H_0, let

$$\nu_0(g_1 \cdots g_l) = \nu_0(g_1) \cdots \nu_0(g_l).$$

   For $\sum_{g \in H_0} c_g g \in \sigma_0(\boldsymbol{k}[G_0])$, let

$$\nu_0\left(\sum_{g \in H_0} c_g g\right) = \sum_{g \in H_0} c_g \nu_0(g).$$

We can check with an easy computation that for each $g \in H_0$, $\nu_0(g)$ satisfies the same relations as g. The only relations among the elements of H_0 are

$$E_d(x)^a E_d(x)^b = E_d(x)^{a+b}$$

for all $d \in \mathbb{N}$, $x \in X$, and $a, b \in \boldsymbol{k}$. We must check that $\nu_0(E_d(x)^a)\nu_0(E_d(x)^b) = \nu_0(E_d(x)^{a+b})$. This is clear for $d = 0, 1$, so suppose $d > 1$. Then

$$\begin{aligned}\nu_0(E_d(x)^n)\nu_0(E_d(x)^m) &= E_{d-1}(x-1)^n \sum_{j=0}^{n} \binom{n}{j} \epsilon_d(x)^j \cdot E_{d-1}(x-1)^m \sum_{j=0}^{m} \binom{m}{j} \epsilon_d(x)^j \\ &= E_{d-1}(x-1)^{n+m} \sum_{j=0}^{n+m} \binom{n+m}{j} \epsilon_d(x)^j.\end{aligned}$$

So $\nu_0(E_d(x)^n)$ satisfies the same relations as $E_d(x)^n$.

It is more difficult to check that ν_0 does indeed map to $\boldsymbol{k}((H_1))$.

Lemma 3.2.2. *For every $s \in \boldsymbol{k}[G_0]$, $\nu_0(\sigma_0(s))$ is an element of the Hahn series field $\boldsymbol{k}((H_1))$.*

Proof. Let $\sigma_0(s) = \sum_{g \in H_0} c_g g$. We will show that $\nu_0\left(\sum_{g \in H_0} c_g g\right)$ is a valid sum in $\boldsymbol{k}((H_1))$. Since each $g \in H_0$ is a finite product of generators, ν_0 is well defined on monomials. So it suffices to check that $\left(c_g \nu_0(g) : g \in H_0\right)$ is summable, i.e., that

1. For each $h \in H_1$ there are only finitely many $g \in H_0$ such that $c_g \neq 0$ and $h \in \mathrm{Supp}(\nu_0(g))$.
2. $\displaystyle\bigcup_{g \in \mathrm{Supp}(\sigma_0(s))} \mathrm{Supp}(\nu_0(g))$ is reverse well-ordered in H_1.

First we introduce some notation and setup. Let $x_1 > x_2 > \cdots > x_p$ be the elements of X that appear in s, and let d be order of the largest derivative that appears in s.

Enumerate $\mathrm{Supp}(\sigma_0(s))$ as $(g_i : i < \delta)$, a reverse well-ordered sequence in H_0. By Remark 3.1.1, we can write

$$g_i = \prod_{j=1}^{p} E_0(x_j)^{\alpha_{i,j}} \cdot E_1(x_j)^{\beta_{i,j}} E_2(x_j)^{n_{i,j,2}} \cdots E_d(x_j)^{n_{i,j,d}}$$

with $\alpha_{i,j}, \beta_{i,j} \in \boldsymbol{k}$ and $n_{i,j,2}, \ldots, n_{i,j,d} \in \mathbb{N}$ for $j = 1, \ldots, p$. Then

$$\begin{aligned}
\nu_0(g_i) = \prod_{j=1}^{p} \Big(& E_0(x_j)^{\alpha_{i,j}} \cdot E_0(x_j - 1)^{\beta_{i,j}} E_1(x_j - 1)^{\beta_{i,j}} \\
& \cdot E_1(x_j - 1)^{n_{i,j,2}} (1 + \epsilon_2(x_j))^{n_{i,j,2}} \cdots E_{d-1}(x_j - 1)^{n_{i,j,d}} (1 + \epsilon_d(x_j))^{n_{i,j,d}} \Big) \\
= \prod_{j=1}^{p} \Big(& E_0(x_j)^{\alpha_{i,j}} \cdot E_0(x_j - 1)^{\beta_{i,j}} E_1(x_j - 1)^{\beta_{i,j} + n_{i,j,2}} E_2(x_j - 1)^{n_{i,j,3}} \cdots E_{d-1}(x_j - 1)^{n_{i,j,d}} \\
& \cdot (1 + \epsilon_2(x_j))^{n_{i,j,2}} \cdots (1 + \epsilon_{d-1}(x_j))^{n_{i,j,d}} \Big).
\end{aligned}$$

We will use the following observations about the final line $(1+\epsilon_2(x_j))^{n_{i,j,2}} \cdots (1+\epsilon_{d-1}(x_j))^{n_{i,j,d}}$ above, which follow from the fact that $\epsilon_2, \ldots, \epsilon_d \in \boldsymbol{k}((H_1'))$:

(A) All the exponents are integers.

(B) Only $E_1(x_j - 1), \ldots, E_d(x_j - 1)$ may appear for $j = 1, \ldots, p$. No E_0 generator may appear. Thus, the exponents of the E_0 generators are fixed across all monomials of $\nu_0(g_i)$.

To finish the setup, we introduce notation for tuples of exponents in g_i for $i < \delta$. Let

- $\overline{\alpha_i} = (\alpha_{i,1}, \alpha_{i,2}, \ldots, \alpha_{i,p})$

- $\overline{\beta_i} = (\beta_{i,1}, \beta_{i,2}, \ldots, \beta_{i,p})$
- $\overline{n_{i,l}} = (n_{i,1,l}, n_{i,2,l}, \ldots, n_{i,p,l})$ for $l = 2, \ldots, d$.

Now we will prove (1). It suffices to show that if $h \in \operatorname{Supp}(\nu_0(g_0))$, then $h \in \operatorname{Supp}(g_i)$ for only finitely many $i < \delta$.

So suppose $h \in \operatorname{Supp}(\nu_0(g_0))$. Then by Observation (B), h must be of the form

$$h = \prod_{j=1}^{p} E_0(x_j)^{\alpha_{0,j}} E_0(x_j - 1)^{\beta_{0,j}} E_1(x_j - 1)^{\beta_{0,j}+N_{j,1}} E_2(x_j - 1)^{N_{j,2}} \cdots E_d(x_j - 1)^{N_{j,d}}$$

with $(N_{j,1}, \ldots, N_{j,d}) \in \mathbb{Z}^d$ by Observation (A).

Similarly, if $h \in \operatorname{Supp}(\nu_0(g_i))$ for $i > 0$, then again by Observations (A) and (B), h must be of the form

$$h = \prod_{j=1}^{p} E_0(x_j)^{\alpha_{i,j}} E_0(x_j - 1)^{\beta_{i,j}} E_1(x_j - 1)^{\beta_{i,j}+M_{j,1}} E_2(x_j - 1)^{M_{j,2}} \cdots E_d(x_j - 1)^{M_{j,d}}$$

with $(M_{j,1}, \ldots, M_{j,d}) \in \mathbb{Z}^d$. So we must have $\alpha_{i,j} = \alpha_{0,j}$ and $\beta_{i,j} = \beta_{0,j}$ for $j = 1, \ldots, p$.

By the way the order on H_1 is defined, since $g_i < g_0$, we must have

$$(\overline{\alpha_i}, \overline{\beta_i}, \overline{n_{i,2}}, \overline{n_{i,3}}, \ldots, \overline{n_{i,d}}) < (\overline{\alpha_0}, \overline{\beta_0}, \overline{n_{0,2}}, \overline{n_{0,3}}, \ldots, \overline{n_{0,d}})$$

in the lexicographic order on $\boldsymbol{k}^{2p} \times \mathbb{N}^{p(d-1)}$. Since the first parts of each of these sequences are equal, we must have

$$(\overline{n_{i,2}}, \overline{n_{i,3}}, \ldots, \overline{n_{i,d}}) < (\overline{n_{0,2}}, \overline{n_{0,3}}, \ldots, \overline{n_{0,d}})$$

in the lexicographic order on $\mathbb{N}^{p(d-1)}$. Any reverse well-ordered sequence in $\mathbb{N}^{p(d-1)}$ with the lexicographic order must be finite, so there can be only finitely many $i < \delta$ with $(\overline{n_{i,2}}, \overline{n_{i,3}}, \ldots, \overline{n_{i,d}}) < (\overline{n_{0,2}}, \overline{n_{0,3}}, \ldots, \overline{n_{0,d}})$, i.e., only finitely many i with $h \in \operatorname{Supp}(\nu_0(g_i))$. This finishes the proof of (1).

For (2), let $\varnothing \neq B \subset \bigcup_{\iota<\alpha} \operatorname{Supp}(\nu_0(g_\iota))$. We will show B has a greatest element by building an increasing sequence $b_0, b_1, b_2, \ldots$ in B and showing it must terminate. In the base case, let $B_0 = B \neq \varnothing$. Given $\varnothing \neq B_i \subset B$, we define $\gamma_i \in H_0$, $b_i \in B$, and $B_{i+1} \subset B$ as follows: Since $\operatorname{Supp}(\sigma_0(s))$ is reverse well-ordered and $B_i \neq \varnothing$, there must be a greatest element $g \in \operatorname{Supp}(\sigma_0(s))$ such that $B_i \cap \nu_0(g) \neq \varnothing$.

1. Let $\gamma_i := g$.
2. Let $b_i := \max\big(B_i \cap \operatorname{Supp}(\nu_0(\gamma_i))\big)$, which exists since $\operatorname{Supp}(\nu_0(\gamma_i))$ is reverse well-ordered.
3. Let $B_{i+1} := \{b \in B : b > b_i\}$.

If $B_{i+1} \neq \varnothing$, we continue. Note that $b_i \in B_i \setminus B_{i+1}$, so $B_{i+1} \subsetneq B_i$. So when we continue, we will have $\gamma_{i+1} < \gamma_i$ and $b_{i+1} > b_i$. If $B_{i+1} = \varnothing$, then b_i is the largest element of B and the sequence terminates.

We must show that the sequence $b_0, b_1, \dots$ terminates. Again by Remark 3.1.1, write

$$\gamma_i = \prod_{j=1}^{p} E_0(x_j)^{\alpha_{i,j}} \cdot E_1(x_j)^{\beta_{i,j}} \cdot E_2(x_j)^{n_{i,j,1}} \cdots E_d(x_j)^{n_{i,j,d}}$$

with $\alpha_{i,j,k}, \dots, \alpha_{i,j,0}, \beta_{i,j} \in \boldsymbol{k}$ and $n_{i,j,2}, \dots, n_{i,j,d} \in \mathbb{N}$ for $j = 1, \dots, p$. Then, again by Observations (A) and (B), we can write b_i as

$$b_i = \prod_{j=1}^{p} E_0(x_j)^{\alpha_{i,j}} E_0(x_j - 1)^{\beta_{i,j}} \cdot E_1(x_j - 1)^{\beta_{i,j}+N_{i,j,1}} E_2(x_j - 1)^{N_{i,j,2}} \cdots E_d(x_j - 1)^{N_{i,j,d}}$$

with $N_{i,j,1}, \dots, N_{i,j,d} \in \mathbb{Z}$.

We will now show that the inequalities $\gamma_i < \gamma_0$ and $b_i > b_0$ for $i > 0$ imply that the sequence $b_0, b_1, \dots$ terminates. First, since $\gamma_i < \gamma_0$, we must have that

$$(\overline{\alpha_i}, \overline{\beta_i}) \leq (\overline{\alpha_0}, \overline{\beta_0})$$

in the lexicographic order on $\boldsymbol{k}^{2p}$. Since $b_i > b_0$, we must have

$$(\overline{\alpha_i}, \overline{\beta_i}) \geq (\overline{\alpha_0}, \overline{\beta_0})$$

in the lexicographic order on $\boldsymbol{k}^{2p}$. Thus $\alpha_{i,j} = \alpha_{0,j}$ and $\beta_{i,j} = \beta_{0,j}$ for all $j = 1, \dots, p$.

So once again, $\gamma_i < \gamma_0$ implies

$$(\overline{n_{i,2}}, \overline{n_{i,3}}, \dots, \overline{n_{i,d}}) < (\overline{n_{0,2}}, \overline{n_{0,3}}, \dots, \overline{n_{0,d}})$$

in the lexicographic order on $\mathbb{N}^{p(d-1)}$. Since $\mathrm{Supp}(\sigma_0(s))$ is reverse well-ordered, and any reverse well-ordered sequence in $\mathbb{N}^{p(d-1)}$ with the lexicographic order must be finite, the sequence $\gamma_1, \gamma_2, \dots$ must terminate after finitely many steps. Thus, the sequence $b_0, b_1, \dots$ terminates too, and its final element is the largest element of B. This completes the proof of (2). □

Thus the map $\nu_0 : \sigma_0(\boldsymbol{k}[G_0]) \to \boldsymbol{k}((H_1))$ is well defined.

The next corollary and remark follow from computations in the proof of Lemma 3.2.2, but we state them separately for clarity and completeness.

Corollary 3.2.3. *If $s \in \sigma_0(\boldsymbol{k}[G_0])$ and $g_1 > g_2$ are two monomials of s, then* $\mathrm{Lm}(\nu_0(g_1)) > \mathrm{Lm}(\nu_0(g_2))$.

Proof. Let $x_1 > x_2 > \cdots > x_p$ be the elements of X that appear in g_1 or g_2, let k be largest such that $E_0(x_j + k)$ appears in either g_1 or g_2 for some x_j, and let d be largest such that some E_d generator appears in g_1 or g_2. Then we can write

$$g_i = \prod_{j=1}^{p} E_0(x_j)^{\alpha_{i,j}} \cdot E_1(x_j)^{\beta_{i,j}} \cdot E_2(x_j)^{n_{i,j,2}} \cdots E_d(x_j)^{n_{i,j,d}}$$

with $\alpha_{i,j}, \beta_{i,j} \in \boldsymbol{k}$ and $n_{i,j,2}, \ldots, n_{i,j,d} \in \mathbb{N}$ for $j = 1, \ldots, p$ and $i = 1, 2$. Then

$$\mathrm{Lm}(\nu_0(g_i)) = \prod_{j=1}^{p} E_0(x_j)^{\alpha_{i,j}} E_0(x_j - 1)^{\beta_{i,j}} E_1(x_j - 1)^{\beta_{i,j}+n_{i,j,2}} \cdots E_{d-1}(x_j - 1)^{n_{i,j,d}}.$$

Since $g_1 > g_2$, we must have

$$(\overline{\alpha_1}, \overline{\beta_1}, \overline{n_{1,2}}, \overline{n_{1,3}}, \ldots, \overline{n_{1,d}}) > (\overline{\alpha_2}, \overline{\beta_2}, \overline{n_{2,2}}, \overline{n_{2,3}}, \ldots, \overline{n_{2,d}})$$

in the lexicographic order on $\boldsymbol{k}^{p(d+1)}$. So we also have $\mathrm{Lm}(\nu_0(g_1)) > \mathrm{Lm}(\nu_0(g_2))$. □

Remark 3.2.4. Suppose g_1 and g_2 are as in Corollary 3.2.3, and also that

$$(\overline{\alpha_1}, \overline{\beta_1}) > (\overline{\alpha_2}, \overline{\beta_2})$$

in the lexicographic order on $\boldsymbol{k}^{2p}$. By observation (B) from the proof of Lemma 3.2.2, the exponents of the E_0 generators are fixed across all monomials of $\nu_0(g_i)$ for $i = 1, 2$. Thus, every monomial of $\nu_0(g_1)$ is greater than every monomial of $\nu_0(g_2)$, i.e.,

$$\mathrm{Supp}(\nu_0(g_1)) > \mathrm{Supp}(\nu_0(g_2)).$$

Corollary 3.2.5. *ν_0 is order-preserving and thus injective.*

Proof. Let $s \in \sigma_0(\boldsymbol{k}[G_0])$ and let $g_1 = \mathrm{Lm}(s)$. We will show that

$$\mathrm{Lt}(\nu_0(s)) = \mathrm{Lc}(s) \cdot \mathrm{Lt}(\nu_0(g_1))$$

and thus $\nu_0(s) > 0$ if and only if $\mathrm{Lc}(s) > 0$ if and only if $s > 0$.

This is clear if s is a single term, so suppose s has some other monomial g_2. By Lemma 3.2.3, we know $\mathrm{Lm}(\nu_0(g_1)) > \mathrm{Lm}(\nu_0(g_2))$. Since this holds for any $g_2 \neq g_1$, we must have

$$\mathrm{Lm}(\nu_0(s)) = \mathrm{Lm}(\nu_0(g_1))$$

and

$$\mathrm{Lt}(\nu_0(s)) = \mathrm{Lc}(s) \cdot \mathrm{Lt}(\nu_0(g_1))$$

as desired. □

3.3 Showing $\boldsymbol{k}[G_0\Lambda_0]$ is ordered

To define an order on $\boldsymbol{k}[G_0]$, we embedded it into $\boldsymbol{k}((H_0))$, which induced an order on $\boldsymbol{k}[G_0]$. The embedding was defined based on intuition from "rewriting" expressions involving the derivatives of E as polynomials in $E_0, E_1, \ldots$. We would like to replicate this idea to define an order on $\boldsymbol{k}[G_0\Lambda_0]$, but the generators $\log E'(x)^a$ of Λ_0 are an obstacle. Recall that the difference-differential equation for E' is $E'(x) = E(x)E'(x-1)$. Taking log of both sides, we get

$$\log E'(x) = E(x-1) + \log E'(x-1).$$

There is not a clear (to us) way of "rewriting" $\log E'(x)$ in terms of $E_0(x-1), E_1(x-1), \ldots$. However, we can "approximate" $\log E'(x)$ by

$$\log E'(x) \approx E_0(x-1) + E_1(x-1).$$

We get this approximation by identifying $E(x-1) = E_0(x-1)$, and since $E'(x-1) < E(x-1)^2$ implies

$$\log E'(x-1) < 2E(x-2) < E'(x-2) = E_1(x-1).$$

In this section, we will define an embedding

$$\rho_0 : \boldsymbol{k}[G_0\Lambda_0] \to \boldsymbol{k}((H_1))$$

so that

1. ρ_0 is "true" on elements of $\boldsymbol{k}[G_0]$, meaning that if $s \in \boldsymbol{k}[G_0]$, then $\rho_0(s) = \nu_0(\sigma_0(s))$
2. $\rho_0(\log E'(x)) = E_0(x-1) + E_1(x-1)$.

We can extend ρ_0 to generators of the form $\log E'(x)^b$ of Λ_0 using the Taylor series for $(1+(\cdot))^b$. We will call ρ_0 the "approximation map" because its effect will be to approximate every $t \in \boldsymbol{k}[G_0\Lambda_0]$ well enough by elements of $\boldsymbol{k}((H_1))$ that the sign of t is determined by the ordering and separation assumptions on $\boldsymbol{k}$ and X and the order on $\boldsymbol{k}((H_1))$.

It is reasonable to approximate $\log E'(x)$ by $E_0(x-1) + E_1(x-1)$ because the approximation does not affect the leading monomial $E_0(x-1)$. We could work to find an element of $\boldsymbol{k}((H_1))$ that better approximates $\log E'(x)$, but we will show that our approximation is close enough to determine how $\boldsymbol{k}[G_0\Lambda_0]$ should be ordered.

To summarize, we define the "approximation map" $\rho_0 : \boldsymbol{k}[G_0\Lambda_0] \to \boldsymbol{k}((H_1))$ by

1. $\rho_0(s) = \nu_0(\sigma_0(s))$ for $s \in \boldsymbol{k}[G_0]$
2. $\rho_0(\log E'(x)^b) = E_0(x-1)^b \sum_{k=0}^{\infty} \binom{b}{k} \left(\frac{E_1(x-1)}{E_0(x-1)}\right)^k$
3. Extend ρ_0 to products and sums so that it is a $\boldsymbol{k}$-algebra homomorphism, i.e., for $g_1, \ldots, g_n \in G_0\Lambda_0$ and $c_1, \ldots, c_n \in \boldsymbol{k}$, define

a) $\rho_0(g_1 \cdots g_n) = \rho_0(g_1) \cdots \rho_0(g_n)$

b) $\rho_0(c_1 g_1 + \cdots + c_n g_n) = c_1 \rho_0(g_1) + \cdots c_n \rho_0(g_n)$.

We can check that ρ_0 is well defined in the same way we checked that σ_0 is well defined: it respects the relations among generators of G_0 and Λ_0, and for each $s \in \boldsymbol{k}[G_0\Lambda_0]$, $\rho_0(s)$ is a valid sum in $\boldsymbol{k}((H_1))$.

We can write any $t \in \boldsymbol{k}[G_0\Lambda_0] = \boldsymbol{k}[G_0][\Lambda_0]$ as

$$t = s_1 g_1 + \cdots + s_n g_n$$

with $s_i \in \boldsymbol{k}[G_0]$ and $g_i \in \Lambda_0$ for $i = 1, \ldots, n$ and $g_{i_1} \neq g_{i_2}$ for $i_1 \neq i_2$.

Remark 3.3.1. Given $s \in \boldsymbol{k}[G_0\Lambda_0]$, we will define a sum $\operatorname{Init}(s)$, which we intend to be an initial subsum of s. In Lemma 3.3.2, we will show that $\operatorname{Init}(s) \neq 0$, and therefore

$$\operatorname{Lm}\big(\rho_0(s)\big) = \operatorname{Lm}\big(\operatorname{Init}(s)\big).$$

From this, we will conclude that ρ_0 is injective.

Let $s = s_1 g_1 + \cdots + s_n g_n \in \boldsymbol{k}[G_0\Lambda_0]$ with $s_i \in \boldsymbol{k}[G_0]$ and $g_i \in \Lambda_0$. Let $x_1 > x_2 > \cdots > x_p$ list the elements of X appearing in s. Let d be the largest derivative appearing in any s_i.

We will define $\operatorname{Init}(s)$ by looking at $\rho_0(s)$ and figuring out what form its largest monomials could take. First, we look at the images of the s_i's under ρ_0. By definition, $\rho_0(s_i) = \nu_0(\sigma_0(s_i))$. By Remark 3.1.1, for each $i = 1, \ldots, n$ write

$$\operatorname{Lm}(\sigma_0(s_i)) = \prod_{j=1}^{p} E_0(x_j)^{\alpha_{i,j}} \cdot E_1(x_j)^{\beta_{i,j}} \cdot E_2(x_j)^{n_{i,j,2}} \cdots E_d(x_j)^{n_{i,j,d}}$$

with $\alpha_{i,j}, \beta_{i,j} \in \boldsymbol{k}$ and $n_{i,j,2}, \ldots, n_{i,j,d} \in \mathbb{N}$ for each $j = 1, \ldots, p$. Let t_i be the initial subsum of $\sigma_0(s_i)$ with monomials of the form

$$\prod_{j=1}^{p} E_0(x_j)^{\alpha_{i,j}} \cdot E_1(x_j)^{\beta_{i,j}} \cdot E_2(x_j)^{m_{i,j,2}} \cdots E_d(x_j)^{m_{i,j,d}}$$

with $m_{i,j,2}, \ldots, m_{i,j,d} \in \mathbb{N}$ and all other exponents the same as in $\operatorname{Lm}(\sigma_0(s_i))$. Then t_i is a finite sum because any reverse well-ordered sequence of tuples of exponents in $\mathbb{N}^{p(d-1)}$ must be finite.

For any monomial M of t_i, we have

$$\nu_0(M) = \prod_{j=1}^{p} E_0(x_j)^{\alpha_{i,j}} E_0(x_j - 1)^{\beta_{i,j}} E_1(x_j - 1)^{\beta_{i,j} + n_{i,j,2}}$$
$$E_2(x_j - 1)^{n_{i,j,3}} \cdots E_{d-1}(x_j - 1)^{n_{i,j,d}} (1 + \epsilon_2(x_j))^{n_{i,j,2}} \cdots (1 + \epsilon_{d-1}(x_j))^{n_{i,j,d}}$$

If $M_1 \in \mathrm{Supp}(\sigma_0(s_i)) \setminus \mathrm{Supp}(t_i)$, then by Remark 3.2.4, $\mathrm{Supp}(\nu_0(M)) > \mathrm{Supp}(\nu_0(M_1))$, i.e., every monomial of $\nu_0(M)$ is greater than every monomial of $\nu_0(M_1)$.

Now we look at the images of the g_i's under ρ_0. Let $g_i = \prod_{j=1}^{p} \log E'(x_j)^{b_{i,j}}$ for each $i = 1, \ldots, n$. Then

$$\rho_0(g_i) = \prod_{j=1}^{p} E_0(x_j - 1)^{b_{i,j}} \sum_{k=0}^{\infty} \binom{b_{i,j}}{k} \left(\frac{E_1(x_j - 1)}{E_0(x_j - 1)}\right)^k.$$

Observe that the leading monomial of $\rho_0(g_i)$ is $\prod_{j=1}^{p} E_0(x_j - 1)^{b_{i,j}}$. If

$$M_2 \in \mathrm{Supp}(\rho_0(g_i)) \setminus \{\mathrm{Lm}(\rho_0(g_i))\}$$

then the exponent of $E_0(x_j - 1)$ in M_2 is strictly less than $b_{i,j}$, for some $j = 1, \ldots, p$. So

$$\mathrm{Supp}(\mathrm{Lm}(\rho_0(g_i))) > \mathrm{Supp}(M_2)$$

Altogether, this shows that

$$\mathrm{Supp}\big(\nu_0(t_i) \cdot \mathrm{Lm}(\rho_0(g_i))\big) > \mathrm{Supp}\big(\rho_0(s_i g_i) - \nu_0(t_i) \cdot \mathrm{Lm}(\rho_0(g_i))\big)$$

i.e., $\nu_0(t_i) \cdot \mathrm{Lm}(\rho_0(g_i))$ is an initial subsum of $\rho_0(s_i g_i)$.

Now we select the indices i for which the initial subsum $\nu_0(t_i) \cdot \mathrm{Lm}(\rho_0(g_i))$ of $\rho_0(s_i g_i)$ may possibly contribute to the leading monomial of $\rho_0(s)$. Define $I_0 \subset \{1, \ldots, n\}$ to be the set of indices such that

$$(\overline{\alpha_i}, \overline{\beta_i} + \overline{b_i})$$

is maximal in the lexicographic order on $\boldsymbol{k}^{2p}$ (where the addition $\overline{\beta_i} + \overline{b_i}$ is coordinatewise). If $i_0 \in I_0$ and $i_1 \notin I_0$, then every monomial of $\nu_0(t_{i_0}) \cdot \mathrm{Lm}(\rho_0(g_{i_0}))$ is greater than every monomial of $\nu_0(t_{i_1}) \cdot \mathrm{Lm}(\rho_0(g_{i_1}))$ by Remark 3.2.4.

Finally, we define

$$\mathrm{Init}(s) := \sum_{i \in I_0} \nu_0(t_i) \cdot \mathrm{Lm}(\rho_0(g_i)).$$

Observe that if $s \in \boldsymbol{k}[G_0\Lambda_0]$ and $\log E'(x)^b$ appears in s, then the sign of $\mathrm{Init}(s)$ is only (possibly) affected by the leading monomial of $\rho_0(\log E'(x)^b)$.

Lemma 3.3.2. *Suppose X, and $\boldsymbol{k}$ satisfy the order and separation assumptions in Remark 3.1.2. Then ρ_0 is injective.*

Proof. Let $s \in \boldsymbol{k}[G_0\Lambda_0]$. If $\mathrm{Init}(s) \neq 0$, then it follows from the definition of $\mathrm{Init}(s)$ in Remark 3.3.1 that $\mathrm{Init}(s)$ is an initial subsum of $\rho_0(s)$. Thus

$$\mathrm{Lm}\big(\mathrm{Init}(s)\big) = \mathrm{Lm}\big(\rho_0(s)\big).$$

Like the proof of Lemma 3.1.3, we will find a monomial of $\mathrm{Init}(s)$ with a nonzero coefficient.

Following the notation of Remark 3.3.1, we may assume without loss of generality that $\alpha_{i,j} = \beta_{i,j} + b_{i,j} = 0$ for all $i \in I_0$, $j = 1, \ldots, p$, by rescaling or factoring out common terms. Let q_i be the (finite) number of terms in t_i, and let $c_{i,l}$ be the coefficient of the lth monomial of t_i. Then we can write

$$\mathrm{Init}(s) = \sum_{i \in I_0} \sum_{l=1}^{q_i} c_{i,l} \prod_{j=1}^{p} E_1(x_j - 1)^{\beta_{i,j} + m_{i,j,l,2}} (1 + \epsilon_2(x_j))^{m_{i,j,l,2}}$$
$$E_2(x_j - 1)^{m_{i,j,l,3}} (1 + \epsilon_3(x_j))^{m_{i,j,l,3}} \cdots E_{d-1}(x_j - 1)^{m_{i,j,l,d}} (1 + \epsilon_d(x_j))^{m_{i,j,l,d}}.$$

with $c_{i,l} \neq 0$ for all i, l. Note that $\beta_{i,j}$ depends on i and j, but not on l. Let

$$\mathrm{Init}(s)_{i,l} := \prod_{j=1}^{p} E_1(x_j - 1)^{\beta_{i,j} + m_{i,j,l,2}} (1 + \epsilon_2(x_j))^{m_{i,j,l,2}}$$
$$E_2(x_j - 1)^{m_{i,j,l,3}} (1 + \epsilon_3(x_j))^{m_{i,j,l,3}} \cdots E_{d-1}(x_j - 1)^{m_{i,j,l,d}} (1 + \epsilon_d(x_j))^{m_{i,j,l,d}}.$$

If $\mathrm{Supp}\,(\mathrm{Init}(s)_{i_1,l_1}) \cap \mathrm{Supp}\,(\mathrm{Init}(s)_{i_2,l_2}) \neq \varnothing$, then $\beta_{i_1,j}$ and $\beta_{i_2,j}$ must be in the same $\mathbb{Z}$-orbit for all $j = 1, \ldots, p$. So without loss of generality, we may assume $\beta_{i,j} \in \mathbb{N}$ for all $i \in I_0$, $j = 1, \ldots, p$.

We will now proceed by induction on d. If $d = 1$, then $\mathrm{Init}(s)$ has just one term, and its coefficient is nonzero, so we are done. The base case of our induction will be $d = 2$.

