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by

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Committee in charge:

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Abstract

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This thesis is composed of six chapters, which mainly deals with embedding continuous paths in Brownian motion. It is adapted from two publications [123, 124], joint with Jim Pitman.

We ask if it is possible to find some particular continuous paths of unit length in linear Brownian motion. Beginning with a discrete version of the problem, we derive the asymptotics of the expected waiting time for several interesting patterns. These suggest corresponding results on the existence/non-existence of continuous paths embedded in Brownian motion.

By various stochastic analysis arguments (path decomposition, Itô excursion theory, potential theory...), we are able to prove some of these existence and non-existence results:

Z	e	$V(b^\lambda)$	m	R
Embedding into B	No	No	Yes	Yes

where e is a normalized Brownian excursion, $V(b)$ is the Vervaat transform of Brownian bridge ending at λ, m is a Brownian meander, and R is the three dimensional Bessel process of unit length.

The question of embedding a Brownian bridge into Brownian motion is more chanllenging. After explaining why some simple or traditional approaches do not work, we make use of recent work of Last and Thorisson on shift couplings of stationary random measures to prove the result. These can be applied after a thorough analysis of the Slepian zero set $\{t \geq 0; B_t = B_{t+1}\}$.

We also discuss the potential theoretical aspect of embedding continuous paths in a random process. A list of open problems is presented.

Contents

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Chapter 1

Introduction and main results

In this thesis, we are interested in continuous paths embedded in a linear Brownian motion. It is mainly adapted from two publications [123, 124], written jointly with Jim Pitman. Before explaining further, let us mention briefly how this problem was brought into our eyes.

For a continuous function $f \in \mathcal{C}([0,1])$, Vervaat [149] defined the following path transform

$$V(f)(t) := f(\tau(f) + t \mod 1) + f(1)1(t + \tau(f) \geq 1) - f(\tau(f)),$$

where $\tau(f)$ corresponds to the first time at which the minimum of f is attained. For $\lambda \in \mathbb{R}$, let $b^\lambda := (b^\lambda_u; 0 \leq u \leq 1)$ be a Brownian bridge ending at λ, and $e := (e_u; 0 \leq u \leq 1)$ be a normalized Brownian excursion. Vervaat [149] showed the following striking identity in law:

$$V(b^0) \stackrel{(d)}{=} e. \tag{1.1}$$

Extending the Vervaat result, Lupu et al. [102] proved the following path decomposition.

Theorem 1.0.1 *[102] Let $\lambda < 0$. Given Z^λ the time of the first return to 0 by $V(b^\lambda)$, whose density is given by*

$$f_{Z^\lambda}(t) := \frac{|\lambda|}{\sqrt{2\pi t(1-t)^3}} \exp\left(-\frac{\lambda^2 t}{2(1-t)}\right),$$

the path is decomposed into two (conditionally) independent pieces:

- $(V(b^\lambda)_u; 0 \leq u \leq Z^\lambda)$ *is a Brownian excursion of length* Z^λ*;*
- $(V(b^\lambda)_u; Z^\lambda \leq u \leq 1)$ *is a first passage bridge through level* λ *of length* $1 - Z^\lambda$*.*

Soon after we finished [102], Jim Pitman sent out the paper to Patrick Fitzsimmons for comment. In response, Fitzsimmons raised the following question:

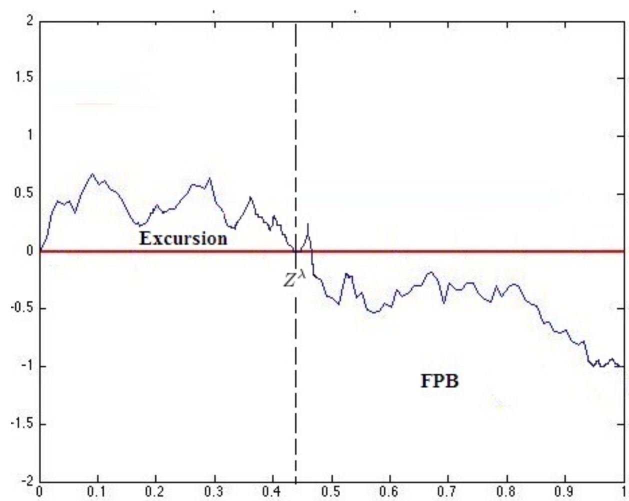


Fig 1. Vervaat bridge = Excursion + First passage bridge.

I wonder if a path fragment as pictured in Fig 1 is a piece of Brownian path occurring naturally in the wild. By occurring naturally in the wild, I mean that a piece of Brownian path $(B_t - B_T; T \leq t \leq T+1)$ for certain random time T.

In other words, Fitzsimmons asked whether we are able to find the Vervaat bridge $V(B^{\lambda,br})$ in Brownian motion by a spacetime shift. This motivated our study of embedding some continuous-time stochastic processes $(Z_u; 0 \leq u \leq 1)$ into a Brownian path $(B_t; t \geq 0)$, without time-change or scaling, just by a random translation of origin in spacetime.

Question 1.0.2 *Given some distribution of a process Z with continuous paths, does there exist a random time T such that $(B_{T+u} - B_T; 0 \leq u \leq 1)$ has the same distribution as $(Z_u, 0 \leq u \leq 1)$?.*

The question of whether external randomization is allowed to construct such a random time T, is of no importance here. In fact, we can simply ignore Brownian motion on $[0,1]$, and consider only random times $T \geq 1$. Then $(B_t; 0 \leq t \leq 1)$ provides an independent random element which is adequate for any randomization, see e.g. Kallenberg [75, Theorem 6.10].

Note that a continuous-time process whose sample paths have different regularity, e.g. fractional Brownian motion with Hurst parameter $H \neq \frac{1}{2}$, cannot be embedded into Brownian motion. Given $(B_t; t \geq 0)$ a linear Brownian motion, we define $g_1 := \sup\{t < 1; B_t = 0\}$ to be the time of last exit from 0 before $t = 1$, and $d_1 := \inf\{t > 1; B_t = 0\}$ to be the first hitting time of 0 after $t = 1$. The following processes, derived from Brownian motion, are of special interest.

- Brownian bridge, which can be defined by

$$\left(b_u^0 := \frac{1}{\sqrt{g_1}} B_{ug_1}; 0 \leq u \leq 1\right),$$

and its reflected counterpart $(|b_u^0|; 0 \le u \le 1)$.

- Normalized Brownian excursion defined by

$$\left(e_u := \frac{1}{\sqrt{d_1 - g_1}} |B_{g_1 + u(d_1 - g_1)}|; 0 \le u \le 1\right).$$

- Brownian meander defined as

$$\left(m_u := \frac{1}{\sqrt{1 - g_1}} |B_{g_1 + u(1 - g_1)}|; 0 \le u \le 1\right).$$

- Brownian co-meander defined as

$$\left(\widetilde{m}_u := \frac{1}{\sqrt{d_1 - 1}} |B_{d_1 - u(d_1 - 1)}|; 0 \le u \le 1\right).$$

- The three-dimensional Bessel process

$$\left(R_u := \sqrt{(B_u)^2 + (B'_u)^2 + (B''_u)^2}; 0 \le u \le 1\right),$$

where $(B'_t; t \ge 0)$ and $(B''_u; u \ge 0)$ are two independent copies of $(B_t; t \ge 0)$.

- The first passage bridge through level $\lambda \ne 0$, defined by

$$(F_u^{\lambda,br}; 0 \le u \le 1) \stackrel{(d)}{=} (B_u; 0 \le u \le 1) \quad \text{conditioned on} \quad \tau_\lambda = 1,$$

where $\tau_\lambda := \inf\{t \ge 0; B_t = \lambda\}$ is the first time at which Brownian motion hits $\lambda \ne 0$. Note that for $\lambda < 0$, $(F_u^{|\lambda|,br}; 0 \le u \le 1) \stackrel{(d)}{=} (-F_u^{\lambda,br}; 0 \le u \le 1)$, and $(F_{1-u}^{\lambda,br} + |\lambda|; 0 \le u \le 1)$ is distributed as a three dimensional Bessel bridge ending at $|\lambda| > 0$, see e.g. Biane and Yor [16].

- The Vervaat transform of Brownian motion, defined as

$$\left(V_u := \begin{cases} B_{\tau+u} - B_\tau & \text{for} \quad 0 \le u \le 1 - \tau \\ B_{\tau-1+u} + B_1 - B_\tau & \text{for} \quad 1 - \tau \le u \le 1 \end{cases} ; 0 \le u \le 1\right),$$

where $\tau := \operatorname{argmin}_{0 \le t \le 1} B_t$, and the Vervaat transform of Brownian bridge with end-point $\lambda \in \mathbb{R}$

$$\left(V_u^\lambda := \begin{cases} b_{\tau+u}^\lambda - b_\tau^\lambda & \text{for} \quad 0 \le u \le 1 - \tau \\ b_{\tau-1+u}^\lambda + \lambda - b_\tau^\lambda & \text{for} \quad 1 - \tau \le u \le 1 \end{cases} ; 0 \le u \le 1\right),$$

where $(b_u^\lambda; 0 \le u \le 1)$ is a Brownian bridge ending at $\lambda \in \mathbb{R}$ and $\tau := \operatorname{argmin}_{0 \le t \le 1} b_t^\lambda$. Recall from (1.1) that $(V_u^0; 0 \le u \le 1) \stackrel{(d)}{=} (e_u; 0 \le u \le 1)$. For $\lambda < 0$, $(V_u^{|\lambda|}; 0 \le u \le 1)$ has the same distribution as $(V_{1-u}^\lambda + |\lambda|; 0 \le u \le 1)$.

The Brownian bridge, meander, excursion and the three-dimensional Bessel process are well-known. The definition of the co-meander is found in Yen and Yor [157, Chapter 7]. The first passage bridge is studied by Bertoin et al. [14]. The Vervaat transform of Brownian bridges and of Brownian motion are extensively discussed in Lupu et al. [102]. According to the above definitions, the distributions of the Brownian bridge, excursion and (co-)meander can all be achieved in Brownian motion provided some Brownian scaling operation is allowed.

Note that the distributions of all these processes are singular with respect to Wiener measure. So it is a non-trivial question whether copies of them can be found in Brownian motion just by a shift of origin in spacetime. Otherwise, for a process $(Z_t, 0 \leq t \leq 1)$ whose distribution is absolutely continuous with respect to that of $(B_t, 0 \leq t \leq 1)$, for instance the Brownian motion with drift $Z_t := \vartheta t + B_t$ for a fixed ϑ, the distribution of Z can be easily obtained as that of $(B_{T+t} - B_T, 0 \leq t \leq 1)$ for a suitable stopping time $T+1$ by *Rost's filling scheme*. We refer readers to Chapter 3 for further development.

The question raised here has some affinity to the question of embedding a given one-dimensional distribution as the distribution of B_T for a random time T. This *Skorokhod embedding problem* traces back to Skorokhod [137] and Dubins [38] – who found integrable stopping times T such that the distribution of B_T coincides with any prescribed one with zero mean and finite second moment. Monroe [112, 113] considered embedding of a continuous-time process into Brownian motion, and showed that every semi-martingale is a time-changed Brownian motion. Rost [132] studied the problem of embedding a one-dimensional distribution in a Markov process with randomized stopping times. We refer readers to the excellent survey of Obloj [117] and references therein.

Let $X_t := (B_{t+u} - B_t; 0 \leq u \leq 1)$ for $t \geq 0$ be the moving-window process associated to Brownian motion. In Question 1.0.2, we are concerned with the possibility of embedding a given distribution on $\mathcal{C}[0,1]$ as that of X_T for some random time T. We start with a list of continuous-time processes that cannot be embedded into Brownian motion by a shift of origin in spacetime.

Theorem 1.0.3 *(Impossibility of embedding of normalized excursion, reflected bridge, Vervaat transform of Brownian motion, first passage bridge and Vervaat bridge) For each of the following five processes $Z := (Z_u; 0 \leq u \leq 1)$, there is no random time T such that $(B_{T+u} - B_T; 0 \leq u \leq 1)$ has the same distribution as Z:*

1. *the normalized Brownian excursion $Z = (e_u; 0 \leq u \leq 1)$;*
2. *the reflected Brownian bridge $Z = (|b_u^0|; 0 \leq u \leq 1)$;*
3. *the Vervaat transform of Brownian motion $Z = (V_u; 0 \leq u \leq 1)$;*
4. *the first passage bridge through level $\lambda \neq 0$, $Z = (F_u^{\lambda,br}; 0 \leq u \leq 1)$;*
5. *the Vervaat transform of Brownian bridge with endpoint $\lambda \in \mathbb{R}$, $Z = (V_u^{\lambda}; 0 \leq u \leq 1)$.*

Note that in Theorem 1.0.3 (4) (5), it suffices to consider the case of $\lambda < 0$ by time-reversal. As we will see later in Theorem 1.0.6, Theorem 1.0.3 is an immediate consequence of the fact that typical paths of these processes cannot be found in Brownian motion. The next theorem shows the possibility of embedding into Brownian motion some continuous-time processes whose distributions are singular with respect to Wiener measure.

Theorem 1.0.4 *(Embeddings of bridge, meander, co-meander and 3-d Bessel process) For each of the following four processes* $Z := (Z_u, 0 \leq u \leq 1)$ *there is some random time* T *such that* $(B_{T+u} - B_T; 0 \leq u \leq 1)$ *has the same distribution as* Z*:*

1. *the bridge* $Z = (b^0; 0 \leq u \leq 1)$.
2. *the meander* $Z = (m_u; 0 \leq u \leq 1)$.
3. *the co-meander* $Z = (\widetilde{m}_u; 0 \leq u \leq 1)$.
4. *the three-dimensional Bessel process* $Z = (R_u; 0 \leq u \leq 1)$.

We emphasize that the problem of embedding Brownian bridge b^0 into Brownian motion is more difficult than it appears to be. As a natural candidate, the bridge-like process as below was considered:

$$(B_{F+u} - B_F; 0 \leq u \leq 1), \tag{1.2}$$

where

$$F := \inf\{t \geq 0; B_{t+1} - B_t = 0\}. \tag{1.3}$$

This bridge-like process bears some resemblance to Brownian bridge. At least, it starts and ends at 0, and is some part of a Brownian path in between. However, the study of the bridge-like process seems to be challenging as we will explain in Chapter 5. Alternatively, we make use of Palm theory of stationary random measures to prove Theorem 1.0.4 (1).

In Question 1.0.2, we seek to embed a particular continuous-time process Z of unit length into a Brownian path. The distribution of X resides in the infinite-dimensional space $\mathcal{C}_0[0, 1]$ of continuous paths $(w(t); 0 \leq t \leq 1)$ starting from $w(0) = 0$. So a closely related problem is whether a given subset of $\mathcal{C}_0[0, 1]$ is hit by the path-valued moving-window process $X_t := (B_{t+u} - B_t; 0 \leq u \leq 1)$ indexed by $t \geq 0$. We formulate this problem as follows.

Question 1.0.5 *Given a Borel measurable subset* $S \subset \mathcal{C}_0[0, 1]$, *can we find a random time* T *such that* $X_T := (B_{T+u} - B_T; 0 \leq u \leq 1) \in S$ *with probability one?*

Question 1.0.5 involves scanning for patterns in a continuous-time process. By the general theory of stochastic processes, assuming that the underlying Brownian motion B is defined on a complete probability space, $\{\exists T \geq 0$ such that $(B_{T+u} - B_T; 0 \leq u \leq 1) \in S\}$ is measurable. See e.g. Dellacherie [33, T32, Chapter I], Meyer and Dellacherie [34, Section 44, Chapter III], and Bass [7, 6]. Assume that

$$\mathbb{P}(\exists T \geq 0 \text{ such that } (B_{T+u} - B_T; 0 \leq u \leq 1) \in S) > 0.$$

Then there exists some fixed $M > 0$ and $p > 0$ such that

$$\mathbb{P}(\exists T : 0 \leq T \leq M \text{ and } (B_{T+u} - B_T; 0 \leq u \leq 1) \in S) = p > 0.$$

We start the process afresh at $M + 1$, and then also

$$\mathbb{P}(\exists T : M + 1 \leq T \leq 2M + 1 \text{ and } (B_{T+u} - B_T; 0 \leq u \leq 1) \in S) = p > 0.$$

By repeating the above procedure, we obtain a sequence of i.i.d. Bernoulli(p) random variables. Therefore, the probability that a given measurable set $S \subset \mathcal{C}_0[0,1]$ is hit by the path-valued process generated by Brownian motion is either 0 or 1:

$$\mathbb{P}[\exists T \geq 0 \text{ such that } (B_{T+u} - B_T; 0 \leq u \leq 1) \in S] = 0 \text{ or } 1. \tag{1.4}$$

Using various stochastic analysis tools, we are able to show that

Theorem 1.0.6 *(Impossibility of embedding of excursion, reflected bridge, Vervaat transform of Brownian motion, first passage bridge and Vervaat bridge paths) For each of the following five sets of paths S, almost surely, there is no random time $T \geq 0$ such that $(B_{T+u} - B_T; 0 \leq u \leq 1) \in S$:*

1. *the set of excursion paths, which first return to 0 at time 1,*

$$S = \mathcal{E} := \{w \in \mathcal{C}_0[0,1]; w(t) > w(1) = 0 \textit{ for } 0 < t < 1\};$$

2. *the set of reflected bridge paths,*

$$S = \mathcal{RBR}^0 := \{w \in \mathcal{C}_0[0,1]; w(t) \geq w(1) = 0 \textit{ for } 0 \leq t \leq 1\};$$

3. *the set of paths of Vervaat transform of Brownian motion with a floating negative endpoint,*

$$S = \mathcal{VB}^- := \{w \in \mathcal{C}_0[0,1]; w(t) > w(1) \textit{ for } 0 \leq t < 1 \textit{ and } \inf\{t > 0; w(t) < 0\} > 0\};$$

4. *the set of first passage bridge paths at fixed level $\lambda < 0$,*

$$S = \mathcal{FP}^\lambda := \{w \in \mathcal{C}_0[0,1]; w(t) > w(1) = \lambda \textit{ for } 0 \leq t < 1\};$$

5. *the set of Vervaat bridge paths ending at fixed level $\lambda < 0$,*

$$S = \mathcal{VB}^\lambda := \{w \in \mathcal{FP}^\lambda; \inf\{t > 0; w(t) < 0\} > 0\} = \{w \in \mathcal{VB}^-; w(1) = \lambda\}.$$

Observe that for each $\lambda < 0$, $\mathcal{VB}^\lambda$ is a subset of $\mathcal{VB}^-$ and $\mathcal{FP}^\lambda$. Then Theorem 1.0.6 (5) follows immediately from Theorem 1.0.6 (3) or (4). As we will see in Chapter 3, Theorem 1.0.6 (5) is also reminiscent of Theorem 1.0.6 (1) in the proof.

It is obvious that for the following two sets of paths S, there is a random time $T \geq 0$ such that $(B_{T+u} - B_T; 0 \leq u \leq 1) \in S$ almost surely:

- the set of positive paths,

$$S = \mathcal{M} := \{w \in \mathcal{C}_0[0,1]; w(t) > 0 \text{ for } 0 < t \leq 1\};$$

- the set of bridge paths, which ends at $\lambda \in \mathbb{R}$,

$$S = \mathcal{BR}^\lambda := \{w \in \mathcal{C}_0[0,1]; w(1) = \lambda\}.$$

The case of positive paths is easily treated by excursion theory, as discussed in Chapter 4. The bridge paths are obtained by simply taking $T := \inf\{t > 0; B_{t+1} = B_t + \lambda\}$, see also Chapter 5 for related discussion. In both cases, $T + 1$ is a stopping time relative to the Brownian filtration. For a general measurable $S \subset \mathcal{C}_0[0,1]$, it is easily shown that if there is a random time T such that $(B_{T+u} - B_T; 0 \leq u \leq 1) \in S$ almost surely, then for each $\epsilon > 0$ this can be achieved by a random time T such that $T + 1 + \epsilon$ is a stopping time relative to the Brownian filtration.

Here we restrict ourselves to continuous paths in linear Brownian motion. However, the problem is also worth considering in the multi-dimensional case, as discussed briefly in Chapter 6.

At first glance, neither Question 1.0.2 nor Question 1.0.5 seems to be tractable. To gain more intuition, we study the analogous problem in the random walk setting. We deal with simple symmetric random walks $SW(n)$ of length n with increments ± 1 starting at 0. A typical question is how long it would take, in a random walk, to observe a pattern in a collection of patterns of length n satisfying some common properties. More precisely,

Question 1.0.7 *Given for each $n \in \mathbb{N}$ a collection $\mathcal{A}^n$ of patterns of length $L(\mathcal{A}^n)$, what is the asymptotics of the expected waiting time $\mathbb{E}T(\mathcal{A}^n)$ until some element of $\mathcal{A}^n$ is observed in a random walk?*

We are not aware of any previous study on pattern problems in which some natural definition of the collection of patterns is made for each $n \in \mathbb{N}$.

Nevertheless, this question fits into the general theory of runs and patterns in a sequence of discrete trials. This theory dates back to work in 1940s by Wald and Wolfowitz [151] and Mood [114]. Since then, the subject has become important in various areas of science, including industrial engineering, biology, economics and statistics. In the 1960s, Feller [45] treated the problem probabilistically by identifying the occurrence of a single pattern as a renewal event. By the generating function method, the law of the occurrence times of a single pattern is entirely characterized. More advanced study, of the occurrence of patterns in a collection, developed in 1980s by two different methods. Guibas and Odlyzko [64], and Breen et al. [23] followed the steps of Feller [45] by studying the generating functions in pattern-overlapping regimes. An alternative approach was adopted by Li [100], and Gerber and Li [55] using martingale arguments. We also refer readers to the book of Fu and Lou [50] for the Markov chain embedding approach for multi-state trials.

Techniques from the theory of patterns in an i.i.d. sequence provide general strategies to Question 1.0.7. Here we focus on some special cases where the asymptotics of the expected waiting time is computable. As we will see later, these asymptotics help us predict the existence or non-existence of some particular paths in Brownian motion. The following result answers Question 1.0.7 in some particular cases.

Theorem 1.0.8 *Let $T(\mathcal{A}^n)$ be the waiting time until some pattern in $\mathcal{A}^n$ appears in the simple random walk. Then*

1. *for the set of discrete positive excursions of length $2n$, whose first return to 0 occurs at time $2n$,*

$$\mathcal{E}^{2n} := \{w \in SW(2n); w(i) > 0 \text{ for } 1 \leq i \leq 2n-1 \text{ and } w(2n) = 0\},$$

we have

$$\mathbb{E}T(\mathcal{E}^{2n}) \sim 4\sqrt{\pi} n^{\frac{3}{2}}; \tag{1.5}$$

2. *for the set of positive walks of length $2n+1$,*

$$\mathcal{M}^{2n+1} := \{w \in SW(2n+1); w(i) > 0 \text{ for } 1 \leq i \leq 2n+1\},$$

we have

$$\mathbb{E}T(\mathcal{M}^{2n+1}) \sim 4n; \tag{1.6}$$

3. *for the set of discrete bridges of length n, which end at λ_n for some $\lambda \in \mathbb{R}$, where $\lambda_n := [\lambda\sqrt{n}]$ if $[\lambda\sqrt{n}]$ and n have the same parity, and $\lambda_n := [\lambda\sqrt{n}] + 1$ otherwise,*

$$\mathcal{BR}^{\lambda,n} := \{w \in SW(n); w(n) = \lambda_n\},$$

we have

$$c^{\lambda}_{\mathcal{BR}} n \leq \mathbb{E}T(\mathcal{BR}^{\lambda,n}) \leq C^{\lambda}_{\mathcal{BR}} n \quad \text{for some } c^{\lambda}_{\mathcal{BR}} \text{ and } C^{\lambda}_{\mathcal{BR}} > 0; \tag{1.7}$$

4. *for the set of negative first passage walks of length n, ending at λ_n with $\lambda < 0$,*

$$\mathcal{FP}^{\lambda,n} := \{w \in SW(n); w(i) > w(n) = \lambda_n \text{ for } 0 \leq i \leq n-1\},$$

we have

$$\sqrt{\frac{\pi}{2\lambda^2}} \exp\left(\frac{\lambda^2}{2}\right) n \leq \mathbb{E}T(\mathcal{FP}^{\lambda,n}) \leq \sqrt{\frac{4}{\lambda}} \exp\left(\frac{\lambda^2}{2}\right) n^{\frac{5}{4}}. \tag{1.8}$$

Now we explain how the asymptotics in Theorem 1.0.8 suggest answers to Question 1.0.2 and Question 1.0.5 in some cases. Formula (1.5) tells that it would take on average $n^{\frac{3}{2}} \gg n$ steps to observe an excursion in a simple random walk. In view of the usual scaling of random walks to converge to Brownian motion, the time scale appears to be too large. Thus we should not expect to find the excursion paths $\mathcal{E}$ in Brownian motion. However, in (1.6) and (1.7), the typical waiting time to observe a positive walk or a bridge has the same order n involved in the time scaling for convergence in distribution to Brownian motion. So we can anticipate to observe the positive paths $\mathcal{M}$ and the bridge paths $\mathcal{BR}^{\lambda}$ in Brownian motion. Finally in (1.8), there is an exponent gap in evaluating the expected waiting time for first passage walks ending at $\lambda_n \sim [\lambda\sqrt{n}]$ with $\lambda < 0$. In this case, we do not know whether it would take asymptotically n steps or much longer to first observe such patterns. This prevents us from predicting the existence of the first passage bridge paths $\mathcal{FP}^{\lambda}$ in Brownian motion.

The scaling arguments used in the last paragraph are quite intuitive but not rigorous since we are not aware of any theory which would justify the existence or non-existence of continuous paths in Brownian motion by taking limits from the discrete setting. We hope that this problem will be taken care of in future work.

Organization of the thesis: The rest of the work is organized as follows.

- Chapter 2 treats the asymptotic behavior of the expected waiting time for discrete patterns. There Theorem 1.0.8 is proved.
- Chapter 3 is devoted to the analysis of continuous paths/processes in Brownian motion. Proofs of Theorem 1.0.4 and Theorem 1.0.6 $(2)-(4)$ are provided.
- Chapter 4 explores the local structure of the Slepian zero set $\{t \in [0,1]; S_t = 0\}$, or $\{t \in [0,1]; X_t \in \mathcal{BR}^0\}$. There a path decomposition is given.
- Chapter 5 deals with the proof of Theorem 1.0.4 (1): embedding a Brownian bridge into Brownian motion.
- Chapter 6 discusses the potential theory of continuous paths in Brownian motion.

