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A study of dimension 5 Ore extensions

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by

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ABSTRACT OF THE DISSERTATION

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Susan Michelle Elle

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Professor Daniel Rogalski, Chair

In order to study AS-regular algebras of dimension 5, we consider dimension 5 graded iterated Ore extensions generated in degree one. We present an interesting example of an Ore extension with two generators and a degree type first discussed by Floystad and Vatne. We classify the possible degrees of relations and structure of the free resolution for extensions with 3 and 4 generators. We show that every known type of algebra of dimension 5 can be realized by an Ore extension and we consider which of these types cannot be realized by an enveloping algebra. We then investigate the possible bigrading of Ore extensions with degree types that cannot be realized

by an enveloping algebra and show that there is no AS-regular algebra with minimal relations of degrees 2, 2, and 3.

1 Introduction

1.1 Motivation and history

The goal of this dissertation is to explore specific Artin-Schelter regular algebras of dimension 5. The study of Artin-Schelter (AS) regular algebras was introduced by Artin and Schelter in 1987 [AS]. AS-regular algebras are noncommutative polynomial rings that in some sense generalize commutative polynomial rings while maintaining some very nice properties, such as having finite dimension. In particular, these properties allow AS-regular algebras to be used to construct a noncommutative equivalent of projective schemes. Research in noncommutative algebraic geometry and its applications to fields such as mathematical physics relies heavily on analyzing specific examples of quantum $\mathbb{P}^n$'s, which can be constructed algebraically by forming the noncommutative projective scheme $\mathrm{Proj}(A)$ where A is a noetherian AS-regular algebra of global dimension $n+1$. Thus, the classification of AS-regular algebras, and less generally the explicit construction of examples of such algebras, is an extremely active area of current research in the field.

Under mild assumptions, iterated Ore extensions are AS-regular algebras with additional nice properties. For example, they have the same K-vector space basis as commutative polynomial rings, $\{x_1^{e_1} \cdots x_n^{e_n}\}$, and multiplication in these algebras is somewhat well understood. As a result, their classification provides a natural starting place for the classification of AS-regular algebras in higher dimensions, which remain very poorly understood in general.

An AS-regular algebra of dimension 2 which is generated in degree one is isomorphic to either the Jordan plane $J = \dfrac{K\langle x_1, x_2\rangle}{\langle x_2x_1 - x_1x_2 - x_1^2\rangle}$, or a quantum plane $\mathcal{O}_q = \dfrac{K\langle x_1, x_2\rangle}{\langle x_2x_1 - qx_1x_2\rangle}$, $q \neq 0$. The possible families of relations of AS-regular algebras of dimension 3 which are generated in degree one were completely classified by Artin, Tate, and Ven den Bergh [AS],[ATVdB].

The classification of AS-regular algebras of dimension 4 remains an active area of research. Restricting to AS-regular algebras which are domains and generated in degree 1, the possible *relation types* i.e. the number and degrees of the minimal set of relations generating the ideal of relations, are known. If the algebra is assumed to be *bigraded*, also called $\mathbb{Z}^2$*-graded*, i.e. if each generator has degree $(1, 0)$ or $(0, 1)$ and each relation is $\mathbb{Z}\times\mathbb{Z}$-homogeneous, then the possible families of relations are known in most cases [LPWZ], [RZ], [ZZ].

A number of interesting patterns have arisen for AS-regular algebras. For any possible relation type of an algebra of dimension 4 or less, the Hilbert series of the algebra is unique, there is an enveloping algebra of a graded Lie algebra with the given relation type, and there is a $\mathbb{Z}^2$-graded algebra with the given relation type. These properties fail for AS-regular algebras of dimension 5.

Although the classification of AS-regular algebras of dimension 5 is also an active area of research, progress in the area has been slow. In 2011, Floystad and Vatne listed the possible relation types of an AS-regular algebra of dimension 5 with 2 degree one generators under mild assumptions and provided an example of an AS-regular algebra with a relation type that could not possibly be realized by an enveloping algebra [FV]. Building on their work, Wang and Wu used A_∞-algebra techniques to find many families of algebras of dimension 5 with two generators, including an Ore extension with 3 degree four relations and 2 degree five relations, i.e. relation type (4,4,4,5,5) [WW]. This relation type provides another example of something that cannot be realized by an enveloping algebra and is the first example where algebras with the same Hilbert series can have different resolution types. Zhou and Lu [ZL] classified the possible families of relations of these algebras under the additional assumption that they were $\mathbb{Z}^2$-graded and found that there is no bigraded algebra with relation type (4,4,4,5), although it remains an open question whether there is any AS-regular algebra with this relation type. There has not yet been a careful treatment of the classification of dimension 5 algebras with 3 or 4 generators.

1.2 Outline of the dissertation

In chapter 2, we provide background definitions and theorems that we will need throughout the dissertation.

In chapters 3-6 we classify possible types of Ore extensions of dimension 5. We list the possible degrees of variables in chapter 3 before presenting a specific example of an Ore extension with 2 degree one generators in chapter 4. We then proceed to examine Ore extensions with 4 (chapter 5) and 3 (chapter 6) degree one generators. We list all possible relation types and explore which of these types can correspond to the enveloping algebra of a graded Lie algebra.

We then explore bigraded AS-regular algebras for the relation types that cannot be realized by an enveloping algebra. In chapter 7, we show that there are

[3,1]-bigraded Ore extensions with relation type (2,2,2,2,2,3), i.e. there are bigraded algebras with 3 degree (1,0) generators and 1 degree (0,1) generator. However, there are no [2,2]-bigraded extensions, i.e. extensions with 2 generators of degree (1,0) and 2 of degree (0,1). In chapter 8, we show that there are no bigraded AS-regular algebras, Ore extensions or otherwise, that have relation type (2,2,3).

We conclude with a list of partial families of Ore extensions of various degree types as a means of providing additional examples for future investigation.

2 Preliminaries

2.1 Graded algebras

A K-*algebra*, A, is a ring with identity which is also a vector space over K, i.e. $k(a_1a_2) = (ka_1)a_2 = a_1(ka_2)$ for all $a_1, a_2 \in A$ and $k \in K$. A is *graded*, more specifically $\mathbb{N}$-*graded*, if $A = \bigoplus_{n=0}^{\infty} A_n$ as K-spaces and $A_nA_m \subseteq A_{n+m}$ for all n and m. And A is *connected* if $A_0 = K$. An element $a \in A$ is *homogeneous* if there exists n such that $a \in A_n$ and an ideal is called homogeneous if it is generated by homogeneous elements.

We say that A is *finitely generated* over K if there exist elements $x_1, \cdots, x_b \in A$ such that the set $\{x_{i_1} \cdots x_{i_m} | 1 \leq i_j \leq b,\ m \geq 1\} \cup \{1\}$ spans A as a K-space. It can be shown that a connected graded K-algebra A is finitely generated if and only if there exists a degree preserving surjection $\phi : K\langle x_1, \cdots, x_b\rangle \to A$ for some weighting of the variables $\deg(x_i) \geq 1$ for all $1 \leq i \leq b$, in which case A has presentation $A \cong \dfrac{K\langle x_1, \cdots, x_b\rangle}{I}$ where I is a homogeneous ideal.

If A is a finitely generated K-algebra, the *Hilbert series* of A is $h_A(t) = \sum_{n=0}^{\infty} (\dim_K A_n)t^n$ where A_n is the nth graded piece of A. This then allows us to define the *Gelfand-Kirillov (GK) dimension* as GK.dim $A = \lim_{n\to\infty} \sup \log_n \dim_K V^n$ where V is any finite dimensional K-subspace of A which generates A. This definition is independent of the choice of V.

A right A-module M is graded if $M = \bigoplus_{n\in\mathbb{Z}} M_n$ and $M_iA_j \subseteq M_{i+j}$. We then define the *shift* module, $M(i)$, to be the graded module isomorphic to M but which has shifted grading such that $M(i)_n = M_{i+n}$. A module homomorphism, $\phi : M \to N$, is graded if $\phi(M_n) \subseteq N_n$ for all $n \in \mathbb{Z}$. If $\mathrm{Hom}_{gr-A}(M, N)$ is the set of all graded homomorphisms from M to N, then we shall define $\underline{\mathrm{Hom}}_A(M, N) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{gr-A}(M, N(n))$. There is a natural inclusion $\underline{\mathrm{Hom}}_A(M, N) \subseteq \mathrm{Hom}_A(M, N)$ and it can be shown that these are equal whenever M is finitely generated as an A-module. Later, we will find it convenient to use the fact that

Lemma 2.1.1 ([Rog, Lemma 1.28 (1)])**.** $\underline{Hom}_A(\bigoplus_{i=1}^{m} A(s_i), A) \cong \bigoplus_{i=1}^{m} A(-s_i)$.

We also recall that A is *noetherian* if every right and left submodule of A is finitely generated or, equivalently, if every right and left submodule of A satisfies the ascending chain condition. We will say A is a *domain* if whenever $a_1a_2 = 0,\ a_1, a_2 \in A$, then either $a_1 = 0$ or $a_2 = 0$ and we will say that an element $a \in A$ is *normal* if $aA = Aa$.

2.2 Artin-Schelter regular algebras

A module P over a ring A is *projective* if, whenever there is surjective homomorphism $f : M \to N$ and homomorphism $g : P \to N$, there exists $h : P \to M$ such that $f \circ h = g$. It can be shown that this is equivalent to saying that P is projective if and only if there exists Q such that $P \bigoplus Q$ is a free module. A *projective resolution* of an A-module M is an exact sequence of modules

$$\cdots \to P_n \stackrel{d_{n-1}}{\to} P_{n-1} \to \cdots \to P_1 \stackrel{d_0}{\to} P_0 \stackrel{\epsilon}{\to} M \to 0,$$

where each P_i is projective. A *free resolution* is a projective resolution as above for which all P_i are free. Since free modules are projective and a free resolution always exists, it follows that every module has a projective resolution, although this need not be unique.

For graded modules M and N, we may then define abelian groups $\underline{\mathrm{Ext}}^i_A(M,N)$ by taking a projective resolution of M, truncated at the last step,

$$\cdots \to P_n \overset{d_{n-1}}{\to} P_{n-1} \to \cdots \to P_1 \overset{d_0}{\to} P_0 \to 0,$$

applying the contravariant functor $\underline{\mathrm{Hom}}_A(-,N)$ to get

$$\cdots \leftarrow \underline{\mathrm{Hom}}_A(P_n,N) \overset{d_n^*}{\leftarrow} \cdots \leftarrow \underline{\mathrm{Hom}}_A(P_1,N) \overset{d_1^*}{\leftarrow} \underline{\mathrm{Hom}}_A(P_0,N) \overset{d_0^*}{\leftarrow} 0,$$

and taking the ith homology, i.e. $\underline{\mathrm{Ext}}^i_A(M,N) = \ker d_i^*/\mathrm{Im}\ d_{i-1}^*$. Up to isomorphism, these groups are independent of the choice of projective resolution of M. Also, $\underline{\mathrm{Ext}}^0_A(M,N) \cong \underline{\mathrm{Hom}}_A(M,N)$ (see [Rot, Corollary 6.57, Theorem 6.61].)

The *projective dimension*, proj.dim(M), is the smallest n such that there is a projective resolution of M of length n, and ∞ if no such n exists. If A is connected, graded, and finitely generated over K, the *global dimension*, or more specifically the *right global dimension* of A, r.gl.dim(A), is the supremum of the projective dimensions of all graded right A-modules, and the left global dimension, l.gl.dim(A), is the supremum of the projective dimensions of all graded left A-modules.

We shall say that $B \subseteq M$ is a *minimal generating set* of the graded module M if B consists of homogeneous elements which generate M and no proper subset of B does the same. If such a set exists, say $B = \{r_i\}$, $\deg(r_i) = a_i$, then we may construct a *minimal surjection* of a graded free module onto M, $\phi : \bigoplus_i A(-a_i) \to M$, by mapping 1 in the ith copy of the sum to r_i. We call a free resolution

$$\cdots \to P_n \overset{d_{n-1}}{\to} P_{n-1} \to \cdots \to P_1 \overset{d_0}{\to} P_0 \overset{\epsilon}{\to} M \to 0$$

a *minimal free resolution* if ϵ and each d_i are minimal surjections onto their images. While minimal resolutions need not exist, we will be most interested in the case where $M = K = A_0$, in which case the minimal resolution does exist. Any two minimal free resolutions are isomorphic as complexes (see [Rog, Lemma 1.24 (2)]), and so there is no harm in referring to the minimal free resolution of a module as though it were unique.

Lemma 2.2.1 ([Rog, Proposition 1.30]). *If A is connected, graded, and finitely generated over K then*

$$r.gl.dim(A) = proj.dim(K_A) = proj.dim({}_AK) = l.gl.dim(A),$$

and this equals the length of the minimal free resolution of the module K_A.

Here K_A refers to $K = A_0$ viewed as a right A-module while ${}_AK$ refers to K viewed as a left A-module.

We now have the tools necessary to define Artin-Schelter regular algebras, the motivating objects of study:

Definition 2.2.2. A connected graded algebra $A = \bigoplus_{i=0}^{\infty} A_i$ is *Artin-Schelter (AS) regular* of dimension d if

1. A has finite global dimension d;

2. A has finite Gelfand-Kirillov dimension;

3. A is *AS-Gorenstein*, i.e.

$$\underline{\mathrm{Ext}}^i_A(K, A) = \begin{cases} 0 & i \neq d \\ {}_AK(l) & i = d \end{cases}$$

for some shift of grading $l \in \mathbb{Z}$.

Although we use the term "dimension" to refer specifically to the global dimension of A, and "global dimension" itself refers to the maximum of the right and left global dimensions of A, r.gl.dim(A)=l.gl.dim(A)=gl.dim(A) for any AS-regular algebra A, so there is no potential for confusion. For all known examples of AS-regular algebras, we also have GK.dim(A)=gl.dim.(A), although it is not known if this holds in general.

Recall that if A is a finitely generated K-algebra, then $A \cong \dfrac{K\langle x_1, \cdots, x_b\rangle}{I}$ and $B \subseteq I$ is called a *minimal generating set* of I if B consists of homogeneous elements which generate I and no proper subset of B does the same.

Lemma 2.2.3 ([Rog, Lemma 2.11])**.** *Suppose* $A \cong \dfrac{K\langle x_1, \cdots, x_b\rangle}{I}$ *and* $B = (r_1, \cdots, r_n)$, $\deg(r_i) = a_i$ *is a minimal generating set of* I*. Then the minimal free resolution of* K_A *and of* ${}_AK$ *begins*

$$\cdots \to \bigoplus_{i=1}^{n} A(-a_i) \to \bigoplus_{i=1}^{b} A(-\deg(x_i)) \to A \to K \to 0. \tag{2.1}$$

We are now ready to present the free resolution of the trivial module K_A for an AS-regular algebra A of (global) dimension d.

Theorem 2.2.4. *Suppose* $A \cong \dfrac{K\langle x_1, \cdots, x_b\rangle}{I}$ *and* $B = (r_1, \cdots, r_n)$, $\deg(r_i) = a_i$ *is a minimal generating set of* I. *Then the minimal free resolution of* K_A *is*

$$0 \to A(-l) \to \bigoplus_{i=1}^{b} A(-l+\deg(x_i)) \to \bigoplus_{i=1}^{n} A(-l+a_i) \to \cdots$$
$$\cdots \to \bigoplus_{i=1}^{n} A(-a_i) \to \bigoplus_{i=1}^{b} A(-\deg(x_i)) \to A \to K \to 0.$$

Proof. Consider the free resolution of K_A which, by Lemma 2.2.1, is of length d=gl.dim.(A):

$$0 \to F_d \to F_{d-1} \to \cdots \to F_1 \to F_0 \to K \to 0. \tag{2.2}$$

This can be written as

$$0 \to \bigoplus_{i=1}^{m_d} A(s_{id}) \to \cdots \to \bigoplus_{i=1}^{m_1} A(s_{i1}) \to \bigoplus_{i=1}^{m_0} A(s_{i0}) \to K \to 0$$

where every $s_{ij} \in \mathbb{Z}$ and indicates the shift in grading of A. Applying Lemma 2.2.3, this becomes

$$0 \to \bigoplus_{i=1}^{m_d} A(s_{id}) \to \cdots \to \bigoplus_{i-1}^{n} A(-a_i) \to \bigoplus_{i=1}^{b} A(-\deg(x_i)) \to A \to K \to 0.$$

We now apply $\underline{\mathrm{Hom}}_A(-, A)$ to this complex with K dropped and use Lemma 2.1.1 to get the complex

$$0 \leftarrow \bigoplus_{i=1}^{m_d} A(-s_{id}) \leftarrow \cdots \leftarrow \bigoplus_{i-1}^{n} A(a_i) \leftarrow \bigoplus_{i=1}^{b} A(\deg(x_i)) \leftarrow A \leftarrow 0.$$

Since A is AS-Gorenstein, the homology of this complex is 0 everywhere but the leftmost step, where it is ${}_AK(l)$. Thus, we may extend this complex to an exact sequence

$$0 \leftarrow K(l) \leftarrow \bigoplus_{i=1}^{m_d} A(-s_{id}) \leftarrow \cdots \leftarrow \bigoplus_{i-1}^{n} A(a_i) \leftarrow \bigoplus_{i=1}^{b} A(\deg(x_i)) \leftarrow A \leftarrow 0.$$

Applying a shift, we may then write the free resolution of the module ${}_AK$, which we write from right to left for convenience.

$$\begin{aligned} 0 \to A(-l) \to \bigoplus_{i=1}^{b} A(-l+\deg(x_i)) \to \bigoplus_{i-1}^{n} A(-1+a_i) \to \cdots \\ \cdots \to \bigoplus_{i=1}^{m_{d-2}} A(-l-s_{i,d-2}) \to \bigoplus_{i=1}^{m_{d-1}} A(-l-s_{i,d-1}) \to \bigoplus_{i=1}^{m_d} A(-l-s_{id}) \to K \to 0. \end{aligned} \tag{2.3}$$

We compare Equation (2.1), the start of the free resolution for ${}_AK$, and Equation (2.3) to find that $m_d = 1,\ s_{1d} = -l,\ m_{d-1} = b,\ s_{i,d-1} = -l + \deg(x_i),\ m_{d-2} = n,\ s_{i,d-2} = -l + a_i$ and the free resolution can be written as

$$\begin{aligned} 0 \to A(-l) \to \bigoplus_{i=1}^{b} A(-l+\deg(x_i)) \to \bigoplus_{i=1}^{n} A(-l+a_i) \to \cdots \\ \cdots \to \bigoplus_{i=1}^{n} A(-a_i) \to \bigoplus_{i=1}^{b} A(-\deg(x_i)) \to A \to K \to 0, \end{aligned}$$

as was to be shown. □

More generally, it is known that the free resolution of K is symmetric up to a shift in the grading: if $F_j = \bigoplus_{i=1}^{m_j} A(s_{ij})$ then $F_{d-j} = \bigoplus_{i=1}^{m_j} A(-l+s_{ij})$, but we do not need this fact since our focus is on dimension 5 AS-regular algebras.

A finitely generated algebra A is said to be *generated in degree one* if there exists a presentation $A \cong \dfrac{K\langle x_1, \cdots, x_b\rangle}{I}$ where $\deg(x_i) = 1$ for all $1 \leq i \leq b$.

Corollary 2.2.5. *Let A be a dimension 5 AS-regular generated in degree one where $A \cong \dfrac{K\langle x_1, \cdots, x_b\rangle}{I}$ and let $B = (r_1, \cdots, r_n)$, $\deg(r_i) = a_i$ be a minimal generating set of I. Then the minimal free resolution of K_A is*

$$0 \to A(-l) \to A(-l+1)^b \to \bigoplus_{i=1}^{n} A(-l+a_i) \to$$
$$\to \bigoplus_{i=1}^{n} A(-a_i) \to A(-1)^b \to A \to K \to 0.$$

Proof. This is a straightforward application of Theorem 2.2.4, taking $d = 5$ and $\deg(x_i) = 1,\ 1 \leq i \leq b$. □

The minimal free resolution of the trivial module K is often described via *graded Betti numbers* where $\beta_{i,j}$ is equal to the number of copies of $A(-j)$ appearing in the ith step of the resolution.

Lemma 2.2.6 ([Rog, page 14])**.** *The Hilbert series of A, $h_A(t)$, is equal to $\dfrac{1}{q(t)}$ where $q(t) = \sum_{i,j}(-1)^i \beta_{i,j} t^j$ and where $\beta_{i,j}$ are the graded Betti numbers of the minimal free resolution of K.*

Proof. For any complex of finite dimensional vector spaces

$$0 \to V_n \to \cdots \to V_1 \to V_0 \to 0,$$

the alternating sum of the dimensions of the vector spaces is the alternating sum of the homology of the complex:

$$\sum_i (-1)^i \dim_K V_i = \sum_i (-1)^i \dim_K H_i.$$

Applying this to Equation (2.2), the free resolution of K, where the homology is always 0, we get $\sum_i (-1)^i \dim_K (F_i)_n = 0$, which we can write as

$$0 = -h_K(t) + h_{F_0}(t) - h_{F_1}(t) + \cdots + (-1)^d h_{F_d}(t).$$

Now each $F_j = \bigoplus_{i=1}^{m_j} A(s_{ij})$ and $h_{A(s)}(t) = h_A(t)t^{-s}$ by the definition of the shift operator, so we can write

$$0 = -1 + \sum_{i=1}^{m_0} h_A(t)t^{-s_{i0}} - \sum_{i=1}^{m_1} h_A(t)t^{-s_{i1}} + \cdots + (-1)^d \sum_{i=1}^{m_d} h_A(t)t^{-s_{id}}.$$

Solving the equation for $h_A(t)$, we get $h_A(t) = \dfrac{1}{q(t)}$ where $q(t) = \sum_{i,j} (-1)^i \beta_{i,j} t^j$. □

Although not used in the dissertation, a fact which also restricts the possible Hilbert series of AS-regular algebras is the following:

Lemma 2.2.7 ([Rog, Lemma 2.7 (1)])**.** *If A is AS-regular then $h_A(t) = \dfrac{1}{q(t)}$ for some polynomial $q(t) \in \mathbb{Z}[t]$ with constant term 1, all roots of $q(t)$ are roots of unity in $\mathbb{C}$, and GK.dim(A) is equal to the multiplicity of the root 1.*

We have now introduced the different invariants that have historically been used to discuss the possible *classification* of types of AS-regular algebras. Most generally, we can classify algebras by their Hilbert series, although there are algebras with fundamentally different structures that share the same series. More refined, we can use their *relation type* (the number and degree of the relations in the minimal generating set of I, which we will denote by $(a_1, \cdots, a_n)$ where $a_1 \leq \cdots \leq a_n$). More refined still, we can refer to the *resolution type*, or the set of graded Betti numbers of A. The most concrete option for classifying AS-regular algebras, and one beyond the scope of this dissertation, would be to list the possible *families of relations* for the algebras by explicitly writing the possible coefficients of the relations. For example, an AS-regular algebra of dimension 2 which is generated in degree one is isomorphic to $\dfrac{K\langle x_1, x_2\rangle}{\langle r\rangle}$ where $r = x_2x_1 - qx_1x_2$, $0 \neq q$ (in which case the algebra is called the *quantum plane*) or $r = x_2x_1 - x_1x_2 - x_1^2$ (and the algebra is called the *Jordan plane*). We note that, by the symmetry of the free resolution, classifying dimension 5 AS-regular algebras by their relation type is sufficient to also classify them by their resolution type.

2.3 Enveloping algebras and the diamond condition

We will now develop some of the definitions necessary for the discussion of Ore extensions, the main object of study in this dissertation.