If $d = 2$, then

$$\mathrm{Init}(s) = \sum_{i \in I_0} \sum_{l=1}^{q_i} c_{i,l} \prod_{j=1}^{p} E_1(x_j - 1)^{\beta_{i,j} + m_{i,j,l,2}} (1 + \epsilon_2(x_j))^{m_{i,j,l,2}}.$$

Recall that $\epsilon_2(x_j) = \frac{E_2(x_j - 1)}{E_1(x_j - 1)}$. So for any i, l, the sum of the exponents of $E_1(x_j - 1)$ and $E_2(x_j - 1)$ in any monomial of $\mathrm{Init}(s)_{i,l}$ is $\beta_{i,j} + m_{i,j,l,2}$. If this value differs for some pairs i_1, l_1 and i_2, l_2 and some $j = 1, \ldots, p$, then

$$\mathrm{Supp}(\mathrm{Init}(s)_{i_1,l_1}) \cap \mathrm{Supp}(\mathrm{Init}(s)_{i_2,l_2}) = \varnothing.$$

Additionally, since the monomials of t_i are enumerated in decreasing order, we have

$$\mathrm{Supp}(\mathrm{Init}(s)_{i,l_1}) > \mathrm{Supp}(\mathrm{Init}(s)_{i,l_2})$$

for all $i \in I_0$ and $l_1 < l_2$.

So it will suffice to find a monomial with a nonzero coefficient among the sum

$$S := \sum_{i \in I_1} c_{i,1} \prod_{j=1}^{p} E_1(x_j - 1)^{\beta_{i,j} + m_{i,j,1,2}} (1 + \epsilon_2(x_j))^{m_{i,j,1,2}}$$

where $I_1 \subset I_0$ is set of indices at which

$$(\beta_{i,1} + m_{i,1,1,2}, \dots, \beta_{i,p} + m_{i,p,1,2})$$

is maximal in the lexicographic order on $\mathbb{N}^p$. Let $N_j := \beta_{i,j} + m_{i,j,l,2}$ for any $i \in I_1$. Without loss of generality, we may assume that for each $j = 1, \dots, p$, there are some $i_1, i_2 \in I_1$ such that $\beta_{i_1,j} \neq \beta_{i_2,j}$, since if $\beta_{i,j}$ is fixed across all $i \in I_1$ for some j, then so is $m_{i,j,1,2}$, and we can just factor out

$$E_1(x_j - 1)^{\beta_{i,j}+m_{i,j,1,2}}(1 + \epsilon_2(x_j))^{m_{i,j,1,2}}$$

from each term in S without affecting the sign of S.

To simplify notation for the rest of the $d = 2$ case, we drop the second index from the coefficients $c_{i,1}$ and the last two indices from the exponents $m_{i,j,1,2}$.

If $|I_1| = 1$, then we are done, since S would be a sum of a single monomial with a nonzero coefficient. So suppose $|I_1| > 1$. We will now find a monomial of S with nonzero coefficient. We will do this by inductively eliminating terms from S until we are left with a single monomial.

1. Let j_1 be the smallest index for which there are $i_1, i_2 \in I_1$ such that $m_{i_1,j_1} \neq m_{i_2,j_1}$. Such an index exists because if $i_1 \neq i_2$, then there is some j such that $\beta_{i_1,j} \neq \beta_{i_2,j}$ (since we assumed $g_{i_1} \neq g_{i_2}$ for $i_1 \neq i_2$), and thus $m_{i_1,j} \neq m_{i_2,j}$.

2. Let $n_1 = \max_{i \in I_1}(m_{i,j_1})$.

Then for $k = 0, \dots, n_1$, the coefficient of

$$\left(\prod_{j=1}^{p} E_1(x_j - 1)^{N_j}\right) \cdot \epsilon_2(x_{j_1})^k$$

is

$$\sum_{i \in I_1} c_i \binom{m_{i,j_1}}{k}.$$

If the m_{i,j_1} are all distinct for $i \in I_1$, then we are done because these coefficients cannot all be zero, as the c_i's are nonzero.

If the m_{i,j_1} are not all distinct, then we proceed by induction: Given I_l, j_l, and n_l with $m_{i,j_l} \neq n_l$ for some $i \in I_l$, we define I_{l+1}, j_{l+1}, and n_{l+1} as follows:

1. Let $I_{l+1} := \{i \in I_l : m_{i,j_l} = n_l\}$. Then $|I_{l+1}| < |I_l|$ by our choice of j_l.

2. Let j_{l+1} be the smallest index for which there are $i_1, i_2 \in I_{l+1}$ such that $m_{i_1,j_{l+1}} \neq m_{i_2,j_{l+1}}$, which exists by the same reasoning as above.

3. Let $n_{l+1} = \max_{i \in I_{l+1}}(m_{i,j_{l+1}})$.

Then for each $k = 1, \ldots, n_{l+1}$, the coefficient of

$$\left(\prod_{j=1}^{p} E_1(x_j - 1)^{N_j}\right) \cdot \left(\epsilon_2(x_{j_1})^{n_1} \cdots \epsilon_2(x_{j_l})^{n_l}\right) \cdot \epsilon_2(x_{j_{l+1}})^k$$

is

$$\sum_{i \in I_{l+1}} c_i \binom{n_1}{n_1} \cdots \binom{n_l}{n_l} \cdot \binom{m_{i,j_{l+1}}}{k} = \sum_{i \in I_{l+1}} c_i \binom{m_{i,j_{l+1}}}{k}.$$

If $|I_{l+1}| = 1$, or if the values of $m_{i,j_{l+1}}$ for $i \in I_{l+1}$ are all distinct, then we find a nonzero coefficient. Since $|I_{l+1}| < |I_l| < \cdots < |I_0| \leq n$, this algorithm must terminate. So we find a term of S with nonzero coefficient, which means $\mathrm{Init}(s) \neq 0$. This proves the Lemma in the $d = 2$ case.

Now, we will show that if the Lemma holds in case $d - 1$, then it holds in case d. Let

$$(\ell_1, \ldots, \ell_p) := \max\{(m_{i,1,l,d}, \ldots, m_{i,p,l,d}) : i \in I_0, l \in \mathbb{N}\}$$

in the lexicographic order on $\mathbb{N}^p$. Let $\mathcal{I}$ be the set of pairs (i, l) at which this maximum is achieved, i.e., at which $(m_{i,1,l,d}, \ldots, m_{i,p,l,d}) = (\ell_1, \ldots, \ell_p)$.

Recall that

$$\mathrm{Init}(s)_{i,l} = \prod_{j=1}^{p} E_1(x_j - 1)^{\beta_{i,j} + m_{i,j,l,2}} (1 + \epsilon_2(x_j))^{m_{i,j,l,2}}$$
$$E_2(x_j - 1)^{m_{i,j,l,3}} (1 + \epsilon_3(x_j))^{m_{i,j,l,3}} \cdots E_{d-1}(x_j - 1)^{m_{i,j,l,d}} (1 + \epsilon_d(x_j))^{m_{i,j,l,d}}.$$

By Lemma 3.2.1, the only place $E_d(x_j - 1)$ appears in this expression is in $\epsilon_d(x_j)$. Furthermore, $E_d(x_j - 1)$ can only ever appear with exponent 1 in monomials of $\epsilon_d(x_j)$. Let $\epsilon_{d*}(x_j)$ be the subsum of $\epsilon_d(x_j)$ with monomials in which $E_d(x_j - 1)$ appears.

Let S_d be the subsum of $\mathrm{Init}(s)$ such that for every monomial in $\mathrm{Supp}(S_d)$ and for each $j = 1, \ldots, p$, the exponent of $E_d(x_j - 1)$ is ℓ_j. Then we can write

$$S_d = \sum_{(i,l) \in \mathcal{I}} c_{i,l} \prod_{j=1}^{p} E_1(x_j - 1)^{\beta_{i,j} + m_{i,j,l,2}} (1 + \epsilon_2(x_j))^{m_{i,j,l,2}}$$
$$E_2(x_j - 1)^{m_{i,j,l,3}} \cdot (1 + \epsilon_3(x_j))^{m_{i,j,l,3}} \cdots E_{d-2}(x_j - 1)^{m_{i,j,l,d-1}} (1 + \epsilon_{d-1}(x_j))^{m_{i,j,l,d-1}}$$
$$E_{d-1}(x_j - 1)^{\ell_j} \cdot \epsilon_{d*}(x_j)^{\ell_j}.$$

By definition, the last line $E_{d-1}(x_j - 1)^{\ell_j} \cdot \epsilon_{d*}(x_j)^{\ell_j}$ above must be the same for all $(i, l) \in \mathcal{I}$, so we can factor it out to write

$$S_d = \left(\prod_{j=1}^{p} E_{d-1}(x_j - 1)^{\ell_j} \cdot \epsilon_{d*}(x_j)^{\ell_j}\right) \cdot S'_d$$

where

$$S'_d = \sum_{(i,l)\in\mathcal{I}} c_{i,l} \prod_{j=1}^{p} E_1(x_j-1)^{\beta_{i,j}+m_{i,j,l,2}}(1+\epsilon_2(x_j))^{m_{i,j,l,2}}$$
$$E_2(x_j-1)^{m_{i,j,l,3}}(1+\epsilon_3(x_j))^{m_{i,j,l,3}} \cdots E_{d-2}(x_j-1)^{m_{i,j,l,d-1}}(1+\epsilon_{d-1}(x_j))^{m_{i,j,l,d-1}}.$$

By induction, we can find a monomial of S'_d with nonzero coefficient, which gives a monomial of $\mathrm{Init}(s)$ with nonzero coefficient. □

Remark 3.3.3. Lemma 3.3.2 shows that ρ_0 is injective, so the order on $\boldsymbol{k}((H_1))$ induces an order on $\boldsymbol{k}[G_0\Lambda_0]$. However, we need the induced order to be compatible with how we intend to interpret elements of $\boldsymbol{k}[G_0\Lambda_0]$ as germs of functions. Because ρ_0 is defined using the intuition that we can "approximate" $\log E'(x) = E(x-1) + \log E'(x-1)$ by $E_0(x-1) + E_1(x-1)$, it may not be immediately clear that the order induced by $\boldsymbol{k}((H_1))$ via ρ_0 is the one we want.

If $s \in \boldsymbol{k}[G_0\Lambda_0]$ and $\log E'(x)^b$ appears in s, then by the way $\mathrm{Init}(s)$ is defined in Remark 3.3.1, then the only term of $\rho_0\left(\log E'(x)^b\right)$ that has any possibility of contributing to $\mathrm{Init}(s)$ is $\mathrm{Lm}\left(\rho_0\left(\log E'(x)^b\right)\right)$. The "error" in the approximation $\rho_0\left(\log E'(x)^b\right)$ of $\log E'(x)^b$ appears in every term *except the leading monomial*. So the sign of S is not distorted by the approximation.

Corollary 3.3.4. *Λ_0 is ordered lexicographically on the exponents of its generators, i.e., if $x_1 > x_2 > \cdots > x_p \in X$ and $b_1, b_2, \ldots, b_p \in \boldsymbol{k}^\times$, then*

$$\prod_{j=1}^{p} \log E'(x_j)^{b_j} > 1$$

if and only if $b_1 > 0$.

If $s \in \boldsymbol{k}[\Lambda_0]$, then $s > 0$ if and only if the coefficient of its largest monomial is positive.

Proof. Tracing through the proof of Lemma 3.3.2, we first want to determine the sign of

$$\prod_{j=1}^{p} \log E'(x_j)^{b_j} - 1.$$

If $b_1 > 0$, then $\mathrm{Init}(s) = \prod_{j=1}^{p} E_0(x_j-1)^{b_j} > 0$, and if $b_1 < 0$, then $\mathrm{Init}(s) = -1 < 0$.

If $g_1 > g_2 > \cdots > g_n \in \Lambda_0$ and $c_1, c_2, \ldots, c_n \in \boldsymbol{k}^\times$, then we want to determine the sign of

$$s = \sum_{i=1}^{n} c_i g_i.$$

In this case, $\mathrm{Init}(s) = c_1 \mathrm{Lm}(\rho_0(g_1))$, so the sign of s is determined by the sign of c_1. □

Chapter 4

A logarithmic-exponential series field constructed from E-monomials

In this chapter, we adapt the construction of the logarithmic-exponential series in [8] to build a logarithmic-exponential series field starting with monomials involving E and its derivatives. The first part of the logarithmic-exponential series construction in [8] begins with the multiplicative group $x^{\boldsymbol{k}}$ of monomials and inductively adds new monomials for increasing levels of exponentiation to end up with an exponential field $\boldsymbol{k}((x^{-1}))^e$. The second part of the construction uses an embedding $\varphi : \boldsymbol{k}((x^{-1}))^e \to \boldsymbol{k}((x^{-1}))^e$ such that every element in the image of φ has a logarithm in $\boldsymbol{k}((x^{-1}))^e$. If this approach is adapted naively to monomials of the form

$$\prod_{j=1}^{p} E(x_j)^{a_{j,0}} \cdot E'(x_j)^{a_{j,1}} \cdots E^{(d_j)}(x_j)^{a_{j,d}}$$

for $x_1 > \cdots > x_p \in X$ and $a_{j,l} \in \boldsymbol{k}$, three problems arise.

1. First, a logarithm of $E^{(d)}(x)$ for $d \geq 1$ would not arise naturally from the first part of the construction.

2. Second, some monomials of the form above are "small" relative to others. For example, as functions we have

$$\exp\left(\frac{E'(x)^{1/2}}{E(x)^{1/2}}\right) = \exp\left(E'(x-1)^{1/2}\right) < E(x)$$

   since $E'(x-1)^{1/2} < E(x-1)$. So it does not make sense to add $\exp\left(\frac{E'(x)^{1/2}}{E(x)^{1/2}}\right)$ as a new monomial over a field of coefficients that contains $E(x)$.

3. Third, it does not always make sense to take infinite sums of some "small" monomials. For example,

$$\frac{E'(x)^2}{E(x)E''(x)} = \frac{1}{1 + \frac{E''(x-1)}{E'(x-1)^2}} < 1$$

but it does not make sense to sum the infinite reverse well-ordered family

$$\left\{\left(\frac{E'(x)^2}{E(x)E''(x)}\right)^n : n \in \mathbb{N}\right\}$$

because each of these monomials is approximately 1.

To fix these issues, we include monomials for $\log E'$ and for exp of some "small" infinite monomials at the very beginning of the construction. Then at successive stages, we add new monomials only for exp applied to "large" monomials. We also only ever allow finite sums of monomials whose quotient is "small." The result is that in our adaptation of the first part of the logarithmic-exponential series construction, we build a ring with a partially defined exponential function. The second stage of our construction is similar to [8] in that we build embeddings between countably many partial exponential rings constructed as in stage one and then show that every element of each of these rings has a multiplicative inverse, a logarithm, and an exponential under a finite sequence of embeddings.

4.1 Part 1: Building partial exponential rings

Let X and $\boldsymbol{k}$ satisfy the ordering and separation assumptions in Remark 3.1.2, so that $\boldsymbol{k}[G_0\Lambda_0]$ is totally ordered by Lemma 3.3.2. Assume also that $\boldsymbol{k} \models T_{\mathrm{an}}(\exp, \log)$.

We start this section by pinning down the starting group of monomials $\Gamma_{X,0}$, from which we will build a partial exponential ring. We want $\Gamma_{X,0}$ to include $G_0\Lambda_0$ along with new monomials to represent exp of the "small" positive purely infinite elements of $\boldsymbol{k}[G_0\Lambda_0]$. We illustrate the issue with another example:

Example 4.1.1. Treating $E(x)$ as a function, we can compute that

$$\exp\left(\frac{E''(x)}{E'(x)}\right) > \exp\left(\frac{E'(x)}{E(x)}\right) = \exp\big(E'(x-1)\big) > E(x)^n$$

for any $n \in \mathbb{N}$, since $E'(x-1) > nE(x-1)$. However,

$$\exp\left(\frac{E''(x)}{E'(x)} - \frac{E'(x)}{E(x)}\right) = \exp\left(\frac{E''(x-1)}{E'(x-1)}\right) < E(x).$$

So even though the "small" monomial issue does not arise with $\exp\left(\frac{E''(x)}{E'(x)}\right)$ or $\exp\left(\frac{E'(x)}{E(x)}\right)$ separately, it does not makes sense to take a group containing both of these as monomials over a field of coefficients that contains $E(x)$.

The examples point toward the following definition of which elements of G_0 are "small" enough to cause a problem.

Definition 4.1.2. *Let $g \in G_0$ and write*

$$g = \prod_{j=1}^{p} E(x_j)^{a_{j,0}} \cdot E'(x_j)^{a_{j,1}} \cdots E^{(d)}(x_j)^{a_{j,d}}$$

with $p, d \in \mathbb{N}$, $x_1 > \cdots > x_p \in X$, and $a_{j,0}, \ldots, a_{j,d} \in \boldsymbol{k}$. Let $\xi_{x_j} = a_{j,0} + \cdots + a_{j,d}$. We say g is small if $\xi_{x_1} = \cdots = \xi_{x_p} = 0$.

Note that 1 is small, and if $g_1, g_2 \in G_0$ are both small, then so is $g_1 g_2$. So the small elements of G_0 form a subgroup.

Since it does not make sense to add new monomials for $\exp(g)$ with small infinite $g \in \boldsymbol{k}[G_0\Lambda_0]$ at later stages, we could try to include such monomials as part of our starting group of monomials $\Gamma_{X,0}$. But the examples above show that $\exp(g)$ would intersperse with elements of $G_0\Lambda_0$ for some such g, and that could make it challenging to figure out how to order $\Gamma_{X,0}$. To prevent $\Gamma_{X,0}$ from becoming too complicated, we will only include monomials $\exp(g)$ for certain relatively simple small infinite $g \in G_0\Lambda_0$. Define

$$T_X := \left\langle c \prod_{j=1}^{p} \log E'(x_j)^{a_{j,-1}} \frac{E''(x_j)^{a_{j,2}} \cdots E^{(d)}(x_j)^{a_{j,d}}}{E'(x_j)^{a_{j,2}+\cdots+a_{j,d}}} : c \in \boldsymbol{k}; p, d, a_{j,l} \in \mathbb{N}; x \in X; \right.$$
$$\left. \exists(j,l)\big(a_{j,l} \neq 0\big); \big(a_{j,-1} = 1\big) \to \exists(k,l) \neq (j,-1)\big(a_{k,l} \neq 0\big) \right\rangle$$

where T_X is generated additively. The final conditions ensure that $\boldsymbol{k} \not\subset T_X$ and $c \log E'(x) \notin T_X$ for any $c \in \boldsymbol{k}$, $x \in X$, since we already know how to exponentiate such elements—we assumed $\boldsymbol{k}$ is an exponential field, and we want to define exp of $c \log E'(x)$ to be $E'(x)^c$.

If X and $\boldsymbol{k}$ satisfy the ordering and separation assumptions in 3.1.2, then T_X is totally ordered because it is a subgroup of $\boldsymbol{k}[G_0\Lambda_0]$. Let $e_{T_X}(T_X)$ be a multiplicative copy of T_X via an order preserving isomorphism $e_{T_X} : T_X \to e_{T_X}(T_X)$.

Define

$$\Gamma_{X,0} := G_0\Lambda_0 e_{T_X}(T_X).$$

We will show how to order $\Gamma_{X,0}$ over the course of the next few lemmas.

Remark 4.1.3. For consistency of notation, we will always express elements of T_X as sums of the standard monomials of the group ring $\boldsymbol{k}[G_0\Lambda_0]$ instead of as sums of the generators of T_X itself.

Lemma 4.1.4. *Suppose X and $\boldsymbol{k}$ satisfy the ordering and separation assumptions in Remark 3.1.2. If $t \in T_X \cap \boldsymbol{k}[G_0]$, then the leading term of $\sigma_0(t)$ has a positive integer power of $E_1(x)$ as a generator, for some $x \in X$.*

Proof. Let $t = \sum_{i=1}^{n} c_i g_i \in T_X \cap \boldsymbol{k}[G_0]$ with $c_i \neq 0$. Let $x_1 > x_2 > \cdots > x_p$ list the elements of X appearing in t, and let d be the largest derivative appearing in any g_i. Then we can write

$$g_i = \prod_{j=1}^{p} \frac{E''(x_j)^{a_{i,j,2}} \cdots E^{(d)}(x_j)^{a_{i,j,d}}}{E'(x_j)^{a_{i,j,2}+\cdots+a_{i,j,d}}}$$

$$\sigma_0(g_i) = \prod_{j=1}^{p} E_0(x_j)^{a_{i,j,2}+\cdots+a_{i,j,d}-(a_{i,j,2}+\cdots+a_{i,j,d})} E_1(x_j)^{2a_{i,j,2}+\cdots+da_{i,j,d}-(a_{i,j,2}+\cdots+a_{i,j,d})}$$

$$\left(\sum_{k_2=0}^{a_{i,j,2}} \binom{a_{i,j,2}}{k_2} \left(\frac{E_2(x_j)}{E_1(x_j)}\right)^{k_2}\right) \cdots \left(\sum_{k_d=0}^{a_{i,j,d}} \binom{a_{i,j,d}}{k_d} \left(\cdots + \frac{E_2(x_j)\cdots E_d(x_j)}{E_1(x_j)^{d-1}}\right)^{k_d}\right).$$

Note that $\sigma_0(g_i)$ is a finite sum because $a_{i,j,2}, \ldots, a_{i,j,d}$ are natural numbers for each $j = 1, \ldots, p$. In every monomial of $\sigma_0(g_i)$, the exponent of $E_0(x_j)$ is 0 and the exponent of $E_1(x_j)$ is a natural number for all $j = 1, \ldots, p$.

If $d = 2$, then the leading monomial of $\sigma_0(t)$ is

$$\prod_{j=1}^{p} E_1(x_j)^{2a_{i,j,2}}$$

for the single index i at which $\overline{a_{i,2}}$ is maximized. So we may assume $d > 2$.

We will trace through the proof of Lemma 3.1.3 to show that we find a monomial with $E_1(x_j)$ as a generator for some $j = 1, \ldots, p$. Let I be as in Lemma 3.1.3, and let

$$A_j = a_{i,j,2} + 2a_{i,j,3} + \cdots + (d-1)a_{i,j,d}$$

for all $i \in I$. Without loss of generality we may assume $I = \{1, \ldots, n\}$. In Lemma 3.1.3, we proceeded with the argument using some fixed j. Here we will use $j = 1$.

First suppose $a_{1,1,d}, \ldots, a_{n,1,d} \in \mathbb{N}$ are all distinct. By possibly renumbering $g_1, \ldots, g_n$, we may assume $a_{1,1,d} < \cdots < a_{n,1,d}$. Note that this means $a_{i,1,d} \geq i-1$. The coefficient $\sum_{i=1}^{n} c_i \binom{a_{i,1,d}}{k_d}$ of

$$\left(\frac{E_2(x_1)\cdots E_d(x_1)}{E_1(x_1)^{d-1}}\right)^{k_d} \prod_{j=1}^{p} E_1(x_j)^{A_j}$$

cannot be 0 for each $k_d = 0, \ldots, n-1$ because then we would have $c_1 = \cdots = c_n = 0$. So some coefficient must be nonzero for some $k_* \leq n-1 \leq a_{n,1,d}$. If $p > 1$ or if $(d-1)k_* < A_1$ then this monomial has some power of $E_1(x_j)$ as a generator, and we are done. So suppose $p = 1$ and $(d-1)k_* = A_1$.

Since $p = 1$, we remove the index corresponding to j in the following computations to lighten notation.

Since $(d-1)k_* = A = a_{n,2} + 2a_{n,3} + \cdots + (d-1)a_{n,d}$, we must have the following:

1. $k_* = n-1 = a_{n,d}$ is the smallest value of k_d such that $\sum_{i=1}^n c_i\binom{a_{i,d}}{k_d} \neq 0$, and $0 \neq \sum_{i=1}^n c_i\binom{a_{i,d}}{k_d} = c_n$

2. $a_{n,2} = 0 = a_{n,3} = \cdots = a_{n,d-1} = 0$.

So $A = (d-1)(n-1)$. Observe also that since $a_{1,d} < \cdots < a_{n,d}$, we must have $a_{i,d} = i-1$. In particular $a_{n-1,d} = n-2$. Since

$$(d-1)(n-1) = A = a_{n-1,2} + \cdots + (d-2)a_{n-1,d-1} + (d-1)(n-2)$$

we have that $a_{n-1,d-1} \leq \frac{d-1}{d-2} \leq 2$.

We will use the fact that $0 \leq a_{i,d-1} \leq 2$ to show that the coefficient of the following monomial with $E_1(x)$ as a generator is nonzero:

$$M := \left(\frac{E_2(x)\cdots E_d(x)}{E_1(x)^{d-1}}\right)^{n-2}\left(\frac{E_2(x)\cdots E_{d-1}(x)}{E_1(x)^{d-2}}\right)E_1(x)^A.$$

Observe that $\frac{E_2(x)\cdots E_{d-1}(x)}{E_1(x)^{d-2}}$ appears only in $\left(\cdots + \frac{E_2(x)\cdots E_l(x)}{E_1(x)^{l-1}}\right)$ for $l = d-1, d$. Its coefficient is 1 if $l = d-1$, and its coefficient is d if $l = d$. So the coefficient of M is

$$\sum_{i=1}^n c_i\left(\binom{a_{i,d}}{n-2}\binom{a_{i,d-1}}{1} + \binom{a_{i,d}}{n-1}\binom{n-1}{1}d\right).$$

The first term of the summand comes from building M as a product of the monomials

$$\left(\frac{E_2(x)\cdots E_d(x)}{E_1(x)^{d-1}}\right)^{n-2} \text{ from } \binom{a_{i,d}}{n-1}\left(\cdots + \frac{E_2(x)\cdots E_d(x)}{E_1(x)^{d-1}}\right)^{n-2}$$
$$\left(\frac{E_2(x)\cdots E_{d-1}(x)}{E_1(x)^{d-2}}\right) \text{ from } \binom{a_{i,d-1}}{1}\left(\cdots + \frac{E_2(x)\cdots E_{d-1}(x)}{E_1(x)^{d-2}}\right)^{1}.$$

The second term of the summand comes from seeing M as a monomial of

$$\binom{a_{i,d}}{n-1}\left(\cdots + d\frac{E_2(x)\cdots E_{d-1}(x)}{E_1(x)^{d-2}} + \cdots + \frac{E_2(x)\cdots E_d(x)}{E_1(x)^{d-1}}\right)^{n-1}.$$

Since $a_{i,d} = i-1$, most of the terms in the sum are 0. We can calculate

$$\binom{a_{i,d}}{n-2}\binom{a_{i,d-1}}{1} + \binom{a_{i,d}}{n-1}\binom{n-1}{1}d = 0 \text{ for } i = 1, \ldots, n-1$$
$$\binom{a_{n-1,d}}{n-2}\binom{a_{n-1,d-1}}{1} + \binom{a_{n-1,d}}{n-1}\binom{n-1}{1}d \leq 1\cdot 2 + 0\cdot(n-1)d = 2$$
$$\binom{a_{n,d}}{n-2}\binom{a_{n,d-1}}{1} + \binom{a_{n,d}}{n-1}\binom{n-1}{1}d = (n-1)\cdot 0 + 1\cdot(n-1)d = (n-1)d.$$

Altogether, the coefficient of M is $\delta \cdot c_{n-1} + (n-1)d \cdot c_n$ with $\delta \in \{0,1,2\}$. Since

$$0 = \sum_{i=1}^{n} c_i \binom{a_{i,d}}{n-2} = c_{n-1} + (n-1)c_n$$

and $d > 2$, we must have $\delta \cdot c_{n-1} + (n-1)d \cdot c_n \neq 0$. This finishes the proof in the case that $a_{1,d}, \ldots, a_{n,d} \in \mathbb{N}$ are all distinct.

Suppose $a_{i,1,d} \in \mathbb{N}$ are not all distinct for $i = 1, \ldots, n$. We will modify the inductive argument of Lemma 3.1.3 to find a monomial with some $E_1(x_j)$.

Let $1 < q < n$ be the number of distinct values among $a_{i,1,d}$. Let $(P_1, \ldots, P_n)$ partition $\{1, \ldots, n\}$ so that for any $i_1, i_2 \in P_l$, $\alpha_l := a_{i_1,1,d} = a_{i_2,1,d}$ and $\alpha_{l_1} \neq \alpha_{l_2}$ for $l_1 \neq l_2$. Consider the sums

$$\sum_{i \in P_l} \frac{E_1(x_j)^{\alpha_l}}{E^{(d)}(x_j)^{\alpha_l}} g_i$$

for $l = 1, \ldots, q$. Each of these sums is an element of T_X with strictly fewer terms than t, and the monomials in each sum have one less multiplicand, since $E^{(d)}(x_1)$ does not appear.

We proceed by induction. As in Lemma 3.1.3, assume that for each $l = 1, \ldots, q$ we can find a leading term $b_l t_l$ of $\sigma_0\left(\sum_{i \in P_l} \frac{E'(x_j)^{\alpha_l}}{E^{(d)}(x_j)^{\alpha_l}} g_i\right)$, where $0 \neq b_l \in \boldsymbol{k}$ and $t_l \in G_0$. Assume further that each t_l has some $E_1(x_j)$ as a generator. By the same reasoning as in Lemma 3.1.3, the sum

$$\sum_{l=1}^{q} \sigma_0\left(\frac{E^{(d)}(x_1)^{\alpha_l}}{E'(x)^{\alpha_l}}\right) b_l t_l = \sum_{l=1}^{q} E_1(x_1)^{(d-1)\alpha_l} b_l t_l \sum_{k_d=0}^{\alpha_l} \binom{\alpha_l}{k_d} \left(\cdots + \frac{E_2(x_1)\cdots E_d(x_1)}{E_1(x_1)^{d-1}}\right)^{k_d}$$

has a nonzero monomial M. Our additional hypothesis guarantees that M has some $E_1(x_j)$ as a generator. Since $\operatorname{Lm}\big(\sigma_0(t)\big) \geq M$, $\operatorname{Lm}\big(\sigma_0(t)\big)$ must have $E_1(x_{j'})$ as a generator for some $j' \leq j$. ☐

Lemma 4.1.5. *Suppose X and $\boldsymbol{k}$ satisfy the ordering and separation assumptions in Remark 3.1.2. If $t \in T_X \cap \boldsymbol{k}[G_0]$, then no term of $\rho_0(t)$ is of the form $E_0(x-1)^n$ for any $x \in X$, $n \in \mathbb{N}$.*

Proof. As in the previous lemma, if $t = \sum_{i=1}^{n} c_i g_i \in T_X \cap \boldsymbol{k}[G_0]$, then for some $p, d \in \mathbb{N}$ we can write

$$g_i = \prod_{j=1}^{p} \frac{E''(x_j)^{a_{i,j,2}} \cdots E^{(d)}(x_j)^{a_{i,j,d}}}{E'(x_j)^{a_{i,j,2}+\cdots+a_{i,j,d}}}$$

$$\sigma_0(g_i) = \prod_{j=1}^{p} E_0(x_j)^{a_{i,j,2}+\cdots+a_{i,j,d}-(a_{i,j,2}+\cdots+a_{i,j,d})} E_1(x_j)^{2a_{i,j,2}+\cdots+da_{i,j,d}-(a_{i,j,2}+\cdots+a_{i,j,d})}$$

$$\left(\sum_{k_2=0}^{a_{i,j,2}} \binom{a_{i,j,2}}{k_2} \left(\frac{E_2(x_j)}{E_1(x_j)}\right)^{k_2}\right) \cdots \left(\sum_{k_d=0}^{a_{i,j,d}} \binom{a_{i,j,d}}{k_d} \left(\cdots + \frac{E_2(x_j)\cdots E_d(x_j)}{E_1(x_j)^{d-1}}\right)^{k_d}\right).$$

As noted above, $\sigma_0(g_i)$ is a finite sum because $a_{i,j,2},\dots,a_{i,j,d}$ are natural numbers for each $j=1,\dots,p$, and the exponent of $E_0(x_j)$ is 0 in every term of $\sigma_0(g_i)$. Additionally, in every term of $\sigma_0(g_i)$ the sum of the exponents of generators composed with x_j is $A_j = a_{i,j,2}+2a_{i,j,3}+\cdots+a_{i,j,d}$.