Chapter 2

Expected waiting time for discrete patterns

Consider the expected waiting time for some collection of patterns

$$\mathcal{A}^n \in \{\mathcal{E}^{2n}, \mathcal{M}^{2n+1}, \mathcal{BR}^{\lambda,n}, \mathcal{FP}^{\lambda,n}\},$$

as defined in Chapter 1, except that we now encode a simple walk with m steps by its sequence of increments, with each increment a ± 1. We call such an increment sequence a *pattern* of length m. For each of these collections $\mathcal{A}^n$, all patterns in the collection have a common length, say $L(\mathcal{A}^n)$. We are interested in the asymptotic behavior of $\mathbb{E}T(\mathcal{A}^n)$ as $L(\mathcal{A}^n) \to \infty$.

We start by recalling the general strategy to compute the expected waiting time for discrete patterns in a simple random walk. From now on, let $\mathcal{A}^n := \{A_1^n, \cdots, A_{\#\mathcal{A}^n}^n\}$, where A_i^n is a sequence of signs ± 1 for $1 \le i \le \#\mathcal{A}^n$. That is,

$$A_i^n := A_{i1}^n \cdots A_{iL(\mathcal{A}^n)}^n, \quad \text{where } A_{ik}^n = \pm 1 \text{ for } 1 \le k \le L(\mathcal{A}^n).$$

Let $T(A_i^n)$ be the waiting time until the end of the first occurrence of A_i^n, and let $T(\mathcal{A}^n)$ be the waiting time until the first of the patterns in $\mathcal{A}^n$ is observed. So $T(\mathcal{A}^n)$ is the minimum of the $T(A_i^n)$ over $1 \le i \le \#\mathcal{A}^n$.

Define the matching matrix $M(\mathcal{A}^n)$, which accounts for the overlapping phenomenon among patterns within the collection $\mathcal{A}^n$. The coefficients are given by

$$M(\mathcal{A}^n)_{ij} := \sum_{l=0}^{L(\mathcal{A}^n)-1} \frac{\epsilon_l(A_i^n, A_j^n)}{2^l} \quad \text{for } 1 \le i, j \le \#\mathcal{A}^n, \tag{2.1}$$

where $\epsilon_l(A_i^n, A_j^n)$ is defined for $A_i^n = A_{i1}^n \cdots A_{iL(\mathcal{A}^n)}^n$ and $A_j^n = A_{j1}^n \cdots A_{jL(\mathcal{A}^n)}^n$ as

$$\epsilon_l(A_i^n, A_j^n) := \begin{cases} 1 & \text{if } A_{i1}^n = A_{j1+l}^n, \cdots, A_{iL(\mathcal{A}^n)-l}^n = A_{jL(\mathcal{A}^n)}^n \\ 0 & \text{otherwise,} \end{cases} \tag{2.2}$$

for $0 \leq l \leq L(\mathcal{A}^n) - 1$. Note that in general for $i \neq j$, $M(\mathcal{A}^n)_{ij} \neq M(\mathcal{A}^n)_{ji}$ and hence the matching matrix $M(\mathcal{A}^n)$ is not necessarily symmetric. The following result, which can be read from Breen et al. [23] is the main tool to study the expected waiting time for the collection of patterns.

Theorem 2.0.9 *[23]*

1. *The matching matrix $M(\mathcal{A}^n)$ is invertible and the expected waiting times for patterns in $\mathcal{A}^n := \{A_1^n, \cdots, A_{\#\mathcal{A}^n}^n\}$ are given by*

$$\left(\frac{1}{\mathbb{E}T(A_1^n)}, \cdots, \frac{1}{\mathbb{E}T(A_{\#\mathcal{A}^n}^n)}\right)^T = \frac{1}{2^n} M(\mathcal{A}^n)^{-1} (1, \cdots, 1)^T; \tag{2.3}$$

2. *The expected waiting time till one of the patterns in $\mathcal{A}^n$ is observed is given by*

$$\frac{1}{\mathbb{E}T(\mathcal{A}^n)} = \sum_{l=1}^{\#\mathcal{A}^n} \frac{1}{\mathbb{E}T(A_l^n)} = \frac{1}{2^n}(1, \cdots, 1) M(\mathcal{A}^n)^{-1}(1, \cdots, 1)^T. \tag{2.4}$$

In Section 2.1, we apply the previous theorem to obtain the expected waiting time for discrete excursions $\mathcal{E}^{2n}$, i.e. Theorem 1.0.8 (1). The same problem for positive walks $\mathcal{M}^{2n+1}$, bridge paths $\mathcal{BR}^{0,2n}$ and first passage walks $\mathcal{FP}^{\lambda,n}$ through $\lambda_n \sim \lambda\sqrt{n}$, i.e. Theorem 1.0.8 (2)−(4), is studied in Sections 2.2−2.4. Finally, we discuss the problem of the exponent gap for some discrete patterns in Section 2.5.

2.1 Expected waiting time for discrete excursions

For $n \in \mathbb{N}$, the number of discrete excursions of length $2n$ is equal to the $n-1^{th}$ Catalan number, see e.g. Stanley [139, Exercise 6.19 (i)]. That is,

$$\#\mathcal{E}^{2n} = \frac{1}{n}\binom{2n-2}{n-1} \sim \frac{1}{4\sqrt{\pi}} 2^{2n} n^{-\frac{3}{2}}. \tag{2.5}$$

Note that discrete excursions never overlap since the starting point and the endpoint are the only two minima. We have then $\epsilon(E_i^n, E_j^n) = \delta_{ij}$ for $1 \leq i, j \leq \#\mathcal{E}^{2n}$ by (2.2). Thus, the matching matrix defined as in (2.1) for discrete excursions $\mathcal{E}^{2n}$ has the simple form

$$M(\mathcal{E}^{2n}) = I_{\#\mathcal{E}^{2n}} \quad (\#\mathcal{E}^{2n} \times \#\mathcal{E}^{2n} \text{ identity matrix}).$$

According to Theorem 2.0.9,

$$\forall 1 \leq i \leq \#\mathcal{E}^{2n},\ \mathbb{E}T(E_i^n) = 2^{2n} \quad \text{and} \quad \mathbb{E}T(\mathcal{E}^n) = \frac{2^{2n}}{\#\mathcal{E}^{2n}} \sim 4\sqrt{\pi} n^{\frac{3}{2}}, \tag{2.6}$$

where $\#\mathcal{E}^{2n}$ is given as in (2.5). This is (1.5). □

2.2 Expected waiting time for positive walks

Let $n \in \mathbb{N}$. It is well-known that the number of non-negative walks of length $2n+1$ is $\binom{2n}{n}$, see e.g. Larbarbe and Marckert [91] and van Leeuwen [148] for modern proofs. Thus the number of positive walks of length $2n+1$ is given by

$$\#\mathcal{M}^{2n+1} = \binom{2n}{n} \sim \frac{1}{\sqrt{\pi}} 2^{2n} n^{-\frac{1}{2}}. \tag{2.7}$$

Note that a positive walk of length $2n+1$ is uniquely determined by

- its first $2n$ steps, which is a positive walk of length $2n$;
- its last step, which can be either $+1$ or -1.

As a consequence,

$$\#\mathcal{M}^{2n} = \frac{1}{2}\#\mathcal{M}^{2n+1} \sim \frac{1}{\sqrt{\pi}} 2^{2n-1} n^{-\frac{1}{2}}. \tag{2.8}$$

Now consider the matching matrix $M(\mathcal{M}^{2n+1})$ defined as in (2.1) for positive walks $\mathcal{M}^{2n+1}$. $M(\mathcal{M}^{2n+1})$ is no longer diagonal since there are overlaps among positive walks. The following lemma presents the particular structure of this matrix.

Lemma 2.2.1 *$M(\mathcal{M}^{2n+1})$ is a multiple of some right stochastic matrix (whose row sums are equal to 1). The multiplicity is*

$$1 + \sum_{l=1}^{2n} \frac{k(\mathcal{M}^l)}{2^l} \sim \frac{2}{\sqrt{\pi}} \sqrt{n}. \tag{2.9}$$

Proof: Let $1 \le i \le \#\mathcal{M}^{2n+1}$ and consider the sum of the i^{th} row

$$\begin{aligned} \sum_{j=1}^{\#\mathcal{M}^{2n+1}} M(\mathcal{M}^{2n+1})_{ij} &:= \sum_{j=1}^{\#\mathcal{M}^{2n+1}} \sum_{l=0}^{2n} \frac{\epsilon_l(M_i^{2n+1}, M_j^{2n+1})}{2^l} \\ &= \sum_{l=0}^{2n} \frac{1}{2^l} \sum_{j=1}^{\#\mathcal{M}^{2n+1}} \epsilon_l(M_i^{2n+1}, M_j^{2n+1}), \end{aligned} \tag{2.10}$$

where for $0 \le l \le 2n$ and $M_i^{2n+1}, M_j^{2n+1} \in \mathcal{M}^{2n+1}$, $\epsilon_l(M_i^{2n+1}, M_j^{2n+1})$ is defined as in (2.2). Note that $\epsilon_0(M_i^{2n+1}, M_j^{2n+1}) = 1$ if and only if $i = j$. Thus,

$$\sum_{j=1}^{\#\mathcal{M}^{2n+1}} \epsilon_0(M_i^{2n+1}, M_j^{2n+1}) = 1. \tag{2.11}$$

In addition, for $1 \le l \le 2n$,

$$\sum_{j=1}^{\#\mathcal{M}^{2n+1}} \epsilon_l(M_i^{2n+1}, M_j^{2n+1}) =$$
$$\#\{M_j^{2n+1} \in \mathcal{M}^{2n+1}; M_{i1}^{2n+1} = M_{j1+l}^{2n+1}, \cdots, M_{i2n+1-l}^{2n+1} = M_{j2n+1}^{2n+1}\}.$$

Note that given $M_{i1}^{2n+1} = M_{j1+l}^{2n+1}, \cdots, M_{i2n+1-l}^{2n+1} = M_{j2n+1}^{2n+1}$, which implies that $M_{j1+l}^{2n+1} \cdots M_{j2n+1}^{2n+1}$ is a positive walk of length $2n - l + 1$, we have

$$M_j^{2n+1} \in \mathcal{M}^{2n+1} \Longleftrightarrow M_{j1}^{2n+1} \cdots M_{jl}^{2n+1} \text{ is a positive walk of length } l.$$

Therefore, for $1 \le l \le 2n$,

$$\sum_{j=1}^{\#\mathcal{M}^{2n+1}} \epsilon_l(M_i^{2n+1}, M_j^{2n+1}) = k(\mathcal{M}^l). \tag{2.12}$$

In view of (2.10), (2.11) and (2.12), we obtain for all $1 \le i \le \#\mathcal{M}^{2n+1}$, the sum of i^{th} row of $M(\mathcal{M}^{2n+1})$ is given by (2.9). Furthermore, by (2.7) and (2.8), we know that $k(\mathcal{M}^l) \sim \frac{1}{\sqrt{2\pi}} 2^l l^{-\frac{1}{2}}$ as $l \to \infty$, which yields the asymptotics $\frac{2}{\sqrt{\pi}}\sqrt{n}$. □

Now by Theorem 2.0.9 (1), $M(\mathcal{M}^{2n+1})$ is invertible and the inverse $M(\mathcal{M}^{2n+1})^{-1}$ is as well the multiple of some right stochastic matrix. The multiplicity is

$$\left(1 + \sum_{l=1}^{n-1} \frac{k(\mathcal{M}^l)}{2^l}\right)^{-1} \sim \frac{\sqrt{\pi}}{2\sqrt{n}}.$$

Then using (2.4), we obtain

$$\mathbb{E}T(\mathcal{M}^{2n+1}) = \frac{2^{2n+1}}{\left(1 + \sum_{l=1}^{n-1} \frac{k(\mathcal{M}^l)}{2^l}\right)^{-1} \#\mathcal{M}^{2n+1}} \sim 4n. \tag{2.13}$$

This is (1.6). □

2.3 Expected waiting time for bridge paths

In this part, we deal with the expected waiting time for the set of discrete bridges. In order to simplify the notations, we focus on the set of bridges of length $2n$ which end at $\lambda = 0$, that is $\mathcal{BR}^{0,2n}$. Note that the result in the general case for $\mathcal{BR}^{\lambda,n}$, where $\lambda \in \mathbb{R}$, can be derived in a similar way.

Using Theorem 2.0.9, we prove a weaker version of (1.7): there exist $\widetilde{c}^0_{\mathcal{BR}}$ and $C^0_{\mathcal{BR}} > 0$ such that

$$\widetilde{c}^0_{\mathcal{BR}} n^{\frac{1}{2}} \leq \mathbb{E}T(\mathcal{BR}^{0,n}) \leq C^0_{\mathcal{BR}} n. \tag{2.14}$$

Compared to (1.7), there is an exponent gap in (2.14) and the lower bound is not optimal. Nevertheless, the lower bound of (1.7) follows a soft argument by scaling limit, Proposition 2.5.3. We defer the discussion to Section 2.5. It is standard that the number of discrete bridges of length $2n$ is

$$\#\mathcal{BR}^{0,2n} = \binom{2n}{n} \sim \frac{1}{\sqrt{\pi}} 2^{2n} n^{-\frac{1}{2}}. \tag{2.15}$$

Denote $\mathcal{BR}^{0,2n} := \{BR^{2n}_1, \cdots, BR^{2n}_{\#\mathcal{BR}^{0,2n}}\}$ and $M(\mathcal{BR}^{0,2n})$ the matching matrix of $\mathcal{BR}^{0,2n}$. We first establish the LHS estimate of (2.14). According to (2.3), we have

$$(1, \cdots, 1) M(\mathcal{BR}^{0,2n}) \left(\frac{1}{\mathbb{E}T(BR^{2n}_1)}, \cdots, \frac{1}{\mathbb{E}T(BR^{2n}_{\#\mathcal{BR}^{0,2n}})} \right)^T = \frac{\#\mathcal{BR}^{0,2n}}{2^{2n}}. \tag{2.16}$$

Note that the matching matrix $\mathcal{M}(\mathcal{BR}^{0,2n})$ is non-negative with diagonal elements

$$M(\mathcal{BR}^{0,2n})_{ii} \geq \epsilon_0(BR^{2n}_i, BR^{2n}_i) = 1,$$

for $1 \leq i \leq \#\mathcal{BR}^{0,2n}$. As a direct consequence, the column sums of $M(\mathcal{BR}^{0,2n})$ is larger or equal to 1. Then by (2.4) and (2.16),

$$\mathbb{E}T(\mathcal{BR}^{0,2n}) \geq \frac{2^{2n}}{\#\mathcal{BR}^{0,2n}} \sim \sqrt{\pi n},$$

where $\#\mathcal{BR}^{0,2n}$ is defined as in (2.15). Take then $\widetilde{c}^0_{\mathcal{BR}} = \sqrt{\pi}$.

Now we establish the RHS estimate of (2.14). In view of (2.16), it suffices to work out an upper bound for the column sums of $M(\mathcal{BR}^{0,2n})$. Similarly as in (2.10), for $1 \leq j \leq \#\mathcal{BR}^{0,2n}$,

$$\sum_{i=1}^{\#\mathcal{BR}^{0,2n}} M(\mathcal{BR}^{0,2n})_{ij} = 1 + \sum_{l=1}^{2n-1} \frac{1}{2^l} \sum_{i=1}^{\#\mathcal{BR}^{0,2n}} \epsilon_l(BR^{2n}_i, BR^{2n}_j), \tag{2.17}$$

and

$$\begin{aligned} \sum_{i=1}^{\#\mathcal{BR}^{0,2n}} \epsilon_l(BR^{2n}_i, BR^{2n}_j) &= \#\{BR^{2n}_i \in \mathcal{BR}^{0,2n}; BR^{2n}_{i1} = BR^{2n}_{j1+l}, \cdots, BR^{2n}_{in-l} = BR^{2n}_{jn}\}. \\ &= \#\{\text{discrete bridges of length } l \text{ which end at } \sum_{k=1}^{n-l} BR^{2n}_{jk}\} \\ &= \binom{l}{\frac{l+\sum_{k=1}^{n-l} BR^{2n}_{jk}}{2}} \leq \binom{l}{[\frac{l}{2}]}, \end{aligned} \tag{2.18}$$

where the last inequality is due to the fact that $\binom{l}{k} \leq \binom{l}{[l]/2}$ for $0 \leq k \leq l$. By (2.17) and (2.18), the column sums of $M(\mathcal{BR}^{0,2n})$ are bounded from above by

$$1 + \sum_{l=0}^{2n-1} \frac{1}{2^l} \binom{l}{[\frac{l}{2}]} \sim \frac{4}{\sqrt{\pi}} n^{\frac{1}{2}}.$$

Again by (2.4) and (2.16),

$$\mathbb{E}T(\mathcal{BR}^{0,2n}) \leq 2^{2n} \frac{4n^{\frac{1}{2}}/\sqrt{\pi}}{\#\mathcal{BR}^{0,2n}} \sim 4n.$$

Hence we take $C^0_{\mathcal{BR}} = 4$. □

2.4 Expected waiting time for first passage walks

We consider the expected waiting time for first passage walks through $\lambda_n \sim \lambda\sqrt{n}$ for $\lambda < 0$. Following Feller [45, Theorem 2, Chapter III.7], the number of patterns in $\mathcal{FP}^{\lambda,n}$ is

$$\#\mathcal{FP}^{\lambda,n} = \frac{\lambda_n}{n} \binom{n}{\frac{n+\lambda_n}{2}} \sim \lambda \exp\left(-\frac{\lambda^2}{2}\right) \sqrt{\frac{2}{\pi}} 2^n n^{-1}. \tag{2.19}$$

For $\mathcal{FP}^{\lambda,n} := \{FP^n_1, \cdots, FP^n_{\#\mathcal{FP}^{\lambda,n}}\}$ and $M(\mathcal{FP}^{\lambda,n})$ the matching matrix for $\mathcal{FP}^{\lambda,n}$, we have, by (2.3), that

$$(1, \cdots, 1) M(\mathcal{FP}^{\lambda,n}) \left(\frac{1}{\mathbb{E}T(FP^n_1)}, \cdots, \frac{1}{\mathbb{E}T(FP^n_{\#\mathcal{FP}^{\lambda,n}})} \right)^T = \frac{\#\mathcal{FP}^{\lambda,n}}{2^n}. \tag{2.20}$$

The LHS bound of (1.8) can be derived in a similar way as in Section 2.3.

$$\mathbb{E}T(\mathcal{FP}^{\lambda,n}) \geq \frac{2^n}{\#\mathcal{FP}^{\lambda,n}} \sim \sqrt{\frac{\pi}{2\lambda^2}} \exp\left(\frac{\lambda^2}{2}\right) n,$$

where $\#\mathcal{FP}^{\lambda,n}$ is defined as in (2.19). We get the lower bound of (1.8).

For the upper bound of (1.8), we aim to obtain an upper bound for the column sums of $M(\mathcal{FP}^{\lambda,n})$. Note that for $1 \leq j \leq k^{\lambda}_{\mathcal{FP}^n}$,

$$\sum_{i=1}^{\#\mathcal{FP}^{\lambda,n}} M(\mathcal{FP}^{\lambda,n})_{ij} = 1 + \sum_{l=1}^{n-1} \frac{1}{2^l} \sum_{i=1}^{\#\mathcal{FP}^{\lambda,n}} \epsilon_l(FP_i, FP_j) \tag{2.21}$$

and

$$\sum_{i=1}^{\#\mathcal{FP}^{\lambda,n}} \epsilon_l(FP^n_i, FP^n_j) = \#\{FP^n_i \in \mathcal{FP}^{\lambda,n}; FP^n_{i1} = FP^n_{j1+l}, \cdots, FP^n_{in-l} = FP^n_{jn}\}.$$

Observe that $\{FP_i^n \in \mathcal{FP}^{\lambda,n}; FP_{i1}^n = FP_{j1+l}^n, \cdots, FP_{in-l}^n = FP_{jn}^n\} \neq \emptyset$ if and only if $\sum_{k=1}^{l} FP_{jk}^n < 0$ (otherwise $\sum_{k=1}^{n-l} FP_{ik}^n = \sum_{k=1+l}^{n} FP_{jk}^n = \lambda_n - \sum_{k=1}^{l} FP_{jk}^n < \lambda_n$, which implies $FP_i^n \notin \mathcal{FP}^{\lambda,n}$). Then given $FP_{i1}^n = FP_{j1+l}^n, \cdots, FP_{in-l}^n = FP_{jn}^n$ and $\sum_{k=1}^{l} FP_{jk}^n < 0$,

$$FP_i^n \in \mathcal{FP}^{\lambda,n} \Longleftrightarrow$$

$$FP_{in-l+1}^n \cdots FP_{in}^n \text{ is a first passage walk of length } l \text{ through } \sum_{k=1}^{l} FP_{jk}^n < 0.$$

Therefore, for $1 \leq l \leq n-1$ and $1 \leq j \leq k_{\mathcal{FP}^n}^{\lambda}$,

$$\sum_{i=1}^{\#\mathcal{FP}^{\lambda,n}} \epsilon_l(FP_i^n, FP_j^n) = 1_{\sum_{k=1}^{l} FP_{jk}^n < 0} \frac{|\sum_{k=1}^{l} FP_{jk}^n|}{l} \binom{l}{\frac{l+\sum_{k=1}^{l} FP_{jk}^n}{2}}. \tag{2.22}$$

From the above discussion, it is easy to see for $1 \leq j \leq \#\mathcal{FP}^{\lambda,n}$,

$$\sum_{i=1}^{\#\mathcal{FP}^{\lambda,n}} M(\mathcal{FP}^{\lambda,n})_{ij} \leq \sum_{i=1}^{\#\mathcal{FP}^{\lambda,n}} M(\mathcal{FP}^{\lambda,n})_{ij^*},$$

where $FP_{j^*}^n$ is defined as follows: $FP_{j^*k}^n = -1$ if $1 \leq k \leq \lambda_n - 1$; $\lambda_n - 1 < k \leq n-1$ and $k - \lambda_n$ is odd; $k = n$. Otherwise $FP_{j^*k}^n = 1$.

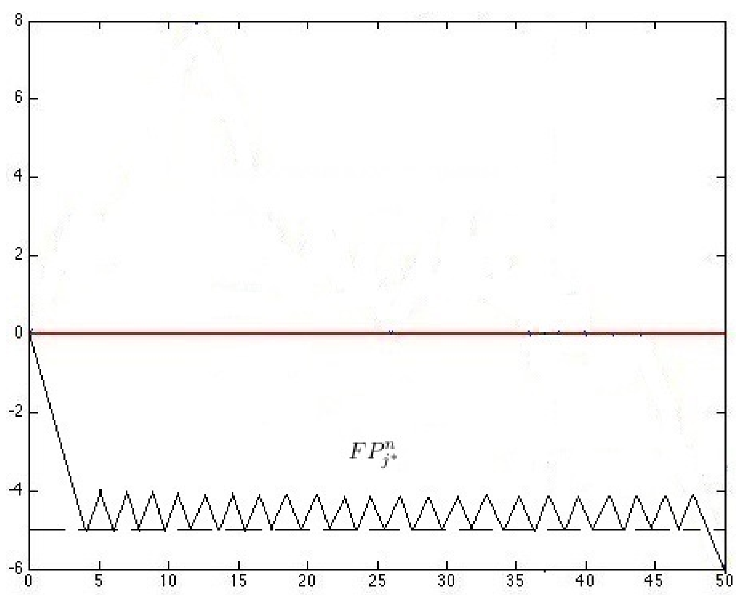


Fig 2. Extreme patterns $FP_{j^*}^n$.

The rest of this part is devoted to estimating $\sum_{i=1}^{\#\mathcal{FP}^{\lambda,n}} M(\mathcal{FP}^{\lambda,n})_{ij^*}$. By (2.21) and (2.22),

$$\sum_{i=1}^{\#\mathcal{FP}^{\lambda,n}} M(\mathcal{FP}^{\lambda,n})_{ij^*} = \sum_{l=0}^{|\lambda_n|-1} \frac{1}{2^l} + \sum_{\substack{l=\lambda_n \\ l-|\lambda_n| \text{ odd}}}^{n-1} \frac{|\lambda_n|-1}{l\cdot 2^l}\binom{l}{\frac{l-|\lambda_n|+1}{2}} + \sum_{\substack{l=\lambda_n \\ l-|\lambda_n| \text{ even}}}^{n-1} \frac{|\lambda_n|-2}{l\cdot 2^l}\binom{l}{\frac{l-|\lambda_n|+2}{2}}$$
$$\leq 2 + |\lambda_n| \sum_{l=|\lambda_n|}^{n-1} \frac{1}{2^l l}\binom{l}{[\frac{l}{2}]} \sim \sqrt{\frac{8\lambda}{\pi}} n^{\frac{1}{4}}.$$

Thus, the column sums of $M(\mathcal{FP}^{\lambda,n})$ are bounded from above by $\sqrt{\frac{8\lambda}{\pi}}n^{\frac{1}{4}}$. By (2.4) and (2.20),

$$\mathbb{E}T(\mathcal{FP}^{\lambda,n}) \leq \frac{2^n\sqrt{8\lambda/\pi}n^{\frac{1}{4}}}{\#\mathcal{FP}^{\lambda,n}} \sim \sqrt{\frac{4}{\lambda}}\exp\left(\frac{\lambda^2}{2}\right)n^{\frac{5}{4}}.$$

This is the upper bound of (1.8). □

2.5 Exponent gaps for $\mathcal{BR}^{\lambda,n}$ and $\mathcal{FP}^{\lambda,n}$

It can be inferred from (2.14) (resp. (1.8)) that the expected waiting time for $\mathcal{BR}^{\lambda,n}$ where $\lambda \in \mathbb{R}$ (resp. $\mathcal{FP}^{\lambda,n}$ where $\lambda < 0$) is bounded from below by order $n^{\frac{1}{2}}$ (resp. n) and from above by order n (resp. $n^{\frac{5}{4}}$). The exponent gap in the estimates of first passage walks $\mathcal{FP}^{\lambda,n}$ is frustrating, since we do not know whether the waiting time is exactly of order n, or is of order $\gg n$. This prevents the prediction of the existence of first passage bridge patterns $\mathcal{FP}^{\lambda}$ in Brownian motion.