Definition 2.3.1. A finite dimensional *Lie algebra* is a finite dimensional vector space over a field K with $\text{char}K \neq 2$, together with a multiplication given by the Lie bracket $[\ ,\ ]$ which satisfies

1. *Bilinearity*: $[k_1x_1 + k_2x_2, x_3] = k_1[x_1, x_3] + k_2[x_2, x_3]$ and $[x_3, k_1x_1 + k_2x_2] = k_1[x_3, x_1] + k_2[x_3, x_2]$;

2. *Alternating property*: $[x_i, x_i] = 0$; and

3. *Jacobi identity*: $[x_k,[x_j,x_i]]+[x_i,[x_k,x_j]]+[x_j,[x_i,x_k]]=0$.

A finite dimensional Lie algebra is graded if $L=\bigoplus_{i=1}^{\infty} L_i$ as K-spaces and $[L_i,L_j]\subseteq L_{i+j}$ for all i and j.

Definition 2.3.2. The universal *enveloping algebra* of a finite dimensional graded Lie algebra, L, is an associative algebra $U(L)=\dfrac{K\langle x_1,\cdots,x_n\rangle}{\langle\{x_jx_i-x_ix_j-[x_j,x_i]=0\mid i<j\}\rangle}$, where [,] denotes the Lie bracket in L and $x_1,\cdots,x_n$ are a K-basis for L.

We now present a basic review of the diamond lemma. A more thorough introduction to the material, as well as a proof of Theorem 2.3.3, can be found in Bergman's paper [Ber, Section 1].

Suppose A is an associative algebra with unity over a field K and that we have a presentation of A by a family X of generators and a family S of relations. In practice, we care about the ideal generated by S, call it I. We have $A\cong\dfrac{K\langle X\rangle}{I}$ where $K\langle X\rangle$ is the free associative K-algebra on $\langle X\rangle$ and $\langle X\rangle$ is the free semigroup with 1 on X. Recall that a subset, $B\subseteq S$, is a minimal generating set for I if B generates I and no proper subset of B does the same. Fix a total ordering on $\langle X\rangle$ with the property that if $w<v$ then $uw<uv$ and $wu<vu$ for all $u\in\langle X\rangle$. Such an ordering will be called a *semigroup total ordering*. Every relation $\sigma\in S$ can be written in the form $W_\sigma=f_\sigma$ where W_σ is a monomial and is larger than any of the monomials in f_σ. We call W_σ the *leading term* (denoted LT) of the relation σ. We can assume that the leading term is always monic since K is a field. We can also take S such that all leading terms are distinct (since otherwise we could subtract a scalar multiple of one relation from another to get two relations with different leading terms which generate the same ideal).

A word w is *irreducible* under S if it does not contain any W_σ as a subword. Otherwise, w contains some W_σ, say $w = uW_\sigma v$ and we consider the K-linear *reduction* map $r_{uW_\sigma v} : K\langle X\rangle \to K\langle X\rangle$ which sends $uW_\sigma v$ to $uf_\sigma v$ and fixes all other elements of $\langle X\rangle$. A finite sequence of reductions $r_1 \cdots r_n$ ($r_i = r_{u_i W_{\sigma_i} v_i}$) is *final* on w if $r_1 \cdots r_n(w)$ is irreducible. The word w is *reduction unique* if its images under all final sequences of reductions are the same.

A 5-tuple (σ, τ, u, v, w) with $\sigma, \tau \in S$ and $u, v, w \in \langle X\rangle$ is an *overlap ambiguity* if $u, v, w \neq 1$, $W_\sigma = uv$, and $W_\tau = vw$ and an *inclusion ambiguity* if $1 \neq \sigma \neq \tau$, $W_\sigma = v$, and $W_\tau = uvw$. An ambiguity is *resolvable* if there exist compositions of reductions s and s' such that $s(r_{W_\sigma w}(uvw)) = s'(r_{uW_\tau}(uvw))$ (in the case of an overlap ambiguity) or $s(r_{uW_\sigma w}(uvw)) = s'(r_{W_\tau}(uvw))$ (in the case of an inclusion ambiguity). A set S of relations satisfies the *diamond condition* if all reduction ambiguities (overlap and inclusion) are resolvable, in which case we say that S is a *Gröbner basis* of I. A Gröbner basis is *reduced* if all leading terms are monic and no element in the basis has a monomial which contains the leading term of any other element in the basis. Throughout the rest of this paper, any reference to a Gröbner basis will mean a reduced Gröbner basis.

Theorem 2.3.3. *[Ber, Theorem 1.2] Let $\leq$ be a semigroup total ordering having the descending chain condition and let S be a set of relations where the leading term of each relation is monic and distinct from the leading term of any other relation. Then the following conditions are equivalent:*

1. *S satisfies the diamond condition;*

2. *All elements of $K\langle X\rangle$ are reduction unique under S;*

3. A set of representatives for the elements of the algebra $A \cong \dfrac{K\langle X\rangle}{I}$ *determined by the generators* X *and the ideal* I *generated by the relations* S *is given by the* K*-submodule* $K\langle X\rangle_{irr}$ *spanned by the* S*-irreducible monomials of* $\langle X\rangle$*.*

We can make A in the preceding theorem graded if, to each $x \in X$, we associate some value $\deg(x) \in \mathbb{Z}_{\geq 0}$ and require that I be homogeneous. We then define *graded lexicographic order* as a total order on $\langle X\rangle$ where $w_1 > w_2$ if $\deg(w_1) > \deg(w_2)$ or if $\deg(w_1) = \deg(w_2)$ and $w_1 = a_1 a_2 \cdots a_j$ comes before $w_2 = b_1 b_2 \cdots b_k$ in the lexicographic order. For the rest of this dissertation, we will use graded lexicographic order. This has the descending chain condition. If $\langle X\rangle = \{x_1, \cdots, x_n\}$, it is sometimes convenient to consider the case when the (lexicographic) order is taken to be

$$x_n > x_{n-1} > \cdots > x_1,$$

and we will assume that this is the order on the variables for the rest of this chapter.

Now that we have an ordering on the variables, we may define an enveloping algebra in terms of its presentation.

Theorem 2.3.4. *U is the universal enveloping algebra of some finite dimensional graded Lie algebra if and only if, labeling generators so that* $\deg(x_1) \geq \cdots \geq \deg(x_n)$ *and taking graded lexicographic order with* $x_n > \cdots > x_1$*, it has presentation* $U \cong \dfrac{K\langle x_1 \cdots x_n\rangle}{\{r_{ji}\}}$ *where for each* $j > i$*, there is a unique homogeneous relation* r_{ji} *given by*

$$r_{ji} : x_j x_i = x_i x_j + \sum_{\substack{k \mid \deg(x_k) = \\ \deg(x_j) + \deg(x_i)}} a_{ji}^k x_k,\ a_{ji}^k \in K$$

and where the relations satisfy the diamond condition.

Proof. Suppose an algebra U has the presentation described. Define L to be generated as a K-vector space by $\langle x_1, \cdots, x_n\rangle$ and define a multiplication on the generators of L by

$$[x_j, x_i] = \begin{cases} \sum_k a_{ji}^k x_k & j > i, \\ \sum_k -a_{ji}^k x_k & j < i, \\ 0 & j = i. \end{cases}$$

This multiplication can be extended bilinearly to general elements in L. We claim that L is the desired graded Lie algebra. The multiplication satisfies bilinearity and has the alternating property by construction. That the Jacobi identity is satisfied is equivalent to the fact that all ambiguities in U resolve. (This is the Poincaré-Birkhoff-Witt theorem, see [Ber, proof of Theorem 3.1]). Finally, since this multiplication is degree preserving, L is a graded Lie algebra with enveloping algebra U.

Conversely, suppose L is a finite dimensional graded Lie algebra and impose graded lexicographic order on the generators with $x_n > \cdots > x_1$. The universal enveloping algebra of L is defined by the relations $x_j x_i - x_i x_j - [x_j, x_i] = 0$ where $[\ ,\ ]$ denotes the Lie bracket, so $[x_j, x_i] = \sum_{m=1}^{n} a_m x_m$. The additional restriction that the Lie algebra be graded forces $\deg([x_j, x_i]) = \deg(x_i) + \deg(x_j)$ so in particular we can take only the x_m such that $\deg(x_m) = \deg(x_i) + \deg(x_j)$. In this case, we must have that $\deg(x_m) > \deg(x_j)$ so the leading term of the relation is $x_j x_i$ and the relations of the enveloping algebra can be written in the form

$$r_{ji} : x_j x_i = x_i x_j + \sum_{\substack{k \mid \deg(x_k) = \\ \deg(x_j) + \deg(x_i)}} a_k x_k, \ a_k \in K.$$

That these relations satisfy the diamond condition is equivalent to the fact that the

Lie bracket satisfies the Jacobi identity. □

2.4 AS-Ore extensions

We are finally ready to define Ore extensions. The interested reader may find [GW, Chapters 1 and 2] a useful reference for some additional background.

Definition 2.4.1. Let R be a ring. A *basic Ore extension* $R[x, \sigma, \delta]$ is a ring with elements which can be written uniquely in the form $f(x) = \sum_{i=0}^{n} a_i x^i,\ a_i \in R$ and multiplication satisfying $xr = \sigma(r)x + \delta(r)$ for all $r \in R$, where σ is an endomorphism of R and δ is a σ*-derivation* of R, i.e. $\delta(r_1r_2) = \sigma(r_1)\delta(r_2) + \delta(r_1)r_2$ for all $r_1, r_2 \in R$.

An *iterated Ore extension* $R[x_1, \sigma_1, \delta_1][x_2, \sigma_2, \delta_2] \cdots [x_n, \sigma_n, \delta_n]$ is a basic Ore extension where for all $j \geq 1$, σ_j and δ_j are a ring endomorphism and a σ_j-derivation of $R_{(j-1)} := R[x_1, \sigma_1, \delta_1][x_2, \sigma_2, \delta_2] \cdots [x_{j-1}, \sigma_{j-1}, \delta_{j-1}]$, respectively. Elements in this extension can be uniquely written in the form $\sum_{i=0}^{n} a_i x_1^{i_1} \cdots x_n^{i_n},\ a_i \in R$.

For this dissertation, an *Ore extension* should be taken to mean an iterated Ore extension where the variables $(x_1, \cdots, x_n)$ have degrees $(\deg(x_1), \cdots, \deg(x_n))$, $\deg(x_i) \in \mathbb{Z}_{\geq 1}$, with $\sigma_j(x_i)$ homogeneous of degree $\deg(x_i)$ and $\delta_j(x_i)$ homogeneous of degree $\deg(x_j) + \deg(x_i)$ for all $n \geq j > i \geq 1$. In particular, such a ring is $\mathbb{N}$-graded.

We refer to $(\deg(x_{i_1}), \cdots, \deg(x_{i_n}))$ as the *degree type* of an Ore extension and require that the degrees be listed in ascending order so that the expression is unique. It is also worth noting that these terms are not standard: what we call a "basic Ore extension" is most commonly known as an Ore extension, but we find it convenient to reserve that term for extensions with the additional properties listed above.

Since both enveloping algebras and Ore extensions have the same basis as a weighted commutative polynomial ring with the same variables and degrees, $\{x_1^{e_1}, \cdots, x_n^{e_n} \mid e_i \in \mathbb{N}\}$, the Hilbert series of these algebras is known: $h(t) = \dfrac{1}{\prod_{i=1}^{n}(1 - t^{\deg(x_i)})}$.

What follows are slightly modified versions of the theorems in Cohn's book Algebra Vol. 2 [Coh, Chapter 12, Theorem 1].

Theorem 2.4.2. *If R is a domain, σ an endomorphism of R, and δ a σ-derivation, then there exists a basic Ore extension $P = R[x, \sigma, \delta]$.*

Proof. Consider the set R^N of finite sequences

$$(c_i) = (c_1, c_2, \cdots),\ c_j \in R$$

as a left R-module and the group homomorphism

$$x : (c_i) \to (\delta(c_i) + \sigma(c_{i-1})),\ \text{where } c_{-1} \text{ is defined to be } 0.$$

Since R is a domain, it acts faithfully on R^N so we may identify R with its image in $\mathrm{End}(R^N)$ and take P to be the subring of $\mathrm{End}(R^N)$ generated by R and x. We claim P is the required Ore extension since for arbitrary $a \in R$,

$$\begin{aligned}
xa(c_i) &= x(ac_i) \\
&= \delta(ac_i) + \sigma(ac_{i-1}) \\
&= \sigma(a)\delta(c_i) + \delta(a)(c_i) + \sigma(a)\sigma(c_{i-1}),\ \text{while} \\
(\sigma(a)x + \delta(a))(c_i) &= \sigma(a)x(c_i) + \delta(a)(c_i) \\
&= \sigma(a)\sigma(c_{i-1}) + \sigma(a)\delta(c_i) + \delta(a)(c_i).
\end{aligned}$$

Hence, $xa = \sigma(a)x + \delta(a)$ in P and it follows that every element of P can be written in the form $a_0 + a_1x + \cdots + a_nx^n,\ a_i \in R$. This expression is unique since

$$(a_0 + a_1x + \cdots + a_nx^n)(1, 0, 0, \cdots) = (a_0, a_1, \cdots, a_n, 0, 0, \cdots)$$

so distinct polynomials represent different elements of P. □

Conversely we have,

Theorem 2.4.3. *Let R be a non-trivial ring and let P be a ring containing R with element $x \in P$ such that elements of P can be written uniquely in the form $f = \sum_{i=0}^{n} a_i x^i$, $a_i \in R$, and satisfy a relation $xa = \sigma(a)x + \delta(a)$ for some $\sigma(a), \delta(a) \in R$. Then σ is an endomorphism, δ a σ-derivation, and $P \cong R[x, \sigma, \delta]$ is a basic Ore extension.*

Proof. We can compute

$$\begin{aligned}
x(a+b) &= \sigma(a+b)x + \delta(a+b) \text{ or alternatively} \\
x(a+b) &= xa + xb \\
&= \sigma(a)x + \delta(a) + \sigma(b)x + \delta(b) \\
&= (\sigma(a) + \sigma(b))x + (\delta(a) + \delta(b)),\ \text{while} \\
x(ab) &= \sigma(ab)x + \delta(ab) \text{ or alternatively} \\
x(ab) &= (\sigma(a)x + \delta(a))b \\
&= (\sigma(a)\sigma(b))x + (\sigma(a)\delta(b) + \delta(a)b).
\end{aligned}$$

By the uniqueness of the form, σ must be an endomorphism of R, δ a σ-derivation, and $P = R[x; \sigma, \delta]$ a basic Ore extension. □

We now consider a presentation for (graded iterated) Ore extensions over a field K. Since we want these to be K-algebras, we are interested in the case where K is central, which means that σ_1 is the identity and δ_1 is the zero mapping. We note that the following two theorems would also hold for ungraded iterated Ore extensions with the term "homogeneous" removed from the proofs, but these are of lesser interest to us. Recall our notation established in Definition 2.4.1: $K_{(j)} := K[x_1] \cdots [x_j, \sigma_j, \delta_j]$.

Theorem 2.4.4. *If K is a field and $P = K[x_1][x_2, \sigma_2, \delta_2] \cdots [x_n, \sigma_n, \delta_n]$ is a (graded iterated) Ore extension, then P has presentation*

$$P \cong \frac{K\langle x_1 \cdots x_n \rangle}{\langle \{r_{ji}\} \rangle}$$

where for each $j > i$, there is a unique homogeneous relation r_{ji}, given by

$$r_{ji} : x_j x_i = \sigma_j(x_i) x_j + \delta_j(x_i),\ \sigma_j(x_i) \text{ and } \delta_j(x_i) \in K_{(j-1)},$$

and these relations satisfy the diamond condition.

Proof. Let K be a field and suppose P is an iterated Ore extension: $P = K[x_1][x_2, \sigma_2, \delta_2] \cdots [x_n, \sigma_n, \delta_n]$. Given $j > i$, $x_j x_i = \sigma_j(x_i) x_j + \delta_j(x_i)$ where σ_j is an endomorphism of $K_{(j-1)}$ and δ_j is a σ-derivation of $K_{(j-1)}$ and every monomial in the equation has degree equal to $\deg(x_j) + \deg(x_i)$. Since these relations allow any element in P to be written as a linear combination of terms of the form $kx_1^{e_1} \cdots x_n^{e_n}$, the leading term of any additional relation would be of this form, which would contradict the fact that $\{x_1^{e_1} \cdots x_n^{e_n}\}$ is a K-basis for the Ore extension P. Thus, there cannot be any additional relations so each r_{ji} is unique, all reduction ambiguities must resolve, and the diamond condition is satisfied. □

We can also prove a converse:

Theorem 2.4.5. *If K is a field and $P = \dfrac{K\langle x_1 \cdots x_n\rangle}{\langle\{r_{ji}\}\rangle}$ where for each $j > i$, there is a unique homogeneous relation r_{ji} given by $x_j x_i = \sigma_j(x_i)x_j + \delta_j(x_i)$, $\sigma_j(x_i)$ and $\delta_j(x_i) \in \dfrac{K\langle x_1, \cdots, x_{j-1}\rangle}{\langle\{r_{ji}\}\rangle}$, and these relations satisfy the diamond condition, then $P \cong K[x_1][x_2, \sigma_2, \delta_2] \cdots [x_n, \sigma_n, \delta_n]$ is a (graded iterated) Ore extension.*

Proof. Assume that we have the unique relations $\{r_{ji}\}$ satisfying the diamond condition and for the purpose of induction assume that

$$R = \frac{K\langle x_1 \cdots x_{m-1}\rangle}{\langle\{r_{ji}\}\rangle} \cong K[x_1][x_2, \sigma_2, \delta_2] \cdots [x_{m-1}, \sigma_{m-1}, \delta_{m-1}]$$

is an Ore extension. Then any monomial in $\dfrac{K\langle x_1 \cdots x_m\rangle}{\langle\{r_{ji}\}\rangle}$ has the form $kx_m^{c_1}s_1x_m^{c_2}s_2 \cdots x_m^{c_q}s_q$, $s_i \in R$ where by induction each s_i can be taken to have the form $kx_1^{f_1} \cdots x_{m-1}^{f_{m-1}}$ since $\{x_1^{e_1} \cdots x_{m-1}^{e_{m-1}}\}$ is a basis for R. By repeated application of the relations $x_m x_i = \sigma_m(x_i)x_m + \delta_m(x_i)$, the monomial can be written in the form $s'_a x_m^a + \cdots + s'_1 x_m + s'_0$, $s'_i \in R$. This representation is unique since the $\{r_{ji}\}$ satisfy the diamond condition by assumption. Thus $\{x_1^{e_1}, \cdots, x_m^{e_m}\}$ is a K-basis by Theorem 2.3.3. Since the relations were also chosen to be homogeneous, $R[x_m, \sigma_m, \delta_m]$ is an Ore extension by Theorem 2.4.3 and by induction, $P \cong K[x_1][x_2, \sigma_2, \delta_2] \cdots [x_n, \sigma_n, \delta_n]$ is an Ore extension. □

Since Ore extensions for which σ_j is an automorphism for all j have especially nice properties, we also find the following result helpful:

Theorem 2.4.6. *In an Ore extension $K[x_1, \sigma_1, \delta_1] \cdots [x_n, \sigma_n, \delta_n]$, for any $1 \leq j \leq n$, if σ_j is injective then it is an automorphism of $K_{(j-1)}$.*

Proof. Let $K^i_{(j-1)}$ denote the ith graded piece of $K_{(j-1)}$, i.e. the set of all degree i homogeneous polynomials in $K_{(j-1)}$. $K^i_{(j-1)}$ has finite K-basis $\{x_1^{f_1}x_2^{f_2}\cdots x_{j-1}^{f_{j-1}} \mid f_1\deg(x_1)+f_2\deg(x_2)+\cdots+f_{j-1}\deg(x_{j-1})=i\}$. By the definition of an Ore extension, we know that σ_j preserves degree on the generators and hence on all of $K_{(j-1)}$. Any injective map from a finite dimensional vector space to itself must also be surjective by the rank-nullity theorem, so for any i and j, $\sigma_j|_{K^i_{(j-1)}} : K^i_{(j-1)} \to K^i_{(j-1)}$ is bijective, and thus σ_j is an automorphism of $K_{(j-1)}$ for all j. □

We are motivated to study Ore extensions because they provide examples of AS-regular algebras. It is a fact that the universal enveloping algebra of a graded Lie algebra is Artin-Schelter regular [FV, Theorem 2.1]. From the presentations provided, it is also clear that any such enveloping algebra is also a specific example of an Ore extension.

Theorem 2.4.7 ([AST, Proposition 2]). *An Ore extension $K[x_1]\cdots[x_n,\sigma_n,\delta_n]$ where σ_j is an automorphism for all $1 \le j \le n$ is AS-regular.*

Motivated by the study of AS-regular algebras, our goal in much of the rest of this dissertation is to classify the possible relation and resolution types of all dimension 5 "Ore extensions," by which we mean "graded iterated Ore extensions with injective (and thus bijective) $\sigma_j's$, generated in degree one." For the sake of brevity, we will wish to have any easy way to refer to such algebras.

Definition 2.4.8. An *AS-Ore* extension is a graded iterated Ore extension with σ_j injective for every $1 \le j \le n$ and which is generated in degree one as a K-algebra.

We note that this definition is in no way standard (and in particular, there are AS-regular algebras that are not generated in degree one and which we have chosen not to study at this time). We also note that the enveloping algebra of a graded Lie algebra which is generated in degree one is also an AS-Ore extension.

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3 Possible degree types of AS-Ore extensions

Our goal in this section is to list the 7 possible degree types for an Ore extension generated in degree one with 5 variables. We note that when considering fully general Ore extensions, we may either order the variables by descending degree ($\deg(x_1) \leq \cdots \leq \deg(x_n)$) at the expense of fully controlling the lexicographic order in the Ore extension $K[x_{i_1}]\cdots[x_{i_n}, \sigma_{i_n}, \delta_{i_n}]$, or we may assume that $x_5 > \cdots > x_1$ in the lexicographic order at the expense of controlling the degrees of these variables. We transfer freely between these two conventions depending on which is more convenient in each situation and the convention we use does not affect the validity of any theorems we prove for general extensions.

Lemma 3.0.9. *If* $A = K[x_1][x_2, \sigma_2, \delta_2]\cdots[x_n, \sigma_n, \delta_n]$ *is an AS-Ore extension and* $\deg(x_k) \neq 1$ *then there exist* i *and* j *with* $\deg(x_i) + \deg(x_j) = \deg(x_k)$.

Proof. Assume $\deg(x_k) > 1$. If the algebra is generated in degree one, $x_k = f(\hat{X})$ where $\hat{X} = \{x \in X \mid \deg(x) = 1\}$, $X = \{x_1, \cdots, x_n\}$, and $f(\hat{X})$ is a (noncommutative) polynomial in variables from $\hat{X}$. This gives the relation $0 = x_k - f(\hat{X})$. By Theorem 2.4.4, $A \cong \dfrac{K\langle x_1, \cdots x_n\rangle}{I}$ where $I = \langle \{r_{ji}\}_{j>i}\rangle$, so any relation is generated by the $\{r_{ji}\}$ and we have that, in the free algebra, $x_k - f(\hat{X}) = \sum_{i,j} p_{ji}(X) r_{ji} q_{ji}(X)$ where $p_{ji}(X)$ and $q_{ji}(X)$ are (noncommutative) polynomials in X and the r_{ji} are the generators of I and hence have degree greater than zero.

