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Limitations of Trace Invariants to the Inverse Spectral Problem

DISSERTATION

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by

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ABSTRACT OF THE DISSERTATION

Limitations of Trace Invariants to the Inverse Spectral Problem

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Doctor of Philosophy in Mathematics

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In this work we study inverse spectral problems for bounded domains, smooth closed manifolds, and semiclassical Schrödinger operators, with particular concern towards the latter. A central tool in the analysis of inverse spectral problems are trace invariants, however these are not without limitations. We show that there exist pairs of non-isometric potentials for the 1D semiclassical Schrödinger operator whose spectra agree up to $O(h^\infty)$, and hence have the same semiclassical trace invariants, yet all corresponding eigenvalues differ no less than exponentially. This result was conjectured in the work of Guillemin and Hezari [GH12], where they prove a very similar result for the ground state eigenvalues, however cannot remove the possibility of a subsequence $h_k \to 0$ where the ground state eigenvalues may agree.

Chapter 1

Introduction to the Semiclassical Schrödinger Operator

1.1 Preliminary Spectral Theory

We begin with the basic setting for which we construct the spectral theory.

Definition 1.1. *Let $\mathcal{H}$ be a vector space over $\mathbb{C}$ equipped with an inner product $\langle \cdot, \cdot \rangle$. If $\mathcal{H}$ is complete with respect to the topology induced by the norm, we say $\mathcal{H}$ is a* ***Hilbert space****.*

From here on out, we assume that the Hilbert space $\mathcal{H}$ is separable, that is, $\mathcal{H}$ possesses a countable dense subset. Operators on $\mathcal{H}$ fall into two categories, bounded and unbounded. Fundamental definitions change and are considerably less intuitive in the latter case, and so we will initially consider only bounded operators.

1.1.1 Bounded Operators

Definition 1.2. *Let $A : \mathcal{H} \to \mathcal{H}$ be a linear operator on a Hilbert space. We say that A is* ***bounded*** *if*

$$||A||_{op} := \sup_{\substack{x\in\mathcal{H} \\ ||x||\leq 1}} ||Ax|| < \infty.$$

We call $||\cdot||_{op}$ the ***operator norm****.*

Moreover, we will study a particular type of linear operator on $\mathcal{H}$.

Definition 1.3. *Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert space. The* ***adjoint*** *of a bounded operator $A : \mathcal{H} \to \mathcal{H}$ is the operator A^* satisfying $\langle Ax, y\rangle = \langle x, A^*y\rangle$ for all $x, y \in \mathcal{H}$. We say the bounded operator $A : \mathcal{H} \to \mathcal{H}$ is* ***self-adjoint*** *if $A^* = A$, or equivalently $\langle Ax, y\rangle = \langle x, Ay\rangle$ for all $x, y \in \mathcal{H}$.*

We are interested the eigenvalues/eigenvectors of self-adjoint A, and the functional calculus of A. The latter being the structure of "taking functions of an operator". To build intuition, let us first consider the analytic functional calculus for self-adjoint operators over finite dimensional Hilbert spaces, starting with a structure theorem from self-adjoint operators on finite dimensional Hilbert spaces.

Theorem 1.1 (The Spectral Theorem for Finite Dimensional Hilbert Spaces, [Art91])**.** *Let A be a self-adjoint operator on a finite dimensional Hilbert space. Then there exists a unitary matrix U and real diagonal matrix such that $A = UDU^{-1}$. Moreover, the j^{th} column of U and j^{th} diagonal entry of D are the j^{th} eigenvector and eigenvalue of A.*

Let $f : U \to \mathbb{C}$ be an analytic function $f(z) = \sum_{j=0}^{\infty} a_j z^j$, whose domain contains all of the eigenvalues of A. Without loss of generality identify $\mathcal{H} = \mathbb{C}^m$, since $\mathcal{H}$ is finite dimensional.

We define the operator $f(A) : \mathcal{H} \to \mathcal{H}$ by

$$f(A) = \sum_{j=0}^{\infty} a_i j A^j,$$

where the limit is understood in the sense of the operator norm, $||X||_{op} := \sup_{\substack{u \in \mathcal{H} \\ ||x|| \leq 1}} ||Xu||$. Indeed, $f(A)$ is well defined, taking $A = UDU^{-1}$, we have

$$\left|\left|\sum_{j=M}^{N} a_j A^j\right|\right|_{op} = \left|\left|U\left(\sum_{j=M}^{N} a_j D^j\right)U^{-1}\right|\right|_{op} = \left|\left|\left(\sum_{j=M}^{N} a_j D^j\right)\right|\right|_{op} \leq \max_k \sum_{j=M}^{N} a_j \lambda_k^j < \epsilon,$$

for M, N sufficiently large, since f is analytic over a set containing all eigenvalues $\{\lambda_k\}$. Moreover, exchanging limits we have that

$$f(A) = U \sum_{j=0}^{\infty} a_j D^j U^{-1} = U \begin{pmatrix} f(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & f(\lambda_m) \end{pmatrix} U^{-1}.$$

Remark 1.1. *The above display equation reveals an important property of the functional calculus. Since the columns of U are the orthonormal eigenvectors of A, and that U is unitary ($U^{-1} = U^*$) we have that*

$$f(A)\psi_j = f(\lambda_j)\psi_j,$$

where ψ_j is the j^{th} eigenvector of A. Thus, for any $x \in \mathcal{H}$, we have

$$f(A)x = \sum_{j=1}^{m} f(\lambda_j)\langle x, \psi_j\rangle\psi_j.$$

Now, for finite dimensional operators A, eigenvalues are those complex numbers λ such that $A - \lambda I$ is not invertible. By rank-nullity we have that $A - \lambda I$ not being invertible is equivalent to $A - \lambda I$ being not-injective, and equivalent to $A - \lambda I$ being not surjective. We would like to study the analog of eigenvalues, and hence the functional calculus for linear operators on infinite dimensional spaces, however non-injectivity of an operator is not equivalent to non-surjectivity of an operator. Take for example the left shift operator and its adjoint $\sigma, \sigma^* : \ell^2(\mathbb{R}) \to \ell^2(\mathbb{R})$

$$\sigma(a_1, a_2, a_3, \dots) = (a_2, a_3, a_4, \dots)$$
$$\sigma^*(a_1, a_2, a_3, \dots) = (0, a_1, a_2, \dots).$$

The map σ is not injective, but is surjective, and σ^* is not surjective, but is injective.

It turns out, for the infinite dimensional setting we study those complex numbers λ which cause $A - \lambda I$ to be non-invertible.

Definition 1.4. *Let A be an operator on $\mathcal{H}$. A complex number is said to be in the resolvent set $\rho(A)$ of A if $A - \lambda I$ is a bijection with bounded inverse. The map $R_\lambda(A) = (\lambda I - A)^{-1}$ is said to be the **resolvent** of A at λ. If $\lambda \notin \rho(A)$, then λ is in the **spectrum** $\mathrm{Spec}(A)$ of A.*

*An $x \neq 0$ satisfying $Ax = \lambda x$ for some $\lambda \in \mathbb{C}$ is called an **eigenvector**, or in the case $\mathcal{H} = L^2(\mathbb{R}^n)$ an **eigenfunction** of A; λ is called the corresponding **eigenvalue**. If λ is an eigenvalue, then $A - \lambda I$ is not injective, so $\lambda \in \mathrm{Spec}(A)$. The set of all eigenvalues is called the **point spectrum** of A.*

*If λ is not an eigenvalue and if the range of $A - \lambda I$ is not dense, then λ is said to be in the **residual spectrum**.*

We will omit the precise discussion of the functional calculus for bounded self-adjoint operators on a Hilbert space, and move directly to the general case of self-adjoint operators.

1.1.2 Unbounded Operators

Many operators that are of interest to mathematical physicists have a natural space of definition that is not a Hilbert space, but rather a densely defined linear subspace.

Definition 1.5. *Let $\mathcal{H}$ be a Hilbert space. We say that A is an **operator** on $\mathcal{H}$ if it is a linear map from a dense subspace $Dom(A) \subseteq \mathcal{H}$ to $\mathcal{H}$. Let $\tilde{A}$ be a linear operator on $\mathcal{H}$ with domain $Dom(\tilde{A}) \supsetneq Dom(A)$ and $\tilde{A} = A$ on $Dom(A)$. We call $\tilde{A}$ and **extension** of A.*

Now, if an operator A is bounded, and hence continuous, on its dense domain of definition it can be extended uniquely to a bounded operator over $\mathcal{H}$. Thus, we are naturally interested in those densely defined operators which are unbounded. Let us consider an important example.

Example 1.1. *Let $V \in C^\infty(\mathbb{R}^n)$ with $\lim_{|x|\to\infty} V(x) = +\infty$ and $V(x) = O(|x|^N)$, for some N, and consider the operator $P : C_0^\infty(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, given by $P = -\Delta + V(x)$. Note that P is densely defined, that is, $C_0^\infty(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that $||\varphi||_2 \leq 1$. Then consider the sequence of translates $\varphi_n(x) = \varphi(x - n)$ satisfying $||\varphi_n|| \leq 1$. Then by the compact support of φ, Jensen's inequality, and a change of variables*

$$||P\varphi_n|| \geq C \left| \int V(x+n)\phi \, dx - \int \Delta\phi \, dx \right|.$$

The left term above grows unboundedly positive by the assumption on V, and the right term is a fixed constant. Thus $||P\varphi_n||$ diverges to infinity. And so,

$$\sup_{\substack{f \in \mathbb{C}_0^\infty(\mathbb{R}^n) \\ ||f|| \leq 1}} ||Pf|| = \infty,$$

and we have that P is unbounded.

The operator from the above example satisfies $\langle Pf, g\rangle = \langle f, Pg\rangle$, for $f, g \in C_0^\infty(\mathbb{R}^n)$, however this is not enough to say that the operator P is "self adjoint", when taking analogy from the definition of self-adjoint for operators defined over the entirety of $L^2(\mathbb{R}^n)$.

Definition 1.6. *Let A be a densely defined operator on the Hilbert space $\mathcal{H}$. Let $Dom(A^*)$ be the set of all $\phi \in \mathcal{H}$ such that there exists an $\eta \in \mathcal{H}$ with*

$$\langle A\psi, \phi \rangle = (\psi, \eta).$$

For each $\psi \in Dom(A^*)$, we define $A^*\phi = \eta$. A^* is called the ***adjoint*** *of A,*

Returning to the above example, we see that many elements of $L^2(\mathbb{R}^n)$ are missing from the domain $\mathrm{Dom}(P) = C_0^\infty(\mathbb{R}^n)$. Indeed, consider the *Schwartz space* $\mathcal{S} = \{\varphi \in C^\infty(\mathbb{R}^n) : \sup_{\mathbb{R}^n} |x^\alpha \partial^\beta \varphi| < \infty\}$, and let $\chi_n \in \mathbb{C}_0^\infty(\mathbb{R}^n)$ be the cutoff functions satisfying $\chi_n(x) = 1$ if $|x| \leq n$, and $\chi_n(x) = 0$ if $|x| \geq n + 1$. Then let $\varphi \in \mathcal{S}$, and note that $\chi_n \varphi \in \mathrm{Dom}(P)$, $\chi_n \varphi \xrightarrow{L^2(\mathbb{R}^n)} \varphi$, and $P\chi_n\varphi \xrightarrow{L^2(\mathbb{R}^n)} P\varphi$. In this sense, φ is missing from the domain of P, which motivates the following definition.

Definition 1.7. *We say that a linear operator A on $\mathcal{H}$ is* ***closed*** *if its graph $\Gamma(A) = \{(x, Ax) : x \in Dom(A)\}$ is closed in $\mathcal{H} \times \mathcal{H}$. Or equivalently, A is closed if for every sequence (x_n) in $Dom(A)$ such that $x_n \to x$ and $Ax_n \to y$ for some $x, y \in \mathcal{H}$ we necessarily have $x \in Dom(A)$ and $Ax = y$. An operator A is* ***closable*** *if $\overline{\Gamma(A)}$ is the graph of an operator. We denote this operator $\overline{A}$ and call it the* ***closure*** *of A. Note that $\overline{\Gamma(A)} = \Gamma(\overline{A})$*

When is a densely defined operator closeable?

Proposition 1.1 ([RS81])**.** *A is closable if and only if $Dom(A^*)$ is dense, in which case $\overline{A} = A^{**}$.*

A very important class of operators which we will consider are inspired by the definition of bounded self-adjoint operators.

Definition 1.8. *A densely defined operator A on $\mathcal{H}$ is called* ***symmetric*** *if $\langle Af, g\rangle = \langle f, Ag\rangle$ for all $f, g \in Dom(A)$.*

An immediate consequence of the above definition is that if A is symmetric, we have that $\mathrm{Dom}(A) \subseteq \mathrm{Dom}(A^*)$, and hence A is closable. Now, it is not always the case that this subset inclusion is equality. In the example 1.1.2, we have that P is symmetric, hence $\mathrm{Dom}(P) \subseteq \mathrm{Dom}(P^*)$. However, this inequality is strict, because $e^{-x^2} \in \mathrm{Dom}(P^*) \setminus \mathrm{Dom}(P)$. Indeed, by integration by parts,

$$\langle P\phi, e^{-x^2}\rangle = \langle \phi, -\Delta e^{-x^2} + V(x)e^{-x^2}\rangle,$$

holding for each $\phi \in \mathrm{Dom}(P) = C_0^\infty(\mathbb{R}^n)$.

Definition 1.9. *A densely defined operator A on $\mathcal{H}$ is* ***self-adjoint*** *if A is both symmetric and $Dom(A) = Dom(A^*)$. The linear operator A is called* ***essentially self-adjoint*** *if it possess a unique self-adjoint extension, or equivalently, A is closable and its unique closure $\overline{A}$ is self adjoint. If A is closed, a subset $C \subseteq Dom(A)$ is called a* **core** *for A if $\overline{A|_C} = A$.*

The principle benefit of an operator being essentially self adjoint is that to specify the self adjoint extension uniquely, one need not give the explicit domain, which may be difficult to specify, but a core of the operator.

Just like in the bounded and finite dimensional cases, the spectrum of a self-adjoint operators are purely real, and they have the following remarkable structure theorem.

Theorem 1.2 (The Spectral Theorem for Unbounded Self-Adjoint Operators, [RS81])**.** *Let A be a self-adjoint operator on a separable Hilbert space $\mathcal{H}$, with domain $Dom(A)$. Then there is a measure space (M, μ) with μ a finite measure, a unitary operator $U : \mathcal{H} \to L^2(M, d\mu)$, and a real-valued function f on M which is finite a.e. so that*

(a) $\psi \in Dom(A)$ if and only if $f(\cdot)(U\psi)(\cdot) \in L^2(M, d\mu)$

(b) *If* $\phi \in U(Dom(A)), then (UAU^{-1}\phi)(m) = f(m)\phi(m)$.

Remark 1.2. *In the finite dimensional case, say* $\mathcal{H} \cong \mathbb{C}^n$*, note that* A *admits an orthonormal basis of eigenvectors. Interpreting the statement of the spectral theorem, we have that* $M = \{1, 2 \ldots, n\}$*,* μ *is the uniform atomic measure over* M*,* $f(m)$ *is the* m^{th} *eigenvalue of* A*, and if* ψ_m *is the* m^{th} *eigenvector of* A*,* $U(a\psi_m) = a\chi_{\{m\}}$.

In light of the spectral theorem and the above remark, there is a natural way to define Borel functions of a self-adjoint operator. Let $h : \mathbb{R} \to \mathbb{C}$ be Borel, then

$$h(A) = U^{-1}T_{h(f)}U,$$

Where $T_{h(f)}$ is the multiplication by $h(f(\cdot))$ operator on $L^2(M, d\mu)$. This is understood as the **Borel functional calculus**.

Theorem 1.3 (Spectral Theorem - Functional Calculus Form, [RS81])**.** *Let* A *be a self-adjoint operator on* $\mathcal{H}$*. Then there exists a unique map* $\hat{\phi}$ *from the bounded Borel functions on* $\mathbb{R}$ *into the bounded operators on* $\mathcal{H}$ *with the following properties:*

(a) $\hat{\phi}$ *is a* $*$*-homomorphism, meaning*

$$\hat{\phi}(fg) = \hat{\phi}(f)\hat{\phi}(g) \qquad \hat{\phi}(\lambda f) = \lambda\hat{\phi}(f)$$
$$\hat{\phi}(1) = I \qquad \hat{\phi}(\overline{f}) = \hat{\phi}(f)^*$$

(b) $\hat{\phi}$ *is norm continuous, meaning* $\left|\left|\hat{\phi}(f)\right|\right|_{op} \leq ||f||_\infty$

(c) *Let* $f_n(x)$ *be a sequence of bounded Borel functions with* $f_n(x) \to x$ *pointwise and* $|f_n(x)| \leq |x|$ *for all* x *and* n*. Then, for any* $\psi \in Dom(A)$*,* $\lim_{n\to\infty} \hat{\phi}(f_n)\psi = A\psi$.

