## Title

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# EXISTENCE, UNIQUENESS, AND STABILITY OF SLOWLY OSCILLATING PERIODIC SOLUTIONS FOR DELAY DIFFERENTIAL EQUATIONS WITH NONNEGATIVITY CONSTRAINTS*

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**Abstract.** Deterministic dynamical system models with delayed feedback and nonnegativity constraints arise in a variety of applications in science and engineering. Under certain conditions oscillatory behavior has been observed and it is of interest to know when this behavior is periodic. Here we consider one-dimensional delay differential equations with nonnegativity constraints as prototypes for such models. We obtain sufficient conditions for the existence of slowly oscillating periodic solutions (SOPS) of such equations when the delay/lag interval is long and the dynamics depend only on the current and delayed state. Under further assumptions, including possibly longer delay intervals and restricting the dynamics to depend only on the delayed state, we prove uniqueness and exponential stability for such solutions. To prove these results, we develop a theory for studying perturbations of these constrained SOPS. We illustrate our results with simple examples of biochemical reaction network models and an Internet rate control model.

**Key words.** delay differential equation, state constraints, one-dimensional Skorokhod problem, slow oscillation, periodic solution, Browder's fixed point theorem, variational equation, exponential stability

**AMS subject classifications.** 34K11, 34K13, 34K20, 92B25

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## 1. Introduction.

Dynamical system models with delay in the dynamics arise in a variety of applications in science and engineering. Examples include Internet congestion control models where the finiteness of transmission speeds leads to a delay in receipt of congestion signals or prices [55, 56, 58, 57, 59, 67], neuronal models where the spatial distribution of neurons can result in a propagation delay [4, 23], epidemiological models where incubation periods result in delayed transmission of disease [7], and biochemical models of gene regulation where transcription and translation processes can lead to a delay in signaling effects [1, 6, 46]. The books by Erneux [18], Gopalsamy [19], and Smith [63] provide several examples and references for delay differential equations that arise in applications. Oftentimes the quantities of interest in such systems are nonnegative. For instance, rates and prices in Internet models, proportions of a population that are infected, and concentrations of ions or molecules are all nonnegative. In a delay differential equation model for such systems, sometimes the right-hand side of the equation (here called the drift function) may naturally constrain all components to be nonnegative (see, e.g., section 3.2 of [63]), but sometimes (e.g., because of the delay) the dynamics need to be modified when one of the components of the current state becomes zero, to prevent that com-

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ponent from becoming negative. This can be thought of as imposing a regulating control at the boundary, which creates a discontinuity in the right-hand side of the differential equation.

In many cases, oscillatory (especially periodic) behavior can be important for the functioning of such systems. While there is a considerable mathematical literature on oscillatory solutions of unconstrained delay differential equations (see, e.g., [19, 20, 22]), there is limited mathematical literature studying oscillatory solutions for constrained delay differential equations with discontinuous dynamics at the boundary. Some examples tied to specific applications include a biochemical application studied in Mather et al. [46], where a simple biochemical reaction network model exhibits oscillatory behavior; and an Internet rate control model in which the existence of oscillatory behavior is shown numerically to arise from an unstable equilibrium solution [47]. Even a one-dimensional delay differential equation with a nonnegativity constraint is an interesting nonlinear system whose natural state descriptor is infinite-dimensional because of the need to track position over the delay/lag period. The behavior of the constrained system can be quite different from that of the analogous unconstrained system. For example, as we show in section 4.1, in the case of dynamics that are linear in the unconstrained context, the additional nonnegativity constraint can turn an equation with unbounded oscillatory solutions into one with bounded periodic solutions.

As a first step toward studying oscillatory solutions of constrained delay differential equations, we provide sufficient conditions for existence, uniqueness, and stability of periodic solutions for prototypical one-dimensional delay equations with nonnegativity constraints of the form

$$x(t) = x(0) + \int_0^t f(x_s)ds + y(t), \qquad t \geq 0, \tag{1.1}$$

where $x$ is a continuous function on $[-\tau, \infty)$ that takes values in the nonnegative real numbers, $\tau \in (0, \infty)$ is the fixed length of the delay interval, $x_s$ is a continuous function on the delay interval $[-\tau, 0]$ defined by $x_s(u) = x(s + u)$ for $-\tau \leq u \leq 0$, $f$ is a real-valued continuous function defined on these continuous path segments, and $y$ is a continuous nondecreasing function that is constant on intervals where $x$ is positive. Indeed, $y$ is a control that increases the minimal amount to keep $x$ nonnegative. As we shall see later in section 2, $y$ is characterized by these properties and is a continuous functional of $x(0) + \int_0^{\cdot} f(x_s)ds$. We show in Lemma 2.2 that this formulation is equivalent to one in which $y$ is specified to be constant on intervals where $x$ is positive and is almost everywhere differentiable at times where $x$ is zero, where at such times $t$, $\frac{dy(t)}{dt} = \max(-f(x_t), 0)$. While these alternative formulations are equivalent, our formulation has certain advantages that we exploit, related to the continuous functional property of $y$ (see Appendix A). Given $f$, we refer to (1.1) as a delay differential equation with reflection (at the boundary), or DDER.

In this work, we focus on slowly oscillating periodic solutions (SOPS) of the DDER (1.1). Here, *slowly oscillating* refers to the fact that the solution oscillates about an equilibrium point and the time spent above/below the equilibrium point per oscillation is greater than the length of the delay interval (see Definition 3.2). There is a substantial literature on SOPS of *unconstrained* delay differential equations dating back to the 1960s, when Jones [25] established the existence of a SOPS to the so-called Wright's equation—a certain nonlinear delay differential equation analyzed in [78]. There are a number of subsequent contributions on the existence (see, e.g., [4, 11, 15, 21, 23, 26, 49, 50, 51, 60, 64, 65, 70]) and uniqueness and stability (see, e.g., [10, 27, 28, 42, 43, 53, 72, 71, 73, 79, 80, 81, 82]) of such periodic solutions, with more recent results focused on SOPS of delay differential equations with state-dependent delays (see, e.g., [2, 29, 31, 32, 35, 37, 38, 39, 40, 41, 36, 68, 74]). For a more in-depth discussion of these results we refer the reader to Chapter XV of [14] as well as the recent survey by Walther [75].

For our results on existence of SOPS for *constrained* delay differential equations, we restrict $f$ to be a function that depends only on the current and delayed states, i.e.,

$$f(x_t) = g(x(t), x(t-\tau)), \qquad t \geq 0, \tag{1.2}$$

where $g$ is a real-valued, locally Lipschitz continuous function on the nonnegative quadrant that is differentiable at $(L, L)$, where $L > 0$ is the equilibrium point, and $g$ satisfies a type of negative feedback condition. Our assumptions on $g$ are similar to those imposed by Atay [4], although we allow somewhat relaxed boundedness assumptions on $g$ since, a priori, our nonnegativity constraint imposes a lower bound on solutions of the DDER. For our results on uniqueness and stability of SOPS, we further restrict the function $f$ to depend only on the delayed state, i.e.,

$$f(x_t) = h(x(t-\tau)), \qquad t \geq 0, \tag{1.3}$$

where $h$ is a real-valued continuously differentiable function on the nonnegative real numbers satisfying a negative feedback condition. Our conditions on $h$ and our proof of uniqueness and stability are inspired by an approach used by Xie [80, 81] to prove the uniqueness and stability of SOPS for unconstrained equations. While the general outline of our approach for proving uniqueness and stability is similar to the one used in [80, 81], substantial new difficulties arise due to the discontinuous dynamics at the boundary. In particular, we develop a theory for understanding perturbations of solutions in the constrained environment, which may be of independent interest. We note that the stability results obtained here and in [80, 81] are local. In the unconstrained setting, results have been obtained (see [30] and Chapter 10 of [3], as well as references therein) on the global behavior of the unconstrained delay differential equation under additional assumptions. However, these results assume that $h$ is a monotone function (i.e., $h'(s) < 0$ for all $-\infty < s < \infty$), which is not a condition we impose here.

The paper is organized as follows. A precise definition for a solution of a DDER is given in section 2. Here our nonnegativity constraint is described and its relation to the one-dimensional Skorokhod problem is explained (a formulation of the one-dimensional Skorokhod problem is detailed in Appendix A). We also explain a parallel formulation using delay differential equations with discontinuous right-hand side. Our two main theorems on the existence of SOPS and on the uniqueness and stability of SOPS are stated in section 3. In section 4, our results are illustrated using simple examples of biochemical reaction network models and an Internet congestion control model.

Our proof of the existence of SOPS is presented in section 5. A version of Browder's fixed point theorem, which implies the existence of a nonejective fixed point, is used to show the existence of a nonconstant fixed point for a function that maps any initial condition $x_0$ in a certain set to the value of $x_t$ at the first time it returns to that set. Such a fixed point corresponds to a SOPS of the DDER. Our proof follows an outline similar to those used in prior works [4, 11, 15, 21, 23, 25, 49, 60, 70] for unconstrained systems. The main difference here is the presence of the lower boundary, which leads to some technical difficulties but also prevents unbounded oscillations, thereby allowing for a less restrictive class of functions $g$.

Our proof of the stability and uniqueness of SOPS is presented in section 6. For sufficiently large delays, we can define a variant of a Poincaré map associated with a SOPS of the DDER. Under our assumptions on $h$, its derivative operator evaluated at the initial condition of the SOPS will have norm less than one, which is sufficient to prove that the SOPS is exponentially stable. Uniqueness of the associated SOPS then follows from its exponential stability and an application of theorems for fixed point indices. Our proof follows the general outline of the method used in [80, 81] to prove analogous results for unconstrained systems. However, the boundary constraint prevents the use of established theory on the stability of periodic solutions to delay differential equations. To understand perturbations of the constrained system, a variational equation (VE) along constrained solutions is developed (see section 6.4 and Appendix B). Solutions of this VE may be discontinuous, which leads to significant technical difficulties that do not appear in [80, 81].

We shall use the following notation throughout this paper. For a positive integer $n$, let $\mathbb{R}^n$ denote $n$-dimensional Euclidean space and let $\mathbb{R}^n_+ = \{v \in \mathbb{R}^n : v_i \geq 0$ for $i = 1, \ldots, n\}$ denote the closed nonnegative orthant in $\mathbb{R}^n$. Given $v \in \mathbb{R}^n$, let $|v|$ denote the Euclidean norm of $v$. When $n = 1$, we suppress the $n$ and write $\mathbb{R}$ for the real numbers and $\mathbb{R}_+$ for the nonnegative real numbers. For $r, s \in \mathbb{R}$, let $r^+ = \max(r, 0)$, $r^- = \max(-r, 0)$ and let $r \vee s = \max(r, s)$, $r \wedge s = \min(r, s)$. For a real number $r$, we say $r$ is positive (resp., nonnegative, negative, nonpositive) if $r > 0$ (resp., $r \geq 0$, $r < 0$, $r \leq 0$).

Let $\tau \in (0, \infty)$ denote a constant *delay*. For an interval of the form $I = [-\tau, 0]$, $[0, \infty)$, or $[-\tau, \infty)$, we will refer to the following sets of functions mapping $I$ into the real numbers. We let $\mathcal{D}_I$ denote the set of functions from $I$ into $\mathbb{R}$ that have finite left and right limits at each finite value in $I$ (at the left endpoint of $I$ we only require a finite right limit and at a finite right endpoint we only require a finite left limit). We let $\mathcal{C}_I$ denote the subset of continuous functions from $I$ into $\mathbb{R}$ and we let $\mathcal{C}_I^+$ denote the further subset of continuous functions from $I$ into $\mathbb{R}_+$. We endow $\mathcal{D}_I$ and its subsets with the topology of uniform convergence on compact intervals in $I$. Given $x \in \mathcal{D}_I$ and a compact interval $J$ in $I$, we define the finite supremum norm:

$$\|x\|_J = \sup_{t \in J} |x(t)| < \infty.$$

For $x \in \mathcal{D}_I$, we say that $x$ is increasing (resp., nondecreasing, decreasing, nonincreasing) on $I$ if $x(s) < x(t)$ (resp., $x(s) \leq x(t)$, $x(s) > x(t)$, $x(s) \geq x(t)$) for all $s, t \in I$ satisfying $s < t$. For $r \in \mathbb{R}$, we write $x \equiv r$ to denote the function $x \in \mathcal{C}_I$ that is identically equal to $r$ on $I$, where the interval $I$ will be clear from context. For $x \in \mathcal{D}_{[-\tau,\infty)}$ and $t \geq 0$, let $x_t \in \mathcal{D}_{[-\tau,0]}$ be defined by $x_t(s) = x(t + s)$, $-\tau \leq s \leq 0$.

We let $L^1(\mathbb{R}_+)$ denote the Banach space of Lebesgue measurable functions $f : \mathbb{R}_+ \to \mathbb{R}$ with finite $L^1$-norm,

$$\|f\|_{L^1(\mathbb{R}_+)} = \int_0^\infty |f(s)| ds < \infty,$$

in which functions equal almost everywhere are identified. Given Banach spaces $X$ and $Y$, we let $\mathcal{L}(X, Y)$ denote the vector space of bounded linear operators from $X$

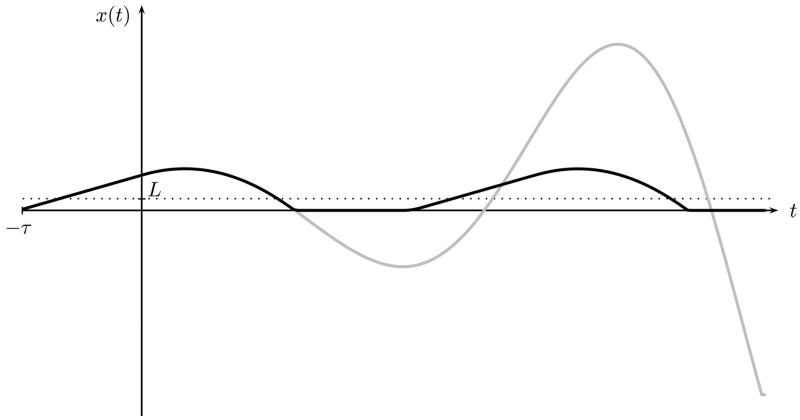


FIG. 1. *Solutions of a DDER (in black) and of an unconstrained delay differential equation (in gray) with identical initial conditions and both with linear drift function $f(\varphi) = L - \varphi(-\tau)$, where $f$ is interpreted as a function on either $\mathcal{C}^+_{[-\tau,0]}$ or $\mathcal{C}_{[-\tau,0]}$, depending on the equation.*

into $Y$. We shall use $\|\cdot\|$ to denote the norm on $X$ or $Y$, depending on the context. For $A \in \mathcal{L}(X,Y)$, we let $\|A\| = \sup\{\|Ax\| : x \in X, \|x\| = 1\}$ denote the operator norm of $A$. For an open subset $U$ of $X$ that contains the zero element and a function $f : U \to Y$, we say that $f(x) = o(\|x\|)$ if $\lim_{\|x\|\to 0}\|f(x)\|/\|x\| = 0$.

**2. Delay differential equations with reflection.** In this section, we define a solution of a DDER and explain its relation to the one-dimensional Skorokhod problem and to a delay differential equation with discontinuous right-hand side. Throughout this section, fix a delay $\tau \in (0,\infty)$ and a continuous function $f : \mathcal{C}^+_{[-\tau,0]} \to \mathbb{R}$.

DEFINITION 2.1. *A solution of the DDER associated with $f$ is a continuous function $x \in \mathcal{C}^+_{[-\tau,\infty)}$ such that there exists $y \in \mathcal{C}^+_{[0,\infty)}$ such that*

(i) *$(x, y)$ satisfies* (1.1),
(ii) *$y(0) = 0$ and $y$ is nondecreasing, and*
(iii) *$\int_0^t x(s)dy(s) = 0$ for all $t \geq 0$.*

See Figure 1 for an example of a solution of the DDER and the corresponding solution of the unconstrained delay differential equation.

*Remark* 2.1. We say a function $y : [0,\infty) \to \mathbb{R}$ has a *point of increase (to the right)* at time $t \geq 0$ if $y(t + s) > y(t)$ for all $s > 0$ sufficiently small. The condition $\int_0^t x(s)dy(s) = 0$ for all $t \geq 0$ in Definition 2.1 can be interpreted as follows: $y$ can have a point of increase (to the right) at time $t$ only if $x(t) = 0$.

*Remark* 2.2. It will be assumed throughout the paper that given $\varphi \in \mathcal{C}^+_{[-\tau,0]}$, there exists a unique solution $x$ of the DDER with $x_0 = \varphi$. We do not prescribe any further conditions than continuity on $f$; however, usually additional assumptions are required to guarantee existence and uniqueness of solutions. For example, if $f$ is locally Lipschitz continuous and a condition for nonexplosion of solutions in finite time is imposed, then existence and uniqueness of solutions hold. In particular, under Assumptions 3.1 and 3.2 or under Assumption 3.3, which are the respective assumptions for our main results, Theorems 3.4 and 3.8, given $\varphi \in \mathcal{C}^+_{[-\tau,0]}$, there exists a unique solution $x$ to the DDER with $x_0 = \varphi$.

In the following, we relate solutions of the DDER to the one-dimensional Skorokhod problem, which was introduced by Skorokhod [62] to constrain a continuous function to be nonnegative. The one-dimensional Skorokhod problem and its associated one-dimensional Skorokhod map are defined in Appendix A, where some relevant properties are discussed. Roughly speaking, given $z \in \mathcal{C}_{[0,\infty)}$ satisfying $z(0) \geq 0$, the solution to the one-dimensional Skorokhod problem is a pair $(x, y) \in \mathcal{C}^+_{[0,\infty)} \times \mathcal{C}^+_{[0,\infty)}$ satisfying $x(0) = z(0)$ and $x(t) = z(t) + y(t)$ for $t \geq 0$, where $x$ is a constrained version of $z$ and $y$ is a nondecreasing continuous control that acts only when $x$ is at the zero boundary and increases the minimal amount to keep $x$ nonnegative. It is a well-known fact in the theory of the one-dimensional Skorokhod problem that given $z \in \mathcal{C}_{[0,\infty)}$ satisfying $z(0) \geq 0$, there is a unique solution $(x, y)$ and $y$ is given by

$$y(t) = \sup_{0 \leq s \leq t} (z(s))^-, \qquad t \geq 0. \tag{2.1}$$

The one-dimensional Skorokhod map $(\Phi, \Psi) : \mathcal{C}_{[0,\infty)} \to \mathcal{C}^+_{[0,\infty)} \times \mathcal{C}^+_{[0,\infty)}$ maps $z$ to the unique solution $(x, y)$ of the one-dimensional Skorokhod problem, i.e., $(x, y) = (\Phi, \Psi)(z)$. Despite the requirement in the formulation of the one-dimensional Skorokhod problem that $z$ satisfy $z(0) \geq 0$, the one-dimensional Skorokhod map $(\Phi, \Psi)$ is in fact well defined on all of $\mathcal{C}_{[0,\infty)}$ (see (A.1)–(A.2)).