Every monomial of $\rho_0(t)=\nu_0(\sigma_0(t))$ is an element of $\mathrm{Supp}(\nu_0(M))$ for some $i=1,\dots,n$ and some monomial M of $\sigma_0(g_i)$. We can write any monomial M of $\sigma_0(g_i)$ as

$$M=\prod_{j=1}^{p} E_1(x_j)^{N_{j,1}}\cdots E_d(x_j)^{N_{j,d}}$$

with $N_{j,1},\dots,N_{j,d}\in\mathbb{N}$ and

$$\sum_{k=1}^{d} N_{j,k}=A_j$$

for each $j=1,\dots,p$. Now

$$\nu_0(M)=\prod_{j=1}^{p} E_0(x-1)^{N_{j,1}}E_1(x-1)^{N_{j,1}+N_{j,2}}(1+\epsilon_2(x_j))^{N_{j,2}}\cdots E_{d-1}(x_j)^{N_{j,d}}(1+\epsilon_d(x_j))^{N_{j,d}}.$$

By Lemma 3.2.1, we know any $\epsilon_l(x_j)$ for $l=2,\dots,d$, $j=1,\dots,p$ contributes 0 to the sum of the exponents and involves only $E_1(x_j-1),\dots,E_{l+1}(x_j-1)$. So the sum of the exponents of $E_1(x_j-1),\dots,E_d(x_j-1)$ is always $\sum_{k=1}^{d} N_{j,k}=A_j$ for each $j=1,\dots,p$. Since at least one of these sums must be nonzero, $\nu_0(M)$ cannot have any generators of the form $E_0(x-1)^n$. □

Now we will show that there is a unique way to order $\Gamma_{X,0}$ that respects the ordering defined in Lemma 3.1.3 and the way we wish to define log. However, since we cannot define a logarithm on $G_0\Lambda_0$ yet, we instead use a function $\widehat{\log}:\rho_0(G_0\Lambda_0)\to\bigcup_{n\in\mathbb{N}}H_n$ to approximate the way we will define the logarithm. Later in this chapter, we will define $\log(g\cdot\ell)$ to be an infinite sum, and in Corollary 4.4.5 $\widehat{\log}(\rho_0(g\cdot\ell))$ is shown to correspond to an initial subsum of $\rho_{d+1}(\log(g\cdot\ell))$ where d is the largest derivative occurring in g.

Define $\widehat{\log}:\rho_0(G_0\Lambda_0)\to\bigcup_{n\in\mathbb{N}}H_n$ as follows:

1. $\widehat{\log}(c)=\log c$ for $c\in\boldsymbol{k}$.
2. $\widehat{\log}(E_0(x-1+k)^a=aE_0(x-2+k)$ for $k=0,1$.
3. $\widehat{\log}(E_1(x-1)^a)=aE_0(x-3)$
4. Extend $\widehat{\log}$ to products by $\widehat{\log}(g_1\cdots g_p)=\widehat{\log}(g_1)+\cdots+\widehat{\log}(g_p)$.
5. Extend $\widehat{\log}$ to sums by truncating after the leading term, i.e., $\widehat{\log}(s)=\widehat{\log}(\mathrm{Lt}(s))$.

Since $E_d(x-1)$ never appears in $\mathrm{Lm}(\rho_0(g\cdot\ell))$ for $d\geq 2$, this fully defines $\widehat{\log}$.

Lemma 4.1.6. *Suppose X and $\boldsymbol{k}$ satisfy the ordering and separation assumptions in Remark 3.1.2. Then there is a unique way to order $\Gamma_{X,0}$ that respects the ordering defined in Lemma 3.1.3 and $\widehat{\log}$.*

Proof. Let $g\cdot\ell\in G_0\Lambda_0$ and $0\neq t\in T_X$. We will compare $g\cdot\ell$ with $e_{T_X}(t)$ by comparing $\widehat{\log}\big(\rho_0(g\cdot\ell)\big)$ with $\rho_0(t)$, and we will show that this uniquely determines an order on $\Gamma_{X,0}$.

First, $\widehat{\log}(\rho_0(g\cdot\ell))=\widehat{\log}(\mathrm{Lt}(\rho_0(g\cdot\ell)))$, which is a finite sum of terms of the form $aE_0(y-n)$ for $a\in\boldsymbol{k}$, $y\in X$, and $n=1,\ldots,d+2$, where d is the highest order of derivative appearing in g. Since $E_0(y-1)\in H_1$, any term of the form $aE_0(y-1)$ is comparable to $\rho_0(t)$. It is not immediately clear how to compare other terms to elements of $\boldsymbol{k}((H_1))$, however.

Write $t=s_1\ell_1+\cdots+s_n\ell_n$ with $s_1,\ldots,s_n\in\boldsymbol{k}[G_0]$ and $\ell_1,\ldots,\ell_n\in\Lambda_0$. By Lemmas 4.1.4 and 4.1.5, we know $\mathrm{Lm}\big(\sigma_0(s_i)\big)=E_1(x)^k h$ for some $x\in X$, $k\in\mathbb{N}^{>0}$, and some $1\neq h\in H_0$. So

$$\begin{aligned}\mathrm{Lm}\big(\rho_0(s_i)\big)&=\mathrm{Lm}\big(\nu_0(E_1(x)^k\cdot h)\big)\\&=E_0(x-1)^kE_1(x-1)^k\mathrm{Lm}\big(\nu_0(h)\big).\end{aligned}$$

By the definition of T_X, each $\mathrm{Lm}(\rho_0(\ell_i))$ is of the form $\prod_{j=1}^{p}E_0(x_j-1)^{b_j}$ with $b_j\in\mathbb{N}$.

Recall that in Remark 3.3.1, given $s\in\boldsymbol{k}[G_0\Lambda_0]$ we defined $\mathrm{Init}(s)$, and in Lemma 3.3.2, we showed that $\mathrm{Init}(s)\neq 0$, so that $\mathrm{Lm}\big(\rho_0(s)\big)=\mathrm{Lm}\big(\mathrm{Init}(s)\big)$. Consider $\mathrm{Init}(t)$. Every monomial of $\mathrm{Init}(t)$ has a positive integer power of $E_0(x-1)$ for some $x\in X$. Therefore, we should interpret every monomial of $\mathrm{Supp}\big(\mathrm{Init}(t)\big)$ as being strictly larger than terms of the form $\widehat{\log}\big(E_d(y-1)^a\big)=aE_0(y-d-2)$ for all $d\in\mathbb{N}$, $y\in X$, and $a\in\boldsymbol{k}$. This gives a unique way to order $\Gamma_{X,0}$ compatible with the order on $\boldsymbol{k}[G_0\Lambda_0]$ and that respects $\widehat{\log}$. □

We now introduce the subgroup of $\Gamma_{X,0}$ of small elements.

Definition 4.1.7. *Let $\Gamma_{X,0,small}:=\{g\cdot\ell : g\in G_0$ is small, $\ell\in\Lambda_0\}$.*

$\Gamma_{X,0,small}$ is a subgroup of $\Gamma_{X,0}$ because the small elements of G_0 form a subgroup.

Corollary 4.1.8. *$\Gamma_{X,0,small}$ is a convex subgroup of $\Gamma_{X,0}$.*

Proof. Let $g_0\in G_0$ be small and $\ell_0\in\Lambda_0$ with $g_0\cdot\ell_0>1$. Let $1<g_1\cdot\ell_1\cdot e_{T_X}(t_1)\in\Gamma_{X,0}$, and suppose $g_1\cdot\ell_1\cdot e_{T_X}(t_1)<g_0\cdot\ell_0$. We will show $g_1\cdot\ell_1\cdot e_{T_X}(t_1)\in\Gamma_{X,0,small}$, i.e., $t_1=0$ and g_1 is small.

Following the way $\Gamma_{X,0}$ is ordered via Lemma 4.1.6, $1<g_1\cdot\ell_1\cdot e_{T_X}(t_1)<g_0\cdot\ell_0$ means

$$0<\rho_0(t_1)+\widehat{\log}\big(\rho_0(g_1\cdot\ell_1)\big)<\widehat{\log}\big(\rho_0(g_0\cdot\ell_0)\big).$$

We can write $\mathrm{Lm}\big(\rho_0(g_0 \cdot \ell_0)\big)$ as

$$\prod_{j=1}^{p} E_0(x_j)^{\alpha_j} E_0(x_j-1)^{\beta_j+b_j} E_1(x_j-1)^{\beta_j+N_{j,1}}.$$

Since g_0 is small, we must have $\alpha_j = 0$. So $\widehat{\log}\big(\rho_0(g_0 \cdot \ell_0)\big)$ can be no larger than $E_0(x-d)$ for some $x \in X$, $d \geq 2$ (in the sense of Lemma 4.1.6).

By Lemmas 4.1.4 and 4.1.5, if $t_1 \neq 0$ then $\mathrm{Lm}(\rho_0(t_1))$ must be of the form $E_0(x-1)^k \cdot h$ with $1 \neq h \in H_1$. Since $\widehat{\log}\big(\rho_0(g_1 \cdot \ell_1)\big)$ is a single generator with exponent 1, $\mathrm{Lm}(\rho_0(t_1)) \neq \widehat{\log}\big(\rho_0(g_1 \cdot \ell_1)\big)$.

If $\mathrm{Lm}(\rho_0(t_1)) > \widehat{\log}\big(\rho_0(g_1 \cdot \ell_1)\big)$, then we must have $\mathrm{Lc}(\rho_0(t)) > 0$ because $\rho_0(t_1) + \widehat{\log}\big(\rho_0(g_1 \cdot \ell_1)\big) > 0$. But this contradicts $\rho_0(t_1) + \widehat{\log}\big(\rho_0(g_1 \cdot \ell_1)\big) < \widehat{\log}\big(\rho_0(g_0 \cdot \ell_0)\big)$.

So $\mathrm{Lm}(\rho_0(t_1)) < \widehat{\log}\big(\rho_0(g_1 \cdot \ell_1)\big)$. We can write $\mathrm{Lm}\big(\rho_0(g_1 \cdot \ell_1)\big)$ as

$$\prod_{j=1}^{p} E_0(x_j)^{\alpha'_j} E_0(x_j-1)^{\beta'_j+b'_j} E_1(x_j-1)^{\beta'_j+N'_{j,1}}.$$

Since $\alpha_j = 0$ and $\mathrm{Lt}\big(\widehat{\log}(\rho_0(g_1 \cdot \ell_1))\big) < \mathrm{Lt}\big(\widehat{\log}(\rho_0(g_0 \cdot \ell_0))\big)$, we must have $\alpha'_j = 0$ too, i.e., g_1 is small. But then the only way to have $\mathrm{Lm}(\rho_0(t_1)) < \widehat{\log}\big(\rho_0(g_1 \cdot \ell_1)\big)$ is if $t_1 = 0$. Thus $g_1 \cdot \ell_1 \cdot e_{T_X}(t_1) = g_1 \cdot \ell_1 \in \Gamma_{X,0,small}$. □

Since $\Gamma_{X,0,small}$ is a convex subgroup of $\Gamma_{X,0}$, the quotient group $\Gamma_{X,0}/\Gamma_{X,0,small}$ is an ordered group.

We will now define a modification of the Hahn series field that disallows infinite sums of monomials whose quotient is in $\Gamma_{X,0,small}$. We will also add an additional restriction that will be necessary in Section 5.3, when we define a derivation on the field of transexponential-sublogarithmic series. Even though we have not yet discussed how to define the derivation, we include the following example now, to motivate this restriction.

Example 4.1.9. Let x be the germ of the identity function, and let

$$X = \left\{ E(x) + r + \frac{1}{E(x^r)} : r \in (0,1) \cap \mathbb{R} \right\}.$$

Let $\boldsymbol{k}$ be any ordered exponential field such that $\boldsymbol{k}$ and X satisfy the ordering and separation assumptions of Remark 3.1.2 (for example, $\boldsymbol{k} = \mathbb{R}$ works). Then we would want to define

$$\frac{d}{dx} E\left(E(x) + r + \frac{1}{E(x^r)}\right) = E'\left(E(x) + r + \frac{1}{E(x^r)}\right)\left(E'(x) - \frac{r x^{r-1} E'(x^r)}{E(x^r)^2}\right).$$

Now, $E'\left(E(x) + r + \frac{1}{E(x^r)}\right)$ will be in any structure built from X, for all $r \in (0,1)$. However, due to the finitary nature of the full sublogarithmic-transexponential series construction, there is no structure in which $\{E(x^r) : r \in C\}$ can appear in a sum for any infinite $C \subset (0,1)$. This is because $(x^r)_{r\in(0,1)}$ all have different growth rates. So we cannot allow any sum s with generators built from infinitely many elements of X because the "derivative" of such an element will not be a valid sum.

Because of the issue that arises in Example 4.1.9, any element s of our modified Hahn series construction must only involve generators built from some finite subset $X_0 \subset X$. A similar issue may arise if the derivations of exponents and coefficients in some sum s do not all lie in one structure. So we will also set up a framework to be able to restrict how exponents and coefficients from $\boldsymbol{k}$ may appear in a sum.

Remark 4.1.10. The construction introduced in the remainder of this section prevents problematic sums of the form

$$\sum_{n\in\mathbb{N}} \left(\frac{E'(x)}{E(x)E''(x)}\right)^n$$

from arising. It also removes other sums that are not obviously problematic. So it is possible that this construction may be more restrictive than it needs to be.

Definition 4.1.11. *Let $\boldsymbol{k}$ be an ordered field, let $(\boldsymbol{k}_\alpha)_{\alpha\in\mathcal{A}}$ be a family of subfields of $\boldsymbol{k}$, and let X be a set. Let $\Gamma = \Gamma(\boldsymbol{k}, X)$ be an ordered multiplicative group depending on $\boldsymbol{k}$ and X, with convex subgroup $H = H(\boldsymbol{k}, X)$. Assume $\boldsymbol{k}[H]$ is an ordered ring. Define*

$$\boldsymbol{k}((\Gamma))_H$$

to be the ring whose elements are sums of the form

$$s = \sum_{M\in\Gamma(\boldsymbol{k}_\alpha, X_0)} c_M M$$

for some finite subset $X_0 \subset X$ and $c_M \in \boldsymbol{k}_\alpha$, for some $\alpha \in \mathcal{A}$, where

1. *for each coset $w \in \Gamma(\boldsymbol{k}_\alpha, X_0)/H$, $\{M \in w : c_M \neq 0\}$ is finite, and*
2. $\mathrm{Supp}(s) = \{M \in \Gamma(\boldsymbol{k}_\alpha, X_0) : c_M \neq 0\}$ *is reverse well-ordered.*

For $s \in \boldsymbol{k}((\Gamma))_H$, define $\mathrm{Lv}(s)$ to be the coset $\mathrm{Lm}(s)H$ of Γ/H. Define $s > 0$ in $\boldsymbol{k}((\Gamma))_H$ if and only if

$$\sum_{M\in\mathrm{Lv}(s)} c_M \frac{M}{\mathrm{Lm}(s)} > 0$$

in $\boldsymbol{k}[H]$. For $w \in \Gamma/H$, we will write $s|_w$ to denote the finite subsum of s with monomials in w.

Since $\mathrm{Supp}(s)$ is reverse well-ordered in Γ, $\{MH : M \in \mathrm{Supp}(s)\}$ is reverse well-ordered in H. Since H is convex, if $M < \mathrm{Lm}(s)$ in Γ then $MH \leq \mathrm{Lv}(s)$ in Γ/H. Note that $\boldsymbol{k}((\Gamma))_H$ is not a field because sums of elements with more than one element in the leading coset may not have multiplicative inverses.

Assume $\boldsymbol{k}$ and X satisfy the ordering and separation assumptions in Remark 3.1.2, and let $(\boldsymbol{k}_\alpha)_{\alpha\in\mathcal{A}}$ be any collection of subfields of $\boldsymbol{k}$. By Lemma 4.1.6, $\Gamma_{X,0}$ is an ordered group depending on X and $\boldsymbol{k}$. By Lemma 4.1.8, $\Gamma_{X,0,small}$ is a convex subgroup of $\Gamma_{X,0}$. By Lemma 3.3.2, $\boldsymbol{k}[\Gamma_{X,0,small}]$ is ordered. So we can construct $K_{X,0} := \boldsymbol{k}((\Gamma_{X,0}))_{\Gamma_{X,0,small}}$. Notice that the order on $K_{X,0}$ extends the order on $\boldsymbol{k}[G_0\Lambda_0]$ from Lemma 3.3.2.

Define

$$\begin{aligned}
A_{X,0} &:= \{s \in K_{X,0} : \mathrm{Supp}(s) > \Gamma_{X,0,small}\}\\
B_{X,0} &:= \{s \in K_{X,0} : \forall M \in \mathrm{Supp}(s), \exists M_0 \in \Gamma_{X,0,small} \text{ with } M \leq M_0\}\\
B^*_{X,0} &:= T_X \oplus \boldsymbol{k} \oplus \{s \in K_{X,0} : \mathrm{Supp}(s) < \Gamma_{X,0,small}\}.
\end{aligned}$$

Define $e_{X,0} : B^*_{X,0} \to (K_{X,0})^{>0}$ by

$$e_{X,0}(t + r + \epsilon) = e_{T_X}(t)\exp(r)\sum_{n=0}^{\infty}\frac{\epsilon^n}{n!}$$

for $t \in T_X$, $r \in \boldsymbol{k}$, and $\epsilon \in B_{X-m,0}$ with $\mathrm{Supp}(\epsilon) < \Gamma_{X,0,small}$. Notice that the only elements of $\boldsymbol{k}[\Gamma_{X,0,small}]$ on which it makes sense to define $e_{X,0}$ are $T_X \oplus \boldsymbol{k}$. For example, there is no element of $K_{X,0}$ to represent the exponential of the “large” infinitesimal $\frac{E(x)}{E'(x)} = \frac{1}{E'(x-1)}$.

Additionally, we introduce a family

$$(K_{\alpha,X_0})_{(\alpha,X_0)\in\mathcal{A}\times[X]^{<\omega}}$$

of subfields of $K_{X,0}$. Given $\alpha \in \mathcal{A}$ and X_0 a finite subset of X, define K_{α,X_0} to consist of sums of the form $s = \sum_{M\in\Gamma(\boldsymbol{k}_\alpha,X_0)} c_M M$ with all $c_M \in \boldsymbol{k}_\alpha$.

We now adapt the notion of the *first extension* of a pre-exponential ordered field from [8]: An *almost pre-exponential ordered ring* (K, A, B, B^*, e) consists of an ordered ring K, an additive subgroup A of K, a convex subgroup B of K with $K = A \oplus B$, a subgroup B^* of B, and a strictly increasing homomorphism $e : B^* \to (K)^{>0}$. We will also assume we are given a family $(K_\beta)_{\beta\in\mathcal{B}}$ of subfields of K.

Define the *first extension* $(K', A', B', (B^*)', e')$ of an almost pre-exponential ordered ring (K, A, B, B^*, e):

1. Take a multiplicative copy $e(A)$ of the ordered additive abelian group A with order-preserving isomorphism $e_A : A \to e(A)$.

2. Define $K' = K((e(A)))_{\{1\}}$ using the family $(K_\alpha)_{\alpha\in\mathcal{A}}$ of subfields of K, where $\{1\}$ is the trivial subgroup of $e(A)$. Then K' is a subring of the usual Mal'cev-Neumann series ring over K with monomial group $e(A)$.

3. Let $A' = \{s \in K' : \mathrm{Supp}(s) > 1\}$, and $B' = \{s \in K' : \mathrm{Supp}(s) \leq 1\}$, so that $K' = A' \oplus B'$ and B' is a convex subring of K'.

4. Let $(B^*)' = A \oplus B^* \oplus \mathfrak{m}(B')$, and extend e to $e' : (B^*)' \to (K')^{>0}$ by
$$e(a+b+\epsilon) = e_A(a)e(b)\sum_{n=0}^{\infty}\frac{\epsilon^n}{n!}$$
for $a \in A$, $b \in B^*$, and $\epsilon \in \mathfrak{m}(B')$.

5. Associate with the first extension the family $(K'_\beta)_{\beta\in\mathcal{B}}$ of subfields of K', where K'_β is such that for every term $ce(a)$ of every element of K'_β, we have $c, a \in K_\beta$.

Then $(K', A', B', (B^*)', e')$ is an almost pre-exponential ordered ring, and e' is defined on all of $A \oplus B^* \subset K$.

Starting with $(K_{X,0}, A_{X,0}, B_{X,0}, B^*_{X,0}, e_{X,0})$ and $(K_{\alpha,X_0})_{(\alpha,X_0)\in\mathcal{A}\times[X]^{<\omega}}$, define
$$(K_{X,n+1}, A_{X,n+1}, B_{X,n+1}, B^*_{X,n+1}, e_{X,n+1})$$
to be the first extension of $(K_{X,n}, A_{X,n}, B_{X,n}, B^*_{X,n}, e_{X,n})$. Given $(K_{\beta,n})_{\beta\in\mathcal{B}}$ associated with $(K_{X,n}, A_{X,n}, B_{X,n}, B^*_{X,n}, e_{X,n})$, define $(K_{\beta,n+1})_{\beta\in\mathcal{B}}$ to be $(K'_{\beta,n})_{\beta\in\mathcal{B}}$. Then
$$B^*_{X,n+1} = A_{X,n} \oplus \cdots \oplus A_{X,0} \oplus B^*_{X,0} \oplus \mathfrak{m}(B_{X,1}) \oplus \cdots \oplus \mathfrak{m}(B_{X,n+1}).$$
Let $K_X = \bigcup_{n\in\mathbb{N}} K_{X,n}$, $B^*_X = \bigcup_{n\in\mathbb{N}} B^*_{X,n}$, and let $e_X : B^*_X \to (K_X)^{>0}$ be the common extension of all the $e_{X,n}$.

Define an increasing sequence of multiplicative subgroups of K_X starting with $\Gamma_{X,0}$ and taking $\Gamma_{X,n+1} := \Gamma_{X,n}e_X(A_{X,n})$. Since $\Gamma_{X,n}$ is convex in $\Gamma_{X,n+1}$, by induction $\Gamma_{X,0,small}$ is a convex subgroup of $\Gamma_{X,n+1}$ for all $n \in \mathbb{N}$.

Lemma 4.1.12. *Let $\boldsymbol{k}$ be an ordered field, Γ an ordered group with convex subgroup G_1 such that Γ is the internal direct product of G_1 and another subgroup G_2. Let H be a convex subgroup of G_1. Then $\boldsymbol{k}((G_1))_H((G_2)) \cong \boldsymbol{k}((\Gamma))_H$.*

Proof. The isomorphism is given by
$$\sum_{M\in G} c_M M \mapsto \sum_{M_2\in G_2}\left(\sum_{M_1\in G_1} c_{M_1M_2}M_1\right)M_2.$$ □

We know $K_{X,0} = \boldsymbol{k}((\Gamma_{X,0}))_{\Gamma_{X,0,small}}$. Assume $K_{X,n} = \boldsymbol{k}((\Gamma_{X,n}))_{\Gamma_{X,0,small}}$. Then using Lemma 4.1.12,
$$K_{X,n+1} = K_{X,n}((e(A_{X,n}))) = \boldsymbol{k}((\Gamma_{X,n}))_{\Gamma_{X,0,small}}((e(A_{X,n}))) = \boldsymbol{k}((\Gamma_{X,n+1}))_{\Gamma_{X,0,small}}.$$
So by induction, $K_{X,n} = \boldsymbol{k}((\Gamma_{X,n}))_{\Gamma_{X,0,small}}$ for all $n \in \mathbb{N}$. In the next section, it will be helpful to use this expression for $K_{X,n}$, rather than the inductive definition above.

4.2 Part 2: Building a logarithmic-exponential field

Suppose X and $\boldsymbol{k}$ satisfy the ordering and separation assumptions of Remark 3.1.2. Since X is assumed to be a subset of an ordered field, we may define

$$X - m := \{x - m : x \in X\}$$

for any $m \in \mathbb{N}$. Note that if X and $\boldsymbol{k}$ satisfy the ordering and separation assumptions, so do $X - m$ and $\boldsymbol{k}$.

In the previous section, we showed how to build a partial exponential ring K_X starting with X and $\boldsymbol{k}$. In this section, we would like to construct a logarithmic-exponential field, starting from K_X. We motivate the construction in this section with several examples.

Example 4.2.1. The difference equation for E tells us that the logarithm of $E(x)$ should be $E(x-1)$. There is no element $E(x-1)$ of K_X, but if we build K_{X-1} using $X-1$ instead of X, then $E(x-1)$ and $e_{X-1}\big(E(x-1)\big)$ are both elements of K_{X-1}, and we can identify them with $\log E(x)$ and $E(x)$ respectively.

Example 4.2.2. Although $\frac{E(x)}{E'(x)}$ does not have an exponential in K_X, if we identify $\frac{E(x)}{E'(x)}$ with $\frac{1}{E'(x-1)} \in K_{X-1}$, then

$$e_{X-1}\left(\frac{1}{E'(x-1)}\right) = \sum_{n=0}^{\infty} \frac{1}{n!E'(x-1)^n} \in K_{X-1}.$$

Example 4.2.3. The natural way to represent the multiplicative inverse of $E'(x) + E(x)$ is by the following computation:

$$\frac{1}{E'(x)+E(x)} = \frac{1}{E'(x)\left(1+\frac{E(x)}{E'(x)}\right)} = \frac{1}{E'(x)} \sum_{n=0}^{\infty} \left(-\frac{E(x)}{E'(x)}\right)^n.$$

Since $E'(x)$ and $E(x)$ are in the same coset of $\Gamma_{X,0,small}$, the infinite sum $\sum_{n=0}^{\infty}\left(-\frac{E(x)}{E'(x)}\right)^n$ is not an element of K_X. However, if we identify $E(x)$ with $e_{X-1}\big(E(x-1)\big)$, $E'(x)$ with $e_{X-1}\big(E(x-1)\big)E'(x-1)$, and $\frac{E(x)}{E'(x)}$ with $\frac{1}{E'(x-1)}$, then we have a multiplicative inverse

$$\frac{1}{e_{X-1}\big(E(x-1)\big)E'(x-1)} \sum_{n=0}^{\infty} \left(-\frac{1}{E'(x-1)}\right)^n \in K_{X-1}$$

of $e_{X-1}\big(E(x-1)\big)E'(x-1) + e_{X-1}\big(E(x-1)\big)$.

In this section, we will construct embeddings $\varphi_m : K_{X-m} \to K_{X-m-1}$ for $m \in \mathbb{N}$ which formally "identify" elements of K_{X-m} with the corresponding elements of K_{X-m-1}. We will show that for all $s \in K_{X-m}$, the image of s under finitely many embeddings has a multiplicative inverse, a logarithm, and an exponential in some K_{X-m-j}.

4.3 Defining φ_m

We begin defining $\varphi_{m,0} : K_{X-m,0} \to K_{X-m-1,1}$ as follows:

1. $\varphi_{m,0}\big(\log E'(x)^b\big) = E(x-1)^b \sum_{n=0}^{\infty} \binom{b}{n} \left(\frac{\log E'(x-1)}{E(x-1)}\right)^n$
   Since $\left(\frac{\log E'(x-1)}{E(x-1)}\right)^n < \Gamma_{X-m-1,0,small}$ for $n > 0$, this sum is allowed in $K_{X-m-1,0}$.
2. $\varphi_{m,0}\big(E(x)^a\big) = e_{X-m-1}\big(E(x-1)\big)^a$
3. $\varphi_{m,0}\big(E'(x)^a\big) = e_{X-m-1}\big(E(x-1)\big)^a E'(x-1)^a$
4. $\varphi_{m,0}\big(E^{(d)}(x)^a\big) = e_{X-m-1}(E(x-1))^a E'(x-1)^{da} \sum_{n=0}^{\infty} \binom{a}{n} \delta_d(x)^n$
   for $d > 1$, where $\delta_d(x) = \frac{B_d(x-1) - E'(x-1)^d}{E'(x-1)^d} = \frac{\binom{d}{2} E'(x-1)^{d-2} E''(x-1) + \cdots + E^{(d)}(x-1)}{E'(x-1)^d}$.
   Since $\mathrm{Supp}(\delta_d(x)^n) < \Gamma_{X-m-1,0,small}$ for $n > 0$, this sum is allowed in $K_{X-m-1,0}$.
5. Extend $\varphi_{m,0}$ to $\boldsymbol{k}[G_m\Lambda_m]$ so that it is a $\boldsymbol{k}$-algebra homomorphism.

In particular, we have now defined $\varphi_{m,0}$ on T_{X-m}. Before we extend $\varphi_{m,0}$ further, we will show that $\varphi_{m,0}$ is order preserving on $\boldsymbol{k}[\Gamma_{X-m,0,small}]$. Ultimately, this follows from the results of Section 3 and Section 4.1, but the orderings on $\boldsymbol{k}[\Gamma_{X-m,0,small}]$ and $K_{X-m-1,0}$ are defined using the maps σ_l, ν_l, and ρ_l for $l = m, m+1$, which are rather delicate. For this reason, it takes some maneuvering to use the results proven so far to show that $\varphi_{m,0}$ is order preserving. We show that $\varphi_{m,0}$ is order preserving on $\boldsymbol{k}[\Gamma_{X-m,0,small}]$ in Lemma 4.3.6.