(d) *If* $f_n \to f$ *pointwise and if the sequence* $||f_n||_\infty$ *is bounded, then* $\hat{\phi}(f_n) \to \hat{\phi}(f)$ *strongly.*

(e) *If* $A\psi = \lambda\psi$, *then* $\hat{\phi}(f) = f(\lambda)\psi$

(f) *If* $f \geq 0$, *then* $\hat{\phi}(f) \geq 0$.

We are primarily interested in taking the complex exponential of self-adjoint operators.

Theorem 1.4 (Stone's Theorem, [CR21])**.** *Let* A *be a self-adjoint operator on the Hilbert space* $\mathcal{H}$*. There exists a unique* C^0*-unitary group* $(U_t)_{t\in\mathbb{R}}$ *such that*

(i) $U_t : Dom(A) \to Dom(A)$,

(ii) *for all* $u \in Dom(A)$, $U_t u \in C^1(\mathbb{R}, \mathcal{H}) \cap C^0(\mathbb{R}, Dom(A))$

(iii) *for all* $u \in Dom(A)$, $\frac{d}{dt}U_t u = iAU_t u = iU_t Au$.

We denote $U_t = e^{itA}$.
Conversely, if $(U_t)_{t\in\mathbb{R}}$ *is a* C^0*-unitary group, then there exists a unique self-adjoint operator* A *such that, for all* $t \in \mathbb{R}$, $U_t = e^{itA}$, *with domain*

$$Dom(A) = \left\{ u \in \mathcal{H} : \sup_{0<t\leq 1} t^{-1}||U_t u - u|| < +\infty \right\}.$$

Stone's theorem gives us a way of reformulating certain partial differential equation in terms of the action of a 1-D group of unitary bounded operators. For example, consider the following Scrhödinger-like initial value problem

$$\begin{cases} i\partial_t\psi(x,t) = H\psi(x,t) \\ \psi(x,0) = f(x) \\ H = -\Delta + V(x) \end{cases}$$

where $V : \mathbb{R}^n \to \mathbb{R}$ is chosen to guarantee H is self-adjoint. Then, by Stone's theorem

$$i\partial_t(e^{-itH}f(x)) = He^{-itH}f(x),$$

and so $\psi(x,t) = e^{-itH}f(x)$ solves the given initial value problem.

1.2 The Semiclassical Schrödinger Operator

1.2.1 Connections to Quantum Mechanics

Before introducing the main operator of interest, the semiclassical Schrödinger operator, we begin with a discussion of the elementary theory of quantum mechanics. The goal is to provide a probabilistic interpretation of the time-evolution of a particle under the influence of an external force. That is, suppose we have a particle of mass m, subject to a potential energy $V(x)$. We wish to determine the probability of the particle existing within a certain subset of $W \subseteq \mathbb{R}^n$ at a time t. This is accomplished through the *time-dependent Schrödinger equation*,

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\Delta\psi + V(x)\psi, \tag{1.1}$$

where $\hbar$ is a small universal constant referred to as Planck's constant, and Δ is the non-positive Laplacian on $\mathbb{R}^n$, $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$. We call the complex solutions to the above equation, $\psi(x,t)$, wave functions, and assume that they are L^2 normalized in space, i.e. $\int_{\mathbb{R}^n} |\psi|^2\, dx = 1$. For a fixed point in time, the wave functions give rise to our desired probability measure on $\mathbb{R}^n$, $\Psi_t(W) = \int_W |\psi(x,t)|^2\, dx$.

To physical quantities of a quantum system we associate self-adjoint operators, called observables. For example, position is an observable, with the corresponding operator Q_j :

$\mathrm{Dom}(Q_j) \subseteq L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, given by $Q_j f = x_j f(x)$. To an observable A, we may compute its expectation $\langle A \rangle_{\psi(x,t)} = \langle A\psi(x,t), \psi(x,t)\rangle$. This computation provides the intuition behind the naming of the position operator. Computing the expectation of Q_j,

$$\langle Q_j \rangle_{\psi(x,t)} = \int x_j |\psi(x,t)|^2 \, dx.$$

This is the expected value of the quantum particle's location in the j^{th} coordinate.

In what follows we are interested in what are called *stationary states*, that is, solutions to (1.1) of the form

$$\psi(x,t) = e^{-\frac{itE}{\hbar}} \psi(x,0), \tag{1.2}$$

for some real constant E. Note that for any stationary state $\psi(x,t)$, we have

$$\Psi_t(W) = \int_W |\psi(x,t)|^2 \, dx = \int_W |\psi(x,0)|^2 \, dx = \Psi_0(W)$$
$$\langle A \rangle_{\psi(x,t)} = \int A\psi(x,t)\overline{\psi(x,t)} \, dx = \int A\psi(x,0)\overline{\psi(x,0)} \, dx = \langle A \rangle_{\psi(x,0)}.$$

Interpreted physically, the former says that the probability of finding a particle in the measurable set W does not change with time, and the latter says that the expected value of any observable A is independent of time. If we plug (1.2) into (1.1) we arrive at the *time-independent Schrödinger equation*:

$$E\psi(x,t) = \left(-\frac{\hbar^2}{2m}\Delta + V(x)\right)\psi(x,t). \tag{1.3}$$

Define the *time-independent Schrödinger operator* $P : \mathrm{Dom}(P) \subseteq L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ by

$$P = -\frac{\hbar^2}{2m}\Delta + V(x). \tag{1.4}$$

With this in mind, we notice that (1.3) factors into the following eigenvalue equation:

$$E\psi(x) = P\psi(x).$$

Now, by Theorem 1.4 (Stone's Theorem), we have that any solution to (1.1) is given by the Schrödinger propagator $\psi(x,t) = e^{-\frac{it}{\hbar}P}\psi(x,0)$, and thus we may reduce the study of the time-dependent Schödinger equation to the study of the eigenfunctions and eigenvalues of P. Indeed, if P admits an orthonormal basis $\{\psi_j\}$ of eigenfunctions with associated eigenvalues $\{\lambda_j\}$ for $L^2(\mathbb{R}^n)$, by the Borel functional calculus (Theorem 1.3), we may represent solutions to the Schrödginer equation (1.1) as

$$\psi(x,t) = \sum_{j=0}^{\infty} a_j e^{-\frac{it}{\hbar}\lambda_j}\psi_j,$$

where $\psi(x,0) = \sum_{j=0}^{\infty} a_j\psi_j(x)$. Note that the previous equality, and the equality above are understood in the sense of L^2.

1.2.2 Definition and Basic Properties of the Semiclassical Schrödinger Operator

Consider the family of operators indexed by the parameter $h > 0$, $H : C_0^\infty(\mathbb{R}) \to L^2(\mathbb{R})$ defined by $H = -h^2\frac{d^2}{dx^2} + V(x)$, where

$$V \in C^\infty(\mathbb{R}), \lim_{|x|\to\infty} V(x) = +\infty. \tag{1.5}$$

Notice that H is the time-independent Schrödinger operator (1.4), with $m = \frac{1}{2}$, and letting $\hbar = h$ vary as a continuous, positive parameter. The operators H are symmetric, and moreover, essentially self adjoint, which follows as a consequence of Sears theorem:

Theorem 1.5 ([BS12], §2.1)**.** *Let V satisfy $V(x) \geq -Q(x)$, where $Q > 0$ and even, satisfying*

$$\int_{\mathbb{R}} \frac{1}{\sqrt{Q(2x)}}\, dx = \infty.$$

Then H is essentially self adjoint.

Indeed, taking $Q(x) = C$, for some $C > 0$, we have that H is essentially self adjoint, with the unique self-adjoint extension $H^* : \mathrm{Dom}(H^*) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$. From here on out, we will use H to denote the unique self adjoint extension formed by taking the closure of the original operator.

Definition 1.10. *Let $h > 0$, and $V \in C^\infty(\mathbb{R})$ with $\lim_{|x|\to\infty} V(x) = +\infty$ and satisfying the hypothesis of Theorem 1.5. The operator $H : Dom(H) \subseteq L^2(\mathbb{R}) \to L^2(\mathbb{R})$ which is the unique self-adjoint extension of the operator $\phi \mapsto -h^2\phi'' + V\phi$ with core $C_0^\infty(\mathbb{R})$, is called the **semiclassical Schrödinger operator** with **potential** V.*

We now consider the spectrum of H.

Theorem 1.6 ([BS12], §2.3)**.** *If V satisfies* (1.5)*, then the spectrum of H is discrete, with an associated orthonormal complete system of L^2 eigenfunctions ψ_k, $k = 0, 1, 2, \ldots$ whose eigenvalues λ_k satisfy $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$.*

The eigenvalues and eigenfunctions above depend on the parameter h, we omit this dependence in the notation for simplicity. Now, each eigenspace is of rank 1, and so the above inequalities of eigenvalues are strict. This follows from the following general characterization from Sturm-Liouville theory.

Theorem 1.7 ([BS12], §2.3)**.** *Let $V(x) \geq \epsilon > 0$ for $x \geq a$. Then for any solution y to $-y'' + Vy = 0$, one of the two limits holds*

a) $\lim_{x\to\infty} y(x) = +\infty$

b) $\lim_{x\to\infty} y(x) = 0$

A solution satisfying b) *exists and is unique (up to a constant factor).*

From here on out we restrict ourself to potentials V that satisfy the following growth conditions:

$$\begin{cases} |V^{(k)}| \le C_k(1+x^2)^{\frac{k}{2}} \quad \text{for each } k \in \mathbb{Z}_{\ge 0} \\ V(x) \ge c(1+x^2)^{\frac{k}{2}} \quad \text{for } |x| \ge R, \quad \text{for some } R > 0 \end{cases} \tag{1.6}$$

We conclude with an important statement on the regularity of eigenfunctions of H.

Proposition 1.2 ([Zwo22]). *Let u_j denote the j^{th} eigenfunction of the semiclassical Schrödinger operator H, with potential satisfying growth conditions* (1.6)*. Then u_j is a member of the Schwartz space.*

1.2.3 The Classical Harmonic Oscillator and Quantization

Elementary Hamiltonian Mechanics

We call the Semiclassical Schrödinger operator semiclassical due to the connection it facilitates between quantum and classical dynamics. To see this, we first review the fundamental of Hamiltonian mechanics. Consider an object of mass m, with displacement relative to equilibrium given by $x(t)$, subject to a force $F(x(t))$. Newton's law tells us that

$$m\ddot{x} = F(x(t)).$$

We may rewrite this second order ODE as a system of equations, introducing the momentum variable $\xi = m\dot{x}$,

$$\begin{cases} \dot{x} &= \frac{1}{m}\xi \\ \dot{\xi} &= F(x(t)), \end{cases} \tag{1.7}$$

which is known as Hamilton's equations of motion. If $F(x)$ is a smooth conservative vector field, with potential function $V(x) = -\partial F(x)$, we may associate to the above differential equation a function $H : \mathbb{R}^{2n} \to \mathbb{R}$, given by $H(x,\xi) = \frac{\xi^2}{2m} + V(x)$, referred to as a *Hamiltonian*, which satisfies

$$\begin{cases} \dot{x} &= \partial_\xi H \\ \dot{\xi} &= -\partial_x H. \end{cases}$$

By construction, we have that solutions to Hamilton's equations (1.7) are constant along H, that is $\frac{d}{dt}H(x(t),\xi(t)) = 0$, and the value H takes on the given solution $H(x(t),\xi(t)) = E$ is known as a constant of motion. This fact is known as the conservation of energy, where E represents the total energy of the traveling mass, $\frac{\xi^2}{2m}$ is the kinetic energy, and $V(x)$ is the potential energy.

The Hamiltonian H defines a vector field associated to Hamilton's equations, $X_H = (\partial_\xi H, -\partial_x H)$, along with a flow $\Phi_H^t : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ taking $(x_0,\xi_0) \mapsto (x(t),\xi(t))$, where $x(t),\xi(t)$ are solutions to Hamilton's equations (1.7) with initial condition (x_0,ξ_0).

The Harmonic Oscillator

We turn to a fundamental example from classical mechanics, the harmonic oscillator. Consider a massive object moving along a frictionless track, subject to a restoring force inversely proportional to its distance from equilibrium. The displacement of this object from equilibrium, $x(t)$, is modeled using Newton's law

$$m\frac{d^2x}{dt^2} = -kx, \tag{1.8}$$

where m is the mass of the object, and k is the spring constant. The hamiltonian associated to the above differential equation is $H(x,\xi) = \frac{\xi^2}{2m} + \frac{kx^2}{2}$, with vector field $X_H = \left(\frac{\xi}{m}, -kx\right)$. The flow of X_H in phase-space, i.e. position-momentum space, is given by matrix exponentiation

$$\Phi_H^t = \exp(tA), \quad A = \begin{pmatrix} 0 & \frac{1}{m} \\ -k & 0 \end{pmatrix}.$$

Note that $A^{2n} = \left(-\frac{k}{m}\right)^n I$, and $A^{2n+1} = \left(-\frac{k}{m}\right)^n A$, hence

$$\Phi_H^t = I\sum_{n=0}^{\infty}\frac{(-1)^n\left(\sqrt{\frac{k}{m}}\right)^{2n}}{(2n)!} + \sqrt{\frac{m}{k}}A\sum_{n=0}^{\infty}\frac{(-1)^n\left(\sqrt{\frac{k}{m}}\right)^{2n+1}}{(2n+1)!} = \begin{pmatrix} \cos\sqrt{\frac{k}{m}}t & \frac{1}{\sqrt{mk}}\sin\sqrt{\frac{k}{m}}t \\ -\sqrt{km}\sin\sqrt{\frac{k}{m}}t & \cos\sqrt{\frac{k}{m}}t \end{pmatrix}$$

And viewed in terms of initial conditions (x,ξ),

$$\Phi_H^t(x,\xi) = \left(\cos\left(\sqrt{\frac{k}{m}}t\right)x + \frac{1}{\sqrt{km}}\sin\left(\sqrt{\frac{k}{m}}t\right)\xi, -\sqrt{km}\sin\left(\sqrt{\frac{k}{m}}t\right)x + \cos\left(\sqrt{\frac{k}{m}}t\right)\xi\right),$$

we see that Φ_H^t parameterizes the ellipse $H(x,\xi) = \frac{\xi^2}{2m} + \frac{kx^2}{2} = E$. Note that the x coordinate of Φ_H^t is the solution to (1.8) with initial condition $(x, \xi/m)$, simple harmonic motion.

Quantization

Given a sufficiently regular Hamiltonian $f : \mathbb{R}^{2n} \to \mathbb{R}$ there is a way of associating a linear operator on an appropriate function space.

Definition 1.11. *Let $f \in \mathcal{S}(\mathbb{R}^{2n})$. To f we associate the operators $f^w(x, hD), f(x, hD) : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$, given by*

$$f^w(x, hD)u(x) := \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y, \xi\rangle} f\left(\frac{x+y}{2}, \xi\right) u(y)\, dy d\xi,$$
$$f(x, hD)u(x) := \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y, \xi\rangle} f\left(x, \xi\right) u(y)\, dy d\xi.$$

We call $f^w(x, hD)$, $f(x, hD)$ the ***Weyl quantization*** *and* ***standard quantization*** *of the* ***symbol*** *f.*

The standard quantization of a symbol f can be realized as the conjugation of f with the following transformation.

Definition 1.12. *Let $\mathcal{F}_h, \mathcal{F}_h^{-1} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ be given by*

$$(\mathcal{F}_h\varphi)(\xi) := \int_{\mathbb{R}^n} e^{-\frac{i}{h}\langle x, \xi} \varphi(x)\, dx,$$
$$(\mathcal{F}_h^{-1}\psi)(x) := \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x, \xi} \psi(\xi)\, d\xi.$$

We call $\mathcal{F}_h, \mathcal{F}_h^{-1}$ the ***semiclassical Fourier transform*** *and the* ***semiclassical inverse Fourier transform****.*

Indeed, conjugating a symbol f by the semiclassical Fourier transform yields the standard quantization, $f(x, hD)u = \mathcal{F}_h^{-1}(f(x, \cdot)(F_h u)(\cdot))$.