From (2.4), we see that the most precise way to compute $\mathbb{E}T(\mathcal{BR}^{\lambda,n})$ and $\mathbb{E}T(\mathcal{FP}^{\lambda,n})$ consists in evaluating the sum of all entries in the inverse matching matrices $M(\mathcal{BR}^{\lambda,n})^{-1}$ and $M(\mathcal{FP}^{\lambda,n})^{-1}$. But the task is difficult since the structures of $M(\mathcal{BR}^{\lambda,n})$ and $M(\mathcal{FP}^{\lambda,n})$ are more complex than the structures of $M(\mathcal{E}^{2n})$ and $M(\mathcal{M}^{2n+1})$. We do not understand well the exact form of the inverse matrices $M(\mathcal{BR}^{\lambda,n})^{-1}$ and $M(\mathcal{FP}^{\lambda,n})^{-1}$.

The technique used in Section 2.3 and Section 2.4 is to bound the column sums of the matching matrix $M(\mathcal{BR}^{\lambda,n})$ (resp. $M(\mathcal{FP}^{\lambda,n})$). More precisely, we have proved that

$$\mathcal{O}(1) \leq \text{ column sums of } M(\mathcal{BR}^{\lambda,n}) \leq \mathcal{O}(n^{\frac{1}{2}}) \quad \text{for each fixed } \lambda \in \mathbb{R}; \tag{2.23}$$

$$\mathcal{O}(1) \leq \text{ column sums of } M(\mathcal{FP}^{\lambda,n}) \leq \mathcal{O}(n^{\frac{1}{4}}) \quad \text{for each fixed } \lambda < 0. \tag{2.24}$$

For the bridge pattern $\mathcal{BR}^{0,2n}$, the LHS bound of (2.23) is obtained by any excursion path of length $2n$, while the RHS bound of (2.23) is achieved by the sawtooth path with consecutive ± 1 increments. In the first passage pattern $\mathcal{FP}^{\lambda,n}$ where $\lambda < 0$, the LHS bound of (2.24) is achieved by some excursion-like path, which starts with an excursion and goes linearly to $\lambda\sqrt{n} < 0$ at the end. The RHS bound of (2.24) is given by the extreme pattern defined in Section 2.4, see Fig 2.

However, the above estimations are not accurate, since there are only few columns in $\mathcal{BR}^{\lambda,n}$ which sum up either to $\mathcal{O}(1)$ or to $\mathcal{O}(n^{\frac{1}{2}})$, and few columns of $\mathcal{FP}^{\lambda,n}$ which sum up either to $\mathcal{O}(1)$ or to $\mathcal{O}(n^{\frac{1}{4}})$.

Open problem 2.5.1

1. *Determine the exact asymptotics for* $\mathbb{E}T(\mathcal{BR}^{\lambda,n})$ *where* $\lambda \in \mathbb{R}$, *as* $n \to \infty$.
2. *Determine the exact asymptotics for* $\mathbb{E}T(\mathcal{FP}^{\lambda,n})$ *where* $\lambda < 0$, *as* $n \to \infty$.

As we prove below, for $\lambda \in \mathbb{R}$, $\mathbb{E}T(\mathcal{BR}^{\lambda,n}) \asymp n$ by a scaling limit argument. Nevertheless, to obtain this result only by discrete analysis would be of independent interest. The following table provides the simulations of the expected waiting time $\mathbb{E}T(\mathcal{FP}^{-1,n})$ for some large n.

n	100	200	500	1000	2000	5000	10000
$\mathbb{E}T(\mathcal{FP}^{n}_{-1})$	179.805	358.249	893.041	1800.002	3682.022	8549.390	12231.412
Estimated ζ		0.9945	0.9968	1.0112	1.0375	1.0205	1.0335

TABLE 1. Estimation of ζ by $\log \frac{\mathbb{E}T(\mathcal{FP}^{n_2}_{-1})}{\mathbb{E}T(\mathcal{FP}^{n_1}_{-1})} / \log(\frac{n_2}{n_1})$, where n_2 is the next to n_1 in the table.

The result suggests that $\mathbb{E}T(\mathcal{FP}^{-1,n})$ be linear, but possibly with some log-correction. Yuval Peres made the following conjecture:

Conjecture 2.5.2 *[118] For* $\lambda < 0$, *there exist* $c^{\lambda}_{\mathcal{FP}}$ *and* $C^{\lambda}_{\mathcal{FP}} > 0$ *such that*

$$c^{\lambda}_{\mathcal{FP}} n \ln n \le \mathbb{E}T(\mathcal{FP}^{\lambda,n}) \le C^{\lambda}_{\mathcal{FP}} n \ln n. \tag{2.25}$$

This is consistent with Theorem 1.0.6 (4), that we cannot find a first passage bridge with fixed negative endpoint in Brownian motion.

Now let us focus on the lower bound (1.7) of expected waiting time for bridge pattern $\mathcal{BR}^0$. For $n \in 2\mathbb{N}$, we run a simple random walk $(RW_k)_{k\in\mathbb{N}}$ until the first level bridge of length n appears. That is, we consider

$$(RW_{F_n+k} - RW_{F_n})_{0\le k\le n}, \quad \text{where } F_n := \inf\{k \ge 0; RW_{k+n} = RW_k\}. \tag{2.26}$$

For simplicity, let RW_k for non-integer k be defined by the usual linear interpolation of a simple random walk. For background on the weak convergence in $\mathcal{C}[0,1]$, we refer to Billingsley [17, Chapter 2].

Proposition 2.5.3

$$\left(\frac{RW_{F_n+nu} - RW_{F_n}}{\sqrt{n}}; 0 \le u \le 1\right) \text{ converges weakly in } \mathcal{C}[0,1] \text{ to the bridge-like process}$$

$$(B_{F+u} - B_F; 0 \le u \le 1), \quad \text{where } F := \inf\{t > 0; B_{t+1} - B_t = 0\}.$$

The process $(S_t := B_{t+1} - B_t; t \geq 0)$ is a stationary Gaussian process, first studied by Slepian [138] and Shepp [135]. The following result, which we will prove in Chapter 4, is needed for the proof of Proposition 2.5.3.

Lemma 2.5.4 *For each fixed $t \geq 0$, the distribution of $(S_u; t \leq u \leq t+1)$ is mutually absolutely continuous with respect to the distribution of*

$$(\widetilde{B}_u := \sqrt{2}(\xi + B_u); t \leq u \leq t+1), \tag{2.27}$$

where $\xi \sim \mathcal{N}(0,1)$. In particular, the distribution of the Slepian zero set restricted to $[t, t+1]$, i.e. $\{u \in [t, t+1]; S_u = 0\}$ is mutually absolutely continuous with respect to that of $\{u \in [t, t+1]; \xi + B_u = 0\}$, the zero set of Brownian motion starting at $\xi \sim \mathcal{N}(0,1)$.

Proof of Proposition 2.5.3: Let $\mathbb{P}^{\mathbf{W}}$ be Wiener measure on $\mathcal{C}[0,\infty)$. Let $\mathbb{P}^{\mathbf{S}}$ (resp. $\mathbb{P}^{\widetilde{\mathbf{W}}}$) be the distribution of the Slepian process S (resp. the distribution of $\widetilde{B}$ defined as in (2.27)). We claim that

$$F := \inf\{t \geq 0; w_{t+1} = w_t\},$$

is a functional of the coordinate process $w := \{w_t; t \geq 0\} \in \mathcal{C}[0,\infty)$ that is continuous $\mathbb{P}^{\mathbf{W}}$ a.s. Note that the distribution of $(x_t := w_{t+1} - w_t; t \geq 0)$ under $\mathbb{P}^{\mathbf{W}}$ is the same as that of $(w_t; t \geq 0)$ under $\mathbb{P}^{\mathbf{S}}$. In addition, $x \in \mathcal{C}[0,\infty)$ is a functional of $w \in \mathcal{C}[0,\infty)$ that is continuous $\mathbb{P}^{\mathbf{W}}$ a.s. By composition, it is equivalent to show that

$$F' := \inf\{t \geq 0; w_t = 0\},$$

is a functional of $w \in \mathcal{C}[0,\infty)$ that is continuous $\mathbb{P}^{\mathbf{S}}$ a.s. Consider the set

$$\mathcal{Z} := \{w \in \mathcal{C}[0,\infty); F' \text{ is not continuous at } w\} = \cup_{p \in \mathbb{Q}} \mathcal{Z}_p,$$

where $\mathcal{Z}_p := \{w \in \mathcal{C}[0,\infty); F' \in [p, p+1] \text{ and } F' \text{ is not continuous at } w\}$. It is obvious that $\mathbb{P}^{\widetilde{\mathbf{W}}}(\mathcal{Z}) = 0$ and thus $\mathbb{P}^{\widetilde{\mathbf{W}}}(\mathcal{Z}_p) = 0$ for all $p \geq 0$. By Lemma 2.5.4, $\mathbb{P}^{\mathbf{S}}$ is locally absolutely continuous relative to $\mathbb{P}^{\widetilde{\mathbf{W}}}$, which implies that $\mathbb{P}^{\mathbf{S}}(\mathcal{Z}_p) = 0$ for all $p \geq 0$. As a countable union of null events, $\mathbb{P}^{\mathbf{S}}(\mathcal{Z}) = 0$, and the claim is proved. Thus, the mapping

$$\Xi_F : \mathcal{C}[0,\infty) \ni (w_t; t \geq 0) \longrightarrow (w_{F+u} - w_F; 0 \leq u \leq 1) \in \mathcal{C}[0,1]$$

is continuous $\mathbb{P}^{\mathbf{W}}$ a.s. According to Donsker's theorem [36], see e.g. Billingsley [17, Section 10] or Kallenberg [75, Chapter 16], the linearly interpolated simple random walks

$$\left(\frac{RW_{[nt]}}{\sqrt{n}}; t \geq 0\right) \text{ converges weakly in } \mathcal{C}[0,1] \text{ to } (B_t; t \geq 0),$$

So by the continuous mapping theorem, see e.g. Billingsley [17, Theorem 5.1],

$$\Xi_F \circ \left(\frac{RW_{[nt]}}{\sqrt{n}}; t \geq 0\right) \text{ converges weakly to } \Xi_F \circ (B_t; t \geq 0). \quad \square$$

Note that $T(\mathcal{BR}^{0,n}) = F_n + n$. Following the above analysis, we know that $T(\mathcal{BR}^{0,n})/n$ converges weakly to $F+1$, where $T(\mathcal{BR}^{0,n})$ is the waiting time until an element of $\mathcal{BR}^{0,n}$ occurs in a simple random walk and F is the random time defined as in (1.2). As a consequence,

$$\liminf_{n\to\infty} \mathbb{E}\frac{T(\mathcal{BR}^{0,n})}{n} \geq \mathbb{E}F + 1, \quad \text{since } \mathbb{E}F < \infty.$$

In particular, $\mathbb{E}F \leq C^0_{\mathcal{BR}} - 1 = 3$ as in Section 2.3. See also Chapter 4 for further discussion on first level bridges and the structure of the Slepian zero set.

Chapter 3

Continuous paths in Brownian motion

This chapter is devoted to the proof of Theorem 1.0.4 (2) − (4) and Theorem 1.0.6 regarding continuous paths and the distribution of continuous-time processes embedded in Brownian motion.

In Section 3.1, we show that there is no normalized excursion in a Brownian path, i.e. Theorem 1.0.6 (1). A slight modification of the proof allows us to exclude the existence of the Vervaat bridges with negative endpoint, i.e. Theorem 1.0.6 (5). Furthermore, we prove in Section 3.2 that there is even no reflected bridge in Brownian motion, i.e. Theorem 1.0.6 (2). In Section 3.3 and 3.4, we show that neither the Vervaat transform of Brownian motion nor first passage bridges with negative endpoint can be found in Brownian motion, i.e. Theorem 1.0.6 (3) (4). We make use of the potential theory of *additive Lévy processes*, which is recalled in Section 3.3. Finally in Section 3.5, we provide a proof for the existence of Brownian meander, co-meander and three-dimensional Bessel process in Brownian motion, i.e. Theorem 1.0.4, using the filling scheme.

3.1 No normalized excursion in a Brownian path

In this section, we provide two proofs for Theorem 1.0.6 (1), though similar, from different viewpoints. The first proof is based on a fluctuation version of *Williams' path decomposition* of Brownian motion, originally due to Williams [154], and later extended in various ways by Millar [109, 108], and Greenwood and Pitman [63]. We also refer readers to Pitman and Winkel [125] for a combinatorial explanation and various applications.

Theorem 3.1.1 *[154, 63] Let $(B_t; t \geq 0)$ be standard Brownian motion and ξ be exponentially distributed with rate $\frac{1}{2}\vartheta^2$, independent of $(B_t; t \geq 0)$. Define $M := \operatorname{argmin}_{[0,\xi]} B_t$, $H := -B_M$ and $R := B_\xi + H$. Then H and R are independent exponential variables, each with the same rate ϑ. Furthermore, conditionally given H and R, the path $(B_t; 0 \leq t \leq \xi)$ is decomposed into two independent pieces:*

- $(B_t; 0 \le t \le M)$ *is Brownian motion with drift* $-\vartheta < 0$ *running until it first hits the level* $-H < 0$*;*
- $(B_{\xi-t} - B_\xi; 0 \le t \le \xi - M)$ *is Brownian motion with drift* $-\vartheta < 0$ *running until it first hits the level* $-R < 0$*.*

Now we introduce the notion of first passage process, which will be used in the proof of Theorem 1.0.6 (1). Given a càdlàg process $(Z_t; t \ge 0)$ starting at 0, we define the first passage process $(\tau_{-x}; x \ge 0)$ associated to X to be the first time that the level $-x < 0$ is hit:

$$\tau_{-x} := \inf\{t \ge 0; Z_t < -x\} \quad \text{for } x > 0.$$

When Z is Brownian motion, the distribution of the first passage process is well-known:

Lemma 3.1.2

1. *Let* $\boldsymbol{W}$ *be Wiener measure on* $\mathcal{C}[0,\infty)$*. Then the first passage process* $(\tau_{-x}; x \ge 0)$ *under* $\boldsymbol{W}$ *is a stable*$(\frac{1}{2})$ *subordinator, with*
$$\mathbb{E}^{\boldsymbol{W}}[\exp(-\alpha\tau_{-x})] = \exp(-x\sqrt{2\alpha}) \quad \textit{for } \alpha > 0.$$
2. *For* $\vartheta \in \mathbb{R}$*, let* $\boldsymbol{W}^\vartheta$ *be the distribution on* $\mathcal{C}[0,\infty)$ *of Brownian motion with drift* ϑ*.*

   Then for each fixed $L > 0$*, on the event* $\tau_{-L} < \infty$*, the distribution of the first passage process* $(\tau_{-x}; 0 \le x \le L)$ *under* $\boldsymbol{W}^\vartheta$ *is absolutely continuous with respect to that under* $\boldsymbol{W}$*, with density* $D_L^\vartheta := \exp(-\vartheta L - \frac{\vartheta^2}{2}\tau_{-L})$*.*

Proof: The part (1) of the lemma is a well known result of Lévy, see e.g. Bertoin et al. [14, Lemma 4]. The part (2) is a direct consequence of Girsanov's theorem, see e.g. Revuz and Yor [130, Chapter VIII] for background. $\square$

Proof of Theorem 1.0.6 (1)**:** Suppose by contradiction that $\mathbb{P}(T < \infty) > 0$, where T is a random time at which some excursion appears. Take ξ exponentially distributed with rate $\frac{1}{2}$, independent of $(B_t; t \ge 0)$. We have then

$$\mathbb{P}(T < \xi < T + 1) > 0. \tag{3.1}$$

Now $(T, T+1)$ is inside the excursion of Brownian motion above its past-minimum process, which straddles ξ. See Figure 2. Define

- $(\tau_{-x}; x \ge 0)$ to be the first passage process of $(B_{\xi+t} - B_\xi; t \ge 0)$.

By the strong Markov property of Brownian motion, $(B_{\xi+t} - B_\xi; t \ge 0)$ is still Brownian motion. Thus, $(\tau_{-x}; x \ge 0)$ is a stable$(\frac{1}{2})$ subordinator by Lemma 3.1.2 (1). Also, define

- $(\sigma_{-x}; x \ge 0)$ to be the first passage process derived from the process $(B_{\xi-t} - B_\xi; 0 \le t \le \xi - M)$ followed by an independent Brownian motion with drift -1 running forever.

According to Theorem 3.1.1, $(B_{\xi-t} - B_\xi; 0 \leq t \leq \xi - M)$ is Brownian motion with drift -1 running until it first hits the level $-R < 0$. Then $(\sigma_{-x}; x \geq 0)$ is the first passage process of Brownian motion with drift -1, whose distribution is absolutely continuous on any compact interval $[0, L]$, with respect to that of $(\tau_{-x}; 0 \leq x \leq L)$ by Lemma 3.1.2 (2).

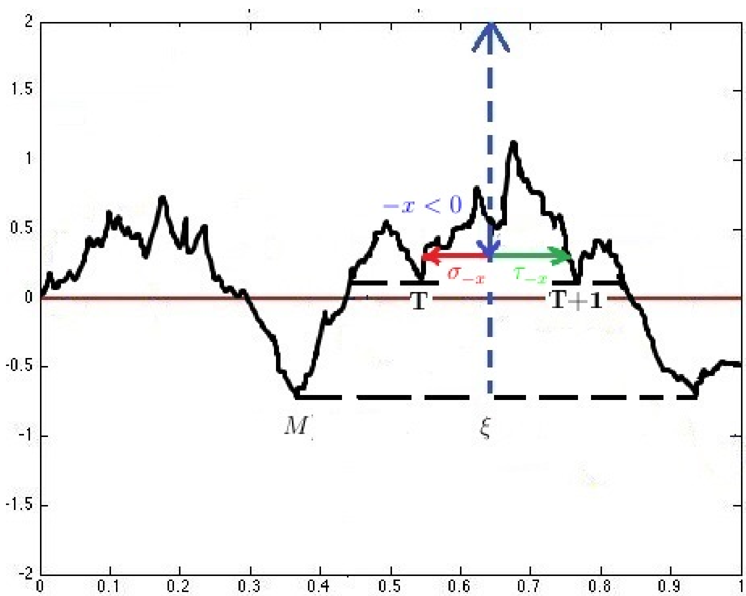


Fig 3. No excursion of length 1 in a Brownian path.

Thus, the distribution of $(\sigma_{-x} + \tau_{-x}; 0 \leq x \leq L)$ is absolutely continuous relative to that of $(\tau_{-2x}; 0 \leq x \leq L)$. It is well known that a real stable($\frac{1}{2}$) process does not hit points, see e.g. Bertoin [11, Theorem 16, Chapter II.5]. As a consequence,

$$\mathbb{P}(\sigma_{-x} + \tau_{-x} = 1 \text{ for some } x \geq 0) = 0,$$

which contradicts (3.1). $\square$

Proof of Theorem 1.0.6 (5)**:** Impossibility of embedding the Vervaat bridge paths $\mathcal{VB}^\lambda$ with endpoint $\lambda < 0$. We borrow the notations from the preceding proof. Observe that, for fixed $\lambda < 0$,

$$\mathbb{P}(\sigma_{-x} + \tau_{-x+\lambda} = 1 \text{ for some } x \geq 0) = 0.$$

The rest of the proof is just a duplication of the preceding one. $\square$

In the rest of this section, we give yet another proof of Theorem 1.0.6 (1), which relies on Itô's excursion theory, combined with Bertoin's self-similar fragmentation theory. For general background on fragmentation processes, we refer to the monograph of Bertoin [12]. The next result, regarding a normalized Brownian excursion, follows from Bertoin [13, Corollary 2].

Theorem 3.1.3 *[13] Let $e := (e_u; 0 \leq u \leq 1)$ be normalized Brownian excursion and $F^e := (F^e_t; t \geq 0)$ be the associated interval fragmentation defined as $F^e_t := \{u \in (0,1); e_u > t\}$. Introduce*

- $\lambda := (\lambda_t; t \geq 0)$ *the length of the interval component of* F^e *that contains* U*, independent of the excursion and uniformly distributed;*
- $\xi := \{\xi_t; t \geq 0\}$ *a subordinator, the Laplace exponent of which is given by*

$$\Phi^{ex}(q) := q\sqrt{\frac{8}{\pi}} \int_0^1 t^{q-\frac{1}{2}}(1-t)^{-\frac{1}{2}} = q\sqrt{\frac{8}{\pi}} B(q+\frac{1}{2}, \frac{1}{2}); \tag{3.2}$$

Then $(\lambda_t; t \geq 0)$ *has the same law as* $(\exp(-\xi_{\rho_t}); t \geq 0)$*, where*

$$\rho_t := \inf\left\{u \geq 0; \int_0^u \exp\left(-\frac{1}{2}\xi_r\right) dr > t\right\}. \tag{3.3}$$

Alternative proof of Theorem 1.0.6 (1)**:** Consider the reflected process $(B_t - \underline{B}_t; t \geq 0)$, where $\underline{B}_t := \inf_{0 \leq u \leq t} B_u$ is the past-minimum process of the Brownian motion. For $\mathbf{e}$ the first excursion of $B - \underline{B}$ that contains some excursion pattern $\mathcal{E}$ of length 1, let $\Lambda_{\mathbf{e}}$ be the length of such excursion, and $\mathbf{e}^*$ be the normalized Brownian excursion. Following Itô's excursion theory, see e.g. Revuz and Yor [130, Chapter XII], $\Lambda_{\mathbf{e}}$ is independent of the distribution of the normalized excursion $\mathbf{e}^*$. As a consequence, the fragmentation associated to $\mathbf{e}^*$ produces an interval of length $\frac{1}{\Lambda_{\mathbf{e}}}$.

Now choose U uniformly distributed on $[0, 1]$ and independent of the Brownian motion. According to Theorem 3.1.3, there exists a subordinator ξ characterized as in (3.2) and a time-change ρ defined as in (3.3) such that $(\lambda_t; t \geq 0)$, the process of the length of the interval fragmentation which contains U, has the same distribution as $(\exp(-\xi_{\rho_t}); t \geq 0)$. Note that $(\lambda_t; t \geq 0)$ depends only on the normalized excursion $\mathbf{e}^*$ and U, so $(\lambda_t; t \geq 0)$ is independent of $\Lambda_{\mathbf{e}}$. It is a well known result of Kesten [76] that a subordinator without drift does not hit points. Therefore,

$$\mathbb{P}\left(\lambda_t = \frac{1}{\Lambda_{\mathbf{e}}} \text{ for some } t \geq 0\right) = 0,$$

which yields the desired result. □

3.2 No reflected bridge in a Brownian path

This section is devoted to proving Theorem 1.0.6 (2). The main difference between Theorem 1.0.6 (1) and (2) is that the strict inequality $B_{T+u} > B_T$ for all $u \in (0, 1)$ is relaxed by the permission of equalities $B_{T+u} = B_T$ for some $u \in (0, 1)$. Thus, there are paths in $\mathcal{C}[0, 1]$ which may contain reflected bridge paths but not excursion paths. Nevertheless, such paths form a null set for Wiener measure. Below is a slightly stronger version of this result.

Lemma 3.2.1 *Almost surely, there are no random times* $S < T$ *such that* $B_T = B_S$*,* $B_u \geq B_S$ *for* $u \in (S, T)$ *and* $B_v = B_w = B_S$ *for some* $S < v < w < T$*.*

Proof: Consider the following two sets

$$\mathcal{T} := \{\text{there exist } S \text{ and } T \text{ which satisfy the conditions in the lemma}\}$$

and

$$\mathcal{U} := \bigcup_{s,t \in \mathbb{Q}} \{B \text{ attains its minimum for more than once on } [s,t]\}.$$

It is straightforward that $\mathcal{T} \subset \mathcal{U}$. In addition, it is well-known that almost surely Brownian motion has a unique minimum on any fixed interval $[s,t]$ for all $s,t \in \mathbb{R}$. As a countable union of null events, $\mathbb{P}(\mathcal{U}) = 0$ and thus $\mathbb{P}(\mathcal{T}) = 0$. $\Box$

Remark 3.2.2 The previous lemma has an interesting geometric interpretation in terms of Brownian trees, see e.g. Pitman [119, Section 7.4] for background. Along the lines of the second proof of Theorem 1.0.6 (1) in Section 3.1, we only need to show that the situation in Lemma 3.2.1 cannot happen in a Brownian excursion either of an independent and diffuse length or of normalized unit length. But this is just another way to state that Brownian trees have only binary branch points, which follows readily from Aldous' stick-breaking construction of the continuum random trees, see e.g. Aldous [2, Section 4.3] and Le Gall [99].

According to Theorem 1.0.6 (1) and Lemma 3.2.1, we see that almost surely, there are neither excursion paths of length 1 nor reflected bridge paths of any length with at least two intermediate returns in Brownian motion. To prove the desired result, it suffices to exclude the possibility of reflected bridge paths with exactly one reflection. This is done by the following lemma.

Lemma 3.2.3 *Assume that $0 \leq S < T < U$ are random times such that $B_S = B_T = B_U$ and $B_u > B_S$ for $u \in (S,T) \cup (T,U)$. Then the distribution of $U - S$ is absolutely continuous with respect to the Lebesgue measure.*

Proof: Suppose by contradiction that the distribution of $U - S$ is not absolutely continuous with respect to the Lebesgue measure. Then there exists $p, q \in \mathbb{Q}$ such that $U - S$ fails to have a density on the event $\{S < p < T < q < U\}$. In fact, if $U - S$ has a density on $\{S < p < T < q < U\}$ for all $p, q \in \mathbb{Q}$, Radon-Nikodym theorem guarantees that $U - S$ has a density on $\{S < T < U\} = \cup_{p,q \in \mathbb{Q}} \{S < p < T < q < U\}$.

Note that on the event $\{S < p < T < q < U\}$, U is the first time after q such that the Brownian motion B attains $\inf_{u \in [p,q]} B_u$ and obviously has a density. Again by Radon-Nikodym theorem, the distribution of $U - S$ has a density on $\{S < p < T < q < U\}$, which leads to a contradiction. $\Box$

Remark 3.2.4 The previous lemma can also be inferred from a fine study on local minima of Brownian motion. Neveu and Pitman [116] studied the renewal structure of local extrema in a Brownian path, in terms of Palm measure, see e.g. Kallenberg [75, Chapter 11].

More precisely, denote

- $\mathcal{C}$ to be the space of continuous paths on $\mathbb{R}$, equipped with Wiener measure $\mathbf{W}$;
- E to be the space of excursions with lifetime ζ, equipped with Itô measure $\mathbf{n}$.