Equating polynomials, we find that the monomial x_k must appear in the right side of this equation, so there exist fixed i and j and monomials m_p, m_r, and m_q of p_{ji}, r_{ji}, and q_{ji} with $x_k = m_p m_r m_q$. Since $\deg(m_r) > 0$, we get that m_p and m_q must be scalars and $a x_k$ is a monomial of r_{ji} where $0 \neq a \in K$. So r_{ji} is a relation with leading term $x_j x_i$ and has now been shown to have a scalar multiple of x_k as a term. Since r_{ji} is also homogeneous, this means that $\deg(x_i) + \deg(x_j) = \deg(x_k)$. $\square$

Corollary 3.0.10. *There is no AS-Ore extension with Hilbert series*

$$h(t) = \frac{1}{(1-t)^k \prod_{j=1}^{n-k} (1-t^{i_j})}, \; i_j > 2 \text{ for all } j, \; k < n.$$

Proof. Suppose this is possible. Choose k such that $\deg(x_k)$ is minimal amongst variables with degree greater than 1. By Lemma 3.0.9, $\deg(x_k) = \deg(x_i) + \deg(x_j)$ for some i and j. If $\deg(x_i) = \deg(x_j) = 1$ then this equation says $\deg(x_k) = 2$, but an Ore extension with the given the Hilbert series cannot have any variables of degree two. Otherwise, we can assume that $\deg(x_i) > 1$. Since x_k was chosen to have smallest degree greater than 1, the equation now becomes $\deg(x_k) = \deg(x_i) + \deg(x_j) \geq \deg(x_k) + 1$, which is impossible. Thus, no Ore extension generated in degree one can have this Hilbert series. $\square$

Lemma 3.0.11. *There is no AS-Ore extension with Hilbert series*

$$h(t) = \frac{1}{(1-t)^2(1-t^2)^2 \prod_{j=1}^{n-4}(1-t^{i_j})}, \; i_j \geq 2 \textit{ for all } j.$$

Proof. To find a contradiction, assume that such an extension, A, exists. By Theorem 2.4.4, there exists an ordering on the variables such that $A \cong \dfrac{K\langle x_1, \cdots, x_n\rangle}{\langle\{r_{ji}\}\rangle}$. Let x_n and x_{n-1} be degree one variables and let x_{n-2} and x_{n-3} be distinct degree two variables in $K\langle x_{n-1}, x_n\rangle$ and so of the form

$$x_{n-2} = a_1 x_{n-1}^2 + a_2 x_{n-1}x_n + a_3 x_n x_{n-1} + a_4 x_n^2,$$

$$x_{n-3} = b_1 x_{n-1}^2 + b_2 x_{n-1}x_n + b_3 x_n x_{n-1} + b_4 x_n^2.$$

Without loss of generality, we can assume that $x_n > x_{n-1}$ in the ordering. Also, we must have that $x_{n-2} < x_n$ and $x_{n-3} < x_n$ since an Ore extension cannot have x_{n-2} or x_{n-3} as the leading term of a relation. Since an Ore extension has no leading term of the form x_n^2 and a unique term of the form $x_n x_{n-1}$, we get that $a_4 = b_4 = 0$ and one of a_3 and b_3 is 0. Without loss of generality, assume that $b_3 = 0$. Then the relation $x_{n-3} = b_1 x_{n-1}^2 + b_2 x_{n-1}x_n$ has a leading term inconsistent with an Ore extension. □

Theorem 3.0.12. *For an AS-Ore extension with 5 variables, one of the following options represents the possible degree type of the extension.*

1. $(1, 1, 2, 3, 5)$,
2. $(1, 1, 2, 3, 4)$,
3. $(1, 1, 2, 3, 3)$,

4. (1, 1, 1, 2, 3),

5. (1, 1, 1, 2, 2),

6. (1, 1, 1, 1, 2),

7. (1, 1, 1, 1, 1).

Proof. Clearly an Ore extension with no variables of degree one cannot be generated in degree one.

Similarly there is no Ore extension generated in degree one with just 1 degree one variable. For if there were such an Ore extension and x_k were of minimal degree amongst the remaining 4 variables, Lemma 3.0.9 says that $\deg(x_k) = \deg(x_i) + \deg(x_j) \geq 1 + \deg(x_k)$ and such an inequality is impossible.

If the Ore extension has exactly 2 degree one variables then Corollary 3.0.10 implies that there is at least one variable of degree two. If x_k is of minimal degree amongst the remaining 2 variables then Lemma 3.0.9 tells us that $\deg(x_k) \in \{2, 3\}$ since these are the only possible combinations of $\deg(x_i) + \deg(x_j)$. By Lemma 3.0.11, there can be at most one variable of degree two, so $\deg(x_k) = 3$. Again by Lemma 3.0.9, the final and largest degree variable, x_l, satisfies $\deg(x_l) = \deg(x_i) + \deg(x_j)$ for some i and j so $\deg(x_l) \in \{3, 4, 5\}$. Thus, the list of possible degree types for an Ore extension with exactly 2 degree one variables is

1. (1, 1, 2, 3, 5),

2. (1, 1, 2, 3, 4),

3. (1, 1, 2, 3, 3).

If the Ore extension has exactly 3 degree one variables then it must have at least one degree two variable and the remaining variable, by Lemma 3.0.9, must be of degree two or three. Thus, the list of possible degree types in this case is

4. $(1, 1, 1, 2, 3)$,

5. $(1, 1, 1, 2, 2)$.

If the Ore extension has exactly 4 degree one variables, then Lemma 3.0.9 tells us that the remaining variable must be degree two and the possible degree type is

6. $(1, 1, 1, 1, 2)$.

Finally, it is possible for the Ore extension to have 5 degree one variables and degree type

7. $(1, 1, 1, 1, 1)$. □

While this result technically only restricts the possible degree types of AS-Ore extensions, it is also true that, for each of the 7 possible options listed, there exists an AS-Ore extension with the given type. A commutative ring in five variables is an example of an algebra with type (1,1,1,1,1). For options 2-6, there are enveloping algebras with variables of appropriate degrees (see [FV, Section 3], Theorem 5.2.1, Theorem 6.1.2, Theorem 6.2.2). Finally, we construct an AS-Ore extension with degree type (1,1,2,3,5) in the next section (Theorem 4.0.14).

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4 An AS-Ore extension with degree type (1,1,2,3,5)

For AS-regular algebras of dimension at most 4, it is known that every Hilbert series has a unique relation type and every relation type can be realized by the enveloping algebra of a graded Lie algebra. In their paper, Floystad and Vatne asked whether this held in dimension 5 and constructed, as a counter example, an AS-regular algebra with 2 degree one generators and Hilbert series $h(t) = \dfrac{1}{(1-t)^2(1-t^2)(1-t^3)(1-t^5)}$. By looking at the shifts in the free resolution, they prove that there is no enveloping algebra of a graded Lie algebra with this Hilbert series [FV, Proposition 3.4 and Theorem 4.2]. Based on the presentation of an enveloping algebra given above, we provide an alternate proof of this result.

Proposition 4.0.13. *There is no enveloping algebra of a graded Lie algebra which is generated in degree one with* $h(t) = \dfrac{1}{(1-t)^2(1-t^2)(1-t^3)(1-t^5)}$.

Proof. Assume to the contrary that there is such an algebra. Then there are variables $(x_5, x_4, x_3, x_2, x_1)$ with respective degrees $(1, 1, 2, 3, 5)$. Consider the possible terms in the relations. We will list only those required to show the contradiction.

$$r_{54} : x_5x_4 = x_4x_5 + a_1x_3$$
$$r_{52} : x_5x_2 = x_2x_5$$
$$r_{42} : x_4x_2 = x_2x_4$$
$$r_{32} : x_3x_2 = x_2x_3 + b_1x_1.$$

Here a_1 and b_1 must be nonzero for this algebra to be generated in degree one (by the proof of Lemma 3.0.9) since these are the only relations of degree two and five respectively. Additionally, the middle two relations cannot contain any additional terms since they are degree four and this algebra contains no variable of degree exactly 4. Now consider:

$$\begin{aligned} x_5(x_4x_2) &= x_5x_2x_4 \\ &= x_2x_5x_4 \\ &= x_2x_4x_5 + a_1x_2x_3, \text{ while} \\ (x_5x_4)x_2 &= x_4x_5x_2 + a_1x_3x_2 \\ &= x_4x_2x_5 + a_1x_2x_3 + a_1b_1x_1 \\ &= x_2x_4x_5 + a_1x_2x_3 + a_1b_1x_1. \end{aligned}$$

In order for this overlap to resolve, $a_1b_1 = 0$, which is impossible if this algebra is generated in degree one. □

It is natural to ask whether every relation type can be realized by a generalization of an enveloping algebra, in particular an AS-Ore extension. In the literature for algebras of dimension 5 there are currently only two known relation types that cannot be realized by an enveloping algebra. One has Hilbert series $h(t) = \frac{1}{(1-t)^2(1-t^2)(1-t^3)^2}$, relation type $(4,4,4,5,5)$, and can be realized by an AS-Ore extension [WW, Section 5.2].

In support of the hypothesis that all relation types can be realized by an AS-Ore extension, we present an example of an AS-Ore extension with the other known relation type that cannot be realized by an enveloping algebra. This is equivalent to finding an example of an AS-Ore extension with the appropriate Hilbert series since Floystad and Vatne have already classified the possible relation types of algebras with two generators [FV, Theorem 5.6], and the relation type of an algebra with this Hilbert series is unique.

The following example was found by writing the general relations provided by Theorem 2.4.4 and using the mathematical software program Mathematica to solve the large system of equations that result from setting overlap ambiguities equal to 0. This proved to be an overwhelming project for the computer and some coefficients were ultimately assumed to be 0 to make the computations possible, as our goal was to prove the existence of such an algebra rather than to completely classify the possible families of relations.

Theorem 4.0.14. *The following relations define an AS-Ore extension (an iterated Ore extension which is graded, generated in degree one, and has each* σ_i *an injection) which has*

$h(t) = \dfrac{1}{(1-t)^2(1-t^2)(1-t^3)(1-t^5)}$ *and relation type (3,4,7):*

$$r_{21} : x_2x_1 = -x_1x_2$$

$$r_{32} : x_3x_2 = x_1 + bx_2x_3$$

$$r_{31} : x_3x_1 = -x_1x_3$$

$$r_{43} : x_4x_3 = x_2 + bx_3x_4$$

$$r_{42} : x_4x_2 = b^2x_2x_4$$

$$r_{41} : x_4x_1 = x_1x_4$$

$$r_{54} : x_5x_4 = x_3 + x_4x_5$$

$$r_{53} : x_5x_3 = -x_3x_5$$

$$r_{52} : x_5x_2 = -x_2x_5 - b^2x_3x_3$$

$$r_{51} : x_5x_1 = x_1x_5 + cx_3x_3x_3,$$

where $b = e^{\frac{4\pi i}{3}}$ *and* $c = \frac{2b^2}{1-b+b^2}$.

Proof. Let the degrees of $(x_5, x_4, x_3, x_2, x_1)$ be $(1, 1, 2, 3, 5)$ with lexicographic order $x_5 > \cdots > x_1$ so that the leading terms are as presented.

We note that, for the given degrees of these variables, each of the above relations is homogeneous. To check that this is an Ore extension, we then check that all reduction ambiguities resolve. All ambiguities have the form $x_kx_jx_i$ where $k > j > i$ and there are a total of 10 such ambiguities for this set of relations. A computation shows that, for the given choice of b and c, all overlaps resolve. We carry out this computation in Mathematica and the code for these and future computations can be found on the author's website [Ell, Section 1]. Thus, $\{x_1^{e_1} \cdots x_n^{e_n}\}$ is a basis for this algebra and this is Ore by Theorem 2.4.5.

It remains to check that this is generated in degree one and that σ_j is injective for all j. To see that this algebra is generated in degree one, note that r_{32}, r_{43}, and r_{54} can be solved for x_1, x_2, and x_3 respectively and so everything may be expressed in terms of the degree one generators x_4 and x_5. Note that for all $1 \leq i < j \leq 5$, $\sigma_j(x_i) = a_{ji}x_i$ where a_{ji} is a root of unity. Thus, $\sigma_j^{n_j}$ is the identity map for some n_j and so each homomorphism is injective. Thus, this algebra is an AS-Ore extension.

That the algebra has the desired Hilbert series is now immediate from the fact that is has the same basis, and therefore the same Hilbert series, as the weighted commutative polynomial ring with the same variables and degrees. We again note that the relation type of an algebra with this Hilbert series is known to be (3,4,7) [FV, Theorem 5.6], although the computations proving it in this case are also included in the online code. □

An algebra A is called *polynomial identity* or PI if there exists a nonzero polynomial $f(x_1, \cdots, x_m) \in K\langle x_1, \cdots, x_m\rangle$ such that $f(a_1, \cdots, a_m) = 0$ for all $a_1, \cdots, a_m \in A$. Any commutative algebra is PI, satisfying $f(x_1, x_2) = x_1x_2 - x_2x_1$. For all relation types of AS-regular algebras of dimension at least 4, there is an example of an algebra which is PI. Thus, it is natural to ask whether the examples we discover are PI.

An enveloping algebra which is generated in degree 1 and which is not already commutative is not PI [Pas, Theorem 1.3]. On the other hand, if R is PI and σ injective then the Ore extension $R[x, \sigma, \delta]$ is PI if and only if there is a nonconstant polynomial in the center of $R[x, \sigma, \delta]$ [LM, Theorem 2.7], in which case we say that the center of $R[x, \sigma, \delta]$, $Z(R[x, \sigma, \delta])$ is *nontrival.*

Theorem 4.0.15. *The above example of an AS-Ore extension is PI.*

Proof. Let $R = K[x_1][x_2, \sigma_2, \delta_2] \cdots [x_5, \sigma_5, \delta_5]$ be as defined by the relations in Theorem 4.0.14. Note that $K[x_1]$ is PI since it is commutative. Thus it will suffice to show that the center of $K_{(i)}$ is nontrivial for $2 \leq i \leq 5$.

$$\begin{aligned} x_2x_2x_1 &= x_2(-x_1x_2) \\ &= x_1x_2x_2 \end{aligned}$$

so $x_2^2 \in Z(K_{(2)})$.

$$\begin{aligned} x_3x_3x_1 &= x_3(-x_1x_3) \\ &= x_1x_3x_3 \\ x_3^6x_2 &= x_3^5(x_1 + bx_2x_3) \\ &= x_3^4[(-1+b)x_1x_3 + b^2x_2x_3x_3] \\ &= x_3^3[(1-b+b^2)x_1x_3x_3 + b^3x_2x_3x_3x_3] \\ &= x_3^2[(-1+b-b^2+b^3)x_1x_3x_3x_3 + b^4x_2x_3x_3x_3x_3] \\ &= x_3[(1-b+b^2-b^3+b^4)x_1x_3x_3x_3x_3 + b^5x_2x_3x_3x_3x_3x_3 \\ &= (-1+b-b^2+b^3-b^4+b^5)x_1x_3x_3x_3x_3x_3 + b^6x_2x_3x_3x_3x_3x_3x_3 \\ &= x_2x_3^6 \end{aligned}$$

so $x_3^6 \in Z(K_{(3)})$.

$$\begin{aligned}
x_4x_1 &= x_1x_4 \\
x_4^3x_2 &= x_4^2(b^2x_2x_4) \\
&= x_4(b^4x_2x_4x_4) \\
&= b^6x_2x_4x_4x_4 \\
&= x_2x_4x_4x_4 \\
x_4^3x_3 &= x_4^2(x_2 + bx_3x_4) \\
&= x_4[(b+b^2)x_2x_4 + b^2x_3x_4x_4] \\
&= (b^2+b^3+b^4)x_2x_4x_4 + b^3x_3x_4x_4x_4 \\
&= x_3x_4x_4x_4
\end{aligned}$$

so $x_4^3 \in Z(K_{(4)})$. We will now show that a power of x_5 is also in the center, leaving some of the details in the calculation to the reader.

$$\begin{aligned}
x_5x_5x_1 &= x_5(x_1x_5 + cx_3x_3x_3) \\
&= x_1x_5x_5 \\
x_5x_5x_2 &= x_5(-x_2x_5 - b^2x_3x_3) \\
&= x_2x_5x_5 \\
x_5x_5x_3 &= x_5(-x_3x_5) \\
&= x_3x_5x_5 \\
x_5x_5x_4 &= x_5(x_3 + x_4x_5) \\
&= x_4x_5x_5
\end{aligned}$$

so $x_5^2 \in Z(K_{(5)} = R)$ and R is PI. □

This, together with existing results in the field, completes the classification of degree types of dimension 5 AS-Ore extensions with 2 generators, with one exception:

Question 4.0.16. *Is there an AS-Ore extension with degree type (1,1,2,3,3) and relation type (4,4,4,5)?*

Our initial computations suggest that there is no such extension. In particular, we have found that this is not a possible relation type for an Ore extension $K[x_1][x_2, \sigma_2, \delta_2][x_3, \sigma_3, \delta_3][x_4, \sigma_4, \delta_4][x_5, \sigma_5, \delta_5]$ where $(x_5, x_4, x_3, x_2, x_1)$ have respective degrees (1,1,2,3,3) and we have tested many Ore extensions with a different ordering on the variables $K[x_{i_1}] \cdots [x_{i_5}, \sigma_{i_5}, \delta_{i_5}]$, although we have not tested all possible orderings of the variables or carefully checked any of these computations.

More generally, the possible relation types of dimension 5 AS-regular algebras which are generated in degree one with 2 generators and which are noetherian domains with GK dimension at least 4 are known, with the same exception:

Question 4.0.17. *Is there a dimension 5 AS-regular algebra which is generated in degree one by 2 generators and which has relation type (4,4,4,5)?*

Answering this question is of interest in the general classification of AS-regular algebras, but would be especially intriguing if it provided an example of a relation type that cannot be realized by any AS-Ore extension.

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5 A classification of relation types for AS-Ore extensions with 4 degree one generators

We now begin the process of attempting to classify all possible resolution types of dimension 5 AS-Ore extensions generated in degree one, beginning with the case where the algebra has 4 degree one generators. It will be convenient to alternate between thinking of the algebra as an Ore extension with the presentation given by Theorem 2.4.4, $A \cong \dfrac{K\langle x_1, \cdots, x_n\rangle}{\langle\{r_{ji}\}\rangle}$, and as an algebra presented in terms of its degree one generators, $A \cong \widetilde{A}\colon = \dfrac{K\langle x_1, \cdots, x_b\rangle}{I}$. We will fix the notation that A refers to the algebra viewed as an AS-Ore extension presented by 5 generators and that $\widetilde{A}$ is an algebra isomorphic to A, viewed as generated in degree one. We get it from A by changing the ordering on the variables so that $x_i > x_j$ whenever $\deg(x_i) > 1$ and $\deg(x_j) = 1$, making a choice that allows us to solve the Ore relations for all variables that are not degree one, and writing the remaining relations in terms of the degree one generators. Changing the ordering of the variables may change the Gröbner basis of the algebra, but will of course not change the Hilbert series. By the construction of

$\widetilde{A}$, we also note that its minimal generating set cannot contain more elements of a particular degree than what the minimal generating set of A has.

By Corollary 2.2.5, the free resolution of any dimension 5 regular algebra generated in degree one is

$$0 \to A(-l) \to A(-l+1)^b \to \bigoplus_{i=1}^{n} A(-l+a_i) \to \bigoplus_{i=1}^{n} A(-a_i) \to A(-1)^b \to A \to K \to 0$$

where b represents the number of degree one generators, l the total shift of the resolution, n the number of relations in the minimal generating set of the ideal I, and a_i the homogeneous degree of the ith relation of a fixed minimal generating set of I. By the symmetry of this free resolution, the resolution type is uniquely determined by the a_i together with the Hilbert series of the algebra since the series will determine the value of l. Thus, it suffices to classify the possible relation types $(a_1, \cdots, a_n)$, $a_1 \leq \cdots \leq a_n$, of dimension 5 AS-Ore algebras.

Recall by Theorem 3.0.12 that an AS-Ore extension, A, with 5 variables and 4 degree one generators will have degree type (1,1,1,1,2). Since the relations for an Ore extension come from the $\{r_{ji}\}$ as described in Theorem 2.4.4 (and there are no additional relations since overlaps resolve by the same theorem), A will have 6 degree two relations in the Gröbner basis, 4 relations of degree three, and no relations of degree four or larger.

Let $\widetilde{A}$ denote the same algebra, A, viewed as an algebra generated in degree one via the process explained above where 1 degree two relation of A will be used to express the degree two variable in terms of the generators and the remaining 5 will be part of the minimal generating set of $\widetilde{A}$. Since A has 4 degree three relations, $\widetilde{A}$ will have at most 4 degree three relations. It is possible that $\widetilde{A}$ will have fewer than

4 K-independent degree three relations or for these relations to be consequences of overlaps that fail to resolve rather than part of the minimal generating set. Since A has no relations of degree more than three, any relations of degree more than three in $\widetilde{A}$ must be consequences of overlaps that fail to resolve and so not part of the minimal generating set of $\widetilde{A}$. Thus, the only candidates for the relation type of an AS-Ore extension with degree type (1,1,1,1,2) are:

1. (2,2,2,2,2),
2. (2,2,2,2,2,3),
3. (2,2,2,2,2,3,3),
4. (2,2,2,2,2,3,3,3),
5. (2,2,2,2,2,3,3,3,3).

In the next theorems, we will classify all possible relation types of an AS-Ore extension with the given degree type. We prove that types (4) and (5) above are impossible, that types (1) and (3) can be realized by enveloping algebras, and that type (2) can be realized by an AS-Ore extension but not by an enveloping algebra. This differs slightly from the comment in [FV, Section 3] where examples of enveloping algebras of types (1) and (3) are explicitly presented but the reader is encouraged to also check that there is an example of an enveloping algebras of type (2).

In order to prove that certain relation types are impossible, we need to know more about the specific leading terms and the overlaps that come from the degree two relations. We use a simplified version of Hilbert driven Gröbner basis computation, a technique used by Rogalski and Zhang to study $\mathbb{Z}^2$-graded dimension 4 algebras with 3 generators [RZ] and later used by Zhou and Lu to classify possible families of relations of $\mathbb{Z}^2$-graded dimension 5 algebras with 2 generators [ZL].

Let A be an AS-Ore extension with degree type (1,1,1,1,2). Then A has Hilbert series $h_A(t) = \dfrac{1}{(1-t)^4(1-t^2)} = 1 + 4t + 11t^2 + 24t^3 + O(t^4)$. The idea of Hilbert driven Gröbner basis computation is to construct $\widetilde{A}$ by viewing it as a free algebra on its degree one generators modulo an ideal I, and to identify the generators of I by comparing the Hilbert series of the constructed algebra against the known Hilbert series. In order to do this one dimension at a time, we will use a *monomial algebra* which we get by replacing each relation in the Gröbner basis with just the leading term of the relation. The details of this construction can be found in [ZL, Section 2], along with the proof that the monomial algebra will have the same Hilbert series as the original algebra [ZL, Lemma 2.1]. Let $\widetilde{A}_0$ denote the free algebra on four generators. Then $h_{\widetilde{A}_0}(t) = 1 + 4t + 16t^2 + 64t^3 + O(t^4)$ and $h_{\widetilde{A}_0}(t) - h_{\widetilde{A}}(t) = 5t^2 + O(t^3)$. Thus, I must contain 5 degree two relations. Although we already knew this, the method can be used to find the number of relations in the basis of higher degrees, although the analysis does depend on which leading terms we choose for our relations.

Let x_1 be the degree two variable and without loss of generality, list the degree one variables so that $x_2 < x_3 < x_4 < x_5$. The degree two LT's (leading terms) in $\widetilde{A}$ must come from the $\{r_{ji}\}$ relations of the Ore extension, A, and so must belong to the list $\{x_3x_2, x_4x_3, x_4x_2, x_5x_4, x_5x_3, x_5x_2\}$. If the set of degree two LTs in I is $\{x_3x_2, x_4x_3, x_4x_2, x_5x_4, x_5x_3\}$, then let us denote the monomial algebra which has these leading terms as its relations by $\widetilde{A}_2$ since the Hilbert series agrees with that of $\widetilde{A}$ up to dimension 2. Then $h_{\widetilde{A}_2} - h_{\widetilde{A}} = 4t^3 + O(t^4)$ so there must be 4 degree three relations in the Gröbner basis. The calculations for the difference of these Hilbert series were done in Mathematica, and the code is available online [Ell, Section 2].