Now, if we relax the requirement that our symbol is Schwarz class, we do not know the exact mapping properties of the associated quantization. However, for the familiar example of the symbol $f(x, \xi) = \xi^2 + x^2$, where f is the Hamiltonian of the harmonic oscillator with mass $m = \frac{1}{2}$ and spring constant $k = 2$, we recover a well known self-adjoint operator. Let us compute its Weyl quantization.

$$f^w u = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle} \left(\xi^2 + \left(\frac{x+y}{2}\right)^2\right) u(y)\, dyd\xi.$$

The first term simplifies to

$$\frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle} \xi^2 u(y)\, dyd\xi = -h^2 \Delta u(x),$$

where we used the fact that $\mathcal{F}_h^{-1}(\xi_j \mathcal{F}_h(g)) = -hD_{x_j} g(x)$. The second term simplifies as follows

$$\frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle} \left(\frac{x^2}{4} + \frac{xy}{2} + \frac{y^2}{4}\right) u(y)\, dyd\xi = \frac{x^2}{4} + \frac{x^2}{2} + \frac{x^2}{4} = x^2.$$

Thus, the Weyl quantization of the classical harmonic oscillator is $f^w = -h^2\Delta + x^2$. Note that the standard quantization of the classical harmonic oscillator yields the same result.

Definition 1.13. *Let $H_0 : Dom(H_0) \subseteq L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ be given by $H_0 u = -h^2\Delta u + x^2 u$, where x^2 is understood as the multiindex summation $x^2 := (x_1^2 + \ldots x_n^2)$. We call H_0 the* ***semiclassical harmonic oscillator****.*

The symbol $p(x) = \xi^2 + V(x)$ of the semiclassical Schrödinger operator tells us a lot of interesting information about the associated eigenvalues and eigenfunctions. We will conclude this section with theorem regarding the asymptotic distribution of eigenvalues of the semiclassical Schrödinger operator $H = p(x, hD)$, using this correspondence.

Theorem 1.8 (Weyl's Law, [Zwo22])**.** *Let $0 \le a < b$, and let E denote an eigenvalue of the semiclassical Schrödinger operator $H = p(x, hD)$, where $p(x) = \xi^2 + V(x)$, and $V(x)$ is smooth potential satisfying the growth conditions* (1.6)*. Then*

$$\#\{E : a \le E \le b\} = \frac{1}{(2\pi h)^n}(|\{a \le p(x,\xi) \le b\}| + o(1))$$

as $h \to 0$. Here $|\{a \le p(x,\xi) \le b\}|$ denotes the volume of the set between the level curves of the hamiltonian $p(x,\xi) = a$ and $p(x,\xi) = b$.

1.2.4 Bohr-Sommerfeld Rules

We are interested in approximating the eigenvalues of a semiclassical Schrödinger operator as we let the semiclassical parameter tend towards zero.

Definition 1.14. *Let $f(h)$ be a function of the semiclassical parameter h. We say that f is of **order** $O(h^\infty)$, or $f = O(h^\infty)$as $h \to 0$, if for each positive integer N there exists a constant C_N such that*

$$|f| \le C_N h^N \qquad \text{for all } h > 0.$$

One can recover the eigenvalues of the semiclassical Schrödinger operator $H = p(x, hD)$ up to order $O(h^\infty)$ by way of an asymptotic expansion in h, known as the Bohr-Sommerfeld rules.

Theorem 1.9 ([CdV05])**.** *Let E_n denote the n^{th} eigenvalue (index $n \in \{1, 2, \dots\}$) of the 1-D Schrödinger operator $H = p(x, hD) = -h^2 \frac{d^2}{dx^2} + V(x)$, with potential satisfying growth conditions (1.6). Let $C > 0$, then $E_n \in (0, C]$ satisfies the **Bohr-Sommerfeld rules**,*

$$\mathcal{S}_h(E_n) - 2\pi n h = O(h^\infty),$$

where

$$\mathcal{S}_h(E) = \sum_{j=0}^{\infty} \mathcal{S}_j(E) h^j. \tag{1.9}$$

We call S_h the semiclassical action, and the terms are given by

- $S_0(E) = \int_{\gamma_E} \xi \, dx = \int_{p^{-1}([0,E])} dx d\xi$ *is the action integral.*

- $S_1(E) = \pi$.

- $S_{2j+1} = 0$ *for* $j > 0$.

- $S_{2j} = \sum_{2 \leq l \leq L(2j)} \frac{(-1)^{l-1}}{(l-1)!} \left(\frac{d}{dE}\right)^{l-2} \int_{\gamma_E} P_{2j,l}(x, \xi) \, dt$

where γ_E *level set* $p(x, \xi) = E$*, parameterized in the direction of the Hamiltonian flow, an the* $P_{j,l}$ *are universal polynomials evaluated on the partial derivatives of the hamiltonian* $\partial^\alpha p$.

We will be interested in the eigenvalues of perturbed harmonic oscillators, and so to build intuition, let us first consider the Bohr-Sommerfeld rules for the unperturbed harmonic oscillator. Recall, the Hamiltonian for this operator is $p(x, \xi) = x^2 + \xi^2$. Computing the action integral:

$$S_0(E) = 2 \int_{-\sqrt{E}}^{\sqrt{E}} \sqrt{E - x^2} \, dx = \pi E$$

Thus the Bohr-Sommerfeld rules tell us that

$$\pi E_n + h\pi + \sum_{j=2}^{\infty} S_j(E_n) h^j - 2\pi n h = O(h^\infty). \tag{1.10}$$

As computed in Lemma 3.2, the n^{th} eigenvalue of the semiclassical harmonic oscillator is $h(2n+1)$. However, the result of Colin de Verdiere indexes eigenvalues starting at $n = 1$, so in the above discussion we take $E_n = h(2n-1)$. Thus we have cancellation in (1.10), and all other action terms vanish to order $O(h^\infty)$,

$$\sum_{j=2}^{\infty} S_j(E_n) h^j = O(h^\infty).$$

And so we have the rather simple expression of the semiclassical action of the quantum harmonic oscillator, $S_h(E_n) = \pi E_n + h\pi + O(h^\infty)$.

Now, consider perturbations to the harmonic oscillator, with hamiltonian

$$p(x,\xi) = x^2 + \gamma(x) + \xi^2, \quad \gamma \in C_0^\infty(\mathbb{R} \setminus \{0\}),$$

where γ is small enough so as to maintain monotonicity in a neighborhood of its support. For $E < \inf\{|x| : x \in \operatorname{supp}(\gamma)\}$ we have that principle action integral S_0 of the perturbed harmonic oscillator coincides with the unperturbed harmonic oscillator, and hence, eigenvalues of the perturbed oscillator agree with the unperturbed oscillator modulo $O(h^\infty)$.

Chapter 2

Inverse Spectral Problems

Given a sufficiently regular set Ω, for example a smooth bounded subset of euclidean space or the entirety a Riemannian manifold, and self-adjoint operator $A : \mathrm{Dom}(A) \subseteq L^2(\Omega) \to L^2(\Omega)$, when does the spectrum of A prescribe the geometry of A, or another geometrical aspect of the operator A? Questions of this form are known as inverse spectral geometry problems. Interest in these types of questions grew rapidly after Mark Kac's seminal work *Can One Hear the Shape of the Drum?* [Kac66], where he explores the following question:

> **The Inverse Spectral Problem for Bounded Plane Domains**
>
> Let $\Gamma_1, \Gamma_2 \subseteq \mathbb{R}^2$ be two simple closed curves bounding domains Ω_1, Ω_2. If the spectrum of the Laplacians $\Delta_1 : \mathrm{Dom}(\Delta_1) \subseteq L^2(\Omega_1) \to L^2(\Omega_1)$ and $\Delta_2 : \mathrm{Dom}(\Delta_2) \subseteq L^2(\Omega_2) \to L^2(\Omega_2)$ with either Dirichlet or Neumann boundary conditions are equal, is it necessarily the case that $\Omega_1 = \Omega_2$ up to isometry?

This is known as the *inverse spectral problem for bounded plane domains.* Before we begin on the discussion of the state of the art on the above problem and related questions, let us connect the problem to its provocative title.

Consider the bounded plane domain Ω and the wave initial value problem with Dirichlet boundary conditions

$$\begin{cases} \frac{\partial^2 F}{\partial t^2} = c^2 \Delta F \\ F|_{\partial\Omega} = 0 \\ F(x,0) = f(x) \\ F_t(x,0) = g(x). \end{cases} \tag{2.1}$$

Interpreted physically, solutions to this problem model the motion of a membrane stretched taught and held fixed along the boundary curve $\partial\Omega$, subject to initial profile $f(x)$ and velocity $g(x)$. Of interest to the physicist are the *normal modes*, those solutions to (2.1) of the form

$$F(x,t) = U(x)e^{ic\omega t}.$$

The oscillation of the membrane in a normal mode produces compression waves of frequency ω, which is then propagated to the human ear and transduced as a tone of frequency ω. If we plug the normal mode into the the wave equation (2.1), we arrive at the Laplace eigenvalue equation on Ω with Dirichlet boundary conditions

$$\begin{cases} -\Delta_\Omega U = \omega_\Omega^2 U \\ U|_{\partial\Omega} = 0. \end{cases}$$

And so, the eigenvalues of the Laplacian on Ω with Dirichlet boundary conditions are the square of the tones of which the drum Ω produces.

2.1 Bounded Domains in Euclidean Space

Let us broaden Kac's question to higher dimensions and introduce some notation. Let $0 \leq \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$ be the eigenvalues of the nonnegative Euclidean Laplacian on the smooth bounded domain $\Omega \subseteq \mathbb{R}^n$, $\Delta_\Omega = -\left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right)$, with either Dirichlet or Neumann boundary conditions.

A natural starting point for this problem is to ask what quantitative aspects of the geometry are spectral invariants? A first result of this sort, known as Weyl's law for the Laplacian on bounded Euclidean domains, preceded Kac's paper by roughly 50 years. Weyl's law is the following asymptotic statement

$$N(\lambda) = \#\{\lambda_j : \lambda_j \leq \lambda\} = \frac{\omega_n \mathrm{Vol}(\Omega)}{(2\pi)^n} \lambda^{\frac{n}{2}} (1 + o(1)) \qquad \text{as } \lambda \to +\infty \tag{2.2}$$

where $N(\lambda)$ is the number of eigenvalues less than λ and ω_n is the volume of the unit ball. This was first conjectured by Hilbert, and then several years later proven by Hermann Weyl in the case for planar domains [Wey11]. The above asymptotic shows that the volume of the domain is a spectral invariant. Moreover, the perimeter [Ple54] of the domain and Euler characteristic [MJS67] were proven to be spectral invariants. These geometric quantities, among others, are contained in the asymptotic expansion of what is known as the heat trace of Δ_Ω.

It turns out that Kac's inverse spectral problem cannot be answered in the affirmative. Gordon, Webb, and Wolpert gave examples of distinct planar domains with the same spectrum [GWW92]. These examples are non-convex and non-smooth, and to the knowledge of the author it remains an open question if there exists a pair of non-isometric smooth and convex domains with the same eigenvalues.

2.1.1 Heat Trace Invariants

An elementary tool in the study of inverse spectral geometry is the heat trace. Before we begin, we need a definition from the theory of distributions.

Definition 2.1. *Let $\mathcal{D}(\Omega)$ be the vector space of functions $C_0^\infty(\Omega)$ endowed with the following notion of convergence: $\varphi_m \in C_0^\infty(\Omega)$ converges to $\varphi \in C_0^\infty(\Omega)$ if there exists a compact subdomain $\overline{B} \subseteq \Omega$ such that $\operatorname{supp}\varphi_n \subseteq \overline{B}$ and that $\frac{\partial^k \varphi_m}{\partial x^k} \to \frac{\partial^k \varphi}{\partial x^k}$ uniformly on $\overline{B}$ for all $0 \leq |k| < \infty$. The space of **distributions** over Ω, $\mathcal{D}'(\Omega)$, is the space of continuous linear functionals over $D(\Omega)$.*

Definition 2.2. *Let λ_j be the eigenvalues of the Laplacian Δ_Ω on the bounded domain $\Omega \subseteq \mathbb{R}^n$ with either Dirichlet or Neumann boundary conditions. The **heat trace** of Δ_Ω is the distribution $\mathrm{Tr}\ e^{-t\Delta_\Omega} \in \mathcal{D}'(\mathbb{R}_{>0})$ given by*

$$\mathrm{Tr}\ e^{-t\Delta_\Omega} := \sum_{j=0}^{\infty} e^{-t\lambda_j},$$

where convergence of the above sum is understood in the space of distributions. Meaning, for each $\varphi(t) \in \mathcal{C}_0^\infty(\mathbb{R}_{>0})$, $\int_0^\infty \sum_{j=0}^\infty e^{-t\lambda_j}\varphi(t)\,dt = \lim_{N\to\infty} \int_0^\infty \sum_{j=0}^N e^{-t\lambda_j}\varphi(t)\,dt$.

Note that the heat trace distribution is well defined as a consequence of Weyl's law (2.2).

For planar domains ($n = 2$) Kac reformulated the area spectral invariant of Weyl and the perimeter spectral invariant of Pleijel in terms of small t asymptotic of the heat trace.

Theorem 2.1 ([Kac66])**.** *For smooth planar domains, the heat trace admits the following asymptotic expansion*

$$\mathrm{Tr}\ e^{-t\Delta_\Omega} \sim t^{-1} \sum_{j=0}^{\infty} a_j t^{j/2}, \quad t \to 0_+,$$

with first coefficients

$$a_0 = \frac{vol(\Omega)}{2\pi}$$
$$a_1 = \frac{per(\Omega)}{4\sqrt{2\pi}}.$$

where vol(Ω) *is the volume of the domain* Ω*, and per*(Ω) *is the length of* $\partial\Omega$*.*

We note here that the heat trace in dimensions $n \geq 2$ admits the following expansion,

$$\text{Tr } e^{-t\Delta_\Omega} \sim t^{-n/2} \sum_{j=0}^{\infty} a_j t^{j/2}, \quad t \to 0_+,$$

where again the first and second coefficients are proportional to the n-dimensional volume of Ω, and the $(n-1)$-dimensional volume of $\partial\Omega$.

The most basic inverse spectral result for bounded plane domains is the spectral uniqueness of balls.

Remark 2.1. *Let* $\Omega \subseteq \mathbb{R}^n$ *be a smooth, bounded plane domain. Then* $\text{Spec}(\Delta_\Omega) = \text{Spec}(\Delta_B)$*, for some ball* $B \subseteq \mathbb{R}^n$ *if and only if* $\Omega = B$ *up to isometry. To see this, we use the isoperimetric inequality:*

$$\text{per}(\Omega) \geq n\,(\text{vol}(\Omega))^{(n-1)/n}\,\omega_n^{1/n},$$

where ω_n *is the volume of the unit ball, and vol*$(\cdot)$*, per*$(\cdot)$ *are the* n *and* $(n-1)$*-dimensional volume measures on the interior and boundary of* Ω*, respectively. Note that the above inequality is equality only when* Ω *is a ball. Using the first two heat trace invariants, we must have that the perimeter and volume of* Ω *are equal to that of the ball* B*. But then equality holds in the isoperimetric inequality for* Ω*, and thus* Ω *must be a ball, and hence isomorphic to* B*.*

The above result is quite strong, it states that a specific geometry is spectrally unique amongst the quite broad class of smooth functions. In fact, the result can be further strengthened to the case of Lipschitz domains. Brown [Bro93] establishes asymptotic formulas analogous to Theorem 2.1 for Lipschitz domains with boundaries of integer Hausdorff dimension.

For over a half century there has not been a new result as strong as Remark 2.1, until in 2022 Hezari and Zelditch prove that nearly circular ellipses are spectrally unique amongst all smooth bounded plane domains [HZ22], making no assumptions about analyticity, symmetry, or closeness to the ellipse on the class of domains. Their result combines the theory of billiard dynamics and microlocal analysis, in particular using an asymptotic expansion of the wave trace $\operatorname{Tr}\cos(t\sqrt{\Delta_\Omega})$.