Given a solution $x$ of the DDER, (1.1) can be rewritten, for $t \geq 0$, as

$$x(t) = z(t) + y(t), \tag{2.2}$$

$$z(t) = x(0) + \int_0^t f(x_s) ds. \tag{2.3}$$

It follows from the conditions on $x$ and $y$ in Definition 2.1 that $(x|_{[0,\infty)}, y)$ is a solution of the one-dimensional Skorokhod problem for $z$. Therefore, $y$ is unique and given by (2.1). In the notation of the one-dimensional Skorokhod map,

$$(x|_{[0,\infty)}, y) = (\Phi, \Psi)(z). \tag{2.4}$$

LEMMA 2.2. *Suppose that $x$ is a solution of the DDER. Then $x$ is locally Lipschitz continuous on $[0, \infty)$ and so is absolutely continuous there. For the almost every $t > 0$ at which $x$ is differentiable, its derivative satisfies*

$$\frac{dx(t)}{dt} = \begin{cases} f(x_t) & \text{if } x(t) > 0, \\ 0 & \text{if } x(t) = 0. \end{cases} \tag{2.5}$$

*Furthermore, $x$ is continuously differentiable at all $t > 0$ such that $x(t) > 0$.*

*Proof.* By the continuity of $t \to f(x_t)$, $z$ defined in (2.3) is locally Lipschitz continuous on $[0, \infty)$ and continuously differentiable on $[0, \infty)$. By (2.4) and Proposition A.2, $x$ inherits the local Lipschitz property from $z$ on $[0, \infty)$. Hence $x$ is absolutely continuous on $[0, \infty)$ and differentiable at almost every $t > 0$. Consider $t > 0$ such that $x(t) > 0$. Then $y$ is constant in a neighborhood of $t$, so by (2.2) and (2.3), $x$ is continuously differentiable there and $\frac{dx(t)}{dt} = f(x_t)$. Now consider $t > 0$ such that $x$ is differentiable and $x(t) = 0$. By considering derivatives from the left and the right and using the nonnegativity of $x$, we see that $\frac{dx(t)}{dt} = 0$. ∎

The following lemma is an immediate consequence of Lemma 2.2.

LEMMA 2.3. *For a solution $x$ of the DDER define $\dot{x} : [0, \infty) \to \mathbb{R}$ by*

$$\dot{x}(t) = \begin{cases} f(x_t) & \text{if } x(t) > 0, \\ 0 & \text{if } x(t) = 0. \end{cases} \tag{2.6}$$

*Then*

$$x(t) = x(0) + \int_0^t \dot{x}(s)ds, \qquad t \geq 0. \tag{2.7}$$

*Remark* 2.3. Note that we are abusing the usual "dot" notation here—when $x(t) = 0$, the derivative of $x$ only exists at almost every such $t$. We find it convenient to have a notation for the right member in (2.6).

Delay differential equations with discontinuous right-hand sides are often used in engineering models (see, e.g., [56, 58, 57, 59]) to account for state constraints. Consider, for example, the equation

$$\frac{dx(t)}{dt} = \begin{cases} f(x_t) & \text{if } x(t) > 0, \\ f(x_t)^+ & \text{if } x(t) = 0, \end{cases} \tag{2.8}$$

where a solution of (2.8) is any absolutely continuous function $x \in \mathcal{C}^+_{[-\tau,\infty)}$ satisfying (2.8) at the almost every $t \in (0, \infty)$ where $x$ is differentiable. In the following lemma we provide a one-to-one correspondence between solutions of (2.8) and solutions of the DDER.

LEMMA 2.4. *A function $x \in \mathcal{C}^+_{[-\tau,\infty)}$ is a solution of* (2.8) *if and only if $x$ is a solution of the DDER associated with $f$.*

*Proof.* Suppose that $x$ is a solution of (2.8). Define $y \in \mathcal{C}^+_{[0,\infty)}$ by $y(t) = \int_0^t 1_{\{x(s)=0\}} f(x_s)^- ds$ for all $t \geq 0$. Then $x$ and $y$ satisfy (1.1), $y$ is nondecreasing, $y(0) = 0$, and $y$ can have a point of increase only when $x$ is zero. Therefore, $x$ is a solution of the DDER. Conversely, suppose that $x$ is a solution of the DDER. By Lemma 2.2, $x$ is absolutely continuous and (2.8) holds when $x(t) > 0$. At $t > 0$ such that $x(t) = 0$ and $x$ is differentiable, by (2.2), (2.3), and (2.5), $y$ is differentiable at $t$ with $\frac{dy(t)}{dt} = -f(x_t) \geq 0$, where the inequality follows because $y$ is nondecreasing. Thus, $\frac{dx(t)}{dt} = 0 = f(x_t)^+$ at such $t$, which proves that $x$ is a solution of (2.8). ∎

**3. Main results.** In this section, we define a SOPS and present our main results on sufficient conditions for the existence, uniqueness, and stability of SOPS to the DDER.

**3.1. Slowly oscillating periodic solutions.** In order to define a SOPS, we assume that there is a positive equilibrium point for the DDER, which is defined as follows.

DEFINITION 3.1. *A point $L > 0$ is an* equilibrium point *of the DDER if $x \equiv L$ on $[-\tau, \infty)$ is a solution of the DDER.*

A solution $x$ of the DDER that oscillates about an equilibrium point $L$ such that the times when $x$ is at the equilibrium point are separated by more than the delay $\tau$ is called *slowly oscillating*. Throughout this paper, we consider periodic solutions with this property, which we denote with an asterisk: $x^*$. We focus on the situation where there is exactly one equilibrium point $L$, which will be ensured either by Assumptions 3.1 and 3.2 or by Assumption 3.3.

DEFINITION 3.2. *A solution $x$ of the DDER is called a* periodic solution with period $p > 0$ *if*

$$x(t + p) = x(t) \textit{ for all } t \geq -\tau. \tag{3.1}$$

*Suppose $L > 0$ is an equilibrium point of the DDER. A periodic solution $x^*$ of the DDER is a SOPS if there exist points $q_0 \geq -\tau$, $q_1 > q_0 + \tau$ and $q_2 > q_1 + \tau$ such that*

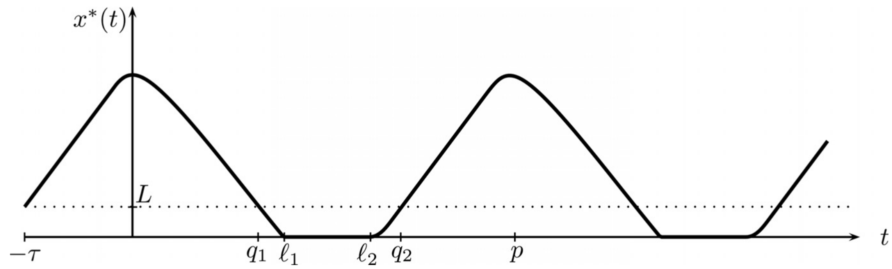


FIG. 2. *An example of a SOPS where* $q_0 = -\tau$. *In the figure,* $\ell_1 = \inf\{t \geq q_1 : x^*(t) = 0\}$ *and* $\ell_2 = \inf\{t \geq \ell_1 : x^*(t) > 0\}$.

(3.1) *holds with* $p = q_2 - q_0$, *and*

$$\begin{aligned} x^*(q_0) &= L, & \\ x^*(t) &> L, & q_0 < t < q_1, \\ 0 \leq x^*(t) &< L, & q_1 < t < q_2. \end{aligned} \tag{3.2}$$

See Figure 2 for an example of a SOPS of the DDER when $q_0 = \tau$.

**3.2. Existence of SOPS.** To establish the existence of a SOPS, we assume that $f$ depends only on the current and delayed states of the system

$$f(\varphi) = g(\varphi(0), \varphi(-\tau)), \qquad \varphi \in \mathcal{C}^+_{[-\tau,0]}, \tag{3.3}$$

where $g : \mathbb{R}^2_+ \to \mathbb{R}$ is a continuous function satisfying two assumptions. The first assumption is used to establish the existence of an equilibrium point and to specify regularity properties of $g$.

*Assumption* 3.1. The function $g : \mathbb{R}^2_+ \to \mathbb{R}$ is locally Lipschitz continuous, there is a constant $L > 0$ such that $g(L, L) = 0$, $g$ is differentiable at $(L, L)$,

$$A = -\partial_1 g(L, L) \geq 0, \qquad B = -\partial_2 g(L, L) > 0, \tag{3.4}$$

and $B > A \geq 0$. Here $\partial_i g$ denotes the first partial derivative with respect to the $i$th argument of $g$, $i = 1, 2$.

The condition (3.4) imposes a negative feedback condition on the local linearization about the equilibrium; for this linearization, the condition $B > A$ is known to be necessary for the equilibrium solution to be unstable. The following is a global negative feedback type of condition.

*Assumption* 3.2. For all $r, s \in \mathbb{R}_+$,

(i) $(g(r, s) - g(r, L))(s - L) < 0$ if $s \neq L$, and

(ii) $(g(r, s) - g(L, s))(r - L) \leq 0$ if $r \neq L$.

*Remark* 3.1. From Assumptions 3.1 and 3.2 it follows that (i) if $r \geq L$ and $s > L$, then $g(r, s) < 0$, and (ii) if $r \leq L$ and $s < L$, then $g(r, s) > 0$. Furthermore, $g(r, r)(r - L) \leq g(L, r)(r - L) < 0$ for all $r \neq L$, which ensures that $L$ is the unique equilibrium point of the DDER.

In [4], a third set of assumptions bounding $g(L, \cdot)$ and providing linear growth conditions on $g$ in both arguments is imposed in part to prevent unbounded oscillations. The presence of the lower boundary in (1.1) prevents unbounded oscillations and a version of the third set of assumptions is instead a consequence of Assumptions 3.1 and 3.2, as follows.

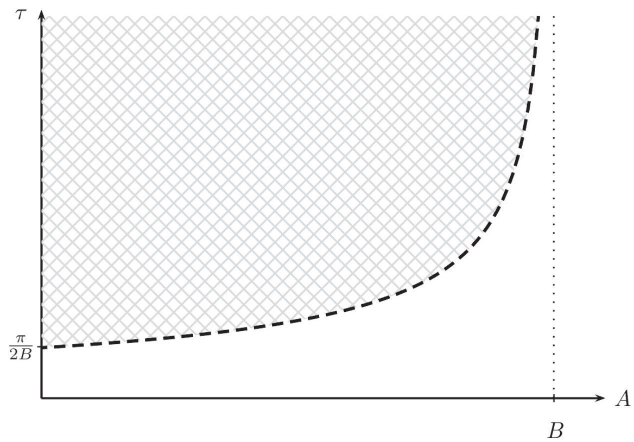


FIG. 3. *For a fixed parameter $B > 0$, $u \equiv 0$ is an unstable equilibrium solution to* (3.6) *for parameters $\tau$ and $A$ in the crosshatched region, which is given by $\{(A, \tau) : A \in [0, B)$ and $\tau > \tau_0\}$, where $\tau_0$ is the function of $A$ and $B$ given by* (3.8).

LEMMA 3.3. *Under Assumptions* 3.1 *and* 3.2, *there exists $G \in (0, \infty)$ such that $g(L, s) \leq G$ for all $s \in \mathbb{R}_+$. Additionally, there exist constants $\kappa_1, \kappa_2 \in (0, \infty)$ such that*

$$|g(r, s)| \leq \kappa_1 |r - L| + \kappa_2 |s - L|, \qquad 0 \leq r, s \leq L + \tau G. \tag{3.5}$$

*Proof.* The existence of $G$ follows because $g(L, \cdot)$ is continuous on $[0, L]$ and is negative on $(L, \infty)$. The existence of $\kappa_1$ and $\kappa_2$ follows because $g$ is locally Lipschitz continuous with $g(L, L) = 0$. ∎

Under Assumption 3.1, consider the linear delay differential equation obtained by linearizing $g$ about its equilibrium point and centering about the equilibrium:

$$\frac{du(t)}{dt} = -Au(t) - Bu(t - \tau). \tag{3.6}$$

Equation (3.6) has characteristic equation

$$\lambda + A + Be^{-\lambda\tau} = 0. \tag{3.7}$$

Let $\theta_0$ be the unique solution in $[\pi/2, \pi)$ to $\cos\theta_0 = -A/B$, which we write as $\theta_0 = \cos^{-1}(-A/B)$, and define

$$\tau_0 = \frac{\theta_0}{\sqrt{B^2 - A^2}}. \tag{3.8}$$

If $\tau > \tau_0$, the characteristic equation (3.7) will have a solution $\lambda$ with positive real part, from which it follows that the equilibrium solution $u \equiv 0$ of (3.6) is unstable (see Theorem 4.7 in [63] or the discussion beginning at the bottom of p. 134 in [24]). See Figure 3 for a depiction illustrating when the equlibrium solution is unstable (for fixed $B$). This, along with the negative feedback condition in Assumption 3.2, will allow us to prove the following result on the existence of a SOPS.

THEOREM 3.4. *Under Assumptions* 3.1 *and* 3.2, *if $\tau_0$ is given by* (3.8), *then for any $\tau > \tau_0$, there exists a SOPS of the DDER.*

The proof of Theorem 3.4 is given in section 5.

**3.3. Uniqueness and stability of SOPS.** To establish uniqueness and stability of SOPS, we impose more restrictive conditions on $f$; in particular, we assume that $f$ depends only on the delayed state:

$$f(\varphi) = h(\varphi(-\tau)), \qquad \varphi \in \mathcal{C}^+_{[-\tau,0]}, \tag{3.9}$$

where $h : \mathbb{R}_+ \to \mathbb{R}$ is a continuous function that satisfies two sets of assumptions. The first set of assumptions imply Assumptions 3.1 and 3.2 that are used in proving the existence of a SOPS. It also includes assumptions on the regularity of $h$ and its asymptotic behavior at infinity.

*Assumption* 3.3. The function $h : \mathbb{R}_+ \to \mathbb{R}$ is continuously differentiable on $\mathbb{R}_+$; there are constants $\alpha > 0$, $\beta > 0$ such that $\lim_{s\to\infty} h(s) = -\alpha$, $h(0) = \beta$; and there is a constant $L > 0$ such that $h(L) = 0$, $h'(L) < 0$, and $(s - L)h(s) < 0$ for all $L \neq s \in \mathbb{R}_+$.

The following lemma is an immediate consequence of Assumption 3.3 and Lemma 2.3.

LEMMA 3.5. *Under Assumption* 3.3, $H = \sup\{|h(s)| : s \in \mathbb{R}_+\} < \infty$. *If $x$ is a solution of the DDER associated with $h$, then $x$ is uniformly Lipschitz continuous on* $[0,\infty)$ *with Lipschitz constant $H$:*

$$|x(t) - x(s)| \leq H|t - s|, \qquad 0 \leq s, t < \infty. \tag{3.10}$$

On setting $g(r, s) = h(s)$ for $r, s \geq 0$, Assumption 3.3 implies that $g$ satisfies Assumptions 3.1 (with $A = 0$ and $B = -h'(L)$) and 3.2. Also, $\tau_0$ in (3.8) is given by

$$\tau_0 = -\frac{\pi}{2h'(L)} > 0. \tag{3.11}$$

Theorem 3.4 ensures that for each $\tau > \tau_0$ there exists a SOPS of the DDER. For our proof of the uniqueness and stability of a SOPS, we assume that $h'(s)$ converges to zero sufficiently fast as $s \to \infty$, as follows.

*Assumption* 3.4. The function $h : \mathbb{R}_+ \to \mathbb{R}$ is continuously differentiable on $\mathbb{R}_+$, its derivative $h'$ is in $L^1(\mathbb{R}_+)$, and $m = \sup\{|sh'(s)| : s \in \mathbb{R}_+\} < \infty$.

In [80], the stronger condition that $sh'(s) \to 0$ as $s \to \pm\infty$ is imposed. However, the presence of the lower boundary in (1.1) allows us to relax this condition.

The following lemma is an immediate consequence of Assumption 3.4 and the fundamental theorem of calculus.

LEMMA 3.6. *Under Assumption* 3.4, $K_h = \sup\{|h'(s)| : s \in \mathbb{R}_+\} < \infty$ *and so $h$ is uniformly Lipschitz continuous with Lipschitz constant $K_h$:*

$$|h(s) - h(r)| = \left|\int_r^s h'(u)du\right| \leq K_h|s - r|, \qquad 0 \leq r, s < \infty. \tag{3.12}$$

We can now present our main result on the uniqueness and stability of SOPS. We first define a notion of uniqueness for SOPS.

DEFINITION 3.7. *We say a SOPS $x^*$ of the DDER is* unique up to time translation *if given any other SOPS $x^\dagger$ of the DDER, there exists $t^\dagger \in [0, p)$ such that $x^*(t) = x^\dagger(t^\dagger + t)$ for $t \geq 0$.*

*Remark* 3.2. Given any SOPS $x^*$ of the DDER, there is a family of SOPS that are equivalent up to time translation. In particular, for any $t^\dagger \geq 0$, the function $x^\dagger \in \mathcal{C}^+_{[-\tau,\infty)}$ given by $x^\dagger(t) = x^*(t^\dagger + t)$ for all $t \geq -\tau$ is a SOPS of the DDER.

THEOREM 3.8. *Suppose that Assumptions* 3.3 *and* 3.4 *hold and that $\tau_0$ is given by* (3.11). *Then there exists $\tau^* \geq \tau_0$ such that for any $\tau > \tau^*$, there exists a SOPS of the DDER and it is unique up to time translation. Furthermore, the SOPS satisfies* *the following property, which we call exponential stability: there are positive constants* $\varepsilon$*,* $\gamma$*,* $K_\gamma$*, and* $K_\rho$ *such that for any member* $x^*$ *of the family of equivalent (up to time translation) SOPS and for* $p$ *equal to the period of* $x^*$*, if* $\varphi \in \mathcal{C}^+_{[-\tau,0]}$ *satisfies* $\|\varphi - x^*_\sigma\|_{[-\tau,0]} < \varepsilon$ *for some* $\sigma \in [0, p)$*, then there is a* $\rho \in (-p, p)$ *satisfying*

$$|\rho| \le K_\rho \|\varphi - x^*_\sigma\|_{[-\tau,0]}, \tag{3.13}$$

*and such that*

$$\|x_t - x^*_{t+p+\sigma+\rho}\|_{[-\tau,0]} \le K_\gamma e^{-\gamma t} \|\varphi - x^*_\sigma\|_{[-\tau,0]}, \tag{3.14}$$

*for all* $t \ge 0$*, where* $x$ *denotes the unique solution of the DDER with* $x_0 = \varphi$*.*

The proof of Theorem 3.8 is given in section 6.