Lemma 4.3.1. *Assume $\boldsymbol{k}$ and X satisfy the ordering and separation assumptions in Remark 3.1.2. If $t \in T_{X-m} \cap \boldsymbol{k}[G_m]$, then*

$$\sigma_{m+1}\big(\varphi_{m,0}(t)\big) = \nu_m\big(\sigma_m(t)\big).$$

Proof. It suffices to show the result for expressions of the form $\frac{E^{(d)}(x)}{E'(x)} \in T_{X-m} \cap G_m$ because σ_{m+1}, $\varphi_{m,0}$, ν_m, and σ_m are $\boldsymbol{k}$-algebra homomorphisms. First, note that σ_{m+1} is defined on $\varphi_{m,0}\left(\frac{E^{(d)}(x)}{E'(x)}\right)$ because

$$\varphi_{m,0}\left(\frac{E^{(d)}(x)}{E'(x)}\right) = \frac{e_{X-m-1}\big(E(x-1)\big)B_d(x-1)}{e_{X-m-1}\big(E(x-1)\big)E'(x-1)} = \frac{B_d(x-1)}{E'(x-1)} \in \boldsymbol{k}[G_{m+1}].$$

If x is the germ of a function and we view all the following expressions as functions, we have

$$\begin{aligned}
\frac{E^{(d)}(x)}{E'(x)} &= \sigma_m\left(\frac{E^{(d)}(x)}{E'(x)}\right) && \text{by definition of } E_0, E_1, \ldots\\
&= \nu_m\left(\sigma_m\left(\frac{E^{(d)}(x)}{E'(x)}\right)\right) && \text{by Lemma 2.1.4}\\
\frac{E^{(d)}(x)}{E'(x)} &= \varphi_{m,0}\left(\frac{E^{(d)}(x)}{E'(x)}\right) && \text{by the Bell polynomial expression for } E^{(d)}\\
&= \sigma_{m+1}\left(\varphi_{m,0}\left(\frac{E^{(d)}(x)}{E'(x)}\right)\right) && \text{by definition of } E_0, E_1, \ldots
\end{aligned}$$

Since $\nu_m\left(\sigma_m\left(\frac{E^{(d)}(x)}{E'(x)}\right)\right)$ and $\sigma_{m+1}\left(\varphi_{m,0}\left(\frac{E^{(d)}(x)}{E'(x)}\right)\right)$ are expressed using the same standard basis $E_0(x-1), E_1(x-1), \ldots, E_d(x-1)$ for monomials in H_{m+1}, they must also be equal in the sense of $\boldsymbol{k}((H_{m+1}))$. Since the computations of $\nu_m\left(\sigma_m\left(\frac{E^{(d)}(x)}{E'(x)}\right)\right)$ and $\sigma_{m+1}\left(\varphi_{m,0}\left(\frac{E^{(d)}(x)}{E'(x)}\right)\right)$ are the same regardless of whether x is the germ of a function, these two expressions must be equal formally. □

Remark 4.3.2. We can extend Lemma 4.3.1 beyond $T_{X-m} \cap \boldsymbol{k}[G_m]$ to include monomials of the form $E(x)^a E'(x)^{-a}$ for any $a \in \boldsymbol{k}$ and products of such monomials with elements of $T_{X-m} \cap \boldsymbol{k}[G_m]$, since

$$\varphi_{m,0}\left(E(x)^a E'(x)^{-a}\right) = e_{X-m-1}\big(aE(x-1) - aE(x-1)\big)E'(x-1)^{-a} = E'(x-1)^{-a}.$$

However, we cannot extend Lemma 4.3.1 any further because monomials in $\Gamma_{X-m,0,small}$ of other forms either involve logarithms, or their image under $\varphi_{m,0}$ is an infinite sum.

Define $\mathfrak{s}_{d,N,a}(x) := E(x)^a \dfrac{E'(x)^{da}}{E(x)^{da}} \displaystyle\sum_{k=0}^{N} \binom{a}{k} \left(\frac{E(x)^{d-1}E^{(d)}(x)}{E'(x)^d} - 1\right)^k$ for $d \geq 2$.

Lemma 4.3.3. *The first N terms (at least) of $\sigma_m\big(E^{(d)}(x)^a\big)$ and $\sigma_m(\mathfrak{s}_{d,N,a}(x))$ are equal.*

Proof. Notice that $\mathfrak{s}_{d,N,a}(x)$ is obtained by paralleling the definition of σ_m. $E(x)^a$ corresponds to $E_0(x)^a$, $\frac{E'(x)^{da}}{E(x)^{da}}$ corresponds to $E_1(x-1)^{da}$, and $\frac{E(x)^{d-1}E^{(d)}(x)}{E'(x)^d} - 1$ corresponds to

$$\frac{E_1(x)^{d-1}E_2(x) + \cdots + E_1(x)\cdots E_d(x)}{E_1(x)^d} = \frac{E_0(x)^d\big(E_1(x)^d + E_1(x)^{d-1}E_2(x) + \cdots + E_1(x)\cdots E_d(x)\big)}{E_0(x)^d E_1(x)^d} - 1.$$

If $d = 2$, then the sum of the first N terms of $\sigma_m(E^{(d)}(x)^a)$ is exactly $\sigma_m(\mathfrak{s}_{d,N,a}(x))$. The result also immediately holds for $N = 1$. So assume $d > 2$ and $N > 1$.

Let $\zeta_d(x) = \frac{E_1(x)^{d-1}E_2(x)+\cdots+E_1(x)\cdots E_d(x)}{E_1(x)^d}$. Since $\frac{E_2(x)}{E_1(x)}$ is the largest monomial of $\zeta_d(x)$, the largest monomial of $\zeta_d(x)^n$ is $\left(\frac{E_2(x)}{E_1(x)}\right)^n$, which has $E_1(x)^n$ as its denominator. The second largest monomial of $\sum_{k=0}^{\infty}\binom{a}{k}\zeta_d(x)^k$ is $\frac{E_2(x)^2}{E_1(x-1)^2}$, which appears in $\zeta_d(x)$ since $d > 2$, and has $E_1(x)^2$ as its denominator. So the result holds for $N = 2$.

For $N > 2$, the Nth monomial g of $\sum_{k=0}^{\infty}\binom{a}{k}\delta_d(x)^k$ has $E_1(x)^l$ as its denominator for some $l < N$, so it must appear before $\zeta_d(x)^N$. Thus the first N terms of $\sigma_m\big(E^{(d)}(x)\big)$ and $\sigma_m(\mathfrak{s}_{d,N,a}(x))$ agree. □

Corollary 4.3.4. *Assume $\boldsymbol{k}$ and X satisfy the ordering and separation assumptions in Remark 3.1.2. Let $s = c_1g_1 + \cdots + c_ng_n \in \boldsymbol{k}[G_m]$, and write*

$$g_i = \prod_{j=1}^{p} E(x_j)^{a_{i,j,0}} \cdots E^{(d)}(x_j)^{a_{i,j,d}}$$

with $x_1 > \cdots > x_p$. Suppose $\xi_{x_j}(g_i) = 0$ for all $j = 1, \ldots, p$. Then for every N_0 there exists some N_1 such that the first N_0 terms (at least) of $\sigma_m(s)$ and $\sigma_m(c_1u_1 + \cdots + c_nu_n)$ are equal, where

$$u_i = \prod_{j=1}^{p} E(x_j)^{a_{i,j,0}} E'(x_j)^{a_{i,j,1}} \mathfrak{s}_{2,N_1,a_{i,j,2}}(x_j) \cdots \mathfrak{s}_{d,N_1,a_{i,j,d}}(x_j).$$

Proof. Let $N_0 \in \mathbb{N}$. Let t be the finite initial subsum of $\sigma_m(s)$ defined as in Remark 3.3.1. Enumerate $\operatorname{Supp}(\sigma_m(g_i))$ as $(h_{i,k})_{k\in\mathbb{N}}$ with $h_{i,k} > h_{ik+1}$. Let $n_i \in \mathbb{N}$ be smallest such that

$$\operatorname{Supp}(t) \cap \{h_{i,k} : k > n_i\} = \varnothing.$$

Let $N_1 = \max(n_1, \ldots, n_p)$. Let

$$u_i = \prod_{j=1}^{p} E(x_j)^{a_{i,j,0}} E'(x_j)^{a_{i,j,1}} \mathfrak{s}_{2,N_1,a_{i,j,2}}(x_j) \cdots \mathfrak{s}_{d,N_1,a_{i,j,d}}(x_j).$$

So it suffices to show that the first N_0 terms of $\nu_m(\sigma_m(s))$ and $\nu_m(\sigma_m(c_1u_1 + \cdots + c_nu_n))$ agree. By Lemma 4.3.3, the first N_1 terms of $\sigma_m(u_i)$ and $\sigma_m(g_i)$ agree. By our choice of N_1, the first N_0 terms of $\sigma_m(c_1u_1 + \cdots + c_nu_n)$ and $\sigma_m(s)$ agree. So at least the first N_0 terms of $\nu_m(\sigma_m(c_1u_1 + \cdots + c_nu_n))$ and $\nu_m(\sigma_m(s))$ agree, and we are done. □

Remark 4.3.5. Observe that

1. $\xi_x(g) = a$ for every monomial g of $\mathfrak{s}_{d,N,a}$,

2. $\varphi_m(\mathfrak{s}_{d,N,a})$ is a finite sum.

Let g_i and u_i be as in Lemma 4.3.4. By the first observation, no monomial of $\varphi_{m,0}(u_i)$ has any generator of the form $e_{X-m-1}(E(x-1))^a$. So $\mathrm{Supp}(\varphi_m(u_i)) \subset G_{m+1}$. This along with the second observation means we have $\varphi_m(u_i) \in \boldsymbol{k}[G_{m+1}]$. So σ_{m+1} is defined on $\varphi_m(u_i)$. In fact, by Lemma 4.3.1 and Remark 4.3.2,

$$\sigma_{m+1}(\varphi_m(u_i)) = \nu_m(\sigma_m(u_i)).$$

So $\sigma_{m+1}(\varphi_m(c_1u_1 + \cdots + c_nu_n)) = \nu_m(\sigma_m(c_1u_1 + \cdots + c_nu_n))$.

Lemma 4.3.6. *$\varphi_{m,0}$ is order preserving on $\boldsymbol{k}[\Gamma_{X-m,0,small}]$.*

Proof. Let $s = s_1\ell_1 + \cdots + s_n\ell_n$ with $s_i \in \boldsymbol{k}[G_m]$ and $\ell_i \in \Lambda_m$. We will show $s > 0$ if and only if $\varphi_{m,0}(s) > 0$.

Recall from Lemma 3.3.2 that the sign of $\mathrm{Init}(s)$ determines the sign of s. We will show that $\varphi_{m,0}(s)|_{\mathrm{Lv}(\varphi_{m,0}(s))} \in \boldsymbol{k}[G_{m+1}]$, which means the sign of $\varphi_{m,0}(s)$ is determined by the sign of $\sigma_{m+1}\left(\varphi_{m,0}(s)|_{\mathrm{Lv}(\varphi_{m,0}(s))}\right)$. We will then show that $\sigma_{m+1}\left(\varphi_{m,0}(s)|_{\mathrm{Lv}(\varphi_{m,0}(s))}\right) = \mathrm{Init}(s)$.

Let t_i be as in Lemma 3.3.2, and let $N_{i,0} = |\mathrm{Supp}(t_i)|$. Let $N_0 = \max(N_{1,0}, \ldots, N_{p,0})$. For each $i = 1, \ldots, p$, let $N_{i,1}$ and $c_{i,1}u_{i,1} + \cdots + c_{i,k_i}u_{i,k_i}$ be given by Lemma 4.3.4, using N_0. Define $u_i := c_{i,1}u_{i,1} + \cdots + c_{i,k_i}u_{i,k_i}$.

By Remark 4.3.5,

$$\sigma_{m+1}\big(\varphi_{m,0}(u_i)\big) = \nu_m\big(\sigma_m(u_i)\big) = \rho_0(u_i).$$

If we write $\ell_i = \prod_{j=1}^p \log E'(x_j)^{b_{i,j}}$, then $\mathrm{Lm}\big(\varphi_{m,0}(\ell_i)\big) = \prod_{j=1}^p E(x_j-1)^{b_{i,j}}$, so

$$\sigma_{m+1}\big(\mathrm{Lm}(\varphi_{m,0}(\ell_i))\big) = \prod_{j=1}^p E_0(x_j-1)^{b_{i,j}}.$$

Following the proof of Lemma 3.3.2, $\mathrm{Init}(s) \neq 0$ is the initial subsum of

$$\rho_m(u_1)\prod_{j=1}^p E(x_j-1)^{b_{1,j}} + \cdots + \rho_m(u_n)\prod_{j=1}^p E(x_j-1)^{b_{n,j}} = \sigma_{m+1}\big(\varphi_{m,0}(u_1)\mathrm{Lm}(\varphi_{m,0}(\ell_1)) + \cdots + \varphi_{m,0}(u_n)\mathrm{Lm}(\varphi_{m,0}(\ell_n))\big)$$

so both these expressions have the same sign as $\mathrm{Init}(s)$. Recall from Remark 3.3.1 that $\mathrm{Init}(s)$ is defined by having tuples of exponents of $E_0(x_1-1), \ldots, E_0(x_p-1)$ maximized. By the way σ_{m+1} is defined, if $g \in \boldsymbol{k}[G_{m+1}]$ and $\xi_{x_j}(g) = a$, then the exponent of $E_0(x_j-1)$ in $\sigma_{m+1}(g)$ is a. So $\mathrm{Init}(s)$ must be the image under σ_{m+1} of $\big(\varphi_{m,0}(u_1)\mathrm{Lm}(\varphi_{m,0}(\ell_1)) + \cdots + m,0(u_n)\mathrm{Lm}(\varphi_{m,0}(\ell_n))\big)|_w$, where

$$w = \mathrm{Lv}\big(\varphi_{m,0}(u_1)\mathrm{Lm}(\varphi_{m,0}(\ell_1)) + \cdots + m,0(u_n)\mathrm{Lm}(\varphi_{m,0}(\ell_n))\big).$$

Since for any

$$g \in \mathrm{Supp}\Big(\varphi_{m,0}(u_i)\big(\varphi_{m,0}(\ell_i) - \mathrm{Lm}(\varphi_{m,0}(\ell_i))\big)\Big)$$

$\sigma_{m+1}(g) < \mathrm{Init}(s)$, we must have $\mathrm{Lv}\big(\varphi_{m,0}(s)\big) = w$. This completes the proof. ∎

We would now like to extend $\varphi_{m,0}$ to all of $\Gamma_{X-m,0}$ by defining

$$\varphi_{m,0}\big(e_{X-m}(t)\big) = e_{X-m-1}\big(\varphi_{m,0}(t)\big)$$

for $t \in T_{X-m}$, but we must check that e_{X-m-1} is defined on $\varphi_{m,0}(t)$.

Lemma 4.3.7. *Assume* $\boldsymbol{k}$ *and* X *satisfy the ordering and separation assumptions in Remark 3.1.2. If* $t \in T_{X-m}$*, then* $e_{X-m-1}\big(\varphi_{m,0}(t)\big)$ *is defined in* $K_{X-m-1,1}$.

Proof. Since any element of T_{X-m} is a finite sum of generators, it suffices to show that e_{X-m-1} is defined on the generators of T_{X-m}. Let

$$t = c\prod_{j=1}^{p} \log E'(x_j)^{a_{j,-1}} \frac{E''(x_j)^{a_{j,2}} \cdots E^{(d)}(x_j)^{a_{j,d}}}{E'(x_j)^{a_{j,2}+\cdots+a_{j,d}}}$$

be a generator of T_{X-m}, with $x_1 > \cdots > x_p \in X - m$ and $a_{j,-1}, a_{j,2}, \ldots, a_{j,d} \in \mathbb{N}$. Notice that for $d \geq 1$,

$$\varphi_{m,0}(E^{(d)}(x)) = e_{X-m-1}(E(x-1))B_d(x-1)$$

which matches the difference-differential equation for $E^{(d)}(x)$. For each term of $B_d(x-1)$, $\xi_{x-1} \geq 1$, with $\xi_{x-1} = 1$ only for the smallest term $E^{(d)}(x-1)$. So for each term of $\delta_d(x) = \frac{B_d(x-1)-E'(x-1)^d}{E'(x-1)^d}$, $-(d-1) \leq \xi_{x-1} \leq -1$, with $\xi_{x-1} = -(d-1)$ only for the smallest term $\frac{E^{(d)}(x-1)}{E'(x-1)^d}$.

By the definition of $\varphi_{m,0}$, we can write

$$\varphi_{m,0}\left(c\prod_{j=1}^{p} \log E'(x_j)^{a_{j,-1}} \frac{E''(x_j)^{a_{j,2}} \cdots E^{(d)}(x_j)^{a_{j,d}}}{E'(x_j)^{a_{j,2}+\cdots+a_{j,d}}}\right) =$$

$$c\prod_{j=1}^{p} E(x_j-1)^{a_{j,-1}} E'(x_j-1)^{a_{j,2}+2a_{j,3}+\cdots+(d-1)a_{j,d}}$$

$$\left(\sum_{n=0}^{a_{j,-1}} \binom{a_{j,-1}}{n} \left(\frac{\log E'(x_j-1)}{E(x_j-1)}\right)^n\right) \left(\sum_{n=0}^{a_{j,2}} \binom{a_{j,2}}{n} \delta_2(x_j)^n\right) \cdots \left(\sum_{n=0}^{a_{j,d}} \binom{a_{j,d}}{n} \delta_d(x_j)^n\right).$$

This is a finite sum. For every term in the sum, $\xi_{x_j-1} \geq 0$ for all $j = 1, \ldots, p$. For every term except the smallest term, $\xi_{x_1-1} + \cdots + \xi_{x_p-1} > 0$. Therefore, every term except the smallest term is in $A_{X-m-1,0}$. The smallest term is

$$c\prod_{j=1}^{p} E(x_j-1)^{a_{j,-1}} E'(x_j-1)^{a_{j,2}+2a_{j,3}+\cdots+(d-1)a_{j,d}}$$

$$\binom{a_{j,-1}}{a_{j,-1}} \left(\frac{\log E'(x_j-1)}{E(x_j-1)}\right)^{a_{j,-1}} \binom{a_{j,2}}{a_{j,2}} \left(\frac{E''(x_j-1)}{E'(x_j-1)^2}\right)^{a_{j,2}} \cdots \binom{a_{j,d}}{a_{j,d}} \left(\frac{E^{(d)}(x_j-1)}{E'(x_j-1)^d}\right)^{a_{j,d}} =$$

$$c\prod_{j=1}^{p} \log E'(x_j-1)^{a_{j,-1}} \frac{E''(x_j-1)^{a_{j,2}} \cdots E^{(d)}(x_j-1)^{a_{j,d}}}{E'(x_j-1)^{a_{j,2}+\cdots+a_{j,d}}}$$

and is an element of T_{X-m-1}. Thus $e_{X-m-1}(t) \in K_{X-m-1,1}$. □

Remark 4.3.8. In the proof of Lemma 4.3.7, observe that the smallest term of $\varphi_{m,0}(t)$ is exactly t with each x_j replaced by $x_j - 1$.

We can now extend $\varphi_{m,0} : K_{X-m,0} \to K_{X-m-1,1}$ to all of $\Gamma_{X-m,0}$.

6. $\varphi_{m,0}\big(e_{X-m}(t)\big) = e_{X-m-1}(\varphi_{m,0}(t))$
7. Extend $\varphi_{m,0}$ to $\boldsymbol{k}[\Gamma_{X-m,0}]$ so that it is a $\boldsymbol{k}$-algebra homomorphism.

Finally, we would like to define

$$\varphi_{m,0}\left(\sum_{M\in\Gamma_{X-m,0}} c_M M\right) = \sum_{M\in\Gamma_{X-m,0}} c_M \varphi_{m,0}(M)$$

but we must check that $\sum_{M\in\Gamma_{X-m,0}} c_M\varphi_{m,0}(M)$ is a valid sum in $K_{X-m-1,1}$. We will check this using the next two lemmas.

Lemma 4.3.9. *Let $M_i = g_i \cdot \ell_i \cdot e_{X-m}(t_i) \in \Gamma_{X-m,0}$ for $i = 1, 2$. If M_1 and M_2 are in different cosets of $\Gamma_{X-m,0}/\Gamma_{x-m,0,small}$ and $M_1 > M_2$, then* $\mathrm{Supp}(\varphi_{m,0}(M_1)) > \mathrm{Supp}(\varphi_{m,0}(M_2))$.

Proof. Let $x_1 > \cdots > x_p$ list the elements of X that appear in M_1, M_2. We can write

$$\varphi_{m,0}(M_i) = e_{X-m-1}\left(\varphi_{m,0}(t_i) + \sum_{j=1}^{p} \xi_{x_j}(M_i)E(x_j - 1)\right) s_i$$

for some $s \in K_{X-m-1,0}$. So it suffices to show that

$$\varphi_{m,0}(t_1) + \sum_{j=1}^{p} \xi_{x_j}(M_1)E(x_j-1) > \varphi_{m,0}(t_2) + \sum_{j=1}^{p} \xi_{x_j}(M_2)E(x_j-1).$$

$M_1 > M_2$ means

$$\rho_m(t_1) + \sum_{j=1}^{p} \xi_{x_j}(M_1)E_0(x_j-1) > \rho_m(t_2) + \sum_{j=1}^{p} \xi_{x_j}(M_2)E_0(x_j-1)$$

$$\text{i.e.,}\quad \rho_m(t_1 - t_2) > \sum_{j=1}^{p} \big(\xi_{x_j}(M_2) - \xi_{x_j}(M_1)\big)E_0(x_j-1).$$

Express $t_1 - t_2 = h_1\lambda_1 + \cdots + h_n\lambda_n$ as a sum in $\boldsymbol{k}[G_m\Lambda_m]$, with $h_i \in G_m$ and $\lambda_i \in \Lambda_m$. Let $\mathrm{Init}(t_1 - t_2)$ be the initial subsum of $\rho_m(t_1 - t_2)$, defined in Remark 3.3.1. We know $\nu_m(\sigma_m(h_{i,1})) = \sigma_{m+1}(\varphi_{m,0}(h_{i,1}))$ by Lemma 4.3.1. We also know the only term of $\rho_m(\lambda_i)$

that can contribute to $\mathrm{Init}(t_1 - t_2)$ is $\mathrm{Lm}\big(\rho_m(\lambda_i)\big)$ for all $i = 1, \ldots, n$. Let $\mathrm{Lm}\big(\rho_m(\lambda_i)\big) = \prod_{j=1}^{p} E_0(x_j - 1)^{b_{i,j}}$. So we must have

$$\sum_{i=1}^{n} \sigma_{m+1}(\varphi_{m,0}(h_i)) \prod_{j=1}^{p} E_0(x_j - 1)^{b_{i,j}} > \sum_{j=1}^{p} \big(\xi_{x_j}(M_2) - \xi_{x_j}(M_1)\big) E_0(x_j - 1).$$

Since σ_{m+1} is order preserving, we must have

$$\varphi_{m,0}(h_i) \prod_{j=1}^{p} E(x_j - 1)^{b_{i,j}} > \sum_{j=1}^{p} \big(\xi_{x_j}(M_2) - \xi_{x_j}(M_1)\big) E(x_j - 1).$$

Just as in Lemma 4.3.6, $\varphi_{m,0}(t_1 - t_2)|_{\mathrm{Lv}(\varphi_{m,0}(t_1 - t_2))}$ is a subsum of $\varphi_{m,0}(h_i) \prod_{j=1}^{p} E(x_j - 1)^{b_{i,j}}$. So

$$\varphi_{m,0}(t_1 - t_2) > \sum_{j=1}^{p} \big(\xi_{x_j}(M_2) - \xi_{x_j}(M_1)\big) E_0(x_j - 1)$$

as desired. □

Lemma 4.3.10. *Assume* $\boldsymbol{k}$ *and* X *satisfy the ordering and separation assumptions in Remark 3.1.2. If* $s \in K_{X-m}$*, then* $\varphi_{m,0}(s)$ *is an element of* $K_{X-m-1,1}$*.*

Proof. Let $s = \sum_{M \in \Gamma_{X-m,0}} c_M M$. We will show $\varphi_{m,0}\left(\sum_{M \in \Gamma_{X-m,0}} c_M M\right)$ is a valid sum in $K_{X-m-1,1}$. Since each $M \in \Gamma_{X-m,0}$ is a finite product of generators, $\varphi_{m,0}$ is well defined on monomials. So it suffices to check that $(c_M M : M \in \Gamma_{X-m,0})$ is summable in the sense of Definition 4.1.11, i.e.,

1. For each coset $N\Gamma_{X-m-1,0,small}$ of $\Gamma_{X-m-1,1}/\Gamma_{X-m-1,0,small}$, there are only finitely many $M \in \Gamma_{X-m,0}$ such that $c_M \neq 0$ and $N\Gamma_{X-m-1,0,small} \cap \mathrm{Supp}(\varphi_{m,0}(M)) \neq \varnothing$.
2. $\displaystyle\bigcup_{M \in \mathrm{Supp}(s)} \mathrm{Supp}(\varphi_{m,0}(M))$ is reverse well-ordered in $\Gamma_{X-m-1,1}$.

We start with (1). Let $N = g \cdot \ell \cdot e_{X-m-1}(t) \cdot e_{X-m-1}(\alpha) \in \Gamma_{X-m-1,1}$ with $g \in G_{m+1}$, $\ell \in \Lambda_{m+1}$, $t \in T_{X-m-1}$, and $\alpha \in A_{X-m-1,0}$. Note that t and α are fixed across all elements of $N\Gamma_{X-m-1,0,small}$. Let $M_i = g_i \cdot \ell_i \cdot e_{X-m}(t_i) \in \Gamma_{X-m,0}$ for $i = 1, 2$, and let $x_1 > \cdots > x_p$ list the elements of X that appear in M_1, M_2. Suppose $N\Gamma_{X-m-1,0,small} \cap \mathrm{Supp}(\varphi_{m,0}(M_i)) \neq \varnothing$ for $i = 1, 2$.

We can write

$$\varphi_{m,0}(M_i) = e_{X-m-1}\left(\varphi_{m,0}(t_i) + \sum_{j=1}^{p} \xi_{x_j}(M_i) E(x_j - 1)\right) s_i$$

for some $s_i \in K_{X-m-1,0}$. M_1 and M_2 must be in the same coset of $\Gamma_{X-m,0}/\Gamma_{X-m,0,small}$, or else by Lemma 4.3.9 we would have

$$\varphi_{m,0}(t_1) + \sum_{j=1}^{p} \xi_{x_j}(M_1)E(x_j - 1) \neq \varphi_{m,0}(t_2) + \sum_{j=1}^{p} \xi_{x_j}(M_2)E(x_j - 1)$$

which contradicts that $N\Gamma_{X-m-1,0,small} \cap \mathrm{Supp}(\varphi_{m,0}(M_i)) \neq \varnothing$ for both $i = 1, 2$. Since there are only finitely many M in the same coset as M_1, M_2 such that $c_M \neq 0$, we have proven (1).

Now we prove (2). Let $B \subset \bigcup_{M \in \mathrm{Supp}(s)} \mathrm{Supp}(\varphi_{m,0}(M))$. We will find a largest element of B. Since $s|_w$ is finite for any coset $w \in \Gamma_{X-m,0}/\Gamma_{X-m,0,small}$, $B \cap \mathrm{Supp}(\varphi_{m,0}(s|_w))$ has a largest element. By Lemma 4.3.9, if $M_1 > M_2$ then $\mathrm{Supp}(\varphi_{m,0}(M_1)) > \mathrm{Supp}(\varphi_{m,0}(M_1))$. So the largest element of B is the largest element of $B \cap \mathrm{Supp}\left(\varphi_{m,0}\left(s|_{\mathrm{Lv}(s)}\right)\right)$. □

We repeat the full definition of $\varphi_{m,0} : K_{X-m,0} \to K_{X-m-1,1}$:

1. $\varphi_{m,0}\big(\log E'(x)^b\big) = E(x-1)^b \sum_{n=0}^{\infty} \binom{b}{n} \left(\frac{\log E'(x-1)}{E(x-1)}\right)^n$
   Since $\left(\frac{\log E'(x-1)}{E(x-1)}\right)^n < \Gamma_{X-m-1,0,small}$ for $n > 0$, this sum is allowed in $K_{X-m-1,0}$.
2. $\varphi_{m,0}\big(E(x)^a\big) = e_{X-m-1}\big(E(x-1)\big)^a$
3. $\varphi_{m,0}\big(E'(x)^a\big) = e_{X-m-1}\big(E(x-1)\big)^a E'(x-1)^a$
4. $\varphi_{m,0}\big(E^{(d)}(x)^a\big) = e_{X-m-1}(E(x-1))^a E'(x-1)^{da} \sum_{n=0}^{\infty} \binom{a}{n} \delta_d(x)^n$
   for $d > 1$, where $\delta_d(x) = \frac{B_d(x-1) - E'(x-1)^d}{E'(x-1)^d} = \frac{\binom{d}{2}E'(x-1)^{d-2}E''(x-1) + \cdots + E^{(d)}(x-1)}{E'(x-1)^d}$
   Since $\mathrm{Supp}(\delta_d(x)^n) < \Gamma_{X-m-1,0,small}$ for $n > 0$, this sum is allowed in $K_{X-m-1,0}$.
5. Extend $\varphi_{m,0}$ to $\boldsymbol{k}[G_m\Lambda_m]$ so that it is a $\boldsymbol{k}$-algebra homomorphism.
6. $\varphi_{m,0}\big(e_{X-m}(t)\big) = e_{X-m-1}(\varphi_{m,0}(t))$
7. $\varphi_{m,0}(g_1 \cdots g_n) = \varphi_{m,0}(g_1) \cdots \varphi_{m,0}(g_n)$ for generators $g_1, \ldots, g_n \in \Gamma_{X-m,0}$
8. $\varphi_{m,0}\left(\sum_{M \in \Gamma_{X-m,0}} c_M M\right) = \sum_{M \in \Gamma_{X-m,0}} c_M \varphi_{m,0}(M)$.