Then the Palm measure of all local minima is the image of $\frac{1}{2}(\mathbf{n} \times \mathbf{n} \times \mathbf{W})$ by the mapping $E \times E \times \mathcal{C} \ni (e, e', w) \to \tilde{w} \in \mathcal{C}$ given by

$$\tilde{w}_t = \begin{cases} w_{t+\zeta(e')} & \text{if} \quad t \leq -\zeta(e'), \\ e'_{-t} & \text{if} \quad -\zeta(e') \leq t \leq 0, \\ e_t & \text{if} \quad 0 \leq t \leq \zeta(e), \\ w_{t-\zeta(e)} & \text{if} \quad t \geq \zeta(e). \end{cases}$$

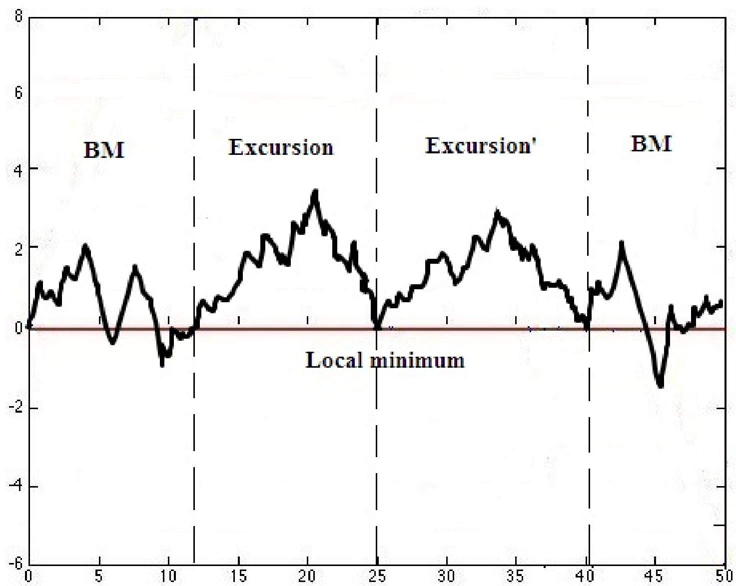


Fig 4. Structure of local minima in Brownian motion.

See Fig 4. Using the notations of Lemma 3.2.3, an in-between reflected position T corresponds to a Brownian local minimum. Then the above discussion implies that $U - S$ is the sum of two independent random variables with densities and hence is diffuse. See also Tsirelson [147] for the i.i.d. uniform sampling construction of Brownian local minima, which reveals the diffuse nature of $U - S$.

3.3 No Vervaat tranform of Brownian motion in a Brownian path

In the current section, we aim to prove Theorem 1.0.6 (3). That is, there is no random time T such that

$$(B_{T+u} - B_T; 0 \leq u \leq 1) \in \mathcal{VB}^-.$$

A similar argument shows that there is no random time T such that

$$(B_{T+u} - B_T; 0 \le u \le 1) \in \mathcal{VB}^+,$$

where

$$\mathcal{VB}^+ := \{w \in \mathcal{C}[0,1]; w(t) > 0 \text{ for } 0 < t \le 1 \text{ and } \sup\{t < 1; w(t) < w(1)\} < 1\}.$$

Observe that $(V_u; 0 \le u \le 1)$ is supported on $\mathcal{VB}^+ \cup \mathcal{VB}^-$. Thus, the Vervaat transform of Brownian motion cannot be embedded into Brownian motion.

In Section 3.1, we showed that for each fixed $\lambda < 0$, there is no random time T such that $(B_{T+u} - B_T; 0 \le u \le 1) \in \mathcal{VB}^\lambda$. However, there is no obvious way to pass from the non-existence of the Vervaat bridges to that of the Vervaat transform of Brownian motion, due to an uncountable number of possible final levels.

To get around the problem, we make use of an additional tool – potential theory of *additive Lévy processes*, developed by Khoshnevisan et al. [81, 83, 84, 85, 80]. We now recall some results of this theory that we need in the proof of Theorem 1.0.6 (3). For a more extensive overview of the theory, we refer to the survey of Khoshnevisan and Xiao [79].

Definition 3.3.1 *An N-parameter, $\mathbb{R}^d$-valued additive Lévy process $(Z_{\boldsymbol{t}}; \boldsymbol{t} \in \mathbb{R}_+^N)$ with Lévy exponent $(\Psi^1, \ldots, \Psi^N)$ is defined as*

$$Z_{\boldsymbol{t}} := \sum_{i=1}^N Z_{t_i}^i \quad \textit{for } \boldsymbol{t} = (t_1, \ldots, t_N) \in \mathbb{R}_+^N, \tag{3.4}$$

where $(Z_{t_1}^1; t_1 \ge 0), \ldots, (Z_{t_N}^N; t_N \ge 0)$ are N independent $\mathbb{R}^d$-valued Lévy processes with Lévy exponent $\Psi^1, \ldots, \Psi^N$.

The following result regarding the range of additive Lévy processes is due to Khoshnevisan et al. [85, Theorem 1.5], [80, Theorem 1.1], and Yang [156, 155, Theorem 1.1].

Theorem 3.3.2 *[85, 156, 80] Let $(Z_{\boldsymbol{t}}; \boldsymbol{t} \in \mathbb{R}_+^N)$ be an additive Lévy process defined as in* (3.4)*. Then*

$$\mathbb{E}[\mathrm{Leb}(Z(\mathbb{R}_+^N))] > 0 \Longleftrightarrow \int_{\mathbb{R}^d} \prod_{i=1}^N \mathrm{Re}\left(\frac{1}{1 + \Psi^i(\zeta)}\right) d\zeta < \infty,$$

where $\mathrm{Leb}(\cdot)$ *is the Lebesgue measure on $\mathbb{R}^d$, and* $\mathrm{Re}(\cdot)$ *is the real part of a complex number.*

The next result, which is read from Khoshnevisan and Xiao [82, Lemma 4.1], makes a connection between the range of an additive Lévy process and the polarity of single points. See also Khoshnevisan and Xiao [79, Lemma 3.1].

Theorem 3.3.3 *[81, 82] Let $(Z_{\boldsymbol{t}}; \boldsymbol{t} \in \mathbb{R}_+^N)$ be an additive Lévy process defined as in* (3.4)*. Assume that for each $\boldsymbol{t} \in \mathbb{R}_+^N$, the distribution of $Z_{\boldsymbol{t}}$ is mutually absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^d$. Let $z \in \mathbb{R}^d \setminus \{0\}$, then*

$$\mathbb{P}(Z_{\boldsymbol{t}} = z \textit{ for some } \boldsymbol{t} \in \mathbb{R}_+^N) > 0 \Longleftrightarrow \mathbb{P}(\mathrm{Leb}(Z(\mathbb{R}_+^N) > 0) > 0.$$

Note that $\mathbb{P}(\text{Leb}(Z(\mathbb{R}_+^N) > 0) > 0$ is equivalent to $\mathbb{E}[\text{Leb}(Z(\mathbb{R}_+^N))] > 0$. Combining Theorem 3.3.2 and Theorem 3.3.3, we have:

Corollary 3.3.4 *Let* $(Z_{\boldsymbol{t}}; \boldsymbol{t} \in \mathbb{R}_+^N)$ *be an additive Lévy process defined as in* (3.4). *Assume that for each* $\boldsymbol{t} \in \mathbb{R}_+^N$, *the distribution of* $Z_{\boldsymbol{t}}$ *is mutually absolutely continuous with respect to Lebesgue measure on* $\mathbb{R}^d$. *Let* $z \in \mathbb{R}^d \setminus \{0\}$, *then*

$$\mathbb{P}(Z_{\boldsymbol{t}} = z \textit{ for some } \boldsymbol{t} \in \mathbb{R}_+^N) > 0 \Longleftrightarrow \int_{\mathbb{R}^d} \prod_{i=1}^N \text{Re}\left(\frac{1}{1+\Psi^i(\zeta)}\right) d\zeta < \infty.$$

Proof of Theorem 1.0.6 (3)**:** We borrow the notations from the proof of Theorem 1.0.6 (1) in Section 3.1. It suffices to show that

$$\mathbb{P}(\sigma_{-t_1} + \tau_{-t_2} = 1 \text{ for some } t_1, t_2 \geq 0) = 0, \tag{3.5}$$

where $(\sigma_{-t_1}; t_1 \geq 0)$ is the first passage process of Brownian motion with drift -1, and $(\tau_{-t_2}; t_2 \geq 0)$ is a stable($\frac{1}{2}$) subordinator independent of $(\sigma_{-t_1}; t_1 \geq 0)$.

Let $Z_{\mathbf{t}} = Z^1_{t_1} + Z^2_{t_2} := \sigma_{-t_1} + \tau_{-t_2}$ for $\mathbf{t} = (t_1, t_2) \in \mathbb{R}^2_+$. By Definition 3.3.1, Z is a 2-parameter, real-valued additive Lévy process with Lévy exponent (Ψ^1, Ψ^2) given by

$$\Psi^1(\zeta) = \sqrt[4]{1+4\zeta^2} \exp\left[-i\frac{\arctan(2\zeta)}{2}\right] - 1,$$

and

$$\Psi^2(\zeta) = \sqrt{|\zeta|}(1 - i \,\text{sgn}\, \zeta),$$

for $\zeta \in \mathbb{R}$, which is derived from the formula in Cinlar [28, Chapter 7, Page 330] and Lemma 3.1.2 (2). Hence,

$$\text{Re}\left(\frac{1}{1+\Psi^1(\zeta)}\right) = \frac{1}{\sqrt[4]{1+4\zeta^2}} \sqrt{\frac{1}{2}\left(1 + \frac{1}{\sqrt{1+4\zeta^2}}\right)},$$

and

$$\text{Re}\left(\frac{1}{1+\Psi^2(\zeta)}\right) = \frac{1+\sqrt{|\zeta|}}{1+2\sqrt{|\zeta|}+2|\zeta|}.$$

Clearly, $\Xi : \zeta \to \text{Re}\left(\frac{1}{1+\Psi^1(\zeta)}\right)\text{Re}\left(\frac{1}{1+\Psi^2(\zeta)}\right)$ is not integrable on $\mathbb{R}$ since $\Xi(\zeta) \sim \frac{1}{4|\zeta|}$ as $|\zeta| \to \infty$. In addition, for each $\mathbf{t} \in \mathbb{R}^2_+$, Z_t is mutually absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$. Applying Corollary 3.3.4, we obtain (3.5). □

3.4 No first passage bridge in a Brownian path

We prove Theorem 1.0.6 (4), i.e. there is no first passage bridge in Brownian motion by a spacetime shift. The main difference between Vervaat bridges with fixed endpoint $\lambda < 0$ and first passage bridges ending at $\lambda < 0$ is that the former start with an excursion piece, while the latter return to the origin infinitely often on any small interval $[0, \epsilon]$, $\epsilon > 0$. Thus, the argument used in Section 3.1 to prove the non-existence of Vervaat bridges is not immediately applied in case of first passage bridges. Nevertheless, the potential theory of additive Lévy processes helps to circumvent the difficulty.

Proof of Theorem 1.0.6 (4)**:** Suppose by contradiction that $\mathbb{P}(T < \infty) > 0$, where T is a random time that some first passage bridge through a fixed level appears. Take ξ exponentially distributed with rate $\frac{1}{2}$, independent of $(B_t; t \geq 0)$. We have then

$$\mathbb{P}(T < \xi < T + 1) > 0. \tag{3.6}$$

Now $(T, T+1)$ is inside the excursion of Brownian motion below its past-maximum process, which straddles ξ. See Figure 5. Define

- $(\tau_{-x}; x \geq 0)$ to be the first passage process of $(B_{\xi+t} - B_\xi; t \geq 0)$.

By strong Markov property of Brownian motion, $(B_{\xi+t} - B_\xi; t \geq 0)$ is still Brownian motion. Thus, $(\tau_{-x}; x \geq 0)$ is a stable($\frac{1}{2}$) subordinator. Let $M := \text{argmax}_{[0,\xi]} B_t$. By a variant of Theorem 3.1.1, $(B_{\xi-t} - B_\xi; 0 \leq t \leq \xi - M)$ is Brownian motion with drift 1 running until it first hits the level $B_M - B_\xi > 0$, independent of $(\tau_{-x}; x \geq 0)$.

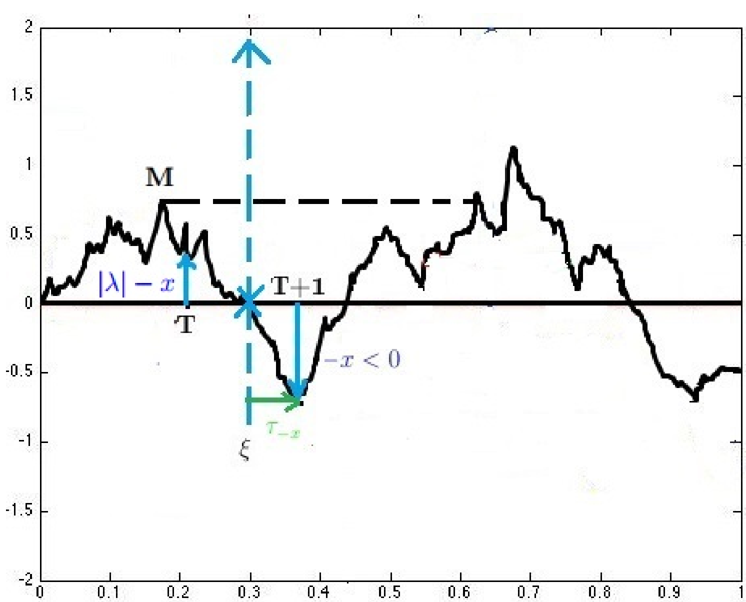


Fig 5. No first passage bridge of length 1 in a Brownian path.

As a consequence, (3.6) implies that

$$\mathbb{P}(\tau_{-x} = l \text{ and } B^{\uparrow}_{1-l} = |\lambda| - x \text{ for some } (x,l) \in \mathbb{R}_+ \times [0,1]) > 0, \tag{3.7}$$

where $(B^{\uparrow}_t; t \geq 0)$ is Brownian motion with drift 1, independent of $\frac{1}{2}$-stable subordinator $(\tau_{-x}; x \geq 0)$. By setting $t_1 := x$ and $t_2 := 1 - l$, we have:

$$\begin{aligned}
&\mathbb{P}(\tau_{-x} = l \text{ and } B^{\uparrow}_{1-l} = |\lambda| - x \text{ for some } (x,l) \in \mathbb{R}_+ \times [0,1]) \\
&= \mathbb{P}(\tau_{-t_1} + t_2 = 1 \text{ and } B^{\uparrow}_{t_2} + t_1 = |\lambda| \text{ for some } (t_1,t_2) \in \mathbb{R}_+ \times [0,1]) \\
&\leq \mathbb{P}[(\tau_{-t_1}, t_1) + (t_2, B^{\uparrow}_{t_2}) = (1, |\lambda|) \text{ for some } (t_1,t_2) \in \mathbb{R}^2_+]
\end{aligned} \tag{3.8}$$

Let $Z_{\mathbf{t}} = Z^1_{t_1} + Z^2_{t_2} := (\tau_{-t_1}, t_1) + (t_2, B^{\uparrow}_{t_2})$ for $\mathbf{t} = (t_1,t_2) \in \mathbb{R}^2_+$. By Definition 3.3.1, Z is a 2-parameter, $\mathbb{R}^2$-valued additive Lévy process with Lévy exponent (Ψ^1, Ψ^2) given by

$$\Psi^1(\zeta_1,\zeta_2) := \sqrt{|\zeta_1|} - i(\sqrt{|\zeta_1|}\operatorname{sgn}\zeta_1 + \zeta_2),$$

and

$$\Psi^2(\zeta_1,\zeta_2) := \frac{\zeta_2^2}{2} - i(\zeta_1 + \zeta_2),$$

for $(\zeta_1,\zeta_2) \in \mathbb{R}^2$. Hence,

$$\begin{aligned}
&\operatorname{Re}\left(\frac{1}{1+\Psi^1(\zeta_1,\zeta_2)}\right)\operatorname{Re}\left(\frac{1}{1+\Psi^2(\zeta_1,\zeta_2)}\right) \\
&= \frac{(1+\sqrt{|\zeta_1|})\left(1+\frac{\zeta_2^2}{2}\right)}{\left[(1+\sqrt{|\zeta_1|})^2 + (\sqrt{|\zeta_1|}\operatorname{sgn}\zeta_1 + \zeta_2)^2\right]\left[\left(1+\frac{\zeta_2^2}{2}\right)^2 + (\zeta_1+\zeta_2)^2\right]} := \Xi(\zeta_1,\zeta_2).
\end{aligned}$$

Observe that $\zeta \to \Xi(\zeta_1,\zeta_2)$ is not integrable on $\mathbb{R}^2$, which is clear by passage to polar coordinates $(\zeta_1,\zeta_2) = (\rho\cos\theta, \sqrt{\rho}\sin\theta)$ for $\rho \geq 0$, $\theta \in [0, 2\pi)$. In addition, for each $\mathbf{t} \in \mathbb{R}^2_+$, $Z_{\mathbf{t}}$ is mutually absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^2$. Applying Corollary 3.3.4, we know that

$$\mathbb{P}(Z_{\mathbf{t}} = (1, |\lambda|) \text{ for some } \mathbf{t} \in \mathbb{R}^2_+) = 0.$$

Combining with (3.8), we obtain:

$$\mathbb{P}(\tau_{-x} = l \text{ and } B^{\uparrow}_{1-l} = |\lambda| - x \text{ for some } (x,l) \in \mathbb{R}_+ \times [0,1]) = 0,$$

which contradicts (3.7). □

It is not hard to see that the above argument, together with those in Section 3.2 works for Bessel bridge of any dimension.

Corollary 3.4.1 *(Impossibility of embedding of reflected bridge paths/Bessel bridge) For each fixed $\lambda > 0$, almost surely, there is no random time T such that*

$$(B_{T+u} - B_T; 0 \leq u \leq 1) \in \mathcal{RBR}^{\lambda} := \{w \in \mathcal{C}[0,1]; w(t) \geq 0 \text{ for } 0 \leq t \leq 1 \text{ and } w(1) = \lambda\}.$$

In particular, there is no random time $T \geq 0$ such that $(B_{T+u} - B_T; 0 \leq u \leq 1)$ has the same distribution as Bessel bridge ending at λ.

3.5 Meander, co-meander and 3-d Bessel process in a Brownian path

We prove Theorem 1.0.4 in this section using Itô's excursion theory, combined with *Rost's filling scheme* [24, 131] solution to the Skorokhod embedding problem.

The existence of Brownian meander in a Brownian path is assured by the following well-known result, which can be read from Maisoneuve [104, Section 8], with explicit formulas due to Chung [27]. An alternative approach was provided by Greenwood and Pitman [62], and Pitman [121, Section 4 and 5]. See also Biane and Yor [16, Theorem 6.1], or Revuz and Yor [130, Exercise 4.18, Chapter XII].

Theorem 3.5.1 *[104, 62, 16] Let $(e^i)_{i\in\mathbb{N}}$ be the sequence of excursions, whose length exceeds 1, in the reflected process $(B_t - \underline{B}_t; t \geq 0)$, where $\underline{B}_t := \inf_{0\leq u\leq t} B_u$ is the past-minimum process of the Brownian motion. Then $(e^i_u; 0 \leq u \leq 1)_{i\in\mathbb{N}}$ is a sequence of independent and identically distributed paths, each distributed as Brownian meander $(m_u; 0 \leq u \leq 1)$.*

Let us recall another basic result due to Imhof [69], which establishes the absolute continuity relation between Brownian meander and the three-dimensional Bessel process. Their relation with Brownian co-meander is studied in Yen and Yor [157, Chapter 7].

Theorem 3.5.2 *[69, 157] The distributions of Brownian meander $(m_u; 0 \leq u \leq 1)$, Brownian co-meander $(\widetilde{m}_u; 0 \leq u \leq 1)$ and the three-dimensional Bessel process $(R_u; 0 \leq u \leq 1)$ are mutually absolutely continuous with respect to each other. For $F : \mathcal{C}[0,1] \to \mathbb{R}^+$ a measurable function,*

1. $\mathbb{E}[F(m_u; 0 \leq u \leq 1)] = \mathbb{E}\left[\sqrt{\frac{\pi}{2}}\frac{1}{R_1}F(R_u; 0 \leq u \leq 1)\right]$;

2. $\mathbb{E}[F(\widetilde{m}_u; 0 \leq u \leq 1)] = \mathbb{E}\left[\frac{1}{R_1^2}F(R_u; 0 \leq u \leq 1)\right]$.

According to Theorem 3.5.1, there exist $T_1, T_2, \cdots$ such that

$$m^i := (B_{T_i+u} - B_{T_i}; 0 \leq u \leq 1) \tag{3.9}$$

form a sequence of i.i.d. Brownian meanders. Since Brownian co-meander and the three-dimensional Bessel process are absolutely continuous relative to Brownian meander, it is natural to think of *von Neumann's acceptance-rejection algorithm* [150], see e.g. Rubinstein and Kroese [133, Section 2.3.4] for background and various applications.

However, von Neumann's selection method requires that the Radon-Nikodym density between the underlying probability measures is essentially bounded, which is not satisfied in the cases suggested by Theorem 3.5.2. Nevertheless, we can apply the filling scheme of Chacon and Ornstein [24] and Rost [131].

We observe that sampling Brownian co-meander or the three-dimensional Bessel process from i.i.d. Brownian meanders $(m^i)_{i\in\mathbb{N}}$ fits into the general theory of Rost's filling scheme applied to the Skorokhod embedding problem. In the sequel, we follow the approach of Dellacherie and Meyer [35, Section $63-74$, Chapter IX], which is based on the seminal work of Rost [131], to construct a stopping time N such that m^N achieves the distribution of $\widetilde{m}$ or R. We need some notions from potential theory for the proof.

Definition 3.5.3

1. *Given a Markov chain $X := (X_n)_{n\in\mathbb{N}}$, a function f is said to be excessive relative to X if*
$$(f(X_n))_{n\in\mathbb{N}} \text{ is } \mathcal{F}_n - \text{supermartingale},$$
where $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is the natural filtration of X.

2. *Given two positive measures μ and λ, μ is said to be a balayage/sweeping of λ if*
$$\mu(f) \leq \lambda(f) \quad \text{for all bounded excessive functions } f.$$

Proof of Theorem 1.0.4: Let μ^m (resp. μ^R) be the distribution of Brownian meander (resp. the three-dimensional Bessel process) on the space $(\mathcal{C}[0,1],\mathcal{F})$. By the filling scheme, the sequence of measures $(\mu_i^m,\mu_i^R)_{i\in\mathbb{N}}$ is defined recursively as

$$\mu_0^m := (\mu^m - \mu^R)^+ \quad \text{and} \quad \mu_0^R := (\mu^m - \mu^R)^-, \tag{3.10}$$

and for each $i \in \mathbb{N}$,

$$\mu_{i+1}^m := (\mu_i^m(1)\cdot\mu^m - \mu_i^R)^+ \quad \text{and} \quad \mu_{i+1}^R := (\mu_i^m(1)\cdot\mu^m - \mu_i^R)^-, \tag{3.11}$$

where $\mu_i^m(1)$ is the total mass of the measure μ_i^m. It is not hard to see that the bounded excessive functions of the i.i.d. meander sequence are constant μ^m a.s. Since μ^R is absolutely continuous with respect to μ^m, for each μ^m a.s. constant function c, $\mu^R(c) = \mu^m(c) = c$. Consequently, μ^R is a balayage/sweeping of μ^m by Definition 3.5.3. According to Dellacherie and Meyer [35, Theorem 69],

$$\mu_\infty^R = 0, \quad \text{where } \mu_\infty^R :=\downarrow \lim_{i\to\infty} \mu_i^R.$$

Now let d_0 be the Radon-Nikodym density of μ_0^m relative to μ^m, and for $i>0$, d_i be the Radon-Nikodym density of μ_i^m relative to $\mu_{i-1}^m(1)\cdot\mu^m$. We have

$$\begin{aligned}\mu^R &= (\mu^R - \mu_0^R) + (\mu_0^R - \mu_1^R) + \cdots \\ &= (\mu^m - \mu_0^m) + (\mu_0^m(1)\cdot\mu^m - \mu_1^m) + \cdots \\ &= (1-d_0)\mu^m + d_0\mu^m(1)\cdot(1-d_1)\mu^m + \cdots .\end{aligned} \tag{3.12}$$

Consider the stopping time N defined by

$$N := \inf\left\{n \geq 0; -\sum_{i=0}^{n} \log d_i(m^i) > \xi\right\}, \tag{3.13}$$

where $(d_i)_{i\in\mathbb{N}}$ is the sequence of Radon-Nikodym densities defined as in the preceding paragraph, $(m^i)_{i\in\mathbb{N}}$ is the sequence of i.i.d. Brownian meanders defined as in (3.9), and ξ is exponentially distributed with rate 1, independent of $(m^i)_{i\in\mathbb{N}}$.

From the computation of (3.12), for all bounded measurable function f and all $k \in \mathbb{N}$,

$$\begin{aligned}\mathbb{E}[f(m^N); N = k] &= \mathbb{E}[f(m^k); -\sum_{i=0}^{k-1}\log d_i(m^i) \leq \xi < -\sum_{i=0}^{k}\log d_i(m^i)] \\ &= \mathbb{E}[d_0(m^0)\cdots d_{k-1}(m^{k-1})f(m^k)(1-d_k(m^k))] \\ &= (\mu^m_{k-1}(1)\cdot\mu^m - \mu^m_k)f \\ &= (\mu^R_{k-1} - \mu^R_k)f,\end{aligned}$$

where $(\mu^m_i, \mu^R_i)_{i\in\mathbb{N}}$ are the filling measures defined as in (3.10) and (3.11). By summing over all k, we have

$$\mathbb{E}[f(m^N); N < \infty] = \mu^R f.$$

That is, m^N has the same distribution as R. As a summary,

$$(B_{T_N+u} - B_{T_N}; 0 \leq u \leq 1) \text{ has the same distribution as } (R_u; 0 \leq u \leq 1),$$

where $(T_i)_{i\in\mathbb{N}}$ are defined by (3.9) and N is the stopping time as in (3.13). Thus we achieve the distribution of the three-dimensional Bessel process in Brownian motion. The embedding of Brownian co-meander into Brownian motion is obtained in the same vein. □

Remark 3.5.4 Note that the stopping time N defined as in (3.13) has infinite mean, since

$$\mathbb{E}N = \sum_{i\in\mathbb{N}}\mu^m_i(1) = \infty.$$

The problem whether Brownian co-meander or the three-dimensional Bessel process can be embedded in finite expected time, remains open. More generally, Rost [132] was able to characterize all stopping distributions of a continuous-time Markov process, given its initial distribution.