**4. Applications.** In this section, we illustrate our results with some simple examples of deterministic models of biochemical reaction networks and an Internet congestion control model. Both biochemical reaction network examples model the concentration of a single protein. In the first example, the delay arises due to delayed degradation, whereas in the second example, the delayed dynamics are due to a lengthy production time. The Internet congestion control example models a pricing mechanism that controls the rate at which data packets are transmitted from a source to a link. Here the delayed dynamics are due to the finiteness of transmission speeds.

**4.1. Biochemical reaction network with delayed protein degradation.** We consider a simple biochemical model for the production and degradation of a protein $X$ in which $X$ may be degraded by either of two mechanisms, one of which involves a delay. Fix $\tau > 0$ and $k_1, k_2, k_3, k_4, k_5 > 0$. In the model, $X$ is produced by components external to the system at rate $k_1$ and each molecule of $X$ degrades at rate $k_2$, which is represented by the following reactions:

$$\emptyset \overset{k_1}{\to} X, \qquad X \overset{k_2}{\to} \emptyset, \tag{4.1}$$

where $\emptyset$ denotes "nothing" (or a quantity external to the system). Furthermore, $X$ is a transcription factor activating the production of a protein $P$, i.e., when $X$ is present in the system it attaches to the promoter region of the DNA template for $P$ at rate $k_3$, thus initiating the production process for $P$. The production process for $P$ is a multistage process, including lengthy transcription and translation stages, which leads to a delay in its production. After a molecule of $P$ is produced, it quickly combines with a molecule of $X$ (if one is available), at rate $k_4$, and the resulting complex is eliminated from the system, or otherwise $P$ rapidly degrades, at rate $k_5$. Thus, $X$ can be degraded by the production of a molecule of $P$ and the subsequent removal of a molecule of $X$ by a molecule of $P$. The reactions involving $P$ are the following:

$$\emptyset \overset{k_3(X)}{\Rightarrow} P, \qquad X + P \overset{k_4}{\to} \emptyset, \qquad P \overset{k_5}{\to} \emptyset, \tag{4.2}$$

where the double arrow indicates a delayed reaction and the term above the double arrow indicates that the reaction is initiated at a rate proportional to the number of molecules of $X$ in the system. Let $x(t)$ and $p(t)$ denote the respective concentrations of proteins $X$ and $P$ at time $t$. The reactions (4.1) and (4.2) suggest the following deterministic dynamics for $x$ and $p$:

$$\frac{dx(t)}{dt} = k_1 - k_2 x(t) - k_4 x(t) p(t),$$
$$\frac{dp(t)}{dt} = k_3 x(t - \tau) - k_4 x(t) p(t) - k_5 p(t).$$

The above equations naturally preserve the nonnegativity constraints for the concentrations $x(t)$ and $p(t)$—whenever $x(t) = 0$, $\frac{dx(t)}{dt} > 0$ holds, and whenever $p(t) = 0$, $\frac{dp(t)}{dt} \geq 0$ holds.

Consider the situation where $k_4$ and $k_5$ are very large constants compared with $k_1, k_2, k_3$, and $k_4$ is considerably larger than $k_5$. As an approximation, we consider the formal limit when $k_4 \to \infty$ first and then $k_5 \to \infty$. In this formal limit, whenever a molecule of $P$ is produced, it *instantly* combines with a molecule of $X$ (provided there is one) and the resulting complex is eliminated from the system; otherwise $P$ *instantly* degrades. As a consequence, the net effect of the reactions in (4.2) is that a molecule of $P$ does not remain in the system for a positive amount of time and whenever there are molecules of $X$ in the system, they are eliminated via delayed degradation at a rate equal to the rate at which molecules of $P$ are produced, i.e., at rate $k_3 x(t-\tau)$. In this limiting case, $x$ satisfies the following delay differential equation with discontinuous right-hand side:

$$\frac{dx(t)}{dt} = \begin{cases} k_1 - k_2 x(t) - k_3 x(t-\tau) & \text{if } x(t) > 0, \\ (k_1 - k_2 x(t) - k_3 x(t-\tau))^+ & \text{if } x(t) = 0. \end{cases} \tag{4.3}$$

By Lemma 2.4, solutions of (4.3) are in one-to-one correspondence with solutions of the DDER associated with $f(\varphi) = g(\varphi(0), \varphi(-\tau)) = k_1 - k_2\varphi(0) - k_3\varphi(-\tau)$. If $k_3 > k_2$, then $g$ satisfies Assumptions 3.1 and 3.2 with equilibrium point $L = \frac{k_1}{k_2+k_3}$. Then by Theorem 3.4, for

$$\tau > \frac{\cos^{-1}(-k_2/k_3)}{\sqrt{k_3^2 - k_2^2}},$$

there exists a SOPS of the DDER.

In [6], Bratsun et al. analyzed a similar deterministic biochemical reaction model with delayed degradation of a protein molecule and linear $g$, but without the nonnegativity constraint. They observed that for sufficiently large delays, solutions exhibited unbounded oscillations. Here we have shown that the addition of a nonnegativity constraint can turn an equation with unbounded oscillatory solutions into one with bounded periodic solutions that obey the natural constraints of the model.

**4.2. Biochemical reaction network with delayed autorepression.** We consider a simple model for a biochemical reaction network in which the quantity of an autorepressor protein (a protein that inhibits its own production) is affected by three factors: production, enzymatic degradation, and dilution. In this case, the dimerized form of the protein represses its own transcription process. The production of the protein monomer, which includes transcription, translation, protein folding, etc., is lengthy (see Figure 4). To simplify the analysis, rather than considering each step as a separate reaction, we treat the production of a protein as one process with a time delay. In simple biochemical reaction network models of genetic circuits, it has been observed that this type of delayed autorepression can lead to oscillations in protein concentration (see, e.g., [5, 34, 48, 61, 66, 69]) and in [46], Mather et al. use analytical tools to demonstrate that these models can exhibit what they term *degrade-and-fire* oscillations, where the "period" of oscillation can be long relative to the delay length. (The term "period" is used in a rough sense as they do not establish the existence of nonconstant *periodic* solutions.) In the following, we show that, for sufficiently long delays, there exists a SOPS and, moreover, for possibly longer delays the SOPS is unique and exponentially stable. In addition, we show that asymptotically as the delay

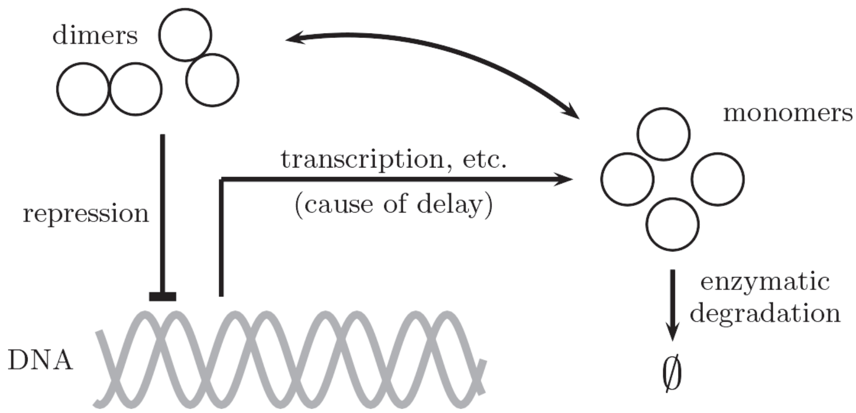


FIG. 4. *A depiction of the simple biochemical reaction network described in Example* 4.2.

increases, the period of oscillation can be made arbitrarily large relative to the delay by choosing production and degradation parameters appropriately (see (4.4) below).

Fix $\tau > 0$ and $a, b, c, C_0, R_0 > 0$. In the simple model proposed by Mather et al. [46], the deterministic dynamics of the system are described by the following delay differential equation:

$$\frac{dx(t)}{dt} = \frac{aC_0^2}{(C_0 + x(t-\tau))^2} - \frac{bx(t)}{R_0 + x(t)} - cx(t), \qquad t \geq 0.$$

Here $x(t)$ represents the concentration of the protein monomer at time $t$. The first term on the right is the production rate with delayed negative feedback—the squared term in the denominator arises because the dimerized form of the protein represses transcription. The second and third terms represent the effects of enzymatic degradation and dilution, respectively. In the above model, the nonnegativity of the protein concentration is ensured by the form of the delay differential equation: if $x(t) = 0$, then $\frac{dx(t)}{dt} > 0$. In [46], $R_0$ and $c$ are very small and the authors consider the limiting case of the deterministic system where $R_0 = 0$ and $c = 0$, which corresponds to the setting where the enzymatic degradation rate is constant and the effects of dilution are negligible. However, in this formal limit, the delay differential equation loses the inherent nonnegativity of its solutions, and the equation must be modified at the boundary, i.e., when $x(t) = 0$. One way to account for the nonnegativity constraint is to consider solutions of the DDER associated with (3.9) and

$$h(\varphi(-\tau)) = \frac{aC_0^2}{(C_0 + \varphi(-\tau))^2} - b, \qquad \varphi \in \mathcal{C}^+_{[-\tau,0]}.$$

If $a > b$, then $h$ satisfies Assumptions 3.3 and 3.4 with $\alpha = b$, $\beta = a-b$ and equilibrium point $L = C_0(\sqrt{a/b} - 1)$. By Theorem 3.4, for $\tau > \frac{\pi}{4} C_0 \sqrt{a/b^3}$, there exists a SOPS of the DDER. Furthermore, by Theorem 3.8, there is a $\tau^* \geq \frac{\pi}{4} C_0 \sqrt{a/b^3}$ such that for $\tau > \tau^*$, the SOPS is unique and exponentially stable. Now for each $\tau > \tau_0$, let $x^\tau$ denote a SOPS of the DDER with delay $\tau$ and let $p^\tau$ denote its period. Then according to Corollary 6.13,

$$\lim_{\tau \to \infty} \frac{p^\tau}{\tau} = \frac{a}{b} + 1. \qquad (4.4)$$

Thus, given the freedom to choose the constants $a > b > 0$, the period of oscillation can be made arbitrarily large relative to the delay, asymptotically as $\tau \to \infty$.

**4.3. Internet congestion control with delayed transmission rates.** Deterministic delay differential equations have been used as approximate (fluid) models for the dynamics of data transmission rates and prices in Internet congestion control models [67], where the finiteness of transmission speeds leads to delayed dynamics. In such models, control protocols are often designed to steer transmission rates toward equilibrium points using a (delayed) negative feedback mechanism. There is a considerable body of work on obtaining sufficient conditions for stability of equilibrium points for such models (see, e.g., [13, 56, 58, 57, 59, 83]). In the following, we analyze a simple one-dimensional model and apply our results to understand conditions under which the equilibrium point is *not* stable and there exists periodic oscillatory behavior. Despite the fact that control protocols are typically designed to prevent such sustained oscillations, it is useful to understand the model when the protocol fails to stabilize the equilibrium point.

We consider the one-dimensional case of a model introduced in [55] and further analyzed in [56, 58, 57, 59]. In this simple model, packets are transmitted from a single source, with nonnegative transmission rate $r(t)$ at time $t$, through a single link with capacity $c \in (0, 1)$ at rate $r(t - \tau_f)$, where $\tau_f > 0$ is referred to as the *forward delay*. At time $t$, the link "charges" packets a nonnegative *price* $x(t)$. The price is sent back to the source, which at time $t$ observes the delayed price $x(t - \tau_b)$, where $\tau_b > 0$ is referred to as the *backward delay*. The transmission rate $r(t)$ is then given as a function of the delayed price $x(t - \tau_b)$. The control protocol, which governs the dynamics of the pricing mechanism, is designed to achieve a specified equilibrium transmission rate. In [55], the dynamics of the price $x$ are given by the following differential dynamics which depend on the delayed transmission rate $r(t - \tau_f)$:

$$\frac{dx(t)}{dt} = \begin{cases} \dfrac{r(t-\tau_f)}{c} - 1 & \text{if } x(t) > 0, \\ \left(\dfrac{r(t-\tau_f)}{c} - 1\right)^+ & \text{if } x(t) = 0. \end{cases}$$

The transmission rate at time $t$ is a function of the delayed price $x(t - \tau_b)$:

$$r(t) = \exp\left(-ax(t - \tau_b)\right), \qquad t \geq 0.$$

Here $a > 0$ is a constant referred to as the *gain*. Upon substitution, we see that the dynamics of the price $x$ are described by the following delay differential equation with discontinuous right-hand side:

$$\frac{dx(t)}{dt} = \begin{cases} h(x(t-\tau)) & \text{if } x(t) > 0, \\ h(x(t-\tau))^+ & \text{if } x(t) = 0, \end{cases} \tag{4.5}$$

where $\tau = \tau_f + \tau_b$ (referred to as the *round-trip time*) and

$$h(\varphi(-\tau)) = \frac{\exp(-a\varphi(-\tau))}{c} - 1, \qquad \varphi \in \mathcal{C}^+_{[-\tau,0]}. \tag{4.6}$$

By Lemma 2.4, solutions of (4.5) are in one-to-one correspondence with solutions of the DDER associated with $f(\varphi) = h(\varphi(-\tau))$. Note that $h$ satisfies Assumptions 3.3 and 3.4 with equilibrium point $L = -\log(c)/a$, so by Theorem 3.4, if $\tau > \pi/2a$, there exists a SOPS of the DDER. Furthermore, by Theorem 3.8, there is a $\tau^* \geq \pi/2a$ such that if $\tau > \tau^*$, then the SOPS is unique and exponentially stable. As noted above, Internet congestion control protocols are typically designed to prevent such oscillatory behavior. Indeed, in [55] it is noted that as the delay $\tau$ increases, solutions to (2.8) with $h$ as in (4.6) exhibit sustained oscillatory behavior. To counteract this behavior, $a$ is often designed so as to depend on the delay $\tau$, e.g., in [55] it is observed that if $a$ is allowed to depend on $\tau$ such that $a(\tau) = b/\tau$ for $b \in (0, \pi/2)$, then the equilibrium solution is exponentially stable for all $\tau > 0$.

**5. Existence of SOPS.** In this section we prove Theorem 3.4, which provides sufficient conditions for the existence of SOPS to the DDER. The general outline is to define an appropriate cone in $\mathcal{C}^+_{[-\tau,0]}$ and a return map on the cone such that nonconstant fixed points of the return map are in one-to-one correspondence with SOPS of the DDER. This approach was first implemented by Jones [25] to prove the existence of SOPS to a certain unconstrained delay differential equation. Subsequently, numerous authors adopted this general approach for establishing the existence of SOPS of unconstrained delay differential equations under various conditions on the drift $g$ (see, e.g., [4, 11, 15, 21, 23, 49, 60, 70]). We adopt this approach as well, though some new difficulties need to be addressed in our context because of the lower boundary constraint and the associated discontinuous dynamics.

Throughout this section, we assume that $f$ is of the form exhibited in (3.3) and that Assumptions 3.1 and 3.2 hold.

**5.1. Browder's fixed point theorem.**

DEFINITION 5.1. *Let $X$ be a topological space, $f : X \to X$ a continuous function, and $x_0 \in X$ a fixed point of $f$. Then $x_0$ is an* ejective fixed point *if there exists an open neighborhood $U$ of $x_0$ such that for every $x \in U \setminus \{x_0\}$, there exists a positive integer $n = n(x)$ such that the $n$th iterate of $f$, $f^n(x)$, is not in $U$.*

The following is a version of Browder's fixed point theorem [8] and is a special case of Corollary 1.1 in [50], which gives sufficient conditions for the existence of nonejective fixed points. Recall that given a topological space $X$, a function $f : X \to X$ is compact if the closure of $f(V)$ is compact whenever $V \subset X$ is bounded.

THEOREM 5.2. *Let $K$ be a closed, bounded, convex, infinite-dimensional subset of a Banach space. Suppose that $f : K \to K$ is a continuous, compact function. Then $f$ has a fixed point in $K$ that is not ejective.*

Briefly, our proof of Theorem 3.4 proceeds as follows. We first perform a spatial shift and rescale time in (1.1) so that the equilibrium solution is at the origin and the delay interval $[-\tau, 0]$ is normalized to $[-1, 0]$. We show that it suffices to prove the existence of a corresponding SOPS for this normalized equation. We denote solutions of the normalized equation with a hat: $\hat{x}$. Existence will be proved by finding a suitable set $\widetilde{\mathcal{K}}$ in the Banach space $\mathcal{C}_{[-1,0]}$ and proving that if $\hat{x}$ is a nonconstant solution of the normalized equation such that $\hat{x}_0 \in \widetilde{\mathcal{K}}$, then $\hat{x}$ is slowly oscillating and $\hat{x}_t \in \widetilde{\mathcal{K}}$ for some $t > 0$. We will define a function $\Lambda$ on $\widetilde{\mathcal{K}}$ that maps the unique constant solution to itself and nonconstant $\hat{x}_0 \in \widetilde{\mathcal{K}}$ to $\hat{x}_t$, where $t$ is the first time after time zero that $\hat{x}_t \in \widetilde{\mathcal{K}}$. An element of $\widetilde{\mathcal{K}}$ that is mapped by $\Lambda$ to itself corresponds to a periodic solution (which may be constant). Under Assumptions 3.1 and 3.2, for $\tau > \tau_0$, the unique constant solution will be an ejective fixed point of $\Lambda$ and Browder's fixed point theorem will imply the existence of a nonejective fixed point, which will correspond to a SOPS.

**5.2. Normalized solutions.** It will be convenient to work with normalized solutions of the DDER (1.1), obtained from a solution $x$ of the DDER by subtracting off $L$ and rescaling time so that the delay is of length one. The normalized solutions will satisfy a normalized version of (1.1). We work with this normalized equation here as well as in the proof of uniqueness and stability (sections 6.1 and 6.2). There is no loss of generality in this as there is a one-to-one correspondence between solutions of the normalized equation and those of the original DDER, as we will show below in Lemma 5.5.