Corollary 4.3.11. *$\varphi_{m,0}$ is order preserving on $K_{X-m,0}$.*

Proof. Let $s \in K_{X-m,0}$. By Lemma 4.3.9, if $M_1 \in \operatorname{Supp}\left(s|_{\mathrm{Lv}(s)}\right)$ and $M_2 \notin \operatorname{Supp}\left(s|_{\mathrm{Lv}(s)}\right)$, then $\operatorname{Supp}(\varphi_{m,0}(M_1)) > \operatorname{Supp}(\varphi_{m,0}(M_1))$. So the sign of $\varphi_{m,0}(s)$ is determined by the sign of $\varphi_{m,0}\left(s|_{\mathrm{Lv}(s)}\right)$. The sign of $\varphi_{m,0}\left(s|_{\mathrm{Lv}(s)}\right)$ is the same as the sign of $\varphi_{m,0}\left(\frac{s|_{\mathrm{Lv}(s)}}{\mathrm{Lm}(s)}\right)$, and $\frac{s|_{\mathrm{Lv}(s)}}{\mathrm{Lm}(s)} \in \boldsymbol{k}[\Gamma_{X-m,0,small}]$. Apply Lemma 4.3.6 to finish the proof. □

Now given $\varphi_{m,n} : K_{X-m,n} \to K_{X-m-1,n+1}$, define $\varphi_{m,n+1} : K_{X-m,n+1} \to K_{X-m-1,n+2}$ as follows:

$$\varphi_{m,n+1}\left(\sum f_a e_{X-m}(a)\right) = \sum \varphi_{m,n}(f_a) e_{X-m}(\varphi_{m,n}(a))$$

for $f_a \in K_{X-m,n}$ and $a \in A_{X-m,n}$. This is a valid sum in $K_{X-m-1,n+2}$, and since $\varphi_{m,n}$ is order preserving, so is $\varphi_{m,n+1}$. Let $\varphi_m : K_{X-m} \to K_{X-m-1}$ be the common extension of all $\varphi_{m,n}$, which is also order preserving.

4.4 Finding logarithms, exponentials, and multiplicative inverses

In the remainder of this section, we will show that for every $s \in K_{X-m}$, there is some $l \in \mathbb{N}$ such that

1. $e_{X-m-l-1}\big((\varphi_{m+l} \circ \cdots \circ \varphi_m)(s)\big)$ is defined in $K_{X-m-l-1}$ (Lemma 4.4.7),
2. if $s > 0$ then there is some $s' \in K_{X-m-l-1}$ such that

$$(\varphi_{m+l} \circ \cdots \circ \varphi_m)(s) = e_{X-m-l-1}(s')$$

   i.e., this element has a logarithm (Lemma 4.4.4), and
3. $(\varphi_{m+l} \circ \cdots \circ \varphi_m)(s)$ has a multiplicative inverse in $K_{X-m-l-1}$ (Lemma 4.4.10).

We find multiplicative inverses with the more general result that if $s_1, \ldots, s_n \in K_{X-m}$ with $\operatorname{Supp}(s_i) < 1$, then there is $l \in \mathbb{N}$ such that

$$\sum_{j_1,\ldots,j_n=0}^{\infty} a_{j_1,\ldots,j_n} \prod_{i=1}^{n} (\varphi_{m+l} \circ \cdots \circ \varphi_m)(s_i)^{j_i} \in K_{X-m-l-1}$$

for all $(a_{j_1,\ldots,j_n})_{\bar{j}\in\mathbb{N}^n} \in \boldsymbol{k}^\omega$ (Lemma 4.4.9). This result will be necessary in order to show our construction is closed under restricted analytic functions.

Lemma 4.4.1. *If $s = e_{X-m}(s')$ for some $s' \in B^*_{X-m}$, then $\varphi_m(s) = e_{X-m-1}\big(\varphi_m(s')\big)$.*

Proof. Since $s' \in B^*_{X-m,n}$ for some $n \in \mathbb{N}$, we can write

$$s' = a + t + r + \epsilon$$

with $a \in A_{X-m,n-1} \oplus \cdots \oplus A_{X-m,0}$, $t \in T_{X-m}$, $r \in \boldsymbol{k}$, and $\epsilon \in B_{X-m,0} \oplus \cdots \oplus B_{X-m,n}$ with $\operatorname{Supp}(\epsilon) < \Gamma_{X-m,0,small}$. So

$$\begin{aligned}\varphi_m(s) &= \varphi_m\left(e_{X-m}(a)e_{X-m}(t)e_{X-m}(r)\cdot\sum_{k=0}^{\infty}\frac{\epsilon^k}{k!}\right)\\ &= e_{X-m-1}\big(\varphi_m(a)\big)e_{X-m-1}\big(\varphi_m(t)\big)e_{X-m-1}\big(\varphi_m(r)\big)\sum_{k=0}^{\infty}\frac{\varphi_m(\epsilon)^k}{k!}\\ &= e_{X-m-1}\big(\varphi_m(a+t+r+\epsilon)\big)\\ &= e_{X-m-1}\big(\varphi_m(s')\big)\end{aligned}$$

as claimed. □

Lemma 4.4.2. *If $M \in \Gamma_{X-m,0}$, then there is $s' \in K_{X-m-2}$ such that $\varphi_{m+1}\big(\varphi_m(M)\big) = e_{X-m-2}(s')$.*

Proof. Let $M = g \cdot \ell \cdot e_{X-m}(t)$ with $g \in G_m$, $\ell \in \Lambda_m$, and $t \in T_{X-m}$. Since

$$\varphi_{m+1}\big(\varphi_m(e_{X-m}(t))\big) = e_{X-m-2}\big(\varphi_{m+1}(\varphi_m(t))\big)$$

and φ_m and φ_{m+1} are ring homomorphisms, it suffices to show that for any $x \in X-m$, $d \in \mathbb{N}$, $a, b \in \boldsymbol{k}$, we can find $s_1, s_2 \in K_{X-m-2}$ such that

$$\begin{aligned}\varphi_{m+1}\left(\varphi_m\left(E^{(d)}(x)^a\right)\right) &= e_{X-m-2}(s_1)\\ \varphi_{m+1}\left(\varphi_m\left(\log E'(x)^b\right)\right) &= e_{X-m-2}(s_2).\end{aligned}$$

First,

$$\varphi_m\left(E^{(d)}(x)^a\right) = e_{X-m-1}(E(x-1))^a E'(x-1)^{da}\sum_{k=0}^{\infty}\binom{a}{k}\delta_d(x)^k.$$

Recall that $\operatorname{Supp}(\delta_d(x)) < \Gamma_{X-m-1,0,small}$, so

$$\operatorname{Supp}\left(\sum_{k=1}^{\infty}\binom{a}{k}\delta_d(x)^k\right) < \Gamma_{X-m-1,0,small}$$

and thus $\operatorname{Supp}\left(\left(\sum_{k=1}^{\infty}\binom{a}{k}\delta_d(x)^k\right)^l\right) < \Gamma_{X-m-1,0,small}$ for all $0 < l \in \mathbb{N}$. So we can express

$$E'(x-1)^{da}\sum_{k=0}^{\infty}\binom{a}{k}\epsilon^k = e_{X-m-1}\left(da\log E'(x-1) + \sum_{l=1}^{\infty}\frac{(-1)^{l+1}}{l}\left(\sum_{k=1}^{\infty}\binom{a}{k}\delta_d(x)^k\right)^l\right).$$

By Lemma 4.4.1, we can take $s_1 = \varphi_{m+1}\left(da\log E'(x-1) + \sum_{l=1}^{\infty}\frac{(-1)^{l+1}}{l}\left(\sum_{k=1}^{\infty}\binom{a}{k}\delta_d(x)^k\right)^l\right)$.

Next we will find s_2. We can calculate

$$\begin{aligned}\varphi_{m+1}\left(\varphi_m\left(\log E'(x)^b\right)\right) &= \varphi_{m+1}\left(E(x-1)^b\sum_{k=0}^{\infty}\binom{b}{k}\left(\frac{\log E'(x-1)}{E(x-1)}\right)^k\right)\\ &= e_{X-m-2}\big(E(x-2)\big)^b\sum_{k=0}^{\infty}\binom{b}{k}\frac{\big(E(x-2)+\log E'(x-2)\big)^k}{e_{X-m-2}\big(E(x-2)\big)^k}.\end{aligned}$$

Since $\operatorname{Supp}\left(\sum_{k=1}^{\infty}\binom{b}{k}\frac{\left(E(x-2)+\log E'(x-2)\right)^k}{e_{X-m-2}\left(E(x-2)\right)^k}\right) < \Gamma_{X-m-2,0,small}$, we can now bring the sum inside e_{X-m-2} to get $\varphi_{m+1}\left(\varphi_m\left(\log E'(x)^b\right)\right) = e_{X-m-2}(s_2)$ where

$$s_2 = bE(x-2) + \sum_{l=1}^{\infty}\frac{(-1)^{l+1}}{l}\left(\sum_{k=1}^{\infty}\binom{b}{k}\frac{\big(E(x-2)+\log E'(x-2)\big)^k}{e_{X-m-2}\big(E(x-2)\big)^k}\right)^l. \qquad \square$$

Lemma 4.4.3. *Let $s \in \boldsymbol{k}[\Gamma_{X-m,0,small}]$ and let d be the value of the largest derivative appearing in s. Let $\tilde{s} = (\varphi_{m+d}\circ\cdots\circ\varphi_m)(s)$. Then $\tilde{s}|_{\mathrm{Lv}(\tilde{s})}$ is a single term of the form*

$$ce_{X-m-d-1}(\alpha)\prod_{j=1}^{p}E'(x_j-d-1)^{a_j}$$

with $c \in \boldsymbol{k}^\times$ $\alpha \in A_{X-m-d-1,0}\oplus\cdots\oplus A_{X-m-d-1,d}$ and $a_j \in \boldsymbol{k}$.

Proof. In Lemma 4.3.6, we showed that $\varphi_{m,0}(s)|_{\mathrm{Lv}(\varphi_{m,0}(s))} \in \boldsymbol{k}[G_{m+1}]$, and

$$\mathrm{Lm}\big(\rho_m(s)\big) = \mathrm{Lm}\Big(\sigma_{m+1}\left(\varphi_{m,0}(s)|_{\mathrm{Lv}(\varphi_{m,0}(s))}\right)\Big).$$

Write

$$\mathrm{Lm}\big(\rho_m(s)\big) = \prod_{j=1}^{p}E_0(x_j-1)^{a_{j,0}}E_1(x_j-1)^{a_{j,1}}\cdots E_d(x_j-1)^{a_{j,d}}.$$

We now inductively define a sequence $s = s_0, s_1, \ldots, s_d$ with $s_n \in \boldsymbol{k}[\Gamma_{X-m-n,0,small}]$ for $n = 0,\ldots,d$. Let $s_0 = s$. Let $s_1 = \frac{\varphi_{m,0}(s_0)|_{\mathrm{Lv}(\varphi_{m,0}(s_0))}}{\prod_{j=1}^{p}E(x_j-1)^{a_{j,0}}} \in \boldsymbol{k}[\Gamma_{X-m-1,0,small}]$, so that

$$\begin{aligned}\mathrm{Lm}\big(\sigma_{m+1}(s_1)\big) &= \prod_{j=1}^{p}E_1(x_j-1)^{a_{j,1}}\cdots E_d(x_j-1)^{a_{j,d}}\\ \mathrm{Lm}\big(\rho_{m+1}(s_1)\big) &= \prod_{j=1}^{p}E_0(x_j-2)^{a_{j,1}}E_1(x_j-2)^{a_{j,1}+a_{j,2}}E_2(x_j-2)^{a_{j,3}}\cdots E_{d-1}(x_j-2)^{a_{j,d}}.\end{aligned}$$

Now given s_n for $n < d$ such that

$$\begin{aligned}\operatorname{Lm}\bigl(\rho_{m+n}(s_n)\bigr) = \prod_{j=1}^{p} & E_0(x_j-n-1)^{a_{j,1}+\cdots+a_{j,n}} E_1(x_j-n-1)^{a_{j,1}+\cdots+a_{j,n+1}} \\ & E_2(x-n-1)^{a_{j,n+2}} \cdots E_{d-n}(x_j-n-1)^{a_{j,d}}\end{aligned}$$

let

$$s_{n+1} = \frac{\varphi_{m+n,0}(s_n)|_{\operatorname{Lv}(\varphi_{m+n,0}(s_n))}}{\prod_{j=1}^{p} E(x_j-n-1)^{a_{j,1}+\cdots+a_{j,n}}}.$$

Then $s_{n+1} \in \boldsymbol{k}[\Gamma_{X-m-n-1,0,small}]$ and

$$\begin{aligned}\operatorname{Lm}\bigl(\sigma_{m+n+1}(s_{n+1})\bigr) &= \sigma_{m+n+1}\left(\frac{\varphi_{m+n,0}(s_n)|_{\operatorname{Lv}(\varphi_{m+n,0}(s_n))}}{\prod_{j=1}^{p} E_0(x_j-n-1)^{a_{j,1}+\cdots+a_{j,n}}}\right) \\ &= \frac{\sigma_{m+n+1}\bigl(\varphi_{m+n,0}(s_n)|_{\operatorname{Lv}(\varphi_{m+n,0}(s_n))}\bigr)}{\prod_{j=1}^{p} E_0(x_j-n-1)^{a_{j,1}+\cdots+a_{j,n}}} \\ &= \prod_{j=1}^{p} E_1(x_j-n-1)^{a_{j,1}+\cdots+a_{j,n+1}} \cdots E_{d-n}(x_j-n-1)^{a_{j,d}}\end{aligned}$$

by Lemma 4.3.6. So

$$\begin{aligned}\operatorname{Lm}\bigl(\rho_{m+n+1}(s_{n+1})\bigr) &= \operatorname{Lm}\bigl(\sigma_{m+n+1}(\operatorname{Lv}(\varphi_{m+n+1,0}(s_{n+1})))\bigr) \\ &= \prod_{j=1}^{p} E_0(x_j-n-2)^{a_{j,1}+\cdots+a_{j,n+1}} E_1(x_j-n-2)^{a_{j,1}+\cdots+a_{j,n+1}+a_{j,n+2}} \\ &\qquad E_2(x_j-n-2)^{a_{j,n+3}} \cdots E_{d-n-1}(x_j-n-2)^{a_{j,d}}\end{aligned}$$

and we can continue the induction.

Note that

$$\operatorname{Lm}\bigl(\rho_{m+d}(s_d)\bigr) = \prod_{j=1}^{p} E_0(x_j-d-1)^{a_{j,1}+\cdots+a_{j,d}} E_1(x_j-d-1)^{a_{j,1}+\cdots+a_{j,d}}.$$

Let $\operatorname{Init}(s_d)$ be as in Remark 3.3.1. Since $s_d \in \boldsymbol{k}[G_{m+d}]$ involves no log generators, and the only possible exponents of $E_l(x_j-d-1)$ for $l \geq 2$ in $\operatorname{Init}(s_d)$ are natural numbers, the preimage of $\operatorname{Init}(s_d)$ under σ_{m+d+1} must be the single term

$$c\prod_{j=1}^{p} E'(x_j-d-1)^{a_{j,1}+\cdots+a_{j,d}} = \varphi_{m+d}(s_d)|_{\operatorname{Lv}(\varphi_{m+d}(s_d))}$$

for some $0 \neq c \in \boldsymbol{k}$. Let $t_0 = (\varphi_{m+d} \circ \cdots \circ \varphi_{m+1}) \left(\prod_{j=1}^{p} E(x_j - 1)^{a_{j,0}}\right) \in A_{X-m-d-1,d}$, and for $l = 1, \ldots, d-1$, let $t_l = (\varphi_{m+d} \circ \cdots \circ \varphi_{m+l+1}) \left(\prod_{j=1}^{p} E(x_j - l - 1)^{a_{j,1}+\cdots+a_{j,l}}\right) \in A_{X-m-d-1,d-l}$. Then

$$\begin{aligned}
\tilde{s}|_{\mathrm{Lv}(\tilde{s})} &= (\varphi_{m+d} \circ \cdots \circ \varphi_m)(s|_{\mathrm{Lv}(s)})|_{\mathrm{Lv}(\tilde{s})} \\
&= (\varphi_{m+d} \circ \cdots \circ \varphi_{m+1}) \left(s_1 \prod_{j=1}^{p} E(x_j - 1)^{a_{j,0}} \right) \Bigg|_{\mathrm{Lv}(\tilde{s})} \\
&= (\varphi_{m+d} \circ \cdots \circ \varphi_{m+1})(s_1)|_{\mathrm{Lv}(\tilde{s})} \cdot t_0 \\
&= (\varphi_{m+d} \circ \cdots \circ \varphi_{m+2}) \left(s_2 \prod_{j=1}^{p} E(x_j - 2)^{a_{j,1}} \right) \Bigg|_{\mathrm{Lv}(\tilde{s})} \cdot t_0 \\
&= (\varphi_{m+d} \circ \cdots \circ \varphi_{m+2})(s_2)|_{\mathrm{Lv}(\tilde{s})} \cdot t_0 t_1 \\
&\vdots \\
&= \varphi_{m+d} \left(s_d \prod_{j=1}^{p} E(x_j - d)^{a_{j,1}+\cdots+a_{j,d}} \right) \Bigg|_{\mathrm{Lv}(\tilde{s})} t_0 \cdots t_{d-2} \\
&= c \prod_{j=1}^{p} E'(x_j - d - 1)^{a_{j,1}+\cdots+a_{j,d}} \cdot t_0 \cdots t_{d-1}.
\end{aligned}$$

So $\tilde{s}|_{\mathrm{Lv}(\tilde{s})}$ is a single term of the specified form. □

The following lemma shows that we can eventually find a logarithm of any positive element of K_{X-m}, after applying enough embeddings.

Lemma 4.4.4. *For all $s \in (K_{X-m})^{>0}$, there is $l \in \mathbb{N}$ and $s' \in K_{X-m-l-1}$ such that*

$$(\varphi_{m+l} \circ \cdots \circ \varphi_m)(s) = e_{X-m-l-1}(s').$$

Proof. Let $\mathrm{Lm}(s) = e_{X-m}(\alpha + t) \cdot g \cdot \ell$ with $\alpha \in A_{X-m,n} \oplus \cdots \oplus A_{X-m,0}$, $t \in T_{X-m}$, $g \in G_m$, and $\ell \in \Lambda_m$. Let $x_1 > \cdots > x_p$ list the elements of X that appear in g or ℓ and let $a_j = \xi_{x_j}(g)$ for $j = 1, \ldots, p$. Then we can write

$$s = s_0 \cdot e_{X-m}(\alpha + t) \prod_{j=1}^{p} E(x_j)^{a_j}$$

with $\mathrm{Lv}(s_0) = \Gamma_{X-m,0,small}$. Let d be the largest derivative that appears in $s_0|_{\mathrm{Lv}(s_0)}$. Then

$$(\varphi_{m+d} \circ \cdots \circ \varphi_m) \left(e_{X-m}(\alpha + t) \prod_{j=1}^{p} E(x_j)^{a_j} \right) =$$
$$e_{X-m-d-1} \left((\varphi_{m+d} \circ \cdots \circ \varphi_m)(\alpha + t) + \sum_{j=1}^{p} a_j (e_{X-m-d-1})^{\circ(d+1)} \big(E(x_j - d - 1)\big) \right).$$

By Lemma 4.4.3, we can write

$$(\varphi_{m+d} \circ \cdots \circ \varphi_m)(s_0) = \left(c e_{X-m-d-1}(\alpha') \prod_{j=1}^{p} E'(x_j - d - 1)^{a_{j,1}+\cdots+a_{j,d}} \right) (1+\epsilon)$$

with $\mathrm{Supp}(\epsilon) < \Gamma_{X-m-d-1,0,small}$. Note that $c > 0$ since $s > 0$ and all φ_k are order preserving for $k \in \mathbb{N}$. So $\log c \in \boldsymbol{k}$ is defined, and we can express $(\varphi_{m+d} \circ \cdots \circ \varphi_m)(s) = e_{X-m-d}(s')$ where

$$s' = \log c + \alpha' + \sum_{j=1}^{p} (a_{j,1} + \cdots + a_{j,d}) \log E'(x_j - d - 1) + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \epsilon^k$$
$$+ (\varphi_{m+d} \circ \cdots \circ \varphi_m)(\alpha + t) + \sum_{j=1}^{p} a_j (e_{X-m-d-1})^{\circ(d+1)} \big(E(x_j - d - 1) \big).$$

Since $\mathrm{Supp}(\epsilon) < \Gamma_{X-m-d-1,0,small}$, $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \epsilon^k \in K_{X-m-d-1}$. □

Corollary 4.4.5. *The order induced in Lemma 4.1.6 on $\Gamma_{X,0}$ using $\widehat{\log}$ matches the order given by defining*

$$g \cdot \ell > e_{T_X}(t) \text{ if and only if } s' > \rho_{d+1}\big((\varphi_d \circ \cdots \circ \varphi_0)(t) \big)$$

where s' is such that $(\varphi_{m+d} \circ \cdots \circ \varphi_m)(g \cdot \ell) = e_{X-m-d-1}(s')$.

Proof. Find s' by applying Lemma 4.4.4 to $g \cdot \ell$. The sum $\sum_{j=1}^{p} a_j (e_{X-m-d-1})^{\circ(d+1)} \big(E(x_j - d - 1) \big) + \alpha'$ is an initial subsum of s' since $\alpha = t = 0$. Tracing through the proof of Lemma 4.4.4, α' arises from s_0, which is a single monomial. We can express $\mathrm{Lm}(\rho_0(s_0)) = \prod_{j=1}^{p} E_0(x_j - 1)^{b_j} E_1(x_j - 1)^{c_j}$. So tracing through the proof of Lemma 4.4.3,

$$\alpha' = \sum_{j=1}^{p} \big(b_j (e_{X-m-d-1})^{\circ d} (E(x_j - d - 1)) + c_j (e_{X-m-d-1})^{\circ(d-1)} (E(x_j - d - 1)) + \cdots \big)$$

where the $+\cdots$ contains lesser terms. Then

$$\sum_{j=1}^{p} \big(a_j (e_{X-m-d-1})^{\circ(d+1)} (E(x_j - d - 1)) +$$
$$b_j (e_{X-m-d-1})^{\circ d} (E(x_j - d - 1)) + c_j (e_{X-m-d-1})^{\circ(d-1)} (E(x_j - d - 1)) \big)$$

exactly matches $\widehat{\log}(\rho_0(g \cdot \ell))$ if we identify $E_0(x_j - n)$ with $(e_{X-m-d-1})^{\circ(d+1-n)}(E(x_j - d - 1))$ for $n = 0, 1, 2$. □

Lemma 4.4.6. *If $s \in K_{X-m}$ and $e_{X-m}(s)$ is defined, so are $e_{X-m}\big(\pm e_{X-m}(s) \big)$.*

Proof. Suppose $s \in B^*_{X-m,n}$, so we can write $s = a+t+c+\epsilon$ with $\mathrm{Supp}(a) > \Gamma_{X-m,0,small}$ (or $a = 0$), $t \in T_{X-m}$, $c \in \boldsymbol{k}$, and $\mathrm{Supp}(\epsilon) < \Gamma_{X-m,0,small}$ (or $\epsilon = 0$). Note that $e_{X-m}(a+t)\epsilon^k \notin \Gamma_{X-m,0,small}$ for all $k \in \mathbb{N}$. So we can separate

$$e_{X-m}(a+t+c+\epsilon) = e_{X-m}(c)\sum_{k=0}^{\infty}\frac{e_{X-m}(a+t)\epsilon^k}{k!} = a' + \epsilon'$$

with $\mathrm{Supp}(a') > \Gamma_{X-m,0,small}$ and $\mathrm{Supp}(\epsilon') < \Gamma_{X-m,0,small}$. So $e_{X-m}(a+t+c+\epsilon) \in B^*_{X-m,n+1}$. The same argument shows $e_{X-m}(-a-t-c-\epsilon) \in B^*_{X-m,n+1}$. □

Corollary 4.4.7. *For all $s \in K_{X-m}$, there is $l \in \mathbb{N}$ such that*

$$e_{X-m-l-1}\big((\varphi_{m+l} \circ \cdots \circ \varphi_m)(s)\big)$$

is defined.

Proof. Let $s \in K_{X-m}$, and write $s = u|s|$ with $u = \pm 1$. By Lemma 4.4.4, we know there is some $l \in \mathbb{N}$ and $s' \in K_{X-m-l-1}$ such that

$$(\varphi_{m+l} \circ \cdots \circ \varphi_m)(|s|) = e_{X-m-l-1}(s').$$

By Lemma 4.4.6, $e_{X-m-l-1}\big(u \cdot e_{X-m-l-1}(s')\big)$ is defined. So

$$\begin{aligned} e_{X-m-l-1}\big(u \cdot e_{X-m-l-1}(s')\big) &= e_{X-m-l-1}\big(u(\varphi_{m+l} \circ \cdots \circ \varphi_m)(|s|)\big) \\ &= e_{X-m-l-1}\big((\varphi_{m+l} \circ \cdots \circ \varphi_m)(s)\big) \end{aligned}$$

is defined. □

Lemma 4.4.8. *Let $s_1, \ldots, s_n \in K_{X-m}$ with $\mathrm{Supp}(s_i) < 1$. Then there is $l \in \mathbb{N}$ such that*

$$\sum_{j_1,\ldots,j_n=0}^{\infty} a_{j_1,\ldots,j_n}\prod_{i=1}^{n}(\varphi_{m+l} \circ \cdots \circ \varphi_m)(s_i)^{j_i} \in K_{X-m-l-1}$$

for any $(a_{j_1,\ldots,j_n})_{j\in\mathbb{N}^n} \in (\boldsymbol{k}_\alpha)^\omega$, for any $\alpha \in \mathcal{A}$.

Proof. It suffices to show that for some $l \in \mathbb{N}$,

$$\mathrm{Supp}\big((\varphi_{m+l} \circ \cdots \circ \varphi_m)(s_i)\big) < \Gamma_{X-m-l-1,0,small}$$

for all $i = 1, \ldots, n$. If $\mathrm{Supp}\big((\varphi_{m+l} \circ \cdots \circ \varphi_m)(s_i)\big) < \Gamma_{X-m-l-1,0,small}$ for some l, then $\mathrm{Supp}\big((\varphi_{m+l'} \circ \cdots \circ \varphi_m)(s_i)\big) < \Gamma_{X-m-l'-1,0,small}$ for all $l' > l$. So we will find l_i that works for i and take $l = \max(l_1, \ldots, l_n)$.

If $\mathrm{Supp}(s_i) < \Gamma_{X-m,0,small}$, then we can use $l = 0$, so we may assume $\mathrm{Lv}(s_i) = \Gamma_{X-m,0,small}$. Let d_i be the largest derivative that appears in $s_i|_{\mathrm{Lv}(s_i)}$. Let $x_1 > \cdots > x_p$ list the elements of X that appear in $s_1|_{\mathrm{Lv}(s_1)}, \dots, s_n|_{\mathrm{Lv}(s_n)}$. Let $\widetilde{s_i} = (\varphi_{m+d} \circ \cdots \circ \varphi_m)(s_i)$. Since $\mathrm{Supp}(s_i) < 1$, we must have $\mathrm{Supp}(\widetilde{s_i}) < 1$. By Lemma 4.4.3, we can write

$$\widetilde{s_i}|_{\mathrm{Lv}(\widetilde{s_i})} = c_i e_{X-m-d-1}(\alpha_i) \prod_{j=1}^{p} E'(x_j - d - 1)^{a_{i,j}}$$

with $\alpha_i \in A_{X-m-d-1,0} \oplus \cdots \oplus A_{X-m-d-1,d}$. No monomial of this form can be in $\Gamma_{X-m-d-1,0,small}$, so we must have $\mathrm{Supp}(\widetilde{s_i}) < \Gamma_{X-m-d-1,0,small}$. □

Corollary 4.4.9. *Let $s_1, \dots, s_n \in K_{X-m}$ with $\mathrm{Supp}(s_i) < 1$. Let $f \in \mathbb{R}\{X_1, \dots, X_n\}$, and let $(r_1, \dots, r_n) \in \boldsymbol{k}_\alpha \cap [-1,1]^n$ for some $\alpha \in \mathcal{A}$. Then there is $l \in \mathbb{N}$ such that*

$$f\big(r_1 + (\varphi_{m+l} \circ \cdots \circ \varphi_m)(s_1), \dots, r_n + (\varphi_{m+l} \circ \cdots \circ \varphi_m)(s_n)\big) \in K_{X-m-l-1}.$$

Proof. The coefficient of any monomial

$$\prod_{i=1}^{n} (\varphi_{m+l} \circ \cdots \circ \varphi_m)(s_i)^{j_i}$$

can be shown to converge because f converges in a neighborhood of $[-1,1]^n$. □

Corollary 4.4.10. *For all $s \in K_{X-m}$, there is $l \in \mathbb{N}$ such that $(\varphi_{m+l} \circ \cdots \circ \varphi_m)(s)$ has a multiplicative inverse in $K_{X-m-l-1}$.*

Proof. We can write $s = \mathrm{Lm}(s)(1+\epsilon)$, with $\mathrm{Supp}(\epsilon) < 1$. By Lemma 4.4.9, there is $l \in \mathbb{N}$ such that

$$\sum_{k=0}^{\infty} \big(-(\varphi_{m+l} \circ \cdots \circ \varphi_m)(\epsilon)\big)^k$$

is defined in $K_{X-m-l-1}$. Then

$$(\varphi_{m+l} \circ \cdots \circ \varphi_m)\left(\frac{1}{\mathrm{Lm}(s)}\right) \cdot \sum_{k=0}^{\infty} \big(-(\varphi_{m+l} \circ \cdots \circ \varphi_m)(\epsilon)\big)^k$$

the multiplicative inverse of $(\varphi_{m+l} \circ \cdots \circ \varphi_m)(s)$ in $K_{X-m-l-1}$. □

Definition 4.4.11. *Let $D_X(\boldsymbol{k})$ be the direct limit of the directed system*

$$(K_{X-m}, \varphi_{m+l} \circ \cdots \circ \varphi_m)_{l,m\in\mathbb{N}}.$$

Corollary 4.4.12. *$D_X(\boldsymbol{k})$ can be made into a model of $T_{\mathrm{an}}(\exp, \log)$.*

Proof. $D_X(\boldsymbol{k})$ is an ordered ring that is closed under log by Lemma 4.4.4, closed under exp by Lemma 4.4.7, closed under restricted analytic functions by Corollary 4.4.9, and is a field by Lemma 4.4.10. So $D_X(\boldsymbol{k})$ can be made into a model of $T_{\mathrm{an}}(\exp, \log)$. □

Lemma 4.4.13. *For all $s \in D_X(\boldsymbol{k})_{+\infty}$ and $x \in X$, there are $n_1, n_2 \in \mathbb{N}$ such that*

$$\log^{\circ n_1} E(x) < s < \exp^{\circ n_2} E(x).$$

Proof. Identify s with a corresponding element s' of some $K_{X-m,n}$, and let $x \in X$. Then $(e_{X-m})^{\circ k}\big(E(x-m)\big) \in A_{X-m,k}$ for all $k \in \mathbb{N}$. So

$$s' < (e_{X-m})^{\circ(n+1)}\big(E(x)\big) \in e_{X-m}(A_{X-m,n})$$

and thus $s' < \exp^{\circ(n+1-m)} E(x)$ in $D_X(\boldsymbol{k})$.