2.1.2 Wave Trace Invariants

One of the most fruitful techniques in the study of inverse spectral problems is the analysis of singularity expansions of the wave trace $\operatorname{Tr}\cos(t\sqrt{\Delta_\Omega})$.

Definition 2.3. *Let λ_j be the eigenvalues of the Laplacian Δ_Ω on the smooth bounded domain $\Omega \subseteq \mathbb{R}^n$. The **wave trace** of Δ_Ω is the distribution $\operatorname{Tr}\cos(t\sqrt{\Delta_\Omega}) \in \mathcal{D}'(\mathbb{R}_{>0})$ given by*

$$\operatorname{Tr}\cos(t\sqrt{\Delta_\Omega}) := \sum_{j=0}^{\infty} \cos(t\sqrt{\lambda_j}).$$

Guillemin and Melrose [GM79] show that the singular support of the wave trace is contained in the closure of the length spectrum of Ω, denoted $\operatorname{Lsp}(\Omega)$, which is the set of all lengths of periodic billiard trajectories on Ω. Additionally, they provide a singularity expansion near lengths of simple, non-degenerate periodic billiard trajectories T.

Theorem 2.2 ([GM79]).

$$\operatorname{Tr}\cos(t\sqrt{\Delta_\Omega})$$
$$= Re\left[i^{\sigma_T}\frac{T^\#}{\sqrt{|\det(I-P_T)|}}(t-T+i0)^{-1}\left(1+\sum_{j=1}^{\infty}a_j(t-T)^j\log(t-T+i0)\right)\right]+S(t) \tag{2.3}$$

where S is smooth near T, $T^\#$ is the primitive length of T, and σ_T is the Maslov index of γ_T (see [GM79]). Simple trajectories meaning there exists a single periodic trajectory of this length, up to time reversal, and non-degenerate meaning the trajectory γ_T of length T intersects the boundary transversally, and that P_T, the linearized Poincaré map does not have eigenvalue 1, where the Poincaré maps is the derivative of the first return map.

The singularity expansion (2.3) has been used primarily to study analytic domains, or piecewise-analytic symmetric domains. This is due to the fact that the *wave invariants* a_j are polynomials in the Taylor coefficients centered at the points of reflection of the billiard trajectory. Leveraging this , Zelditch [Zel09] proves that piecewise-analytic plane domains possessing a $\mathbb{Z}_2$ symmetry, along with a few other conditions, are spectrally determined in this class. Specifically, if Ω_1, Ω_2 satisfy

1. Ω_i is simply connected, symmetric about the x-axis, and $\partial\Omega_i$ is analytic on $\{y \neq 0\}$
2. There is a non-degenerate vertical bouncing ball orbit γ of length T such that both T and $2T$ are simple lengths in the length spectrum
3. The endpoints of γ are not critical points of the curvature of $\partial\Omega_i$

then if $\operatorname{Spec}(\Omega_1) = \operatorname{Spec}(\Omega_2)$, we must have $\Omega_1 = \Omega_2$, up to isometry. The proof recovers the wave trace invariants from the common spectrum using the expansion (2.3) along the bouncing ball orbit, and then determines the Taylor coefficients from the wave invariants.

Most results of this type require strong assumptions on the domain under consideration, such as analyticity, symmetry, convexity, etc. For example, for bounded analytic domains $\Omega \subseteq \mathbb{R}^n$ with $\pm$ reflection symmetries across all axes, with one axis height fixed, and some generic non-degeneracy conditions, Hezari and Zelditch [HZ12] prove that domains of this type are spectrally determined using similar techniques to that of [Zel09]. A natural question to ask is if wave trace invariants can be used to solve the inverse spectral problem for smooth, but not necessarily analytic domains. It turns out that one cannot use these invariants.

2.1.3 Wave Trace Invariant Limitations

Zelditch in [Zel04] asks the question, does there exist a pair of non-isometric smooth planar domains with the same wave trace invariants? Fulling-Kuchment in [FK05] answer the question in the affirmative. The non-isometric smooth planar domains Ω, Ω' satisfying this property are called Penrose-Lifshits mushrooms (Figure 2.1). Specifically, the associated wave traces are equal modulo a smooth function, $\mathrm{Tr}(\cos(t\sqrt{\Delta_\Omega})) - \mathrm{Tr}(\cos(t\sqrt{\Delta_{\Omega'}})) \in C^\infty(\mathbb{R})$. This means that the wave traces possess the same singular structure, hence, the same wave trace invariants, rendering wave trace methods unable to distinguish these domains.

The Penrose-Lifshits mushrooms are constructed as follows, take an ellipse and flatten it along its major axis. Smooth out the flattening at either end with perturbations A, B. To obtain Ω, in between the foci of the ellipse, insert a perturbation C. To obtain Ω', reflect C about the midpoint of the foci to get C'. That these two domains possess the same wave-trace invariants follows from an analysis of the billiard trajectories. The idea behind this result is that a billiard trajectory γ on the ellipse falls into one of three categories:

(1) γ crosses the x-axis between the two foci, in which case γ is periodically tangent to a hyperbolic caustic

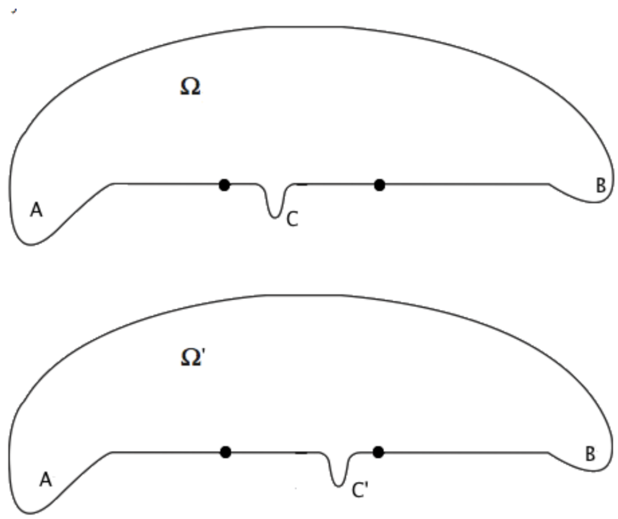

Figure 2.1: Two non-isometric domains with the same wave trace invariants.

(2) γ crosses the x-axis outside of the two foci, and so γ is periodically tangent to a elliptic caustic

(3) γ only crosses the x-axis through the foci.

With this characterization in mind, we see that if γ is a trajectory of type (1) in Ω, then that exact same geodesic lies within Ω', since γ does not pass between the foci. A similar argument holds for trajectories of type (3). If γ is a trajectory of type (2) in Ω, then the reflection about the vertical axis of γ is a geodesic of type (2) in Ω', and vice versa. This tells us that the points of reflection of the periodic geodesics of a given length of Ω and Ω' have the same Taylor coefficients, and hence the same wave trace invariants.

2.2 Closed Manifolds

An analogous question can be asked for spaces more general than bounded domains in $\mathbb{R}^n$. Let M be a smooth closed manifold, which is a compact smooth manifold without boundary. Moreover, equip M with a Riemannian metric g, which is a smooth symmetric covariant 2-

tensor field that is positive definite in the sense of $g_p(v,v) \geq 0$ for each $p \in M$ and each $v \in T_pM$, with equality if and only if $v = 0$. If M is expressed locally in coordinates (x^i), we may write the metric in the form $g = g_{ij}dx^i \otimes dx^j$, adopting the Einstein summation convention. Alternatively, in local coordinates one may view the metric as a matrix

$$g(v,w) = v^T(g_{ij})w.$$

The metric g gives rise to notion of divergence, and hence a Laplacian on M.

Definition 2.4. *Let (M,g) be a Riemannian Manifold with or without boundary, and let (x^i) be any smooth local coordinates on an open set $U \subseteq M$. The coordinate representation of the* ***Laplacian*** *or* ***Laplace-Beltrami*** *operator is as follows*

$$\Delta_g u = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i} \left(g^{ij} \sqrt{\det g} \frac{\partial u}{\partial x^j} \right),$$

where g^{ij} is the ij^{th} component of the inverse of the component matrix g.

This leads us to the analogous inverse spectral problem

The Inverse Spectral Problem for Closed Manifolds

Let $(M_1,g_1),(M_2,g_2)$ be two Riemannian manifolds. If $\text{Spec}(\Delta_{g_1}) = \text{Spec}(\Delta_{g_2})$, is it necessarily the case that there exists an isometry between M_1 and M_2? An isometry between Riemannian manifolds is a diffeomorphism $F : M_1 \to M_2$ that that preserves the metric, i.e. $g_1(v,w) = g_2(F(v),F(w))$.

It appears that the only spectral uniqueness results for closed manifolds are for surfaces of revolution, and there are two known approaches to the problem. The first approach is to separate variables in the eigenvalue equation of Δ_g, and obtain eigenvalue equations of a Schrödinger operator in terms of an arc length, and spherical harmonics in the rotation parameter. Then one asks if you can determine the metric from the joint spectrum. Bérard [Bér76] and Gurarie [Gur95] show that the joint spectrum of a smooth surface of revolution determines the metric. The second approach, taken by Zelditch [Zel98], is to consider analytic surfaces of revolution (satisfying a non-degeneracy condition) and show that the singularity expansion of the wave trace determines the quantum normal form of Δ_g, which in turn determines the metric.

2.2.1 Trace Invariant Limitations

The wave trace (2.3) singularity expansion of Guillemin and Melrose [GM79] was initially explored in the setting of a closed manifold M (without boundary) first by Chazarain [Cha74] and then Duistermatt and Guillemin [DG75], where they provide an explicit formula for the leading order term. A more detailed analysis of the higher order terms appears in the set of lecture notes by Zelditch [Zel99]. Common to these formulas is the simplifying assumption that one expands the wave trace around simple lengths in the length spectrum, that is lengths which correspond to a single geodesic. Expanding along lengths corresponding to many distinct geodesics is a complicated, and virtually unexplored problem. However, Zelditch [Zel98] studies these singularity expansions for surfaces of revolution of simple type, which possess many one-parameter families of closed geodesics of a common length. Zelditch studies analytic surfaces of revolution with a single closed equatorial geodesic. However, if we remove the analyticity assumption, does Zelditch's result still hold?

Conjecture 2.1. *There exists a pair of smooth, non-isomorphic surfaces of revolution with the same wave-trace invariants.*

In chapter 4 we explore the above conjecture, by way of constructing surfaces of revolution analogous to those of Fulling-Kuchment [FK05].

2.3 Inverse Potential Problems

We now let the underlying domain be the entirety of Euclidean space, and consider the semiclassical Schrödinger operator. Let us ask the analogous question, can one hear the shape of the potential? Or from the perspective of quantum mechanics, do the energy levels of the stationary states of a quantum system determine the potential? Since we are concerning ourselves with the semiclassical Schrödinger operator, we often only make assumptions on the small h asymptotic behavior of the spectrum.

Definition 2.5. *Let $j = 1, 2$, and H_j be two semiclassical Schrödinger operators with spectra* $\mathrm{Spec}(H_j) = \{E_{j,0}, E_{j,1}, E_{j,2}, \dots\}$*, and let $N = 1, 2, \dots, \infty$. Their spectra* ***agree up to order*** *$o(h^N)$, or $\mathrm{Spec}(H_1) = \mathrm{Spec}(H_2) + o(h^N)$, if there exists $E > 0$, such that for $k = 0, 1, 2, \dots$*

$$\lim_{h \to 0^+} \frac{E_{1,k} - E_{2,k}}{h^N} = 0,$$

uniformly for $E_{\cdot,k} < E$, in the case that $N \neq \infty$. In the case that $N = \infty$ the above limit must hold uniformly for $E_{\cdot,k} < E$ and for all N.

We also consider the following stronger asymptotic classification.

Definition 2.6. *Let $N = 1, 2, \dots, \infty$. The spectra of the semiclassical Schrödinger operators H_1, H_2* ***agree up to order*** *$O(h^N)$, or $\mathrm{Spec}(H_1) = \mathrm{Spec}(H_2) + O(h^N)$, if there exists an $E > 0$ such that for each $k = 0, 1, 2, \dots$ we have a positive constant C_k satisfying*

$$\sup_{\{E_{\cdot,k} < E\}} |E_{1,k} - E_{2,k}| \le C_k h^N, \quad \text{for all } 0 < h,$$

in the case that $N \neq \infty$. In the case that $N = \infty$ the above inequality must hold for all N with constants $C_{k,N}$

With these definitions in mind, we may phrase the corresponding inverse problem.

The Inverse Spectral Problem for Semiclassical Schrödinger Operators Let $j = 1, 2$, and let $V_j \in C^\infty(\mathbb{R}^n; \mathbb{R})$, with $\lim_{|x|\to\infty} V_j(x) = \infty$ define semiclassical Schrödinger operators $H_j = -h^2\Delta + V_j(x)$. If $\mathrm{Spec}(H_1) \sim \mathrm{Spec}(H_2)$, must it be the case that $V_1(x) = V_2(x)$, up to isometry? The equivalence $\sim$ above may be taken to be equivalence modulo $o(h^N)$ or $O(h^N)$, for some $N = 1, \ldots, \infty$, or equality.

Before discussing the above question, we briefly consider the analogous non-semiclassical problem, that is, do the eigenvalues of the operator $-\Delta + V(x)$, with the above assumptions on V, uniquely determine V? McKean and Trubowitz [MT82] answered this question in the negative, providing an infinite-dimensional family of smooth potentials isospectral to the harmonic oscillator $-\Delta + x^2$.

Returning to the semiclassical problem, there have been several positive results for radially symmetric potentials. These results are analogous to the spectral uniqueness of balls in the class of bounded domains. Colin de Verdiére [CdV11] proves that a one-dimensional semiclassical Schrödginer operator with smooth potential having a non-degenerate unique local minimum (and root) at $x = 0$, satisfying a symmetry defect, is determined by its spectrum up tor order $o(h^2)$ in an interval $(-\infty, E]$, for some $E > 0$, amongst all smooth potentials. In particular, even potentials are determined by their spectral modulo $o(h^2)$. Guillemin and Wang [GW12] explore this result and show in the case of dimension 2, radially symmetric potentials are spectrally determined amongst all smooth potentials by their low-lying eigenvalues, up to order $o(h^2)$. Datchev, Hezari, and Ventura [DHV11] prove spectral uniqueness up to $o(h^2)$ of smooth radial potentials in dimension $n \geq 2$, with a more general result that if the potential is radially symmetric in a ball around the minimum, then the the potential is spectrally determined within this ball.

There are no other known results that are as strong as the case for radially symmetric potentials; all other known results determine the potential only among a class of potentials defined by an analyticity or symmetry requirement. Guillemin and Uribe [GU07] consider potentials V in $\mathbb{R}^n$ with unique non-degenerate local minimum at 0 is symmetric with respect to all coordinate axes, $V^{-1}([0,\epsilon])$ is compact for some $\epsilon > 0$, and that the square root of all 2nd order Taylor coefficients are linearly independent over $\mathbb{Q}$. They show that the Taylor coefficients of potentials of this form are determined by their low-lying eigenvalues, and hence if V is analytic, then the low lying eigenvalues determine V. Hezari [Hez09] replaces the symmetry assumption with, in the case dimension $n = 1$, $V'''(0) \neq 0$, and for $n \geq 2$ the potential $V(x) = f(x_1^2, \dots, x_n^2) + x_n^3 g(x_1^2, \dots, x_n^2)$ for some smooth f, g.

2.3.1 Semiclassical Trace Invariants

The proofs of [Hez09], [CdV11], [DHV11], and [GW12] rely on spectral invariants derived from trace formulas analogous to (2.3). In particular, the wave invariants used in [Hez09] are obtained by localizing the Schrödinger propagator to the low-lying eigenvalues by way of a cutoff $\Theta(x) \in C_0^\infty(\mathbb{R})$ which is equal to 1 on a neighborhood of 0

$$\mathrm{Tr}(\Theta(P)e^{-\frac{it}{h}P}) = \sum_{j=0}^{\infty} a_j(t)h^j + O(h^\infty),$$

where $P = -\frac{h^2}{2}\Delta + V(x)$ is the semiclassical Schrödinger operator associated to V, and the sum is understood in the sense of distributions. For appropriate potentials, Hezari provides explicit formulas for the a_j, and shows that the a_j is a polynomial in the Taylor coefficients of V at its minimum.