In our setting, let $(P_t)_{t\geq 0}$ be the semi-group of the moving window process $X_t := (B_{t+u} - B_t; 0 \leq u \leq 1)$ for $t \geq 0$, and μ^W be its initial distribution, corresponding to Wiener measure on $\mathcal{C}[0,1]$. Following Rost [132], for any distribution μ on $\mathcal{C}[0,1]$, one can construct the continuous-time filling measures $(\mu_t, \mu^W_t)_{t\geq 0}$ and a suitable stopping time T such that

$$\mu - \mu_t + \mu^W_t = \mu^W P_{t\wedge T}.$$

Thus, the distribution μ is achieved if and only if $\mu_\infty = 0$, where $\mu_\infty :=\downarrow \lim_{t\to\infty}\mu_t$. In particular, Brownian motion with drift $(\vartheta t + B_t; 0 \leq t \leq 1)$ for a fixed ϑ, can be obtained for a suitable stopping time $T + 1$.

Chapter 4

The bridge-like process and the Slepian zero set

Recall from (1.2) the definition of the bridge-like process, which serves as a candidate for Brownian bridge embedded in a Brownian path. It is natural to ask the following question:

Question 4.0.5

1. *Is the bridge-like process defined as in* (1.2) *a standard Brownian bridge?*
2. *If not, is the distribution of standard Brownian bridge absolutely continuous with respect to that of the bridge-like process?*

Let $\mathcal{C}_0[0,1]$ be the set of continuous paths $(w_t; 0 \leq t \leq 1)$ starting at $w_0 = 0$, and $\mathcal{B}$ be the Borel $\sigma-$field of $\mathcal{C}_0[0,1]$. To provide a context for the above questions, we observe that

$$F := \inf\{t \geq 0; X_t \in \mathcal{BR}^0\}, \tag{4.1}$$

where

$$X_t := (B_{t+u} - B_t; 0 \leq u \leq 1) \quad \text{for } t \geq 0, \tag{4.2}$$

is the moving-window process associated to Brownian motion $(B_t; t \geq 0)$, and

$$\mathcal{BR}^0 := \{w \in \mathcal{C}_0[0,1]; w(1) = 0\}$$

is the set of bridges with endpoint 0.

Note that the moving-window process X is a stationary Markov process, with transition kernel $P_t : (\mathcal{C}_0[0,1], \mathcal{B}) \to (\mathcal{C}_0[0,1], \mathcal{B})$ for $t \geq 0$ given by

$$P_t(w, d\widetilde{w}) = \begin{cases} \mathbb{P}^{\mathbf{W}}(d\widetilde{w}) & \text{if } \; t \geq 1, \\ 1(\widetilde{w} = (w_{t+u} - w_t; u \leq 1-t) \otimes \widetilde{w}')\mathbb{P}^{\mathbf{W}_t}(d\widetilde{w}') & \text{if } \; t < 1, \end{cases}$$

where $\mathbb{P}^{\mathbf{W}}$ (resp. $\mathbb{P}^{\mathbf{W}_t}$) is Wiener measure on $\mathcal{C}_0[0,1]$ (resp. $\mathcal{C}_0[0,t]$), and $\otimes$ is the usual path concatenation. Note that $\mathbb{P}^{\mathbf{W}}$ is invariant with respect to $(P_t; t \geq 0)$. Moreover, X_{t+l} and X_t are independent for all $t \geq 0$ and $l \geq 1$.

For a suitably nice continuous-time Markov process $(Z_t; t \geq 0)$, there have been extensive studies on the post-T process $(Z_{T+t}; t \geq 0)$ with some random time T which is

- a stopping time, see e.g. Hunt [68] for Brownian motion, Blumenthal [21], and Dynkin and Jushkevich [42] for general Markov processes;
- an honest time, that is the time of last exit from a predictable set, see e.g. Meyer et al. [107], Pittenger and Shih [127, 128], Getoor and Sharpe [60, 59, 61], Maisonneuve [104] and Getoor [57];
- the time at which X reaches its ultimate minimum, see e.g. Williams [153] and Jacobsen [71] for diffusions, Pitman [120] for conditioned Brownian motion and Millar [109, 108] for general Markov processes.

Question 4.0.5 is related to decomposition/splitting theorems of Markov processes. We refer to the survey of Millar [110], which contains a unified approach to most if not all of the above cases. See also Pitman [121] for a presentation in terms of point processes and further references. Moreover, if Z is a semi-martingale and T is an honest time, the semi-martingale decomposition of the post-T process was investigated in the context of progressive enlargement of filtrations, by Barlow [5], Yor [158], Jeulin and Yor [72] and in the monograph of Jeulin [73]. The monograph of Mansuy and Yor [105] offers a survey of this theory.

The study of the bridge-like process is challenging, because the random time F as in (1.3) does not fit into any of the above classes. We even do not know whether this bridge-like process is Markov, or whether it enjoys the semi-martingale property. Note that if the answer to Question 4.0.5 (2) is positive, then we can apply *Rost's filling scheme* [24, 131] as in Section 3.5 to sample Brownian bridge from a sequence of i.i.d. bridge-like processes in Brownian motion by iteration of the construction (1.2).

While we are unable to answer either of the above questions about the bridge-like process, we are able to prove Theorem 1.0.4 (1). That is, there is a random time $T > 0$ such that $(B_{T+u} - B_T; 0 \leq u \leq 1) \stackrel{(d)}{=} (b_u^0; 0 \leq u \leq 1)$. In terms of the moving-window process, it is equivalent to find a random time $T \geq 0$ such that $X_T \stackrel{(d)}{=} b^0$.

Our proof relies on Last and Thorisson's result [94, 95] on the *Palm measure* of local times of the moving-window process. The idea of embedding *Palm/Revuz measures* arose earlier in the work of Bertoin and Le Jan [10], and the connection between Palm measures and Markovian bridges was made by Fitzsimmons et al. [48]. The existence of local times follows from the Brownian structure of the zero set of the *Slepian process* $S_t := B_{t+1} - B_t$ for $t \geq 0$.

This triggers the study of the Slepian zero set on $[0, 1]$, that is $\{u \in [0, 1]; S_u = 0\}$. The problem here involves level crossings of a stationary Gaussian process. We refer to the surveys of Blake and Lindsey [20], Abrahams [1], Kratz [90], as well as the books of Cramér and Leadbetter [32, Chapter 10], Azaïs and Wschebor [4, Chapter 3] for further development.

Berman [9] studied general criteria for stationary Gaussian processes to have local times. In particular, he proved that if $(Z_t; t \geq 0)$ is a stationary Gaussian process with covariance

$R^Z(t)$ and $1 - R^Z(t) \underset{t \to 0}{\sim} |t|^\alpha$ for some $0 < \alpha < 2$, then Z has local times $(L_t^x; x \in \mathbb{R}, t \geq 0)$ such that for any Borel measurable set $C \subset \mathbb{R}$ and $t \geq 0$,

$$\int_0^t 1(Z_s \in C) ds = \int_C L_t^x dx.$$

It is not hard to see that the Slepian process has covariance $R^S(t) := \max(1 - |t|, 0)$, which obviously fits into the above category. See also the survey of Geman and Horowitz [53] for further development on Gaussian occupation measures.

Below is the plan for this chapter. In Section 4.1, we present some analysis of random walks related to Question 4.0.5. In Section 4.2, after recalling some results for the Slepian process due to Slepian [138] and Shepp [135], we provide a path decomposition for the Slepian process on $[0, 1]$, Theorem 4.2.1. The proof of Theorem 4.2.1 is given in Section 4.4. In Section 4.3, we deal with the local absolute continuity between the distribution of the Slepian process and that of Brownian motion with random starting point. Finally in Section 4.5, we study a *Palm-Itô measure* associated to the gaps between Slepian zeros, with comparison to the well-known *Itô's excursion law* [70].

4.1 Random walk approximation

In this section, we consider the discrete analog of the bridge-like process. Namely, for an even positive integer n, we run a simple symmetric random walk $(RW_k)_{k \in \mathbb{N}}$ until the first level bridge of length n appears. That is, we consider the process

$$(RW_{F_n + k} - RW_{F_n})_{0 \leq k \leq n}, \quad \text{where } F_n := \inf\{k \geq 0; RW_{k+n} = RW_k\}. \tag{4.3}$$

Recall the invariance principle from Proposition 2.5.3. Further, we may consider Knight's [86, 87] embedding of random walks in Brownian motion. Endow the space $\mathcal{C}[0, \infty)$ with the topology of uniform convergence on compact sets. Fix $n \in \mathbb{N}$. Let $\tau_0^{(n)} := 0$ and $\tau_{k+1}^{(n)} := \inf\{t > \tau_k^{(n)}; |B_t - B_{\tau_k^{(n)}}| = 2^{-n}\}$ for $k \in \mathbb{N}$. Note that $\left(RW_k^{(n)} := 2^n B_{\tau_k^{(n)}}\right)_{k \in \mathbb{N}}$ is a simple random walk. In addition, the sequence of linearly interpolated random walks

$$\left(\frac{RW_{2^{2n}t}^{(n)}}{2^n}; t \geq 0\right) \text{ converges almost surely in } \mathcal{C}[0, \infty) \text{ to } (B_t; t \geq 0).$$

It is not hard to see that $F(w) := \inf\{t \geq 0; w_{t+1} = w_t\}$ is not continuous at all paths $w \in \mathcal{C}_0[0, \infty)$. But from the proof of Proposition 2.5.3, F is $\mathbb{P}^{\mathbf{W}}$-a.s. continuous, where $\mathbb{P}^{\mathbf{W}}$ is Wiener measure on $\mathcal{C}[0, \infty)$. Thus, the convergence of Proposition 2.5.3 is almost sure in the context of Knight's construction of simple random walks.

Now we focus on the discrete bridge defined as in (4.3). Note that the support of the first level bridge is all bridge paths since the first n steps starting from 0 can be any path.

For $n = 2$, the bridge $(RW_{F_2+k} - RW_{F_2})_{0 \le k \le 2}$ obviously has uniform distribution on the two possible paths, one positive and one negative. However, the first level bridge of length n is not uniform for $n > 2$. Using the Markov chain matrix method, we can compute the exact distribution of this first level bridge for $n = 4$ and 6. By up-down symmetry, we only need to be concerned with those paths whose first step is $+1$.

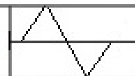
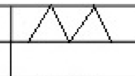
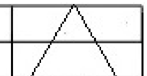
Bridge patterns			
Distribution	8/56	9/56	11/56

TABLE 1. The distribution of the first level bridge as in (4.3) for $n = 4$.

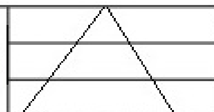
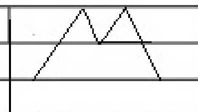
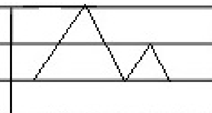
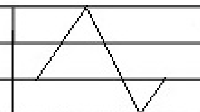
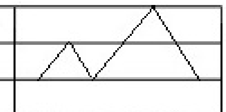
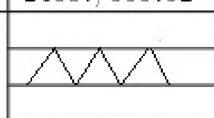
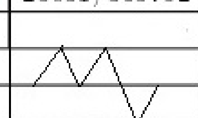
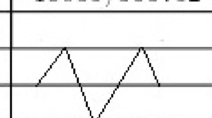
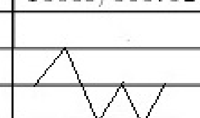
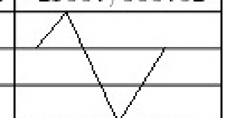
Bridge patterns					
Distribution	24687/365792	23051/365792	18059/365792	13088/365792	21337/365792
Bridge patterns					
Distribution	17241/365792	14336/365792	14745/365792	16384/365792	19968/365792

TABLE 2. The distribution of the first level bridge as in (4.3) for $n = 6$.

The numerical results in Table 1 and 2 give us that the first level bridge fails to be uniform, at least, for $n = 4$ and 6. By elementary algebraic computation, it is not hard to check that this is true for all $n > 2$. Now it is natural to ask whether the first level bridge could be asymptotically uniform. To this end, we compute the ratio of extremal probabilities of the first level bridge for some small n's.

n	2	4	6	8
max/min	1.000	1.375	1.886	2.580

TABLE 3. The ratio max/min probability of the first level bridge of length n.

In Table 3, the ratios max/min of hitting probabilities suggest that the first level bridge might not be asymptotically uniform. Thus, the answer to Question 4.0.5 (1) may be negative, i.e. the bridge-like process defined as in (1.2) is not standard Brownian bridge.

This is further confirmed by the following simulations, which show that as n grows, the empirical distribution of the maximum of the first level bridge does not appear to converge to the *Kolmogorov-Smirnov distribution*, that is the distribution of the supremum of Brownian bridge, see e.g. Billingsley [17, Section 13].

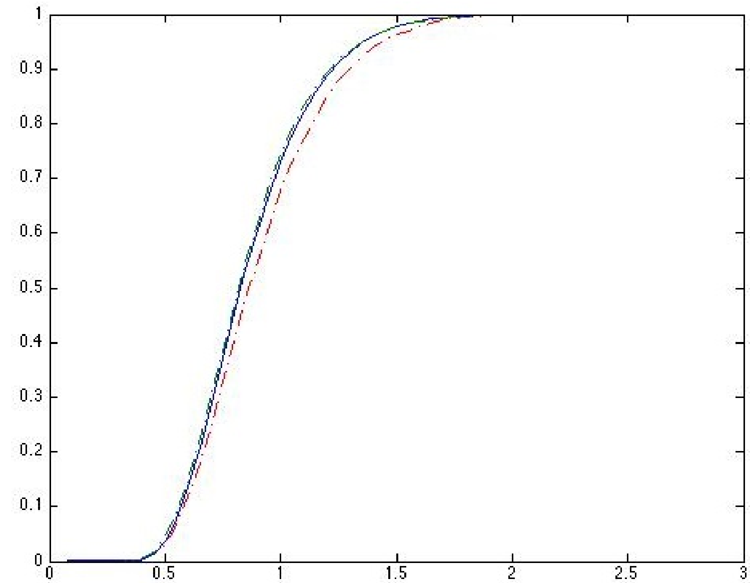


Fig 6. Solid curve: the Kolmogorov-Smirnov CDF; Dashed curve over the solid curve: the empirical CDF of the maximum of scaled uniform bridge of length $n = 10^4$; dashed curve below the solid curve: the empirical CDF of the maximum of the first level bridge of length $n = 10^4$.

n	100	500	1000	2000	5000	10000
CDF(1.3)	0.9361	0.9193	0.9129	0.9117	0.9088	0.9080
Difference	−0.0042	0.0126	0.0190	0.0202	0.0231	0.0239

TABLE 4. 2^{nd} row: the CDFs at 1.3 of the scaled maximum of the first level bridge of length n. 3^{rd} row: the differences between the Kolmogorov-Smirnov CDF evaluated at 1.3 (≈ 0.9319) and those of the 2^{nd} row.

4.2 The Slepian process: Old and New

Let us turn back to the random time F defined as in (1.3). We rewrite it as

$$F := \inf\{t \geq 0; S_t = 0\}, \tag{4.4}$$

where $S_t := B_{t+1} - B_t$ for $t \geq 0$ is a stationary Gaussian process with mean 0 and covariance $\mathbb{E}[S_{t_1} S_{t_2}] = \max(1 - |t_1 - t_2|, 0)$.

Note that $(S_t; t \geq 0)$ is not Markov, since the only nontrivial stationary, Gaussian and Markov process is the Ornstein-Uhlenbeck process, see e.g. Doob [37, Theorem 1.1]. The process $(S_t; t \geq 0)$ was first studied by Slepian [138]. Later, Shepp [135] gave an explicit formula for

$$I(t|x) := \mathbb{P}(F > t | S_0 = x),$$

as a $t-$fold integral when t is an integer and as a $2[t]+2-$fold integral when t is not an integer. Shepp's results are as follows. Let

$$\phi(x) := \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \quad \text{and} \quad \phi_\theta(x) := \frac{1}{\sqrt{\theta}}\phi\left(\frac{x}{\sqrt{\theta}}\right).$$

When $t = n$ is an integer,

$$I(t|x)\phi(x) = \int_{\mathcal{D}'} \det\left[\phi(y_i - y_{j+1})\right]_{0\le i,j\le n} dy_2 \cdots dy_{n+1}, \tag{4.5}$$

where $y_0 = 0$, $y_1 = |x|$ and $\mathcal{D}' := \{|x| < y_2 < \cdots < y_{n+1}\}$. When $t = n + \theta$ where $0 < \theta < 1$,

$$\begin{aligned} I(t|x)\phi(x) = \int_{\mathcal{D}''} \det\left[\phi_\theta(x_i - y_i)\right]_{0\le i,j\le n+1} \\ \times \det\left[\phi_\theta(y_i - x_{j+1})\right]_{0\le i,j\le n} dx_2 \cdots dx_{n+1} dy_0 \cdots dy_{n+1}, \end{aligned} \tag{4.6}$$

where $x_0 = 0$, $x_1 = |x|$ and $\mathcal{D}'' := \{|x| < x_2 < \cdots < x_{n+1}$ and $y_0 < \cdots < y_{n+1}\}$. The distribution of the first passage time F is characterized by

$$\mathbb{P}(F > t) = \int_{\mathbb{R}} I(t|x)\phi(x)dx, \tag{4.7}$$

where $I(t|x)\phi(x)$ is given as (4.5) when t is integer and given as (4.6) when it is not. In particular,

$$\begin{aligned} \mathbb{P}(F > 1) &= \int_{\mathbb{R}} [\Phi(0)\phi(x) - \phi(0)\Phi(x)]dx \\ &= \frac{1}{2} - \frac{1}{\pi}, \end{aligned} \tag{4.8}$$

where $\Phi(x) := \int_{-\infty}^{x} \phi(z)dz$ is the cumulative distribution function of the standard normal distribution.

Here we study the local structure of the Slepian zero set, i.e. $\{t \in [0,1]; S_t = 0\}$, by showing that it is mutually absolutely continuous relative to that of Brownian motion with normally distributed starting point. The main result, which provides a path decomposition of the Slepian process on $[0,1]$, is stated as below.

Theorem 4.2.1 *Let $F := \inf\{t \ge 0; S_t = 0\}$ and $G := \sup\{t \le 1; S_t = 0\}$. Given the quadruple (S_0, S_1, F, G) with $0 < F < G < 1$, the Slepian process $(S_t; 0 \le t \le 1)$ is decomposed into three conditionally independent pieces:*

- $(S_t/\sqrt{2}; 0 \le t \le F)$ *is Brownian first passage bridge from* $(0, S_0/\sqrt{2})$ *to* $(F, 0)$*;*
- $(S_t/\sqrt{2}; F \le t \le G)$ *is Brownian bridge of length* $G - F$*;*
- $(|S_t|/\sqrt{2}; G \le t \le 1)$ *is a three-dimensional Bessel bridge from* $(G, 0)$ *to* $(1, |S_1|/\sqrt{2})$*.*

In addition, the distribution of (S_0, S_1, F, G) *with* $0 < F < G < 1$ *is given by*

$$\mathbb{P}(S_0 \in dx, S_1 \in dy, F \in da, G \in db) = \frac{|xy|}{8\pi^2\sqrt{(b-a)a^3(1-b)^3}} \exp\left(-\frac{x^2}{4a} - \frac{y^2}{4(1-b)} - \frac{(x+y)^2}{4}\right). \quad (4.9)$$

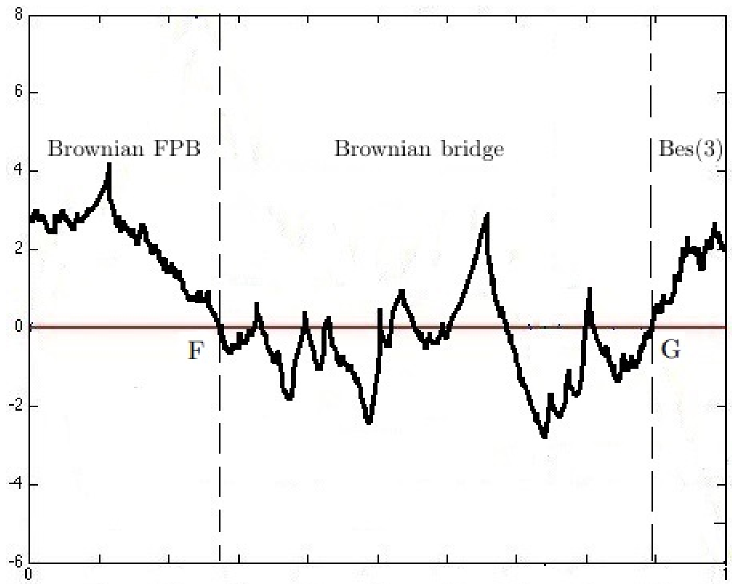


Fig 7. Path decomposition of $(S_t/\sqrt{2}; 0 \le t \le 1)$ with $0 < F < G < 1$.

On the event $\{0 < F < G < 1\}$, the Slepian process is achieved by first creating the quadruple (S_0, S_1, F, G) and then filling in with usual Brownian components. Similarly, on the event $\{F > 1\}$, $(S_t/\sqrt{2}; 0 \le t \le 1)$ is Brownian bridge from $(0, S_0/\sqrt{2})$ to $(1, S_1/\sqrt{2})$ conditioned not to hit 0.

The proof of Theorem 4.2.1 is deferred to Section 4.4. One method relies on Shepp [136]'s result of the absolute continuity between Gaussian measures, where the Slepian process was proved to be mutually absolutely continuous with respect to some modified Brownian motion on $[0, 1]$. As pointed out by Shepp [135], the absolute continuity fails beyond the unit interval. This is why we restrict the study of the Slepian zero set to intervals of length 1. Nevertheless, we have the following conjecture:

Conjecture 4.2.2 *For* $t \ge 0$*, the Slepian zero set on* $[0, t]$*, i.e.* $\{u \in [0, t]; S_u = 0\}$*, is mutually absolutely continuous with respect to that of* $\{u \in [0, t]; \xi + B_u = 0\}$*, the zero set of Brownian motion started at* ξ *with standard normal distribution,* $\xi \sim \mathcal{N}(0, 1)$*, and* ξ *independent of* B*.*

4.3 Local absolute continuity between Slepian zeros and Brownian zeros

As proved by Shepp [136], for each fixed $t \leq 1$, the distribution of the Slepian process $(S_u; 0 \leq u \leq t)$ is mutually absolutely continuous with respect to that of

$$(\widetilde{B}_u := \sqrt{2}(\xi + B_u); 0 \leq u \leq t), \tag{4.10}$$

where $\xi \sim \mathcal{N}(0,1)$ is independent of $(B_u; u \geq 0)$. The Radon-Nikodym derivative is given by

$$\frac{d\mathbb{P}^{\mathbf{S}}}{d\mathbb{P}^{\widetilde{\mathbf{W}}}}(w) := \frac{2}{\sqrt{2-t}} \exp\left(\frac{w_0^2}{4} - \frac{(w_0+w_t)^2}{4(2-t)}\right), \tag{4.11}$$

where $\mathbb{P}^{\mathbf{S}}$ (resp. $\mathbb{P}^{\widetilde{\mathbf{W}}}$) is the distribution of the Slepian process S (resp. the modified Brownian motion $\widetilde{B}$ defined as in (4.10)) on $\mathcal{C}[0,1]$. As a first application, we compute the density of the first passage time F, defined as in (4.4), on the unit interval.

Proposition 4.3.1 *For $w \in \mathcal{C}[0,1]$, let $F := \inf\{t \geq 0; w_t = 0\}$. Then*

$$\mathbb{P}^{\mathbf{S}}(F \in da) = \frac{1}{\pi}\sqrt{\frac{2-a}{a}}da \quad \text{for } 0 \leq a \leq 1, \tag{4.12}$$

Proof: Fix $a \leq 1$. By the change of measure formula (4.11),

$$\begin{aligned}
\mathbb{P}^{\mathbf{S}}(F \in da) &= \mathbb{E}^{\widetilde{\mathbf{W}}}\left[1(F \in da) \cdot \frac{2}{\sqrt{2-a}} \exp\left(\frac{w_0^2}{4} - \frac{(w_0+w_a)^2}{4(2-a)}\right)\right] \\
&= \frac{2}{\sqrt{2-a}} \mathbb{E}^{\widetilde{\mathbf{W}}}\left[1(F \in da) \exp\left(\frac{w_0^2}{4} - \frac{w_0^2}{4(2-a)}\right)\right] \\
&= \frac{2}{\sqrt{2-a}} \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi}} \exp\left(-\frac{x^2}{4}\right) \cdot \mathbb{P}^{\widetilde{\mathbf{W}}_x}(F \in da) \exp\left(\frac{x^2}{4} - \frac{x^2}{4(2-a)}\right) dx, \quad (4.13)
\end{aligned}$$

where $\mathbb{P}^{\widetilde{\mathbf{W}}_x}$ is the distribution of $\widetilde{B}$ conditioned on $\widetilde{B}_0 = x$. It is well-known that

$$\mathbb{P}^{\widetilde{\mathbf{W}}_x}(F \in da) = \frac{|x|}{\sqrt{4\pi a^3}} \exp\left(-\frac{x^2}{4a}\right) da. \tag{4.14}$$

Injecting (4.14) into (4.13), we obtain

$$\begin{aligned}
\mathbb{P}^{\mathbf{S}}(F \in da) &= \frac{1}{2\pi\sqrt{(2-a)a^3}} \int_{\mathbb{R}} |x| \exp\left(-\frac{x^2}{2a(2-a)}\right) dx da \\
&= \frac{1}{\pi}\sqrt{\frac{2-a}{a}} da. \quad \square
\end{aligned}$$

Remark 4.3.2 As a check, from (4.12),

$$\mathbb{P}^{\mathbf{S}}(F \leq 1) = \frac{1}{2} + \frac{1}{\pi} \approx 0.82,$$

which agrees with the formula (4.8) derived from the determinantal expressions (4.5), (4.6) and (4.7). Since the absolute continuity relation does not hold when $t > 1$, we are not able to derive a simple formula for the density of F on $(1, \infty)$.

Next we deal with the local absolute continuity between the distribution of Slepian zeros and that of Brownian motion with normally distributed starting point. The result enables us to prove Proposition 2.5.3, that is the weak convergence of the discrete first level bridges to the bridge-like process as in (1.2). The following is a stronger version of Lemma 2.5.4.