Now we must find n_1 such that

$$\log^{\circ n_1} E(x) < s.$$

As in the proof of Lemma 4.4.4, write

$$s' = s_0 \cdot e_{X-m}(\alpha + t) \prod_{j=1}^{p} E(x_j - m)^{a_j}$$

with $\alpha \in A_{X-m,n} \oplus \cdots \oplus A_{X-m,0}$, $t \in T_{X-m}$, and $\mathrm{Lv}(s_0) = \Gamma_{X-m,0,small}$. Since s is positive and infinite, so is s'.

We now split into cases based on the form of $\mathrm{Lm}(s')$. First, if $\alpha \neq 0$, then we must have $\alpha > 0$ since s' is positive and infinite. Then

$$s' > E(x-m) \in K_{X-m,0}$$

so $s > \log^{\circ m} E(x)$.

Second, if $\alpha = 0$ and $e_{x-m}(t) \prod_{j=1}^{p} E(x_j - m)^{a_j} \neq 1$, then we must have

$$e_{x-m}(\alpha + t) \prod_{j=1}^{p} E(x_j - m)^{a_j} > 1$$

again since s' is infinite. Let y be the smallest element of X appearing in $\mathrm{Lm}(t)$ (or $y = +\infty$ if $t \neq 0$), and let $x_* = \min(y, x_p)$. We can find $0 < a \in \boldsymbol{k}$ such that

$$E(x_* - m)^a < e_{x-m}(\alpha + t) \prod_{j=1}^{p} E(x_j - m)^{a_j}.$$

If $t \neq 0$ or if $p > 1$, then $a = 1$ works. If $t = 0$ and $x_* = x_1$, then let $a = \frac{|a_1|}{2}$. Now since $\mathrm{Lv}(s_0) = \Gamma_{X-m,0,small}$, we must have $s_0 > E(x_* - m)^{-a/2}$. So

$$s' > E(x_* - m)^a \cdot E(x_* - m)^{-a/2} = E(x_* - m)^{a/2}.$$

And thus $s > E(x_* - m)^{a/2} > E(x - m - 1) = \log^{\circ(m+1)} E(x)$.

Third, suppose $s' = s_0$. Let d be the largest derivative appearing in $s_0|_{\mathrm{Lv}(s_0)}$. Let $\widetilde{s_0} = (\varphi_{m+d} \circ \cdots \circ \varphi_m)(s_0)$, which must be positive and infinite since s' is. By Lemma 4.4.3, we can write

$$\widetilde{s_0}|_{\mathrm{Lv}(\widetilde{s_0})} = c e_{X-m-d-1}(\alpha) \prod_{j=1}^{p} E'(x_j - d - 1)^{a_j}$$

with $c > 0$, $\alpha \in A_{X-m-d-1,0} \oplus \cdots \oplus A_{X-m-d-1,d}$, and $a_j \in \boldsymbol{k}$. Since $\widetilde{s_0} \in K_{X-m-d-1}$ is of a form handled by either the first or second case, we get either $\widetilde{s_0} > E(x - m - d - 1)$ or $\widetilde{s_0} > E(x_* - m - d - 1)^{a/2}$. We have

$$s > E(X - m - d - 2) = \log^{\circ(m+d+2)} E(x)$$

in either case. □

Chapter 5

An $\mathcal{L}_{\text{transexp}}$ differential series field

Let $F \models T_{\text{transexp}}$. We will build an increasing sequence $(H_i : i \in \mathbb{N})$ of ordered log-exp differential fields, starting with $H_0 = F((\tau^{-1}))^{le}$ where we take $\tau > F$ for the ordering. We will build this sequence so that

$$M_F := \bigcup_{i \in \mathbb{N}} H_i$$

is a $\mathcal{L}_{\text{an}}(\exp, \log)$-structure closed under E, its derivatives $E^{(d)}$, and their functional inverses. We will then define a derivation on M_F that works like differentiation with respect to τ. Each H_{i+1} will be constructed from H_i using the constructions in Sections 4.1 and 4.2 to add new monomials for E, its derivatives, and L applied to certain elements of H_i.

We do not need to create new monomials for E, its derivatives, and L applied to *all* elements of H_i. For example, there is no need to add new monomials for E composed with elements in the same $\mathbb{Z}$-orbit—if we include $E(x)$ when building H_1, then we automatically have expressions that represent $E(x+k)$, $k \in \mathbb{Z}$. Also, if $x, y \in H_i$ are "too close" in the sense that $x > y$ but $E(x) < E(y)^a$ for some a, then we cannot add both $E(x)$ and $E(y)$ as new elements over a field of coefficients containing a because the separation assumptions of Remark 3.1.2 would not be satisfied. This will not be an issue because we will be able to express $E(x)$ and $E(y)$ in terms of each other.

5.1 Constructing H_{i+1} from H_i

Suppose $H_0, \ldots, H_i$ have been constructed. If $i > 0$, assume we have also constructed order-preserving embeddings $\iota_j : H_j \to H_{j+1}$ for $j = 0, \ldots, i-1$ such that if $j > 0$, $z \in H_{j-1}$, and $E(z)$ is defined in H_j, then

$$\iota_j\big(E(z)\big) = E\big(\iota_{j-1}(z)\big).$$

For each $j = 0, \ldots, i$, let

$$\operatorname{Fin}(H_j) = \big\{ f \in H_j : \exists n \in \mathbb{N}\big(|f| \leq n\big) \big\}.$$

Since $\text{Fin}(H_j)$ is a convex subgroup of H_j, the quotient $H_j/\text{Fin}(H_j)$ inherits an order from H_j. For each $j = 1, \ldots, i-1$, $\iota_j(H_j)/\text{Fin}\left(\iota_j\left(H_j\right)\right)$ is a (non-convex) subgroup of $H_{j+1}/\text{Fin}\left(H_{j+1}\right)$.

Definition 5.1.1. *If $s \in H_{j+1}$ and $s > F$, define $\lambda(s)$ to be the unique (if it exists) one of*

1. $\beta = \left\{E\left(L^{\circ(j+2)}(\tau)\right)\right\}$ *or*
2. $\beta \in H_j/\text{Fin}(H_j)$

such that for some $z \in \beta$ and $n \in \mathbb{N}$, $E(z \pm n)$ are defined in H_{j+1} and

$$E(z-n) < s < E(z+n).$$

Suppose also that for each $s \in (H_{j+1})_{>F}$ for $j = 0, \ldots, i-1$, $\lambda(s)$ exists. We will prove by induction in Lemma 5.1.9 that $\lambda(s)$ exists for all $s \in (H_{i+1})_{>F}$.

The goal of this section is to construct H_{i+1} from H_i as follows: Any coset $F < \alpha \in H_i/\text{Fin}(H_i)$ viewed as a subset of H_i consists only of positive elements of H_i that are infinite relative to F. For each such α, we will define $X_\alpha \subseteq \alpha$ and create new monomials $E^{(d)}(x)$ and $\log E'(x)$ for all $x \in X_\alpha$ and $d \in \mathbb{N}$. We will do this by building a field $C_{\bar{\alpha},i} \vDash T_{\text{an}}(\exp, \log)$ for each finite increasing sequence

$$\bar{\alpha} = \{\alpha_0 < \alpha_1 < \cdots < a_k\} \subset \left(H_i/\text{Fin}(H_i)\right)_{>F}$$

using the constructions in Sections 4.1 and 4.2. We will then define H_{i+1} to be the direct limit of $\left(C_{\bar{\alpha},i} : \alpha \in (H_i/\text{Fin}(H_i))_{>0}\right)$, and show that for each infinite (relative to F) $s \in H_{i+1}$, $\lambda(s)$ exists. Finally, we will define an order preserving embedding $\iota_i : H_i \to H_{i+1}$.

Constructing $C_{\bar{\alpha},i}$

First suppose $\bar{\alpha}$ is the empty sequence. We will build $C_{\varnothing,i}$ in two steps. First, let $X_i' = \{L^{\circ(i+2)}(\tau)\}$, a single element set. Note that X_i' and F trivially satisfy the ordering and separation assumptions of Remark 3.1.2. Since $F \vDash T_{\text{an}}(\exp, \log)$, we can build $D_{X_i'}(F) \vDash T_{\text{an}}(\exp, \log)$, using the one-element family $\{F\}$ of subfields of F.

Second, let $X_i = \{L^{\circ(i+1)}(\tau)\}$, another single element set. We would like to define $C_{\varnothing,i} := D_{X_i}(D_{X_i'}(F))$, so we must check that X_i and $D_{X_i'}(F)$ satisfy the ordering and separation assumptions of Remark 3.1.2. Most of the assumptions are satisfied trivially, but we must show that

$$E\left(L^{\circ(i+1)}(\tau) - m\right) > D_{X_i'}(F)$$

for all $m \in \mathbb{N}$. We have not yet defined how $E\left(L^{\circ(i+1)}(\tau) - m\right)$ compares to elements of $D_{X_i'}(F)$, but by Lemma 4.4.13, any element $s \in D_{X_i'}(F)$ is bounded above by

$$\exp^{\circ n} E\left(L^{\circ(i+2)}(\tau)\right) = \exp^{\circ n} L^{\circ(i+1)}(\tau)$$

for some $n \in \mathbb{N}$. Since we want $E > \exp^{\circ l}$ for any $l \in \mathbb{N}$, we extend the ordering so that

$$\begin{aligned} E\big(L^{\circ(i+1)}(\tau) - m\big) &= \log^{\circ m} E\big(L^{\circ(i+1)}(\tau)\big) \\ &> \log^{\circ m}\left(\exp^{\circ(n+m)}(L^{\circ(i+1)}(\tau))\right) \\ &= \exp^{\circ n}\left(L^{\circ(i+1)}(\tau)\right) \\ &> s \end{aligned}$$

for all $m \in \mathbb{N}$. Since the assumptions of Remark 3.1.2 are now satisfied, we may take

$$C_{\varnothing,i} := D_{X_i}(D_{X_i'}(F)) \vDash T_{\text{an}}(\exp, \log)$$

Again using the one-element family $\{D_{X_i'}(F)\}$ of subfields of $D_{X_i'}(F)$.

Next, we will define the sets $C_{\bar{\alpha},i}$ for $\bar{\alpha} = \{\alpha_0 < \cdots < \alpha_k\} \subset \big(H_i/\text{Fin}(H_i)\big)_{>0}$ by induction, again using the construction $D_X(\boldsymbol{k})$ from Section 4.2. To define $C_{\{\alpha\},i}$ for a singleton $\{\alpha\}$, we use $\boldsymbol{k} = C_{\varnothing,i}$ as the field of coefficients, and the collection $(\boldsymbol{k}_\alpha)_{\alpha\in\mathcal{A}} = \{C_{\varnothing,i}\}$, which we call $\kappa_{\varnothing,i}$. To define $C_{\bar{\alpha},i}$ for $\bar{\alpha} = \{\alpha_0 < \cdots < \alpha_{k-1} < \alpha\}$, we use $\boldsymbol{k} = C_{\{\alpha_0<\cdots<\alpha_{k-1}\},i}$ as the field of coefficients. Let

$$\begin{aligned} \kappa_{\{\alpha_0<\cdots<\alpha_{k-1}\},i} &= \big\{D_X(\boldsymbol{k}) : X \subset X_{\alpha_{k-1}} \text{ finite, } \boldsymbol{k} \in \kappa_{\varnothing,i}\big\} && \text{if } k = 1 \\ \kappa_{\{\alpha_0<\cdots<\alpha_{k-1}\},i} &= \big\{D_X(\boldsymbol{k}) : X \subset X_{\alpha_{k-1}} \text{ finite, } \boldsymbol{k} \in \kappa_{\{\alpha_0<\cdots<\alpha_{k-2}\},i}\big\} && \text{if } k > 1. \end{aligned}$$

We will use $\kappa_{\{\alpha_0<\cdots<\alpha_{k-1}\},i}$ as the family of subsets of $C_{\{\alpha_0<\cdots<\alpha_{k-1}\},i}$ in the construction $C_{\bar{\alpha},i} = D_{X_{\alpha_k}}(C_{\{\alpha_0<\cdots<\alpha_{k-1}\},i})$ from Section 4.2. In both these cases, we will use the same set $X = X_{\alpha_k} \subset \alpha_k$ to build the monomials.

We define X_α for $\alpha \in \big(H_i/\text{Fin}(H_i)\big)_{>F}$ as follows:

Definition 5.1.2. *If $i > 0$ and $\alpha \in \big(\iota_{i-1}(H_{i-1})/\text{Fin}(\iota_{i-1}(H_{i-1}))\big)_{>F}$, then let*

$$X'_\alpha = \iota_{i-1}\big(X_{(\iota_{i-1})^{-1}(\alpha)}\big).$$

If $i > 0$ and $E\big(L^{\circ(i+1)}\big) \in \alpha$, let $X'_\alpha = \big\{E\big(L^{\circ(i+1)}\big)\big\}$. Otherwise, let $X'_\alpha = \varnothing$.

Extend X'_α to a maximal set X_α of representatives from α satisfying the following conditions:

1. *The elements of X_α are not too close: Suppose $i > 0$, $x > y$, and $\frac{1}{x-y}$ is infinite. If*

$$\alpha \leq \lambda\left(\frac{1}{x-y}\right)$$

   then at most one of x, y is in X_α.

2. *The elements of X_α not too far apart: there exists $r : X_\alpha \times X_\alpha \to \mathbb{Q} \cap (0,1)$ such that for all $x, y \in X_\alpha$ with $x > y$, we have $x - y < r(x, y)$*

Remark 5.1.3. The intuition for why the first condition means x and y are "not too close" comes from its equivalence via Lemma 5.1.6 to the ordering and separation assumption from Remark 3.1.2 requiring that $E(x-m) > E(y-m)^a$ whenever $x > y$, $m \in \mathbb{N}$, and a is in the field $\boldsymbol{k}$ of coefficients and exponents. We use the equivalent condition here to make it clear that the same set X_α will satisfy the ordering and separation assumptions regardless of which field of coefficients we pair it with.

Remark 5.1.4. The cases used to define X'_α in Definition 5.1.2 are disjoint. Every infinite monomial in H_{i-1} is bounded below by $\log^{\circ l}(\tau)$ for some $l \in \mathbb{N}$ if $i = 1$, and by

$$\log^{\circ l} E\big(L^{\circ i}(\tau)\big) \in D_{X'_{i-2}}(F)$$

for some $l \in \mathbb{N}$ if $i > 1$ by Lemma 4.4.13. If $E\big(L^{\circ(i+1)}\big) \in \alpha$, then

$$\alpha < \big(\iota_{i-1}(H_{i-1})/\operatorname{Fin}(\iota_{i-1}(H_{i-1}))\big)_{>F}.$$

So the different definitions of X'_α do not conflict.

If $i > 0$ and $\alpha \in \iota_{i-1}(H_{i-1})/\operatorname{Fin}(\iota_{i-1}(H_{i-1}))$, we must check that X'_α satisfies the two conditions in Definition 5.1.2, so that it is possible to extend it to a maximal set X_α that satisfies these conditions.

1. To see that X'_α satisfies Condition (1), let $x > y \in X'_\alpha$ and suppose $\frac{1}{x-y}$ is infinite.

   If $i = 1$, then we must have $\iota_0^{-1}\left(\frac{1}{x-y}\right) \in H_0$. Then

   $$\lambda\left(\frac{1}{x-y}\right) = \big\{E\big(L(L(\tau))\big)\big\}$$

   since H_0 is exponentially bounded and we identify $E\big(E(L(L(\tau))) + n\big)$ with $\exp^{\circ n}(\tau)$ for all $n \in \mathbb{Z}$. So we have $\alpha > \lambda\left(\frac{1}{x-y}\right)$.

   If $i > 1$, then $(\iota_{i-1})^{-1}(x), (\iota_{i-1})^{-1}(y) \in X_{(\iota_{i-1})^{-1}(\alpha)}$ means that

   $$(\iota_{i-2})^{-1}(\alpha) > \lambda\left(\frac{1}{(\iota_{i-1})^{-1}(x-y)}\right).$$

   Since $\iota_{i-1}\big(E(z)\big) = E\big(\iota_{i-2}(z)\big)$, we must have $\iota_{i-2}\big(\lambda(s)\big) = \lambda\big(\iota_{i-1}(s)\big)$ because

   $$E(z-n) < s < E(z+n) \text{ if and only if } \iota_{i-1}\big(E(z-n)\big) < \iota_{i-1}(s) < \iota_{i-1}\big(E(z+n)\big)$$
   $$\text{i.e., } E\big(\iota_{i-2}(z) - n\big) < \iota_{i-1}(s) < E\big(\iota_{i-2}(z) - n\big).$$

   So $\alpha > \iota_{i-2}\left(\lambda\left(\frac{1}{(\iota_{i-1})^{-1}(x-y)}\right)\right) = \lambda\left(\frac{1}{x-y}\right)$.

2. Since $X_{(\iota_{i-1})^{-1}(\alpha)}$ satisfies Condition (2) and ι_{i-1} is order preserving, X'_α satisfies Condition (2).

Since X'_α satisfies Conditions (1) and (2) in Definition 5.1.2, it can be extended to a maximal set X_α that satisfies the two conditions.

In order to define

$$\begin{aligned} C_{\{\alpha_0\},i} &:= D_{X_{\alpha_0}}(C_{\varnothing,i}) && \{\alpha_0\} \text{ a singleton} \\ C_{\bar{\alpha},i} &:= D_{X_{\alpha_k}}(C_{\{\alpha_0<\cdots<\alpha_{k-1}\},i}) && \bar{\alpha} = \{\alpha_0 < \cdots < \alpha_k\}, k > 0 \end{aligned}$$

we must check that the ordering and separation assumptions of Remark 3.1.2 are satisfied in each case. Any X_α is a subset of an ordered field, and each $C_{\varnothing,i}$ and $C_{\{\alpha_0<\cdots<\alpha_{k-1}\},i}$ is constructed to be an ordered exponential field. Note that Condition (2) in Definition 5.1.2 of the sets X_α matches the final ordering and separation assumption of Remark 3.1.2. So all that remains to show are assumptions (3a) and (3b).

First, we will check that for all $x \in X_{\alpha_k}$ and $m \in \mathbb{N}$, we have

$$\begin{aligned} E(x-m) &> C_{\varnothing,i} && \text{if } k = 0 \\ E(x-m) &> C_{\{a_0<\cdots<a_{k-1}\},i} && \text{if } k > 0. \end{aligned}$$

Lemma 5.1.5. *Let $k \in \mathbb{N}$, $\bar{\alpha} = \{\alpha_0 < \cdots < \alpha_k\}$, and $x \in X_{\alpha_k}$.*

1. *If $k = 0$, then $E(x-m) > C_{\varnothing,i}$ for all $m \in \mathbb{N}$.*
2. *If $k > 0$, then $E(x-m) > C_{\{a_0<\cdots<a_{k-1}\},i}$ for all $m \in \mathbb{N}$.*

Proof. If $k = 0$, let $s \in C_{\varnothing,i}$ and $z = L^{\circ(i+1)}(\tau)$. Since $x \in X_{\alpha_0} \subset \alpha_0 \in \big(H_i/\mathrm{Fin}(H_i)\big)_{>F}$, x is positive and infinite. The smallest positive infinite elements of H_i come from $D_{X'_{i-1}}(F)$. Since $z \in X'_{i-1}$, by Lemma 4.4.13 there is some $l \in \mathbb{N}$ such that

$$x > \log^{\circ l} E(z).$$

Since $z \in X_i$, by Lemma 4.4.13 there is some $n \in \mathbb{N}$ such that

$$s < \exp^{\circ n} E(z).$$

We have not yet defined how $E(x-m)$ compares to elements of $C_{\varnothing,i}$, but since we want $E > \exp^{\circ k}$ for all $k \in \mathbb{N}$, we extend the ordering so that

$$\begin{aligned} E(x-m) &= \log^{\circ m} E(x) \\ &> \log^{\circ m} E\big(\log^{\circ l} E(z)\big) \\ &> \log^{\circ m} \exp^{\circ(n+m+l)} \big(\log^{\circ l} E(z)\big) \\ &= \exp^{\circ n} E(z) \\ &> s \end{aligned}$$

for all $m \in \mathbb{N}$.

If $k > 0$, let $s \in C_{\{a_0 < \cdots < a_{k-1}\},i}$. By Lemma 4.4.13, there is some $z \in X_{\alpha_{k-1}}$ and $n \in \mathbb{N}$ such that $s < \exp^{\circ n} E(z)$. We have not yet defined how $E(x-m)$ compares to elements of $C_{\{a_0 < \cdots < a_{k-1}\},i}$, but since $\alpha_k > \alpha_{k-1}$ in $H_i/\mathrm{Fin}(H_i)$, we know $x - m > z + n$ for any $m \in \mathbb{N}$. So we extend the ordering to have

$$E(x-m) > E(z+n) = \exp^{\circ n} E(z) > s$$

for all $m \in \mathbb{N}$. □

To finish checking the assumptions of Remark 3.1.2, we must show that

$$E(x-m) > E(y-m)^a$$

for all $m \in \mathbb{N}$, where

1. if $k = 0$, then $x, y \in X_{\alpha_0}$ with $x > y$ and $a \in C_{\varnothing,i}$, and
2. if $k > 0$, then $x, y \in X_{\alpha_k}$ with $x > y$ and $a \in C_{\{a_0 < \cdots < a_{k-1}\},i}$.

We prove this from the first condition on X_{α_k} from Definition 5.1.2.

Lemma 5.1.6. *Let $i \in \mathbb{N}$ and let $\beta > \bar{\alpha} \in (H_i/\mathrm{Fin}(H_i))_{>F}$, so that $E(x-m) > C_{\bar{\alpha},i}$ for all $x \in X_\beta$ and $m \in \mathbb{N}$. Then the following are equivalent:*

1. *X_β satisfies Condition (1) of Definition 5.1.2.*
2. *For $x, y \in X_\beta$ with $x > y$, $m \in \mathbb{N}$, and $a \in C_{\bar{\alpha},i}$, we have $E(x-m) > E(y-m)^a$.*

Proof. Let $x, y \in X_\beta$ with $x > y$. We will need to extend the partial order on expressions involving E, L, and log so that for any $m \in \mathbb{N}$ and any $a \in (C_{\bar{\alpha},i})_{>1}$,

$$\begin{aligned} E(x-m) > E(y-m)^a &\text{ if and only if } E(x-m-2) > E(y-m-2) + \log a \\ &\text{ if and only if } x - m - 2 > L\big(E(y-m-2) + \log a\big) \\ &\text{ if and only if } x - y > L\big(E(y-m-2) + \log a\big) - L\big(E(y-m-2)\big). \end{aligned}$$

Expanding $L\big(E(y-m-2) + \log a\big)$ using the Taylor series for L around $E(y-m-2)$ gives

$$\begin{aligned} L(E(y-m-2) + \log a) - L(E(y-m-2)) &= \sum_{j=1}^{\infty} \frac{L^{(j)}(E(y-m-2))(\log a)^j}{j!} \\ &= \frac{\log a}{E'(y-m-2)} + \frac{(\log a)^2 E''(y-m-2)}{2E'(y-m-2)^3} + \cdots \end{aligned}$$

which is a valid sum in the structure $D_{X'}(C_{\bar{\alpha},i})$ for any $X' \ni y$ such that X' and $C_{\bar{\alpha},i}$ satisfy the ordering and separation assumptions of Remark 3.1.2.

First suppose $\frac{1}{x-y} \in \text{Fin}(H_i)$. Then Condition (1) is trivially satisfied. Since there is some $n \in \mathbb{N}$ such that $\frac{1}{x-y} < n$, and since $\frac{E(y-m)}{\log a}$ is infinite, we have $\frac{1}{x-y} < \frac{E(y-m)}{\log a}$. By the computations above, this shows that $E(x-m) > E(y-m)^a$.

Now suppose $\frac{1}{x-y} \notin \text{Fin}(H_i)$. Let $z \in \lambda\left(\frac{1}{x-y}\right)$, and let $n \in \mathbb{N}$ be such that

$$E(z-n) < \frac{1}{x-y} < E(z+n).$$

If $E(x-m) \leq E(y-m)^a$ for some $m \in \mathbb{N}$ and $a \in (C_{\bar{\alpha},i})_{>1}$, then

$$E(z+n) > \frac{1}{x-y} > \frac{E'(y-m-2)}{2\log a} > E(y-m-2).$$

So $z+n > y-m-2$, and thus $\lambda\left(\frac{1}{x-y}\right) \geq \beta$. Conversely, if $\lambda\left(\frac{1}{x-y}\right) \geq \beta$, then there is some $m \in \mathbb{N}$ such that $z-n > y-m$. So

$$\frac{1}{x-y} > E(z-n) > E(y-m) > \frac{E'(y-m-2)}{2\log a}$$

and by the computations above, this means $E(x-m) \leq E(y-m)^a$. □

It follows from Lemma 5.1.6 that the ordering and separation assumptions in Remark 3.1.2 are satisfied in the cases $k=0$ and $k>0$. So we can define

$$C_{\{a_0\},i} := D_{X_{\alpha_m}}\left(C_{\varnothing,i}\right)$$
$$C_{\bar{\alpha},i} := D_{X_{\alpha_0}}\left(C_{\{a_0<\cdots<a_{k-1}\},i}\right) \text{ for } k>0.$$

Defining H_{i+1} and $\iota_i : H_i \to H_{i+1}$

We would like to define H_{i+1} as the direct limit of the sets $C_{\bar{\alpha},i}$, so we must show that these sets form a directed system ordered by inclusion.

Lemma 5.1.7. *Let $\bar{\alpha} = \{\alpha_0 < \cdots < \alpha_k\} \subset (H_i/\text{Fin}(H_i))_{>F}$, and let $\beta \in (H_i/\text{Fin}(H_i))_{>F}$ such that $\beta \neq \alpha_j$ for $j = 1, \ldots, k$. Then $C_{\bar{\alpha},i}$ is a substructure of $C_{\bar{\alpha}\cup\beta,i}$.*

Proof. If $\beta > \alpha_k$, then $C_{\bar{\alpha},i} \subset D_{C_{\bar{\alpha},i}}(X_\beta) = C_{\bar{\alpha}\cup\beta,i}$.

If $\beta < \alpha_0$, then

$$\begin{aligned} C_{\varnothing,i} \subset C_{\{\beta\},i} &\Rightarrow C_{\{\alpha_0\},i} \subset C_{\{\beta<\alpha_0\},i} \\ &\Rightarrow C_{\{\alpha_0<\alpha_1\},i} \subset C_{\{\beta<\alpha_0<\alpha_1\},i} \\ &\vdots \\ &\Rightarrow C_{\{\alpha_0<\cdots<\alpha_k\},i} \subset C_{\{\beta<\alpha_0<\cdots<\alpha_k\},i}. \end{aligned}$$

Similarly, if $\alpha_l < \beta < \alpha_{l+1}$, then

$$\begin{aligned} C_{\{\alpha_0<\cdots<\alpha_l\},i} \subset C_{\{\alpha_0<\cdots<\alpha_l<\beta\},i} &\Rightarrow C_{\{\alpha_0<\cdots<\alpha_{l+1}\},i} \subset C_{\{\alpha_0<\cdots<\alpha_l<\beta<\alpha_{l+1}\},i} \\ &\vdots \\ &\Rightarrow C_{\{\alpha_0<\cdots<\alpha_k\},i} \subset C_{\{\alpha_0<\cdots<\alpha_l<\beta<\alpha_{l+1}<\cdots<\alpha_k\},i} \end{aligned}$$

as claimed. □

Corollary 5.1.8. $\big(C_{\bar{\alpha},i} : \bar{\alpha} \in (H_i/\mathrm{Fin}(H_i))_{>F}\big)$ *forms a directed system.*

Build H_{i+1} from H_i as the direct limit of the directed system

$$\big(C_{\bar{\alpha},i} : \bar{\alpha} \in (H_i/\mathrm{Fin}(H_i))_{>F}\big).$$

H_{i+1} can be made into a model of $T_{\mathrm{an}}(\exp, \log)$ because each $C_{\bar{\alpha},i} \models T_{\mathrm{an}}(\exp, \log)$.

Lemma 5.1.9. *For all* $s \in (H_{i+1})_{+\infty}$*,* $\lambda(s)$ *exists.*

Proof. Let $s \in C_{\bar{\alpha},i}$ with $s > F$. We divide into several cases:

1. If s is bounded in $D_{X_i'}(F)$, then

   $$\lambda(s) := \big\{E\big(L^{\circ(i+2)}(\tau)\big)\big\}$$

   works, by Lemma 4.4.13.

2. If $s > D_{X_i'}(f)$ and is bounded in $D_{X_i}\big(D_{X_i'}(F)\big)$, then

   $$\lambda(s) := \beta$$

   for $E\big(L^{\circ(i+1)}(\tau)\big) \in X_\beta \subset \beta \subset H_i$ works, again by Lemma 4.4.13.

3. If $s > C_{\varnothing,i}$, then we may write $\bar{\alpha} = \{\alpha_0 < \cdots < \alpha_k\}$ for some $k \geq 0$. If s is bounded in $C_{\{\alpha_0\},i}$, then

   $$\lambda(s) := \alpha_0$$

   works, by Lemma 4.4.13.

4. Finally, if $s > C_{\{\alpha_0<\cdots<\alpha_l\},i}$ and is bounded in $C_{\{\alpha_0<\cdots<\alpha_{l+1}\},i}$ for some $l < k$, then use

   $$\lambda(s) := \alpha_{l+1}$$

   by Lemma 4.4.13.

So $\lambda(s)$ is defined for all $s > F$ in H_{i+1}. □

We will now define an order preserving embedding $\iota_i : H_i \to H_{i+1}$. If $i = 0$, we think of $\iota_0 : H_0 \to H_1$ as substituting $E(L(\tau))$ for τ. Let $s \in H_0 = F((\tau^{-1}))^{le}$. We follow the notation of [8], in which $F((\tau^{-1}))^e$ is the direct limit of $(K_n : n \in \mathbb{N})$, and $F((\tau^{-1}))^{le}$ is the direct limit of $(L_n : n \in \mathbb{N})$. We may identify s with a unique element of $C_{\varnothing,0} \subset H_1$ as follows:

1. If $s = \sum a_r \tau^r \in K_0$, then define $\iota_0(s) = \sum a_r E(L(\tau))^r$
2. If $s = \sum f_a e(a) \in K_{n+1}$, then define $\iota_0(s) = \sum \iota_0(f_a) e(\iota_0(a))$
3. For any $s \in L_n$, let $\hat{s} = \eta_n(s) \in F((\tau^{-1}))^e$, and express
$$s = \hat{s}\big(\log^{\circ n}(\tau)\big).$$
   a) If $\hat{s} = \sum a_r \tau^r \in K_0$, then define
   $$\iota_{0,n}(\hat{s}) = \sum a_r \log^{\circ n} E(L(\tau))^r.$$
   b) If $\hat{s} = \sum f_a e(a) \in K_{m+1}$, then define
   $$\iota_{0,n}(\hat{s}) = \sum \iota_{0,n}(f_a) e(\iota_{0,n}(a)).$$

   Define $\iota_0(s) = \iota_{0,n}(\hat{s})$.