In [CdV11], [DHV11], and [GW12], the authors take advantage of different spectral invariants arising from a different trace formula. Let $f \in C_0^\infty((-\infty, E])$, for some $E > 0$ guaranteeing that $V^{-1}((-\infty, E])$ is compact. Then, letting H be the semiclassical Schrödinger operator associated to V, we consider the asymptotic expansion

$$\operatorname{Tr} f(H) = \sum_{j=0}^{\infty} \nu_j(f) h^j + O(h^\infty).$$

In [DHV11], the authors take advantage of the first two invariants above, and in [GW12], the authors provide an algorithm for computing the higher order terms.

Remark 2.2. *If H, H' are semiclassical Schrödinger operators satisfying*

$$\operatorname{Spec}(H) = \operatorname{Spec}(H') + O(h^\infty),$$

then they have the same semiclassical invariants. Indeed, let $f \in C_0^\infty((-\infty, E])$, for the $E > 0$ guaranteeing $O(h^\infty)$ agreement of spectra, then

$$|\,Tr\, f(H) - Tr\, f(H')| = \left| \sum_{j=0}^{M_h} f(E_j(h)) - f(E_j'(h))) \right|,$$

where by Weyl's law (Theorem 1.8) we have that $\frac{1}{M_h}$ is a polynomial in h. Since f is Lipschitz, we have

$$|\,Tr\, f(H) - Tr\, f(H')| \leq C \cdot \sum_{j=0}^{M_h} \left| E_j(h) - E_j'(h) \right|.$$

Since $\operatorname{Spec}(H) = \operatorname{Spec}(H') + O(h^\infty)$,

$$|\,Tr\, f(H) - Tr\, f(H')| \leq C_M \cdot M_h h^M,$$

for any M. Finally, since $\frac{1}{M_h}$ is a polynomial in h, for a sufficiently large choice of M we have that $Tr\, f(H) = Tr\, f(H') + O(h^\infty)$, and so H, H' have the same semiclassical invariants. A similar argument holds for the invariants of [Hez09].

Following the above remark, we note that arguments using the above trace invariants fail to distinguish potentials if one is working within a class of potentials that contains pairs of non-isomorphic potentials with spectra agreeing up to $O(h^\infty)$. Guillemin and Hezari provide a counterexample in the spirit of Fulling-Kuchment [FK05],

Theorem 2.3 ([GH12])**.** *There exists a pair of smooth, non-analytic potentials* $V^\pm(x)$, $V(x) \geq 0$, *such that the operators*

$$H_\pm = -h^2 \frac{d^2}{dx^2} + V^\pm(x)$$

satisfy $\mathrm{Spec}(H_+) = \mathrm{Spec}(H_-) + O(h^\infty)$, *yet their ground state eigenvalues differ for* $h > 0$, *except for possibly a subsequence* $h_k \to 0^+$.

The potentials, $V_\pm$ are constructed by taking the harmonic oscillator potential x^2 and adding two bumps, then reflecting the outer most bump across the y-axis (Figure 2.2). Note that the bumps in Figure 2.2 are highly exaggerated to provide visual intuition, we must make the bumps small enough so as to not introduce any new local minima, and to maintain monotonicity away from 0.

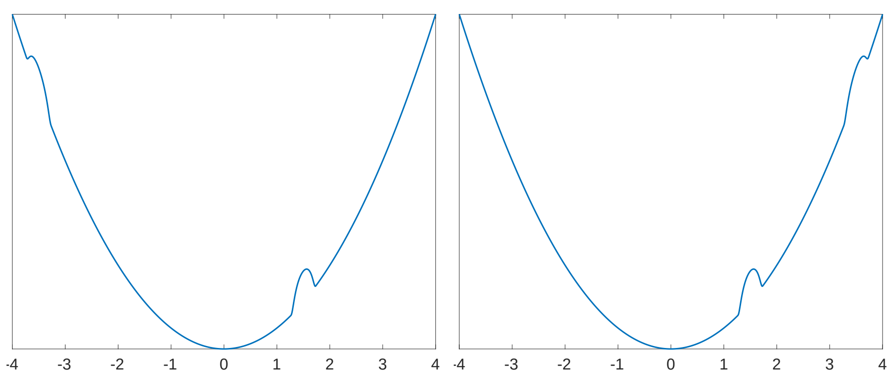

Figure 2.2: Two potential functions $V^\pm$ whose corresponding semiclassical spectra agree up to $O(h^\infty)$, yet whose eigenvalues are distinct (V^- left, V^+ right).

Guillemin and Hezari establish their result by way of analyzing the ground state eigenfunctions of the associated non-semiclassical Schrödinger operator obtained by setting $h = 1$. By taking the perturbations to be small enough, they show by way of Hadamard's variational formula that ground state eigenvalues for $h = 1$ disagree. Then, by the Kato-Rellich theorem ([RS78], Theorem XII.8) the ground state eigenvalues are analytic functions of $h > 0$, and so they can only agree possibly on a discrete set $h_k \to 0$. In Chapter 3, the author improves Theorem 2.3 of Guillemin and Hezari with the follow result.

Theorem 2.4. *There exist pairs of non-isometric potentials $V^{\pm}(x) \in C^{\infty}(\mathbb{R})$ with $V(x) \geq 0$ defining Schrödinger operators $H^{\pm} = -h^2 \frac{d^2}{dx^2} + V^{\pm}(x)$ whose semiclassical spectra agree modulo $O(h^{\infty})$, yet their eigenvalues $E_j^{\pm}$ differ for all $h > 0$, and $j \in \mathbb{Z}_{\geq 0}$. Moreover,*

$$D_j e^{-\frac{d_j}{h}} \leq E_j^- - E_j^+ \leq C_j e^{-\frac{c_j}{h}},$$

for constants $c_j, d_j, C_j, D_j > 0$, holding for all $0 < h < h_j$ with $h_j \to 0$.

Chapter 3

Semiclassical Trace Invariant Limitations

3.1 Exponential Estimates for Eigenfunctions

In this section we will restrict our attention to perturbations of the semiclassical harmonic oscillator:

$$H = -h^2 \frac{d^2}{dx^2} + x^2 + \gamma(x), \quad \gamma \in C_0^\infty(\mathbb{R} \setminus \{0\}),$$

where γ is chosen so that the potential $V(x) = x^2 + \gamma(x)$ has a unique global minimum at 0 and is monotonic away from 0. Denote the semiclassical harmonic oscillator $H_0 = -h^2 \frac{d^2}{dx^2} + x^2$. We will develop upper and lower locally uniform estimates for the ground states of these operators, which we will ultimately transfer to eigenvalue bounds using Hadamard's variational formula. Theorem 2.4 will follow from a judicious choice of γ, analogous to the domains shown in Figure 2.1.

To begin we require the following characterization of the perturbed harmonic oscillator spectra,

$$\mathrm{spec}(H) = \mathrm{spec}(H_0) + O(h^\infty) = \{h, 3h, 5h, 7h, \dots\} + O(h^\infty).$$

Using the methods of quantom-birkhoff normal forms at the bottom of a potential well established by Sjöstrand [Sjö92], or Borh-Sommerfeld quantization to all orders given by Colin de Verdiere in Theorem 1.9, one can see that the above holds. Moreover, any two perturbations of the harmonic oscillator have spectra which agree up to $O(h^\infty)$. This fact is underpinned by the observation that the level sets of the Hamiltonians corresponding to $H^\pm$ enclose the same area, see Figure 3.1. Thus, all semiclassical action terms S_j from (1.9) are the same, when considering eigenvalues in an interval $(0, E]$ bounded away from the support of the perturbation.

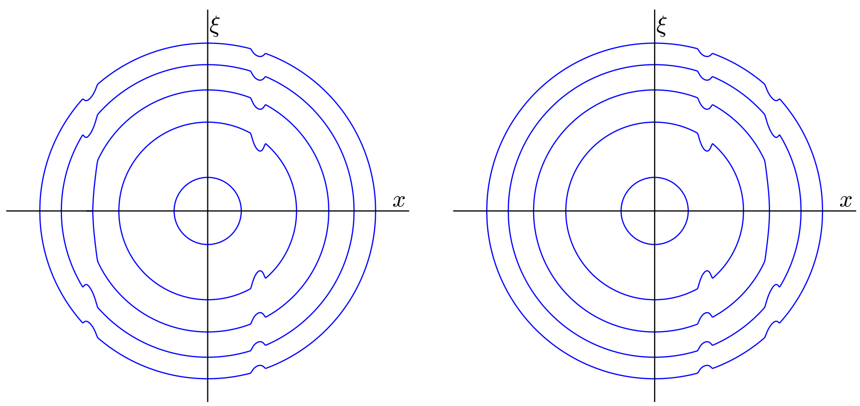

Figure 3.1: Level sets of the Hamiltonians corresponding to $H^\pm$ (H^- left, H^+ right).

Proposition 3.1. *Let* $H = -h^2 \frac{d^2}{dx^2} + V(x)$ *be a semiclassical Schrödinger operator with real valued potential of the form* $V(x) = x^2 + \gamma(x)$*, for some* $\gamma \in C_0^\infty(\mathbb{R})$ *supported away from the origin. Let* $\psi_j(x)$ *denote the* L^2 *normalized eigenfunctions of* H*. Then, for any* $\delta, r, R \in (0, \infty)$ *with* $r < R$*, there exist constants* $C_j, D_j > 0$ *such that for all* $h > 0$

sufficiently small, the following estimates hold uniformly on $r < |x| < R$

$$(i) \qquad |\psi_j(x)| \leq C_j e^{-\frac{1}{h}(1-\delta)^2\left|\int_0^x \sqrt{V(x)}\,dx\right|}$$

$$(ii) \qquad |\psi_j(x)| \geq D_j e^{-\frac{1}{h}(1+\delta)^2\left|\int_0^x \sqrt{V(x)}\,dx\right|}$$

Remark 3.1. *A similar proposition appears in the work of Simon [Sim84], with some important differences. Simon studies the ground state eigenfunctions of the double well potential, in contrast to this paper where we study the single well potential. Additionally, the above result holds for each eigenfunction, whereas in the paper of Simon, the analogous result was only established for the ground state.*

Proof of Proposition 3.1.(i). Define $\tilde{\varphi}(x) = \left|\int_0^x \sqrt{V(t)}\,dt\right|$. Note that for each choice of positive constants σ, r, R, δ there exists a smooth mollifier f which satisfies the following:

(1) on the annulus $r < |x| < R,\ f * \tilde{\varphi} \leq \tilde{\varphi} + \sigma$ and $f * (\tilde{\varphi}') \leq \tilde{\varphi}' + \sigma$

(2) there exists positive constants a, k with $a < r,$ such that for all $|y| \leq a$ and $r < |x| < R,$ we have $\delta(f * \tilde{\varphi})(x) - (f * \tilde{\varphi})(y) \geq k > 0.$

Let $\epsilon > 0$ and define $A = \inf\left\{(1-\epsilon)\frac{\sqrt{V(x)}}{\sqrt{V(x)}+\sigma} : r < |x| < R\right\}$. Finally, define $\varphi = (Af) * \tilde{\varphi}$. We may choose ϵ, σ sufficiently small so that

$$(1-\delta)\tilde{\varphi} \leq (1-\delta)(f * \tilde{\varphi}) \leq A(f * \tilde{\varphi}),$$

and

$$A(f * \tilde{\varphi}) \leq A(\tilde{\varphi} + \sigma) \leq \tilde{\varphi} + \sigma \leq \tilde{\varphi} + \delta\tilde{\varphi} = (1+\delta)\tilde{\varphi}.$$

Thus,

$$(1-\delta)\tilde{\varphi} \leq \varphi \leq (1+\delta)\tilde{\varphi}. \tag{3.1}$$

Also,

$$|\varphi'(x)| \leq A(\sqrt{V(x)} + \sigma) \leq (1-\epsilon)\sqrt{V(x)}. \tag{3.2}$$

Without loss of generality we will smoothly re-define φ such that for $|x| \geq 2R$, $\varphi'(x) = 0$. Recall that E_j is the j^{th} eigenvalue of H, and let ψ be any smooth function. We will now introduce the key object which gives us the upper bound of Proposition 3.1.(i):

$$\left\langle e^{\frac{\varphi}{h}}\psi, (H-E_j)e^{-\frac{\varphi}{h}}\psi \right\rangle = \langle \psi, (V-E_j)\psi\rangle - \left\langle \psi, \left(h\frac{d}{dx} - \varphi'\right)^2 \psi \right\rangle.$$

By the fundamental theorem of calculus, and that φ' is eventually 0, we have that

$$\left\langle \psi, \left[\left(h\frac{d}{dx}\right)\varphi' + \varphi'\left(h\frac{d}{dx}\right)\right]\psi \right\rangle = 0.$$

So

$$\left\langle e^{\frac{\varphi}{h}}\psi, (H-E_j)e^{-\frac{\varphi}{h}}\psi \right\rangle = \left\langle \psi, \left[\left(-h^2\frac{d^2}{dx^2} - (\varphi')^2\right) + V - E_j\right]\psi \right\rangle.$$

By bound (3.2), and since $-\frac{d^2}{dx^2}$ is positive definite we arrive at the following bound:

$$\left\langle e^{\frac{\varphi}{h}}\psi, (H-E_j)e^{-\frac{\varphi}{h}}\psi \right\rangle \geq \langle \psi, [\epsilon V - E_j]\,\psi\rangle\,.$$

Recall that the eigenvalue $E_j = h(2j+1) + O(h^\infty)$, so for all $|x| > a/2$ and h sufficiently small, there exist a $C > 0$ satisfying $\epsilon V(x) - E_j \geq \frac{1}{C} > 0$. Taking ψ supported in $|x| > a/2$, we get

$$||\psi||_2^2 \leq C \left\langle e^{\frac{\varphi}{h}}\psi, (H - E_j)e^{-\frac{\varphi}{h}}\psi \right\rangle. \tag{3.3}$$

Let $\psi = e^{\frac{\varphi}{h}}\eta\psi_j$, where $\eta(x) = 0$ if $|x| < \frac{a}{2}$ and $\eta(x) = 1$ if $|x| > a$, and is smooth. Using the above bound,

$$\int_{|x|>a} e^{\frac{2\varphi}{h}}\psi_j^2\, dx \leq C \left\langle e^{\frac{2\varphi}{h}}\psi_j, (H - E_j)\eta\psi_j \right\rangle.$$

Using the eigenvalue equation, bounding the derivatives of η on $a/2 < |x| < a$, and using monotonicity of φ on $|x| > a/2$, we get

$$\int_{|x|>a} e^{\frac{2\varphi}{h}}\psi_j^2\, dx \leq h^2 C e^{\frac{2\varphi(a)}{h}} \int_{\frac{a}{2}<|x|<a} (K\psi_j'\psi_j + D\psi_j^2).$$

Applying Cauchy-Schwarz, using that $E_j = h(2j+1)+O(h^\infty)$ and the eigenvalue equation to see that for some constant K_j, $||\psi_j'|| = \frac{K_j}{h}+O(h^\infty)$, along with the fact that the eigenfunctions are L^2 normalized we have

$$\int_{|x|>a} e^{\frac{2\varphi}{h}}\psi_j^2\, dx \leq Ce^{\frac{2\varphi(a)}{h}}(K_j h + h^2 D).$$

Finally, for h sufficiently small, we get the following estimate:

$$\int_{|x|>a} e^{\frac{2\varphi}{h}}\psi_j^2\, dx \leq e^{\frac{2\varphi(a)}{h}} C. \tag{3.4}$$

We will now use bound (3.4) to achieve the desired pointwise bound on $r < |x| < R$ by using convexity of $|\psi_j|$ in the forbidden region $|x| > \sqrt{h(2j+1) + O(h^\infty)}$. Without loss of generality assume that $r < x < R$, then for all b satisfying $0 < b < r - a$ and h sufficiently small we have

$$\psi_j^2(x) \leq \frac{1}{2b} \int_{x-b}^{x+b} e^{-2\frac{\varphi}{h}} e^{2\frac{\varphi}{h}} \psi_j^2 \, dt.$$

By the monotonicity of φ in $|x| > a$ and applying bound (3.4) we get

$$\psi_j^2(x) \leq \frac{1}{2b} e^{-2\frac{\varphi(x-b)}{h}} e^{\frac{2\varphi(a)}{h}} C.$$

Let $M = \max_{a<|x|<R} \varphi'(x)$. Then using $-\varphi(x-b) \leq -\varphi(x) + bM$ we get

$$\psi_j^2(x) \leq \frac{C}{2b} e^{-2\frac{\varphi(x)}{h}} e^{2\frac{bM}{h}} e^{\frac{2\varphi(a)}{h}}.$$

By our choice of a which guarantees $\delta\varphi(x) - \varphi(a) \geq k > 0$, we get

$$\psi_j^2(x) \leq \frac{C}{2b} e^{-2(1-\delta)\frac{\varphi(x)}{h}} e^{2\frac{bM-k}{h}}.$$

Taking $b < k/M$ and using bound (3.1) establishes the upper bound of Proposition 3.1:

$$|\psi_j(x)| \leq C e^{-(1-\delta)^2 \frac{\tilde{\varphi}(x)}{h}}.$$

□

Take note that the above lemma only holds for those h satisfying $0 < h < h_j = E_j^{-1}(\epsilon V\left(\frac{a}{2}\right))$, otherwise the estimate (3.3) would not hold. Now, the proof of Proposition 3.1.(ii) is more difficult. We will first need to establish a lower bound for ψ_j on the boundary of the allowed region $\{-\sqrt{E_j}, \sqrt{E_j}\}$. To achieve this we need a fact about the $L^2(\mathbb{R})$ convergence of rescaled eigenfunctions of H.