If instead we start with LTs in I $\{x_3x_2, x_4x_3, x_5x_4, x_5x_3, x_5x_2\}$ or $\{x_3x_2, x_4x_3, x_4x_2, x_5x_4, x_5x_2\}$, then $h_{\widetilde{A}_2} - h_{\widetilde{A}} = 3t^3 + O(t^4)$ and there are 3 degree three relations in the basis. If we start with LTs $\{x_4x_3, x_4x_2, x_5x_4, x_5x_3, x_5x_2\}$, $\{x_3x_2, x_4x_2, x_5x_4, x_5x_3, x_5x_2\}$, or $\{x_3x_2, x_4x_3, x_4x_2, x_5x_3, x_5x_2\}$ then there will be 2 degree three relations by a similar analysis.

So we have found that the Gröbner basis of an AS-Ore extension with the given degree type has 5 degree two relations and 2-4 degree three relations. We could continue the process to see what the possible LTs of degree three are and if the basis has additional relations of higher degree, but it is not useful for our analysis. We remain more interested in the number and degrees of the minimal generators since these completely classify the possible relation types of the algebra, and we know the algebra has no minimal relations of degree greater than three. We still need to investigate, in each case, which of the degree three relations are part of the minimal generating set and which are simply consequences of overlaps that fail to resolve.

5.1 No relation types with 3 or more minimal degree three relations

We will first eliminate the possibility that there are 4 independent degree three relations in the minimal generating set.

Theorem 5.1.1. *There is no AS-Ore extension with degree type* $(1, 1, 1, 1, 2)$ *and minimal relation type* $(2, 2, 2, 2, 2, 3, 3, 3, 3)$.

Proof. Assume to the contrary that there is such an AS-Ore extension $A = K[x_{i_1}]\cdots[x_{i_5}, \sigma_{i_5}, \delta_{i_5}]$. Label the degree two variable x_1 and label the degree one variables so that $x_2 < x_3 < x_4 < x_5$ in the ordering. (We make no assumption about when x_1 is adjoined.) The list of reduced degree 2 monomials is

$$\{x_1, x_2x_2, x_2x_3, x_3x_3, x_2x_4, x_3x_4, x_4x_4, x_2x_5, x_3x_5, x_4x_5, x_5x_5\}.$$

From Theorem 2.4.4, x_j must occur only to the first power in the relation with leading term x_jx_i and x_k should not appear for any $x_k > x_j$. Based on these observations, we will write the most general possible degree two relations:

$$\begin{aligned}
r_{32} &: x_3x_2 = b_1x_1 + b_2x_2x_2 + b_3x_2x_3 \\
r_{42} &: x_4x_2 = e_1x_1 + e_2x_2x_2 + e_3x_2x_3 + e_4x_2x_4 + e_6x_3x_3 + e_7x_3x_4 \\
r_{43} &: x_4x_3 = d_1x_1 + d_2x_2x_2 + d_3x_2x_3 + d_4x_2x_4 + d_6x_3x_3 + d_7x_3x_4 \\
r_{52} &: x_5x_2 = i_1x_1 + i_2x_2x_2 + i_3x_2x_3 + i_4x_2x_4 + i_5x_2x_5 + i_6x_3x_3 + i_7x_3x_4 + i_8x_3x_5 \\
&\qquad + i_9x_4x_4 + i_{10}x_4x_5 \\
r_{53} &: x_5x_3 = h_1x_1 + h_2x_2x_2 + h_3x_2x_3 + h_4x_2x_4 + h_5x_2x_5 + h_6x_3x_3 + h_7x_3x_4 \\
&\qquad + h_8x_3x_5 + h_9x_4x_4 + h_{10}x_4x_5 \\
r_{54} &: x_5x_4 = g_1x_1 + g_2x_2x_2 + g_3x_2x_3 + g_4x_2x_4 + g_5x_2x_5 + g_6x_3x_3 + g_7x_3x_4 \\
&\qquad + g_8x_3x_5 + g_9x_4x_4 + g_{10}x_4x_5.
\end{aligned}$$

(It is worth noting that if x_1 is adjoined late, some of these coefficients must be zero, although this fact will not be needed to complete the contradiction. For example, if the Ore extension is $K[x_2][x_3, \sigma_3, \delta_3][x_1, \sigma_1, \delta_1][x_4, \sigma_4, \delta_4][x_5, \sigma_5, \delta_5]$, then b_1 must be zero since $x_3x_2 = \sigma_3(x_2)x_3 + \delta_3(x_2)$ where δ_3 is a derivation of $K[x_2]$ and thus cannot map x_2 to a term containing x_1.)

Without loss of generality, we can solve for x_1 using the relation with smallest leading term that has a nonzero coefficient of x_1. For example, if $b_1 \neq 0$ then we can solve r_{32} to find that $x_1 = \frac{1}{b_1}(x_3x_2 - b_2x_2x_2 - b_3x_2x_3)$. Consider now the Gröbner basis of the algebra $\widetilde{A}$, viewed as an algebra generated in degree one.

The preceding analysis shows that the list of degree two LTs of $\widetilde{A}$ must be $\{x_3x_2, x_4x_3, x_4x_2, x_5x_4, x_5x_3\}$ since this is the only set of LTs that has 4 degree three relations in the Gröbner basis. In particular, x_5x_2 is not a leading term. But if any of b_1, e_1, or d_1 are non-zero, then x_1 can be expressed in terms of monomials smaller than x_5x_2 and the LT of r_{52} will be x_5x_2, a contradiction. We may therefore assume that $b_1 = e_1 = d_1 = 0$ and we may assume that i_1 is not zero since again this would otherwise make the leading term of r_{52} equal to x_5x_2. Thus $\widetilde{A}$ is obtained by using r_{52} to write $x_1 = \frac{1}{i_1}(x_5x_2 - i_{10}x_4x_5 - \cdots - i_2x_2x_2)$ and substituting this expression for x_1 in the other relations.

We are interested in the coefficient of $x_5x_2x_3$ in the reduction of the overlap $x_5x_3x_2$ in $\widetilde{A}$ since this will provide the contradiction. We compute:

$$\begin{aligned}
x_5(x_3x_2) &= x_5(0x_1 + b_3x_2x_3 + [\text{smaller terms}]) \\
&= b_3x_5x_2x_3 + [\text{smaller terms}], \text{ while} \\
(x_5x_3)x_2 &= (h_1x_1 + h_{10}x_4x_5 + [\text{smaller terms}])x_2 \\
&= (\frac{h_1}{i_1}(x_5x_2 - i_{10}x_4x_5 - [\text{smaller terms}]) + h_{10}x_4x_5 + [\text{smaller terms}])x_2 \\
&= \frac{h_1}{i_1}x_5x_2x_2 + [\text{smaller terms}].
\end{aligned}$$

Note that $x_5x_2x_3$ is a reduced word with respect to the degree two relations of $\widetilde{A}$. Thus $x_5(x_3x_2) - (x_5x_3)x_2 = b_3x_5x_2x_3 +$ [smaller terms] and $b_3 = 0$ if this overlap resolves. However, if $b_3 = 0$ then r_{32} becomes

$$x_3x_2 = \sigma_3(x_2)x_3 + \delta_3(x_2) = 0x_2x_3 + 0x_1 + b_2x_2x_2.$$

This would suggest that $\sigma_3(x_2) = 0$ and σ_3 is not injective, which contradicts the claim that the initial algebra was an AS-Ore extension. Thus, b_3 is not zero, this overlap does not resolve, at least one of the degree three relations in the Gröbner basis of this algebra is a consequence of an overlap, and so there are not 4 degree three relations in the minimal generating set. □

With a little more work, the same technique can be used to show that there cannot be 3 degree three relations in the minimal generating set.

Theorem 5.1.2. *There is no AS-Ore extension generated in degree one with degree type* $(1, 1, 1, 1, 2)$ *and relation type* $(2, 2, 2, 2, 2, 3, 3, 3)$.

Proof. Assume to the contrary. As in the previous theorem, label the degree two variable x_1 and label the degree one variables so that $x_2 < x_3 < x_4 < x_5$ in the ordering. Then the general form of the possible degree two relations is the same as in the previous theorem. We will consider the different cases where the set of degree two leading terms leads to Gröbner bases with at least 3 degree three relations.

Case 1: The set of degree two LTs in $\widetilde{A}$ is $\{x_3x_2, x_4x_3, x_4x_2, x_5x_4, x_5x_3\}$.

By the same argument as that in Theorem 5.1.1, since x_5x_2 is not a leading term, the relations with smaller leading terms must have coefficient in front of x_1 equal to zero and r_{52} must have a nonzero coefficient in front of the x_1. Thus, $b_1 = e_1 = d_1 = 0$, $i_1 \neq 0$, there are 4 degree three relations in the Gröbner basis, and we have already seen in the proof of Theorem 5.1.2 that $x_5x_3x_2$ is an ambiguity that fails to resolve. Taking $x_5(x_3x_2) - (x_5x_3)x_2$ in $\widetilde{A}$ gives a new degree three relation with leading term $x_5x_2x_3$. When evaluating whether other degree three overlaps resolve, we should reduce them modulo this new relation, but this will not be necessary in the calculations that follow since $x_5x_2x_3$ is too small to affect the analysis of whether the overlaps resolve. Now consider the coefficient of $x_5x_2x_4$ in the following reduction of overlaps in $\widetilde{A}$, which we obtain from A by solving r_{52} for x_1:

$$\begin{aligned}
x_5(x_4x_2) &= x_5(0x_1 + e_4x_2x_4 + e_6x_3x_3 + e_7x_3x_4 + [\text{smaller terms}]) \\
&= e_4x_5x_2x_4 + e_6x_5x_3x_3 + e_7x_5x_3x_4 + [\text{small}] \\
&= e_4x_5x_2x_4 + e_6(h_1x_1 + [\text{small}])x_3 + e_7(h_1x_1 + [\text{small}])x_4 + [\text{small}] \\
&= e_4x_5x_2x_4 + e_6(\frac{h_1}{i_1}x_5x_2 - [\text{small}])x_3 + e_7(\frac{h_1}{i_1}x_5x_2 - [\text{small}])x_4 + [\text{small}] \\
&= (e_4 + \frac{e_7h_1}{i_1})x_5x_2x_4 + [\text{small}], \text{while} \\
(x_5x_4)x_2 &= (g_1x_1 + [\text{smaller terms}])x_2 \\
&= \frac{g_1}{i_1}x_5x_2x_2 + [\text{small}]
\end{aligned}$$

So $x_5(x_4x_2) - (x_5x_4)x_2 = (e_4 + \frac{e_7h_1}{i_1})x_5x_2x_4 + [\text{smaller terms}]$.

Similarly, $x_5(x_4x_3) - (x_5x_4)x_3 = (d_4 + \frac{d_7h_1}{i_1})x_5x_2x_4 + [\text{smaller terms}]$. From the relations

$$r_{42} : x_4x_2 = e_1x_1 + e_2x_2x_2 + e_3x_2x_3 + e_4x_2x_4 + e_6x_3x_3 + e_7x_3x_4 \text{ and}$$
$$r_{43} : x_4x_3 = d_1x_1 + d_2x_2x_2 + d_3x_2x_3 + d_4x_2x_4 + d_6x_3x_3 + d_7x_3x_4,$$

we see that

$$\sigma_4(x_2) = e_4x_2 + e_7x_3 \text{ and}$$
$$\sigma_4(x_3) = d_4x_2 + d_7x_3.$$

Thus, σ_4 is injective if and only if $\det \begin{bmatrix} e_4 & e_7 \\ d_4 & d_7 \end{bmatrix} \neq 0$. If we assume that both overlaps resolve, $e_4 = -\dfrac{e_7h_1}{i_1}$ and $d_4 = -\dfrac{d_7h_1}{i_1}$, so $\begin{vmatrix} e_4 & e_7 \\ d_4 & d_7 \end{vmatrix} = \begin{vmatrix} \dfrac{-e_7h_1}{i_1} & e_7 \\ \dfrac{-d_7h_1}{i_1} & d_7 \end{vmatrix} = 0$ and σ_4 is not injective, a contradiction. Thus, at least one of the two overlaps, $x_5x_4x_3$ or $x_5x_4x_2$, fails to resolve. In total, there are at least two overlaps that do not resolve so there are at most 2 degree three relations in the minimal generating set.

Case 2: The set of degree two LTs in $\widetilde{A}$ is $\{x_3x_2, x_4x_3, x_4x_2, x_5x_4, x_5x_2\}$. In this case, x_5x_3 is not a leading term and similar reasoning as that used in Theorem 5.1.1 allows us to conclude that $b_1 = e_1 = d_1 = i_1 = 0$, $h_1 \neq 0$, and there are 3 degree three relations in the Gröbner basis. We solve r_{53} for x_1 and then, working in $\widetilde{A}$, compute $x_5(x_4x_2) - (x_5x_4)x_2 = e_7x_5x_3x_4 +$ [smaller terms] and $x_5(x_4x_3) - (x_5x_4)x_3 = d_7x_5x_3x_4 +$ [smaller terms]. These computations can be done by hand by looking at the largest terms in the reduction, just as in the previous case, but we omit the details. The code used for all calculations in this proof is on the author's website [Ell, Section 3]. If these overlaps both resolve, $e_7 = 0$ and $d_7 = 0$, $\begin{vmatrix} e_4 & e_7 \\ d_4 & d_7 \end{vmatrix} = 0$, and σ_4 is not injective, which is a contradiction. Thus, one of these overlaps must fail to resolve and the minimal generating set has at most 2 relations of degree three.

Case 3: The set of degree two LTs in $\widetilde{A}$ is $\{x_3x_2, x_4x_3, x_5x_4, x_5x_3, x_5x_2\}$. In this case, x_4x_2 is not a leading term so $b_1 = 0$, $e_1 \neq 0$, and there are 3 degree three relations in the Gröbner basis. After solving r_{42} for x_1, $x_4(x_3x_2) - (x_4x_3)x_2 = b_3x_4x_2x_3 +$ [smaller terms]. This computation can be done by hand, but we omit the details here. If σ_3 is injective, b_3 cannot be zero. Thus, at least one overlap fails to resolve and the minimal generating set has at most 2 relations of degree three.

In all cases where the Gröbner basis has at least 3 degree three relations, we find that the minimal generating set has at most 2 degree three relations, so there is no AS-Ore extension with relation type $(2, 2, 2, 2, 2, 3, 3, 3)$. □

5.2 Relation types of enveloping algebras with degree type (1,1,1,1,2)

There exist algebras with relation types with 0, 1, and 2 degree three relations. We begin by considering what relation types can be realized by enveloping algebras.

Theorem 5.2.1. *There are enveloping algebras with degree type (1,1,1,1,2) and* *relation type (2,2,2,2,2) and (2,2,2,2,2,3,3), but no enveloping algebra with relation type (2,2,2,2,2,3).*

Proof. An enveloping algebra with such a degree type can be taken to have $x_5 > x_4 > x_3 > x_2 > x_1$ with deg(x_1)=2 and is then defined by the relations

$$r_{21} : x_2x_1 = x_1x_2$$

$$r_{31} : x_3x_1 = x_1x_3$$

$$r_{41} : x_4x_1 = x_1x_4$$

$$r_{51} : x_5x_1 = x_1x_5$$

$$r_{32} : x_3x_2 = b_1x_1 + x_2x_3$$

$$r_{43} : x_4x_3 = d_1x_1 + x_3x_4$$

$$r_{42} : x_4x_2 = e_1x_1 + x_2x_4$$

$$r_{54} : x_5x_4 = g_1x_1 + x_4x_5$$

$$r_{53} : x_5x_3 = h_1x_1 + x_3x_5$$

$$r_{52} : x_5x_2 = i_1x_1 + x_2x_5.$$

These relations are homogeneous, overlaps resolve (see [Ell, Section 4]), σ_j is the identity for all $1 \leq j \leq n$, δ_j is linear for all $1 \leq j \leq n$, and this is generated in degree one if at least 1 of b_1, d_1, e_1, g_1, h_1, or i_1 is nonzero. In this case, by Theorem 2.3.4, this is an enveloping algebra and it remains to find its relation type.

By the symmetry of the relations, we may assume without loss of generality that b_1 is nonzero and write $x_1 = \dfrac{-x_2x_3 + x_3x_2}{b_1}$. We can now view the algebra as $\widetilde{A}$, something generated in degree one, so that the set of LTs of degree two relations is $\{x_4x_3, x_4x_2, x_5x_4, x_5x_3, x_5x_2\}$. Given this set of degree two LTs, we see from the analysis preceding Theorem 5.1.1 that the Gröbner basis of $\widetilde{A}$ has 2 degree three relations. Further, the degree three overlaps that come from this list of degree two LTs are $x_5x_4x_3$ and $x_5x_4x_2$. We calculate:

$$\begin{aligned} x_5(x_4x_3) - (x_5x_4)x_3 &= (\frac{g_1}{b_1} - \frac{e_1h_1}{b_1^2} + \frac{d_1i_1}{b_1^2})x_2x_3x_3 \\ &\quad + (\frac{-2g_1}{b_1} + \frac{2e_1h_1}{b_1^2} - \frac{2d_1i_1}{b_1^2})x_3x_2x_3 + (\frac{g_1}{b_1} - \frac{e_1h_1}{b_1^2} + \frac{d_1i_1}{b_1^2})x_3x_3x_2, \text{ and} \\ x_5(x_4x_2) - (x_5x_4)x_2 &= (\frac{-g_1}{b_1} + \frac{e_1h_1}{b_1^2} - \frac{d_1i_1}{b_1^2})x_2x_2x_3 \\ &\quad + (\frac{2g_1}{b_1} - \frac{2e_1h_1}{b_1^2} + \frac{2d_1i_1}{b_1^2})x_2x_3x_2 + (\frac{-g_1}{b_1} + \frac{e_1h_1}{b_1^2} - \frac{d_1i_1}{b_1^2})x_3x_2x_2. \end{aligned}$$

As usual, the details for these calculations are omitted here but included in the code posted online [Ell, Section 4]. If $g_1 = \frac{e_1h_1}{b_1} - \frac{d_1i_1}{b_1}$ then both of these overlaps resolve, the degree three relations in the Gröbner basis are independent of overlaps, and the relation type is (2,2,2,2,2,3,3). Otherwise, these are two K-independent overlaps that fail to resolve (with LTs $x_3x_3x_2$ and $x_3x_2x_2$) and the minimal relation type is (2,2,2,2,2). Since any enveloping algebra must have one of these two relation types, the theorem is proven. □

Recall that an enveloping algebra cannot be PI. This gives rise to the following question:

Question 5.2.2. *Are there PI AS-Ore extensions with relation types (2,2,2,2,2) and (2,2,2,2,2,3,3)?*

5.3 AS-Ore extension with relation type (2,2,2,2,2,3)

Although there is no enveloping algebra with relation type (2,2,2,2,2,3), it is possible to construct an AS-Ore extension with this type. Our process for doing this is similar to that used to find the AS-Ore extension of degree type (1,1,2,3,5) in Theorem 4.0.14. We use Theorem 2.4.4 (page 22) to write general relations for the extension and a mathematical program to evaluate the possible values of coefficients that make it so that those relations satisfy the diamond condition. This problem is generally too complex for the computer to handle, so we can set some coefficients equal to zero to simplify the process. In this case, we also have to determine how many degree three relations the Gröbner basis of the algebra generated in degree one has, as well as whether these relations are part of the minimal generating set as opposed to consequences of overlaps that do not resolve.

Theorem 5.3.1. *There is an AS-Ore extension with degree type (1,1,1,1,2) and relation type (2,2,2,2,2,3).*

Proof. Consider the algebra defined by the relations

$$r_{21} : x_2x_1 = x_1x_2$$
$$r_{32} : x_3x_2 = x_1 + x_2x_2 - x_2x_3$$
$$r_{31} : x_3x_1 = x_1x_3$$
$$r_{43} : x_4x_3 = x_2x_2 - x_3x_4$$
$$r_{42} : x_4x_2 = -x_2x_4$$
$$r_{41} : x_4x_1 = x_1x_4$$
$$r_{54} : x_5x_4 = -x_3x_3 - x_4x_5$$
$$r_{53} : x_5x_3 = x_3x_3 - x_3x_5$$
$$r_{52} : x_5x_2 = -x_2x_5 + x_3x_3$$
$$r_{51} : x_5x_1 = x_1x_5.$$

Taking $x_5 > \cdots > x_1$, the leading terms are as presented and all overlaps resolve [Ell, Section 5]. Taking $\deg(x_1) = 2$ and $\deg(x_i) = 1$, $2 \leq i \leq 5$, the relations are homogeneous. Thus by Theorem 2.4.5, this is an Ore extension. It is also generated in degree one since r_{32} can be used to express x_1 in terms of the degree one generators. Finally, $\sigma_j(x_i) = \pm 1$ for all $1 \leq i < j \leq 5$ so these maps are injective and this is an AS-Ore extension.

We can solve r_{32} for x_1 and then view the algebra as $\widetilde{A}$, generated in degree one. From the analysis preceding Theorem 5.1.1, the Gröbner basis of $\widetilde{A}$ has 2 degree three relations and it remains to show that exactly one of these is a consequence of an overlap that fails to resolve. We compute

$$x_5(x_4x_3) - (x_5x_4)x_3 = x_2x_2x_3 - x_2x_3x_3 - x_3x_2x_2 + x_3x_3x_2$$

so this overlap never resolves. Reducing the remaining overlap modulo this additional relation,

$$x_5(x_4x_2) - (x_5x_4)x_2 = 0.$$

Thus, 1 of the degree three relations in the Gröbner basis is minimal and the relation type of this algebra is (2,2,2,2,2,3). □

Theorem 5.3.2. *The example above is a PI algebra.*

Proof. Let $R = K[x_1]\cdots[x_5, \sigma_5, \delta_5]$ be defined by the relations above. Then $K[x_1][x_2, \sigma_2, \delta_2] = K[x_1, x_2]$ is commutative and so PI and it suffices to check that $Z(K_{(i)})$ is nontrivial for $3 \leq i \leq 5$. It can be checked that $x_i^2 \in Z(K_{(i)})$ for $3 \leq i \leq 5$ (see [Ell, Section5]), which completes the proof. □

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6 A classification of relation types for AS-Ore extensions with 3 degree one generators

We now begin the process of classifying the relation types of AS-Ore extensions with 3 degree one generators. Again, by the symmetry of the free resolution, this also provides us with all the information we need to classify all possible resolution types of such algebras. We recall that, by Theorem 3.0.12, there are two possible degree types for algebras with 3 generators: (1,1,1,2,2) and (1,1,1,2,3).