Now suppose $i > 0$. We inductively define ι_i on the generators built from X'_{i-1}, X_{i-1}, and X_α for $\alpha \in \big(H_{i-1}/\mathrm{Fin}(H_{i-1})\big)_{>F}$. For all $d \in \mathbb{N}$ and $a \in F$, define
$$\iota_i\big(E^{(d)}(L^{\circ(i+1)}(\tau))^a\big) = E^{(d)}\big(L^{\circ(i+1)}(\tau)\big)^a \in C_{\varnothing,i}.$$
Since the image of each generator is a single generator, we can extend ι_i so that $\iota_i\big(D_{X'_{i-1}}(F)\big) = D_{X_i}(F) \subset C_{\varnothing,i}$.

For all $d \in \mathbb{N}$ and $a \in D_{X'_{i-1}}(F)$, define
$$\iota_i\big(E^{(d)}(L^{\circ i}(\tau))^a\big) = E^{(d)}\big(E(L^{\circ(i+1)}(\tau))\big)^{\iota_i(a)} \in C_{\{\beta\},i}$$
where β is such that $E(L^{\circ(i+1)}(\tau)) \in X_\beta$. Again, since the image of each generator is a single generator, we can extend ι_i to identify $C_{\varnothing,i-1}$ with an isomorphic copy of itself in $C_{\{\beta\},i}$.

Now assume we have defined ι_i on $C_{\bar{\alpha},i}$ for some $\bar{\alpha} \subset \big(H_{i-1}/\mathrm{Fin}(H_{i-1})\big)_{>F}$. Let $\beta > \bar{\alpha}$. For all $d \in \mathbb{N}$ and $a \in C_{\bar{\alpha},i}$, define
$$\iota_i\big(E^{(d)}(x)^a\big) = E^{(d)}\big(\iota_{i-1}(x)\big)^{\iota_i(a)}.$$
Again, we can extend ι_i to identify $D_{X_\beta}(C_{\bar{\alpha},i-1})$ with an isomorphic copy of itself in $C_{\iota_{i-1}(\bar{\alpha}\cup\{\beta\}),i}$.

Thus, we have identified H_i with an isomorphic copy of itself $\iota_i(H_i) \subset H_{i+1}$. We will often suppress the ι_j maps and identify H_j with this isomorphic copy in H_{i+n}.

5.2 Building a structure closed under the symbols of $\mathcal{L}_{\text{transexp}}$

Let M_F be the direct limit of the directed system given by (H_i, ι_i). Next we will show that M_F is closed under E, its derivatives, L, and the inverses of the derivatives of E.

Lemma 5.2.1. *For all $s \in (H_i)_{>F}$ and $d \in \mathbb{N}$, $E^{(d)}(s) \in H_{i+1}$.*

Proof. Let $s \in \beta \in (H_i/\text{Fin}(H_i))_{>F}$. If $s - l \in X_\beta$ for some $l \in \mathbb{Z}$, then $E^{(d)}(s) \in C_{\{\beta\},i}$ by construction.

If $s - l \notin X_\beta$ for any $l \in \mathbb{Z}$, then we must have $i > 0$, and each $s - l$ must have been excluded from X_β for a reason. Let $m \in \mathbb{N}$ and $x \in X_\beta$ be the unique elements such that $s - m$ and x violate Condition (1) and satisfy Condition (2). For ease of notation we may assume $m = 0$. Let $\beta > \alpha \in (H_i/\text{Fin}(H_i))_{>F}$. By Lemma 5.1.6, either $s > x$ and there is some $1 < a \in C_{\{\alpha\},i}$ and $n \in \mathbb{N}$ such that

$$E(s-n) \leq E(x-n)^a$$

or $x > s$ and there are a and n as above with

$$E(x-n) \leq E(s-n)^a.$$

Following the computation in Lemma 5.1.6, this means that

$$|s-x| < \frac{1}{E(x-n-3)}.$$

Let $k = n + 3$. Since $s - x \in H_i$, we have $\iota_i(s - x) \in C_{\bar{\alpha},i} \subset H_{i+1}$ for some $\bar{\alpha} \in H_i/\text{Fin}(H_i)$. So we can represent $E^{(l)}(s - k)$ by

$$\sum_{n=0}^{\infty} \frac{E^{(l+n)}(x-k)}{n!}(s-x)^n \in C_{\bar{\alpha}\cup\{\beta\},i}$$

for $l \in \mathbb{N}$. Then for $m = 0, \ldots, k-1$, we can represent $E(s - m)$ by

$$\exp^{\circ(k-m)}\left(\sum_{n=0}^{\infty} \frac{E^{(n)}(x-k)}{n!}(s-x)^n\right).$$

Finally, we express $E^{(d)}(s)$ in terms of the expressions we have found for $E^{(l)}(s - k)$ and $E(s - m)$, $l, m \in \mathbb{N}$, using the Bell polynomial difference-differential equations for E. □

We will use the following technical lemma to show that for all positive infinite (relative to F) elements $s \in H_i$, we can identify $L(s)$ with an element of H_{i+1}.

Lemma 5.2.2. *Suppose X and $\boldsymbol{k}$ satisfy the ordering and separation axioms of Remark 3.1.2. Then for any positive infinite (relative to $\boldsymbol{k}$) element $s \in D_X(\boldsymbol{k})$, there are $n_0, n_1, n_2 \in \mathbb{N}$ for which we can express $\log^{\circ n_0}(s)$ in the form*

$$\log^{\circ n_0}(s) = \exp^{\circ n_1} E(x - n_2) + s_0$$

with $\operatorname{Supp}\left(\frac{s_0}{\exp^{\circ n_1} E(x-n_2)}\right) < \Gamma_{X-n_2,0,small}$.

Proof. By Lemma 4.4.3, there is some $m \in \mathbb{N}$ for which s can be identified with an element $cM_0(1+\epsilon) \in K_{X-m}$ with $c > 0$, $\operatorname{Supp}(\epsilon) < \Gamma_{X-m,0,small}$, and M_0 of the form

$$e_{X-m}(\alpha) \prod_{j=1}^{p} E'(x_j - m)^{a_j}$$

with $\alpha \in A_{X-m,k}$ and $x_1 > \cdots > x_p \in X$. We will also call this element s. Then

$$\begin{aligned} \log s &= \log\big(cM_0(1+\epsilon)\big) \\ &= \alpha + \sum_{j=1}^{p} a_j \log E'(x_j - m) + \log c + \sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{l}\epsilon^l \\ &\in K_{X-m}. \end{aligned}$$

If $\alpha = 0$, then we must have $a_1 > 0$ since $s > F$. We can compute that

$$\log^{\circ 2} s = E(x_1 - m - 2) + \log a_1 + \log\left(\frac{\log s - a_1 E(x_1 - m - 1)}{a_1 E(x_1 - m - 1)}\right).$$

So $\log^{\circ 2} s \in K_{X-m-2}$, and $\operatorname{Supp}\left(\frac{\log^{\circ 2} s - E(x_1-m-2)}{E(x_1-m-2)}\right) < \Gamma_{X-m-2,0,small}$. So $n_0 = 2$, $n_1 = 0$, and $n_2 = m+2$ work in the case $\alpha = 0$.

If $0 \neq \alpha \in A_{X-m,0}$, then we can write

$$\alpha = s_1 \prod_{j=1}^{q} E(y_j - m)^{b_j}$$

with $b_1 > 0$ and $\operatorname{Lv}(s_1) = \Gamma_{X-m,0,small}$. By Lemma 4.4.4, there is some m' such that s_1 and $\log s_1$ have logarithms in $K_{X-m-m'}$. Then we can compute that

$$\operatorname{Lm}\Big(\log^{\circ 2}(\varphi_{m+m'-1} \circ \cdots \circ \varphi_m)(\log s)\Big) = (e_{X-m-m'})^{\circ(m'-2)}\big(E(y_1 - m - m')\big)$$

and $\operatorname{Supp}\left(\frac{\log^{\circ 2}(\varphi_{m+m'-1}\circ\cdots\circ\varphi_m)(\log s)}{(e_{X-m-m'})^{\circ(m'-2)}\left(E(y_1-m-m')\right)}\right) < \Gamma_{X-m-m',0,small}$. So we use $n_0 = 3$, $n_1 = m'-2$, and $n_2 = m+m'$.

If $0 \neq \alpha \in A_{X-m,n+1}$, then let $\alpha_{n+1} = \alpha$. There are $c_l, \alpha_l, \epsilon_l$ for $l = 0, \dots, n$ such that

$$\alpha_{l+1} = c_l e_{X-m}(\alpha_l)(1 + \epsilon_l)$$

with $c_l \in K_{X-m,l}$, $\alpha_l \in A_{X-m,l}$, and $\epsilon_l \in \mathfrak{m}\big(B_{X-m,l+1}\big)$. Write

$$\alpha_0 = s_1 \prod_{j=1}^{q} E(y_j - m)^{b_j}$$

with $b_1 > 0$ and $\mathrm{Lv}(s_1) = \Gamma_{X-m,0,small}$. Again by Lemma 4.4.4, there is some m' such that $s_1, \log c_l, \cdots, \log^{\circ l} c_l \in K_{X-m-m'}$ for all $l = 0, \dots, n$. Then

$$\mathrm{Lm}\Big(\log^{\circ(2+(n+1))}(\varphi_{m+m'-1} \circ \cdots \circ \varphi_m)(\log s)\Big) = (e_{X-m-m'})^{\circ(m'-2)}\big(E(y_1 - m - m')\big)$$

and $\mathrm{Supp}\left(\frac{\log^{\circ(2+(n+1))}(\varphi_{m+m'-1}\circ\cdots\circ\varphi_m)(\log s)}{(e_{X-m-m'})^{\circ(m'-2)}\left(E(y_1-m-m')\right)}\right) < \Gamma_{X-m-m',0,small}$. So we use $n_0 = 2+(n+1)+1 = n+4$, $n_1 = m'-2$, and $n_2 = m+m'$. □

Lemma 5.2.3. *For all $s \in (H_i)_{>F}$, $L(s) \in H_{i+1}$.*

Proof. First suppose $i = 0$, so that $s \in F((\tau^{-1}))^{le}$. Then there are some $n, k \in \mathbb{N}$ such that

$$\mathrm{Lm}\,(\log^{\circ n}(s)) = \log^{\circ k}(\tau).$$

So $\log^{\circ n}(s) = r \log^{\circ k}(\tau)(1 + \epsilon)$ with $r > 0$. Then we can represent $L(s)$ as follows:

$$\begin{aligned} L(s) &= L\big(\log^{\circ(n+1)}(s)\big) + n + 1 \\ &= L\big(\log^{\circ(k+1)}(\tau) + \log r + \log(1+\epsilon)\big) + n + 1 \\ &= L\big(\log^{\circ(k+1)}(\tau)\big) + \sum_{l=1}^{\infty} \frac{L^{(l)}\left(\log^{\circ(k+1)}(\tau)\right)}{l!}\big(\log r + \log(1+\epsilon)\big)^l + n + 1. \end{aligned}$$

Now $L\left(\log^{\circ(k+1)}(\tau)\right) = L(\tau)-k-1$, and differentiating, we get $L'\left(\log^{\circ(k+1)}(\tau)\right)\cdot\left(\log^{\circ(k+1)}(\tau)\right)' = L'(\tau)$. So

$$L'\left(\log^{\circ(k+1)}(\tau)\right) = \frac{L'(\tau)}{\left(\log^{\circ(k+1)}(\tau)\right)'} = L'(\tau)\log^{\circ k}(\tau)\cdots\log\tau\cdot\tau.$$

We can continue differentiating to obtain expressions for $L^{(l)}\left(\log^{\circ(k+1)}(\tau)\right)$ in terms of $L'(\tau), \dots, L^{(l)}(\tau)$ and $\log^{\circ k}(\tau), \dots, \log\tau, \tau$, all of which are elements of $C_{\varnothing,i}$. This allows us to finish expressing $L(s)$ as

$$\begin{aligned} L(s) =& n - k + L(\tau) + \big(L'(\tau)\log^{\circ k}(\tau)\cdots\log\tau\cdot\tau\big)\big(\log r + \log(1+\epsilon)\big) \\ &+ \Big(L''(\tau)\log^{\circ k}(\tau)\cdots\log\tau\cdot\tau + L'(\tau)\big(\log^{\circ k}(\tau)\cdots\log\tau\cdot\tau\big)'\Big)\big(\log r + \log(1+\epsilon)\big)^2 \\ &+ \cdots. \end{aligned}$$

Since the lth derivative of $\log^{\circ k}(\tau)\cdots\log\tau\cdot\tau$ is infinitesimal with τ^{l-1} in the denominator for $l>1$, this sum is an element of $C_{\varnothing,0}$. So $L(s)\in H_1$.

Now suppose $i>0$, and let $s\in(H_i)_{>F}$. Then we can identify s with an element, which we also call s, of $C_{\bar\alpha,i-1}$ for some $\bar\alpha\subset H_{i-1}/\mathrm{Fin}(H_{i-1})$. By Lemma 5.2.2, there are $n_0,n_1,n_2\in\mathbb{N}$ such that we can express

$$\log^{\circ n_0}(s)=\exp^{\circ n_1}E(x-n_2)+s_0$$

with $\operatorname{Supp}\left(\frac{s_0}{\exp^{\circ n_1}E(x-n_2)}\right)<\Gamma_{X-n_2,0,small}$ and $x\in X$, where X is either X_i', X_i, or X_α for some $\alpha\in H_{i-1}/\mathrm{Fin}(H_{i-1})$. So identifying $\exp^{\circ n_1}E(x-n_2)$ with $E(x+n_1-n_2)$, we can represent $L(s)$ by

$$\begin{aligned}
L(s)&=L\big(\log^{\circ n_0}(s)\big)+n_0\\
&=L\big(E(x+n_1-n_2)+s_0\big)+n_0\\
&=x+n_0+n_1-n_2+\sum_{l=1}^{\infty}\frac{L^{(l)}\big(E(x+n_1-n_2)\big)}{l!}(s_0)^l\\
&=x+n_0+n_1-n_2+\frac{s_0}{E'(x+n_1-n_2)}+\frac{(s_0)^2E''(x+n_1-n_2)}{E'(x+n_1-n_2)^3}+\cdots\\
&\in K_{X-n_2}.
\end{aligned}$$

So we have identified $L(s)$ with an element of H_{i+1}. □

We will use the next two lemmas to show that for all positive infinite (relative to F) elements $s\in H_i$, we can represent $\left(E^{(d)}\right)^{-1}(s)$ by an element of H_{i+3}.

Lemma 5.2.4. *For any infinite $x\in H_i$ and $0<a\in H_{i+1}$ such that $\frac{\log a}{E(x-1)}$ is infinitesimal, we can compute $\epsilon_{a,d}(x)\in H_{i+1}$ such that*

$$E(x+\epsilon_{a,d}(x))=aE^{(d)}(x)$$

in H_{i+2}.

Proof. Note that $E(x+\epsilon_{a,d}(x))=aE^{(d)}(x)$ if and only if (taking log twice)

$$E(x-2+\epsilon_{a,d}(x))=E(x-2)+\log\left(1+\frac{\log B_d(x-1)+\log a}{E(x-1)}\right).$$

Let $C(x)=\log\left(1+\frac{\log B_d(x-1)+\log a}{E(x-1)}\right)$, an infinitesimal. Now

$$\begin{aligned}
E\big(x-2+\epsilon_{a,d}(x)\big)&=E(x-2)+C(x) &&\text{if and only if}\\
E\big(L(x)+\epsilon_{a,d}(L(x)+2)\big)&=x+C\big(L(x)+2\big) &&\text{if and only if}\\
L(x)+\epsilon_{a,d}\big(L(x)+2\big)&=L\big(x+C(L(x)+2)\big).
\end{aligned}$$

Rearranging terms, we get

$$\epsilon_{a,d}\big(L(x)+2\big) = L\big(x + C(L(x)+2)\big) - L(x) = \sum_{n=1}^{\infty} \frac{C\big(L(x)+2\big)^n L^{(n)}(x)}{n!}.$$

And substituting $E(x-2)$ for x, we get

$$\epsilon_{a,d}(x) = \frac{C(x)}{E'(x-2)} - \frac{C(x)^2 E''(x-2)}{2!E'(x-2)^3} + \dots$$

which is an element of K_{X-2}. □

Lemma 5.2.5. *For all $s \in (H_i)_{>F}$ and $d \in \mathbb{N}$, $\big(E^{(d)}\big)^{-1}(s) \in H_{i+3}$.*

Proof. We will find $f \in H_{i+2}$ such that

$$0 \le \frac{s}{E^{(d)}(f)} - 1 \le \frac{1}{E(f-1)^{1/2}}.$$

Let $\mu = \frac{s}{E^{(d)}(f)} - 1$. This suffices to prove the Lemma because then

$$\begin{aligned}
\big(E^{(d)}\big)^{-1}(s) &= \big(E^{(d)}\big)^{-1}\left(E^{(d)}(f) + \mu E^{(d)}(f)\right) \\
&= \sum_{n=0}^{\infty} \frac{\left(\big(E^{(d)}\big)^{-1}\right)^{(n)}\left(E^{(d)}(f)\right)}{n!}\left(\mu E^{(d)}(f)\right)^n \\
&= f + \frac{\mu E^{(d)}(f)}{E^{(d+1)}(f)} - \frac{\mu^2 E^{(d)}(f)^2 E^{(d+2)}(f)}{2!E^{(d+1)}(f)^3} + \dots \\
&= f + \frac{\mu B_d(f-1)}{B_{d+1}(f-1)} - \frac{\mu^2 B_d(f-1)^2 B_{d+2}(f-1)}{2!B_{d+1}(f-1)^3} + \dots \\
&= f + \mu \frac{B_d(f-1)}{E'(f-1)^{d+1}} \sum_{n=0}^{\infty} \left(1 - \frac{B_{d+1}(f-1)}{E'(f-1)^{d+1}}\right)^n \\
&\quad - \mu^2 \frac{B_d(f-1)^2 B_{d+2}(f-1)}{2!E'(f-1)^{3d+3}} \sum_{n=0}^{\infty} \left(1 - \frac{B_{d+1}(f-1)^3}{E'(f-1)^{3d+3}}\right)^n.
\end{aligned}$$

We will show the final sum is an element of H_{i+3} if $f \in H_{i+2}$. We must show it is a valid sum in some $C_{\bar{\alpha},i+2}$ containing both s and $E^{(d)}(f)$. Let $\bar{\alpha} = \{\alpha_0 < \dots < \alpha_k\}$ and $m \in \mathbb{N}$ be such that $s, E^{(d)}(f) \in K_{X_{\alpha_k}-m}$.

The above expansion of each $\left(\big(E^{(d)}\big)^{-1}\right)^{(n)}\big(E^{(d)}(f)\big) \cdot E^{(d)}(f)^n$ for $n > 0$ is a valid infinite sum involving only integer powers of $E^{(k)}(f-1)$ for $k \in \mathbb{N}$. The sum of exponents in each term is a negative integer, and the largest term of each sum is $\frac{1}{E'(f-1)}$. Also, for each l, there are only finitely many terms with sum of exponents equal to l that can appear in the expansion of $\left(\big(E^{(d)}\big)^{-1}\right)^{(n)}\big(E^{(d)}(f)\big) \cdot E^{(d)}(f)^n$ for some $n \in \mathbb{N}$. So if $0 \le \mu \le \frac{1}{E(f-1)^{1/2}}$, then the final sum representing $\big(E^{(d)}\big)^{-1}(s)$ is summable:

1. For each monomial M appearing in the expression, there are only finitely many $n \in \mathbb{N}$ such that
$$N \in \operatorname{Supp}\left(\mu^n \left(\left(E^{(d)}\right)^{-1}\right)^{(n)} \left(E^{(d)}(f)\right) \cdot E^{(d)}(f)^n\right)$$
for some $N \in M\Gamma_{X_{\alpha_k}-m,0,small}$.

2. $\bigcup_{n>1} \operatorname{Supp}\left(\mu^n \left(\left(E^{(d)}\right)^{-1}\right)^{(n)} \left(E^{(d)}(f)\right) \cdot E^{(d)}(f)^n\right)$ is reverse well-ordered.

So we need only find an element f satisfying the required inequalities.

Define $f := L(s) - \dfrac{d \log E'(L(s)-1) + \frac{1}{E(L(s)-1)^{1/2}}}{E'\big(L(s)-2\big) E\big(L(s)-1\big)}$. We will show that

$$E^{(d)}(f) \leq s \leq \left(1 + \frac{1}{E(L(s)-1)^{1/2}}\right) E^{(d)}(f).$$

Since we can compute $\epsilon_{a,d}(f)$ such that $aE^{(d)}(f) = E\big(f + \epsilon_{a,d}(f)\big)$, this amounts to showing that

$$\begin{aligned}
f + \epsilon_{1,d}(f) &\leq L(s) \leq f + \epsilon_{1+\frac{1}{E(L(s)-1)^{1/2}},d}(f) \\
\epsilon_{1,d}(f) &\leq L(s) - f \leq \epsilon_{1+\frac{1}{E(L(s)-1)^{1/2}},d}(f) \\
\epsilon_{1,d}(f) &\leq \frac{d \log E'(L(s)-1) + \frac{1}{E(L(s)-1)^{1/2}}}{E'\big(L(s)-2\big) E\big(L(s)-1\big)} \leq \epsilon_{1+\frac{1}{E(L(s)-1)^{1/2}},d}(f).
\end{aligned}$$

Let

$$\begin{aligned}
\delta &= L(s) - f \\
&= \frac{d \log E'(L(s)-1) + \frac{1}{E(L(s)-1)^{1/2}}}{E'\big(L(s)-2\big) E\big(L(s)-1\big)} \\
&= \frac{dE(L(s)-2) + d \log E'(L(s)-2) + \frac{1}{E(L(s)-1)^{1/2}}}{E'\big(L(s)-2\big) E\big(L(s)-1\big)}.
\end{aligned}$$

So $\delta^n \approx \frac{d^n}{E'(L(s)-3)^n E(L(s)-1)^n}$. We can compute $\epsilon_{a,d}(f)$ using Lemma 5.2.4. First we compute $E^{(d)}(f-m)$ for $m \geq 1$:

$$\begin{aligned}
E^{(d)}(f-m) &= E^{(d)}\left(L(s) - \delta - m\right) \\
&= \sum_{n=0}^{\infty} \frac{E^{(d+n)}(L(s)-m)}{n!}(-\delta)^n.
\end{aligned}$$

Now compute $C(f)$:

$$\begin{aligned}
C(f) &= \log\left(1+\frac{\log B_d(f-1)+\log a}{E(f-1)}\right)\\
&= \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\left(\frac{\log B_d(f-1)+\log a}{E(f-1)}\right)^k\\
&= \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\left(\frac{dE(x-2)+d\log E'(f-2)+\log a+\log\left(\frac{B_d(f-1)}{E'(f-1)^d}-1\right)}{E(f-1)}\right)^k\\
&= \frac{dE(f-2)+d\log E'(f-2)+\log a}{E(f-1)}+\cdots\\
&= \frac{d\sum_{n=0}^{\infty}\frac{E^{(n)}(L(s)-2)}{n!}\delta^n+d\log\sum_{n=0}^{\infty}\frac{E^{(1+n)}(L(s)-2)}{n!}\delta^n+\log a}{\sum_{n=0}^{\infty}\frac{E^{(n)}(L(s)-1)}{n!}\delta^n}+\cdots\\
&= \frac{dE(L(s)-2)+d\log E'(L(s)-2)+\log a}{E(L(s)-1)}+\cdots
\end{aligned}$$

where the "$\cdots$" has terms with $E(L(s)-1)^n$ in the denominator for $n\geq 2$.

Finally, we compute $\epsilon_{a,d}(f)$:

$$\begin{aligned}
\epsilon_{a,d}(f) =&\frac{C(f)}{E'(f-2)}-\frac{C(f)^2E''(f-2)}{2!E'(f-2)^3}+\cdots\\
=&\frac{\frac{dE(L(s)-2)+d\log E'(L(s)-2)+\log a}{E(L(s)-1)}+\cdots}{\sum_{n=0}^{\infty}\frac{E^{(1+n)}(L(s)-2)}{n!}\delta^n}\\
&-\frac{\left(\frac{dE(L(s)-2)+d\log E'(L(s)-2)+\log a}{E(L(s)-1)}+\cdots\right)^2\sum_{n=0}^{\infty}\frac{E^{(2+n)}(L(s)-2)}{n!}\delta^n}{2!\left(\sum_{n=0}^{\infty}\frac{E^{(1+n)}(L(s)-2)}{n!}\delta^n\right)^3}+\cdots\\
=&\frac{dE(L(s)-2)+d\log E'(L(s)-2)+\log a}{E(L(s)-1)E'(L(s)-2)}+\cdots
\end{aligned}$$

where again the "$\cdots$" has terms with $E(L(s)-1)^n$ in the denominator for $n\geq 2$. Since

$$\frac{E(L(s)-2)^n}{E(L(s)-1)}<\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\left(\frac{1}{E(L(s)-1)^{1/2}}\right)^k=\log\left(1+\frac{1}{E(L(s)-1)^{1/2}}\right)$$

for all $n\in\mathbb{N}$, we have

$$\epsilon_{1,d}(f)<\delta<\epsilon_{1+\frac{1}{E(L(s)-1)^{1/2}},d}(f)$$

as claimed.

Since $L(s)\in H_{i+1}$ by Lemma 5.2.3, we have $f\in H_{i+2}$ by Lemma 5.2.1, and thus $\left(E^{(d)}\right)^{-1}(s)\in H_{i+3}$ again by Lemma 5.2.1. □

So we have shown that if $s \in (H_i)_{>F}$, then $E^{(d)}(s), L(s) \in H_{i+1}$ and $\left(E^{(d)}\right)^{-1}(s) \in H_{i+3}$. If $s < 0$, then we define $E^{(d)}(s) = 0$, and if $s \leq 1$ then we define $L(s) = \left(E^{(d)}\right)^{-1}(s) = 0$. If $0 \leq s \in H_i$ is bounded in F, then there are $r \in F_+$ and $\epsilon \in \mu(H_i)$, the infinitesimals of H_i relative to F, such that $s = r + \epsilon$. Since $F \vDash T_{\text{transexp}}$, $E^{(d)}(r) \in F$ for all $d \in \mathbb{N}$, so we identify $E^{(d)}(s)$ with the series

$$\sum_{n=0}^{\infty} \frac{E^{(d+n)}(r)}{n!}\epsilon^n \in H_i$$

which is an element of H_i by Lemma 4.4.9.

If $s > 1$, then we can define

$$L(s) = \sum_{n=0}^{\infty} \frac{L^{(n)}(r)}{n!}\epsilon^n \in H_i$$

$$\left(E^{(d)}\right)^{-1}(s) = \sum_{n=0}^{\infty} \frac{\left(\left(E^{(d)}\right)^{-1}\right)^{(n)}(r)}{n!}\epsilon^n \in H_i.$$

So M_F can be made into a $\mathcal{L}_{\text{transexp}}$-structure and a model of T_{transexp}.

5.3 A derivation on M_F

There is a derivation ∂_0 on $H_0 = F((\tau^{-1}))^{le}$, which can be thought of as differentiation with respect to τ. Its field of constants is F. We will show that given a derivation ∂_i on H_i that we think of as differentiation with respect to τ, we can extend it to a derivation ∂_{i+1} on H_{i+1}.

Let $i > 0$. We first define how ∂_{i+1} acts on generators built from X'_i. We will extend it to $D_{X'_i}(F)$, then to $C_{\varnothing,i}$, and then to $C_{\bar{\alpha},i}$ inductively for each $\bar{\alpha} \subset \big(H_i/\operatorname{Fin}(H_i)\big)_{>F}$.

The derivation on $D_{X'_i}(F)$

First, let $x'_i = L^{\circ(i+2)}(\tau)$ and let

$$y'_i = \frac{1}{E'\big(L^{\circ(i+1)}(\tau)\big)E'\big(L^{\circ i}(\tau)\big)\cdots E'\big(L(\tau)\big)}$$

which is intended to be the derivative of x'_i. For all $d \in \mathbb{N}$ and $a \in F$, define

$$\partial_{i+1}\big(E^{(d)}(x'_i - m)^a\big) = aE^{(d)}(x'_i - m)^{a-1} \cdot E^{(d+1)}(x'_i - m) \cdot y'_i$$

$$\partial_{i+1}\big(\log E'(x'_i - m)^a\big) = a\log E'(x'_i - m)^{a-1} \cdot \frac{E''(x'_i - m)}{E'(x'_i - m)} \cdot y'_i$$

which are elements of $C_{\{\alpha_i<\cdots<\alpha_1\},i}$ where α_j is such that $E'\big(L^{\circ j}(\tau)\big) \in X_{\alpha_j} \subset \alpha_j$ for $j = 1,\ldots,i$. (Recall that $E'\big(L^{\circ(i+1)}(\tau)-m\big) \in C_{\varnothing,i}$.) Extend ∂_{i+1} to products so that it satisfies the Leibniz rule. Extend ∂_{i+1} to sums in $K_{X_i'-m,0}$ by

$$\partial_{i+1}\left(\sum_{M\in\Gamma_{X_i'-m,0}} c_M M\right) = \sum_{M\in\Gamma_{X_i'-m,0}} c_M\partial_{i+1}(M).$$

Extend the derivation to monomials with exp by defining $\partial_{i+1}(e(a)) = e(a)\partial_{i+1}(a)$. We must show that ∂_{i+1} maps to H_{i+1} and that it is well defined.

Lemma 5.3.1. *For each $s \in K_{X_i'-m}$, we have $\partial_{i+1}(s) \in C_{\{\alpha_i<\cdots<\alpha_1\},i}$ where α_j is such that $E'\big(L^{\circ j}(\tau)\big) \in X_{\alpha_j} \subset \alpha_j$ for $j = 1,\ldots,i$.*

Proof. If $s \in K_{X_i'-m}$, then every monomial of $\partial_{i+1}(s)$ is a product of a monomial of $K_{X_i'-m}$ with y_i'. We will show that if $s \in K_{X_i'-m}$, then $\frac{\partial_{i+1}(s)}{y_i'}$ is a valid sum in $K_{X_i'-m}$ and thus an element of $C_{\varnothing,i}$.