We first need some definitions. Recall that $g$ is assumed to satisfy Assumptions 3.1 and 3.2. Let $\hat{g} : [-L,\infty)^2 \to \mathbb{R}$ be the function defined by

$$\hat{g}(r,s) = g(r+L, s+L), \qquad r,s \in [-L,\infty). \tag{5.1}$$

Then $\hat{g}$ inherits the following properties from $g$: the function $\hat{g}$ is locally Lipschitz continuous, $\hat{g}(0,0) = 0$, and $\hat{g}$ is differentiable at $(0,0)$ with

$$-\partial_1 \hat{g}(0,0) = A \geq 0, \qquad -\partial_2 \hat{g}(0,0) = B > 0,$$

where $B > A \geq 0$ are as in Assumption 3.1. By Assumption 3.2, $\hat{g}$ satisfies the following inequalities:

$$\hat{g}(r,s) > \hat{g}(r,0) \text{ if } -L \leq s < 0, \tag{5.2}$$

$$\hat{g}(r,s) < \hat{g}(r,0) \text{ if } s > 0, \tag{5.3}$$

$$\hat{g}(r,s) \geq \hat{g}(0,s) \text{ if } -L \leq r \leq 0, \tag{5.4}$$

$$\hat{g}(r,s) \leq \hat{g}(0,s) \text{ if } r \geq 0. \tag{5.5}$$

By (5.2) and (5.4), if $r \leq 0$ and $s < 0$, then $\hat{g}(r,s) > 0$, and similarly, by (5.3) and (5.5), if $r \geq 0$ and $s > 0$, then $\hat{g}(r,s) < 0$. Finally, for positive constants $G$, $\kappa_1$, and $\kappa_2$ as in Lemma 3.3, we have $\hat{g}(0,s) \leq G$ for all $s \geq -L$ and

$$|\hat{g}(r,s)| \leq \kappa_1|r| + \kappa_2|s|, \qquad -L \leq r,s \leq \tau G. \tag{5.6}$$

We can now define a solution of a normalized DDER associated with $\hat{g}$, or DDER$^{\text{n}}$ for short.

Definition 5.3. *A solution of the normalized DDER$^{\text{n}}$ associated with $\hat{g}$ is a continuous function $\hat{x} \in \mathcal{C}_{[-1,\infty)}$ such that there exists $\hat{y} \in \mathcal{C}^+_{[0,\infty)}$ such that*

$$\hat{x}(t) = \hat{x}(0) + \tau \int_0^t \hat{g}(\hat{x}(s), \hat{x}(s-1))ds + \hat{y}(t), \qquad t \geq 0, \tag{5.7}$$

*and*

(i) $\hat{x}(t) \geq -L$ *for all* $t \geq 0$,

(ii) $\hat{y}(0) = 0$, $\hat{y}$ *is nondecreasing, and*

(iii) $\int_0^t (\hat{x}(s) + L) d\hat{y}(s) = 0$ *for all* $t \geq 0$.

Given a solution $\hat{x}$ of the DDER$^{\text{n}}$, (5.7) can be rewritten as

$$\hat{x}(t) = \hat{z}(t) + \hat{y}(t), \qquad t \geq 0, \tag{5.8}$$

$$\hat{z}(t) = \hat{x}(0) + \tau \int_0^t \hat{g}(\hat{x}(s), \hat{x}(s-1))ds, \qquad t \geq 0. \tag{5.9}$$

Adding $L$ to either side of (5.8), we see that $(\hat{x}|_{[0,\infty)} + L, \hat{y})$ is a solution of the one-

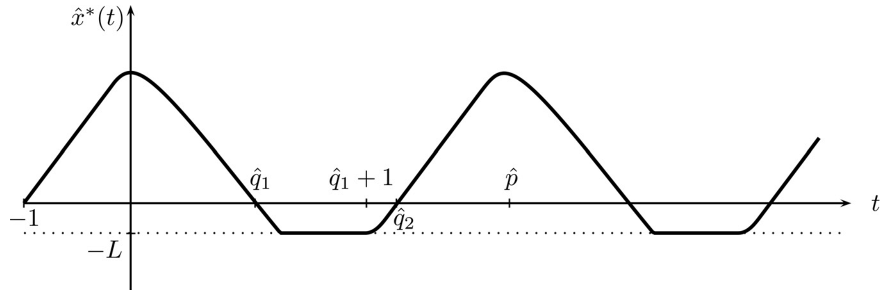


Fig. 5. *An example of a SOPS$^n$ with $\hat{q}_0 = -1$.*

dimensional Skorokhod problem for $\hat{z} + L$ (see Appendix A), so by Proposition A.4,

$$\hat{y}(t) = \sup_{0 \le s \le t} (\hat{z}(s) + L)^-, \qquad t \ge 0. \tag{5.10}$$

From the conditions on $\hat{g}$, zero is the unique equilibrium point for the DDER$^{\rm n}$, i.e., $\hat{x} \equiv 0$ is the only constant solution of the DDER$^{\rm n}$. Next, we define a slowly oscillating periodic solution (SOPS$^{\rm n}$) of the DDER$^{\rm n}$, which we denote with an asterisk: $\hat{x}^*$.

Definition 5.4. *A solution $\hat{x}$ of the DDER$^n$ is called* periodic *if there exists $\hat{p} > 0$ such that*

$$\hat{x}(t + \hat{p}) = \hat{x}(t), \text{ for all } t \ge -1. \tag{5.11}$$

*A periodic solution $\hat{x}^*$ of the DDER$^n$ is called a* slowly oscillating periodic solution (SOPS$^{\rm n}$) *if there exists $\hat{q}_0 \ge -1$, $\hat{q}_1 > \hat{q}_0 + 1$, and $\hat{q}_2 > \hat{q}_1 + 1$ such that* (5.11) *holds with $\hat{p} = \hat{q}_2 - \hat{q}_0$, and*

$$\begin{aligned} \hat{x}^*(\hat{q}_0) &= 0, \\ \hat{x}^*(t) &> 0 \text{ for } \hat{q}_0 < t < \hat{q}_1, \\ -L \le \hat{x}^*(t) &< 0 \text{ for } \hat{q}_1 < t < \hat{q}_2. \end{aligned} \tag{5.12}$$

*See Figure* 5 *for an example of a SOPS$^n$ of the DDER$^n$ when $\hat{q}_0 = -1$.*

The following lemma provides a one-to-one correspondence between solutions of the DDER and solutions of the DDER$^{\rm n}$ as well as between SOPS and SOPS$^{\rm n}$. The proof is a straightforward verification and so we omit it.

Lemma 5.5. *Let $x$ be a solution of the DDER associated with $g$. If $\hat{x} \in \mathcal{C}_{[-1,\infty)}$ is defined by*

$$\hat{x}(t) = x(\tau t) - L, \qquad t \ge -1, \tag{5.13}$$

*then $\hat{x}$ is a solution of the DDER$^n$ associated with $\hat{g}$. Furthermore, if $x$ is a SOPS with period $p$, then $\hat{x}$ is a SOPS$^n$ with period $\tau^{-1}p$. Conversely, let $\hat{x}$ be a solution of the DDER$^n$ associated with $\hat{g}$. If $x \in \mathcal{C}^+_{[-\tau,\infty)}$ is defined by*

$$x(t) = \hat{x}(\tau^{-1} t) + L, \qquad t \ge -\tau, \tag{5.14}$$

*then $x$ is a solution of the DDER associated with $g$. Furthermore, if $\hat{x}$ is a SOPS$^n$ with period $\hat{p}$, then $x$ is a SOPS with period $\tau \hat{p}$.*

By the unique correspondence between solutions $x$ of the DDER and $\hat{x}$ of the DDER$^{\rm n}$ described in Lemma 5.5, $\hat{x}$ inherits the following properties from $x$ that are described in Lemmas 2.2, 2.3, and 2.4. Again the proof is straightforward and we omit it.

LEMMA 5.6. *Suppose that $\hat{x}$ is a solution of the DDER$^n$. Then $\hat{x}$ is locally Lipschitz continuous on $[0,\infty)$ and so is absolutely continuous there. Define $\dot{\hat{x}} : [0,\infty) \to \mathbb{R}$ by*

$$\dot{\hat{x}}(t) = \begin{cases} \tau\hat{g}(\hat{x}(t), \hat{x}(t-1)) & \text{if } \hat{x}(t) > -L, \\ 0 & \text{if } \hat{x}(t) = -L. \end{cases} \tag{5.15}$$

*Then $\frac{d\hat{x}(t)}{dt} = \dot{\hat{x}}(t)$ at the almost every $t > 0$ that $\hat{x}$ is differentiable and*

$$\hat{x}(t) = \hat{x}(0) + \int_0^t \dot{\hat{x}}(s)ds, \qquad t \geq 0. \tag{5.16}$$

*Furthermore, $\hat{x}$ is continuously differentiable on open intervals $I \subset [-1,\infty)$ such that $\hat{x}(t) > -L$ for all $t \in I$. Moreover, a function $\hat{x} \in \mathcal{C}_{[-1,\infty)}$ is a solution of the DDER$^n$ if and only if it is absolutely continuous and at the almost every $t \in (0,\infty)$ where $\hat{x}$ is differentiable,*

$$\frac{d\hat{x}(t)}{dt} = \begin{cases} \tau\hat{g}(\hat{x}(t), \hat{x}(t-1)) & \text{if } \hat{x}(t) > -L, \\ \tau\hat{g}(\hat{x}(t), \hat{x}(t-1))^+ & \text{if } \hat{x}(t) = -L. \end{cases} \tag{5.17}$$

**5.3. Slowly oscillating solutions.** In this section we prove that solutions of the DDER$^n$ with initial condition in a certain subset of $\mathcal{C}_{[-1,0]}$ are slowly oscillating. Throughout this section, let $G$, $\kappa_1$ and $\kappa_2$ be as in Lemma 3.3 so that (5.6) holds. Define

$$\mathcal{K} = \left\{\hat{\varphi} \in \mathcal{C}_{[-1,0]} : \hat{\varphi}(-1) = 0,\ \hat{\varphi}(t) \geq 0 \text{ for all } t \in [-1,0]\right\}, \tag{5.18}$$

$$\widehat{\mathcal{K}} = \{\hat{\varphi} \in \mathcal{K} : \exp(\tau\kappa_1 \cdot)\hat{\varphi}(\cdot) \text{ is nondecreasing on } [-1,0]\}, \tag{5.19}$$

$$\widetilde{\mathcal{K}} = \left\{\hat{\varphi} \in \widehat{\mathcal{K}} : \|\hat{\varphi}\|_{[-1,0]} \leq \tau G\right\}. \tag{5.20}$$

Then $\widetilde{\mathcal{K}}$ is a closed, convex, bounded, infinite-dimensional subset of the Banach space $\mathcal{C}_{[-1,0]}$. The zero element of $\widetilde{\mathcal{K}}$ is the function $\hat{\varphi} \equiv 0$ on $[-1,0]$.

The following lemma will be used to approximate $\hat{g}(r,s)$ when both $r$ and $s$ are in a small neighborhood of zero and are either both positive or both negative.

LEMMA 5.7. *For each $\eta \in (0,1)$, there exists $\delta \in (0,L)$ such that*

$$|\hat{g}(r,s)| \geq \eta|Ar + Bs| \text{ for all } (r,s) \in \mathbb{B}_\delta, \tag{5.21}$$

*where $\mathbb{B}_\delta = \{(r,s) \in \mathbb{R}^2 : rs \geq 0 \text{ and } |r|,|s| \leq \delta\}$.*

*Proof.* In the proof of Lemma 3 in [4], the author proved the above lemma when $A > 0$. Here we prove the lemma when $A = 0$ using a method similar to the one used to prove Lemma 3 in [4].

Fix $\eta \in (0,1)$. For a proof by contradiction, suppose that there does not exist $\delta \in (0,L)$ for which (5.21) holds (when $A = 0$). Let $\{\delta_n\}_{n=1}^\infty$ be a sequence in $(0,L)$ such that $\delta_n \to 0$ as $n \to \infty$. Then there exists a sequence $\{(r_n,s_n)\}_{n=1}^\infty$ such that for each $n$, $(r_n,s_n) \in \mathbb{B}_{\delta_n}$, $s_n \neq 0$ and

$$|\hat{g}(r_n,s_n)| < \eta B|s_n|. \tag{5.22}$$

By the definition of $\mathbb{B}_{\delta_n}$, we can assume that either $r_n$ and $s_n$ are both nonnegative for all $n$ or both nonpositive for all $n$. We consider the case that they are both nonnegative for all $n$, with the case that they are both nonpositive for all $n$ being similar. By (5.5) and (5.22), for each $n$, $s_n > 0$ and

$$-\hat{g}(0, s_n) \leq -\hat{g}(r_n, s_n) < \eta B s_n. \tag{5.23}$$

Substituting the first order approximation $-\hat{g}(0, s_n) = Bs_n + o(s_n)$ into (5.23), dividing by $B|s_n|$ on either side, and letting $n \to \infty$, we arrive at the contradiction $1 \leq \eta$, which proves the lemma in the case $A = 0$. ◻

To prove the existence of SOPS$^{\rm n}$ of the DDER$^{\rm n}$, we first show that solutions of the DDER$^{\rm n}$ with initial conditions in $\widetilde{\mathcal{K}}$ are slowly oscillating. The next lemma is an adaptation of an analogous result in the unconstrained setting, which is detailed in Lemma 4 of [4]. The main difference in the following lemma is the presence of the lower boundary constraint.

LEMMA 5.8. *Suppose $\tau > 1/B$ and $\hat{\varphi} \in \widetilde{\mathcal{K}} \setminus \{0\}$. Let $\hat{x} \in \mathcal{C}_{[-1,\infty)}$ be the unique solution of the DDER$^n$ with $\hat{x}_0 = \hat{\varphi}$. Then there is a positive constant $Q$ (depending only on $\tau$, $\hat{g}$ and $L$), and countably many points $0 < \hat{q}_1 < \hat{q}_2 < \cdots$ such that*

(i) *$\hat{x}(\hat{q}_k) = 0$ for $k = 1, 2, \ldots$,*

(ii) *$0 < \hat{q}_1 < Q$,*
*$1 < \hat{q}_{k+1} - \hat{q}_k < 1 + Q$ for $k = 1, 2, \ldots$,*

(iii) *$\hat{x}(t) > 0$ for $t \in (0, \hat{q}_1)$,*
*$\hat{x}(t) < 0$ for $t \in (\hat{q}_{2k-1}, \hat{q}_{2k})$ for $k = 1, 2, \ldots$,*
*$\hat{x}(t) > 0$ for $t \in (\hat{q}_{2k}, \hat{q}_{2k+1})$ for $k = 1, 2, \ldots$,*

(iv) *the function $\exp(\tau\kappa_1 \cdot)\hat{x}(\cdot)$ is nonincreasing on the intervals $(\hat{q}_{2k-1}, \hat{q}_{2k-1}+1)$ and nondecreasing on the intervals $(\hat{q}_{2k}, \hat{q}_{2k}+1)$, for $k = 1, 2, \ldots$, and*

(v) *$\hat{x}(t) \leq \tau G$ for all $t \geq -1$.*

*Furthermore, if $\lim_{s\to\infty} \hat{g}(0, s)$ exists and is negative, then $Q$ can be chosen to depend only on $\hat{g}$ and $L$.*

*Remark* 5.1. We call $\hat{q}_1, \hat{q}_2, \ldots$ the zeros of $\hat{x}$.

*Proof.* Fix $\tau > 1/B$ and choose $\eta \in (0, 1)$ such that $\eta\tau > 1/B$. Let $\delta \in (0, \tau G \wedge L)$ be such that (5.21) holds. Suppose that $\hat{\varphi} \in \widetilde{\mathcal{K}} \setminus \{0\}$ and $\hat{x}$ is the unique solution of the DDER$^{\rm n}$ with $\hat{x}_0 = \hat{\varphi}$. By (5.19), $\hat{x}(0) = \hat{\varphi}(0) > 0$. Let $\hat{q}_1 = \inf\{t \geq 0 : \hat{x}(t) \leq 0\}$. The negative feedback conditions (5.3) and (5.5) imply that $\hat{x}$ is nonincreasing on $(0, \hat{q}_1)$. In order to show that $\hat{x}$ is eventually zero, we first prove that it reaches the $\delta$ neighborhood of zero. Let $t_1 = \inf\{t \geq 0 : \hat{x}(t) \leq \delta\}$. Suppose that $t_1 > 1$. Then $\hat{x}(0) > \delta$ and $\hat{x}$ is nonincreasing on $[0, t_1]$. For all $t \in [1, t_1]$, the drift $\hat{g}(\hat{x}(t), \hat{x}(t-1))$ is bounded above by

$$-d_1 = \max\{\hat{g}(r, s) : \delta \leq r, s \leq \tau G\} < 0.$$

It then follows from Lemma 5.6 that

$$t_1 \leq 1 + \frac{\tau G - \delta}{\tau d_1}. \tag{5.24}$$

We show that $\hat{q}_1 < t_1 + 2$. For a proof by contradiction, suppose that $\hat{q}_1 \geq t_1 + 2$. Since $\hat{x}$ is nonincreasing on $[t_1, t_1 + 1]$, it follows that $\hat{x}(t-1) \geq \hat{x}(t_1 + 1)$ for all $t \in [t_1 + 1, t_1 + 2]$. Then (5.21) implies that, for all $t \in [t_1 + 1, t_1 + 2]$,

$$\hat{g}(\hat{x}(t), \hat{x}(t-1)) \leq -\eta(A\hat{x}(t) + B\hat{x}(t-1)) \leq -\eta B\hat{x}(t_1 + 1).$$

By Lemma 5.6 and the fact that $\eta\tau > 1/B$, we see that $\hat{x}(t_1+2) \le \hat{x}(t_1+1) - \tau\eta B\hat{x}(t_1+1) < 0$, which contradicts the fact that $\hat{q}_1 \ge t_1+2$. With this contradiction thus obtained, we have

$$\hat{q}_1 < t_1 + 2 \le 3 + \frac{\tau G - \delta}{\tau d_1} \le 3 + \frac{G}{d_1}.$$

Next, we show the derivative of $\hat{x}$ at $\hat{q}_1$ is negative. If $\hat{q}_1 \ge 1$, this follows from Lemma 5.6 and the negative feedback condition implied by (5.3) and (5.5). On the other hand, if $\hat{q}_1 < 1$, then suppose, for a proof by contradiction, that the derivative of $\hat{x}$ at $\hat{q}_1$ is zero. Then $\hat{g}(0, \hat{x}(\hat{q}_1 - 1)) = 0$, which, by the negative feedback condition on $\hat{g}$, implies that $\hat{x}(\hat{q}_1 - 1) = \hat{\varphi}(\hat{q}_1 - 1) = 0$. From the definition of $\widetilde{\mathcal{K}}$, we must have $\hat{x}(t) = \hat{\varphi}(t) = 0$ for all $t \in [-1, \hat{q}_1 - 1]$. Combining this with (5.5) and (5.6) implies $0 \ge \hat{g}(\hat{x}(t), 0) \ge -\kappa_1\hat{x}(t)$ for all $t \in [0, \hat{q}_1]$. Thus $\frac{d\hat{x}(t)}{dt} \ge -\tau\kappa_1\hat{x}(t)$ for all $t \in [0, \hat{q}_1]$ and so $\hat{x}(\hat{q}_1) \ge \exp(-\tau\kappa_1\hat{q}_1)\hat{x}(0) > 0$, contradicting the definition of $\hat{q}_1$. Hence $\frac{d\hat{x}(t)}{dt}|_{t=\hat{q}_1} < 0$.