6.1 Degree type (1,1,1,2,2)

Theorem 6.1.1. *An AS-Ore extension with degree type (1,1,1,2,2) has relation type (2,3,3,3,3,3).*

Proof. By Corollary 2.2.5, a dimension 5 AS-Ore extension generated in degree one by 3 generators has free resolution

$$0 \to A(-l) \to A(-l+1)^b \to \bigoplus_{i=1}^{n} A(-l+a_i) \to \\ \to \bigoplus_{i=1}^{n} A(-a_i) \to A(-1)^b \to A \to K \to 0,$$

where b=3 since there are 3 degree one generators, n represents the number of relations in the minimal generating set, and a_i represents the degree of the ith relation. This algebra has Hilbert series

$$h_A(t) = \frac{1}{q(t)} \text{ where } q(t) = 1 - 3t + \sum_{i=1}^{n} t^{a_i} - \sum_{i=1}^{n} t^{l-a_i} + 3t^{l-1} - t^l,$$

On the other hand, an AS-Ore extension with degree type (1,1,1,2,2) has Hilbert series

$$h_A(t) = \frac{1}{(1-t)^3(1-t^2)^2}, \text{ so } q(t) = 1 - 3t + t^2 + 5t^3 - 5t^4 - t^5 + 3t^6 - t^7.$$

Assume $a_1 \leq \cdots \leq a_n$. Then $l = 7$, $a_1 = 2$, $a_i = 3$ for $2 \leq i \leq 6$ (there are 5 degree three relations), and if there are any other minimal relations, they must cancel in the expression $\sum_{i=1}^{n} t^{a_i} - \sum_{i=1}^{n} t^{l-a_i}$ since they do not appear in the second equation for $q(t)$. This would mean either that $a_i = l - a_i$ (which is impossible since l is odd) or that there are at least two additional relations, a_i and a_j with $a_i + a_j = l$. Although we are interested in the minimal generating set of $\widetilde{A}$, the algebra generated in degree one, we note that any minimal relations of $\widetilde{A}$ must come from the original relations of the algebra viewed as an Ore extension, A. The possible relations for A are described in the presentation of an Ore extension given by Theorem 2.4.4 and an Ore extension with degree type (1,1,1,2,2) can only have relations of degrees two, three, or four. Thus, if there are additional minimal relations that cancel in the Hilbert series, they must be of degree three and four (since something of degree two or lower could only cancel if there were also a relation of degree five or higher, and we know that an Ore extension with this degree type has no such relations which are minimal).

If we label the degree one generators with the order $x_3 < x_4 < x_5$, the algebra, when viewed as $\widetilde{A}$, will have 1 degree two relation with leading term x_5x_4, x_5x_3, or x_4x_3. (The remaining 2 degree two relations will be used to write the x_1 and x_2 in terms of the degree one generators.) If $\widetilde{A}_2$ denotes the monomial algebra generated in degree one that has one of x_5x_4, x_5x_3, or x_4x_3 as a relation and $\widetilde{A}$ denotes the AS-Ore extension with degree type (1,1,1,2,2) viewed as an algebra generated in degree one, then $h_{\widetilde{A}_2}(t) - h_{\widetilde{A}}(t) = 5t^3 + O(t^4)$ [Ell, Section 6], so there can only be 5 degree three relations in the Gröbner basis. Thus, the only relation type for an AS-Ore extension with degree type (1,1,1,2,2) is (2,3,3,3,3,3). □

Theorem 6.1.2. *There is an enveloping algebra with degree type (1,1,1,2,2) and relation type (2,3,3,3,3,3).*

Proof. Consider the algebra defined by relations

$$r_{21} : x_2x_1 = x_1x_2$$

$$r_{32} : x_3x_2 = x_2x_3$$

$$r_{31} : x_3x_1 = x_1x_3$$

$$r_{42} : x_4x_2 = x_2x_4$$

$$r_{41} : x_4x_1 = x_1x_4$$

$$r_{52} : x_5x_2 = x_2x_5$$

$$r_{51} : x_5x_1 = x_1x_5$$

$$r_{43} : x_4x_3 = x_1 + x_3x_4$$

$$r_{54} : x_5x_4 = x_2 + x_4x_5$$

$$r_{53} : x_5x_3 = x_3x_5.$$

Assigning $(x_5, x_4, x_3, x_2, x_1)$ degrees $(1, 1, 1, 2, 2)$, these relations are homogeneous and it can be verified by hand or computer that all overlaps resolve, so this is an Ore extension by Theorem 2.4.5. Additionally, for all $1 \leq i < j \leq 5$, $\sigma_j(x_i)$ is the identity and $\delta_j(x_i)$ is linear, so this is an enveloping algebra by Theorem 2.3.4. It is also generated in degree one. Another quick check in Mathematica shows that the relation type is $(2, 3, 3, 3, 3, 3)$ [Ell, Section 7], although the analysis from Theorem 6.1.1 already indicates that this has to be the case since this is the only possible relation type for an AS-Ore extension with this degree type. □

As before, this example naturally gives rise to the question:

Question 6.1.3. *Is there a PI AS-Ore extension with degree type (1,1,1,2,2)?*

6.2 Degree type (1,1,1,2,3)

Theorem 6.2.1. *An AS-Ore extension with degree type (1,1,1,2,3) has relation type (2,2,3) or (2,2,3,4).*

Proof. Following the logic of Theorem 6.1.1, we know from the free resolution of the algebra that

$$h_A(t) = \frac{1}{q(t)} \text{ where } q(t) = 1 - 3t + \sum_{i=1}^{n} t^{a_i} - \sum_{i=1}^{n} t^{l-a_i} + 3t^{l-1} - t^l.$$

On the other hand, an AS-Ore extension with degree type (1,1,1,2,3) has Hilbert series

$$h_A(t) = \frac{1}{(1-t)^3(1-t^2)(1-t^3)}, \text{ so } q(t) = 1 - 3t + 2t^2 + t^3 - t^5 - 2t^6 + 3t^7 - t^8.$$

So $l = 8$, $a_1 = a_2 = 2$, $a_3 = 3$, and if there are any other minimal relations of $\widetilde{A}$, generated in degree one, they must cancel. In this case we would have $a_i = l - a_j$ (where it is possible that $i = j$). This means that there may possibly be a pair of minimal relations of degree three and five (there cannot be more than one such pair since there is only 1 degree five relation in the Ore extension) and there may possibly be up to 3 relations of degree four (since the Ore extension A has 3 relations of degree four). There cannot be any additional minimal relations of degree two since they would need to cancel with something of degree six and there are no minimal relations of degree six for an AS-Ore extension with this degree type. To conclude that the relation type is either (2,2,3) or (2,2,3,4), it remains to show that there are not 2 independent relations of degree three (and so no additional pair of relations of degree three and five) and that there is at most 1 minimal relation of degree four.

We can write relations based on the fact that this is an Ore extension. Without loss of generality, label the degree one generators so that $x_3 < x_4 < x_5$. We will let x_2 be the degree two variable and x_1 will be the degree three variable, but we make no claim about when these variables are adjoined. (Thus possible orders include $x_1 < x_2 < x_3 < x_4 < x_5$ and $x_3 < x_2 < x_4 < x_1 < x_5$, amongst many others.) We do note, however, that for A to be generated in degree one, x_1 must not be adjoined last (since it must be adjoined by the time it appears in a relation with leading term of the form x_jx_i and hence must be adjoined before x_j), and (similarly) x_2 must not be adjoined last. Then by the choice of ordering of our degree one variables, x_4 and x_3 are both adjoined before x_5, so x_5 is always added last in the iterated Ore extension.

Using the ordering of the degree one variables, we can write the degree two relations. We note that the only thing that could change in the relations below that depends on the order in which the variables is adjoined is that d_1 must be zero if x_2 is added after x_4, since r_{43} should only involve variables that have been added by the time x_4 is adjoined. The degree two relations are:

$$r_{43} : x_4x_3 = d_1x_2 + d_2x_3x_3 + d_3x_3x_4$$

$$r_{53} : x_5x_3 = h_1x_2 + h_2x_3x_3 + h_3x_3x_4 + h_4x_3x_5 + h_5x_4x_4 + h_6x_4x_5$$

$$r_{54} : x_5x_4 = g_1x_2 + g_2x_3x_3 + g_3x_3x_4 + g_4x_3x_5 + g_5x_4x_4 + g_6x_4x_5.$$

We wish to repeat this process for the degree three relations. The list of possible degree three monomials that can appear on the right side of a relation, given the ordering $x_3 < x_4 < x_5$, $x_1 < x_5$, $x_2 < x_5$, is:

$$\{x_1, x_2x_3, x_3x_2, x_2x_4, x_4x_2, x_2x_5, x_3x_3x_3, x_3x_3x_4, x_3x_3x_5, x_3x_4x_4, x_3x_4x_5, x_4x_4x_4\}.$$

Since the ordering of the variables is not known, the leading terms of the degree 3 relations can vary, as can the other allowable monomials. We will write fully general versions of the relations for each possible leading term. It is appropriate to use r_{32a} below when $x_3 > x_2$ and r_{32b} when $x_2 > x_3$. Similarly, r_{42a} applies when $x_4 > x_2$, and r_{42b} when $x_2 > x_4$.

From Theorem 2.4.4, x_j must occur only to the first power in the relation with leading term x_jx_i and x_k should not appear for any $x_k > x_j$. For example in r_{32b} below, the leading term implies that $x_2 > x_3$. If $x_2 > x_4 > x_3$ then the monomial x_4x_2 may appear. Otherwise, $x_4 > x_2 > x_3$ and the monomial x_2x_4 does not appear since x_4 has not yet been adjoined in the Ore extension. The most general possible degree three relations are:

$$r_{32}a: \quad -b_0x_3x_2 = b_1x_1 + b_2x_2x_3$$

$$r_{32}b: \quad -b_2x_2x_3 = b_1x_1 + b_0x_3x_2 + b_3x_4x_2 + b_4x_3x_3x_3 + b_5x_3x_3x_4 + b_6x_3x_4x_4 + b_7x_4x_4x_4$$

$$r_{42}a: \quad -e_0x_4x_2 = e_1x_1 + e_2x_2x_3 + e_3x_2x_4 + e_4x_3x_2 + e_5x_3x_3x_3 + e_6x_3x_3x_4$$

$$r_{42}b: \quad -e_3x_2x_4 = e_1x_1 + e_0x_4x_2 + e_4x_3x_2 + e_5x_3x_3x_3 + e_6x_3x_3x_4 + e_7x_3x_4x_4 + e_8x_4x_4x_4$$

$$r_{52}: \quad x_5x_2 = i_1x_1 + i_2x_2x_3 + i_3x_2x_4 + i_4x_3x_2 + i_5x_4x_2 + i_6x_2x_5 + i_7x_3x_3x_3 + i_8x_3x_3x_4 + i_9x_3x_3x_5 + i_{10}x_3x_4x_4 + i_{11}x_3x_4x_5 + i_{12}x_4x_4x_4 + i_{13}x_4x_4x_5.$$

Note that not all of these coefficients can be nonzero. For example, if $x_2 < x_4$ then $i_5 = 0$ while if $x_4 < x_2$, $i_3 = 0$. These do, however, capture all possible terms that could occur in the Ore relations.

The goal now is to view this as $\widetilde{A}$, generated in degree one, and to try to identify the possible degrees of relations that can occur. The proof depends on which relations are used to solve for x_2. Without loss of generality, we will solve for x_1 and x_2 using the relation with smallest leading term so that the leading terms of the remaining degree two and three relations are easily identifiable.

Case 1: x_2 comes from r_{43}.

In this case, the degree two leading terms are x_5x_4 and x_5x_3. If $\widetilde{A}_2$ is the monomial algebra with these leading terms then $h_{\widetilde{A}_2} - h_{\widetilde{A}} = t^3 + O(t^4)$ [Ell, Section 8] so there is 1 degree three relation in the Gröbner basis of $\widetilde{A}$ and so at most (and exactly) 1 degree three relation in the minimal generating set, as we wished to show. It remains to show that there is no more than 1 degree four relation in the minimal generating set.

Since x_2 comes from r_{43}, x_2 certainly appears in the relation r_{43} and so must have been adjoined before x_4. This means that $x_2 < x_4$ in the order which in turn implies that we should use the relation $r_{42}a$, that e_0 is not zero since it is the leading term of the expression, and that $b_3 = b_5 = b_6 = b_7 = 0$. We can begin to view the algebra as generated in degree one by solving r_{43} to get $x_2 = \frac{1}{d_1}(x_4x_3 - d_3x_3x_4 - d_2x_3x_3)$. We can substitute this into the other relations and move all terms to the right side of the equation:

$$\begin{aligned}
r_{32a} : 0 &= b_0x_3x_2 + b_1x_1 + b_2x_2x_3 \\
&= b_0x_3\frac{1}{d_1}(x_4x_3 - d_3x_3x_4 - d_2x_3x_3) + b_1x_1 \\
&\quad + b_2\frac{1}{d_1}(x_4x_3 - d_3x_3x_4 - d_2x_3x_3)x_3 \\
&= b_1x_1 + \frac{b_2}{d_1}x_4x_3x_3 + [\text{smaller terms}] \\
r_{32b} : 0 &= b_2x_2x_3 + b_1x_1 + b_0x_3x_2 + 0x_4x_2 + b_4x_3x_3x_3 + 0x_3x_3x_4 + 0x_3x_4x_4 \\
&\quad + 0x_4x_4x_4 \\
&= b_2\frac{1}{d_1}(x_4x_3 - d_3x_3x_4 - d_2x_3x_3)x_3 + b_1x_1 + b_4x_3x_3x_3 \\
&= b_1x_1 + \frac{b_2}{d_1}x_4x_3x_3 + [\text{smaller terms}].
\end{aligned}$$

We note that the largest term appearing in the relation for r_{32} is now independent of which version of the relation we use and, repeating the substitution for x_2 in r_{42a}, we can write

$$\begin{aligned}
r_{32} : 0 &= b_1x_1 + \frac{b_2}{d_1}x_4x_3x_3 + [\text{smaller terms}] \\
r_{42} : 0 &= e_1x_1 + \frac{e_0}{d_1}x_4x_4x_3 + [\text{smaller terms}].
\end{aligned}$$

From the original relations, we observe that b_2 is not zero: either $x_2 > x_3$ in the ordering and b_2 is the LT of r_{32} or $x_3 > x_2$ and $\sigma_3(x_2) = b_2 x_2$ and so $b_2 \neq 0$ by the injectivity of σ_3.

If b_1 is not zero, we may solve r_{32} for x_1 and, even after substituting this value into r_{42}, the leading term of r_{42} in $\widetilde{A}$ will be $x_4x_4x_3$. If b_1 is zero then $x_4x_3x_3$ will be a leading term in $\widetilde{A}$. If $\widetilde{A}_3$ is the monomial algebra with LTs $\{x_5x_4, x_5x_3, x_4x_3x_3\}$ or $\{x_5x_4, x_5x_3, x_4x_4x_3\}$ then $h_{\widetilde{A}_3} - h_{\widetilde{A}} = t^4 + O(t^5)$. So the Gröbner basis of $\widetilde{A}$ has 1 degree four relation and so at there is at most 1 degree four relation in the minimal generating set, as we wished to show.

Case 2: x_2 comes from r_{54}.

In this case, the degree two leading terms are x_5x_3 and x_4x_3, $d_1 = g_1 = 0$ (or else x_2 would come from r_{43} or r_{53}), and $h_{\widetilde{A}_2} - h_A = t^3 + O(t^4)$. This means there is at most 1 degree three relation in the minimal generating set and it remains to show that there is also at most 1 degree four relation. From r_{54}, we can write $x_2 = \dfrac{1}{g_1}(x_5x_4 + \text{ [smaller terms]})$. As in the first case, we can substitute this value into the degree three terms to get that the highest terms in the degree three relations of interest are known, independent of which version of r_{42} we use:

$$r_{42} : 0 = e_1x_1 + \frac{e_3}{g_1}x_5x_4x_4 + \text{[smaller terms]}$$
$$r_{52} : 0 = i_1x_1 - \frac{1}{g_1}x_5x_5x_4 + \text{[smaller terms]}.$$

Note that e_3 is not zero: either $x_4 < x_2$ and it e_3 the leading coefficient of r_{42b}, or $x_2 < x_4$ and from r_{42a}, $\sigma_4(x_2) = -\dfrac{e_3}{e_0}x_2 - \dfrac{e_6}{e_0}x_3x_3$. In this case, since $\sigma_4(x_3) = d_3x_3$,

$\sigma_4(x_2 + \frac{e_6}{e_0 d_3^2} x_3x_3) = -\frac{e_3}{e_0}x_2 - \frac{e_6}{e_0}x_3x_3 + \frac{e_6}{e_0}x_3x_3$. By the injectivity of σ_4, $e_3 \neq 0$.

If e_1 is not zero, we may solve r_{42} for x_1 and, even after substituting this value into r_{52}, the LT of r_{52} will be $x_5x_5x_4$. If e_1 is zero then $x_5x_4x_4$ will be a leading term in $\widetilde{A}$. In either case, we calculate that $h_{\widetilde{A}_3} - h_{\widetilde{A}} = t^4 + O(t^5)$, which means that there is at most 1 degree four relation in the minimal generating set, as we wished to prove.

Case 3: x_2 comes from r_{53}.

In this case, the degree two LTs are x_5x_4 and x_4x_3, $d_1 = 0$, and $h_{\widetilde{A}_2} - h_{\widetilde{A}} = 2t^3 - O(t^4)$. We can solve r_{53} to get that $x_2 = \frac{1}{h_1}(x_5x_3 - h_6x_4x_5 + [\text{smaller terms}])$. We can rewrite the remaining degree two relations after substituting the value of x_2 into the equations:

$$r_{43} : x_4x_3 = 0x_2 + d_3x_3x_4 + [\text{smaller terms}]$$
$$r_{54} : x_5x_4 = g_1\frac{1}{h_1}(x_5x_3 + [\text{smaller terms}]) + g_6x_4x_5 + [\text{smaller terms}].$$

We can then compute the degree three overlap in $\widetilde{A}$.

$$\begin{aligned}(x_5x_4)x_3 - x_5(x_4x_3) &= \frac{g_1}{h_1}(x_5x_3 - h_6x_4x_5 + [\text{small}])x_3 - x_5(d_3x_3x_4 + d_2x_3x_3) \\ &= -d_3x_5x_3x_4 + (\frac{g_1}{h_1} - d_2)x_5x_3x_3 + [\text{small}].\end{aligned}$$

Since d_3 is not zero by the injectivity of σ_3, this overlap does not resolve and we know that there is a relation, $x_5x_3x_4 = \frac{g_1 - d_2h_1}{h_1}x_5x_3x_3 + [\text{smaller terms}]$ in the Gröbner basis of $\widetilde{A}$. We conclude that at most 1 of the 2 degree three relations in the Gröbner basis can be minimal and it remains to show that there is at most 1 minimal relation of degree four.

Again substituting the value of x_2, we may re-examine the degree three relations and note that, as with the first case, the LT of r_{32} is independent of which version we use:

$$r_{32} : 0 = b_1 x_1 + \frac{b_2}{h_1} x_5 x_3 x_3 + \text{[smaller terms]}$$
$$r_{52} : 0 = i_1 x_1 - \frac{1}{h_1} x_5 x_5 x_3 + \text{[smaller terms]}.$$

By the same analysis as in case 1, $b_2 \neq 0$. If $b_1 = 0$ then the leading term of r_{32} in $\widetilde{A}$ is $x_5 x_3 x_3$. If $\widetilde{A}_3$ is the monomial algebra with LTs $\{x_5 x_4, x_4 x_3, x_5 x_3 x_3, x_5 x_3 x_4\}$ then $h_{\widetilde{A}_3} - h_{\widetilde{A}} = t^4 + O(t^5)$ and there is at most 1 degree 4 relation in the minimal generating set as desired.

If b_1 is not zero then r_{32} may be solved for x_1 and, even after substituting the value of x_1 into the relation, the leading term of r_{52} in $\widetilde{A}$ is $x_5 x_5 x_3$. Thus, the 2 relations of degree three in the Gröbner basis have LTs $x_5 x_3 x_4$ and $x_5 x_5 x_3$ and in this case, $h_{\widetilde{A}_3} - h_{\widetilde{A}} = 2t^4 + O(t^5)$. We can also compute the degree four overlap (see [Ell, Section 8]):

$$\begin{aligned}(x_5 x_3 x_4) x_3 - x_5 x_3 (x_4 x_3) = (&\frac{g_1 - d_2 h_1}{d_3 h_1} x_5 x_3 x_3 + \text{[small]}) x_3 \\ &- x_5 x_3 (d_3 x_3 x_4 + \text{[small]}) \\ &= -d_3 x_5 x_3 x_3 x_4 + \text{[small]}.\end{aligned}$$

We note that $x_5 x_3 x_3 x_4$ cannot be reduced in $\widetilde{A}_3$ and d_3 is not zero so this overlap fails to resolve. In total, we have found that the Gröbner basis of $\widetilde{A}$ has 2 degree four relations, at least one of which is not minimal.

Thus, in all cases we have shown that $\widetilde{A}$ has at most 1 degree three and at most 1 degree four relation in the minimal generating set, which means that the relation type must be either (2,2,3), or (2,2,3,4). □

Theorem 6.2.2. *There is an enveloping algebra with degree type (1,1,1,2,3) and relation type (2,2,3,4), but not one with relation type (2,2,3).*

Proof. An enveloping algebra can be taken to have $x_5 > x_4 > x_3 > x_2 > x_1$ with $\deg(x_2) = 2$ and $\deg(x_1) = 3$ and is then defined by the relations

$$
\begin{aligned}
r_{21} &: x_2x_1 = x_1x_2 \\
r_{31} &: x_3x_1 = x_1x_3 \\
r_{41} &: x_4x_1 = x_1x_4 \\
r_{51} &: x_5x_1 = x_1x_5 \\
r_{32} &: x_3x_2 = b_1x_1 + x_2x_3 \\
r_{42} &: x_4x_2 = e_1x_1 + x_2x_4 \\
r_{52} &: x_5x_2 = i_1x_1 + x_2x_5 \\
r_{43} &: x_4x_3 = d_1x_2 + x_3x_4 \\
r_{54} &: x_5x_4 = g_1x_2 + x_4x_5 \\
r_{53} &: x_5x_3 = h_1x_2 + x_3x_5.
\end{aligned}
$$

By construction we have that for all $1 \leq i < j \leq 5$, $\sigma_j(x_i)$ is the identity and $\delta_j(x_i)$ is linear. All overlaps resolve, except for $x_5(x_4x_3) - (x_5x_4)x_3 = (b_1g_1 - e_1h_1 + d_1i_1)x_1$, so we will have to choose values of coefficients which make this expression zero. Additionally, to be generated in degree one, we must have that at least 1 of b_1, e_1, i_1 and at least 1 of d_1, g_1, h_1 is nonzero. If this happens, then by Theorem 2.3.4, this is an enveloping algebra.

We can now solve for x_1 and x_2 to view this as $\widetilde{A}$ and analyze the possible degrees of minimal relations. As the process for these computations is quite similar to that seen in previous theorems, we will omit most of the details. Our goal is to show that the relation type is always (2,2,3,4). By the symmetry of the relations, we may assume that b_1 is the coefficient that is not zero.

Case 1: d_1 nonzero.

From the overlap in A, $x_5(x_4x_3) - (x_5x_4)x_3 = (g_1b_1 - e_1h_1 + i_1d_1)x_1$, we conclude that $g_1 = \dfrac{e_1h_1 - i_1d_1}{b_1}$. Solving r_{43} and r_{32} for x_2 and x_1 and substituting these into the remaining relations to view the algebra as generated in degree one, we find that there are degree two relations in $\widetilde{A}$ with LTs x_5x_3 and x_5x_4 and a degree three relation from r_{42} with LT $x_4x_4x_3$. Reduced modulo these relations, r_{51} then has LT $x_4x_3x_3x_3$ and it remains to show that this is minimal in $\widetilde{A}$. The only degree four overlap, given these LTs, is $(x_5x_4)x_4x_3 - x_5(x_4x_4x_3) = 0$. Details for computations in this proof are available online [Ell, Section 9]. Since this overlap resolves, the degree four relation is independent. By Theorem 6.2.1, this means that the relation type must be (2,2,3,4). In particular, we have now shown that there is an enveloping algebra with this relation type. It remains to show that (2,2,3) is never a possible relation type.

Case 2: $d_1 = 0$ and h_1 nonzero.