Lemma 4.3.3 *For each fixed $t \geq 0$, the distribution of $(S_u; t \leq u \leq t+1)$ is mutually absolutely continuous with respect to that of $(\widetilde{B}_u; t \leq u \leq t+1)$ defined as in (4.10). The Radon-Nikodym derivative is given by*

$$\frac{d\mathbb{P}^{\boldsymbol{S}}}{d\mathbb{P}^{\widetilde{\boldsymbol{W}}^t}}(w) = 2\sqrt{\frac{1+t}{2-t}} \exp\left(\frac{w_0^2}{4(1+t)} - \frac{(w_0+w_1)^2}{4(2-t)}\right), \tag{4.15}$$

where $\mathbb{P}^{\widetilde{\boldsymbol{W}}^t}$ is the distribution of $\widetilde{B}$ on $[t, t+1]$. In particular, the distribution of the Slepian zero set restricted to $[t, t+1]$, i.e. $\{u \in [t, t+1]; S_u = 0\}$ is mutually absolutely continuous with respect to that of $\{u \in [t, t+1]; \xi + B_u = 0\}$, the zero set of Brownian motion starting at $\xi \sim \mathcal{N}(0, 1)$.

Proof: It suffices to prove the first part of this lemma. By stationarity of the Slepian process, the distribution of $(S_u; t \leq u \leq t+1)$ is the same as that of $(S_u; 0 \leq u \leq 1)$, which is mutually absolutely continuous relative to $(\widetilde{B}_u; 0 \leq u \leq 1)$ with density given by (4.11). Now we conclude by noting that the distribution of $(\widetilde{B}_u; t \leq u \leq t+1)$ and that of $(\widetilde{B}_u; 0 \leq u \leq 1)$ are mutually absolutely continuous, with Radon-Nikodym derivative

$$\frac{d\mathbb{P}^{\widetilde{\mathbf{W}}}}{d\mathbb{P}^{\widetilde{\mathbf{W}}^t}}(w) := \sqrt{1+t} \exp\left(-\frac{t w_0^2}{4(1+t)}\right). \quad \square$$

As a consequence, all local properties of the Slepian zero set mimic closely those of Brownian motion with normally distributed starting point. In particular, with positive probability, the Slepian process visits the origin on the unit interval. And immediately thereafter, it returns to the origin infinitely often, as does Brownian motion. In addition, it is easy to see that the Radon-Nikodym derivative between the distribution of $\{u \in [0, 1]; S_u = 0\}$ and that of $\{u \in [0, 1]; \xi + B_u = 0\}$ is given by

$$\mathbb{E}\left[\frac{d\mathbb{P}^{\mathbf{S}}}{d\mathbb{P}^{\widetilde{\mathbf{W}}}}(w) \middle| \mathrm{Proj}(w)\right], \tag{4.16}$$

where $\mathrm{Proj}(w) := \{u \in [0,1]; w_u = 0\}$ is the zero set of $w \in \mathcal{C}[0,1]$. In the next subsection, we will see how this conditional expectation, as the Radon-Nikodym derivative, can be made explicit by some sufficient statistics.

4.4 Path decomposition of the Slepian process on $[0,1]$

In this section, we investigate further the local structure of the Slepian zero set by providing two proofs of Theorem 4.2.1.

From the work of Slepian [138], we know that given the i.i.d. normally distributed sequence $(S_n := B_{n+1} - B_n)_{n \in \mathbb{N}}$, for each $n \in \mathbb{N}$, the process $(S_{n+u}/\sqrt{2}; 0 \leq u \leq 1)$ has the same distribution as Brownian bridge from $S_n/\sqrt{2}$ to $S_{n+1}/\sqrt{2}$. For $n \in \mathbb{N}$ and $k \geq 2$, the processes $(S_{n+u}/\sqrt{2}; 0 \leq u \leq 1)$ and $(S_{n+k+u}/\sqrt{2}; 0 \leq u \leq 1)$ are independent. However, the consecutive bridges $(S_{n+u}/\sqrt{2}; 0 \leq u \leq 1)$ and $(S_{n+1+u}/\sqrt{2}; 0 \leq u \leq 1)$ for $n \in \mathbb{N}$, are correlated. The correlation is inferred from the following construction of the Slepian process.

Proposition 4.4.1 *Let $(Z_n)_{n \in \mathbb{N}}$ be a sequence of i.i.d. $\mathcal{N}(0,1)$-distributed random variables, and $(b_t^n; 0 \leq t \leq 1)_{n \in \mathbb{N}}$ be a sequence of i.i.d. standard Brownian bridges independent of $(Z_n)_{n \in \mathbb{N}}$. Define a continuous-time process $(Z_t; t \geq 0)$ as*

$$Z_t := b_{t-n}^{n+1} - b_{t-n}^{n} + (n+1-t)Z_n + (t-n)Z_{n+1} \quad \text{for } n \leq t < n+1, \; n \in \mathbb{N}. \tag{4.17}$$

Then $(Z_t; t \geq 0)$ has the same distribution as the Slepian process $(S_t; t \geq 0)$.

Proof: Note that $(Z_t; t \geq 0)$ and $(S_t; t \geq 0)$ are centered Gaussian processes. It suffices to show that $(Z_t; t \geq 0)$ and $(S_t; t \geq 0)$ have the same covariance function. Let $t_2 \geq t_1 \geq 0$. Recall that $\mathbb{E}[S_{t_1} S_{t_2}] = \max(1 - t_2 + t_1, 0)$. From the construction (4.17) of $(Z_t; t \geq 0)$, we know that Z_{t_1} and Z_{t_2} are independent if $t_1 \in [n, n+1)$ and $t_2 \geq n+2$ for some $n \in \mathbb{N}$. In this case, $\mathbb{E}[Z_{t_1} Z_{t_2}] = 0$. The other cases are:

- $t_1, t_2 \in [n, n+1)$ for some $n \in \mathbb{N}$. Then

$$\begin{aligned}
\mathbb{E}[Z_{t_1} Z_{t_2}] &= \mathbb{E}[b_{t_1-n}^{n+1} b_{t_2-n}^{n+1}] + \mathbb{E}[b_{t_1-n}^{n} b_{t_2-n}^{n}] \\
&\qquad + (n+1-t_1)(n+1-t_2)\mathbb{E}Z_n^2 + (t_1-n)(t_2-n)\mathbb{E}Z_{n+1}^2 \\
&= 2(t_1-n)(n+1-t_2) + (n+1-t_1)(n+1-t_2) + (t_1-n)(t_2-n) \\
&= 1 - t_2 + t_1.
\end{aligned}$$

- $t_1 \in [n, n+1)$ and $t_2 \in [n+1, n+2)$ for some $n \in \mathbb{N}$. Then

$$\mathbb{E}[Z_{t_1} Z_{t_2}] = -\mathbb{E}[b_{t_1-n}^{n+1} b_{t_2-n-1}^{n+1}] + (t_1-n)(n+2-t_2)\mathbb{E}Z_{n+1}^2.$$

  When $t_2 - t_1 \geq 1$, we obtain:

$$\mathbb{E}[Z_{t_1} Z_{t_2}] = -(t_1-n)(n+2-t_2) + (t_1-n)(n+2-t_2) = 0.$$

When $t_2 - t_1 < 1$, we obtain:

$$\mathbb{E}[Z_{t_1} Z_{t_2}] = -(n+1-t_1)(t_2-n-1) + (t_1-n)(n+2-t_2) = 1 - t_2 + t_1.$$

Putting all pieces together, we have $\mathbb{E}[Z_{t_1} Z_{t_2}] = \max(1 - t_2 + t_1, 0) = \mathbb{E}[S_{t_1} S_{t_2}]$. $\square$

Remark 4.4.2 In particular, the proposition shows that given the triple of i.i.d. standard normal variables (S_n, S_{n+1}, S_{n+2}), the two standard Brownian bridges derived from $(S_{n+u}; 0 \leq u \leq 1)$ and $(S_{n+1+u}; 0 \leq u \leq 1)$ by subtracting off the lines between endpoints:

$$\left(\frac{S_{n+u} - (1-u)S_n - uS_{n+1}}{\sqrt{2}}; 0 \leq u \leq 1 \right) \text{ and } \left(\frac{S_{n+1+u} - (1-u)S_{n+1} - uS_{n+2}}{\sqrt{2}}; 0 \leq u \leq 1 \right)$$

are not conditionally independent.

For $w \in \mathcal{C}[0,1]$, let $F := \inf\{t \geq 0; w_t = 0\}$ be the time of first hit to 0, and $G := \sup\{t \leq 1; w_t = 0\}$ be the time of last exit from 0 on the unit interval. From Proposition 4.4.1, the Slepian process on $[0,1]$ can be constructed by first picking independently $S_0, S_1 \sim \mathcal{N}(0,1)$, and then filling in a $\sqrt{2}$-Brownian bridge from S_0 to S_1. This bridge construction provides a proof of Theorem 4.2.1.

Proof of Theorem 4.2.1: The first part of the statement is quite straightforward from Proposition 4.4.1 and the discussion above. To finish the proof, we compute the $\mathbb{P}^{\mathbf{S}}$-joint distribution of the quadruple (w_0, w_1, F, G) on the event $\{0 < F < G < 1\}$.

$$\mathbb{P}^{\mathbf{S}}(w_0 \in dx, w_1 \in dy, F \in da, G \in db) = \frac{dxdy}{2\pi} \exp\left(-\frac{x^2+y^2}{2}\right) \mathbb{P}^{\widetilde{\mathbf{W}}_{x \to y}}(F \in da, G \in db), \tag{4.18}$$

where $\mathbb{P}^{\widetilde{\mathbf{W}}_{x \to y}}$ is the distribution of $\widetilde{B}$ defined as in (4.10), conditioned to start at x and end at y. In addition,

$$\begin{aligned}
&\mathbb{P}^{\widetilde{\mathbf{W}}_{x \to y}}(F \in da, G \in db) \\
&= \frac{|x|}{\sqrt{4\pi(1-a)a^3}} \exp\left(-\frac{y^2}{4(1-a)} - \frac{x^2}{4a} + \frac{(x-y)^2}{4}\right) \cdot \mathbb{P}^{\widetilde{\mathbf{W}}_{x \to y}}(G \in db | F \in da) \\
&= \frac{|x|}{\sqrt{4\pi(1-a)a^3}} \exp\left(-\frac{y^2}{4(1-a)} - \frac{x^2}{4a} + \frac{(x-y)^2}{4}\right) \\
&\qquad \cdot \frac{|y|\sqrt{1-a}}{\sqrt{4\pi(b-a)(1-b)^3}} \exp\left(\frac{y^2}{4(1-a)} - \frac{y^2}{4(1-b)}\right) \\
&= \frac{|xy|}{4\pi\sqrt{(b-a)a^3(1-b)^3}} \exp\left(-\frac{x^2}{4a} - \frac{y^2}{4(1-b)} + \frac{(x-y)^2}{4}\right).
\end{aligned} \tag{4.19}$$

Injecting (4.19) into (4.18), we obtain the formula (4.9). $\square$

By integrating over $S_0 \in dx$ and $S_1 \in dy$ in the formula (4.9), we get:

$$\mathbb{P}(F \in da, G \in db) = \frac{2}{\pi\sqrt{b-a}}\left[\frac{1}{(2+a-b)\sqrt{a(1-b)}} + \frac{1}{\sqrt{(2+a-b)^3}}\arctan\sqrt{\frac{a(1-b)}{2+a-b}}\right], \quad (4.20)$$

and integrating further (4.20) over $G \in db$, we obtain:

$$\mathbb{P}(F \in da) = \frac{1}{\pi}\sqrt{\frac{2-a}{a}}da \quad \text{for } 0 < a < 1,$$

which agrees with the formula (4.12) found in Proposition 2.1.

In the rest of the subsection, we give yet another proof of Theorem 4.2.1. We start by deriving the formula (4.9) from the absolute continuity relation (4.11).

Proof of (4.9) **by** (4.11)**:** We first compute the $\mathbb{P}^{\widetilde{\mathbf{W}}}$-joint distribution of (w_0, w_1, F, G), where $\mathbb{P}^{\widetilde{\mathbf{W}}}$ is the distribution of $\widetilde{B}$ on $[0,1]$ defined as in (4.10).

$$\begin{aligned}
&\mathbb{P}^{\widetilde{\mathbf{W}}}(w_0 \in dx, w_1 \in dy, F \in da, G \in db)\\
&= \frac{dx}{\sqrt{4\pi}}\exp\left(-\frac{x^2}{4}\right)\cdot \mathbb{P}^{\widetilde{\mathbf{W}}}(w_1 \in dy, F \in da, G \in db | w_0 \in dx)\\
&= \frac{dx}{\sqrt{4\pi}}\exp\left(-\frac{x^2}{4}\right)\cdot \frac{|x|da}{\sqrt{4\pi a^3}}\exp\left(-\frac{x^2}{4a}\right)\cdot \mathbb{P}^{\widetilde{\mathbf{W}}}(w_1 \in dy, G \in db | w_0 \in dx, F \in da)\\
&= \frac{dx}{\sqrt{4\pi}}\exp\left(-\frac{x^2}{4}\right)\cdot \frac{|x|da}{\sqrt{4\pi a^3}}\exp\left(-\frac{x^2}{4a}\right)\cdot \frac{|y|dydb}{4\pi\sqrt{(b-a)(1-b)^3}}\exp\left(-\frac{y^2}{4(1-b)}\right) \quad (4.21)\\
&= \frac{|xy|}{16\pi^2\sqrt{(b-a)a^3(1-b)^3}}\exp\left(-\frac{x^2}{4} - \frac{x^2}{4a} - \frac{y^2}{4(1-b)}\right) dx\, dy\, da\, db, \quad (4.22)
\end{aligned}$$

where (4.21) can be read from Revuz and Yor [130, Exercise 3.23, Chapter III]. Now (4.22) combined with (4.11) yields the desired result. □

We need the following elementary result regarding the change of measures.

Lemma 4.4.3 *Assume that $\mathbb{P}$ and $\mathbb{Q}$ are two probability measures on $(\Omega, \mathcal{F})$ such that*

$$\frac{d\mathbb{Q}}{d\mathbb{P}}(w) := f(Z),$$

where $Z := Z(w)$ is a random element and $f(Z)$ is the Radon-Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$. Futhermore,

(1). *Let $A \in \sigma(Z)$ be an event determined by Z, with $\mathbb{P}(A) > 0$ and $\mathbb{Q}(A) > 0$;*

(2). *Let Y be another random element such that under $\mathbb{P}$, Y is independent of Z given A (such random element Y need only be defined conditional on A).*

Then the $\mathbb{Q}$-distribution of Y given A is the same as the $\mathbb{P}$-distribution of Y given A. And under $\mathbb{Q}$, Y is independent of Z given A.

Proof: Take $g, h : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ two bounded measurable functions. First note that

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}[g(Y)|A] &= \frac{\mathbb{P}(A)}{\mathbb{Q}(A)}\mathbb{E}^{\mathbb{P}}[g(Y)f(Z)|A] \\ &= \frac{\mathbb{P}(A)}{\mathbb{Q}(A)}\mathbb{E}^{\mathbb{P}}[f(Z)|A] \cdot \mathbb{E}^{\mathbb{P}}[g(Y)|A] && (4.23) \\ &= \mathbb{E}^{\mathbb{P}}[g(Y)|A], && (4.24)\end{aligned}$$

where (4.23) is due to the $\mathbb{P}$-conditional independence of Y and Z given A. In addition,

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}[g(Y)h(Z)|A] &= \frac{\mathbb{P}(A)}{\mathbb{Q}(A)}\mathbb{E}^{\mathbb{P}}[g(Y)h(Z)f(Z)|A] \\ &= \frac{\mathbb{P}(A)}{\mathbb{Q}(A)}\mathbb{E}^{\mathbb{P}}[h(Z)f(Z)|A] \cdot \mathbb{E}^{\mathbb{P}}[g(Y)|A] && (4.25) \\ &= \mathbb{E}^{\mathbb{Q}}[h(Z)|A] \cdot \mathbb{E}^{\mathbb{Q}}[g(Y)|A], && (4.26)\end{aligned}$$

where (4.25) is again due to the $\mathbb{P}$-conditional independence of Y and Z given A, and (4.26) follows readily from (4.24). □

Proof of Theorem 4.2.1: We borrow the notations from Lemma 4.4.3 in our setting: $\mathbb{P} := \mathbb{P}^{\widetilde{\mathbf{W}}}$, $\mathbb{Q} := \mathbb{P}^{\mathbf{S}}$, $Z := (w_0, w_1, F, G)$ and $A := \{0 < F < G < 1\}$. Conditional on A, define $Y^{(2)}$ to be the scaled bridge on $[F, G]$, that is

$$Y_u^{(2)} := \frac{w_{F+u(G-F)}}{\sqrt{G-F}} \quad \text{for } 0 \leq u \leq 1.$$

It is well-known that under $\mathbb{P}^{\widetilde{\mathbf{W}}}$ and on the event $\{0 < F < G < 1\}$, $(Y_u^{(2)}/\sqrt{2}; 0 \leq u \leq 1)$ is standard Brownian bridge, independent of (w_0, w_1, F, G), see e.g. Revuz and Yor [130, Exercise 3.8, Chapter XII]. Then by Lemma 4.4.3, under $\mathbb{P}^{\mathbf{S}}$ and on the event $\{0 < F < G < 1\}$, $(Y_u^{(2)}/\sqrt{2}; 0 \leq u \leq 1)$ is also standard Brownian bridge, independent of (w_0, w_1, F, G). In addition, define $Y^{(1)} := (w_u; 0 \leq u \leq F)$ and $Y^{(3)} := (w_u; G \leq u \leq 1)$. Similarly, under $\mathbb{P}^{\widetilde{\mathbf{W}}}$ and on the event $\{0 < F < G < 1\}$,

- $Y^{(1)}/\sqrt{2}$ is Brownian first passage bridge from $(0, w_0/\sqrt{2})$ to $(F, 0)$, see e.g. Bertoin et al. [14];
- $|Y^{(3)}|/\sqrt{2}$ is reversed Brownian first passage bridge from $(1, |w_1|/\sqrt{2})$ to $(G, 0)$, that is the three-dimensional Bessel bridge from $(G, 0)$ to $(1, |w_1|/\sqrt{2})$, see e.g. Biane and Yor [16].

Moreover, these two processes are conditionally independent given (w_0, w_1, F, G). It suffices to apply again Lemma 4.4.3 to conclude. □

In view of the Brownian characteristics,it would be interesting to find a construction of the conditioned Slepian process $(S_t/\sqrt{2}; 0 \leq t \leq 1)$ with $\{0 < F < G < 1\}$ by some path transformation of standard Brownian motion/bridge. We leave the interpretation open for future investigation.

4.5 A Palm-Itô measure related to Slepian zeros

To capture the structure of the Slepian zero set, an alternative way is to study the Slepian excursions between consecutive zeros. Let E be the space of excursions defined by

$$E := \{\epsilon \in \mathcal{C}[0,\infty); \epsilon_0 = 0 \text{ and } \epsilon_t = 0 \text{ for all } t \geq \zeta(\epsilon) \in]0,\infty[\},$$

where $\zeta(\epsilon) := \inf\{t > 0; \epsilon_t = 0\}$ is the lifetime of the excursion $\epsilon \in E$. Following Pitman [122], the gaps between zeros of a process $(Z_t; t \geq 0)$ with σ-finite invariant measure can be described by a *Palm-Itô measure* $\mathbf{n}^Z$, defined on the space of excursions E as

$$\mathbf{n}^Z(d\epsilon) := \mathbb{E}\#\{0 < t < 1; Z_t = 0 \text{ and } e(t) \in d\epsilon\},$$

where $e(t)$ is the excursion starting at time $t > 0$ in the process Z. The following result of *last exit decomposition* for stationary processes is read from Pitman [122, Theorem 1(iii)].

Theorem 4.5.1 *[122] Let $\mathbb{P}^{\boldsymbol{Z}}$ govern a stationary process $(Z_t; t \geq 0)$ with σ-finite invariant measure. For $w \in \mathcal{C}[0,1]$, let*

$$G_t := \sup\{u \leq t; w_u = 0\}, \tag{4.27}$$

be the last exit time from 0 before time t, and $e(G_t)$ be the excursion straddling time $t > 0$ in the path. Then

$$\mathbb{P}^{\boldsymbol{Z}}(t - G_t \in da, e(G_t) \in d\epsilon) = da 1(\zeta(\epsilon) > a)\boldsymbol{n}^Z(d\epsilon), \tag{4.28}$$

Remark 4.5.2

1. Theorem 4.5.1 extends a result of Bismut [18], where Z is Brownian motion with invariant Lebesgue measure. In this case, the Palm-Itô measure $\mathbf{n}^Z$ is just Itô's excursion law $\mathbf{n}$.

2. There is an analog of last exit decomposition (4.28) for standard Brownian motion. Let $\mathbb{P}^{\mathbf{W}}$ be Wiener measure on $\mathcal{C}[0,\infty)$, then

$$\mathbb{P}^{\mathbf{W}}(t - G_t \in da, e(G_t) \in d\epsilon) = da \frac{1}{\sqrt{2\pi(t-a)}} 1(\zeta(\epsilon) > a)\mathbf{n}(d\epsilon),$$

   where $\mathbf{n}$ is Itô's excursion law. The result is deduced from Getoor and Sharpe [58], who gave a last exit decomposition for general Markov processes.

Now we apply Theorem 4.5.1 to the Slepian process $(S_t; t \geq 0)$. Let $\mathbb{P}^{\mathbf{S}}$ be the distribution of the Slepian process, we have:

$$\mathbb{P}^{\mathbf{S}}(t - G_t \in da, e(G_t) \in d\epsilon) = da 1(\zeta(\epsilon) > a)\mathbf{n}^S(d\epsilon), \tag{4.29}$$

where

$$\mathbf{n}^S(d\epsilon) := \mathbb{E}\#\{0 < t < 1; S_t = 0 \text{ and } e(t) \in d\epsilon\}. \tag{4.30}$$

As shown in the following lemma, the last exit time G_t is closely related to the first passage time F defined as in (4.4). Here we adopt the convention that $\sup\emptyset := 0$.

Lemma 4.5.3 *Let $t > 0$. Under $\mathbb{P}^{\boldsymbol{S}}$, $t - G_t$ has the same distribution as $F \cdot 1(F \leq t) + t \cdot 1(F > t)$, where G_t is defined by* (4.27) *and F by* (4.4)*.*

Proof: It suffices to observe that $(S_u; 0 \leq u \leq t)$ has the same distribution as $(S_{t-u}; 0 \leq u \leq t)$. This is clear from the covariance function of the Slepian process. □

Proposition 4.5.4 *For $a > 0$,*

$$\boldsymbol{n}^S(\zeta > a) = \mathbb{P}(F \in da)/da, \tag{4.31}$$

where $\boldsymbol{n}^S$ is defined as in (4.30) *and F is defined as in* (4.4)*. In particular,*

$$\boldsymbol{n}^S(\zeta > a) = \frac{1}{\pi}\sqrt{\frac{2-a}{a}} \quad \text{for } 0 < a < 1. \tag{4.32}$$

Proof: It follows from (4.28) that

$$\mathbf{n}^S(\zeta > a)da = \mathbb{P}(t - G_t \in da).$$

According to Lemma 4.5.3,

$$\mathbb{P}(t - G_t \in da) = \mathbb{P}(F \in da) \quad \text{for } t > a.$$

Then (4.31) is a direct consequence of the above two observations. Combining with the formula (4.12), we derive further (4.32). □

From the *Palm-Lévy measure* (4.32), we can see how the Slepian zero set restricted to $[0, 1]$ differs from a plain Brownian zero set, where *Itô's excursion law* is given by

$$\mathbf{n}(\zeta > a) = \sqrt{\frac{2}{\pi a}} \quad \text{for } a > 0.$$

As expected, $\mathbf{n}^S(\zeta > a)$ and $\mathbf{n}(\zeta > a)$ have asymptotically equivalent tails $a^{-\frac{1}{2}}$ when $a \to 0^+$. Observe a constant factor $\sqrt{\pi}$ between them. This is because the invariant Lebesgue measure of reflected Brownian motion is σ-finite and there is no canonical normalization. We also refer readers to Pitman and Yor [126, Section 2] for the Palm-Lévy measure of the gaps between zeros of squared Ornstein-Uhlenbeck processes.

Chapter 5

Brownian bridge embedded in Brownian motion

In this chapter, we prove Theorem 1.0.4 (1); that is embedding Brownian bridge $(b_u^0; 0 \leq u \leq 1)$ into Brownian motion $(B_t; t \geq 0)$ by a random translation of origin in spacetime.

The problem is closely related to the notion of *shift-coupling*, initiated by Aldous and Thorisson [3], and Thorisson [145]. General results of shift-coupling were further developed by Thorisson [143, 146, 144], see also the book of Thorisson [142]. In the special cases of a family of i.i.d. Bernoulli random variables indexed by $\mathbb{Z}^d$ or a spatial Poisson process on $\mathbb{R}^d$, Liggett [101], and Holroyd and Liggett [66] provided an explicit construction of the random shift and computed the tail of its probability distribution. Two continuous processes $(Z_u; u \geq 0)$ and $(Z'_u; u \geq 0)$ are said to be shift-coupled if there are random times $T, T' \geq 0$ such that $(Z_{T+u}; u \geq 0)$ has the same distribution as $(Z'_{T'+u}; u \geq 0)$. From Theorem 1.0.4 (1), we know that $Z := X$, the moving-window process can be shift-coupled with some $\mathcal{C}_0[0,1]$-valued process Z' starting at $Z'_0 := b^0$ for random times $T \geq 0$ and $T' = 0$.

Recently, Hammond et al. [65] constructed local times on the exceptional times of two dimensional dynamical percolation. Further, they showed that at a typical time with respect to local times, the percolation configuration has the same distribution as *Kesten's Incipient Infinite Cluster* [77]. They also made use of Palm theory and the idea was similar in spirit to ours, though the framework is completely different.

Recall that $(X_t; t \geq 0)$ is the moving-window process associated to Brownian motion defined as in (4.2). We aim to find a random time $T \geq 0$ such that X_T has the same distribution as b^0. A general result of Rost [132] implies that such a randomized stopping time $T \geq 0$ exists (relative to the filtration of the moving window process, so $T+1$ would be a randomized stopping time in the Brownian filtration) if and only if

$$\lim_{\alpha \to 0} \sup_{1 \geq g \in \mathcal{S}^\alpha} \left(\int g d\mathbb{P}^{\mathbf{W}^0} - \int g d\mathbb{P}^{\mathbf{W}} \right) = 0, \tag{5.1}$$

where $\mathbb{P}^{\mathbf{W}}$ is Wiener measure on $\mathcal{C}_0[0,1]$ and $\mathbb{P}^{\mathbf{W}^0}$ is Wiener measure pinned to 0 at time 1, that is the distribution of Brownian bridge b^0. For $\alpha > 0$, $\mathcal{S}^\alpha$ is the set of $\alpha-$excessive functions, see e.g. the book of Sharpe [134] for background. However, the criterion (5.1) is difficult to check since $\mathbb{P}^{\mathbf{W}^0}$ is singular with respect to $\mathbb{P}^{\mathbf{W}}$.