Note that the sum of exponents in $\frac{\partial_{i+1}\left(E^{(d)}\left(x_i'-m\right)^a\right)}{y_i'}$ is still a. Using this, we make two observations:

1. If $g \in \Gamma_{X_i'-m,0,small}$, then $\frac{\partial_{i+1}(g)}{y_i'} \in F[\Gamma_{X_i'-m,0,small}]$.

2. If $t \in T_{X_i'-m}$, then $\frac{\partial_{i+1}(t)}{y_i'} \in T_{X_i'-m} \subset F[\Gamma_{X_i'-m,0,small}]$.

So if $M = e_{X_i'-m}(t)g$ is in a coset $w \in \Gamma_{X_i'-m,0}/\Gamma_{X_i'-m,0,small}$, then

$$\frac{\partial_{i+1}(M)}{y_i'} = e_{X_i'-m}(t)\left(\frac{\partial_{i+1}(t)}{y_i'}g + \frac{\partial_{i+1}(g)}{y_i'}\right)$$

and Supp $\left(\frac{\partial_{i+1}(M)}{y_i'}\right)$ is a finite subset of w. So if $s \in K_{X_i'-m,0}$, then $\frac{\partial_{i+1}(s)}{y_i'}$ is a valid sum in $K_{X_i'-m,0}$.

Now assume that for all $l = 0,\ldots,n$, if $s \in K_{X_i'-m,l}$, then $\frac{\partial_{i+1}(s)}{y_i'} \in K_{X_i'-m,l}$. Let $s = \sum c_a e_{X_i'-m}(a) \in K_{X_i-m,n+1}$ where $a, c_a \in A_{X_i'-m,n}$. Then

$$\frac{\partial_{i+1}(s)}{y_i'} = \sum_{a\in A_{X_i'-m,n}} \left(\frac{\partial_{i+1}(c_a)}{y_i'} + c_a\frac{\partial_{i+1}(a)}{y_i'}\right) e_{X_i'-m}(a).$$

By assumption, $\frac{\partial_{i+1}(c_a)}{y_i'} + c_a\frac{\partial_{i+1}(a)}{y_i'} \in K_{X_i'-m,n}$, so $\frac{\partial_{i+1}(s)}{y_i'} \in K_{X_i'-m,n+1}$. □

Lemma 5.3.2. *Let $s \in K_{X_i'-m}$. Then*

$$\varphi_m\left(\frac{\partial_{i+1}(s)}{y_i'}\right) = \frac{\partial_{i+1}(\varphi_m(s))}{y_i'}.$$

Proof. It suffices to prove the maps commute on generators built from X_i'. We can compute

$$\begin{aligned}
\varphi_m\left(\frac{\partial_{i+1}\big(E^{(d)}(x_i')^a\big)}{y_i'}\right) =& \varphi_m\left(aE^{(d)}(x_i')^{a-1}E^{(d+1)}(x_i')\right) \\
=& ae^{aE(x_i'-1)}E'(x_i'-1)^{d(a-1)}\sum_{n=0}^{\infty}\binom{a-1}{n}\left(\frac{B_d(x_i'-1)}{E'(x_i'-1)^d}-1\right)^n \\
&\cdot B_{d+1}(x_i'-1) \\
\frac{\partial_{i+1}(\varphi_m(s))}{y_i'} =& \frac{\partial_{i+1}\left(e^{aE(x_i'-1)}E'(x_i'-1)^{da}\sum_{n=0}^{\infty}\binom{a}{n}\left(\frac{B_d(x_i'-1)}{E'(x_i'-1)^d}-1\right)^n\right)}{y_i'} \\
=& e^{aE(x_i'-1)}\Bigg(aE'(x_i'-1)\cdot E'(x_i'-1)^{da}\sum_{n=0}^{\infty}\binom{a}{n}\left(\frac{B_d(x_i'-1)}{E'(x_i'-1)^d}-1\right)^n \\
&+ daE'(x_i'-1)^{da-1}\sum_{n=0}^{\infty}\binom{a}{n}\left(\frac{B_d(x_i'-1)}{E'(x_i'-1)^d}-1\right)^n \\
&+ \Bigg(E'(x_i'-1)^{da}\sum_{n=1}^{\infty}n\binom{a}{n}\left(\frac{B_d(x_i'-1)}{E'(x_i'-1)^d}-1\right)^{n-1} \\
&\cdot \frac{B_d(x_i'-1)'E'(x_i'-1)^d - dB_d(x_i'-1)E'(x_i'-1)^{d-1}}{E'(x_i'-1)^{2d}}\Bigg)\Bigg).
\end{aligned}$$

To show these two expressions are equal, observe first that $e^{aE(x_i'-1)}$ appears in all monomials of both, so we may divide it out. Next, we will factor

$$a\varphi_m\big(E^{(d)}(x_i')^{a-1}\big) = aE'(x_i'-1)^{d(a-1)}\sum_{n=0}^{\infty}\binom{a-1}{n}\left(\frac{B_d(x_i'-1)}{E'(x_i'-1)^d}-1\right)^n$$

out of the second expression, and show that what remains is equal to $B_{d+1}(x_i'-1)$. Note

$$\begin{aligned}
\sum_{n=0}^{\infty}\binom{a}{n}\left(\frac{B_d(x_i'-1)}{E'(x_i'-1)^d}-1\right)^n &= \left(\sum_{n=0}^{\infty}\binom{a-1}{n}\left(\frac{B_d(x_i'-1)}{E'(x_i'-1)^d}-1\right)^n\right)\cdot\frac{B_d(x_i'-1)}{E'(x_i'-1)^d} \\
\sum_{n=1}^{\infty}n\binom{a}{n}\left(\frac{B_d(x_i'-1)}{E'(x_i'-1)^d}-1\right)^{n-1} &= a\sum_{n=1}^{\infty}\left(\binom{a}{n}-\binom{a-1}{n}\right)\left(\frac{B_d(x_i'-1)}{E'(x_i'-1)^d}-1\right)^{n-1} \\
&= a\sum_{n=0}^{\infty}\left(\binom{a}{n+1}-\binom{a-1}{n+1}\right)\left(\frac{B_d(x_i'-1)}{E'(x_i'-1)^d}-1\right)^n \\
&= a\sum_{n=0}^{\infty}\binom{a-1}{n}\left(\frac{B_d(x_i'-1)}{E'(x_i'-1)^d}-1\right)^n.
\end{aligned}$$

So when we factor out $a\varphi_m\big(E^{(d)}(x_i')^{a-1}\big)$ of the second expression we are left with

$$\begin{aligned}\frac{\frac{\partial_{i+1}(\varphi_m(s))}{y_i'}}{a\varphi_m\big(E^{(d)}(x_i')^{a-1}\big)} =& E'(x_i'-1)B_d(x_i'-1)+d\frac{B_d(x_i'-1)}{E'(x_i'-1)}\\ &+E'(x_i'-1)^d\left(\frac{B_d(x_i'-1)'}{E'(x_i'-1)^d}-d\frac{B_d(x_i'-1)}{E'(x_i'-1)^{d+1}}\right)\\ =& E'(x_i'-1)B_d(x_i'-1)+B_d(x_i'-1)'\\ =& B_{d+1}(x_i'-1).\end{aligned}$$

The computation showing the maps commute on generators of the form $\log E'(x_i')^a$ is very similar. □

Lemma 5.3.2 shows that ∂_{i+1} is well defined on $D_{X_i'}(F)$.

The derivation on $C_{\varnothing,i}$

Let $x_i = L^{\circ(i+1)}(\tau)$ and

$$y_i = \frac{1}{E'\big(L^{\circ i}(\tau)\big)E'\big(L^{\circ(i-1)}(\tau)\big)\cdots E'\big(L(\tau)\big)}$$

which is intended to be the derivative of x_i. For all $d\in\mathbb{N}$ and $a\in D_{X_i'}(F)$, define

$$\begin{aligned}\partial_{i+1}\big(E^{(d)}(x_i-m)^a\big) =& E^{(d)}(x_i-m)^a\left(\partial_{i+1}(a)\log\big(E^{(d)}(x_i-m)\big)+a\frac{E^{(d+1)}(x_i-m)}{E^{(d)}(x_i-m)}y_i\right)\\ \partial_{i+1}\big(\log E'(x_i-m)^a\big) =& \log E'(x_i-m)^a\\ &\left(\partial_{i+1}(a)\log\big(\log E'(x_i-m)\big)+a\frac{E''(x_i-m)}{\log E'(x_i-m)E'(x_i-m)}y_i\right)\end{aligned}$$

which are elements of $C_{\{\alpha_i<\cdots<\alpha_1\},i}$ where α_j is such that $E'\big(L^{\circ j}(\tau)\big)\in X_{\alpha_j}\subset\alpha_j$. Extend ∂_{i+1} to products so that it satisfies the Leibniz rule. Extend ∂_{i+1} to sums in $K_{X_i-m,0}$ by

$$\partial_{i+1}\left(\sum_{M\in\Gamma_{X_i-m,0}}c_M M\right) = \sum_{M\in\Gamma_{X_i-m,0}}\partial_{i+1}(c_M)M+c_M\partial_{i+1}(M).$$

Extend the derivation to monomials with exp by defining $\partial_{i+1}(e(a)) = e(a)\partial_{i+1}(a)$. We must show that ∂_{i+1} maps to H_{i+1} and that it is well defined.

Remark 5.3.3. Compute that

$$\begin{aligned}\log\big(E^{(d)}(x_i-m)\big) =& E(x_i-m-1)+d\log E'(x_i-m-1)\\ &+\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\left(\frac{B_d(x_i-m-1)}{E'(x_i-m-1)^d}-1\right)^k\\ \log\big(\log E'(x_i-m)\big) =& E(x_i-m-2)+\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\left(\frac{\log E'(x_i-m-1)}{E(x_i-m-1)}\right)^k.\end{aligned}$$

So if $s=\sum_{M\in\Gamma_{X_i-m,0}}c_M M$, then every monomial of $\sum_{M\in\Gamma_{X_i-m,0}}c_M\partial_{i+1}(M)$ is a product of a monomial of K_{X_i-m-2} with either y_i' or y_i, i.e., we can split this sum into

$$\sum_{M\in\Gamma_{X_i-m,0}}c_M\partial_{i+1}(M)=s_0y_i'+s_1y_i$$

with $\mathrm{Supp}(s_0),\mathrm{Supp}(s_1)\subset\Gamma_{X_i-m-2,2}$.

Inductively, if $s=\sum_{a\in A_{X_i-m,n}}c_a e_{X_i-m}(a)\in K_{X_i-m,n+1}$, then we have

$$\sum_{a\in A_{X_i-m,n}}c_a\partial_{i+1}\big(e_{X_i-m}(a)\big)=\sum_{a\in A_{X_i-m,n}}c_a e_{X_i-m}(a)\partial_{i+1}(a).$$

Since we can split each $\partial_{i+1}(a)$, we can also split

$$\sum_{a\in A_{X_i-m,n}}c_a\partial_{i+1}\big(e_{X_i-m}(a)\big)=s_0y_i'+s_1y_i$$

with $\mathrm{Supp}(s_0),\mathrm{Supp}(s_1)\subset\Gamma_{X_i-m-2,n+3}$.

Lemma 5.3.4. *For each $s\in K_{X_i-m}$, we have $\partial_{i+1}(s)\in C_{\{\alpha_i<\cdots<\alpha_1\},i}$, where α_j is such that $E'\big(L^{\circ j}(\tau)\big)\in X_{\alpha_j}\subset\alpha_j$.*

Proof. If $s=\sum_{M\in\Gamma_{X_i-m,0}}c_M M\in K_{X_i-m,0}$, then so is $\sum_{M\in\Gamma_{X_i-m,0}}\frac{\partial_{i+1}(c_M)}{y_i'}M$ since Lemmas 5.3.1 and 5.3.2 show each $\frac{\partial_{i+1}(c_a)}{y_i'}\in D_{X_i'}(F)$, the field of coefficients. So we need only worry about

$$\sum_{M\in\Gamma_{X_i-m,0}}c_M\partial_{i+1}(M).$$

Write $\sum_{M\in\Gamma_{X_i-m,0}}c_M\partial_{i+1}(M)=s_0y_i'+s_1y_i$ as in Remark 5.3.3. To prove the Lemma, we must show s_0,s_1 are valid sums in K_{X_i-m-2}.

By the exact same argument as Lemma 5.3.1, the preimage of s_1 is a valid sum in K_{X_i-m}, and thus s_1 is a valid sum in K_{X_i-m-2}.

Now we show s_0 is a valid sum in K_{X_i-m-2}. Let $w \in \Gamma_{X_i-m,0}/\Gamma_{X_i-m,0,small}$ be a coset with representatives appearing in s. Since $s|_w$ is finite, there is some largest derivative d appearing in $s|_w$. Thus the part of $\partial_{i+1}\big(s|_w\big)$ that contributes to s_0 can be split into

$$t_{-1}\log\log E'(x_i-m)+t_0\log E(x_i-m)+\cdots+t_d\log E^{(d)}(x_i-m)$$

where t_{-1} is the image under $\varphi_{m+1}\circ\varphi_m$ of the subsum of $s|_w$ with monomials containing $\log E'(x_i-m)$, and t_j is the image under $\varphi_{m+1}\circ\varphi_m$ subsum of $s|_w$ with monomials containing $E^{(j)}(x_i-m)$. This expression gives a valid sum in K_{X_i-m-2} because each subsum of $s|_w$ is finite. Since $\varphi(s|_v)>\varphi(s|_w)$ if $v>w$, the whole of s_0 is a valid sum in K_{X_i-m-2}.

Now assume the result holds for all elements of $K_{X_i-m,l}$, for $l=0,\ldots,n$. Suppose

$$s=\sum_{a\in A_{X_i-m,n}} c_a e_{X_i-m}(a)\in K_{X_i-m,n+1}$$

where $a,c_a\in A_{X_i-m,n}$. Just as above, $\sum_{a\in A_{X_i-m,n}}\frac{\partial_{i+1}(c_a)}{y_i'}e_{X_i-m}(a)\in K_{X_i-m,n+1}$ by Lemmas 5.3.1 and 5.3.2. By Remark 5.3.3, write

$$\sum_{a\in A_{X_i-m,n}} c_a\partial_{i+1}\big(e_{X_i-m}(a)\big)=s_0y_i'+s_1y_i.$$

Again, $s_1\in K_{X_i-m-2}$ by the same argument as Lemma 5.3.1. And

$$s_0=\sum_{a\in A_{X_i-m}}(\varphi_m\circ\varphi_{m+1})\big(e_{X_i-m}(a)\big)a_0$$

where a_0 is such that $\partial_{i+1}(a)=a_0y_i'+a_1y_i$. By assumption, $a_0\in K_{X_i-m-2}$, so $s_0\in K_{X_i-m-2}$ as well. □

Lemma 5.3.5. *Let $s\in K_{X_i-m}$. Write*

$$\partial_{i+1}(s)=s_0y_i'+s_1y_i$$
$$\partial_{i+1}\big(\varphi_m(s)\big)=t_0y_i'+t_1y_i$$

as in Remark 5.3.3, where $s_0,s_1,t_0,t_1\in C_{\varnothing,i}$ by Lemma 5.3.4. Then $\varphi_m(s_0)=t_0$.

Proof. It suffices to show that the maps commute on generators built from X_i. First let $s=E^{(d)}(x_i)^a$. Let $Y=\left(\frac{B_d(x_i-1)}{E'(x_i-1)^d}-1\right)$. Then

$$
\begin{aligned}
\varphi_m(s_0)y_i' =& \varphi_m\big(E^{(d)}(x_i)^a \log E^{(d)}(x_i)\big)\partial_{i+1}(a) \\
=& E(x_i)^a E'(x_i-1)^{da}\partial_{i+1}(a) \\
& \sum_{n=0}^{\infty}\binom{a}{n}Y^n\left(E(x_i-1)+dE'(x_i-1)+\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}Y^k\right) \\
t_0y_i' =& E(x_i)^a\log E(x_i)\partial_{i+1}(a)\cdot E'(x_i-1)^{da}\sum_{n=0}^{\infty}\binom{a}{n}Y^n \\
& + E(x_i)^a E'(x_i-1)^{da}\log E'(x_i-1)\partial_{i+1}(da)\sum_{n=0}^{\infty}\binom{a}{n}Y^n \\
& + E(x_i)^a E'(x_i-1)\sum_{n=0}^{\infty}\partial_{i+1}\left(\binom{a}{n}\right)Y^n \\
=& E(x_i)^a E'(x_i-1)^{da} \\
& \left(\sum_{n=0}^{\infty}\binom{a}{n}Y^n\big(E(x_i-1)+dE'(x_i-1)\big)\partial_{i+1}(a)+\sum_{n=0}^{\infty}\partial_{i+1}\left(\binom{a}{n}\right)Y^n\right)
\end{aligned}
$$

Matching like terms, all that remains to show is that

$$
\left(\sum_{n=0}^{\infty}\binom{a}{n}Y^n\right)\left(\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}Y^k\right)\partial_{i+1}(a)=\sum_{n=0}^{\infty}\partial_{i+1}\left(\binom{a}{n}\right)Y^n
$$

which follows from the identity

$$
\frac{d}{dX}\binom{X}{n}=\sum_{j=0}^{n-1}\frac{(-1)^{n-j-1}}{n-j}\binom{X}{j}
$$

for binomial coefficients. The argument for $s=\log E'(x_i)^a$ is very similar. □

To see that ∂_{i+1} is well defined on $C_{\varnothing,i}$, write

$$
\begin{aligned}
\partial_{i+1}(s) &= s_0y_i' + s_1y_i \\
\partial_{i+1}\big(\varphi_m(s)\big) &= t_0y_i' + t_1y_i
\end{aligned}
$$

as in Remark 5.3.3, where $s_0, s_1, t_0, t_1 \in C_{\varnothing,i}$ by Lemma 5.3.4. By Lemma 5.3.5, $\varphi(s_0)=t_0$. By the same argument as in Lemma 5.3.2, $\varphi_m(s_1)=t_1$. So ∂_{i+1} commutes with the maps φ_m.

The derivation on $C_{\bar{\alpha},i}$

Before extending the derivation, we will associate to every $s \in H_{i+1}$ a finite sequence $\chi(s) \subset H_{i+1}$. $\chi(s)$ will list all the elements that some $E^{(d)}(\cdot)$ is composed with, including instances in the coefficients and exponents of s.

1. For $s \in D_{X_i'}(F)$, let $\chi(s) := (y_i')$.
2. For $s \in C_{\varnothing,i} \setminus D_{X_i'}(F)$, let $\chi(s) := (y_i', y_i)$.
3. Assume we've defined $\chi(a)$ for all $a \in C_{\bar{\alpha},i}$. Assume also that if $s_0, s_1 \in \boldsymbol{k} \in \kappa_{\bar{\alpha},i}$ but $s_0, s_1 \notin \boldsymbol{k}_0 \subsetneq \boldsymbol{k}$ for any $\boldsymbol{k}_0 \in \kappa_{\bar{\alpha},i}$, then $\chi(s_0) = \chi(s_1)$.

   Let $\beta > \bar{\alpha}$ and $\bar{\alpha}$ minimal such that $s \in C_{\bar{\alpha}\cup\{\beta\},i} \setminus C_{\bar{\alpha},i}$. Then s must be in $D_X(\boldsymbol{k})$ for some minimal finite $X = \{x_1, \ldots, x_p\} \subset X_\beta$ and some minimal $\boldsymbol{k} \in \kappa_{\bar{\alpha},i}$. Define

   $$\chi(s) := \chi(s_0) \frown (\iota_i(x_1), \ldots, \iota_i(x_p))$$

   where $s_0 \in \boldsymbol{k} \in \kappa_{\bar{\alpha},i}$ with $s_0 \notin \boldsymbol{k}_0 \subsetneq \boldsymbol{k}$ for any $\boldsymbol{k}_0 \in \kappa_{\bar{\alpha},i}$.

Now we extend the derivation. Let $\bar{\alpha} \in \big(H_i/\mathrm{Fin}(H_i)\big)_{>F}$, and assume we have defined ∂_{i+1} on $C_{\bar{\alpha},i}$. Let $\beta > \bar{\alpha}$. For all $d \in \mathbb{N}$, $x \in X_\beta$, and $a \in C_{\bar{\alpha},i}$, define

$$\partial_{i+1}\big(E^{(d)}(x-m)^a\big) = E^{(d)}(x-m)^a\left(\partial_{i+1}(a)\log\big(E^{(d)}(x-m)\big) + a\frac{E^{(d+1)}(x-m)}{E^{(d)}(x-m)}\iota_i\big(\partial_i(x)\big)\right)$$

and

$$\begin{aligned}\partial_{i+1}\big(\log E'(x-m)^a\big) = \log E'(x-m)^a\Bigg(&\partial_{i+1}(a)\log\big(\log E'(x-m)\big)\\ &+ a\frac{E''(x-m)}{\log E'(x-m)E'(x-m)}\iota_i\big(\partial_i(x)\big)\Bigg)\end{aligned}$$

which are elements of $C_{\bar{\alpha}\cup\{\beta\}\cup\bar{\gamma},i}$ where $\bar{\gamma}$ is such that $\partial_{i+1}(y) \in C_{\bar{\gamma},i}$ for all

$$y \in \chi\big(E^{(d)}(x-m)^a\big) = \chi\big(\log E'(x-m)^a\big).$$

Extend ∂_{i+1} to products so that it satisfies the Leibniz rule. Extend ∂_{i+1} to sums in $K_{X_i-m,0}$ by

$$\partial_{i+1}\left(\sum_{M\in\Gamma_{X_i-m,0}} c_M M\right) = \sum_{M\in\Gamma_{X_i-m,0}} \partial_{i+1}(c_M)M + c_M\partial_{i+1}(M).$$

Extend the derivation to monomials with exp by defining $\partial_{i+1}(e(a)) = e(a)\partial_{i+1}(a)$. Just like earlier, we must show that ∂_{i+1} maps to H_{i+1} and that it is well defined.

Remark 5.3.6. Let $s = \sum_{M \in \Gamma_{X_\beta - m,0}} c_M M$, and let $\chi(s) = (z_1, \dots, z_q; x_1, \dots, x_p)$. For any $x \in X_\beta$, we can compute $\log E^{(d)}(x - m)$ and $\log\big(\log E'(x-m)\big)$ just as in Remark 5.3.3. So every monomial of $\sum_{M \in \Gamma_{X_\beta - m,0}} c_M \partial_{i+1}(M)$ is a product of a monomial of $K_{X_\beta - m-2}$ with one of $\partial_{i+1}(z_1), \dots, \partial_{i+1}(z_1), \partial_{i+1}(x_1), \dots, \partial_{i+1}(x_p)$. So we can split

$$\sum_{M \in \Gamma_{X_\beta - m,0}} c_M \partial_{i+1}(M) = s_1 \partial_{i+1}(z_1) + \dots + s_p \partial_{i+1}(z_q) + s_{p+1}\partial_{i+1}(x_1) + \dots + s_{p+q}\partial_{i+1}(x_p)$$

with $\operatorname{Supp}(s_j) \subset \Gamma_{X_\beta - m - 2}$ for $j = 1, \dots, p+q$.

Lemma 5.3.7. *For each $s \in K_{X_\beta - m}$, we have*

$$\partial_{i+1}(s) \in C_{\bar\alpha \cup \{\beta\} \cup \bar\gamma, i}$$

for some $\bar\gamma \subset \big(H_i/\operatorname{Fin}(H_i)\big)_{>F}$.

Proof. Let $s \in K_{X_\beta - m}$, and write $\chi(s) = (z_1, \dots, z_q; x_1, \dots, x_p)$. Let $\bar\gamma$ be such that $\partial_{i+1}(z_j), \partial_{i+1}(x_l) \in C_{\bar\alpha \cup \bar\gamma, i}$ for $j = 1, \dots, q$, $l = 1, \dots, p$.

First suppose $s = \sum_{M \in \Gamma_{X_\beta - m,0}} c_M M \in K_{X_\beta - m,0}$. Then for each M, we can write

$$\partial_{i+1}(c_M) = c_{M,1}\partial_{i+1}(z_1) + \dots + c_{M,q}\partial_{i+1}(z_q)$$

with $c_{M,1}, \dots, c_{M,q} \in C_{\bar\alpha, i}$. Since s is a valid sum, so is $\sum_{M \in \Gamma_{X_\beta - m,0}} c_{M,j} M$ for each $j = 1, \dots, q$. Thus

$$\sum_{M \in \Gamma_{X_\beta - m,0}} \partial_{i+1}(c_M) M = \left(\sum_{M \in \Gamma_{X_\beta - m,0}} c_{M,1} M\right) \partial_{i+1}(z_1) + \dots + \left(\sum_{M \in \Gamma_{X_\beta - m,0}} c_{M,q} M\right) \partial_{i+1}(z_q)$$

is a valid sum in $C_{\bar\alpha \cup \{\beta\} \cup \bar\gamma, i}$.

Following Remark 5.3.6, write

$$\sum_{M \in \Gamma_{X_\beta - m,0}} c_M \partial_{i+1}(M) = s_1 \partial_{i+1}(z_1) + \dots + s_p \partial_{i+1}(z_q) + s_{p+1}\partial_{i+1}(x_1) + \dots + s_{p+q}\partial_{i+1}(x_p).$$

To show $\sum_{M \in \Gamma_{X_\beta - m,0}} c_M \partial_{i+1}(M)$ is a valid sum, we must show that each $s_j \in K_{X_\beta - m - 2}$. For $s_{q+1}, \dots, s_{q+p}$, this follows from the argument of Lemma 5.3.1. For $s_{p+1}, \dots, s_{p+q}$, this follows from the argument of Lemma 5.3.4. In fact, each $s_j \in K_{X_\beta - m - 2, 2}$ since $s \in K_{X_\beta - m, 0}$.

Now assume the result holds for all $s \in K_{X_\beta - m, l}$ for $l = 1, \dots, n$. Suppose

$$s = \sum_{a \in A_{X_\beta - m, n}} c_a e_{X_\beta - m}(a) \in K_{X_\beta - m, n+1}.$$

The same argument as above shows

$$\sum_{a\in A_{X_\beta - m,n}} \partial_{i+1}(c_a)e_{X_\beta - m}(a) \in C_{\bar{\alpha}\cup\{\beta\}\cup\bar{\gamma},i}.$$

To show $\sum_{a\in A_{X_\beta - m,n}} c_a e_{X_\beta - m}(a)\partial_{i+1}(a)$ is a valid sum, note that by induction, we can write

$$\partial_{i+1}(a) = a_1\partial_{i+1}(z_1) + \cdots + a_p\partial_{i+1}(z_q) + a_{p+1}\partial_{i+1}(x_1) + \cdots + a_{p+q}\partial_{i+1}(x_p)$$

with each $a_j \in K_{X_\beta - m-2,n+2}$. And

$$\sum_{a\in A_{X_\beta - m,n}} c_a e_{X_\beta - m}\big((\varphi_{m+1}\circ\varphi_m)(a)\big)a_j$$

is a valid sum in $K_{X_\beta - m-2,n+3}$ for each $j = 1,\ldots,p+q$. □

Remark 5.3.8. To show that ∂_{i+1} is well defined on $C_{\bar{\alpha}\cup\{\beta\},i}$, it suffices to show that ∂_{i+1} and φ_m commute on generators, i.e., for each $x \in X_\beta$ and $a \in C_{\bar{\alpha},i}$, if we write

$$\partial_{i+1}\big(\varphi_m(E^{(d)}(x-m))\big) = t_0\partial_{i+1}(a) + t_1\partial_{i+1}(x)$$

then

$$t_0 = \varphi_m\left(E^{(d)}(x-m)^a \log E^{(d)}(x-m)\right)$$
$$t_1 = \varphi_m\left(aE^{(d)}(x-m)^a\frac{E^{(d+1)}(x-m)}{E^{(d)}(x-m)}\right)$$

and similarly for $\log E'(x-m)$. The first equality follows from the computations of Lemma 5.3.5, and the second equality follows from the computations of Lemma 5.3.2. So ∂_{i+1} is well defined on $C_{\bar{\alpha}\cup\{\beta\},i}$.

Having defined ∂_{i+1} on all $C_{\bar{\alpha},i}$, we may extend it to the direct limit H_{i+1}. It is well defined on the direct limit by Lemma 5.1.7. So M_F is a differential field.

5.4 Ordering of germs of terms at $+\infty$

Let $\mathcal{G}$ be the ring of germs at $+\infty$ of functions $f:\mathbb{R}\to\mathbb{R}$, and let $\mathcal{T}(x)$ be the algebra of $\mathcal{L}_{\text{transexp}}$-terms in a single variable x over $\mathbb{R}$. Define $\theta:\mathcal{T}(x)\to\mathcal{G}$ by sending each term $t(x)$ to the germ of the function $x\mapsto t(x)$ at $+\infty$.

Theorem 5.4.1. *$\theta\big(\mathcal{T}(x)\big)$ is totally ordered.*

Proof. Let M_0 be the $\mathcal{L}_{\text{transexp}}$-substructure of $M_{\mathbb{R}}$ generated by τ. Define $\psi: M_0\to\mathcal{G}$ as follows:

1. $\psi(\tau)$ is the germ of the identity function.
2. If $\psi(s_i) = g_i \in \mathcal{G}$ for $i = 1, \ldots, n$ and $\tilde{f} \in \mathcal{L}_{\text{transexp}}$ is an n-ary function symbol corresponding to the function f, then $\psi(\tilde{f}(s_1, \ldots, s_n)) = f(g_1, \ldots, g_n)$.

By Lemma 5.1.6, for each $i \in \mathbb{N}$ and each $\{\alpha_0 < \cdots < \alpha_k\} \subset (H_i/\text{Fin}(H_i))_{>F}$, X_{α_k} and $C_{\{\alpha_0 < \cdots < \alpha_{k-1}\},i}$ satisfy the ordering and separation assumptions of Remark 3.1.2. Thus there can be no relations among the many different monomials of $M_{\mathbb{R}}$ other than those arising from the difference equation $E(x+1) = \exp E(x)$. Therefore, ψ is a map, i.e., it associates to each element of M_0 a single element of $\mathcal{G}$.

Since M_0 is a field, ψ is injective. $\psi(M_0)$ is formally real, and since every $s > 0$ in M_0 is a square, the order on $\psi(M_0)$ coming from M_0 is unique. So $\theta\big(\mathcal{T}(x)\big) = \psi(M_0)$ is ordered. □

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