From the overlap $x_5(x_4x_3) - (x_5x_4)x_3 = (-e_1h_1 + g_1b_1)x_1$ we conclude that $e_1 = \dfrac{g_1b_1}{h_1}$.

Solving r_{53} and r_{32} for x_2 and x_1 and substituting these into the remaining relations to view the algebra as generated in degree 1, we find that there are degree two relations in $\widetilde{A}$ with LTs x_4x_3 and x_5x_4, a degree three overlap that fails to resolve with LT $x_5x_3x_4$, and a degree three relation from r_{52} with LT $x_5x_5x_3$. The monomial algebra with these LTs has 2 degree four relations so it remains to show that exactly one such relation is the consequence of an overlap that does not resolve. One such overlap is $x_5x_3(x_4x_3) - (x_5x_3x_4)x_3 = \frac{g_1}{h_1}x_3x_5x_3x_3 - x_4x_5x_3x_3 - \frac{g_1}{h_1}x_5x_3x_3x_3 + x_5x_3x_3x_4$ and so fails to resolve. The remaining degree four overlap, when reduced modulo the degree two and three relations together with this new relation, is $x_5(x_5x_3x_4) - (x_5x_5x_3)x_4 = 0$. Thus, there is 1 independent degree four relation and the relation type is (2,2,3,4).

Case 3: $d_1 = 0$, $h_1 = 0$, and g_1 nonzero.

Recall that, by the symmetry of the relations, we have assumed that $b_1 \neq 0$, and that $g_1 \neq 0$ if $d_1 = 0$, $h_1 = 0$, and A is generated in degree one. From the overlap in A, $x_5(x_4x_3) - (x_5x_4)x_3 = b_1g_1x_1$, we conclude that there is no enveloping algebra with coefficients with these values since the overlaps of the original Ore relations fail to resolve.

Thus in all cases, the only possible relation type is (2,2,3,4) and so this is the only relation type of an enveloping algebra with variables of degrees (1,1,1,2,3). □

Again, this example naturally gives rise to the question:

Question 6.2.3. *Is there a PI AS-Ore extension with degree type (1,1,1,2,3)?*

We also wish to know if there is an AS-Ore extension of relation type (2,2,3).

Theorem 6.2.4. *There is an AS-Ore extension with relation type (2,2,3).*

Proof. Consider the algebra defined by the relations

$$r_{21} : x_2x_1 = -x_1x_2$$

$$r_{32} : x_3x_2 = x_2x_3$$

$$r_{31} : x_3x_1 = x_1x_3$$

$$r_{43} : x_4x_3 = x_3x_4$$

$$r_{42} : x_4x_2 = x_1 + x_2x_4$$

$$r_{41} : x_4x_1 = -x_1x_4$$

$$r_{54} : x_5x_4 = x_2 + x_4x_5$$

$$r_{53} : x_5x_3 = x_3x_5 + x_4x_4$$

$$r_{52} : x_5x_2 = x_2x_5$$

$$r_{51} : x_5x_1 = x_1x_5.$$

Assigning $(x_5, x_4, x_3, x_2, x_1)$ degrees (1,1,1,2,3), these relations are homogeneous. Using the order $x_5 > x_4 > x_3 > x_2 > x_1$ so that the relations are as presented, all overlaps resolve [Ell, Section 10]. Hence, this is an Ore extension by Theorem 2.4.5. It is also generated in degree one. For all $1 \leq i < j \leq 5$, $\sigma_j(x_i) = \pm 1$ so the σ_j are automorphisms and this algebra is AS-Ore.

We can view this algebra as $\widetilde{A}$, generated in degree one, by solving r_{54} and r_{42} for x_2 and x_1 and plugging these values into the relations. The remaining degree two relations in $\widetilde{A}$ have LTs x_5x_3 and x_4x_3 and there is a degree three relation with LT $x_5x_5x_4$. Reduced modulo these relations, r_{31} and r_{51} become 0 while $r_{41} = -x_4x_4x_4x_5 + x_4x_4x_5x_4 + x_4x_5x_4x_4 - x_5x_4x_4x_4$. So there is 1 degree four relation in the Gröbner basis (which we already knew from the Hilbert series analysis of

Theorem 6.2.1) and it remains to show that this is not minimal. We compute the overlap $x_5x_5(x_4x_3)-(x_5x_5x_4)x_3 = -x_4x_4x_4x_5+x_4x_4x_5x_4+x_4x_5x_4x_4-x_5x_4x_4x_4$. Thus, the degree four relation is a consequence of an overlap that fails to resolve and the relation type is (2,2,3). □

We again ask whether this example is PI.

Theorem 6.2.5. *The above example is not PI.*

Proof. Let $R = K[x_1]\cdots[x_5,\sigma_5,\delta_5]$ be the algebra defined by the above relations. Although not relevant to the proof of this theorem, it is worth noting that $K_{(4)}$ is PI as a relatively low power of x_i is in $Z(K_{(i)})$ for $1 \leq i \leq 4$.

For contradiction, assume that R is PI. Note that the property of being PI passes to factor rings since if $f(a_1,\cdots,a_m) = 0$ in R for any $a_1,\cdots,a_m \in R$ then certainly $f(\bar{a}_1,\cdots,\bar{a}_m) = 0$ in R/I for any $\bar{a}_1,\cdots,\bar{a}_m \in R/I$ where I is any (two sided) ideal of R. Examining the relations, x_1 is normal in R, i.e. $x_1R = Rx_1$, so $\langle x_1\rangle$ is a two sided ideal. Then, again looking at the relations, x_2 is normal in $R/\langle x_1\rangle$. Consider $R/\langle x_1,x_2\rangle$, which is defined by the following relations:

$$x_4x_3 = x_3x_4$$

$$x_5x_4 = x_4x_5$$

$$x_5x_3 = x_3x_5 + x_4x_4$$

$\bar{R} = R/\langle x_1,x_2\rangle \cong K[x_3,x_4][x_5,\sigma_5,\delta_5]$ is an Ore extension by Theorem 2.4.5. Thus, if it is PI, there is a nonconstant polynomial, $p(x_5)$, with coefficients in $K[x_3,x_4]$ which is central in $\bar{R}$. Write $p = q_0 + q_i x_5^i + \cdots + q_n x_5^n$ where $q_j \in K[x_3, x_4]$ for all j and $i > 0$ is the smallest value such that $q_i \neq 0$.

It can be shown inductively that $x_5^n x_3 = x_3 x_5^n + n x_4 x_4 x_5^{n-1}$. Thus,

$$\begin{aligned}
x_3 p(x_5) &= x_3(q_0 + q_i x_5^i + \cdots + q_n x_5^n) \\
&= x_3 q_0 + x_3 q_i x_5^i + \cdots + x_3 q_n x_5^n \\
&= q_0 x_3 + q_i x_3 x_5^i + \cdots + q_n x_3 x_5^n, \text{ while} \\
p(x_5) x_3 &= (q_0 + q_i x_5^i + \cdots + q_n x_5^n) x_3 \\
&= q_0 x_3 + q_i(x_3 x_5^i + i x_4 x_4 x_5^{i-1}) + q_{i+1}(x_3 x_5^{i+1} + (i+1) x_4 x_4 x_5^i) + O(x_5^{i+1}),
\end{aligned}$$

Where $O(x_5^{i+1})$ involves terms where x_5 occurs to at least the power of $i + 1$. Since $i x_4 x_4 x_5^{i-1}$ appears in $p(x_5)x_3$ but not in $x_3 p(x_5)$, these expressions cannot be equal, which contradicts $p \in Z(\bar{R})$. Thus $\bar{R}$, and so also R, must not be PI. □

Since it is not uncommon that an AS-regular algebra fail to be PI, this result in itself is not that interesting. However, the $x_4 x_4$ coefficient in r_{53}, which was chosen to be nonzero as an easy way force the desired relation type, has restricted the center of the algebra in all examples we have tried. We are thus led to ask

Question 6.2.6. *Is there a PI AS-Ore extension with relation type (2,2,3)?*

Portions of this chapter have been accepted for publication in Communications in Algebra.

7 Bigraded Ore extensions with relation type (2,2,2,2,2,3)

We now wish to explore bigraded Ore extensions in more detail. Recall that an algebra A is *bigraded* or $\mathbb{Z}^2$*-graded* if $A = \dfrac{K\langle x_1, \cdots, x_b\rangle}{I}$, $\deg(x_i) \in \{(1,0),(0,1)\}$ for all i, and I is homogeneous in $\mathbb{Z} \times \mathbb{Z}$. We avoid the case of a trivial bigrading by requiring that there be at least one variable of degree (1,0) and at least one of degree (0,1).

We can then investigate the *bigraded Hilbert series* of A, $h_A(u,v) = \sum\limits_{i,j\in\mathbb{N}} (\dim_K A_{i,j}) u^i v^j$ where $A_{i,j}$ has bidegree (i,j). We can recover the Hilbert series of A from the bigraded Hilbert series by replacing u and v with t and collecting terms of the same degree: $h_A(t) = \sum\limits_{n\in\mathbb{N}} (\sum\limits_{i+j=n} \dim_K A_{i,j}) t^n$.

Notice that the Ore extension in Theorem 4.0.14 is bigraded with $\deg(x_4) = (1,0)$ and $\deg(x_5) = (0,1)$, which forces the remaining degrees to be $(\deg(x_3), \deg(x_2), \deg(x_1)) = ((1,1),(2,1),(3,2))$, respectively

Each of the enveloping algebras presented is also bigraded with any nontrivial assignment of the degrees of the generators to $\{(1,0),(0,1)\}$. In fact, the enveloping algebras presented are $\mathbb{Z}^b$ graded where b is the number of degree one generators.

On the other hand, the Ore extension provided in Theorem 5.3.1 is not bigraded for, from the relations

$$r_{43} : x_4x_3 = x_2x_2 - x_3x_4$$

$$r_{54} : x_5x_4 = -x_3x_3 - x_4x_5,$$

we conclude that $\deg(x_2) = \deg(x_3) = \deg(x_4) = \deg(x_5)$ and there is no nontrivial bigrading. Similarly, the Ore extension provided in Theorem 6.2.4 is not bigraded since the relation

$$r_{53} : x_5x_3 = x_3x_5 + x_4x_4$$

forces all 3 degree one generators to have the same bidegree.

7.1 [3,1]-bigraded Ore extension

Although the example in Theorem 5.3.1 is not bigraded, there does exist a bigraded Ore extension with the same relation type.

Theorem 7.1.1. *There is a bigraded Ore extension with degree type (1,1,1,1,2) and relation type (2,2,2,2,2,3).*

Proof. Consider the algebra defined by the relations

$$r_{21} : x_2x_1 = -\frac{1}{g_{10}}x_1x_2$$

$$r_{32} : x_3x_2 = b_2x_2x_2 - x_2x_3$$

$$r_{31} : x_3x_1 = -\frac{1}{g_{10}}x_1x_3$$

$$r_{43} : x_4x_3 = d_2x_2x_2 + d_6x_3x_3 - x_3x_4$$

$$r_{42} : x_4x_2 = e_2x_2x_2 - x_2x_4 + e_6x_3x_3$$

$$r_{41} : x_4x_1 = -\frac{1}{g_{10}}x_1x_4$$

$$r_{54} : x_5x_4 = g_{10}x_4x_5$$

$$r_{53} : x_5x_3 = g_{10}x_3x_5$$

$$r_{52} : x_5x_2 = x_1 + g_{10}x_2x_5$$

$$r_{51} : x_5x_1 = -g_{10}x_1x_5$$

where $0 \neq g_{10}, d_2 \in K$ and $b_2, d_6, e_2, e_6 \in K$. Assigning $(x_5, x_4, x_3, x_2, x_1)$ bidegrees ((0,1),(1,0),(1,0),(1,0),(1,1)), these relations are $\mathbb{Z}^2$-homogeneous. Using the order $x_5 > x_4 > x_3 > x_2 > x_1$ so that the relations are as presented, all overlaps resolve [Ell, Section 11]. Hence, this is an Ore extension by Theorem 2.4.5. It is also generated in degree one. For all $1 \leq i < j \leq 5$, $\sigma_j(x_i) = m_{ij}x_i$ where $m_{ij} \in K^\times$ so the σ_j are automorphisms and this algebra is AS-Ore.

We can solve r_{52} for x_1 and then view the algebra as $\widetilde{A}$, generated in degree one. From the analysis preceding Theorem 5.1.1, the Gröbner basis of $\widetilde{A}$ has 4 degree three relations and it remains to show that exactly 3 of these are a consequence of an overlap that fails to resolve. We compute

$$x_4(x_3x_2) - (x_4x_3)x_2 = 0$$

$$x_5(x_3x_2) - (x_5x_3)x_2 = -g_{10}x_3x_5x_2 + b_2x_5x_2x_2 - x_5x_2x_3$$

$$x_5(x_4x_2) - (x_5x_4)x_2 = e_6g_{10}^2x_3x_3x_5 - g_{10}x_4x_5x_2 + e_2x_5x_2x_2 - x_5x_2x_4$$

$$x_5(x_4x_3) - (x_5x_4)x_3 = -d_2g_{10}^2x_2x_2x_5 + d_2x_5x_2x_2.$$

Since $d_2 \neq 0$ there are 3 relations with distinct leading terms that are consequences of overlaps which fail to resolve, so the relation type of the algebra is (2,2,2,2,2,3). □

We shall say that an algebra with i degree (1,0) generators and j degree (0,1) generators is $[i, j]$-*bigraded.* So the algebra found in the preceding theorem is [3,1]-bigraded. Had we instead assigned (x_5, x_4, x_3, x_2) bidegrees ((1,0),(0,1),(0,1),(0,1)), the algebra would be [1,3]-bigraded, so we can see that the order of i and j in this definition is not informative.

7.2 No [2,2]-bigraded Ore extension

Given the relative ease with which the previous example was found, it may come as a surprise that there is no [2,2]-bigraded algebra with the same relation type. Before proving this, we find it useful to present a lemma that will help us determine which coefficients in the Ore relations must be nonzero.

Lemma 7.2.1. *If σ_j is an automorphism of $A = K[x_1, \sigma_1, \delta_1] \cdots [x_j, \sigma_j, \delta_j]$ and A is* *graded by the degree of the $x's$, write $\sigma_j(x_i)$ as a sum of its linear and nonlinear terms:*

$$\sigma_j(x_i) = \sum_{\{k \mid \deg(x_k)=\deg(x_i)\}} a_k x_k + \sum_k b_k(\prod_l x_l),$$

where each product is a reduced word in A of length at least 2, each $a_k \in K$, each $b_k \in K^{\times}$. Then there exists k such that $a_k \neq 0$.

Proof. Suppose this fails: $\sigma_j(x_i) = \sum_k b_k(\prod_l x_l)$. Let $S = \{x_m \mid \deg(x_m) < \deg(x_i)\}$. Since σ_j is surjective, there exists a reduced polynomial in A, call it p_m, such that $\sigma_j(p_m) = x_m$ for each $x_m \in S$. By degree considerations, each $p_m \in K\langle S\rangle$ and so in particular does not include x_i. Then $x_i - \sum_k b_k(\prod_l p_l) \neq 0$ but $\sigma_j(x_i - \sum_k b_k(\prod_l p_l)) = 0$, which contradicts the injectivity of σ_j and proves the lemma. □

Theorem 7.2.2. *There is no [2,2]-bigraded Ore extension of relation type (2,2,2,2,2,3).*

Proof. For contradiction, suppose A is a [2,2]-bigraded extension. As usual, we will look at the possible Ore relations before passing to the algebra generated in degree one. Without loss of generality we may label the degree one generators so that $x_5 > x_4 > x_3 > x_2$. For now we will also assume that $x_2 > x_1$, although we will have something to say about alternate orderings of the variables later. We will partially write out the degree three relations, restricting to the monomials that will be of interest in the rest of the proof:

$$r_{21} : x_2x_1 = g_1x_1x_2$$

$$r_{31} : x_3x_1 = h_1x_1x_3 + h_2x_1x_2 + \text{[words of length 3, no larger than } x_2x_2x_3\text{]}$$

$$r_{41} : x_4x_1 = i_1x_1x_4 + i_2x_1x_3 + i_3x_1x_2 + O(x_3x_3x_4)$$

$$r_{51} : x_5x_1 = j_1x_1x_5 + j_2x_1x_4 + j_3x_1x_3 + j_4x_1x_2 + O(x_4x_4x_5)$$

We note that by Lemma 7.2.1, we know that $g_1,\ h_1,\ i_1$, and j_1 are not zero. These will end up being the coefficients of the only monomials we care about for our analysis. We will now pass to the algebra $\widetilde{A}$ by solving one of the degree two relations for x_1. Based on the LTs of the degree two relations, the expression for x_1 will have a LT from the list $\{x_5x_4, x_5x_3, x_5x_2, x_4x_3, x_4x_2, x_3x_2\}$ and, as usual, the analysis will depend on which of the degree two relations we use to solve for x_1. For example, if r_{54} is used to solve for x_1 then x_1 will have LT x_5x_4 and r_{51} will become

$$x_5(x_5x_4 + \text{[smaller terms]}) - j_1(x_5x_4 + \text{[smaller terms]})x_5 - \text{[smaller terms]} = 0$$

and will have LT $x_5x_5x_4$, while r_{41} will become

$$x_4(x_5x_4 + \text{[smaller terms]}) - i_1(x_5x_4 + \text{[smaller terms]})x_4 - \text{[smaller terms]} = 0$$

with leading term $x_5x_4x_4$ since $i_1 \neq 0$. We write a table that captures the known LTs of each relation in each case:

	Leading term of x_1					
relation	x_5x_4	x_5x_3	x_5x_2	x_4x_3	x_4x_2	x_3x_2
r_{51}	$x_5x_5x_4$	$x_5x_5x_3$	$x_5x_5x_2$			
r_{41}	$x_5x_4x_4$	$x_5x_3x_4$	$x_5x_2x_4$	$x_4x_4x_3$	$x_4x_4x_2$	
r_{31}		$x_5x_3x_3$	$x_5x_2x_3$	$x_4x_3x_3$	$x_4x_2x_3$	$x_3x_3x_2$
r_{21}			$x_5x_2x_2$		$x_4x_2x_2$	$x_3x_2x_2$

Again, the entries above are found by replacing x_1 in the degree three relations listed above. We note that the leading term then either comes from the LT of the original degree three relation, or from the monomial with coefficient $g_1,\ h_1,\ i_1$, or j_1. Some entries are left blank because the leading term is not known. For example, if r_{54} is used to solve for x_1 then the LT of x_1 is x_5x_4 and the the largest term of r_{31} becomes $x_5x_4x_3$. But this term can be reduced using r_{43} and it is no longer obvious what the leading term of the relation will be without fully writing out all of the other relations used for reductions.

In fact, from the analysis preceding Theorem 5.1.1 (page 43), we know that the Gröbner basis of the algebra where r_{54} is used to solve for x_1 has 2 degree three relations. From the table above, we now also know that their leading terms are $x_5x_5x_4$ and $x_5x_4x_4$ and come from r_{51} and r_{41}. Thus we can conclude that r_{31} and r_{21} must simplify to 0 in $\widetilde{A}$. Again comparing the known number of relations in the Gröbner basis, calculated in the analysis preceding Theorem 5.1.1, against the entries in the table, we can conclude that all of the blanks are actually 0, although this information is not needed for the analysis that follows.

It is also worth briefly mentioning what happens in the case that the ordering is different. The possible orderings for an Ore extension generated in degree one with variables $(x_5, x_4, x_3, x_2, x_1)$ of degrees (1,1,1,1,2) are $x_5 > x_4 > x_3 > x_2 > x_1$, $x_5 > x_4 > x_3 > x_1 > x_2$, $x_5 > x_4 > x_1 > x_3 > x_2$, $x_5 > x_1 > x_4 > x_3 > x_2$. Note that $x_1 > x_5 > x_4 > x_3 > x_2$ is not a possibility since then there will be no degree two relation in which x_1 can appear and the algebra will not be generated in degree one.

If the ordering is $x_5 > x_1 > x_4 > x_3 > x_2$, i.e.

$$A = K[x_2][x_3, \sigma_3, \delta_3][x_4, \sigma_4, \delta_4][x_1, \sigma_1, \delta_1][x_5, \sigma_5, \delta_5],$$

the columns corresponding to x_4x_3, x_4x_2, and x_3x_2 can be left blank since x_1 is adjoined too late in the Ore extension to come from r_{43}, r_{42}, or r_{32}. The other entries will be the same as in the column above, coming directly from the LTs of the degree three relations. For example, with the given ordering, r_{41} has LT x_1x_4. If x_1 comes from x_5x_4 then this becomes $x_5x_4x_4$ and is the leading term of r_{41} in $\widetilde{A}$, just as the entry in the table above suggests.

With a little more work, it can be shown that this table is accurate for the other possible orderings of the variables. Using the fact that each σ_i is bijective, we can prove that each degree three relation has a nonzero coefficient in front of the monomials x_ix_1 and x_1x_i, $2 \leq i \leq 5$, and that the table has the entries listed above whenever the ordering permits x_1 to have the leading term indicated. We omit the details here. The rest of the proof relies only on the degree two relations and is independent of the ordering of x_1 relative to the other variables.

The goal now is to prove that there is no [2,2]-bigraded algebra where exactly one of the relations listed in the table above is in the minimal generating set. Thus, we wish to investigate the degree three overlaps to determine whether any degree three relations may be consequences of these. The degree three overlaps are $x_5x_4x_3$, $x_5x_4x_2$, $x_5x_3x_2$, and $x_4x_3x_2$, although not all of these will necessarily occur since this depends on which relation is used to solve for x_1. The analysis breaks down into cases depending on which bigrading is chosen for the degree one generators. Without loss of generality, we may assume that $\deg(x_5) = (0, 1)$ so the options for the bigrading are

1. $\deg(x_3) = \deg(x_4) = (1, 0);\ \deg(x_2) = \deg(x_5) = (0, 1);$
2. $\deg(x_2) = \deg(x_4) = (1, 0);\ \deg(x_3) = \deg(x_5) = (0, 1);$
3. $\deg(x_2) = \deg(x_3) = (1, 0);\ \deg(x_4) = \deg(x_5) = (0, 1).$

Case 1: $\deg(x_3) = \deg(x_4) = (1, 0)$.

If x_1 comes from r_{54} then, by the table above, the Gröbner basis has degree three relations with LTs $x_5x_5x_4$ and $x_5x_4x_4$. The only possible degree three overlaps are all smaller than this. So both of these relations are independent of overlaps that fail to resolve and the relation type is (2,2,2,2,2,3,3) rather than (2,2,2,2,2,3).

If x_1 comes from r_{53} then, by the table above, the Gröbner basis has degree three relations with LTs $x_5x_5x_3,\ x_5x_3x_4,\ x_5x_3x_3$. Note that $x_5x_5x_3$ is too large to come from an overlap so must be independent. Additionally, $x_5x_3x_3$ and $x_5x_3x_4$ have bidegree (2,1), while the only overlap with this bidegree which is large enough is $x_5x_4x_3$. Thus, one of these leading terms must be independent of overlaps. In total, there are at least (and by Theorem 5.1.2, exactly) 2 independent degree three relations and the relation type is not (2,2,2,2,2,3).

If x_1 comes from r_{52} then, by the table above, the Gröbner basis has LTs $x_5x_5x_2$, $x_5x_2x_4$, $x_5x_2x_3$, $x_5x_2x_2$. The first is too large to come from an overlap and $x_5x_2x_2$ has bidegree (0,3), different from any of the overlaps. Thus, there must be at least 2 total independent degree three relations and the relation type cannot be (2,2,2,2,2,3).