We work around the problem in another way, which relies heavily on Palm theory of stationary random measures. Such theory has been successfully developed by the Scandinavian probability school in the last few decades. The book of Thorisson [142] records much of this important work. For technical purposes, we introduce a two-sided Brownian motion $(\widehat{B}_t; t \in \mathbb{R})$, and let

$$\widehat{X}_t := (\widehat{B}_{t+u} - \widehat{B}_t; 0 \leq u \leq 1) \quad \text{for } t \in \mathbb{R}, \tag{5.2}$$

be the moving-window process associated to the two-sided Brownian motion $\widehat{B}$. Note that $(\widehat{X}_t; t \in \mathbb{R})$ is a stationary Markov process with state space $(\mathcal{C}_0[0,1], \mathcal{B})$. Alternatively, $\widehat{X}$ can be viewed as a random element in the space $\mathcal{C}_0[0,1]^{\mathbb{R}}$, to which we assign the metric ρ by

$$\rho(x, y) := \sum_{n=0}^{\infty} \frac{1}{2^n} \min\left(\sup_{-n \leq t \leq n} ||x_t - y_t||_{\infty}, 1 \right) \quad \text{for } x, y \in \mathcal{C}_0[0,1]^{\mathbb{R}} \tag{5.3}$$

where $\mathcal{C}_0[0,1]$ is equipped with the sup-norm $|| \cdot ||_{\infty}$.

Below is the plan for this section:

In Section 5.1, we provide background on Palm theory of stationary random measures. We define a notion of local times of the $\mathcal{C}_0[0,1]-$valued process $\widehat{X}$ by weak approximation. Furthermore, we show that the 0-marginal of the Palm measure of local times is Brownian bridge.

In Section 5.2, we derive from a result of Last and Thorisson [95] that the Palm probability measure of a jointly stationary random measure associated to $\widehat{X}$ can be obtained by a random time-shift of $\widehat{X}$ itself. In particular, there exists a random time $\widehat{T} \in \mathbb{R}$ such that $\widehat{X}_{\widehat{T}}$ has the same distribution as $(b_u^0; 0 \leq u \leq 1)$.

In Section 5.3, we prove that if some distribution on $\mathcal{C}_0[0,1]$ can be achieved in the moving-window process $\widehat{X}$ associated to two-sided Brownian motion, then we are able to construct a random time $T \geq 0$ such that $\widehat{X}_T$ has that desired distribution. Theorem 1.0.4 (1) follows immediately from the above observations.

In Section 5.4, after presenting some results of Last et al. [94], we construct a random time $T \geq 0$ such that $\widehat{X}_T$ has the same distribution as $(b_u^0; 0 \leq u \leq 1)$. The construction also makes use of the local times defined in Section 5.1. The argument is due to Hermann Thorisson.

5.1 Local times of $\widehat{X}$ and its Palm measure

In this section, we present background on Palm theory of stationary random measures. To begin with, $(\Omega, \mathcal{F}, \mathbb{P})$ is a generic probability space on which random elements are defined.

Let $(Z_t; t \in \mathbb{R}) \in E^{\mathbb{R}}$ be a continuous-time process with a measurable state space $(E, \mathcal{E})$. We further assume that the process Z is path-measurable, that is $E^{\mathbb{R}} \times \mathbb{R} \ni (Z, t) \to Z_t \in E$

is measurable for all $t \in \mathbb{R}$. See e.g. Appendix of Thorisson [141] for more on path-measurability. Let ξ be a random $\sigma-$finite measure on $\mathbb{R}$.

Assume that the pair (Z, ξ) is jointly stationary, that is

$$\theta_s(Z,\xi) \overset{(d)}{=} (Z,\xi) \quad \text{for all } s \in \mathbb{R}, \tag{5.4}$$

where $\theta_s Z := (Z_{s+t}; t \in \mathbb{R})$ and $\theta_s \xi(\cdot) := \xi(\cdot + s)$ are usual time-shift operations. Then the *Palm measure* $\mathbb{P}_{Z,\xi}$ of the jointly stationary pair (Z, ξ) is defined as follows: for $f : E^{\mathbb{R}} \times \mathcal{M}(\mathbb{R}) \to \mathbb{R}$ bounded measurable,

$$\mathbb{P}_{Z,\xi} f := \mathbb{E} \int_0^1 f(\theta_t(Z,\xi))\xi(dt).$$

In the rest of the work, we only need to care about the $E^{\mathbb{R}}$-marginal of $\mathbb{P}_{Z,\xi}$. That is, for $f : E^{\mathbb{R}} \to \mathbb{R}$ bounded measurable,

$$\mathbb{P}_{\xi} f := \mathbb{E} \int_0^1 f(\theta_t Z)\xi(dt). \tag{5.5}$$

By abuse of language, we call $\mathbb{P}_\xi$ defined by (5.5) the *Palm measure* of the stationary random measure ξ. Thus, $\mathbb{P}_\xi$ is a $\sigma-$finite measure on $E^{\mathbb{R}}$. If $\mathbb{P}_\xi 1 = \mathbb{E}\xi[0,1) < \infty$, then the normalized measure $\mathbb{P}_\xi/\mathbb{P}_\xi 1$ is called the *Palm probability measure* of ξ. So far most of the results have been established for $\mathbb{P}_{Z,\xi}$, but they still hold for the marginal $\mathbb{P}_\xi$. We refer readers to Kallenberg [75, Chapter 11], Thorisson [142, Chapter 8], Last [92, 93] and the thesis of Gentner [54] for further development on Palm versions of stationary random measures.

In the sequel, we adapt our problem setting to the above abstract framework. We take the state space $E := \mathcal{C}_0[0,1]$ equipped with its Borel $\sigma-$field $\mathcal{B}$. Recall that $(\widehat{X}_t; t \in \mathbb{R})$, the moving-window process defined as in (5.2), is a random element in the metric space $(\mathcal{C}_0[0,1]^{\mathbb{R}}, \rho)$ with the distance ρ defined by (5.3).

For a Borel measurable set $C \subset \mathbb{R}$, let

$$\mathcal{BR}^C := \{w \in \mathcal{C}_0[0,1]; w(1) \in C\}$$

be the set of bridge paths with endpoint in C. By stationarity of $(\widehat{X}_t; t \in \mathbb{R})$, for each fixed $t \in \mathbb{R}$, $\{u \in [t, t+1]; \widehat{X}_u \in \mathcal{BR}^0\} - \{t\}$ has the same distribution as $\{u \in [0,1]; \widehat{X}_u \in \mathcal{BR}^0\}$, which is mutually absolutely continuous relative to that of $\{u \in [0,1]; \widetilde{B}_u = 0\}$ by Lemma 4.3.3. Here $\widetilde{B}$ is the modified Brownian motion as in (2.27). Inspired from the notion of Brownian local times, we define a random $\sigma-$finite measure Γ on $\mathbb{R}$ as follows: for $n \in \mathbb{N}$ and $C \subset \mathbb{R}$,

$$\Gamma([-n,n] \cap C) := \lim_{\epsilon \to 0} \sqrt{\frac{\pi}{2}} \frac{1}{\epsilon} \int_{[-n,n]\cap C} 1(\widehat{X}_u \in \mathcal{BR}^{[-\epsilon,\epsilon]}) du. \tag{5.6}$$

Let us justify that the random measure Γ as in (5.6) is well-defined. Write $C = \cup_{k \in \mathbb{Z}} C_k$ where $C_k := C \cap [k, k+1]$. We want to show that for each $k \in \mathbb{Z}$,

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{C_k} 1(\widehat{X}_u \in \mathcal{BR}^{[-\epsilon,\epsilon]}) du \quad \text{is well-defined almost surely.}$$

The following lemma is quite straightforward, the proof of which is omitted.

Lemma 5.1.1 *Assume that two random sequences* $(Y_\epsilon)_{\epsilon>0}$ *and* $(Y'_\epsilon)_{\epsilon>0}$ *with a measurable state space* $(G, \mathcal{G})$ *have the same distribution. If* $f : G \to \mathbb{R}$ *is a measurable function satisfying that* $f(Y_\epsilon)$ *converges almost surely as* $\epsilon \to 0$*, then* $f(Y'_\epsilon)$ *converges almost surely as* $\epsilon \to 0$

Observe that for each fixed $k \in \mathbb{Z}$, $\{u \in [k, k+1]; \widehat{X}_u \in \mathcal{BR}^{[-\epsilon,\epsilon]}\}$ has the same distribution as $\{u \in [0,1]; S_u \in [-\epsilon, \epsilon]\} + \{k\}$ where $(S_u; u \geq 0)$ is the Slepian process. By Lemma 5.1.1, it suffices to prove that for each Borel measurable set $C' \subset [0,1]$,

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{C'} 1(S_u \in [-\epsilon, \epsilon]) du \quad \text{is well-defined almost surely.}$$

And this is quite clear from the path decomposition of the Slepian process on $[0,1]$, Theorem 4.2.1. We refer readers to Revuz and Yor [130, Chapter VI] for the existence of Brownian local times by approximation. Now for $n \in \mathbb{N}$,

$$\frac{1}{\epsilon} \int_{[-n,n] \cap C} 1(\widehat{X}_s \in \mathcal{BR}^{[-\epsilon,\epsilon]}) ds = \sum_{k=-n}^{n-1} \frac{1}{\epsilon} \int_{C_k} 1(\widehat{X}_s \in \mathcal{BR}^{[-\epsilon,\epsilon]}) ds$$

converges almost surely as $\epsilon \to 0$.

The random measure Γ defined by (5.6) can be interpreted as the local times of the moving-window process $\widehat{X}$ at the level $\mathcal{BR}^0$. Note that the pair $(\widehat{X}, \Gamma)$ is jointly stationary in the sense of (5.4). Next, we compute explicitly the 0-marginal of the Palm measure of the local times Γ:

Proposition 5.1.2 *Let* $\Pi_0 : \mathcal{C}_0[0,1]^{\mathbb{R}} \ni w \to w_0 \in \mathcal{C}_0[0,1]$ *be the* $0-$*marginal projection. Then the image by* Π_0 *of the* Palm probability measure *of* Γ *as in* (5.6) *is*

$$\mathbb{P}_\Gamma \circ \Pi_0^{-1} = \mathbb{P}^{\boldsymbol{W}^0},$$

where $\mathbb{P}^{\boldsymbol{W}^0}$ *is Wiener measure pinned to* 0 *at time* 1*, that is the distribution of a standard Brownian bridge.*

Proof: Take $f : \mathcal{C}_0[0,1] \to \mathbb{R}$ bounded continuous. By injecting (5.6) into (5.5), we obtain:

$$\begin{aligned}
\mathbb{P}_\Gamma \circ \Pi_0^{-1} f &= \lim_{\epsilon \to 0} \sqrt{\frac{\pi}{2}} \frac{1}{\epsilon} \int_0^1 \mathbb{E}[f(\widehat{X}_t) 1(\widehat{X}_t \in \mathcal{BR}^{[-\epsilon,\epsilon]})] dt \\
&= \lim_{\epsilon \to 0} \sqrt{\frac{\pi}{2}} \frac{1}{\epsilon} \mathbb{E}[f(\widehat{X}_0) 1(\widehat{X}_0 \in \mathcal{BR}^{[-\epsilon,\epsilon]})] \qquad (5.7) \\
&= \lim_{\epsilon \to 0} \sqrt{\frac{\pi}{2}} \frac{1}{\epsilon} \mathbb{E}^{\mathbf{W}}[f(w) 1(w_1 \in [-\epsilon, \epsilon])] \\
&= \lim_{\epsilon \to 0} \mathbb{E}^{\mathbf{W}}[f(w) | w_1 \in [-\epsilon, \epsilon]] \\
&= \mathbb{P}^{\mathbf{W}^0} f, \qquad (5.8)
\end{aligned}$$

where $\mathbb{P}^{\mathbf{W}}$ is Wiener measure on $\mathcal{C}_0[0,1]$. The equality (5.7) is due to stationarity of the moving-window process $\widehat{X}$, and the equality (5.8) follows from the weak convergence to Brownian bridge of Brownian motion, see e.g. Billingsley [17, Section 11]. □

Remark 5.1.3 The measure $P_\Gamma \circ \Pi_0^{-1}$ defined in Proposition 5.1.2 is closely related to the notion of *Revuz measure* of Markov additive functionals. Note that for $s \in \mathbb{R}$ and $t \geq 0$, $\Gamma[s, s+t] = \Gamma[0,t] \circ \theta_s$, i.e. Γ induces a continuous additive functional of the moving-window process $\widehat{X}$.[1] Since $(\widehat{X}_t; t \in \mathbb{R})$ is stationary with respect to $\mathbb{P}^{\mathbf{W}}$,

$$\mathbb{P}_\Gamma \circ \Pi_0^{-1} f := \mathbb{E} \int_0^1 f(\widehat{X}_t)\Gamma(dt) \quad \text{for } f : \mathcal{C}[0,1] \to \mathbb{R} \text{ bounded measurable,}$$

can be viewed as Revuz measure of Γ in the two-sided setting. For further discussions on Revuz measure of additive functionals, we refer readers to Revuz [129], Fukushima [52], and Fitzsimmons and Getoor [47] among others.

5.2 Brownian bridge in two-sided Brownian motion

In this paragraph, we show that there exists a random time $\widehat{T} \in \mathbb{R}$ such that $(\widehat{B}_{\widehat{T}+u} - \widehat{B}_{\widehat{T}}; 0 \leq u \leq 1)$ has the same distribution as Brownian bridge b^0. In terms of the moving-window process $\widehat{X}$, we show that

Proposition 5.2.1 *There exists a random time $\widehat{T} \in \mathbb{R}$ such that $\widehat{X}_{\widehat{T}}$ has the same distribution as $(b^0_u; 0 \leq u \leq 1)$*

As mentioned in the introduction, our proof relies on a recent result of Last and Thorisson [95], which establishes a dual relation between stationary random measures and mass-stationary ones in the Euclidian space. We refer readers to Last and Thorisson [96, 97] for the notion of mass-stationarity, which is an analog to point-stationarity of random point processes.

Before proceeding further, we need the following notations. Recall that $(Z_t; t \in \mathbb{R})$ is a path-measurable process with a state space $(E, \mathcal{E})$ such that $(E^{\mathbb{R}}, \mathcal{E}^{\mathbb{R}})$ is time-invariant, and ξ is a random $\sigma-$finite measure on $\mathbb{R}$. Let N_ξ be a simple point process on $\mathbb{Z}$ such that

$$i \in N_\xi \Longleftrightarrow \xi(i + [0,1)) > 0.$$

Next we associate each $j \in \mathbb{Z}$ to the point of N_ξ that is closest to j, choosing the smaller one when there are two such points. Then we obtain a countable number of sets, each of which contains exactly one point of N_ξ. Let D be the set that contains 0, and S be the vector from $N_\xi-$point in the set D to 0.

[1] This generalizes the definition of continuous additive functionals of one-sided Markov processes, see e.g. the book of Sharpe [134, Chapter IV] and the survey paper of Getoor [56] for background.

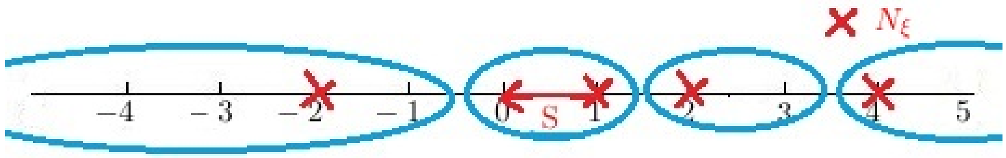


Fig 7. Decomposition of $\mathbb{Z}$ induced by the simple point process N_ξ.

The next result is read from Last and Thorisson [95, Theorem 2]:

Theorem 5.2.2 *[95] Assume that* (1). *the pair* (Z, ξ) *is stationary,* (2). $\mathbb{E}\xi[0,1) < \infty$ *and* (3). $\operatorname{conv}\operatorname{supp}\xi = \mathbb{R}$ *a.s. where* $\operatorname{conv}\operatorname{supp}\xi$ *is the convex hull of the support of* ξ*. Define*

$$Z^0 := \theta_{T-S} Z,$$

where the conditional distribution of $T \in [0,1)$ *given* (Z, ξ) *is*

$$\theta_{-S}\xi(\cdot|[0,1)) := \frac{\theta_{-S}\xi(\cdot \cap [0,1))}{\theta_{-S}\xi([0,1))}.$$

Define

$$d\mathbb{P}^0 := \frac{\theta_{-S}\xi[0,1)}{\#D \cdot \mathbb{E}\xi[0,1)} d\mathbb{P},$$

where $\#D$ *is the cardinality of the set* D*. Then* Z^0 *under* $\mathbb{P}^0$ *is the* Palm probability measure *of* ξ*.*

Thorisson [143, 144] presented a duality between stationary point processes and point-stationary ones in the Euclidian space. In particular, the stationary point process and its (modified) Palm version are the same with some random time-shift. Thus Theorem 5.2.2 is regarded as a generalization of those results in the random diffuse measure setting.

Now we apply Theorem 5.2.2 to $Z := \widehat{X}$ the moving-window process as in (5.2), and $\xi := \Gamma$ the local times as in (5.6). It is straightforward that all assumptions in Theorem 5.2.2 are satisfied. This leads to:

Corollary 5.2.3 *There exists a random time* $T \in \mathbb{R}$ *such that the* Palm probability measure *of* Γ*, i.e.* $\mathbb{P}_\Gamma/\mathbb{P}_\Gamma 1$ *is absolutely continuous with respect to the distribution of* $\theta_T \widehat{X}$*.*

By Proposition 5.1.2, the 0-marginal of the Palm probability measure of Γ is Brownian bridge. If we can show that the Palm probability measure $\mathbb{P}_\Gamma/\mathbb{P}_\Gamma 1$ is achieved by $\theta_{\widehat{T}}\widehat{X}$ for a random time $\widehat{T} \in \mathbb{R}$, then Proposition 5.2.1 follows as a consequence. To this end, we state a general result, the proof of which is deferred.

Theorem 5.2.4 *Let* $(Z_t; t \in \mathbb{R})$ *be a path-measurable process with a state space* $(E, \mathcal{E})$*. Assume that*

1. *Z is ergodic under the time-shift group $(\theta_t; t \in \mathbb{R})$, that is*

$$\mathbb{P}(Z \in H) = 0 \text{ or } 1 \quad \text{for all } H \in \mathcal{I} := \{H' \in \mathcal{E}^{\mathbb{R}}; \theta_t H' = H' \text{ for all } t \in \mathbb{R}\};$$

2. *μ is a probability measure on $(E^{\mathbb{R}}, \mathcal{E}^{\mathbb{R}})$ absolutely continuous with respect to the distribution of $\theta_T Z$ for a random time $T \in \mathbb{R}$.*

Then there exists a random time $\widehat{T} \in \mathbb{R}$ such that $\theta_{\widehat{T}} Z$ is distributed as μ.

Proof of Proposition 5.2.1: We apply Theorem 5.2.4 to $Z := \widehat{X}$ the moving-window process and $\mu := \mathbb{P}_\Gamma / \mathbb{P}_\Gamma 1$ the Palm probability measure of Γ. Observe that the invariant σ-field $\mathcal{I} \subset \cap_{n \in \mathbb{N}} \theta_n^{-1} \mathcal{E}^{\mathbb{R}}$, the tail σ-field of $(\widehat{X}_n; n \in \mathbb{N})$ which are i.i.d. copies of Brownian motion on $[0,1]$. By Kolmogorov's zero-one law, $\mathcal{I}$ is trivial under the distribution of $\widehat{X}$ and the assumption (1) is satisfied. The assumption (2) follows from Corollary 5.2.3. Combining Theorem 5.2.4 and Proposition 5.1.2, we obtain the desired result. □

Remark 5.2.5 In ergodic theory, the process Z is said to be *θ-mixing* if

$$\sup\{\mathbb{P}(Z \in A \cap B) - \mathbb{P}(Z \in A)\mathbb{P}(Z \in B); t \in \mathbb{R}, A \in \mathcal{F}_t, B \in \mathcal{F}^{t+s}\} \to 0 \quad \text{as } s \to \infty,$$

where $\mathcal{F}_t := \sigma(Z_u; u \leq t)$ and $\mathcal{F}^{t+s} := \sigma(Z_u; u \geq t+s)$. See e.g. Bradley [22] for a survey on strong mixing conditions. It is quite straightforward that the moving-window process $\widehat{X}$ is θ-mixing, since $\widehat{X}_{t+l}$ and $\widehat{X}_t$ are independent for all $t \geq 0$ and $l \geq 1$. Consequently, $\widehat{X}$ is ergodic under time-shift $(\theta_t; t \in \mathbb{R})$. In Section 3.5, this notion of θ-mixing plays an important role in one-sided embedding out of two-sided processes.

In the rest of this part, we aim to prove Theorem 5.2.4. We need the following result of Thorisson [146], which provides a necessary and sufficient condition for two continuous-time processes being transformed from one to the other by a random time-shift.

Theorem 5.2.6 *[146] Let $(Z_t; t \in \mathbb{R})$ and $(Z'_t; t \in \mathbb{R})$ be two path-measurable processes with a state space $(E, \mathcal{E})$. Then there exists a random time $\widehat{T} \in \mathbb{R}$ such that $\theta_{\widehat{T}} Z$ has the same distribution as Z' if and only if the distributions of Z and Z' agree on the invariant $\sigma-$field $\mathcal{I}$.*

Proof of Theorem 5.2.4: We apply Theorem 5.2.6 to Z' distributed as μ. Since $(\theta_t; t \in \mathbb{R})$ is ergodic under the distribution of Z,

$$\mathbb{P}(Z \in H) = 0 \text{ or } 1 \quad \text{for all } H \in \mathcal{I}.$$

If $\mathbb{P}(Z \in H) = 0$ for $H \in \mathcal{I}$, then $\mathbb{P}(\theta_T Z \in H) = \mathbb{P}(Z \in \theta_{-T} H) = \mathbb{P}(Z \in H) = 0$. By the absolute continuity between the distribution μ and that of $\theta_T Z$, we have $\mu(H) = 0$. By applying the same argument to the complement of H, $\mathbb{P}(Z \in H) = 1$ for $H \in \mathcal{I}$ implies $\mu(H) = 1$. Thus, the probability distribution μ and that of Z agree on the invariant $\sigma-$field $\mathcal{I}$. Theorem 5.2.6 permits to conclude. □

5.3 From two-sided embedding to one-sided embedding

We explain here how to achieve a certain distribution on $\mathcal{C}_0[0,1]$ in Brownian motion by a random spacetime shift, once this has been done in two-sided Brownian motion. We aim to prove that:

Proposition 5.3.1 *Assume that μ is a probability measure on $(\mathcal{C}_0[0,1],\mathcal{B})$. If $\widehat{X}_{\widehat{T}}$ is distributed as μ for some random time $\widehat{T}\in\mathbb{R}$, then there exists a random time $T\geq 0$ such that $\widehat{X}_T$ is distributed as μ.*

It is not hard to see that Theorem 1.0.4 (1) follows readily from Corollary 5.2.1 and Proposition 5.3.1. In the sequel, we assume path-measurability for any continuous-time processes that are involved. Let $\mathcal{L}(\mathcal{X})$ be the distribution of any random element $\mathcal{X}$. To prove Proposition 5.3.1, we begin with a general statement.

Proposition 5.3.2 *Let $(Z_t;t\in\mathbb{R})$ be a stationary process and θ-mixing as in Remark 5.2.5. Assume that $Z_{\widehat{T}}$ is distributed as μ for some random time $\widehat{T}\in\mathbb{R}$. Given $\epsilon>0$ and $N\in\mathbb{N}$, there exist random times $0\leq T_1<\cdots<T_N$ on some event E_N of probability larger than $1-\epsilon$ such that*

$$||\mathcal{L}(Z_{T_1},\cdots,Z_{T_N}|E_N)-\mu^{\otimes N}||_{TV}\leq\epsilon,$$

where $||\cdot||_{TV}$ is the total variation norm of a measure.

Before proceeding the proof, we need the following lemma known as Blackwell-Dubins' merging of opinions [19]. In that paper, they only proved the result for discrete chains. But the argument can be easily adapted to the continuous setting. We rewrite Blackwell-Dubins' theorem for our own purpose, and leave full details to careful readers.

Lemma 5.3.3 *[19] Let $(Z_t;t\in\mathbb{R})$ and $(Z'_t;t\in\mathbb{R})$ be two path-measurable processes with a state space $(E,\mathcal{E})$. If the distribution of Z' is absolutely continuous with respect to Z, then*

$$||\mathcal{L}(Z'_{t+s};s\geq 0|\mathcal{F}'_t)-\mathcal{L}(Z_{t+s};s\geq 0|\mathcal{F}_t)||_{TV}\to 0 \quad \text{as } t\to\infty,$$

where $\mathcal{F}_t:=\sigma(Z_u;u\leq t)$ and $\mathcal{F}'_t:=\sigma(Z'_u;u\leq t)$.