Continuing, we show that $\hat{x}$ is negative on $(\hat{q}_1, \hat{q}_1 + 1]$ and $\exp(\tau\kappa_1\cdot)\hat{x}(\cdot)$ is nonincreasing on $[\hat{q}_1, \hat{q}_1 + 1]$. Define $\hat{q}_2 = \inf\{t > \hat{q}_1 : \hat{x}(t) \ge 0\}$. We first show that $\hat{q}_2 \ge \hat{q}_1 + 1$. Suppose, for a proof by contradiction, that $\hat{q}_2 \in (\hat{q}_1, \hat{q}_1 + 1)$. Then $\hat{q}_2 - 1 \in (\hat{q}_1 - 1, \hat{q}_1)$, so $\hat{x}(\hat{q}_2 - 1) \ge 0$. By Lemma 5.6, this would imply that $\hat{x}$ is nonincreasing in a neighborhood of $\hat{q}_2$, which contradicts the definition of $\hat{q}_2$. Therefore $\hat{x}$ is negative on $(\hat{q}_1, \hat{q}_1 + 1)$. Now, for $t \in [\hat{q}_1, \hat{q}_1 + 1]$ such that $\hat{x}(t) > -L$, we have $\hat{x}$ is differentiable at $t$ and

$$\begin{aligned}\frac{d}{dt}(\exp(\tau\kappa_1 t)\hat{x}(t)) &= \exp(\tau\kappa_1 t)(\tau\kappa_1\hat{x}(t) + \tau\hat{g}(\hat{x}(t), \hat{x}(t-1))) \\ &\le \tau\exp(\tau\kappa_1 t)(\kappa_1\hat{x}(t) + \hat{g}(\hat{x}(t), 0)) \\ &\le \tau\kappa_1\exp(\tau\kappa_1 t)(\hat{x}(t) + |\hat{x}(t)|) = 0,\end{aligned}$$

where we have used (5.3) and (5.6). For $t \in (\hat{q}_1, \hat{q}_1 + 1]$ such that $\hat{x}(t) = -L$ and $\hat{x}$ is differentiable at $t$, it follows from Lemma 5.6 that

$$\frac{d}{dt}(\exp(\tau\kappa_1 t)\hat{x}(t)) = -\tau\kappa_1 L\exp(\tau\kappa_1 t) < 0.$$

Hence, $\exp(\tau\kappa_1\cdot)\hat{x}(\cdot)$ is nonincreasing on $[\hat{q}_1, \hat{q}_1 + 1]$. Combining this with the fact that $\hat{x}(t) < 0$ for $t \in (\hat{q}_1, \hat{q}_1 + 1)$ we obtain that $\hat{x}(\hat{q}_1 + 1) < 0$.

By employing arguments similar to those above, we can show that $\hat{q}_2$ is bounded, $\hat{x}$ is positive on $(\hat{q}_2, \hat{q}_2 + 1]$, and $\exp(\tau\kappa_1\cdot)\hat{x}(\cdot)$ is nondecreasing on $[\hat{q}_2, \hat{q}_2 + 1]$. For this, let $t_2 = \inf\{t \ge \hat{q}_1 + 1 : \hat{x}(t) \ge -\delta\}$. Suppose $t_2 > \hat{q}_1 + 2$. Then by the definition of $t_2$ and the fact that $\hat{x}$ is bounded below by $-L$, we have $\hat{x}(t) \in [-L, -\delta]$ for all $t \in [\hat{q}_1 + 1, t_2]$. Thus, for all $t \in [\hat{q}_1 + 2, t_2]$, $\hat{g}(\hat{x}(t), \hat{x}(t-1))$ is bounded below by

$$d_2 = \min\{\hat{g}(r, s) : -L \le r, s \le -\delta\} > 0.$$

By (5.17), at the almost every $t \in [\hat{q}_1, t_2]$ that $\hat{x}$ is differentiable, $\frac{d\hat{x}(t)}{dt} \ge \tau d_2$. Since $\hat{x}$ is absolutely continuous, it follows that

$$t_2 \le \hat{q}_1 + 2 + \frac{L - \delta}{\tau d_2} \le \hat{q}_1 + 2 + \frac{LB}{d_2}.$$

We can again use a proof by contradiction, analogous to the one used to prove $\hat{q}_1 < t_1 + 2$, to obtain the bound

$$\hat{q}_2 < t_2 + 2 < 7 + \frac{G}{d_1} + \frac{LB}{d_2}. \tag{5.25}$$

Furthermore, using arguments analogous to the ones used on the interval $[\hat{q}_1, \hat{q}_1+1]$, we have the derivative of $\hat{x}$ at $\hat{q}_2$ is strictly positive, $\hat{x}$ stays positive for all $t \in (\hat{q}_2, \hat{q}_2+1]$, and $\exp(\tau\kappa_1 \cdot)\hat{x}(\cdot)$ is nondecreasing on $[\hat{q}_2, \hat{q}_2+1]$. The main difference here is that we do not need to consider the case that $\hat{x}$ is at the lower boundary when showing that $\exp(\tau\kappa_1 \cdot)\hat{x}(\cdot)$ is nondecreasing.

From the above, we see that $\hat{x}_{\hat{q}_2+1} \in \widehat{\mathcal{K}} \setminus \{0\}$. By (5.5) and (5.6), we have $\hat{g}(\hat{x}(t), \hat{x}(t-1)) \leq \hat{g}(0, \hat{x}(t-1)) \leq G$ for all $t \in [\hat{q}_2, \hat{q}_2+1]$ and so $\|\hat{x}_{\hat{q}_2+1}\|_{[-1,0]} \leq \tau G$. It follows that $\hat{x}_{\hat{q}_2+1} \in \widetilde{\mathcal{K}} \setminus \{0\}$ and by shifting the time origin, the preceding argument can be repeated countably many times to prove that (i)–(iv) hold with $Q = 3 + \max(Gd_1^{-1}, LBd_2^{-1})$. Furthermore, by the negative feedback condition on $\hat{g}$ and the definitions of $\hat{q}_1$ and $\hat{q}_2$, $\hat{x}$ is nonincreasing on $[0, \hat{q}_1]$ and nonpositive on $[\hat{q}_1, \hat{q}_2]$. Therefore $\hat{x}(t) \leq \tau G$ for all $t \in [-1, \hat{q}_2]$ and the argument can be repeated countably many times to complete the proof of (v).

To see that the final line in the lemma holds, suppose that $\lim_{s\to\infty} \hat{g}(0,s) < 0$. Then $\sup\{\hat{g}(0,s) : \delta \leq s < \infty\} < 0$, where we have used continuity and the negative feedback condition on $\hat{g}$. Furthermore, by (5.5),

$$-d_1^\dagger = \sup\{\hat{g}(r,s) : \delta \leq r, s < \infty\} \leq \sup\{\hat{g}(0,s) : \delta \leq s < \infty\} < 0.$$

Then we can use $d_1^\dagger$ in place of $d_1$ in (5.24). Since $d_2$ does not depend on $\tau$, it follows that $Q$ can be chosen to depend only on $\hat{g}$ and $L$. ◻

For the remainder of the section, we fix a delay $\tau$ as in Lemma 5.8. Consider the function $\Lambda : \widetilde{\mathcal{K}} \to \mathcal{C}_{[-1,0]}$ defined by

$$\Lambda(\hat{\varphi}) = \begin{cases} 0 & \text{if } \hat{\varphi} \equiv 0, \\ \hat{x}_{\hat{q}_2+1} & \text{if } \hat{\varphi} \not\equiv 0, \end{cases} \tag{5.26}$$

where $\hat{x}$ denotes the unique solution of the DDER$^n$ with $\hat{x}_0 = \hat{\varphi}$ and $\hat{q}_2$ is the second zero of $\hat{x}$, as in Lemma 5.8. In Lemma 5.11, we will show that $\Lambda$ is a continuous mapping from $\widetilde{\mathcal{K}}$ into itself. It will then follow from Lemma 5.8 that nonconstant fixed points of $\Lambda$ are initial conditions for SOPS$^n$ of the DDER$^n$. The following two lemmas are used in the proof of Lemma 5.11 to show that $\Lambda$ is continuous. The first lemma proves continuity of solutions of the DDER$^n$ in their initial condition on the set $\widetilde{\mathcal{K}}$. For the following lemma, recall that we have equipped $\mathbb{R}^2$ with the Euclidean norm.

LEMMA 5.9. *There exists a positive constant $K_g < \infty$ such that*

$$|\hat{g}(r,s) - \hat{g}(r^\dagger, s^\dagger)| \leq K_g |(r,s) - (r^\dagger, s^\dagger)| \qquad \text{for all } r, s, r^\dagger, s^\dagger \in [-L, \tau G]. \tag{5.27}$$

*Furthermore, whenever $\hat{x}$ and $\hat{x}^\dagger$ are solutions of the DDER$^n$ with $\hat{x}_0, \hat{x}_0^\dagger \in \widetilde{\mathcal{K}}$, then*

$$\|\hat{x} - \hat{x}^\dagger\|_{[-1,t]} \leq 2\exp(2K_g t)\|\hat{x}_0 - \hat{x}_0^\dagger\|_{[-1,0]} \qquad \text{for all } t \geq 0. \tag{5.28}$$

*Proof.* The bound (5.27) is due to the Lipschitz continuity of $g$ on the compact rectangle $[-L, \tau G] \times [-L, \tau G]$. For the proof that (5.28) holds, let $t \geq 0$. Part (v) of Lemma 5.8 implies that $\hat{x}(s), \hat{x}^\dagger(s) \in [-L, \tau G]$ for all $s \in [-1, t]$. Using (5.9), (5.27) and taking supremums over $s \in [0, t]$, we have

$$\|\hat{z} - \hat{z}^\dagger\|_{[0,t]} \leq \|\hat{x}_0 - \hat{x}_0^\dagger\|_{[-1,0]} + K_g \int_0^t \|\hat{x} - \hat{x}^\dagger\|_{[-1,s]} ds.$$

Since $\hat{x}|_{[0,\infty)} + L = \Phi(\hat{z} + L)$, we can apply Proposition A.1 and then extend the supremum norm on the left to the interval $[-1, t]$ to obtain

$$\|\hat{x} - \hat{x}^\dagger\|_{[-1,t]} \le 2\|\hat{x}_0 - \hat{x}_0^\dagger\|_{[-1,0]} + 2K_g \int_0^t \|\hat{x} - \hat{x}^\dagger\|_{[-1,s]} ds.$$

Since $t \ge 0$ was arbitrary, (5.28) follows from Gronwall's inequality. ∎

LEMMA 5.10. *The function $\hat{\varphi} \to \hat{q}_2$, where $\hat{q}_2$ is the second zero of $\hat{x}$ as defined in Lemma 5.8, is continuous as a function from $\widetilde{\mathcal{K}} \setminus \{0\}$ into $[0, \infty)$.*

*Proof.* Fix $\hat{\varphi} \in \widetilde{\mathcal{K}} \setminus \{0\}$. Let $\hat{x}$ denote the associated solution of the DDER$^{\text{n}}$ and let $\hat{q}_1$ and $\hat{q}_2$ be as in Lemma 5.8. By part (iii) of Lemma 5.8, $\hat{q}_2$ is bounded by $1+2Q$, which only depends on $\tau$, $\hat{g}$, and $L$. Choose $\eta \in (0, \frac{1}{2})$ such that $0 < \hat{q}_1 - \eta < \hat{q}_1 + \eta < \hat{q}_2 - \eta$. It follows from (5.28) that we can choose $\varepsilon > 0$ so that whenever $\hat{\varphi}^\dagger \in \widetilde{\mathcal{K}}$ satisfies $\|\hat{\varphi} - \hat{\varphi}^\dagger\|_{[-1,0]} < \varepsilon$ and $\hat{x}^\dagger$ denotes the solution of the DDER$^{\text{n}}$ with $\hat{x}_0^\dagger = \hat{\varphi}^\dagger$, then $\hat{x}^\dagger$ is positive on $[0, \hat{q}_1 - \eta]$, negative on $[\hat{q}_1 + \eta, \hat{q}_2 - \eta]$, and positive on $[\hat{q}_2 + \eta, \hat{p}]$. Then, by the continuity of $\hat{x}^\dagger$ and the fact that the zeros of $\hat{x}^\dagger$ must be separated by at least one, we have $\hat{q}_1^\dagger \in (\hat{q}_1 - \eta, \hat{q}_1 + \eta)$ and $\hat{q}_2^\dagger \in (\hat{q}_2 - \eta, \hat{q}_2 + \eta)$, where $q_1^\dagger$ and $q_2^\dagger$ are the zeros of $\hat{x}^\dagger$, proving the desired continuity result. ∎

LEMMA 5.11. *The function $\Lambda$ is a continuous and compact function that maps $\widetilde{\mathcal{K}}$ into $\widetilde{\mathcal{K}}$.*

*Proof.* By Lemma 5.8, $\Lambda$ maps $\widetilde{\mathcal{K}}$ into itself. Since $\hat{q}_2 < 1 + 2Q$, (5.28) implies that $\Lambda$ is continuous at $\hat{\varphi} \equiv 0$. The continuity of $\Lambda$ at $\hat{\varphi} \not\equiv 0$ follows from (5.28), the bound on $\hat{q}_2$, and Lemma 5.10. The compactness property of $\Lambda$ is implied by the theorem of Arzelà and Ascoli, since $\hat{x}$ is bounded and differentiable on $[\hat{q}_2, \hat{q}_2 + 1]$ with its derivative on the interval uniformly bounded by

$$|\tau \hat{g}(\hat{x}(t), \hat{x}(t-1))| \le \tau \cdot \sup\{|\hat{g}(r,s)| : (r,s) \in [0, \tau G] \times [-L, 0]\} < \infty,$$

where we have used that $\hat{g}$ is continuous on this compact set. ∎

**5.4. Ejective equilibrium solution.** In order to prove the existence of a SOPS, it remains to show that the zero solution of the DDER$^{\text{n}}$ is an ejective fixed point for $\Lambda$. It will then follow from Theorem 5.2 that there exists another fixed point that is nonejective which will correspond to a SOPS$^{\text{n}}$. The following lemma is the analogue of Lemma 6 in [4].

LEMMA 5.12. *Let $\tau_0$ be given by* (3.8). *If $\tau > \tau_0$, then $\hat{\varphi} \equiv 0$ is an ejective fixed point of $\Lambda$.*

*Proof.* As in the proof of Lemma 6 in [4], we claim that there exists $\gamma \in (0, L)$ such that whenever $\hat{x}$ is a solution of the DDER$^{\text{n}}$ with $\hat{x}_0 \in \widetilde{\mathcal{K}} \setminus \{0\}$ and $0 < \hat{q}_1 < \hat{q}_2 < \cdots$ are the zeros of $\hat{x}$ defined as in Lemma 5.8, then

$$\sup\{|\hat{x}(t)| : t \ge \hat{q}_k\} \ge \gamma \qquad \text{for all } k = 1, 2, \ldots. \tag{5.29}$$

Observe that by part (v) of Lemma 5.8, the supremum in (5.29) is bounded, though it is not necessarily achieved. For a proof by contradiction, suppose that for each $\gamma \in (0, L)$, there is a SOPS$^{\text{n}}$ $\hat{x}$ with $\hat{x}_0 \in \widetilde{\mathcal{K}} \setminus \{0\}$ such that $\sup\{|\hat{x}(t)| : t \ge \hat{q}_{2k}\} < \gamma$ for some $k = 1, 2, \ldots$, where $0 < \hat{q}_1 < \hat{q}_2 < \cdots$ denote the zeros of $\hat{x}$. Since $\hat{x}(t) > -\gamma > -L$ for all $t \ge \hat{q}_{2k}$, it follows that $\hat{y}$ is constant on $[\hat{q}_{2k}, \infty)$. Thus, given $\hat{x}_{\hat{q}_{2k}+1}$ (which lies in $\widetilde{\mathcal{K}} \setminus \{0\}$ by Lemma 5.8), the dynamics of $\hat{x}$ on $[\hat{q}_{2k} + 1, \infty)$ are identical in the constrained and unconstrained settings. In particular, this implies that the unique solution $\tilde{x}$ to the related *unconstrained* delay differential equation with $\tilde{x}_0 = \hat{x}_{\hat{q}_{2k}+1} \in \widetilde{\mathcal{K}} \setminus \{0\}$ satisfies $\sup\{|\tilde{x}(t)| : t \geq 0\} < \gamma$. However, this contradicts the first part of the proof of Lemma 6 in [4], yielding the desired contradiction.