If x_1 comes from r_{43} then, by the table above, the Gröbner basis has LTs $x_4x_4x_3$ and $x_4x_3x_3$, both of bidegree (3,0). Since there are no overlaps of this degree, these are part of the minimal generating set and the relation type cannot be (2,2,2,2,2,3).

If x_1 comes from r_{42}, we are required to consider the degree two terms, which we write recalling the chosen bigrading:

$$r_{32} : x_3x_2 = a_1x_1 + a_3x_2x_3$$

$$r_{42} : x_4x_2 = b_1x_1 + b_3x_2x_3 + b_4x_2x_4$$

$$r_{43} : x_4x_3 = c_1x_1 + c_6x_3x_3 + c_7x_3x_4$$

$$r_{52} : x_5x_2 = d_1x_1 + d_2x_2x_2 + d_5x_2x_5$$

$$r_{53} : x_5x_3 = e_1x_1 + e_3x_2x_3 + e_4x_2x_4 + e_8x_3x_5 + e_{10}x_4x_5$$

$$r_{54} : x_5x_4 = f_1x_1 + f_3x_2x_3 + f_4x_2x_4 + f_8x_3x_5 + f_{10}x_4x_5.$$

If x_1 comes from r_{42} and has LT x_4x_2, we may assume that $a_1 = 0$ and $b_1 \neq 0$ and, since $\deg(x_1)$ must then equal (1,1), we may assume $c_1 = d_1 = 0$. By the injectivity of the σ_i, we also know that a_3, b_4, c_7, d_5 are not zero and that $\begin{vmatrix} e_8 & e_{10} \\ f_8 & f_{10} \end{vmatrix} \neq 0$. We can then compute the 3 overlaps in $\widetilde{A}$ with the help of computer software ([Ell, Section 12]):

$$x_4(x_3x_2) - (x_4x_3)x_2 = -c_6x_3x_3x_2 - c_7x_3x_4x_2 + a_3x_4x_2x_3$$

does not resolve, and

$$\begin{aligned} x_5(x_4x_3) - (x_5x_4)x_3 &= (c_6e_{10}^2 - e_{10}f_{10} + c_7e_{10}f_{10})x_4x_4x_5 \\ &\qquad + (\frac{c_6e_1e_{10}}{b_1} + \frac{c_7e_{10}f_1}{b_1} - \frac{e_1f_{10}}{b_1})x_4x_4x_2 + \text{[smaller terms]} \\ x_5(x_3x_2) - (x_5x_3)x_2 &= -d_5e_{10}x_4x_2x_5 - (\frac{e_1}{b_1} + d_2e_{10})x_4x_2x_2 + \text{[smaller terms]}. \end{aligned}$$

Since the 3 degree three LTs in the Gröbner basis, according to the table above, are $x_4x_4x_2,\ x_4x_2x_3,\ x_4x_2x_2$, we conclude that $x_4x_2x_5$ cannot be a LT of a relation and thus that $-d_5e_{10} = 0$. Since $d_5 \neq 0$, $e_{10} = 0$ and the overlaps become

$$\begin{aligned} x_5(x_4x_3) - (x_5x_4)x_3 &= -\frac{e_1f_{10}}{b_1}x_4x_4x_2 + \text{[smaller terms]} \\ x_5(x_3x_2) - (x_5x_3)x_2 &= -\frac{e_1}{b_1}x_4x_2x_2 + \text{[smaller terms]}. \end{aligned}$$

By the injectivity of the σ_i and the fact that $e_{10} = 0$, $f_{10} \neq 0$. If e_1 is not zero then neither of these overlaps resolve, all degree three relations are consequences of overlaps, and the relation type is (2,2,2,2,2). If e_1 is zero then the LTs of the overlaps are smaller than $x_4x_4x_2$ and $x_4x_2x_2$, respectively. (The term $x_4x_2x_2$ never appears in the overlap $x_5x_4x_3$, since these have different bigradings). But there are relations in the Gröbner basis with these leading terms so we conclude that both of these relations must be independent of overlaps that fail to resolve and the relation type is (2,2,2,2,3,3). In either case, the relation type cannot be (2,2,2,2,3).

If x_1 comes from r_{32} then we will use the same degree two relations written above, observing that $a_1 \neq 0$, $c_1 = d_1 = 0$ since $\deg(x_1) = (1,1)$, and $a_3, b_4, c_7, d_5, \begin{vmatrix} e_8 & e_{10} \\ f_8 & f_{10} \end{vmatrix} \neq 0$ by the injectivity of the σ_i. The overlaps are:

$$x_5(x_4x_3) - (x_5x_4)x_3 = k_1x_4x_4x_5 + k_2x_3x_4x_5 + k_3x_3x_3x_5 + k_4x_3x_3x_2 + [\text{smaller terms}]$$

$$x_5(x_4x_2) - (x_5x_4)x_2 = k_5x_3x_2x_5 + k_6x_3x_2x_2 + [\text{smaller terms}], \text{ where}$$

$$k_1 = c_6e_{10}^2 - e_{10}f_{10} + c_7e_{10}f_{10}$$

$$k_2 = c_6e_{10}e_8 + c_6c_7e_{10}e_8 - e_{10}f_8 + c_7^2e_{10}f_8$$

$$k_3 = c_6^2e_{10}e_8 + c_6e_8^2 - c_6e_8f_{10} + c_6c_7e_{10}f_8 - e_8f_8 + c_7e_8f_8$$

$$k_4 = (c_6^2e_1e_{10} + c_6e_1e_8 + c_6c_7e_{10}f_1 + c_7e_8f_1 - c_6e_1f_{10} - e_1f_8)/(a_1)$$
$$+ (b_1c_6c_7e_1e_{10} + b_1c_7^2e_{10}f_1 - b_1c_7e_1f_{10})/a_1^2$$

$$k_5 = -b_1^2d_5e_{10}/a_1^2 - b_1d_5e_8/a_1 + b_1d_5f_{10}/a_1 + d_5f_8$$

$$k_6 = -b_1e_1/a_1^2 - b_1^2d_2e_{10}/a_1^2 - b_1d_2e_8/a_1 + f_1/a_1 + b_1d_2f_{10}/a_1 + d_2f_8).$$

From the table, the LTs of the degree three relations are $x_3x_3x_2$ and $x_3x_2x_2$. We note that in order for the relations to have the correct LTs, we must have $k_1 = k_2 = k_3 = k_5 = 0$. Then, in order to get relation type (2,2,2,2,2,3), and noting that these overlaps have different bigradings, we require that exactly one of k_4 and k_6 be zero. We code this as $k_4k_6 = 0$ and $k_4 + k_6 \neq 0$. We then ask Mathematica to solve this system of equations, together with the requirement that $a_3, b_4, c_7, d_5, \begin{vmatrix} e_8 & e_{10} \\ f_8 & f_{10} \end{vmatrix} \neq 0$, and we find that there is no solution to this system. We thus conclude that (2,2,2,2,2,3) is not a possible relation type.

The other two cases for different bigradings are similar and the details are provided online (see [Ell, Section 12]). In all cases, we find that (2,2,2,2,2,3) is not a possible relation type for an Ore extension which is [2,2]-bigraded. □

8 No bigraded AS-regular algebras with relation type (2,2,3)

Our next goal is to prove that there is no bigraded AS-regular algebra, Ore extension or otherwise, with relation type (2,2,3).

Theorem 8.0.3. *There is no bigraded AS-regular algebra which is a domain generated by 3 degree one generators with relation type (2,2,3).*

Proof. Assume that such an algebra, A, exists and label the generators x_1, x_2, x_3. Without loss of generality we may assume that $\deg(x_1) = \deg(x_2) = (1,0)$ and $\deg(x_3) = (0,1)$. We may also choose to order the variables so that $x_3 > x_2 > x_1$. By the analysis following Corollary 2.2.5 (page 12), the Hilbert series of A must be $\frac{1}{(1-t)^3(1-t^2)(1-t^3)}$. If A_0 represents the free algebra on three generators then $h_{A_0}(t) - h_A(t) = 2t^2 + O(t^3)$ so the algebra must have 2 degree two relations.

As usual, the proof depends on the analysis of these degree 2 relations. Note that there can be no relation of degree (0,2) since this would force the relation to be $kx_3x_3 = 0$ which would violate the assumption that A is a domain. If there is a relation of degree (1,1), it must come from the list of monomials $\{x_3x_2, x_3x_1, x_2x_3, x_1x_3\}$. We note that if x_2x_3 or x_1x_3 is a leading term then it can be checked that this would mean that x_3 is a right zero divisor, which violates the assumption that A is a domain.

If there is at least one relation of degree (2,0) then by [KKZ, Lemma 3.7], the subalgebra B of A generated by x_1 and x_2 is a 2-dimensional AS-regular algebra. In particular this means that there can be at most 1 relation of degree (2,0) and that it must have leading term x_2x_1.

Thus, the possible degree two relations have leading terms from the list $\{x_3x_2, x_3x_1, x_2x_1\}$.

Case 1: The leading terms are x_3x_2 and x_3x_1.
The Hilbert series of A is $h_A(t) = \dfrac{1}{(1-t)^3(1-t^2)(1-t^3)}$. Since $\deg(x_1) = \deg(x_2) = (1,0)$ and $\deg(x_3) = (0,1)$, the $(1-t)^3$ corresponds to $(1-u)^2(1-v)$ in the bigrading. The $(1-t^2)$ may correspond to $(1-u^2),\ (1-uv),$ or $(1-v^2)$. The $(1-t^3)$ may correspond to $(1-u^3),\ (1-u^2v),\ (1-uv^2),$ or $(1-v^3)$. If A_2 denotes the monomial algebra with relations $x_3x_2 = 0$ and $x_3x_1 = 0$, then we can calculate

$$h_{A_2}(u,v) = 2u + v + 4u^2 + 2uv + v^2 + 8u^3 + 4u^2v + 2uv^2 + v^3 + \cdots$$

which we will notate as $h_{A_2}(u,v) = \{\{2,1\},\{4,2,1\},\{8,4,2,1\},\cdots\}$. Comparing the bigraded $h_{A_2}(u,v)$ against $h_A(u,v)$ we find that the $(1-t^2)$ must correspond to $(1-u^2)$ (see [Ell, Section 13]). Further, since x_3x_2 and x_3x_1 do not overlap, any degree three relation in the Gröbner basis will be part of the minimal generating set. Since the relation type is (2,2,3), we wish there to be only 1 degree three relation. The only bigraded Hilbert series of A for which $h_{A_2}(u,v) - h_A(u,v)$ is off by only one degree three term is the one for which $(1-t^3)$ corresponds to $(1-u^3)$. We now know the bigraded Hilbert series of A and can calculate:

$$h_{A_2}(u,v) - h_A(u,v) = \{\{0,0\},\{0,0,0\},\{1,0,0,0\},[\text{higher order terms}]\}.$$

Thus, there is a relation of bidegree (3,0). The list of all possible reduced monomials in A_2 of degree (3,0) is

$$\{x_1x_1x_1, x_1x_1x_2, x_1x_2x_1, x_1x_2x_2, x_2x_1x_1, x_2x_1x_2, x_2x_2x_1, x_2x_2x_2\}.$$

If the algebra is a domain, then it can be checked that the LT of the degree 3 relation must be one of $\{x_2x_1x_1, x_2x_1x_2, x_2x_2x_1, x_2x_2x_2\}$.

If the LT is taken to be $x_2x_2x_2$ and A_3 refers to the monomial algebra with LTs $x_3x_2,\ x_3x_1$, and $x_2x_2x_2$ then

$$h_{A_3}(u,v) - h_A(u,v) = \{\{0,0\},\{0,0,0\},\{0,0,0,0\},\{2,0,0,0,0\},\cdots\}.$$

If it's taken to be $x_2x_1x_1,\ x_2x_1x_2,$ or $x_2x_2x_1$ then

$$h_{A_3}(u,v) - h_A(u,v) = \{\{0,0\},\{0,0,0\},\{0,0,0,0\},\{1,0,0,0,0\},\cdots\}.$$

In all cases, since there can be no overlap of degree (4,0) in an algebra with the leading terms of A_3, we conclude that there must be at least 1 degree four relation in both the basis and the minimal generating set and so the relation type (2,2,3) is impossible.

Case 2: The leading terms are x_3x_1 and x_2x_1.

With these leading terms, $h_{A_2}(u,v) = \{\{2,1\},\{3,3,1\},\{4,6,4,1\},\cdots\}$ and so we can check that $h_A(u,v)$ must have $(1-t^2)$ term corresponding to $(1-uv)$ and can have $(1-t^3)$ term corresponding to $(1-u^2v)$ or $(1-uv^2)$.

In the first case,

$$h_{A_2}(u,v) - h_A(u,v) = \{\{0,0\},\{0,0,0\}\{0,0,1,0\},\cdots\}$$

and there is one relation of degree (1,2) in the Gröbner basis. The list of possible reduced degree (1,2) monomials in A_2 is $\{x_1x_3x_3, x_2x_3x_3, x_3x_2x_3, x_3x_3x_2\}$. Since the first 3 monomials in this list end in x_3 and A is a domain, we require that there be a term that does not end in x_3 and so conclude that the LT must be $x_3x_3x_2$. In this case,

$$h_{A_3}(u,v) - h_A(u,v) = \{\{0,0\},\{0,0,0\},\{0,0,0,0\}\{0,1,0,0,0\},\cdots\}.$$

Thus A has a relation of bidegree (3,1), but the only possible overlap in A of degree 4 is $x_3x_3x_2x_1$ which has bidegree (2,2). We conclude that the relation of degree (3,1) is independent of overlaps and so the relation type cannot be (2,3,3).

Otherwise, the $(1-t^3)$ term corresponds to $(1-uv^2)$,

$$h_{A_2}(u,v) - h_A(u,v) = \{\{0,0\},\{0,0,0\}\{0,1,0,0\},\cdots\},$$

and there is one relation of degree (2,1) in the Gröbner basis. The list of possible degree (2,1) monomials in A_2 is $\{x_1x_1x_3, x_1x_2x_3, x_1x_3x_2, x_2x_2x_3, x_2x_3x_2, x_3x_2x_2\}$ and the LT of any degree three relation must not begin with x_1 since this would violate the fact that A is a domain. If the LT is $x_2x_2x_3,\ x_2x_3x_2$, or $x_3x_2x_2$, then

$$h_{A_3}(u,v) - h_A(u,v) = \{\{0,0\},\{0,0,0\},\{0,0,0,0\}\{0,0,0,1,0\},\cdots\}.$$

Thus, A has a relation of degree (1,3) in the Gröbner basis. However, there are no overlaps of degree (1,3) for A in any of these cases, so this relation is minimal and the relation type cannot be (2,2,3).

Case 3: The leading terms are x_3x_2 and x_2x_1.

In the case, using the fact that the relations are bigraded, we can write the most general possible degree two relations in A:

$$x_2x_1 = a_1x_1x_1 + a_2x_1x_2$$
$$x_3x_2 = b_1x_1x_3 + b_2x_2x_3 + b_3x_3x_1$$

and we can compute the overlap

$$x_3(x_2x_1) - (x_3x_2)x_1 = -b_1x_1x_3x_1 - b_2x_2x_3x_1 + (a_1 - b_3)x_3x_1x_1 + a_2x_3x_1x_2.$$

Note that a_2 cannot be zero since otherwise the relation $(x_2 - a_1x_1)x_1 = 0$ would violate the fact that A is a domain. Thus, $x_3x_1x_2$ is the leading term of a degree three relation in the Gröbner basis of A which is not part of the minimal generating set. If A_3' refers to the monomial algebra with LTs $x_3x_2,\ x_2x_1$, and $x_3x_1x_2$ then, comparing $h_{A_3'}(u,v)$ with the possible series $h_A(u,v)$, we find that the term $(1-t^2)$ must correspond to $(1-uv)$ and the $(1-t^3)$ term to either $(1-u^2v)$ or $(1-uv^2)$.

In the first case, A must have an additional relation of degree (1,2) and the list of possible monomials of this degree is $\{x_1x_3x_3, x_2x_3x_3, x_3x_1x_3, x_3x_3x_1\}$. Of these, the only candidate for a LT in a domain is $x_3x_3x_1$, in which case

$$h_{A_3}(u,v) - h_A(u,v) = \{\{0,0\},\{0,0,0\},\{0,0,0,0\},\{0,2,0,0,0\},\cdots\}.$$

The overlaps for an algebra with these leading terms are $x_3x_1x_2x_1$ and $x_3x_3x_1x_2$, of bidegrees (3,1) and (2,2), respectively. The difference in Hilbert series suggests that A must have 2 relations of degree (3,1), only one of which may be the consequence of an overlap which fails to resolve, and so there must be an independent degree four relation and the relation type cannot be (2,3,3).

Otherwise, the $(1-t^3)$ corresponds to $(1-uv^2)$ and there must be an additional relation of degree (2,1). The list of possible reduced monomials in A_3' of this degree, is

$$\{x_1x_1x_3, x_1x_2x_3, x_1x_3x_1, x_2x_2x_3, x_2x_3x_1, x_3x_1x_1\}.$$

If A is a domain, the degree three LT must not begin with x_1, which leaves 3 possibilities. If the LT is $x_3x_1x_1$ then let A_3 denote the monomial algebra that has LTs $x_3x_2,\ x_2x_1,\ x_3x_1x_2$, and $x_3x_1x_1$. Then

$$h_{A_3}(u,v) - h_A(u,v) = \{\{0,0\},\{0,0,0\},\{0,0,0,0\},\{0,0,0,1,0\},\cdots\}$$

and the only degree four overlap is $x_3x_1x_2x_1$. We conclude that there must be a relation of bidegree (1,3) and that this cannot be the consequence of an overlap, so the relation type of this algebra cannot be (2,2,3). If instead A_3 refers to the algebra with leading terms $x_3x_2,\ x_2x_1,\ x_3x_1x_2$ and $x_2x_2x_3$ or $x_2x_3x_1$ then

$$h_{A_3}(u,v) - h_A(u,v) = \{\{0,0\},\{0,0,0\},\{0,0,0,0\},\{0,1,1,1,0\},\cdots\}$$

while the degree four overlaps are

$$\{x_3x_1x_2x_1, x_2x_2x_3x_2, x_3x_2x_2x_3\} \text{ or } \{x_3x_1x_2x_1, x_3x_2x_3x_1, x_2x_3x_1x_2\}$$

and so again there is a relation of bidegree (1,3) which must be minimal and the relation type cannot be (2,2,3). □

It has been conjectured that, as is the case in AS-regular algebras of dimension 4 or less, every relation type can be realized by a bigraded AS-regular algebra. This result is the first known example where the conjecture fails. As mentioned earlier, it is also true that every AS-regular algebra of dimension 4 or less has a unique relation type for each distinct Hilbert series and can be realized by the enveloping algebra of a graded Lie algebra, both properties that fail in the dimension 5 case. Higher dimensional AS-regular algebras have many unexpected properties and in general remain very poorly understood.

Generalizing the question about bigraded algebras and in light of the fact that not every relation type of an AS-regular algebra can be realized by a bigraded AS-regular algebra, we ask:

Question 8.0.4. *Can every Hilbert series of an AS-regular algebra be realized by a bigraded AS-regular algebra?*

For all Hilbert series that we know of, the answer to the preceding question is yes.

In our computations in the preceding theorem, we relied upon the assumption that the algebra had at most 1 degree three and 1 degree four relation in the minimal generating set in order to restrict the possible Hilbert series of A and hence the cases that we had to consider. It is then natural to ask:

Question 8.0.5. *Is there an AS-regular algebra (bigraded or otherwise) with Hilbert series* $\frac{1}{(1-t)^3(1-t^2)(1-t^3)}$ *that has a relation type other than (2,2,3) or (2,2,3,4)?*

More generally, we are curious about the question:

Question 8.0.6. *Can every AS-regular algebra of dimension 5 (and of higher dimensions) be realized by an AS-Ore extension?*

Two questions that would help us to explore this are:

Question 8.0.7. *Is there an AS-regular algebra of dimension 5 with a Hilbert series considered in this thesis that has a different relation type than that of an Ore extension?*

Question 8.0.8. *Is there an AS-regular algebra of dimension 5 with a different Hilbert series than those considered here?*

A Appendix: Additional examples

We now provide additional examples of AS-Ore extensions for the degree types discussed in this dissertation. These may be of interest to people studying AS-regular algebras. While we made some attempt to provide diversity in the Ore extensions that follow (for example changing which coefficients are nonzero, which relations are used to solve for higher degree variables, and the relation type), we note that all extensions have lexicographic order $x_5 > x_4 > x_3 > x_2 > x_1$, i.e. are of the form $K[x_1][x_2, \sigma_2, \delta_2][x_3, \sigma_3, \delta_3][x_4, \sigma_4, \delta_4][x_5, \sigma_5, \delta_5]$, with the exception of Example A.2.6. This was motivated by convenience, not necessity.

For each example that follows, it is easy to check that the relations are homogeneous with $(x_1, x_2, x_3, x_4, x_5)$ of the degree type listed, that the algebra is generated in degree one, and that each σ_i is injective. We use a computer program to verify that overlaps resolve and that the relation type is as stated in the cases where the relation type is not already determined by the degree type [Ell, Section 14].

A.1 Degree type (1,1,2,3,5)

Example A.1.1.

$x_2x_1 = 1/2(1 - j_6)x_1x_2$

$x_3x_2 = x_1 + x_2x_3$

$x_3x_1 = 2/(1 - j_6)x_1x_3$

$x_4x_3 = x_2 + 2/(1 - j_6)x_3x_4$

$x_4x_2 = 1/2(1 - j_6)x_2x_4 + (3 + 2j_6 - j_6^2)/(4j_4)x_3x_3$

$x_4x_1 = x_1x_4 + (1 + j_6)/2x_2x_2 + (-3 + j_6)/(2j_4)x_3x_3x_3$

$x_5x_4 = x_3 + 1/2(-1 - j_6)x_4x_5$

$x_5x_3 = 1/6(-3j_4 - j_4j_6)x_2 + 1/2(j_4 - j_4j_6)x_3x_4 + 1/2(-1 + j_6)x_3x_5$

$x_5x_2 = 1/2(-j_4 + j_4j_6)x_2x_4 + 1/4(1 - j_6^2)x_2x_5 + 1/2(3 + j_6)x_3x_3$

$+ 1/2(-3j_4 - j_4j_6)x_3x_4x_4$

$x_5x_1 = 1/2(j_4 - j_4j_6)x_1x_4 + 1/2(-1 + j_6)x_1x_5 + 1/2(j_4 - j_4j_6)x_2x_2 + j_4x_2x_3x_4$

$+ j_6x_3x_3x_3$

where $j_6 = -\boldsymbol{i}\sqrt{3}$

or $j_6 = \boldsymbol{i}\sqrt{3}$.

Example A.1.2.

$x_2x_1 = (1 + j_2)x_1x_2$

$x_3x_2 = x_1 + x_2x_3$

$x_3x_1 = 1/(1 + j_2)x_1x_3$

$x_4x_3 = x_2 + 1/(1 + j_2)x_3x_4$

$x_4x_2 = (1 + j_2)x_2x_4 + (j_2^2)/(h_1)x_3x_3$

$x_4x_1 = x_1x_4 - j_2x_2x_2 + (j_2)/(h_1)x_3x_3x_3$

$x_5x_4 = x_3 + (-1 - j_2)x_4x_5$

$x_5x_3 = h_1x_2 + j_2x_3x_5$

$x_5x_2 = (-j_2 - j_2^2)x_2x_5$

$x_5x_1 = j_2x_1x_5$

where $j_2 = -(-1)^{1/3}$

or $j_2 = (-1)^{2/3}$.

Example A.1.3.