Lemma 3.1. *Let $\tilde{H} = -\frac{d^2}{dx^2} + x^2 + h^{-1}\gamma(\sqrt{h}x)$ be the rescaled perturbed semiclassical harmonic oscillator, and $\tilde{\psi}_j(x)$ be the eigenfunctions of $\tilde{H}$. Let $\tilde{H}_0 = -\frac{d^2}{dx^2} + x^2$ be the harmonic* *oscillator, and let $\tilde{\kappa}_j$ denote its eigenfunctions. Then $\tilde{\psi}_j(x) \to \tilde{\kappa}_j(x)$ in $L^2(\mathbb{R})$ as $h \to 0$.*

Proof. Define $(U\phi)(x) := h^{-\frac{1}{2}}\phi(h^{-\frac{1}{2}}x)$, and notice that $\tilde{H} = h^{-1}U^{-1}HU$ (and $\tilde{H}_0 = h^{-1}U^{-1}H_0U$). Let $\tilde{E}_j$ denote the eigenvalues of $\tilde{H}$. The rescaling operator U gives us a relationship between the eigenfunctions and eigenvalues of $\tilde{H}$ and H:

$$E_j = h\tilde{E}_j, \quad \psi_j = U\tilde{\psi}_j.$$

From this, and the fact that $\text{spec}(H) = \{h, 3h, 5h, \dots\} + O(h^\infty)$, we get that

$$\text{spec}(\tilde{H}) = \{1, 3, 5, \dots\} + O(h^\infty).$$

Note by a similar rescaling, we get that $\text{spec}(\tilde{H}_0) = \{1, 3, 5, 7, \dots\}$.

We define the projections $P_{\tilde{H}}$ and P_{H_0} as follows:

$$P_{\tilde{H}} = \frac{1}{2\pi i}\oint_{\partial D(\tilde{E}_j, \epsilon)} (z - \tilde{H})^{-1}\, dz, \qquad P_{H_0} = \frac{1}{2\pi i}\oint_{\partial D(2j+1, \epsilon)} (z - H_0)^{-1}\, dz,$$

where $D(z, \epsilon)$ is the disc centered at z of radius ϵ, sufficiently small so as to contain a single E_j. Note that these projections are rank one, onto the subspaces spanned by $\tilde{\psi}_j$ and $\tilde{\kappa}_j$ respectively, see [BS12]. Now, for all $\epsilon > 0$ and $h > 0$ sufficiently small, we have that $|\tilde{E}_{j-1} - (2j-1)| < \epsilon$, $|\tilde{E}_j - (2j+1)| < \epsilon$, and $|\tilde{E}_{j+1} - (2j+3)| < \epsilon$ so we may write the difference of the projections under the same integrand:

$$P_{\tilde{H}_0} - P_{\tilde{H}} = \frac{1}{2\pi i}\oint_{\partial D(2j+1, \epsilon)} (z - \tilde{H}_0)^{-1}(\tilde{H}_0 - \tilde{H})(z - \tilde{H})^{-1}\, dz.$$

Now, applying this difference to the eigenfunction $\tilde{\psi}_j$ yields:

$$\left|\left|(P_{\tilde{H}_0} - P_{\tilde{H}})\tilde{\psi}_j\right|\right|_2 \leq \frac{1}{2\pi}\left|\left|\oint_{\partial D(2j+1,\epsilon)} (z - \tilde{E}_1(h))^{-1}(z - H_0)^{-1}\, dz\right|\right|_{op}\left|\left|(h^{-1}\gamma(\sqrt{h}x))\tilde{\psi}_j\right|\right|_2.$$

Bounding by the distance to the spectrum along the contour, and that $\gamma \in C_0^\infty(\mathbb{R} \setminus \{0\})$, we get

$$\left|\left|(P_{\tilde{H}_0} - P_{\tilde{H}})\tilde{\psi}_j\right|\right|_2 \leq C\left|\left|h^{-1}\tilde{\psi}_j\right|\right|_{L^2(\mathrm{supp}(\gamma(\sqrt{h}x)))}.$$

One can see that $\lim_{h\to 0}\left|\left|h^{-1}\tilde{\psi}_j\right|\right|_{L^2(\mathrm{supp}(\gamma(\sqrt{h}x)))} = 0$ by Proposition 3.1.(i), and that

$$\lim_{h\to 0} \inf \mathrm{supp}(\gamma(\sqrt{h}x)) = \infty.$$

Some algebra yields the desired results:

$$1 - \langle\tilde{\psi}_j, \tilde{\kappa}_j\rangle^2 = \left|\left|(P_{\tilde{H}_0} - P_{\tilde{H}})\tilde{\psi}_j\right|\right|_2^2,$$

so $\lim_{h\to 0}\langle\tilde{\psi}_j, \tilde{\kappa}_j\rangle = 1$. Finally

$$\left|\left|\tilde{\psi}_j - \tilde{\kappa}_j\right|\right|_2^2 = 2(1 - \langle\tilde{\psi}_j, \tilde{\kappa}_j\rangle),$$

which tends to 0 in the limit as $h \to 0$. □

The above lemma tells us that in the semiclassical limit, the L^2 mass of the perturbed eigenfunctions are distributed in the same way as the non-perturbed eigenfunctions. In fact, a stronger result holds, the perturbed eigenfunctions converge locally uniformly to the non-perturbed eigenfunctions. This is proven along the way to establishing lemma 3.3. However, before we can prove the lemma, we require a bound on the roots of $\tilde{\kappa}_j$, which, in turn requires a theorem of Gershgorin.

Theorem 3.1 ([BB11], §9.1). *Let A be an $n \times n$ matrix and R_i denote the circle in the complex plane with center a_{ii}, and radius $\sum_{j=1, j\neq i}^{n} |a_{ij}|$. The eigenvalues of A are contained within the union of these circles. Moreover, the union of any k of the circles that do not intersect the remaining $(n-k)$ contains precisely k (counting multiplicities) of the eigenvalues.*

Lemma 3.2. *For each $j \in \mathbb{Z}_{\geq 0}$, $\max\{|r| : \tilde{\kappa}_j(r) = 0\} \leq \sqrt{2j-2}$.*

Proof. Let $A^* := (x - \frac{d}{dx})$ and $A := (x + \frac{d}{dx})$ be the creation and annihilation operators associated to $\tilde{H}_0$, satisfying $AA^* - \mathrm{Id} = A^*A + \mathrm{Id} = \tilde{H}_0$. We first aim to show that

$$\tilde{\kappa}_j = \frac{1}{\sqrt{2^j j!}} A^{*j} \tilde{\kappa}_0, \qquad \tilde{\kappa}_0 = \pi^{-\frac{1}{4}} e^{-\frac{x^2}{2}}.$$

First, we have that $[A, A^*] = 2\mathrm{Id}$. Note that $[A, A^{*j}] = 2kA^{*j-1}$, which holds for the base case $j = 1$. Suppose this commutator identity holds for j, then

$$[A, A^{*j+1}] = [A, A^{*j}]A^* + 2A^{*j} = 2(j+1)A^{*j},$$

Satisfying the inductive hypothesis. Now, we have the following two relationships between the creation and annihilation operators, and the eigenfunctions of the harmonic oscillator:

$$A^*\tilde{\kappa}_j = \sqrt{2(j+1)}\tilde{\kappa}_{j+1}, \qquad A\tilde{\kappa}_j = \sqrt{2j}\tilde{\kappa}_{j-1}.$$

The first equation above is an immediate consequence of the commutator identity above. The second, follows from the fact that $A\tilde{\kappa}_0 = 0$, and that

$$A\tilde{\kappa}_j = \frac{1}{\sqrt{2^j j!}}([A, A^{*j}] + A^{*j}A)\tilde{\kappa}_0 = \frac{2j}{\sqrt{2^j j!}} A^{*j-1}\tilde{\kappa}_0$$

Hence, $\tilde{H}_0\tilde{\kappa}_j = (A^*A + \mathrm{Id})\tilde{\kappa}_j = (2j+1)\tilde{\kappa}_j$, that is, $\tilde{\kappa}_j$ satisfies the eigenvalue equation

$\tilde{H}_0\tilde{\kappa}_j = (2j+1)\tilde{\kappa}_j$. By Theorem 1.6, we have that the $\tilde{\kappa}_j$ are a complete, orthonormal sequence of eigenvectors.

More explicitly,

$$\tilde{\kappa}_j(x) = \frac{1}{\sqrt{2^j j!\sqrt{\pi}}} P_j(x) e^{-\frac{x^2}{2}},$$

where $P_j(x)$ are the physicist's Hermite polynomials, which are defined by the recurrence relation $P_{j+1}(x) = 2xP_j(x) - \frac{d}{dx}P_j(x)$, $P_0(x) = 1$. Indeed, from the form of $\tilde{\kappa}_j$,

$$P_{j+1}(x)e^{-\frac{x^2}{2}} = A^{*j+1}e^{-\frac{x^2}{2}} = A^* P_j(x)e^{-\frac{x^2}{2}} = \left(2xP_j(x) - \frac{d}{dx}P_j(x)\right) e^{-\frac{x^2}{2}}.$$

By induction, we show that $\frac{d}{dx}P_j(x) = 2jP_{j-1}(x)$, and so we have the modified recurrence relation

$$P_{j+1}(x) = 2xP_j(x) - 2jP_{j-1}(x). \tag{3.5}$$

Indeed, the base case holds by computation, and the inductive step is as follows

$$\frac{d}{dx}P_{j+1} = \frac{d}{dx}(2xP_j(x) - \frac{d}{dx}P_j(x)) = \frac{d}{dx}(2xP_j(x) - 2jP_{j-1}(x)).$$

Taking derivatives and using the inductive hypothesis we arrive at

$$\frac{d}{dx}P_{j+1} = 2P_j(x) + 2j\left(2xP_{j-1} - \frac{d}{dx}P_{j-1}\right) = 2P_j(x) + 2jP_j(x) = 2(j+1)P_j(x)$$

Define the tridiagonal symmetric matrices

$$C_j = \begin{pmatrix} 0 & \sqrt{\frac{1}{2}} & 0 & 0 & 0 & \cdots & 0 & 0 \\ \sqrt{\frac{1}{2}} & 0 & \sqrt{\frac{2}{2}} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \sqrt{\frac{2}{2}} & 0 & \sqrt{\frac{3}{2}} & 0 & \cdots & 0 & 0 \\ \vdots & & & \ddots & & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \sqrt{\frac{j-1}{2}} \\ 0 & 0 & 0 & 0 & 0 & \cdots & \sqrt{\frac{j-1}{2}} & 0 \end{pmatrix}.$$

We claim that $P_j(x) = 2^j \det(xI - C_j)$. To see this, we show that the expression satisfies the recurrence relation (3.5). Expanding along the the bottom row of $(xI - C_j)$, we get

$$P_j(x) = 2^j x \det(xI - C_{j-1}) + 2^j \sqrt{\frac{j-1}{2}} \det A_{j,j-1},$$

where $A_{j,j-1}$ is the $(j, j-1)^{\text{th}}$ cofactor of $\det(xI - C_j)$. Expanding along the rightmost column of $A_{j,j-1}$, we get

$$P_j(x) = 2^j x \det(xI - C_{j-1}) - 2^j \frac{j-1}{2} \det(xI - C_{j-2}).$$

Finally, recognize that this satisfies the recurrence relation (3.5):

$$P_j(x) = 2x \cdot \left(2^{j-1} \det(xI - C_{j-1})\right) - 2(j-1)\left(2^{j-2} \det(xI - C_{j-2})\right).$$

Thus the roots of $P_j(x)$ are exactly the eigenvalues of C_j. Applying Theorem 3.1 we may bound the eigenvalues of C_j, hence the roots of P_j which are exactly the roots of $\tilde{\kappa}_j$:

$$\max\{|r| : \tilde{\kappa}_j(r) = 0\} \leq \sqrt{\frac{j-2}{2}} + \sqrt{\frac{j-1}{2}} \leq \sqrt{2j-2}.$$

□

Lemma 3.3. *For all h sufficiently small, $|\psi_j(\pm\sqrt{E_j})| \geq \frac{D_j}{\sqrt{h}} \geq 1$.*

Proof. We will first get the bound on the rescaled eigenfunction $\tilde{\psi}_j$, then transfer this bound to ψ_j. Let $\theta = \tilde{\psi}_j - \tilde{\kappa}_j$. Lemma 3.1 establishes that $\theta \to 0$ in $L^2(\mathbb{R})$. Now, let $I \subseteq \mathbb{R}$ be any bounded interval. Then $\theta'' \to 0$ in $L^2(I)$, since

$$\begin{aligned} ||\theta''||_{L^2(I)} \leq & \left|\left|h^{-1}\gamma(\sqrt{h}x)\tilde{\psi}_j(x)\right|\right|_{L^2(\mathbb{R})} + \left|\left|x^2(\tilde{\psi}_j(x) - \tilde{\kappa}_j(x))\right|\right|_{L^2(I)} \\ & + \left|\left|\tilde{E}_j\tilde{\psi}_j(x) - (2j-1)\tilde{\kappa}_j(x)\right|\right|_{L^2(\mathbb{R})}, \end{aligned}$$

where the first term vanishes by Proposition 3.1.(i) and that $\lim_{h\to 0} \inf \operatorname{supp}(\gamma(\sqrt{h}x)) = \infty$, the second term vanishes due to Lemma 3.1, and the third term vanishes due to lemma 3.1 and since $\lim_{h\to 0} \tilde{E}_j = 2j - 1$.

Let $\eta \in C_0^\infty(\mathbb{R})$ and consider the following identity: $\eta(\theta')^2 + \eta\theta\theta'' - \frac{1}{2}\theta^2\eta'' = \frac{d}{dx}(\eta\theta\theta' - \frac{1}{2}\theta^2\eta')$. Integrating yields

$$\int \eta(\theta')^2 \, dx = \frac{1}{2}\int \eta''\theta^2 \, dx - \int \eta\theta\theta'' \, dx,$$

which tends to 0 as $h \to 0$ since $\theta, \theta'' \to 0$ in $L^2(I)$, thus $\theta' \to 0$ in $L^2(I)$. Now let $x \in [b, c] = I$. Then, for each $a < b$, by the fundamental theorem of calculus

$$|\theta(x) - \theta(a)| \leq \sqrt{x-a}||\theta'||_{L^2([a,c])}.$$

By a 2-ϵ argument, since $\lim_{a\to-\infty}\theta(a) = 0$ and that $\lim_{h\to 0}||\theta'||_{L^2([a,c])} = 0$ we must have that $\lim_{h\to 0}||\theta(x)||_{L^\infty(I)} = 0$, or equivalently $\tilde{\psi}_j \to \tilde{\kappa}_j$ in $L^\infty(I)$.