Proof of Proposition 5.3.2: We proceed by induction over $N\in\mathbb{N}$. By stationarity of Z, for each $s\in\mathbb{R}$, $\theta_s Z$ has the same distribution as Z. Let $\widehat{T}_{\theta_s}$ be the random time constructed from $\theta_s Z$ just as $\widehat{T}$ is constructed from Z. Therefore, $(\theta_s Z)_{\widehat{T}_{\theta_s}}=Z_{\widehat{T}_{\theta_s}+s}$ is distributed as μ. Let $t\in\mathbb{R}$ be the $\frac{\epsilon}{2}$-quantile of $\widehat{T}$, that is $\mathbb{P}(\widehat{T}\geq t)\geq 1-\frac{\epsilon}{2}$. Define $T_1:=\widehat{T}_{\theta_{-t}}-t$. Observe that for $A\in\mathcal{E}$,

$$\mathbb{P}(Z_{T_1}\in A)-\frac{\epsilon}{2}\leq\mathbb{P}(Z_{T_1}\in A\text{ and }T_1\geq 0)\leq\mathbb{P}(Z_{T_1}\in A).$$

As a consequence, on the event $E_1 := \{T_1 \geq 0\}$ of probability larger than $1 - \frac{\epsilon}{2} > 1 - \epsilon$,

$$||\mathcal{L}(Z_{T_1}|E_1) - \mu||_{TV} \leq \frac{\epsilon/2}{1-\epsilon/2} < \epsilon.$$

Suppose that there exist $0 \leq T_1 < \cdots < T_N$ on some event E_N of probability larger than $1 - \frac{\epsilon}{4}$ such that

$$||\mathcal{L}(Z_{T_1}, \cdots, Z_{T_N}|E_N) - \mu^{\otimes N}||_{TV} \leq \frac{\epsilon}{4}. \tag{5.9}$$

Note that the distribution of the conditioned moving-window process $(Z_s; s \in \mathbb{R}|E_N)$ is absolutely continuous with respect to that of the original one $(Z_s; s \in \mathbb{R})$. By Lemma 5.3.3,

$$||\mathcal{L}(Z_{t+s}; s \geq 0|E_N, \mathcal{F}_t) - \mathcal{L}(Z_{t+s}; s \geq 0|\mathcal{F}_t)||_{TV} \to 0 \quad \text{as } t \to \infty. \tag{5.10}$$

By triangle inequality, we have for $t', t'' \geq 0$,

$$\begin{aligned}
&||\mathcal{L}(Z_{t'+t''+s}; s \geq 0|E_N) - \mathcal{L}(Z_s; s \geq 0)||_{TV} \\
&\quad \leq ||\mathcal{L}(Z_{t'+t''+s}; s \geq 0|E_N) - \mathcal{L}(Z_{t'+t''+s}; s \geq 0|E_N, \mathcal{F}_{t'})||_{TV} \\
&\qquad + ||\mathcal{L}(Z_{t'+t''+s}; s \geq 0|E_N, \mathcal{F}_{t'}) - \mathcal{L}(Z_{t'+t''+s}; s \geq 0|\mathcal{F}_{t'})||_{TV} \\
&\qquad\qquad + ||\mathcal{L}(Z_{t'+t''+s}; s \geq 0|\mathcal{F}_{t'}) - \mathcal{L}(Z_s; s \geq 0)||_{TV}.
\end{aligned} \tag{5.11}$$

By θ-mixing property, the first and the third term of (5.11) goes to 0 as $t'' \to \infty$. By (5.10), the second term of (5.11) goes to 0 as $t' \to \infty$. Therefore,

$$\lim_{t \to \infty} ||\mathcal{L}(Z_{t+s}; s \geq 0|E_N) - \mathcal{L}(Z_s; s \geq 0)||_{TV} = 0 \quad \text{as } t \to \infty. \tag{5.12}$$

Pick $t_N \geq 0$ such that $\mathbb{P}(T_N \geq t_N) \leq \frac{\epsilon}{8}$ and

$$||\mathcal{L}(Z_{t_N+s}; s \geq 0|E_N) - \mathcal{L}(Z_s; s \geq 0)||_{TV} \leq \frac{\epsilon}{8}.$$

By a similar argument as in the case of $N = 1$, there exists a random time $T_{N+1} \in \mathbb{R}$ such that $\mathbb{P}(T_{N+1} \geq t_N) \geq 1 - \frac{\epsilon}{8}$ and

$$||\mathcal{L}(Z_{T_{N+1}}|E_N \cap \{T_{N+1} \geq t_N\}) - \mu||_{TV} \leq \frac{\epsilon}{8} + \frac{\epsilon}{8} = \frac{\epsilon}{4}. \tag{5.13}$$

Let $E_{N+1} := E_N \cap \{T_{N+1} > T_N\}$. Since

$$\mathbb{P}(T_{N+1} > T_N) \geq \mathbb{P}(T_{N+1} \geq t_N) - \mathbb{P}(T_N > t_N) \geq 1 - \frac{\epsilon}{4},$$

we get $\mathbb{P}(E_{N+1}) \geq 1 - \frac{\epsilon}{2} > 1 - \epsilon$. By (5.9) and (5.13),

$$||\mathcal{L}(Z_{T_1}, \cdots, Z_{T_N}|E_{N+1}) - \mu^{\otimes N}||_{TV} \leq \frac{\epsilon}{2} \quad \text{and} \quad ||\mathcal{L}(Z_{T_{N+1}}|E_{N+1}) - \mu||_{TV} \leq \frac{\epsilon}{2}.$$

We obtain immediately that $||\mathcal{L}(Z_{T_1}, \cdots, Z_{T_{N+1}}|E_{N+1}) - \mu^{\otimes N+1}||_{TV} \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. □

By applying Lemma 5.3.3 in the first step, we deduce from Proposition 5.3.2 that:

Corollary 5.3.4 *Let $(Z_t; t \in \mathbb{R})$ be a stationary process and θ-mixing as in Remark 5.2.5. Assume that $Z_{\widehat{T}}$ is distributed as μ for some random time $\widehat{T} \in \mathbb{R}$. Given $\epsilon > 0$, $N \in \mathbb{N}$ and E_0 an event of positive probability, there exist random times $0 \leq T_1 < \cdots < T_N$ on some event E_N of probability larger than $1 - \epsilon$ such that*

$$||\mathcal{L}(Z_{T_1}, \cdots, Z_{T_N} | E_0, E_N) - \mu^{\otimes N}||_{TV} \leq \epsilon.$$

Now let us recall some elements of *von Neumann's acceptance-rejection algorithm* [150]. Assume that μ and ν are two probability measures such that the Radon-Nikodym derivative $f := \frac{d\nu}{d\mu}$ is essentially bounded under μ. Let $(Z_n)_{n \in \mathbb{N}} \sim \mu^{\otimes \mathbb{N}}$ be a sequence of i.i.d. random variables distributed as μ. Then

$$Z_T \sim \nu \quad \text{with } T := \inf \left\{ i \in \mathbb{N}; U_i \leq \frac{f(Z_i)}{\operatorname{ess\,sup} f} \right\},$$

where $(U_n)_{n \in \mathbb{N}}$ is a sequence of i.i.d. uniform-$[0, 1]$ random variables independent of $(Z_n)_{n \in \mathbb{N}}$. It is well-known that the total variation between the N^{th} updated distribution and the target one is of geometric decay, i.e.

$$||\mathcal{L}(Z_{T \wedge N}) - \nu||_{TV} \leq 2 \left(1 - \frac{1}{\operatorname{ess\,sup} f} \right)^N.$$

If the sample size N is large enough, a good portion of the target distribution ν can be sampled from $(Z_1, \cdots, Z_N) \sim \mu^{\otimes N}$ à la von Neumann. The following lemma is a slight extension of the above result to the quasi-i.i.d. case. The proof is quite standard, and thus is omitted.

Lemma 5.3.5 *Assume that $||\mathcal{L}(Z_1 \cdots Z_N) - \mu^{\otimes N}||_{TV} \leq \epsilon$ for some $\epsilon > 0$ and $N \in \mathbb{N}$. Then*

$$||\mathcal{L}(Z_{T_N}) - \nu||_{TV} \leq \epsilon + 2 \left(1 - \frac{1}{\operatorname{ess\,sup} f} \right)^N,$$

where $T_N := \inf\{i \leq N; U_i \leq f(Z_i)/\operatorname{ess\,sup} f\} \wedge N$.

Proof of Proposition 5.3.1: We use the same notation as in the proof of Proposition 5.3.2. Let $t \in \mathbb{R}$ be the $\frac{1}{2}$−quantile of $\widehat{T}$, and define $T_1 := \widehat{T}_{\theta_{-t}}$. By taking $T_1 \geq 0$ as the stopping rule, we obtain a $\frac{1}{2}$ portion of μ. The idea now is to get the remaining $\frac{1}{2}$ portion of μ by filling-type argument. Note that the target distribution $\mathcal{L}(\widehat{X}_{T_1} | T_1 < 0)$ is absolutely continuous with respect to μ with the Radon-Nikodym density

$$f_1 := \frac{d\mathcal{L}(\widehat{X}_{T_1} | T_1 < 0)}{d\mu},$$

which is bounded by 2. As indicated in Remark 5.2.5, the moving-window process $\widehat{X}$ is stationary and θ-*mixing*. We apply Corollary 5.3.4 to $(\widehat{X}_t; t \in \mathbb{R} | T_1 < 0)$: for any fixed

$N \in \mathbb{N}$, there exist random times $0 \leq T_1 < \cdots < T_N$ on some event E_N of probability larger than $\frac{3}{4}$ such that

$$||\mathcal{L}(\widehat{X}_{T_1}, \cdots, \widehat{X}_{T_N} | T_1 < 0, E_N) - \mu^{\otimes N}||_{TV} \leq \frac{1}{4}.$$

By Lemma 5.3.5, there is a random integer $M \leq N$ such that

$$||\mathcal{L}(\widehat{X}_{T_M} | T_1 < 0, E_N) - \mathcal{L}(\widehat{X}_{T_1} | T_1 < 0)||_{TV} \leq \frac{1}{4} + 2\left(1 - \frac{1}{\operatorname{ess\,sup} f_1}\right)^N.$$

By taking $N \in \mathbb{N}$ such that $(1 - 1/\operatorname{ess\,sup} f_1)^N \leq \frac{1}{8}$, we retrieve a $\frac{1}{2}$ portion of the targeted $\mathcal{L}(\widehat{X}_{T_1} | T_1 < 0)$. Restricted to the sub-probability space $\{T_1 < 0\}$, we obtain a remaining $\frac{1}{4}$ portion of μ. Repeat the algorithm, and we finally achieve the desired $(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots)$ distribution μ. $\square$

5.4 An explicit embedding of Brownian bridge into Brownian motion

In this section, we present a constructive proof of Theorem 1.0.4 (1) due to Hermann Thorisson. Recall that $\widehat{X}$ is the moving-window process associated to a two-sided Brownian motion.

Theorem 5.4.1 *[140] Let Γ be the random σ-finite measure defined as in (5.6). Define*

$$T := \inf\{t > 0; \Gamma[0, t] = t\}. \tag{5.14}$$

Then $T < \infty$ almost surely, and $\widehat{X}_T$ has the same distribution as a standard Brownian bridge $(b_u^0; 0 \leq u \leq 1)$.

To proceed further, we need the following notions regarding transports of random measures, initiated by Holroyd and Peres [67], and Last and Thorisson [96].

Definition 5.4.2 *[67, 96] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a generic probability space, equipped with a flow $(\theta_t; t \in \mathbb{R})$.*

1. *A measurable mapping $\tau : \Omega \times \mathbb{R} \to \mathbb{R}$ is called an allocation rule if*

$$\tau(\theta_t w, s - t) = \tau(w, s) - t \quad \textit{for } s, t \in \mathbb{R} \quad \mathbb{P} \textit{ a.s.}$$

2. *An allocation rule τ is said to balance two random measures ξ and η if*

$$\int_{\mathbb{R}} 1(\tau(s) \in \cdot)\xi(ds) = \eta \quad \mathbb{P} \textit{ a.s.}$$

Triggered by the work of Liggett [101], and Holroyd and Liggett [66] on transporting counting measures on $\mathbb{Z}^d$ to the Bernoulli random measure, allocation rules of counting measures on $\mathbb{Z}^d$ to an ergodic point process have received much attention, see e.g. Holroyd and Peres [67], Chatterjee et al. [26], and Last and Thorisson [96] among others. The following result of Last et al. [94, Theorem 5.1] gives a balancing allocation rule for diffuse random measures on the line.

Theorem 5.4.3 *[94] Let ξ and η be invariant orthogonal diffuse random measures on $\mathbb{R}$ with finite intensities. Assume that*

$$\mathbb{E}[\xi[0,1]|\mathcal{I}] = \mathbb{E}[\eta[0,1]|\mathcal{I}] \quad a.s., \tag{5.15}$$

where $\mathcal{I}$ is the invariant σ-field. Then the mapping

$$\tau(s) := \inf\{t > s; \xi[s,t] = \eta[s,t]\} \quad \textit{for all } s \in \mathbb{R}$$

is an allocation rule balancing ξ and η.

Corollary 5.4.4 *Let Γ be the random σ-finite measure defined as in* (5.6)*. Define*

$$T(s) := \inf\{t > s; \Gamma[s,t] = t - s\} \quad \textit{for all } s \in \mathbb{R}. \tag{5.16}$$

Then $(T(s); s \in \mathbb{R})$ is an allocation rule balancing the Lebesgue measure $\mathcal{L}^1$ on $\mathbb{R}$ and Γ.

Proof: We want to apply Theorem 5.4.3 to $\xi := \mathcal{L}^1$ and $\eta := \Gamma$. We need to check the conditions. First it is obvious that $\mathcal{L}^1$ and Γ are invariant diffuse measures on the real line. Note that the measure Γ is supported on the set $\{t \in \mathbb{R}; \widehat{X}_t \in \mathcal{BR}^0\}$ almost surely. The distribution of $\{t \in \mathbb{R}; \widehat{X}_t \in \mathcal{BR}^0\}$ is the same as that of $\{t \in \mathbb{R}; S_t = 0\}$, which has null Lebesgue measure almost surely. Therefore, the measures $\mathcal{L}^1$ and Γ are orthogonal. In the proof of Proposition 5.2.1, we know that $\mathcal{I}$ is trivial under the distribution of $\widehat{X}$. In addition, $\mathbb{E}\Gamma[0,1] = 1$ by the computation as in Proposition 5.1.2. Thus, we have the condition (5.15) in the case of $\xi := \mathcal{L}^1$ and $\eta := \Gamma$. $\square$

In terms of Palm measures, Last and Thorisson [96, Theorem 4.1] gave a necessary and sufficient condition for an allocation rule to balance two random measures. See also Last et al. [94, Theorem 2.1].

Theorem 5.4.5 *[96] Consider two random measures ξ and η on $\mathbb{R}$ and an allocation rule τ. Then τ balances ξ and η if and only if*

$$\mathbb{P}_\xi(\theta_{\tau(0)} w \in \cdot) = \mathbb{P}_\eta,$$

where $\mathbb{P}_\xi$ (resp. $\mathbb{P}_\eta$) is the Palm measure of ξ (resp. η).

Proof of Theorem 5.4.1: Applying Theorem 5.4.5 to $\xi := \mathcal{L}^1$, $\eta := \Gamma$ and $\tau := T$, we get:

$$\mathbb{P}_{\mathcal{L}^1}(\theta_T \widehat{X} \in \cdot) = \mathbb{P}_\Gamma.$$

where T is defined by (5.14). Note that for a stationary process, the Palm version of the Lebesgue measure is the stationary process itself. In particular, $\mathbb{P}_{\mathcal{L}^1}$ is the distribution of the moving-window process $\widehat{X}$. By Proposition 5.1.2, the 0-marginal of $\mathbb{P}_\Gamma$ is standard Brownian bridge. Theorem 5.4.1 follows readily from these facts.. □

Finally, let us derive a simple consequence of Theorem 5.4.1. Let $(b_u^{ps}; 0 \leq u \leq 1)$ be the *pseudo Brownian bridge* defined by

$$b_u^{ps} := \frac{B_{u\tau_1}}{\sqrt{\tau_1}} \quad \text{for all } 0 \leq u \leq 1,$$

where $\tau_1 := \inf\{t \geq 0; L_t > 1\}$ is the inverse local times of Brownian motion. Biane et al. [15] proved that the distribution of the pseudo Brownian bridge is mutually absolutely continuous relative to that of standard Brownian bridge. That is, for $f : \mathcal{C}_0[0,1] \to \mathbb{R}$ a bounded measurable function,

$$\mathbb{E}[f(b_u^{ps}; 0 \leq u \leq 1)] = \mathbb{E}\left[\sqrt{\frac{2}{\pi}} L_1^{br} f(b_u^0; 0 \leq u \leq 1)\right],$$

where L_1^{br} is the local time of Brownian bridge at level 0 up to time 1.

From Theorem 5.4.1, we are able to find a sequence of i.i.d. Brownian bridges by iteration of the construction (5.14). According to Section 3.5, we can apply Rost's filling scheme [24, 131] to sample distributions which are absolutely continuous relative to that of Brownian bridge. In particular,

Corollary 5.4.6 *There exists a random time $T \geq 0$ such that $(B_{T+u} - B_T; 0 \leq u \leq 1)$ has the same distribution as $(b_u^{ps}; 0 \leq u \leq 1)$.*

Chapter 6

Potential theory for continuous-time paths

In Question 1.0.5, we ask for any Borel measurable subset S of $\mathcal{C}_0[0,1]$ whether S is hit by the moving-window process $X_t := (B_{t+u} - B_t; 0 \leq u \leq 1)$ for $t \geq 0$, at some random time T. Related studies of the moving window process appear in several contexts. Knight [88, 89] introduced the prediction processes, where the whole past of the underlying process is tracked to anticipate its future behavior. The relation between Knight's prediction processes and our problems is discussed briefly at the end of the section. Similar ideas are found in stochastic control theory, where certain path-dependent stochastic differential equations were investigated, see e.g. the monograph of Mohammed [111] and Chang et al. [25]. More recently, Dupire [39] worked out a functional version of Itô's calculus, in which the underlying process is path-valued and notions as time-derivative and space-derivative with respect to a path, are proposed. We refer readers to the thesis of Fournié [49] as well as Cont and Fournié [29, 30, 31] for further development.

Indeed, Question 1.0.5 is some issue of potential theory. In Benjamini et al. [8] a potential theory was developed for transient Markov chains on any countable state space E. They showed that the probability for a transient chain to ever visit a given subset $S \subset E$, is estimated by $\text{Cap}_M(S)$ – the *Martin capacity* of the set S. See also Mörters and Peres [115, Section 8.3] for a detailed exposition.

As pointed out by Steven Evans [43], such a framework still works well for our discrete patterns. For $0 < \alpha < 1$, define the $\alpha-$potential of the discrete patterns/strings of length n as

$$\begin{aligned} G^{\alpha}(\epsilon', \epsilon'') :&= \sum_{k=0}^{\infty} \alpha^k P^k(\epsilon', \epsilon'') \\ &= \sum_{k=0}^{n-1} \left(\frac{\alpha}{2}\right)^k 1\{\sigma_k(\epsilon') = \tau_k(\epsilon'')\} + \frac{1}{1-\alpha}\left(\frac{\alpha}{2}\right)^k, \end{aligned}$$

where $\epsilon', \epsilon'' \in \{-1, 1\}^n$, and $P(\cdot, \cdot)$ is the transition kernel of discrete patterns/strings of length n in a simple random walk, and σ_k (resp. τ_k): $\{-1,1\}^n \to \{-1,1\}^{n-k}$ the restriction operator to the last $n-k$ strings (resp. to the first $n-k$ strings). The following result is a direct consequence of the first/second moment method, and we leave the detail to readers.

Proposition 6.0.7 *[43] Let T be an $\mathbb{N}-$valued random variable with $\mathbb{P}(T > n) = \alpha^n$, independent of the simple random walk. For $\mathcal{A}^n$ a collection of discrete patterns of length n, we have*

$$\frac{1}{2}\frac{2^{-n}}{1-\alpha}\operatorname{Cap}_{\alpha}(\mathcal{A}^n) \leq \mathbb{P}(T(\mathcal{A}^n) < T) \leq \frac{2^{-n}}{1-\alpha}\operatorname{Cap}_{\alpha}(\mathcal{A}^n),$$

where for $A \subset \{-1,1\}^n$,

$$\operatorname{Cap}_{\alpha}(A) := \left[\inf\left\{\sum_{\epsilon',\epsilon'' \in \{-1,1\}^n} G^{\alpha}(\epsilon',\epsilon'')g(\epsilon')g(\epsilon''); g \geq 0, g(A^c) = \{0\} \text{ and } \sum_{\epsilon \in \{0,1\}^n} g(\epsilon) = 1\right\}\right]^{-1}.$$

Now let us mention some previous work regarding the potential theory for path-valued Markov processes. There has been much interest in developing a potential theory for the Ornstein-Uhlenbeck process in the Wiener space $\mathcal{C}_0[0,\infty)$, defined as

$$Z_t := U(t,\cdot) \quad \text{for } t \geq 0,$$

where $U(t,\cdot) := e^{-t/2}W(e^t,\cdot)$ is the Ornstein-Uhlenbeck Brownian sheet. Note that the continuous-time process $(Z_t; t \geq 0)$ takes values in the Wiener space $\mathcal{C}_0[0,\infty)$ and starts at $Z_0 := W(1,\cdot)$ as standard Brownian motion. Following from Williams [152], a Borel measurable set $S \subset \mathcal{C}_0[0,\infty)$ is said to be quasi-sure if $\mathbb{P}(\forall t \geq 0, Z_t \in S) = 1$, which is known to be equivalent to

$$\operatorname{Cap}_{OU}(S^c) = 0, \tag{6.1}$$

where

$$\operatorname{Cap}_{OU}(S^c) := \int_0^{\infty} e^{-t}\mathbb{P}(\exists T \in [0,t] \text{ such that } Z_T \in S^c)dt \tag{6.2}$$

is the *Fukushima-Malliavin capacity* of S^c, that is the probability that Z hits S^c before an independent exponential random time with parameter 1. Taking advantage of the well-known *Wiener-Itô decomposition* of the Ornstein-Uhlenbeck semigroup, Fukushima [51] provided an alternative construction of (6.1) via the Dirichlet form. The approach allows the strengthening of many Brownian almost sure properties to quasi-sure properties. See also the survey of Khoshnevisan [78] for recent development.

Note that the definition (6.2) can be extended to any (path-valued) Markov process. Within this framework, a related problem to Question 1.0.5 is

Question 1.0.5′: *Given a Borel measurable set $S_{\infty} \subset \mathcal{C}_0[0,\infty)$, is*

$$\begin{aligned}\operatorname{Cap}_{MW}(S_{\infty}) :&= \int_0^{\infty} e^{-t}\mathbb{P}[\exists T \in [0,t] \text{ such that } \Theta_T \circ B \in S_{\infty}]dt \\ &= 0 \text{ or } > 0?\end{aligned}$$

where $(\Theta_t)_{t \geq 0}$ is the family of spacetime shift operators defined as

$$\Theta_t \circ B := (B_{t+u} - B_t; u \geq 0) \quad \text{for all } t \geq 0. \tag{6.3}$$

It is not difficult to see that the set function Cap_{MW} is a Choquet capacity associated to the shifted process $(B_{t+u} - B_t; u \geq 0)$ for $t \geq 0$, or the moving-window process $X_t := (B_{t+u} - B_t; 0 \leq u \leq 1)$ for $t \geq 0$. For a Borel measurable subset S of $\mathcal{C}_0[0,1]$, if $\mathrm{Cap}_{MW}(S \otimes_1 \mathcal{C}_0[0,\infty)) = 0$, where

$$S \otimes_1 \mathcal{C}_0[0,\infty) := \{(w_t 1_{t<1} + (w_1 + w'_t)1_{t \geq 1})_{t \geq 0}; w \in S \text{ and } w' \in \mathcal{C}_0[0,\infty)\} \tag{6.4}$$

is the usual path-concatenation, then

$$\mathbb{P}(\exists T > 0 \text{ such that } X_T \in S] = 0,$$

i.e. almost surely the set S is not hit by the moving-window process X. Otherwise,

$$\mathbb{P}[\exists T \in [0,t] \text{ such that } X_T \in S] > 0 \quad \text{for some } t \geq 0,$$

and an elementary argument leads to $\mathbb{P}[\exists T \geq 0 \text{ such that } X_T \in S] = 1$.

As context for this question, we note that path-valued Markov processes have also been extensively investigated in the superprocess literature. In particular, Le Gall [98] characterized the polar sets for the Brownian snake, which relies on earlier work on the potential theory of symmetric Markov processes by Fitzsimmons and Getoor [46] among others.

There has been much progress in the development of potential theory for symmetric path-valued Markov processes. However, the shifted process, or the moving-window process, is not time-reversible and the transition kernel is more complicated than that of the Ornstein-Uhlenbeck process in Wiener space. So working with a non-symmetric Dirichlet form, see e.g. the monograph of Ma and Röckner [103], seems to be far from obvious.

Open problem 6.0.8

1. *Is there any relation between the two capacities* Cap_X *and* Cap_{MW} *on Wiener space?*
2. *Propose a non-symmetric Dirichlet form for the shifted process* $(\Theta_t \circ B)_{t \geq 0}$*, which permits to compute the capacities of the sets of paths* $\mathcal{E}$*,* $\mathcal{M}$*,* $\mathcal{BR}^{\lambda}$ *...etc.*

This problem seems substantial already for one-dimensional Brownian motion. But it could of course be posed also for higher dimensional Brownian motion, or a still more general Markov process. Following are some well-known examples of non-existing patterns in d-dimensional Brownian motion for $d \geq 2$.

- $d = 2$ (Evans [44]): There is no random time T such that $(B_{T+u} - B_T; 0 \leq u \leq 1)$ has a two-sided cone point with angle $\alpha < \pi$;

- $d = 3$ (Dvoretzky et al. [41]): There is no random time T such that $(B_{T+u} - B_T; 0 \leq u \leq 1)$ contains a triple point;
- $d \geq 4$ (Kakutani [74], Dvoretzky et al. [40]): There is no random time T such that $(B_{T+u} - B_T; 0 \leq u \leq 1)$ contains a double point.

We refer readers to the book of Mörters and Peres [115, Chapter 9 and 10] for historical notes and further discussions on sample path properties of Brownian motion in all dimensions.

Finally, we make some connections between Knight's prediction processes and our problems. For background, readers are invited to Knight [88, 89] as well as the commentary of Meyer [106] on Knight's work. To avoid heavy measure theoretic discussion, we restrict ourselves to the classical Wiener space $(\mathcal{C}_0[0,\infty), \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}^{\mathbf{W}})$, where $(\mathcal{F}_t)_{t \geq 0}$ is the augmented Brownian filtrations satisfying the usual hypothesis of right-continuity.

The prediction process is defined as, for all $t \geq 0$ and S_∞ a Borel measurable set of $\mathcal{C}_0[0,\infty)$,

$$Z_t^{\mathbf{W}}(S_\infty) := \mathbb{P}^{\mathbf{W}}[\Theta_t \circ B \in S_\infty | \mathcal{F}_t],$$

where $\Theta_t \circ B$ is the shifted path defined as in (6.3). Note that $(Z_t^{\mathbf{W}})_{t \geq 0}$ is a strong Markov process, which takes values in the space of probability measure on the Wiener space $(\mathcal{C}_0[0,\infty), \mathcal{F})$. In terms of the prediction process, Question 1.0.5 can be reformulated as

Question 1.0.5″: *Given a Borel measurable set $S \subset \mathcal{C}_0[0,1]$, can we find a random time T such that*

$$\mathbb{E} Z_T^{\mathbf{W}}(S \otimes_1 \mathcal{C}_0[0,\infty)) = 1?$$

where $S \otimes_1 \mathcal{C}_0[0,\infty)$ is defined as in (6.4).

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