We are left to show there exists $\delta > 0$ such that given any $\hat{\varphi} \in \widetilde{\mathcal{K}} \setminus \{0\}$, there exists a positive integer $n$ such that

$$\|\Lambda^n(\hat{\varphi})\|_{[-\tau,0]} > \delta.$$

The proof closely follows the latter half of the proof of Lemma 6 in [4] and we briefly summarize it here. Fix $0 < \delta < \frac{\gamma}{\tau\kappa_2} \wedge \gamma$. Let $\hat{\varphi} \in \widetilde{\mathcal{K}} \setminus \{0\}$, $\hat{x}$ denote the solution of the $\mathrm{DDER^n}$ with $\hat{x}_0 = \hat{\varphi}$, and $0 < \hat{q}_1 < \hat{q}_2 < \cdots$ denote the zeros of $\hat{x}$. By the negative feedback condition on $\hat{g}$, for each $k = 1, 2, \ldots$, we can choose $\sigma_k \in [\hat{q}_k, \hat{q}_k + 1]$ such that $|\hat{x}(\sigma_k)| = \|\hat{x}\|_{[\hat{q}_k, \hat{q}_{k+1}]}$. It suffices to show that $\hat{x}(\sigma_{2k}) > \delta$ for some $k \geq 1$. For a proof by contradiction, suppose that

$$|\hat{x}(\sigma_{2k})| \leq \delta < \gamma \qquad \text{for } k = 1, 2, \ldots. \tag{5.30}$$

By (5.4), (5.6), and our choice of $\delta$, for each $k = 1, 2, \ldots$ and $t \in [\hat{q}_{2k+1}, \hat{q}_{2k+1} + 1]$,

$$0 \geq \tau\hat{g}(\hat{x}(t), \hat{x}(t-1)) \geq \tau\hat{g}(0, \hat{x}(t-1)) \geq -\tau\kappa_2|\hat{x}(t-1)| \geq -\tau\kappa_2\delta > -\gamma.$$

It follows from Lemma 5.6 that $|\hat{x}(\sigma_{2k+1})| < \tau\kappa_2\delta < \gamma$ for all $k = 1, 2, \ldots$. This combined with (5.30) implies that $|\hat{x}(\sigma_k)| < \max(\tau\kappa_2\delta, \delta) < \gamma$ for all $k = 2, 3, \ldots$, which, in light of our definition of $\sigma_k$, contradicts (5.29). With this contradiction thus obtained, we have $\|\hat{x}_{\hat{q}_{2k}+1}\|_{[-1,0]} > \delta$ for some $k \geq 1$, completing the proof. ∎

**5.5. Proof of existence.**

*Proof of Theorem* 3.4. By Browder's fixed point theorem, the mapping $\Lambda : \widetilde{\mathcal{K}} \to \widetilde{\mathcal{K}}$ has a nonejective fixed point. By Lemma 5.12, the constant function $\hat{\varphi} \equiv 0$ is an ejective fixed point of $\Lambda$ and so there must be another fixed point $\hat{\varphi} \in \widetilde{\mathcal{K}} \setminus \{0\}$. Let $\hat{x}$ denote the unique solution of the $\mathrm{DDER^n}$ with $\hat{x}_0 = \hat{\varphi}$. Since the $\mathrm{DDER^n}$ is autonomous, it follows that $\hat{x}$ is periodic with period $\hat{p} = \hat{q}_2 + 1$. Moreover, by Lemma 5.8, $\hat{x}$ is a $\mathrm{SOPS^n}$. Last, it follows from Lemma 5.5 that the associated solution $x$, defined via (5.14), is a SOPS. ∎

**6. Uniqueness and stability of SOPS.** In this section we prove Theorem 3.8, which provides sufficient conditions for the uniqueness and exponential stability of SOPS to the DDER. Both our proof of uniqueness and our proof of exponential stability follow a general outline similar to arguments used by Xie [80, 81], where similar results were proved for the unconstrained setting. However, some substantial additional difficulties arise in our context due to the lower boundary constraint.

Throughout this section we assume that $f$ in (1.1) is of the form exhibited in (3.9) and that $h$ satisfies Assumption 3.3. In sections 6.3–6.7, we additionally assume that $h$ satisfies Assumption 3.4.

Intuitively, our proof proceeds as follows. As the delay $\tau$ increases, the amplitude of an associated SOPS will grow approximately linearly with $\tau$. As a result, for large delays, any SOPS $x^*$ will spend "most" of its time either at the lower boundary or "far" above the equilibrium point. Since $h'$ converges to zero at infinity (see Assumption 3.4), perturbations of $x^*(t-\tau)$, for $x^*(t-\tau)$ large, will have a small effect on the drift. This, together with the fact that a SOPS $x^*$ cannot be linearly perturbed when at the boundary, will imply that $x^*$ is exponentially stable. More specifically, when the delay $\tau$ is sufficiently large, we can construct a variant of a Poincaré map $\Gamma$ defined on a neighborhood of $x_0^*$ in $\mathcal{C}^+_{[-\tau,0]}$, such that $\Gamma$ is continuously

Fréchet differentiable on the neighborhood and its derivative operator evaluated at $x_0^*$, $D\Gamma(x_0^*)$, has norm less than one. By using the approximation

$$\|\Gamma(x_0) - x_0^*\|_{[-\tau,0]} \approx \|D\Gamma(x_0^*)\|\|x_0 - x_0^*\|_{[-\tau,0]} < \|x_0 - x_0^*\|_{[-\tau,0]},$$

for $x_0 \in \mathcal{C}^+_{[-\tau,0]}$ in a small neighborhood of $x_0^*$, we will show that $x^*$ is exponentially stable. Then, by an argument involving fixed point indices, we show that if every SOPS of the DDER is exponentially stable, as is the case for sufficiently large delays $\tau$, then there is at most one SOPS (up to time translation).

There are two main differences between our approach and that of Xie. First, the main technical difference is the argument by Xie relies on previously developed theory for a VE along a solution of an unconstrained delay differential equation, which does not apply in the constrained setting. Here we develop a VE along constrained solutions. This is done for our current setting in section 6.4 and more generally in Appendix B and may be of independent interest. Second, in contrast to [80], we do not allow the drift function to have a linear dependence on the current state $x^*(t)$. In the unconstrained setting, a transformation can be used to reduce the equation to one in which the drift depends only on the delayed state. Here, such a transformation is not readily available to us due to the effect of the lower boundary constraint.

**6.1. Normalized solutions.** As in section 5.2, we normalize solutions of the DDER (1.1) by subtracting off $L$ and rescaling time so that the normalized delay interval is of length one. We will work with solutions of the normalized DDER, or DDER$^{\text{n}}$ for short, in sections 6.1 and 6.2.

Let $\hat{h} : [-L, \infty) \to \mathbb{R}$ be the normalized function defined by

$$\hat{h}(s) = h(s + L), \qquad s \geq -L. \tag{6.1}$$

We note some important properties that $\hat{h}$ inherits from $h$. By Assumption 3.3, $\hat{h}$ is continuously differentiable on $[-L, \infty)$, $\lim_{s\to\infty} \hat{h}(s) = -\alpha$, $\hat{h}(-L) = \beta$, $\hat{h}(0) = 0$, $\hat{h}'(0) < 0$, $s\hat{h}(s) < 0$ for all $s \neq 0$, and $\hat{h}$ is bounded by $H$. On setting $g(r, s) = h(s)$ for all $r, s \geq 0$, Assumption 3.3 implies that Assumptions 3.1 and 3.2 hold for $g$ and that $\hat{g}(r, s) = \hat{h}(s)$ for all $r, s \geq -L$, where $\hat{g}$ is defined in (5.1). Hence, the definitions and results from section 5 hold for $g$ and $\hat{g}$ so defined from $h$.

The following three propositions are used to further describe $\hat{h}$. They are adapted from Remark 1 and Lemma 1 of [52] and Lemma 5 of [80]. The main difference is that $\hat{h}$ is defined only on $[-L, \infty)$, whereas the function $f$ in [52, 80] is defined on the whole real line. The proofs found in [52, 80] can be readily adapted for our current setting. The first proposition is a straightforward consequence of the facts that $\hat{h}$ is continuous, satisfies a negative feedback condition, and has a finite nonzero limit at infinity, $\hat{h}(0) = 0$, and $\hat{h}'(0) < 0$.

Proposition 6.1. *There exists* $C \geq 1$ *such that* $\hat{h}(r_1) \geq C\hat{h}(r_2)$ *for all* $0 \leq r_1 \leq r_2 < \infty$ *and* $\hat{h}(s_1) \leq C\hat{h}(s_2)$ *for all* $-L \leq s_2 \leq s_1 \leq 0$.

Proposition 6.2. *There exists* $d > 0$ *such that*

$$\int_s^r \hat{h}(u)du \geq \frac{d}{C^2}(r - s)|s| \qquad \textit{for all } -L \leq s \leq r \leq 0. \tag{6.2}$$

*Proof.* Define $\hat{H}(s, r) = \int_s^r \hat{h}(u)du$ for all $r, s \in [-L, 0]$. Since $\hat{h}$ is continuous, continuously differentiable, and satisfies a negative feedback condition, $\hat{h}(0) = 0$, and

$\hat{h}'(0) \neq 0$, there exists $d > 0$ such that $|\hat{H}(s,0)| \geq d|s|^2$ for all $s \in [-L,0]$. If $-L \leq s < r \leq 0$, then by Proposition 6.1, $0 \leq \hat{H}(r,0) \leq C|r|\hat{h}(r)$ and $\hat{H}(s,r) \geq C^{-1}(r-s)\hat{h}(r) > 0$, which imply the following:

$$\begin{aligned}\hat{H}(s,r) = \frac{\hat{H}(s,0)}{1+\hat{H}(r,0)\hat{H}(s,r)^{-1}} &\geq \frac{\hat{H}(s,0)}{1+C^2|r|(r-s)^{-1}} \\ &\geq \frac{ds^2(r-s)}{r-s+C^2|r|} \\ &\geq \frac{d}{C^2}(r-s)|s|,\end{aligned}$$

where the last inequality uses the fact that $\frac{|s|}{r-s+C^2|r|} \geq \frac{1}{C^2}$ for all $-L \leq s < r \leq 0$. ∎

PROPOSITION 6.3. *Given $M > m > 0$, there exist two constants $\gamma^{(1)} > 0$ and $\tau^{(1)} \geq \tau_0$ (depending on $m, M$ and $\hat{h}$) such that, if $m \leq r \leq M$, $rs \geq -L$, $|s| \geq \tau|\hat{h}(rs)|$, and $\tau \geq \tau^{(1)}$, then*

$$|s| > \gamma^{(1)}\tau.$$

*Proof.* Briefly, the proof is as follows. Due to the facts that $\hat{h}'(0) < 0$, $s\hat{h}(s) < 0$ for all $s \neq 0$ and $\lim_{s\to\infty}\hat{h}(s) = -\alpha$, it follows that there exist constants $a, b, c \in (0,\infty)$ and $\delta \in (0,L)$ such that $|s(\hat{h}(s))^{-1}| \in (a,b)$ for all $|s| \leq \delta$, and $|\hat{h}(s)| \geq c$ for all $s \in [-L,-\delta] \cup [\delta,\infty)$. Combining, we see that $|s(\hat{h}(s))^{-1}| < b + |s|c^{-1}$ for all $s \in [-L,\infty)$. Thus, if $m \leq r \leq M$, $rs \geq -L$, and $|s| \geq \tau|\hat{h}(rs)|$, we have

$$m\tau \leq |rs(\hat{h}(rs))^{-1}| < b + |rs|c^{-1},$$

and so

$$|s| > c\left(\frac{m\tau - b}{M}\right) \geq \gamma^{(1)}\tau \qquad \text{for all } \tau \geq \tau^{(1)},$$

where $\gamma^{(1)} = \frac{1}{2}cmM^{-1}$ and $\tau^{(1)} = 2m^{-1}b$. ∎

Throughout the remainder of this section let $\hat{x}^*$ denote a SOPS$^{\text{n}}$ with $\hat{q}_0 = -1$ and let $\hat{z}^*$ and $\hat{y}^*$ be defined as in (5.9) and (5.10) with $\hat{g}(r,s) = \hat{h}(s)$ and with $\hat{x}^*, \hat{y}^*$ and $\hat{z}^*$ in place of $\hat{x}, \hat{y}$, and $\hat{z}$, respectively. For the following, note that we can use $G = \sup\{h(s) : s \geq 0\} \in (0,\infty)$ for the $G$ in Lemma 3.3.

LEMMA 6.4. *The SOPS$^n$ $\hat{x}^*$ is continuously differentiable on $[-1,\hat{q}_1]$ and $(\hat{q}_1 + 1, \hat{q}_2]$; increasing on $[-1,0)$, decreasing on $(0,\hat{q}_1]$, nonincreasing on $[\hat{q}_1, \hat{q}_1+1]$, and increasing on $(\hat{q}_1+1,\hat{q}_2]$; and bounded above by $\tau G$. Furthermore, $\hat{x}^*$ and $\hat{z}^*$ satisfy*

$$\hat{x}^*(t) = \hat{z}^*(t) \qquad \textit{for } t \in [0,\hat{q}_1], \tag{6.3}$$

$$\hat{x}^*(t) = \max(\hat{z}^*(t), -L) \qquad \textit{for } t \in [\hat{q}_1, \hat{q}_1+1]. \tag{6.4}$$

*Proof.* By (5.9) and the continuity of $\hat{h}$ and $\hat{x}^*$, $\hat{z}^*$ is continuously differentiable on $[0,\infty)$. Since $\hat{x}^*$ is above the lower boundary on $[0,\hat{q}_1]$, $\hat{y}^*$ is zero there and (6.3) follows from (5.8). By (5.9), with $\hat{g}(r,s) = \hat{h}(s)$, (6.3), and the negative feedback condition on $\hat{h}$, $\hat{x}^*$ is continuously differentiable and decreasing on $(0,\hat{q}_1]$ and $\hat{z}^*$ is nonincreasing on $[0,\hat{q}_1+1]$. Thus, (5.10) implies that $\hat{y}^*(t) = (\hat{z}^*(t)+L)^-$ for all $t \in [0,\hat{q}_1+1]$. Substituting into (5.8), we obtain (6.4) and hence $\hat{x}^*$ is nonincreasing on $[\hat{q}_1, \hat{q}_1 + 1]$. By (5.9) and the negative feedback condition on $\hat{h}$, $\hat{z}^*$ is increasing on $(\hat{q}_1 + 1, \hat{q}_2 + 1)$. From (5.10), we see that $\hat{y}^*$ is constant on $[\hat{q}_1 + 1, \hat{q}_2 + 1]$ and it follows from (5.8) and (5.9) that $\hat{x}^*$ is continuously differentiable on $(\hat{q}_1 + 1, \hat{q}_2 + 1]$ and increasing on $(\hat{q}_1 + 1, \hat{q}_2 + 1)$. Last, the upper bound on $\hat{x}^*$ follows from the facts that $\hat{x}^*$ is periodic with period $\hat{q}_2 + 1$, decreasing on $(0, \hat{q}_1)$, negative on $(\hat{q}_1, \hat{q}_2)$, and continuously differentiable on $[\hat{q}_2, \hat{q}_2 + 1]$ with its derivative bounded by $\tau G$ there. ◻

The following lemma has been adapted from Lemma 3 in [52] for our use.

LEMMA 6.5. *Suppose* $\hat{q}_1 + 1 \le t_1 < t_2 \le \hat{q}_1 + 2$. *Then*

$$\hat{x}^*(t) \le \frac{C(t - t_1)\hat{x}^*(t_2) + (t_2 - t)\hat{x}^*(t_1)}{C(t - t_1) + (t_2 - t)}, \qquad t \in [t_1, t_2]. \tag{6.5}$$

*Proof.* If $t = t_1$ or $t = t_2$, then equality holds in (6.5). Fix $t \in (t_1, t_2)$ and let $t_0 \in (t_1, t)$. By Lemma 6.4 and the periodicity of $\hat{x}^*$, $\hat{x}^*$ is continuously differentiable and above the lower boundary on $[t_0, t_2]$, so the mean value theorem and Lemma 5.6 imply there exist $s_1 \in (t_0, t)$ and $s_2 \in (t, t_2)$ such that

$$\frac{\hat{x}^*(t) - \hat{x}^*(t_0)}{t - t_0} = \tau\hat{h}(\hat{x}^*(s_1 - 1)) \qquad \text{and} \qquad \frac{\hat{x}^*(t_2) - \hat{x}^*(t)}{t_2 - t} = \tau\hat{h}(\hat{x}^*(s_2 - 1)).$$

Applying Proposition 6.1 and the fact that $\hat{x}^*$ is nonincreasing on $[\hat{q}_1, \hat{q}_1 + 1]$, we obtain the inequality

$$\frac{\hat{x}^*(t) - \hat{x}^*(t_0)}{t - t_0} \le C \cdot \frac{\hat{x}^*(t_2) - \hat{x}^*(t)}{t_2 - t},$$

which, after rearranging, yields

$$\hat{x}^*(t) \le \frac{C(t - t_0)\hat{x}^*(t_2) + (t_2 - t)\hat{x}^*(t_0)}{C(t - t_0) + (t_2 - t)}, \qquad t \in [t_0, t_2].$$

Since the above inequality holds for all $t_0 \in (t_1, t)$, (6.5) follows from the continuity of $\hat{x}^*$. ◻

In Lemmas 6.6, 6.7, and 6.8 below, we provide estimates for $\hat{x}^*$ that depend on the size of $\hat{q}_{2,1}$, where we have defined $\hat{q}_{2,1} = \hat{q}_2 - \hat{q}_1 - 1$. The lemmas and their proofs have been adapted from Lemmas 8, 10, and 11 in [80]. The main difference is that the SOPS$^{\text{n}}$ $\hat{x}^*$ is bounded below by $-L$ and the drift term does not have (a linear) dependence on $\hat{x}^*(t)$. In the following, $C$ and $d$ are the constants from Propositions 6.1 and 6.2.

LEMMA 6.6. *If* $\hat{q}_{2,1} \le 1$*, then*

$$\hat{x}^*(t) \ge \frac{|\hat{x}^*(\hat{q}_1 + 1)|}{C^2} \cdot \frac{t - \hat{q}_2}{\hat{q}_{2,1}}, \qquad t \in [\hat{q}_2, \hat{q}_1 + 2], \tag{6.6}$$

$$\hat{x}^*(t) \ge \frac{|\hat{x}^*(\hat{q}_1 + 1)|}{C^2} \cdot \left[\frac{1 - \hat{q}_{2,1}}{\hat{q}_{2,1}} + \frac{\tau d}{C^2}(t - \hat{q}_1 - 2)\right], \qquad t \in [\hat{q}_1 + 2, \hat{q}_2 + 1]. \tag{6.7}$$

*Proof.* Suppose that $\hat{q}_{2,1} \le 1$. Since $\hat{x}^*$ is nonpositive and nonincreasing on $[\hat{q}_1, \hat{q}_1 + 1]$ and $\hat{x}^*(t) > -L$ for all $t \in (\hat{q}_1 + 1, \hat{q}_1 + 2]$, we have the following inequalities:

$$-\hat{x}^*(\hat{q}_1 + 1) = \tau \int_{\hat{q}_1 + 1}^{\hat{q}_2} \hat{h}(\hat{x}^*(s - 1))ds \le C\tau\hat{h}(\hat{x}^*(\hat{q}_2 - 1))\hat{q}_{2,1}, \tag{6.8}$$

$$\hat{x}^*(t) = \tau \int_{\hat{q}_2}^{t} \hat{h}(\hat{x}^*(s - 1))ds \ge C^{-1}\tau\hat{h}(\hat{x}^*(\hat{q}_2 - 1))(t - \hat{q}_2), \qquad t \in [\hat{q}_2, \hat{q}_1 + 2], \tag{6.9}$$

where we have used Lemma 5.6 and Proposition 6.1. Combining (6.8) and (6.9) yields (6.6).