Now, we pass to the unscaled semiclassical operator. Working in the allowed region $C =$

$[-\sqrt{E_j}, \sqrt{E_j}]$:

$$\left|\left|\psi_j(x) - h^{-\frac{1}{2}}\tilde{\kappa}_j(h^{-\frac{1}{2}}x)\right|\right|_{L^\infty(C)} = h^{-\frac{1}{2}}\left|\left|\tilde{\psi}_j(x) - \tilde{\kappa}_j(x)\right|\right|_{L^\infty\left(\left[-\sqrt{\tilde{E}_j},\sqrt{\tilde{E}_j}\right]\right)}.$$

Multiplying the above equation by $\sqrt{h}$, and recalling that $\sqrt{\tilde{E}_j} = \sqrt{2j+1} + O(h^\infty)$ yields

$$\lim_{h\to 0}\left|\left|\sqrt{h}\psi_j(x) - \tilde{\kappa}_j(h^{-\frac{1}{2}}x)\right|\right|_{L^\infty(C)} = 0.$$

Lemma 3.2 establishes that $|\tilde{\kappa}_j(\pm\sqrt{2j+1})| > 0$, and since $\sqrt{E_j} = \sqrt{h(2j+1)} + O(h^\infty)$ for all h sufficiently small there is a constant $D_j > 0$ satisfying,

$$|\tilde{\kappa}_j(\pm h^{-\frac{1}{2}}\sqrt{E_j})| \geq D_j.$$

Thus by the above limit, for sufficiently small h, $|\psi_j(\pm\sqrt{E_j})| \geq \frac{D_j}{\sqrt{h}}$.

□

The final lemma that we require gives us a way to transfer the lower bound of Lemma 3.3 outward from the boundary of the allowed region $[-\sqrt{E_j}, \sqrt{E_j}]$.

Lemma 3.4. *Consider the interval* $I = [x_1, x_2(1+\epsilon)]$ *with* $\sqrt{E_j} \leq x_1$. *Let* $v = \sup_{t\in I}\sqrt{V(t)}$. *Then*

$$|\psi_j(x_2)| \geq e^{-\frac{v}{h}(x_2-x_1)}\left(1 - e^{-2\epsilon\frac{v}{h}x_2}\right)|\psi_j(x_1)|.$$

Proof. Define $\phi(x) = \left(e^{-\frac{v}{h}(x-x_1)} - e^{-2\frac{v}{h}(x_2(1+\epsilon)-x_1)}e^{\frac{v}{h}(x-x_1)}\right)|\psi_j(x_1)|$. Notice that $\phi(x_2(1+\epsilon)) = 0$, and

$$\phi(x_1) = (1 - e^{-2\frac{v}{h}(x_2(1+\epsilon)-x_1)})|\psi_j(x_1)| \leq |\psi_j(x_1)|.$$

Thus, on ∂I, $\phi \leq |\psi_j|$. We claim that on the interior of I, $\phi \leq |\psi_j|$. Let $\eta = |\psi_j| - \phi$ and define $I_0 = \{x : \eta(x) < 0\} \subseteq I$. Now, since $\eta \geq 0$ on ∂I, $\eta = 0$ on ∂I_0. Let $W(x) = V(x) - E_j$ and note by choice of lower bound on I we have that $W \geq 0$ on I. Then on I_0:

$$\frac{d^2}{dx^2}\eta = W\eta - \left(\left(\frac{v}{h}\right)^2 - W\right)\phi,$$

which is less than zero for all h sufficiently small. Thus, η is concave on I_0, and so it attains is minimum on ∂I_0. So $\eta \geq 0$ on I_0, and I_0 is empty, i.e. $\phi \leq |\psi_j|$ on I. With this, evaluating ϕ at x_2 yields $|\psi_j(x_2)| \geq e^{-\frac{v}{h}(x_2-x_1)}(1 - e^{-2\epsilon\frac{v}{h}x_2})|\psi_j(x_1)|$. □

Finally, we are able to prove Proposition 3.1.(ii), which will involve propagating the bound from Lemma 3.3, using Lemma 3.4.

Proof of Proposition 3.1.(ii). Let $\sqrt{E_j} < |x| < R$, and without loss of generality, assume x is positive. Let $0 = x_0 < x_1 = \sqrt{E_j} < x_2 < \cdots < x_n = x$ be a partition of $[0, x]$ such that the upper Darboux sum of $\int_0^x \sqrt{V(x)}\,dx$ satisfies

$$\sum_{i=1}^{n} \sup_{t\in[x_{i-1},x_i]} \sqrt{V(t)}|x_i - x_{i-1}| \leq \left(1 + \frac{\delta}{2}\right) \int_0^x \sqrt{V(t)}\,dt.$$

for some $\delta > 0$. Let $\epsilon > 0$ be small enough such that if we define $D_i = [x_{i-1}, x_i(1+\epsilon))]$, and $v_i = \sup_{t\in D_i} \sqrt{V(t)}$, we get

$$\sum_{i=1}^{n} v_i|x_i - x_{i-1}| \leq (1+\delta) \int_0^x \sqrt{V(t)}\,dt. \tag{3.6}$$

Next, applying Lemma 3.4 iteratively on this partition, we get that

$$|\psi_j(x)| \geq e^{-\sum_{i=2}^{n} \frac{v_i}{h}(x_i - x_{i-1})} \prod_{i=2}^{n} \left(1 - e^{-2\epsilon\frac{v_i}{h}x_i}\right) |\psi_j(x_1)|.$$

Lemma 3.3 guarantees (for $h > 0$ sufficiently small), $|\psi_j(x_1)| = |\psi_j(\sqrt{E_j})| \geq 1$, so we recover the full sum

$$|\psi_j(x)| \geq e^{-\sum_{i=1}^{n} \frac{v_i}{h}(x_i - x_{i-1})} \prod_{i=1}^{n} \left(1 - e^{-2\epsilon \frac{v_i}{h} x_i}\right).$$

Since $v_i = O(1)$ for $i \neq 1$, and $v_1 = O(\sqrt{h})$, for all h sufficiently small, we estimate $\prod_{i=1}^{n} \left(1 - e^{-2\epsilon \frac{v_i}{h} x_i}\right) \geq 2^{-n}$. Using this along with bound (3.6), we get the final desired bound:

$$|\psi_j(x)| \geq D e^{-(1+\delta)^2 \frac{1}{h} \left| \int_0^x \sqrt{V}\, dt \right|}.$$

To get uniformity in $r < |x| < R$, extend the partition. □

3.2 Proof of the Main Result

To prove Theorem 2.4 we will construct the desired potential functions $V^{\pm}$. Let $\alpha, \beta \in C_0^\infty(\mathbb{R})$ with $\text{supp}(\alpha) \subseteq (1,2)$ and $\text{supp}(\beta) \subseteq (3,4)$ and set $V^{\pm}(x) = x^2 + \alpha(x) + \beta(\pm x)$. To recover the sub $O(h^\infty)$ differences between the eigenvalues of the associated operators $H^{\pm} = -h^2 \frac{d^2}{dx^2} + x^2 + \alpha(x) + \beta(\pm x)$, we will apply a variation in $\beta(\pm x)$. Define the following family of operators

$$H_t^{\pm} = -h^2 \frac{d^2}{dx^2} + x^2 + \alpha(x) + t\beta(\pm x),$$

and notice that $H_1^{\pm} = H^{\pm}$. Denote the corresponding eigenfunctions and eigenvalues $\psi_j^{\pm}(t,x)$, $E_j^{\pm}(t)$.

Proof of Theorem 2.4. The following equation, known as Hadamard's variational formula, will be useful

$$\frac{d}{dt}E_j^{\pm}(t) = \int \beta(\pm x)(\psi_j^{\pm}(t,x))^2\,dx.$$

The proof of this formula for the one dimensional case is elementary, see [GH12]. Now, with Hadamard's variational formula and the fundamental theorem of calculus, we have

$$E_j^{\pm}(t) - E_j^{\pm}(0) = \int_0^t \int \beta(\pm x)(\psi_j^{\pm}(s,x))^2\,dx\,ds.$$

Using Proposition 3.1 and the above equation, and that $E_j^{\pm}(1) = E_j^{\pm}$ gives us the following bounds:

$$E_j^{\pm} - E_j^{\pm}(0) \geq \int_0^1 \int \beta(\pm x) D e^{-\frac{(1+\delta)^2}{h}\left|\int_0^x \sqrt{a^2+\alpha(a)+s\beta(\pm a)}\,da\right|}\,dx\,ds \tag{3.7}$$

$$E_j^{\pm} - E_j^{\pm}(0) \leq \int_0^1 \int \beta(\pm x) C e^{-\frac{(1-\delta)^2}{h}\left|\int_0^x \sqrt{a^2+\alpha(a)+s\beta(\pm a)}\,da\right|}\,dx\,ds \tag{3.8}$$

Notice that the minus version of lower bound (3.7) avoids integrating in the exponent through the support of α. This allows us to express the difference of the minus version of (3.7) with the plus version of (3.8) as

$$E_j^{-} - E_j^{+} \geq \int_0^1 \int \beta(x) \left(D e^{-\frac{(1+\delta)^2}{h}\int_0^x \sqrt{a^2+s\beta(a)}\,da} - C e^{-\frac{(1-\delta)^2}{h}\int_0^x \sqrt{a^2+\alpha(a)+s\beta(a)}\,da} \right) dxds$$

Factoring, recalling that β is supported in $(3,4)$, and recognizing that the integrals in the exponent are monotonic in x gives

$$E_j^{-} - E_j^{+}$$
$$\geq \int_0^1 e^{-\frac{(1+\delta)^2}{h}\int_0^4 \sqrt{a^2+s\beta(a)}\,da} \int \beta(x) \left(D - C e^{-\frac{1}{h}\int_0^x (1-\delta)^2\sqrt{a^2+\alpha(a)+s\beta(a)} - (1+\delta)^2\sqrt{a^2+s\beta(a)}\,da} \right) dxds.$$

The bounds of Proposition 3.1 hold for any choice of δ, so we may take δ small enough to satisfy for each $x \in (3,4)$

$$\int_0^x (1-\delta)^2\sqrt{a^2+\alpha(a)+s\beta(a)} - (1+\delta)^2\sqrt{a^2+s\beta(a)}\,da > 0.$$

With this choice of delta, and again using monotonicity of the integral in the exponent, we have

$$E_j^- - E_j^+ \geq e^{-\frac{(1+\delta)^2}{h}\int_0^4 \sqrt{a^2+\beta(a)}\,da}\left(D - Ce^{-\frac{1}{h}\int_0^3 (1-\delta)^2\sqrt{a^2+\alpha(a)}-(1+\delta)^2\sqrt{a^2}\,da}\right)\int \beta(x)\,dx.$$

For all h sufficiently small, this then reduces to the inequality $E_j^- - E_j^+ \geq De^{\frac{-d}{h}}$, for some positive constants $c, C > 0$. To prove the upper bound $E_j^- - E_j^+ \leq Ce^{\frac{-c}{h}}$, a similar argument is used where we instead take the difference of the minus version of (3.8) with the positive version of (3.7). $\square$

Chapter 4

Compact Manifold Trace Invariant Limitations

We ask the same question of compact manifolds: do there exist non-isometric compact manifolds with the same wave-trace invariants? Inspired by the results of Fulling and Kuchment [FK05], we study radially perturbed surfaces of revolution. In this chapter we first collect some useful classical geometric facts about surfaces of revolution, and use these to prove that there exists two radially perturbed surfaces of revolution with the same length spectrum. However, it remains an open question to whether or not these surfaces of revolution possess the same Laplace spectrum, or even the same wave-trace invariants. In the last section of this chapter we detail the process of reducing the study of the Laplace spectrum on surfaces of revolution to the study of the spectrum of an associated 1-dimensional semiclassical Schrödinger operator. It was initially thought that the estimates from Theorem 2.4 could be applied to show that certain surfaces of revolution have the same wave-trace invariants, however such an argument has been found to fail due to limitations on the semiclassical parameter for large eigenvalues. We conclude the chapter with this discussion.

4.1 Surfaces of Revolution

Let $\sigma : (0, \pi) \to \mathbb{R}^+ \times \mathbb{R}$ be a curve with parametric equations $\sigma(s) = (f(s), h(s))$, satisfying $f(0) = f(\pi) = 0$ and that $f'(t) > 0$ on $(0, \pi/2)$, $f'(\pi/2) = 0$, and $f'(t) < 0$ on $(\pi/2, \pi)$. For example, see Figure 4.1. The surface of revolution, symmetric about the z-axis, generated by this curve, is given by the parameterization $X : [0, 2\pi] \times [0, \pi] \to \mathbb{R}^3$, $X(u, v) = (f(v) \cos u, f(v) \sin u, h(v))$.

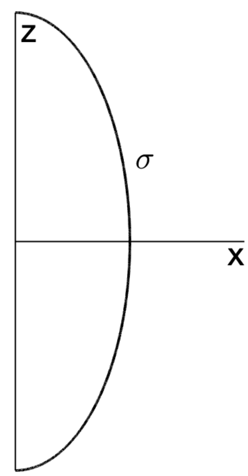

Figure 4.1: A curve which generates a surface of revolution, satisfying desired constraints.

Let g be the pullback the euclidean metric $\overline{g}$ through X (cf. [Lee18], §2):

$$g = X^*\overline{g} = f^2\, du^2 + |\sigma'|^2\, dv^2.$$

Definition 4.1. *Let X be the parameterization defined by the generating curve σ, and let S be the image of X. We call (S, g) the* ***surface of revolution*** *generated by σ.*

Definition 4.2. *A curve of constant v coordinate is called a* ***parallel****, and curves of constant u coordinates are called* ***meridians****.*

In the context of terrestrial navigation, meridians are referred to as longitudinal lines, and parallels, latitudinal lines.

Specified in the coordinates u, v, given by the parameterization X, we recover two geodesic equations (cf. [DCFF92], §3) for the geodesic $\gamma(t) = (u(t), v(t))$

$$\frac{d^2u}{dt^2} + \frac{2ff'}{f^2}\frac{du}{dt}\frac{dv}{dt} = 0 \tag{4.1}$$

$$\frac{d^2v}{dt^2} - \frac{ff'}{|\sigma'|^2}\left(\frac{du}{dt}\right)^2 + \frac{f'f'' + h'h''}{|\sigma'|^2}\left(\frac{dv}{dt}\right)^2 = 0 \tag{4.2}$$

We now state and provide proofs of two requisite lemmas that appear as exercises in ([DCFF92], §3).

Lemma 4.1. *Geodesics on the surface of revolution (X, g) are of constant speed.*

Proof. Consider the time derivative of the speed of the geodesic $\gamma(t) = (u(t), v(t))$,

$$\frac{d}{dt}\left|\left|(u'(t), v'(t))\right|\right|_g = 2ff'\frac{dv}{dt}\left(\frac{du}{dt}\right)^2 + f^2 2\frac{du}{dt}\frac{d^2u}{dt^2} + 2(ff' + hh')\frac{dv}{dt}\left(\frac{dv}{dt}\right)^2 + |\sigma'|^2 2\frac{dv}{dt}\frac{d^2v}{dt^2}$$

Factoring and using (4.1) gives

$$\frac{d}{dt}\left|\left|(u'(t), v'(t))\right|\right|_g = 2\frac{dv}{dt}|\sigma'|^2\left(\frac{d^2}{dt^2} - \frac{ff'}{|\sigma'|^2}\left(\frac{du}{dt}\right)^2 + \frac{ff' + hh'}{|\sigma'|^2}\left(\frac{dv}{dt}\right)^2\right).$$

To finish the proof, we recognize that the parenthetical expression is the left hand side of (4.2) . □

Lemma 4.2. *To each geodesic we assign a constant of motion C, satisfying*

$$f(v(t))\cos\theta(t) = C, \tag{4.3}$$

where $\theta(t)$ is the oriented angle made with γ' and the parallel of radius $f(v(t))$. Equation

(4.3) *is known as Clairaut's relation.*

Proof. Multiplying both sides of equation (4.1) by f^2, and integrating yields

$$f^2 \frac{du}{dt} = \int \frac{d}{dt}\left(f^2(v(t))\frac{du}{dt}\right) dt = C. \tag{4.4}$$

Without loss of generality, by Lemma 4.1 we may assume that γ is of unit speed. Interpreting equation (4.4) using the metric g, we have

$$1 \cdot f \cdot \cos\theta(t) = ||\gamma'||_g \cdot ||(1,0)||_g \cdot \cos\theta = \langle \gamma'(t), (1,0)\rangle_g = f^2 \frac{du}{dt} = C \tag{4.5}$$

where $\theta(t)$ is the angle made with $\gamma'(t)$ and the parallel of radius $f(v(t))$. The above equation is indeed Clairaut's relation. □

In the hemi-spheroidal polar coordinates, Clairaut's relation can be used to reduce the geodesic equations to quadrature, see [GS79]. In our coordinates, we arrive at a similar result.