$x_2x_1 = i_3x_1x_2$

$x_3x_2 = x_1 + x_2x_3$

$x_3x_1 = 1/(i_3)x_1x_3$

$x_4x_3 = x_2 + 1/(i_3)x_3x_4$

$x_4x_2 = i_3x_2x_4 + (-f_6 + f_6i_3)x_3x_3$

$x_4x_1 = x_1x_4 + (1 - i_3)x_2x_2 + f_6x_3x_3x_3$

$x_5x_4 = x_3 - x_4x_5$

$x_5x_3 = x_3x_5$

$x_5x_2 = -x_2x_5 + i_3x_3x_3$

$x_5x_1 = -x_1x_5$

where $i_3 = (-1)^{1/3}$

or $i_3 = -(-1)^{2/3}$.

Example A.1.4.

$x_2x_1 = 1/2(4 - j_6)x_1x_2$

$x_3x_2 = x_1 + x_2x_3$

$x_3x_1 = 2/(4 - j_6)x_1x_3$

$x_4x_3 = x_2 + 3(4 - j_6)/(j_4)x_3x_4$

$x_4x_2 = 1/2(4 - j_6)x_2x_4 + 3(4 - j_6)/(j_4)x_3x_3$

$x_4x_1 = (48 - 24j_6 + 3j_6^2)/(2j_4)x_1x_4 + 1/2(-2 + j_6)x_2x_2$

$+ 3(48 - 4j_4 - 24j_6 + j_4j_6 + 3j_6^2)/(j_4^2)x_3x_3x_3$

$x_5x_4 = x_3 + 1/2(4 - j_6)x_4x_5$

$x_5x_3 = (j_4)/(2)x_2 + 1/2(-2j_4 + j_4j_6)x_3x_4 + 1/2(2 - j_6)x_3x_5+$

$1/3(-3j_7 + j_6j_7)x_4x_4x_4$

$x_5x_2 = 1/2(4j_4 - j_4j_6)x_2x_4 + 1/4(8 - 6j_6 + j_6^2)x_2x_5 + (2 - j_6)x_3x_3$

$+ j_4x_3x_4x_4 + j_7x_4x_4x_4x_4$

$x_5x_1 = 1/6j_4j_6x_1x_4 + 1/8(16 - 20j_6 + 8j_6^2 - j_6^3)x_1x_5 + 1/6(6j_4 - j_4j_6)x_2x_2$

$+ j_4x_2x_3x_4 + j_6x_3x_3x_3 + j_7x_2x_4x_4x_4 - 1/6j_4j_6x_3x_3x_4x_4$

where $j_6 = 3 + \boldsymbol{i}\sqrt{3},\ j_4 = 6 - 3j_6,\ j_7 = 12 - 3j_6$

or $j_6 = 3 - \boldsymbol{i}\sqrt{3},\ j_4 = 6 - 3j_6,\ j_7 = 12 - 3j_6.$

A.2 Degree type (1,1,1,2,2)

Example A.2.1.

$x_2x_1 = x_1x_2$

$x_3x_2 = x_2x_3$

$x_3x_1 = x_1x_3$

$x_4x_3 = x_1 + g_5/2x_3x_3 + x_3x_4$

$x_4x_2 = x_1x_3 + x_2x_4 + 1/2g_5x_3x_3x_3$

$x_4x_1 = x_1x_4$

$x_5x_4 = x_2 + g_3x_3x_3 - h_2x_3x_4 + g_5x_3x_5 + x_4x_5$

$x_5x_3 = h_1x_1 + h_2x_2 + x_3x_5$

$x_5x_2 = h_1x_1x_3 + h_2x_2x_3 + x_2x_5$

$x_5x_1 = x_1x_5.$

Example A.2.2.

$x_2x_1 = x_1x_2$

$x_3x_2 = x_2x_3$

$x_3x_1 = x_1x_3$

$x_4x_3 = x_1 + d_3x_3x_3 + x_3x_4$

$x_4x_2 = x_1x_3 + x_2x_4 + e_7x_3x_3x_3$

$x_4x_1 = f_1x_1x_3 + x_1x_4 + f_7x_3x_3x_3$

$x_5x_4 = x_2 + g_3x_3x_3 + g_4x_3x_4 + g_5x_3x_5 + x_4x_5$

$x_5x_3 = h_1x_1 + x_3x_5$

$x_5x_2 = h_1x_1x_3 + x_2x_5$

$x_5x_1 = f_1h_1x_1x_3 + x_1x_5$

where $h_1 = 0, g_4 = 0$

or $f_7 = e_7 = d_3 = 0, g_4 = -g_5h_1$

or $f_7 = f_1 = 0, d_3 = e_7, g_4 = (2e_7 - g_5)h_1, e_7 = g_5/2$

or $f_1 = 1/2(g_5 \pm \sqrt{-8f_7 + g_5^2}\ ), e_7 = d_3 = 1/2(-f_1 + g_5), g_4 = -f_1h_1.$

Example A.2.3.

$x_2x_1 = x_1x_2$

$x_3x_2 = x_2x_3$

$x_3x_1 = x_1x_3$

$x_4x_3 = x_2 + 1/2(-e_4 + g_5)x_3x_3 + x_3x_4$

$x_4x_2 = e_4x_2x_3 + x_2x_4 + e_7x_3x_3x_3$

$x_4x_1 = x_1x_4$

$x_5x_4 = g_1x_1 + g_2x_2 + g_3x_3x_3 - 2h_3x_3x_4 + g_5x_3x_5 + x_4x_5$

$x_5x_3 = x_1 + h_3x_3x_3 + x_3x_5$

$x_5x_2 = e_4x_1x_3 + x_2x_5 + i_7x_3x_3x_3$

$x_5x_1 = x_1x_5$

where $e_7 = (g_5h_3 - i_7)i_7/(2h_3^2), e_4 = i_7/h_3$

or $i_7 = 0, h_3 = 0, e_4 = 1/2(g_5 \pm \sqrt{-8e_7 + g_5^2}\ )$.

Example A.2.4.

$x_2x_1 = x_1x_2$

$x_3x_2 = -x_1x_3 + b_4x_2x_3$

$x_3x_1 = -b_4x_1x_3$

$x_4x_3 = -x_3x_4$

$x_4x_2 = -x_1x_4 + b_4x_2x_4$

$x_4x_1 = -b_4x_1x_4$

$x_5x_4 = x_2 - h_3/b_4^2x_3x_3 + g_4x_3x_4 + 1/b_4^2x_3x_5 + g_6x_4x_4$

$x_5x_3 = x_1 - b_4x_2 + h_3x_3x_3 + h_4x_3x_4 - g_6x_4x_4 + x_4x_5$

$x_5x_2 = -1/b_4^2x_1x_5 + 1/b_4x_2x_5$

$x_5x_1 = -1/b_4x_1x_5$

where $g_4 = h_4, b_4 = 1$

or $g_4 = h_4, b_4 = -1$

or $b_4 \neq 0, h_4 = h_3 = g_6 = g_4 = 0.$

Example A.2.5.

$x_2x_1 = x_1x_2$

$x_3x_2 = -x_1x_3 + b_4x_2x_3$

$x_3x_1 = -b_4x_1x_3$

$x_4x_3 = -x_3x_4$

$x_4x_2 = -x_1x_4 + b_4x_2x_4$

$x_4x_1 = -b_4x_1x_4$

$x_5x_4 = x_2 - h_3/b_4^2x_3x_3 + g_4x_3x_4 + 1/b_4^2x_3x_5 + g_6x_4x_4$

$x_5x_3 = x_1 - b_4x_2 + h_3x_3x_3 + h_4x_3x_4 - g_6x_4x_4 + x_4x_5$

$x_5x_2 = i_1x_1x_3 + i_2x_1x_4 - 1/b_4^2x_1x_5 - b_4i_1x_2x_3 - b_4i_2x_2x_4 + 1/b_4x_2x_5$

$x_5x_1 = b_4i_1x_1x_3 + b_4i_2x_1x_4 - 1/b_4x_1x_5$

where $i_1 = i_2 = -g_4, g_6 = h_3, h_4 = 0, b_4 = \sqrt{h_3}/\sqrt{h_3 + i_1}, b_4i_2 \neq 0$

or $b_4i_2 \neq 0, i_1 = i_2 = -g_4, g_6 = h_3, h_4 = 0, b_4 = -\sqrt{h_3}/\sqrt{h_3 + i_1}$

or $h_3 = (h_4i_1)/(i_1 - i_2), g_6 = h_3 - h_4, g_4 = h_4 - i_2, b_4 = \pm\sqrt{h_3}/\sqrt{h_3 + i_1}$.

Example A.2.6. This is an example of an AS-Ore extension

$$K[x_3][x_1,\sigma_1,\delta_1][x_4,\sigma_4,\delta_4][x_2,\sigma_2,\delta_2][x_5,\sigma_5,\delta_5].$$

$x_2x_1 = a_1x_1x_1 + x_1x_2$

$x_2x_3 = b_2x_1x_4 + (-b_2 + e_1)x_3x_1 + x_3x_2 + 1/6(-3a_1h_3 - 6b_2h_3 + 6e_1h_3)x_3x_3x_3$

$\quad + b_2h_3x_3x_3x_4$

$x_1x_3 = x_3x_1$

$x_4x_3 = x_3x_4$

$x_2x_4 = e_1x_3x_1 + x_4x_2 + 1/6(-3a_1h_3 - 6b_2h_3 + 6e_1h_3)x_3x_3x_3 + b_2h_3x_3x_3x_4$

$x_4x_1 = x_1x_4$

$x_5x_4 = x_1 + h_3x_3x_3 + x_4x_5$

$x_5x_3 = x_1 + h_3x_3x_3 + x_3x_5$

$x_5x_2 = i_2x_1x_4 + (-a_1 + e_1)x_1x_5 + x_2x_5 + i_1x_3x_1 + i_7x_3x_3x_3 + i_8x_3x_3x_4$

$\quad + 1/2(-a_1h_3 + 2e_1h_3)x_3x_3x_5 + i_{13}x_4x_4x_4$

$x_5x_1 = x_1x_5.$

A.3 Degree type (1,1,1,2,3)

Example A.3.1.

$$x_2x_1 = x_1x_2$$

$$x_3x_2 = x_1 + x_2x_3$$

$$x_3x_1 = x_1x_3$$

$$x_4x_3 = x_2 + x_3x_4$$

$$x_4x_2 = x_1 + x_2x_4$$

$$x_4x_1 = x_1x_4$$

$$x_5x_4 = (h_1 + h_3 + 2h_5)x_2 + (-h_3 - h_5)x_3x_3 + h_3x_3x_4 - x_3x_5 + h_5x_4x_4 + 2x_4x_5$$

$$x_5x_3 = h_1x_2 + (-h_3 - h_5)x_3x_3 + h_3x_3x_4 + h_5x_4x_4 + x_4x_5$$

$$x_5x_2 = (-h_3 - 2h_5)x_1 + x_2x_5$$

$$x_5x_1 = x_1x_5.$$

This has relation type (2,2,3,4).

Example A.3.2.

$x_2x_1 = x_1x_2$

$x_3x_2 = x_1 + x_2x_3$

$x_3x_1 = x_1x_3$

$x_4x_3 = x_2 + x_3x_4$

$x_4x_2 = e_1x_1 + x_2x_4$

$x_4x_1 = x_1x_4$

$x_5x_4 = (g_6h_3 + g_6h_5)x_2 + (e_1^2h_3 - e_1g_6h_3 + e_1^2h_5 - e_1g_6h_5)x_3x_3$

$+ (-h_3 + g_6h_3 - h_5 + e_1^2h_5 + g_6h_5 - e_1g_6h_5)x_3x_4 + (e_1^2 - e_1g_6)x_3x_5$

$+ (-h_5 + g_6h_5)x_4x_4 + g_6x_4x_5$

$x_5x_3 = h_3x_2 + (-h_3 - h_5)x_3x_3 + h_3x_3x_4 + h_5x_4x_4 + x_4x_5$

$x_5x_2 = (-h_3 - h_5 - e_1h_5)x_1 + (-h_3 - e_1^2h_3 + e_1g_6h_3 - h_5 - e_1^2h_5 + e_1g_6h_5)x_2x_3$

$+ (-h_5 - e_1^2h_5 + e_1g_6h_5)x_2x_4 + (-e_1^2 + e_1g_6)x_2x_5$

$x_5x_1 = (-h_3 - e_1^3h_3 + e_1^2g_6h_3 - h_5 - e_1^3h_5 + e_1^2g_6h_5)x_1x_3$

$+ (-h_5 - e_1^3h_5 + e_1^2g_6h_5)x_1x_4 + (-e_1^3 + e_1^2g_6)x_1x_5$

where $g_6 \neq 0, e_1 \neq 0$.

This has relation type (2,2,3,4).

Example A.3.3.

$$x_2x_1 = x_1x_2$$

$$x_3x_2 = x_1 + x_2x_3$$

$$x_3x_1 = x_1x_3 + c_4x_2x_2$$

$$x_4x_3 = x_2 + x_3x_4$$

$$x_4x_2 = x_2x_4$$

$$x_4x_1 = x_1x_4$$

$$x_5x_4 = g_1x_2 + x_4x_5$$

$$x_5x_3 = h_1x_2 + x_3x_5 + h_5x_4x_4$$

$$x_5x_2 = -g_1x_1 + x_2x_5$$

$$x_5x_1 = x_1x_5 - c_4g_1x_2x_2.$$

This has relation type (2,2,3,4).

Example A.3.4.

$x_2x_1 = d_3/(\boldsymbol{i}\sqrt{3} + d_3^2)x_1x_2$

$x_3x_2 = x_2x_3$

$x_3x_1 = d_3x_1x_3$

$x_4x_3 = d_2x_3x_3 + d_3x_3x_4$

$x_4x_2 = (-\boldsymbol{i}\sqrt{3}d_2 + d_2d_3 - d_2d_3^2)/(-\boldsymbol{i}\sqrt{3} + \boldsymbol{i}\sqrt{3}d_3 - d_3^2 + d_3^3)x_2x_3$

$+ d_3/(\boldsymbol{i}\sqrt{3} + d_3^2)x_2x_4$

$x_4x_1 = d_3x_1x_4 - \boldsymbol{i}\sqrt{3}x_2x_2 - \boldsymbol{i}\sqrt{3}g_2x_2x_3x_3$

$x_5x_4 = x_2 + g_2x_3x_3 + (-\boldsymbol{i}\sqrt{3}d_2 + d_2d_3 - d_2d_3^2)/(-\boldsymbol{i}\sqrt{3}d_3 + \boldsymbol{i}\sqrt{3}d_3^2 - d_3^3 + d_3^4)x_3x_5$

$+ 1/(\boldsymbol{i}\sqrt{3} + d_3^2)x_4x_5$

$x_5x_3 = 1/d_3x_3x_5$

$x_5x_2 = x_1 + d_3x_2x_5 + i_5x_3x_3x_3 + (-i_5 + d_3i_5)/(d_2(1 + d_3))x_3x_3x_4$

$+ (-i_5 + 2d_3i_5 - d_3^2i_5)/(d_2^2d_3^2(1 + d_3))x_3x_4x_4$

$x_5x_1 = (\boldsymbol{i}\sqrt{3} + d_3^2)/d_3x_1x_5 + (8\boldsymbol{i}i_5 - 2\boldsymbol{i}d_3i_5 - 3\sqrt{3}d_3i_5)/(\boldsymbol{i}d_2 + 2\sqrt{3}d_2)x_2x_3x_3$

$+ (-1 + d_3)(8\boldsymbol{i}i_5 - 2\boldsymbol{i}d_3i_5 - 3\sqrt{3}d_3i_5)/(d_2(\boldsymbol{i}d_2 + 2\sqrt{3}d_2))x_2x_3x_4$

$- ((2 + d_3)(-5\boldsymbol{i} + 3\sqrt{3} + (\boldsymbol{i} + 2\sqrt{3})d_3)g_2i_5)/((\boldsymbol{i} + 2\sqrt{3})d_2)x_3x_3x_3x_3$

$+ 3(-5\boldsymbol{i}g_2i_5 + 3\sqrt{3}g_2i_5 + \boldsymbol{i}d_3g_2i_5 + 2\sqrt{3}d_3g_2i_5)/(\boldsymbol{i}d_2^2 + 2\sqrt{3}d_2^2)x_3x_3x_3x_4$

where $d_3 = -(-1)^{1/3}$

or $d_3 = (-1)^{2/3}$.

This has relation type (2,2,3,4).

Example A.3.5.

$x_2x_1 = x_1x_2$

$x_3x_2 = x_2x_3$

$x_3x_1 = x_1x_3$

$x_4x_3 = x_3x_4$

$x_4x_2 = x_1 - x_2x_4 - 2g_2x_3x_3x_4$

$x_4x_1 = x_1x_4$

$x_5x_4 = x_2 + g_2x_3x_3 + g_3x_3x_4 + x_4x_5$

$x_5x_3 = h_2x_3x_3 + x_3x_5 + h_5x_4x_4$

$x_5x_2 = i_1x_1 + i_2x_2x_3 - 2i_1x_2x_4 + x_2x_5 + i_5x_3x_3x_3 + i_6x_3x_3x_4 - 2g_2h_5x_3x_4x_4$

$x_5x_1 = (g_3 + i_2)x_1x_3 + x_1x_5 + 2x_2x_2 + 4g_2x_2x_3x_3 + 2g_2^2x_3x_3x_3x_3$

$+ 1/4(16g_2h_2 - 8g_2i_2 + 8i_5)x_3x_3x_3x_4 + 1/2(8g_2i_1 + 4i_6)x_3x_3x_4x_4$

where $i_5 = 0, h_2 = i_2/2$

or $g_2 = i_5/(-2h_2 + i_2)$.

This has relation type (2,2,3,4) if $h_5 = 0$ and type (2,2,3) if $h_5 \neq 0$.

There is another family of solutions similar to this, with a coefficient in front of x_2x_3 in r_{42} which is a complicated root of a polynomial, see the online code.

Example A.3.6.

$x_2x_1 = \boldsymbol{i}x_1x_2$

$x_3x_2 = -\boldsymbol{i}x_2x_3$

$x_3x_1 = x_1x_3$

$x_4x_3 = d_2x_3x_3 - \boldsymbol{i}x_3x_4$

$x_4x_2 = x_1 - (1+\boldsymbol{i})d_2x_2x_3 + \boldsymbol{i}x_2x_4$

$x_4x_1 = d_2x_1x_3 - \boldsymbol{i}x_1x_4$

$x_5x_4 = x_2 + (-1/4+\boldsymbol{i}/4)d_2^3h_5x_3x_3 + 1/2(2h_2 + (2+\boldsymbol{i})d_2^2h_5 - (1-\boldsymbol{i})d_2i_3)x_3x_4$

$+\boldsymbol{i}d_2x_3x_5 + (1/2-\boldsymbol{i}/2)(d_2h_5+i_3)x_4x_4 - \boldsymbol{i}x_4x_5$

$x_5x_3 = h_2x_3x_3 + (1/2-\boldsymbol{i}/2)((-1+\boldsymbol{i})d_2h_5 - \boldsymbol{i}i_3)x_3x_4 - x_3x_5 + h_5x_4x_4$

$x_5x_2 = -\boldsymbol{i}i_3/2x_1 - (1/2-\boldsymbol{i}/2)d_2i_3x_2x_3 + i_3x_2x_4 - \boldsymbol{i}x_2x_5 - (1/2-\boldsymbol{i}/2)d_2i_6x_3x_3x_3$

$+ i_6x_3x_3x_4$

$x_5x_1 = 1/2(2h_2 + d_2^2h_5)x_1x_3 + (-1/2-\boldsymbol{i}/2)i_3x_1x_4 - x_1x_5.$

This has relation type (2,2,3,4) if $h_5 = 0$ and type (2,2,3) if $h_5 \neq 0$.

A.4 Degree type (1,1,1,1,2)

Example A.4.1.

$x_2x_1 = -x_1x_2$

$x_3x_2 = x_1 + x_2x_3$

$x_3x_1 = -x_1x_3$

$x_4x_3 = -2x_2x_3 + 2x_2x_4 + x_3x_4$

$x_4x_2 = x_1 + 2x_2x_3 - x_2x_4$

$x_4x_1 = -x_1x_4$

$x_5x_4 = x_2x_5 + x_3x_5 - x_4x_5$

$x_5x_3 = x_2x_5 + x_3x_5 - x_4x_5$

$x_5x_2 = x_4x_5$

$x_5x_1 = -x_1x_5.$

This has relation type (2,2,2,2,2,3,3) and 2 degree three relations in the Gröbner basis.

Example A.4.2.

$$x_2x_1 = x_1x_2$$

$$x_3x_2 = -x_2x_3$$

$$x_3x_1 = x_1x_3$$

$$x_4x_3 = d_2x_2x_2 + d_6x_3x_3 - x_3x_4$$

$$x_4x_2 = e_2x_2x_2 - x_2x_4 + e_6x_3x_3$$

$$x_4x_1 = x_1x_4$$

$$x_5x_4 = g_1x_1 + g_2x_2x_2 + g_6x_3x_3 + g_9x_4x_4 - x_4x_5$$

$$x_5x_3 = h_2x_2x_2 + h_6x_3x_3 - x_3x_5$$

$$x_5x_2 = x_1 + i_2x_2x_2 - x_2x_5 + i_6x_3x_3 + i_9x_4x_4$$

$$x_5x_1 = x_1x_5.$$

This has 4 degree three relations in the Gröbner basis. The relation type is (2,2,2,2,2,3,3) if $d_2 = 0$ and (2,2,2,2,2,3) if $d_2 \neq 0$.

Example A.4.3.

$x_2x_1 = x_1x_2$

$x_3x_2 = x_2x_3$

$x_3x_1 = x_1x_3$

$x_4x_3 = x_1 + d_6x_3x_3 - x_3x_4$

$x_4x_2 = x_2x_4$

$x_4x_1 = x_1x_4$

$x_5x_4 = g_1x_1 + d_6g_1x_3x_3 - 2g_1x_3x_4 + x_4x_5$

$x_5x_3 = x_3x_5$

$x_5x_2 = i_1x_1 + x_2x_5 + i_6x_3x_3 + i_9x_4x_4$

$x_5x_1 = x_1x_5.$

This has 2 degree three relations in the Gröbner basis. The relation type can be any of (2,2,2,2,2,3,3), (2,2,2,2,2,3), or (2,2,2,2,2) depending on the values of $i_1,\ i_6,\ i_9$. In particular, if $i_1 = i_6 = i_9$, the relation type is (2,2,2,2,2), If $i_1 = i_6 = 0,\ i_9 \neq 0$ or $i_1 = i_9 = 0,\ i_6 \neq 0$, the relation type is (2,2,2,2,2,3). Otherwise, the relation type is (2,2,2,2,2,3,3).

Example A.4.4.

$x_2x_1 = x_1x_2$

$x_3x_2 = x_2x_3$

$x_3x_1 = x_1x_3$

$x_4x_3 = x_1 + d_6x_3x_3 + x_3x_4$

$x_4x_2 = x_2x_4$

$x_4x_1 = x_1x_4$

$x_5x_4 = g_1x_1 - h_3x_2x_4 + g_6x_3x_3 + g_7x_3x_4 + x_4x_5$

$x_5x_3 = h_1x_1 + h_3x_2x_3 + h_6x_3x_3 + x_3x_5 + h_9x_4x_4$

$x_5x_2 = i_1x_1 + x_2x_5$

$x_5x_1 = x_1x_5$

where $g_7 = -2h_6, d_6 = 0$

or $h_9 = h_3 = g_7 = 0, d_6 = h_6/h_1$.

This has 2 degree three relation in the Gröbner basis. The relation type is (2,2,2,2,2,3,3) if $i_1 = 0$ and (2,2,2,2,2) if $i_1 \neq 0$.

For another family of relations that can have relation type (2,2,2,2,2,3) or (2,2,2,2,2,3,3), and which is also bigraded, see Theorem 7.1.1.

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