Letting $t_1 = \hat{q}_1 + 1$ and $t_2 = \hat{q}_2$ in (6.5), we see that

$$\hat{x}^*(t) \le \frac{(\hat{q}_2 - t)\hat{x}^*(\hat{q}_1 + 1)}{C(t - \hat{q}_1 - 1) + (\hat{q}_2 - t)} \le \frac{\hat{x}^*(\hat{q}_1 + 1)}{C\hat{q}_{2,1}}(\hat{q}_2 - t), \qquad t \in [\hat{q}_1 + 1, \hat{q}_2], \tag{6.10}$$

where the last inequality holds because $\hat{x}^*(\hat{q}_1+1) < 0$ and $C \ge 1$. Then by Lemma 5.6, Proposition 6.1, (6.10), and (6.2), we have, for $t \in [\hat{q}_1 + 2, \hat{q}_2 + 1]$,

$$\begin{aligned}
\hat{x}^*(t) - \hat{x}^*(\hat{q}_1 + 2) &= \tau \int_{\hat{q}_1+1}^{t-1} \hat{h}(\hat{x}^*(s))ds \\
&\ge \frac{\tau}{C} \int_{\hat{q}_1+1}^{t-1} \hat{h}\left(\frac{\hat{x}^*(\hat{q}_1 + 1)}{C\hat{q}_{2,1}}(\hat{q}_2 - s)\right) ds \\
&\ge \frac{\tau \hat{q}_{2,1}}{|\hat{x}^*(\hat{q}_1 + 1)|} \int_{\hat{x}^*(\hat{q}_1+1)/C}^{\hat{x}^*(\hat{q}_1+1)(\hat{q}_2-t+1)/C\hat{q}_{2,1}} \hat{h}(u)du \\
&\ge \frac{\tau d}{C^4} |\hat{x}^*(\hat{q}_1 + 1)|(t - \hat{q}_1 - 2).
\end{aligned}$$

The estimate (6.7) then follows from the above inequality and (6.6). ∎

LEMMA 6.7. *If* $1 \le \hat{q}_{2,1} \le 3/2$, *then*

$$\hat{x}^*(\hat{q}_1 + 3) \ge \frac{\tau d}{4C^4} |\hat{x}^*(\hat{q}_1 + 1)|. \tag{6.11}$$

*Proof.* Suppose that $1 \le \hat{q}_{2,1} \le 3/2$. Letting $t_1 = \hat{q}_1 + 1$ and $t_2 = \hat{q}_1 + 2$ in (6.5), we have, for $t \in [\hat{q}_1 + 1, \hat{q}_1 + 2]$,

$$\begin{aligned}
\hat{x}^*(t) &\le \frac{C(t - \hat{q}_1 - 1)\hat{x}^*(\hat{q}_1 + 2) + (\hat{q}_1 + 2 - t)\hat{x}^*(\hat{q}_1 + 1)}{C(t - \hat{q}_1 - 1) + (\hat{q}_1 + 2 - t)} \\
&\le \frac{\hat{x}^*(\hat{q}_1 + 1)}{C}(\hat{q}_1 + 2 - t),
\end{aligned} \tag{6.12}$$

where the last inequality holds because $\hat{x}^*(\hat{q}_1 + 1)$ and $\hat{x}^*(\hat{q}_1 + 2)$ are nonpositive and $C \ge 1$. By Lemma 5.6, Proposition 6.1, (6.12), and (6.2), we have

$$\begin{aligned}
\hat{x}^*(\hat{q}_1 + 3) &= \tau \int_{\hat{q}_2-1}^{\hat{q}_1+2} \hat{h}(\hat{x}^*(s))ds \\
&\ge \frac{\tau}{C} \int_{\hat{q}_2-1}^{\hat{q}_1+2} \hat{h}\left(\frac{\hat{x}^*(\hat{q}_1 + 1)}{C}(\hat{q}_1 + 2 - s)\right) ds \\
&\ge \frac{\tau}{|\hat{x}^*(\hat{q}_1 + 1)|} \int_{\hat{x}^*(\hat{q}_1+1)(2-\hat{q}_{2,1})/C}^{0} \hat{h}(u)du \\
&\ge \frac{\tau d}{C^4} |\hat{x}^*(\hat{q}_1 + 1)|(2 - \hat{q}_{2,1})^2.
\end{aligned} \tag{6.13}$$

The estimate (6.11) then follows since $1 \le \hat{q}_{2,1} \le 3/2$. ∎

LEMMA 6.8. *There exists* $\tau^{(2)} \ge \tau_0$ *(depending only on* $\hat{h}$*) such that if* $\tau > \tau^{(2)}$, *then* $\hat{q}_{2,1} < 3/2$.

*Proof.* Let $\delta \in (0, 1/2)$ and set $r = \delta C^{-1}$. Choose constants $m$ and $M$ satisfying $0 < m < r < M$. Let $\gamma^{(1)} > 0$ and $\tau^{(1)} \ge \tau_0$ be such that Proposition 6.3 holds.

Then set $\tau^{(2)} = \tau^{(1)} \vee LC(\delta\gamma^{(1)})^{-1}$. Fix $\tau > \tau^{(2)}$ and suppose $\hat{q}_{2,1} \geq 3/2$. We will obtain a contradiction. Since $\hat{q}_1 + 1 < \hat{q}_2 - 1 - \delta$, we have $\hat{x}^*(t) \in (-L, 0]$ and $\hat{x}^*$ is increasing and differentiable with $\frac{d\hat{x}^*(t)}{dt} = \tau\hat{h}(\hat{x}^*(t-1))$ at all $t \in [\hat{q}_2 - 1 - \delta, \hat{q}_2]$. Then by Proposition 6.1, we obtain the inequality

$$\hat{x}^*(\hat{q}_2 - 1) < \hat{x}^*(\hat{q}_2 - \delta) = -\tau \int_{\hat{q}_2-\delta}^{\hat{q}_2} \hat{h}(\hat{x}^*(s-1))ds \leq -\tau\delta C^{-1}\hat{h}(\hat{x}^*(\hat{q}_2 - 1)).$$

It follows from the conclusion of Proposition 6.3, with $s = \hat{x}^*(\hat{q}_2 - 1)C\delta^{-1}$, that

$$\hat{x}^*(\hat{q}_2 - 1) \leq -\tau\delta\gamma^{(1)}C^{-1} < -L, \tag{6.14}$$

a contradiction. Therefore, we must have $\hat{q}_{2,1} < 3/2$. ∎

LEMMA 6.9. *There exist $\tau^{(3)} \geq \tau_0$ and $\gamma > 0$ (both depending only on $\hat{h}$) such that if $\tau > \tau^{(3)}$, then*

$$\|\hat{x}^*\|_{[-1,\infty)} \geq \tau\gamma. \tag{6.15}$$

*Proof.* Note that we can define a function $\tilde{h} : \mathbb{R} \to \mathbb{R}$ such that $\tilde{h}(s) = \hat{h}(s)$ for all $s \in [-L, \infty)$, $\tilde{h}$ is continuously differentiable, $s\tilde{h}(s) < 0$ for all $s \neq 0$, $\tilde{h}'(0) < 0$, and $\lim_{s\to-\infty} \tilde{h}(s)$ exists and is positive. Then $-\tilde{h}$ satisfies the condition H1 in [80]. Therefore, we can apply Theorem 12 in [80] (with $\tau$ in place of $\varepsilon^{-1}$) to obtain that there exist $\tau^\dagger \geq \tau_0$ and $\gamma^\dagger > 0$ (depending only on $\tilde{h}$) such that if $\tau > \tau^\dagger$ and $\hat{x}^*$ is a SOPS of the *unconstrained* delay differential equation $\frac{d\hat{x}^*(t)}{dt} = \tau\tilde{h}(\hat{x}^*(t-1))$, then (6.15) is satisfied with $\gamma^\dagger$ in place of $\gamma$.

Now define

$$\tau^{(3)} = \max\left(\tau^\dagger, \tau^{(2)}, \frac{8LC}{|\hat{h}(L/C^2)|}, \frac{16L}{\beta}\right), \tag{6.16}$$

$$\gamma = \min\left(\gamma^\dagger, \frac{|\hat{h}(L/C^2)|}{4C}, \frac{dL}{4C^4}, \frac{\beta}{16}\right), \tag{6.17}$$

and assume that $\tau > \tau^{(3)}$ and $\hat{x}^*$ is a SOPS$^{\text{n}}$. We treat the following two cases separately. First, consider the case that $\hat{x}^*(t) > -L$ for all $t \geq -1$. Then the dynamics of $\hat{x}^*$ are the same as in the unconstrained case, so $\hat{x}^*$ is a solution of the *unconstrained* delay differential equation $\frac{d\hat{x}^*(t)}{dt} = \tau\tilde{h}(\hat{x}^*(t-1))$. By our choice of $\tau^{(3)}$ and $\gamma$, (6.15) is satisfied.

Next, consider the case that $\hat{x}^*(t) = -L$ for some $t \geq -1$. From (5.12), Lemma 6.4, and the periodicity of $\hat{x}^*$, we see that $\hat{x}^*$ has a global minimum at $\hat{q}_1+1$, so $\hat{x}^*(\hat{q}_1+1) = -L$. By our choice of $\tau^{(3)}$ and Lemma 6.8, $\hat{q}_{2,1} < 3/2$. Suppose that $1 \leq \hat{q}_{2,1} < 3/2$. By (6.11) and (6.17), we have $\hat{x}^*(\hat{q}_1 + 3) \geq \frac{\tau dL}{4C^4} \geq \tau\gamma$. Next, suppose that $1/2 \leq \hat{q}_{2,1} \leq 1$. By (6.7) and (6.17), we have $\hat{x}^*(\hat{q}_2 + 1) \geq \frac{\tau dL}{2C^4} \geq \tau\gamma$. Finally, suppose that $0 < \hat{q}_{2,1} \leq 1/2$. For a proof by contradiction, assume that (6.15) does not hold. Then (6.6), (6.7), and the periodicity of $\hat{x}^*$ imply that $\hat{x}^*(t) \geq \frac{L}{C^2}$ for all $\hat{q}_2+1/2 \leq t \leq \hat{q}_2+1$. By periodicity, the estimate holds for all $-1/2 \leq t \leq 0$ as well. It then follows from

(5.9), (6.3), Proposition 6.1, and (6.17) that for $1/2 \le t \le 1$,

$$\begin{aligned}\hat{z}^*(t) &\le \hat{x}^*(0) + \tau \int_{1/2}^{t} \hat{h}(\hat{x}^*(s-1))ds \\ &< \tau\gamma - \frac{\tau}{C}|\hat{h}(L/C^2)|(t-1/2) \\ &\le -\frac{\tau}{C}|\hat{h}(L/C^2)|\,(t-3/4)\,.\end{aligned}$$

From (6.16), we see that $\hat{z}^*(t) \le -L$ for all $t \in [7/8, 1]$ and so $\hat{x}^*(t) = -L$ there. Thus, by (5.8), (5.9), and the fact that $\hat{y}^*$ is nondecreasing,

$$\hat{x}^*(2) \ge \hat{x}^*(15/8) + \tau \int_{15/8}^{2} \hat{h}(-L)ds \ge -L + \frac{\tau\beta}{8} \ge \tau\gamma,$$

a contradiction. With the contradiction thus obtained, (6.15) must hold if $\tau > \tau^{(3)}$, $\hat{q}_{2,1} \le 1/2$, and $\hat{x}^*(t) = -L$ for some $t \ge -1$, completing the proof. ∎

**6.2. Convergence of scaled SOPS$^{\mathrm{n}}$.** In this section we prove the convergence of scaled SOPS$^{\mathrm{n}}$ as $\tau$ goes to infinity. Throughout this section we assume that $h$ satisfies Assumption 3.3.

Define $\tau_0 > 0$ as in (3.11). By Theorem 3.4 and Lemma 5.5, given $\tau > \tau_0$, there exists a SOPS$^{\mathrm{n}}$ $\hat{x}^*$ of the DDER$^{\mathrm{n}}$. Since the DDER$^{\mathrm{n}}$ is autonomous by performing a time shift on $\hat{x}^*$, we can assume that $\hat{q}_0 = -1$. Define the scaled functions $\bar{x}^* \in \mathcal{C}_{[-1,\infty)}$, $\bar{y}^* \in \mathcal{C}^+_{[0,\infty)}$, and $\bar{z}^* \in \mathcal{C}_{[0,\infty)}$ by

$$\bar{x}^*(t) = \tau^{-1}\hat{x}^*(t), \qquad t \ge -1, \tag{6.18}$$

$$\bar{y}^*(t) = \tau^{-1}\hat{y}^*(t), \qquad t \ge 0, \tag{6.19}$$

$$\bar{z}^*(t) = \tau^{-1}\hat{z}^*(t), \qquad t \ge 0. \tag{6.20}$$

By (5.8)–(5.10) and (6.18)–(6.20), $\bar{x}^*$, $\bar{y}^*$, and $\bar{z}^*$ satisfy

$$\bar{x}^*(t) = \bar{z}^*(t) + \bar{y}^*(t), \qquad t \ge 0, \tag{6.21}$$

$$\bar{z}^*(t) = \bar{x}^*(0) + \int_0^t \hat{h}(\tau\bar{x}^*(s-1))ds, \qquad t \ge 0, \tag{6.22}$$

where

$$\bar{y}^*(t) = \sup_{0\le s\le t} (\bar{z}^*(s) + \tau^{-1}L)^-, \qquad t \ge 0. \tag{6.23}$$

By adding $\tau^{-1}L$ to both sides of (6.21), we see that $(\bar{x}^*|_{[0,\infty)} + \tau^{-1}L, \bar{y}^*)$ is a solution of the one-dimensional Skorokhod problem for $\bar{z}^* + \tau^{-1}L$ (see Appendix A).

In the following lemma we prove that $\bar{x}^*$ is uniformly Lipschitz continuous with a Lipschitz constant that depends only on $h$. Note that Lemma 3.5 and (6.1) imply that $H = \sup\{|\hat{h}(s)| : s \in [-L, \infty)\} < \infty$.

LEMMA 6.10. *The scaled functions $\bar{x}^*$, $\bar{y}^*$, and $\bar{z}^*$ satisfy, for $0 \le s < t < \infty$,*

$$|\bar{x}^*(t) - \bar{x}^*(s)| \le H|t-s|, \tag{6.24}$$

$$|\bar{y}^*(t) - \bar{y}^*(s)| \le H|t-s|, \tag{6.25}$$

$$|\bar{z}^*(t) - \bar{z}^*(s)| \le H|t-s|. \tag{6.26}$$

*Since $\bar{x}^*$ is periodic,* (6.24) *in fact holds for $-1 \le s < t < \infty$.*

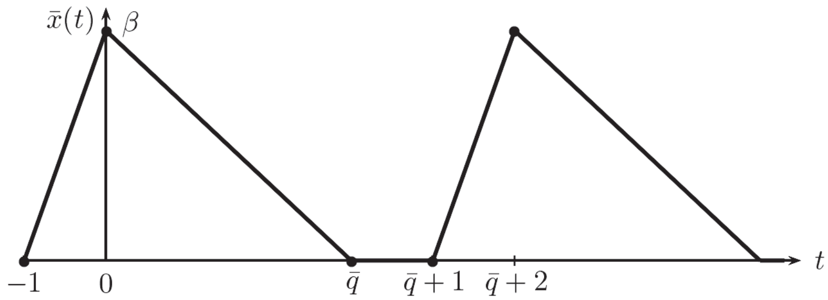


FIG. 6. *Graph of* $\bar{x}$ *as described in* (6.27).

*Proof.* By (6.22) and the bound on $\hat{h}$, (6.26) holds. Then by Proposition A.2 and because $\mathrm{Osc}(\bar{z}^*, [s,t]) \le H|t-s|$ for all $0 \le s \le t < \infty$, it follows that (6.24) and (6.25) hold. ∎

Recall the definitions of $\alpha$ and $\beta$ from Assumption 3.3. Let $\bar{q} = \alpha^{-1}\beta$. Define $\bar{x} \in \mathcal{C}^+_{[-1,\infty)}$ to be a periodic function with period $\bar{q}+2$ satisfying

$$\bar{x}(t) = \begin{cases} \beta(t+1) & \text{for } t \in [-1,0], \\ \beta - \alpha t & \text{for } t \in [0,\bar{q}], \\ 0 & \text{for } t \in [\bar{q},\bar{q}+1]. \end{cases} \tag{6.27}$$

See Figure 6 for a graph of $\bar{x}$. Define $\bar{z} \in \mathcal{C}_{[0,\infty)}$ and $\bar{y} \in \mathcal{C}^+_{[0,\infty)}$ by

$$\bar{z}(t) = \bar{x}(0) + \int_0^t \bar{h}(\bar{x}(s-1))ds, \qquad t \ge 0, \tag{6.28}$$

$$\bar{y}(t) = \sup_{0\le s\le t} (\bar{z}(s))^-, \qquad t \ge 0, \tag{6.29}$$

where

$$\bar{h}(s) = \begin{cases} -\alpha & \text{if } s > 0, \\ \beta & \text{if } s = 0. \end{cases} \tag{6.30}$$

Note that $(\bar{x}|_{[0,\infty)}, \bar{y})$ is the unique solution of the one-dimensional Skorokhod problem for $\bar{z}$ (see Appendix A).

THEOREM 6.11. *Suppose* $\{\tau_n\}_{n=1}^\infty$ *is a sequence in* $(\tau_0,\infty)$ *such that* $\tau_n \to \infty$ *as* $n \to \infty$. *For each* $n$, *let* $\hat{x}^{\tau_n}$ *be a SOPS*$^n$ *with delay* $\tau_n$, *zeros* $-1, \hat{q}_1^{\tau_n}, \hat{q}_2^{\tau_n}, \ldots,$ *and period* $\hat{p}^{\tau_n} = \hat{q}_2^{\tau_n}+1$. *Define* $\bar{x}^{\tau_n}, \bar{y}^{\tau_n}$, *and* $\bar{z}^{\tau_n}$ *as in* (6.18)–(6.20), *but with* $\tau_n$, $\bar{x}^{\tau_n}, \bar{y}^{\tau_n}$, $\bar{z}^{\tau_n}$, $\hat{x}^{\tau_n}$, $\hat{y}^{\tau_n}$, *and* $\hat{z}^{\tau_n}$ *in place of* $\tau$, $\bar{x}^*$, $\bar{y}^*$, $\bar{z}^*$, $\hat{x}^*$, $\hat{y}^*$, *and* $\hat{z}^*$, *respectively. Then* $(\bar{x}^{\tau_n}, \bar{y}^{\tau_n}, \bar{z}^{\tau_n}, \hat{q}_1^{\tau_n}, \hat{q}_2^{\tau_n}, \hat{p}^{\tau_n})$ *converges to* $(\bar{x}, \bar{y}, \bar{z}, \bar{q}, \bar{q}+1, \bar{q}+2)$ *in* $\mathcal{C}_{[-1,\infty)} \times \mathcal{C}_{[0,\infty)} \times \mathcal{C}_{[0,\infty)} \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$ *as* $n \to \infty$.