Proposition 4.1. *Let $\gamma(t_0) = (u_0, v_0)$ and $\gamma(t_1) = (u_1, v_1)$ be points on the geodesic γ so that on $[t_0, t_1]$, $\frac{dv}{dt}$ does not change sign. Then, for $t \in [t_0, t_1]$,*

$$u(t) = u_0 + \int_{t_0}^{t} \frac{C|\sigma'(v(s))|}{f(v(s))\sqrt{f^2(v(s)) - C^2}} \frac{dv}{ds}\, ds \qquad \text{if} \qquad \frac{dv}{dt} \geq 0$$
$$u(t) = u_0 - \int_{t_0}^{t} \frac{C|\sigma'(v(s))|}{f(v(s))\sqrt{f^2(v(s)) - C^2}} \frac{dv}{ds}\, ds \qquad \text{if} \qquad \frac{dv}{dt} \leq 0,$$

where C is the constant of motion given by Clairaut's relation. Moreover, since $v(t)$ is *monotonic on* $[t_0, t_1]$*, we may eliminate the parameter*

$$u(v) = u_0 + \int_{v_0}^{v} \frac{C|\sigma'(v)|}{f(v)\sqrt{f^2(v) - C^2}}\, dv \qquad \textit{if} \qquad \frac{dv}{dt} \geq 0 \tag{4.6}$$

$$u(v) = u_0 - \int_{v_0}^{v} \frac{C|\sigma'(v)|}{f(v)\sqrt{f^2(v) - C^2}}\, ds \qquad \textit{if} \qquad \frac{dv}{dt} \leq 0. \tag{4.7}$$

Proof. Without loss of generality, by Lemma 4.1 we may assume that γ is of unit speed. Then, from the metric

$$1 = ||\gamma||_g = f^2 \left(\frac{du}{dt}\right)^2 + |\sigma'|^2 \left(\frac{dv}{dt}\right)^2$$

Using equation (4.4), we have

$$1 = \frac{C^2}{f^2} + |\sigma'|^2 \left(\frac{dv}{dt}\right)^2$$

With some algebraic manipulation, taking square roots, and again applying equation (4.4), we have

$$\frac{du}{dt} = \left|\frac{dv}{dt}\right| C \frac{|\sigma'|}{f\sqrt{f^2 - C^2}}$$

Integrating, and performing a change of variables, yields the desired expressions. □

4.2 A Pair of Rotationally Symmetric Perturbations

Let $(S_\pm, g_\pm)$ be radially perturbed unit spheres, given by the generating curves

$$\begin{aligned} \sigma_+(v) =& \Big(\sin(v)(1 + \alpha(v) + \beta(v)), \cos(v)(1 + \alpha(v) + \beta(v)) \Big) \\ \sigma_-(v) =& \Big(\sin(v)(1 + \alpha(v) + \beta(\pi - v)),\ \cos(v)(1 + \alpha(v) + \beta(\pi - v)) \Big), \end{aligned} \tag{4.8}$$

where $\alpha \in C_0^\infty([\pi/4, 3\pi/8])$ and $\beta \in C_0^\infty([0, \pi/8])$, (see Figure 4.2).

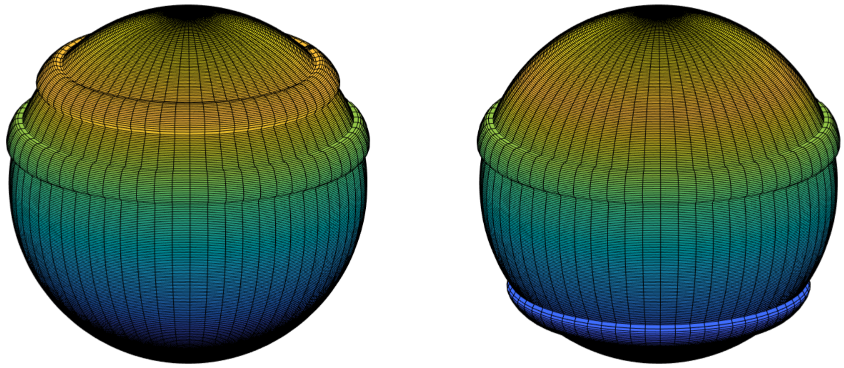

Figure 4.2: A pair of rotationally symmetric smooth perturbations of the sphere (S_+ left, S_- right).

Lemma 4.3. *Let γ be a geodesic on either $S_\pm$. Then γ intersects the equator $X([0, 2\pi], \pi/2)$, and the oriented angle of intersection uniquely determines the constant of motion C, and uniquely determines γ up to rotations in the u coordinate.*

Proof. We will prove the case for S_+, the case for S_- will follow similarly. Let $\gamma(t) = u(t), v(t)$ be specified in u, v coordinates. First, we establish that the geodesic must intersect the equator. To do so, we analyze the critical values of the v coordinate of γ. Let $\frac{dv}{dt} = 0$, assume we are working with a unit speed geodesic, and using the metric equations we have

$$1 = f^2(v(t)) \left(\frac{du}{dt}\right)^2.$$

Equation (4.5) allows us to refine the above equation

$$f(v(t)) = C.$$

Finally, in terms of our explicit surface of revolution

$$\sin(v)(1 + \alpha(v) + \beta(\pm v)) = C.$$

Taking α, β sufficiently small, we have that $\sin(v)(1 + \alpha(v) + \beta(\pm v))$ is strictly increasing for $0 \leq v < \pi/2$, and monotonically decreasing for $\pi/2 < v \leq \pi$. Hence, $v(t)$ has one critical value for $v \in (0, \pi/2)$ and one critical value for $v \in (\pi/2, \pi)$, or the critical value $\pi/2$, trivially, in the case that γ parameterizes the equator ($C = 0$). Thus, since $S_\pm$ is of positive Gaussian curvature, we must have that $v(t)$ oscillates between these two critical values, and hence γ intersects the equator.

Now, let ϕ be the angle of intersection of γ with the equator. Note, that by Clairaut's relation (4.3), we have that the constant of motion $C = \arcsin \phi$. Moreover, any other geodesic that intersects the equator at this angle is locally a portion of a great circle, that when rotated, coincides with the given geodesic. Uniqueness follows from uniqueness of solutions to the geodesic equations. □

4.2.1 Coincidence of Length Spectra

In this section we prove that the length spectra of $S_\pm$ coincide. The proof hinges on a fact of surface of revolution with rotationally symmetric perturbations: passing through such a perturbation, when compared to the unperturbed surface, amounts to a rotation in the azimuth direction. This was observed in [GS79], where they establish this fact of perturbations for a different class of surfaces of revolution.

Proposition 4.2. *The length spectrum of S_+ and S_- coincide.*

Proof. Let $\gamma_\pm(t) = (u_\pm(t), v_\pm(t))$ be unit speed, closed geodesics on $S_\pm$, with common equatorial angle. Assume without loss of generality $\gamma_\pm$ is oriented upwards, that is $\gamma_\pm(0)$ lies on the equator, and $\frac{dv_\pm}{dt}(0) < 0$. Note, the geodesic equations for $S_\pm$ agree for all $v \in [0, \pi] \setminus (\mathrm{supp}(\beta(\cdot)) \cup \mathrm{supp}(\beta(\pi - \cdot)) = [\pi/8, 7\pi/8]$, so if the geodesics do not encounter the perturbation β, $\gamma_\pm$ coincide.

From here on out, we assume that the critical values of $v_\pm(t)$ lie above/below the perturbation β. The case that the critical values of $v_\pm(t)$ lie within the perturbation β follows similarly, and the case that the critical values lie below β is trivial. Note, that between perturbations, the geodesics lie on the surface of a sphere, and hence are locally segments of great circles. Clairaut's relation says that necessarily, between perturbations, the great circle segments make the same angle with the equator. Thus, passing through a perturbation results in a rotation in the azimuth angle u. Quantitatively, from equations (4.6) and (4.7), the difference in rotation after γ_+ passes through β twice is given by

$$\delta_1 = 2\int_0^{\frac{\pi}{8}} \frac{C|\sigma_+'(s)|}{f_+(s)\sqrt{f_+^2(s) - C}} - \frac{C|\sigma_-'(s)|}{f_-(s)\sqrt{f_-^2(s) - C}}\, ds.$$

After γ_+ passes through β, $\gamma_\pm$ both pass through α and receive a common rotation u_α. Finally, γ_- encounters $\beta(\pi - \cdot)$, and after γ_- exits this perturbation, the difference in rotation is

$$\delta_2 = 2\int_{\frac{7\pi}{8}}^{\pi} \frac{C|\sigma_+'(s)|}{f_+(s)\sqrt{f_+^2(s) - C}} - \frac{C|\sigma_-'(s)|}{f_-(s)\sqrt{f_-^2(s) - C}}\, ds.$$

Performing the change of variables $s \mapsto \pi - s$, we have

$$\delta_2 = -2\int_{\frac{\pi}{8}}^{0} \frac{C|\sigma_-'(s)|}{f_-(s)\sqrt{f_-^2(s) - C}} - \frac{C|\sigma_+'(s)|}{f_+(s)\sqrt{f_+^2(s) - C}}\, ds.$$

So $\delta_1 + \delta_2 = 0$, thus after both γ_+, γ_- travel through $\beta(\cdot), \beta(\pi - \cdot)$ (respt.), their u coordinates coincide, and hence both travel along the same great circle, returning to the equator. This process happens finitely many times until the geodesics return to their initial value.

The geodesics $\gamma_\pm$ consist of a sequence of great circles, segments traveling through α, and segments traveling through $\beta(\cdot), \beta(\pi - \cdot)$. In total, by rotational symmetry of the surface, $\gamma_\pm$ will travel along a great circle of common angle, for the same amount of arc length, through α, β at the same angle (and hence contribute to the same amount of arc length). $\square$

4.3 Separating the Laplacian on a Surface of Revolution

In this section we discuss how, through separation of variables, one can separate the Laplacian (Laplace-Beltrami operator) on our surfaces of revolution $(S_\pm, g_\pm)$, and reduce the eigenvalue analysis to a problem of semiclassical analysis. It was thought that one could use this technique, along with the estimates of Theorem 2.4 to show that $(S_\pm, g_\pm)$ have the same wave trace invariants, which was unfortunately incorrect.

We begin with finding the expression of the Laplacian in the elevation/azimuth coordinates. Expressed in coordinates $\frac{\partial}{\partial x^i}$, the Laplacian is defined as

$$\Delta_g \phi = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i} \left(g^{ij} \sqrt{\det g} \frac{\partial \phi}{\partial x^j} \right),$$

using Einstein summation notation. In what follows, we drop the $\pm$ subscripts for ease of notation. Now, In azimuth/elevation coordinates, the Laplacian takes the form

$$\Delta_g \phi = \frac{1}{f(v)|\sigma'(v)|} \left(\frac{\partial}{\partial u} \left(\frac{|\sigma'(v)|}{f(v)} \frac{\partial \phi}{\partial u} \right) + \frac{\partial}{\partial v} \left(\frac{f(v)}{|\sigma'(v)|} \frac{\partial \phi}{\partial v} \right) \right).$$

Expanded yields

$$\Delta_g = \frac{1}{f^2} \frac{\partial^2}{\partial u^2} + \frac{f'|\sigma'| - f|\sigma'|'}{f|\sigma'|^3} \frac{\partial}{\partial v} + \frac{1}{|\sigma'|^2} \frac{\partial^2}{\partial v^2}, \tag{4.9}$$

with

$$\begin{aligned} f = f_\pm(v) &= \sin v(1 + \alpha(v) + \beta_\pm(v)) \\ |\sigma'| = |\sigma'_\pm(v)| &= \sqrt{(1 + \alpha(v) + \beta_\pm(v))^2 + (\alpha'(v) \pm \beta_\pm(v))^2}. \end{aligned}$$

From here on out, we will pass to the unit speed parameter s. Explicitly, let

$$l(s) = \int_0^s |\sigma'(w)|\, dw,$$

and define the unit speed generating curve $\tilde{\sigma} : \left(0, \int_0^\pi |\sigma'(v)|\, dv\right) \to \mathbb{R}_+ \times \mathbb{R}$, given by

$$\tilde{\sigma}(s) = (\tilde{f}(s), \tilde{g}(s)) = (\, f(l^{-1}(s)),\, g(l^{-1}(s))\,).$$

Note l^{-1} is well defined because $|\sigma'| > 0$. This parameterization yields $|\tilde{\sigma}'(s)| = 1$, and so the Laplacian above simplifies to

$$\Delta_{\tilde{g}} = \frac{1}{\tilde{f}^2}\frac{\partial^2}{\partial u^2} + \frac{\tilde{f}'}{\tilde{f}}\frac{\partial}{\partial s} + \frac{\partial^2}{\partial s^2},$$

where the $'$ denote differentiation with respect to the arc-length parameter s. We will drop the tilde notation, again, for simplicity.

We are interested in eigenvalues/eigenfunctions of the negative Laplacian, that is, functions Ψ, λ such that $-\Delta_g \Psi = \lambda\Psi$. By separating variables, that is letting $\Psi(u, s) = U(u)S(s)$, we substitute into (4.9) and obtain two ordinary differential equations

$$U'' = -m^2 U \tag{4.10}$$

$$-f^2 S'' - ff'S' - f^2\lambda S = -m^2 S. \tag{4.11}$$

From (4.10) we have that U are complex exponentials with frequencies $m = 0, \pm 1, \pm 2, \ldots$. Rearranging (4.11) and setting $h = \frac{1}{m}$, we obtain

$$-h^2 S'' - h^2 \frac{f'}{f} S' + \frac{1}{f^2} S = h^2 \lambda S = E(h) S.$$

Conjugating $\hat{H} = -h^2 \frac{d^2}{ds^2} - h^2 \frac{f'}{f}\frac{d}{ds} + \frac{1}{f^2}$ by a multiplication operator does not change the spectrum, and in particular, conjugating $\hat{H}$ by $\sqrt{f}$ yields

$$H = f^{1/2}\hat{H}f^{-1/2} = -h^2\frac{d^2}{ds^2} + \frac{1}{f^2} + h^2\left(\frac{1}{2}\left(\frac{f'}{f}\right)' - \frac{1}{4}\left(\frac{f'}{f}\right)^2\right),$$

which is a perturbed semiclassical Schrödinger operator. Following the analysis of Gurarie [Gur95] we drop the perturbation and study

$$H_h = -h^2\frac{d^2}{ds^2} + \frac{1}{f^2}.$$

Thus, the study of the eigenvalues of Δ_g is reduced to the study of eigenvalues H. That is, there exists a 1-to-1 map $\sigma(i,j) : \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ such that $\lambda_{\sigma(m,j)} = m^2 E_j(m)$, where $E_j(m)$ is the j^{th} eigenvalue of $H_{\frac{1}{m}}$.

Returning to the pair of surfaces $(S_\pm, g_\pm)$, we associate the semiclassical Schródinger operators

$$H^\pm = -h^2\frac{d^2}{ds^2} + \frac{1}{f_\pm^2},$$

where $f_\pm$ are given by (4.8). That $\text{Spec}(H^+) = \text{Spec}(H^-) + O(h^\infty)$ is a consequence of the Bohr-Sommerfeld quantization rules (Theorem 1.9). Note that Theorem 1.9 is for semiclassical Schrödinger operators defined over $\mathbb{R}$, whereas $H^\pm$ are defined over an interval. However, the result of [CdV05] is established by the geometry of f near the global minimum, and so the analysis carries over to the interval case.

We would like to show that $(S_\pm, g_\pm)$ have the same wave trace invariants, that is

$$\sum_{j=0}^{\infty} \cos\left(\sqrt{-\lambda_j^+}t\right) - \sum_{j=0}^{\infty} \cos\left(\sqrt{-\lambda_j^-}t\right) \in C^\infty(\mathbb{R}),$$

or in terms of the eigenvalues of $H_h^\pm$

$$\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\cos\left(m\sqrt{-E_j^+(m)}t\right)-\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\cos\left(m\sqrt{-E_j^-(m)}t\right)\in C^\infty(\mathbb{R}), \tag{4.12}$$

however, we run into an issue when translating over the analysis of Chapter 3 to $H_h^\pm$. The exponential bounds for the j^{th} eigenvalue of $H_h^\pm$ only hold for $0 < h < h_j$. And in particular, the sequence $h_j \to 0^+$. Thus, for a given eigenvalue $E_j^\pm(\cdot)$, we may not obtain bounds for the first finitely many $E_j(1), E_j(2), \ldots, E_j(m_j)$. This amounts to infinitely many terms in (4.12) which we cannot control.

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