*Proof.* For each $n \ge 1$, $\bar{x}^{\tau_n}(-1) = 0$ and (6.24)–(6.26) hold with $\bar{x}^{\tau_n}$, $\bar{y}^{\tau_n}$, and $\bar{z}^{\tau_n}$ in place of $\bar{x}^*$, $\bar{y}^*$, and $\bar{z}^*$, respectively. Thus, on each compact interval in $[-1,\infty)$ (resp., $[0,\infty)$, $[0,\infty)$), the functions $\bar{x}^{\tau_n}$ (resp., $\bar{y}^{\tau_n}$, $\bar{z}^{\tau_n}$), $n \ge 1$, are uniformly bounded and Lipschitz continuous. Therefore, by the theorem of Arzelá and Ascoli and a diagonal sequence argument, there is a subsequence, also denoted $\{\tau_n\}_{n=1}^\infty$, and a triple $(\bar{x}^\dagger, \bar{y}^\dagger, \bar{z}^\dagger)$ in $\mathcal{C}_{[-1,\infty)} \times \mathcal{C}_{[0,\infty)} \times \mathcal{C}_{[0,\infty)}$ such that $\bar{x}^{\tau_n}$, $\bar{y}^{\tau_n}$, and $\bar{z}^{\tau_n}$ converge to $\bar{x}^\dagger$, $\bar{y}^\dagger$, and $\bar{z}^\dagger$ uniformly on compact intervals in $[-1,\infty)$, $[0,\infty)$, and $[0,\infty)$, respectively, as $n \to \infty$. Letting $\hat{g}(r,s) = \hat{h}(s)$, then part (ii) and the last line of Lemma 5.8 imply that $\{(\hat{q}_1^{\tau_n}, \hat{q}_2^{\tau_n})\}_{n=1}^\infty$ is uniformly bounded and hence relatively compact in $\mathbb{R}^2_+$. Therefore, by taking a further subsequence if necessary, we can assume that $\hat{q}_1^{\tau_n}, \hat{q}_2^{\tau_n}$, and $\hat{p}^{\tau_n} = \hat{q}_2^{\tau_n} + 1$ converge to nonnegative real numbers $\bar{q}_1^\dagger, \bar{q}_2^\dagger$ and $\bar{p}^\dagger = \bar{q}_2^\dagger + 1$, respectively, as $n \to \infty$.

By (5.12), Lemma 6.4, and the above convergence results, it follows that $\bar{x}^\dagger$ is periodic with period $\bar{p}^\dagger$, $\bar{q}_2^\dagger \geq \bar{q}_1^\dagger + 1$, $\bar{z}^\dagger(t) = \bar{x}^\dagger(t)$ for all $t \in [0, \bar{q}_1^\dagger]$, $\bar{x}^\dagger(t) \geq 0$ for all $t \geq -1$, $\bar{x}^\dagger(t) = 0$ for all $t \in [\bar{q}_1^\dagger, \bar{q}_2^\dagger]$, $\bar{x}^\dagger$ is nondecreasing on $[-1, 0]$, and $\bar{x}^\dagger$ is nonincreasing on $[0, \bar{q}_1^\dagger]$. Lemma 6.9 implies $\bar{x}^\dagger$ is nontrivial, so there exist $t_1 \in [-1, 0)$ and $t_2 \in (0, \bar{q}_1^\dagger]$ such that

$$\begin{aligned} \bar{x}^\dagger(t) &= 0, \qquad t \in [-1, t_1], \\ \bar{x}^\dagger(t) &> 0, \qquad t \in (t_1, t_2), \\ \bar{x}^\dagger(t) &= 0, \qquad t \in [t_2, \bar{q}_2^\dagger]. \end{aligned} \tag{6.31}$$

Note that by the periodicity of $\bar{x}^\dagger$, $t_2 - t_1$ is the length of the intervals on which $\bar{x}^\dagger$ is positive.

We first show that $t_2 - t_1 > 1$. Suppose for a proof by contradiction that $t_2 - t_1 \leq 1$. By (6.21), (6.22), and the fact that $\bar{y}^{\tau_n}$ is nonnegative, we have, for all $t \in [t_1+1, t_2+1]$,

$$\begin{aligned} \bar{z}^{\tau_n}(t) &= \bar{x}^{\tau_n}(t_1+1) - \bar{y}^{\tau_n}(t_1+1) + \bar{z}^{\tau_n}(t) - \bar{z}^{\tau_n}(t_1+1) \\ &\leq \bar{x}^{\tau_n}(t_1+1) + \int_{t_1+1}^t \hat{h}(\tau_n \bar{x}^{\tau_n}(s-1))ds. \end{aligned}$$

Using bounded convergence, we can pass to the limit as $n \to \infty$ to obtain

$$\bar{z}^\dagger(t) \leq -\alpha(t - t_1 - 1), \qquad t \in [t_1+1, t_2+1].$$

Here we have used that $\bar{x}^\dagger(t_1+1) = 0$ and that for each $s \in (t_1+1, t_2+1)$, $\hat{h}(\tau_n \bar{x}^{\tau_n}(s-1)) \to -\alpha$ as $n \to \infty$. The former follows from (6.31), our assumption $t_2 \leq t_1 + 1$, and the fact that $t_1 + 1 \leq \bar{q}_1^\dagger + 1 \leq \bar{q}_2^\dagger$. The latter follows because $\bar{x}^\dagger$ is positive on $(t_1, t_2)$. Since $\bar{z}^\dagger(t) = \bar{x}^\dagger(t) \geq 0$ for all $t \in [0, \bar{q}_1^\dagger]$, it follows that $\bar{q}_1^\dagger \leq t_1 + 1$ and so $[t_1+1, t_2+1] \subset [\bar{q}_1^\dagger, \bar{q}_1^\dagger + 1]$. Then for each $t \in (t_1+1, t_2+1)$, for all $n$ sufficiently large, $t \in (\hat{q}_1^{\tau_n}, \hat{q}_1^{\tau_n} + 1)$ and $\hat{z}^{\tau_n}(t) = \tau_n \bar{z}^{\tau_n}(t) \leq -L$, and so by (6.4), $\hat{x}^{\tau_n}(t) = -L$. Thus,

$$\lim_{n\to\infty} \tau_n \bar{x}^{\tau_n}(t) = \lim_{n\to\infty} \hat{x}^{\tau_n}(t) = -L, \qquad t \in (t_1+1, t_2+1). \tag{6.32}$$

Now, from (6.21) and the fact that $\hat{x}^{\tau_n}(s) \geq -L$ for all $s \geq -1$, we have, for $t \in [t_1+2, t_2+2]$,

$$\begin{aligned} \bar{x}^{\tau_n}(t) &= \bar{x}^{\tau_n}(t_1+2) + \bar{z}^{\tau_n}(t) - \bar{z}^{\tau_n}(t_1+2) + \bar{y}^{\tau_n}(t) - \bar{y}^{\tau_n}(t_1+2) \\ &\geq -\tau_n^{-1} L + \int_{t_1+2}^t \hat{h}(\tau_n \bar{x}^{\tau_n}(s-1))ds, \end{aligned}$$

where we have used the fact that $\bar{y}^{\tau_n}$ is nondecreasing. Using bounded convergence and (6.32), we can pass to the limit as $n \to \infty$ to obtain

$$\bar{x}^\dagger(t) \geq \beta(t - t_1 - 2), \qquad t \in [t_1+2, t_2+2].$$

Then $\bar{x}^\dagger(t) > 0$ for all $t \in (t_1+2, t_2+2]$ and since $\bar{x}^\dagger$ is continuous, it follows that $\bar{x}^\dagger$ is positive on an interval of length greater than $t_2 - t_1$, which contradicts (6.31). With the contradiction thus obtained, we must have $t_2 - t_1 > 1$.

Proceeding, it follows from (6.3) and (6.22) that

$$\bar{z}^{\tau_n}(t) = \bar{x}^{\tau_n}(t_2) + \int_{t_2}^{t} \hat{h}(\tau_n \bar{x}^{\tau_n}(s-1))ds, \qquad t \in [t_2, t_2+1].$$

Using bounded convergence and the fact that $\bar{x}^\dagger$ is positive on $(t_2 - 1, t_2)$, we can pass to the limit as $n \to \infty$ to obtain

$$\bar{z}^\dagger(t) = -\alpha(t - t_2), \qquad t \in [t_2, t_2+1].$$

Thus, given $t \in (t_2, t_2+1]$, then for all $n$ sufficiently large, $\hat{z}^{\tau_n}(t) = \tau_n \bar{z}^{\tau_n}(t) \le -L$ and so $\hat{q}_1^{\tau_n} < t$. Consequently, $\bar{q}_1^\dagger \le t_2$. Combining this with the fact that $t_2 \le \bar{q}_1^\dagger$ yields $t_2 = \bar{q}_1^\dagger$. Hence, for each $t \in (\hat{q}_1^{\tau_n}, \hat{q}_1^{\tau_n} + 1]$, using (6.4), we have for all $n$ sufficiently large, $\hat{x}^{\tau_n}(t) = -L$, and so

$$\lim_{n\to\infty} \tau_n \bar{x}^{\tau_n}(t) = \lim_{n\to\infty} \hat{x}^{\tau_n}(t) = -L, \qquad t \in (\bar{q}_1^\dagger, \bar{q}_1^\dagger + 1). \tag{6.33}$$

By (6.21) and (6.22), we have, for $t \in [\bar{q}_1^\dagger + 1, \bar{q}_1^\dagger + 2]$,

$$\begin{aligned} \bar{x}^{\tau_n}(t) &= \bar{x}^{\tau_n}(\bar{q}_1^\dagger + 1) + \bar{z}^{\tau_n}(t) - \bar{z}^{\tau_n}(\bar{q}_1^\dagger + 1) + \bar{y}^{\tau_n}(t) - \bar{y}^{\tau_n}(\bar{q}_1^\dagger + 1) \\ &\ge \bar{x}^{\tau_n}(\bar{q}_1^\dagger + 1) + \int_{\bar{q}_1^\dagger + 1}^{t} \hat{h}(\tau_n \bar{x}^{\tau_n}(s-1))ds, \end{aligned} \tag{6.34}$$

where we have used the fact that $\bar{y}^{\tau_n}$ is nondecreasing. Using bounded convergence and (6.33), we can pass to the limit as $n \to \infty$ to obtain

$$\bar{x}^\dagger(t) \ge \beta(t - \bar{q}_1^\dagger - 1), \qquad t \in [\bar{q}_1^\dagger + 1, \bar{q}_1^\dagger + 2]. \tag{6.35}$$

Then for each $t \in (\bar{q}_1^\dagger + 1, \bar{q}_1^\dagger + 2]$, for all $n$ sufficiently large we have $\bar{x}^{\tau_n}(t) > 0$ and so $\hat{q}_2^{\tau_n} < t$. It follows that $\bar{q}_2^\dagger \le \bar{q}_1^\dagger + 1$. Combining this with the fact that $\bar{q}_2^\dagger \ge \bar{q}_1^\dagger + 1$ yields $\bar{q}_2^\dagger = \bar{q}_1^\dagger + 1$ and $\bar{p}^\dagger = \bar{q}_1^\dagger + 2$. By (6.35), $\bar{x}^\dagger$ is positive on $(\bar{q}_2^\dagger, \bar{p}^\dagger] = (\bar{q}_1^\dagger + 1, \bar{q}_1^\dagger + 2]$, so for each closed interval $I$ contained in $(\bar{q}_1^\dagger + 1, \bar{q}_1^\dagger + 2]$, $\bar{x}^{\tau_n}$ is positive on $I$ for all $n$ sufficiently large and so $\bar{y}^{\tau_n}$ is constant on $I$ for such $n$. Since this holds for all closed intervals $I$ in $(\bar{q}_1^\dagger + 1, \bar{q}_1^\dagger + 2]$, $\bar{y}^\dagger$ being continuous must be constant on $[\bar{q}_1^\dagger + 1, \bar{q}_1^\dagger + 2]$. Then taking the limit as $n \to \infty$ in (6.34) yields (6.35), but with equality instead of the inequality. Periodicity implies that $\bar{x}^\dagger(t) = \beta(t+1)$ for all $t \in [-1, 0]$ and so $t_1 = -1$. By (6.3) and (6.22), we have

$$\bar{x}^{\tau_n}(t) = \bar{x}^{\tau_n}(0) + \int_0^t \hat{h}(\tau_n \bar{x}^{\tau_n}(s-1))ds, \qquad t \in [0, \bar{q}_1^\dagger].$$

Then by bounded convergence, we can pass to the limit as $n \to \infty$ to obtain

$$\bar{x}^\dagger(t) = \beta - \alpha t, \qquad t \in [0, \bar{q}_1^\dagger], \tag{6.36}$$

where we have used (6.31) with $t_1 = -1$ and $t_2 = \bar{q}_1^\dagger$. Furthermore, since $\bar{q}_1^\dagger$ is the first time $\bar{x}^\dagger$ hits zero after time $t = 0$, we have $\bar{q}_1^\dagger = \alpha^{-1}\beta$. This completes the proof that $\bar{x}^\dagger = \bar{x}$, $\bar{q}_1^\dagger = \bar{q}$, $\bar{q}_2^\dagger = \bar{q} + 1$, and $\bar{p}^\dagger = \bar{p}$.

We now show that $\bar{y}^\dagger = \bar{y}$ and $\bar{z}^\dagger = \bar{z}$. From (6.31) and (6.33), it follows that $\lim_{n\to\infty} \hat{h}(\tau_n \bar{x}^{\tau_n}(t)) = \bar{h}(\bar{x}(t))$ at all $t \in (-1, \bar{q} + 1) \setminus \{\bar{q}\}$. Since $\bar{x}$ is periodic with period $\bar{p}$, we can repeat this argument countably many times to obtain that $\lim_{n\to\infty} \hat{h}(\tau_n \bar{x}^{\tau_n}(t)) = \bar{h}(\bar{x}(t))$ for all but countably many $t$ in $[-1, \infty)$. Then, using bounded convergence, we can pass to the limit as $n \to \infty$ in (6.22), with $\bar{x}^{\tau_n}$ and $\bar{z}^{\tau_n}$ in place of $\bar{x}^*$ and $\bar{z}^*$, respectively, to obtain

$$\bar{z}^\dagger(t) = \bar{x}(0) + \int_0^t \bar{h}(\bar{x}(s-1))ds, \qquad t \geq 0,$$

and so $\bar{z}^\dagger = \bar{z}$. Last, since $\bar{z}^{\tau_n} \to \bar{z}$ uniformly on compact intervals in $[0,\infty)$,

$$\bar{y}^\dagger(t) = \lim_{n\to\infty} \bar{y}^{\tau_n}(t) = \lim_{n\to\infty} \sup_{0\leq s\leq t} (\bar{z}^{\tau_n}(s) + \tau_n^{-1}L)^- = \sup_{0\leq s\leq t} (\bar{z}(s))^-,$$

and so $\bar{y}^\dagger = \bar{y}$.

We have shown that the sequence $\{(\bar{x}^{\tau_n}, \bar{y}^{\tau_n}, \bar{z}^{\tau_n}, \hat{q}_1^{\tau_n}, \hat{q}_2^{\tau_n}, \hat{p}^{\tau_n})\}_{n=1}^\infty$ is relatively compact in $\mathcal{C}_{[-1,\infty)} \times \mathcal{C}_{[0,\infty)} \times \mathcal{C}_{[0,\infty)} \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$ and that every convergent subsequence converges to $(\bar{x}, \bar{y}, \bar{z}, \bar{q}, \bar{q}+1, \bar{q}+2)$. By a standard real analysis argument, the entire sequence must also converge to $(\bar{x}, \bar{y}, \bar{z}, \bar{q}, \bar{q}+1, \bar{q}+2)$ as $n \to \infty$, which completes the proof of the theorem. ∎

The following corollary is a consequence of the preceding convergence result. A proof can be given using a straightforward argument by contradiction and so we omit it.

COROLLARY 6.12. *For each $t > 0$ and $\varepsilon > 0$, there exists $\tau^{t,\varepsilon} \geq \tau_0$ such that whenever $\tau > \tau^{t,\varepsilon}$ and $\hat{x}^*$ is a SOPS$^n$ of the DDER$^n$ with delay $\tau$ and $\hat{q}_0 = -1$, and $\bar{x}^*, \bar{y}^*$, and $\bar{z}^*$ are defined as in* (6.21)–(6.23)*, then $\|\bar{x}^* - \bar{x}\|_{[-1,t]} < \varepsilon$, $\|\bar{y}^* - \bar{y}\|_{[0,t]} < \varepsilon$, $\|\bar{z}^* - \bar{z}\|_{[0,t]} < \varepsilon$, $|\hat{q}_1 - \bar{q}| < \varepsilon$, $|\hat{q}_2 - \bar{q} - 1| < \varepsilon$, and $|\hat{p} - \bar{q} - 2| < \varepsilon$.*

The next corollary is an immediate consequence of the above corollary and Lemma 5.5.

COROLLARY 6.13. *For each $\tau > \tau_0$, let $x^\tau$ be a SOPS with delay $\tau$ and period $p^\tau$. Then $\tau^{-1}p^\tau \to \alpha^{-1}\beta + 2$ as $\tau \to \infty$.*

In the following lemma we prove that for $\tau$ sufficiently large, any SOPS$^n$ will hit the lower boundary. We provide bounds on both the time it takes to reach the lower boundary from the zero level as well as the time it takes to reach the zero level after leaving the lower boundary.

LEMMA 6.14. *There exists $\tau^{(4)} \geq \tau_0$ such that whenever $\tau > \tau^{(4)}$ and $\hat{x}^*$ is a SOPS$^n$ of DDER$^n$ with delay $\tau$ and $\hat{q}_0 = -1$, there exists $\hat{\ell}_1 \in (\hat{q}_1, \hat{q}_1 + 1)$ such that*

$$0 > \hat{x}^*(t) > -L \qquad \text{for all } t \in (\hat{q}_1, \hat{\ell}_1), \tag{6.37}$$

$$\hat{x}^*(t) = -L \qquad \text{for all } t \in [\hat{\ell}_1, \hat{q}_1 + 1], \tag{6.38}$$

*and*

$$\hat{z}^*(t) > -L \qquad \text{for all } t \in (\hat{q}_1, \hat{\ell}_1), \tag{6.39}$$

$$\hat{z}^*(t) < -L \qquad \text{for all } t \in (\hat{\ell}_1, \hat{q}_1 + 1]. \tag{6.40}$$

*Furthermore, $0 < \hat{\ell}_1 - \hat{q}_1 < \frac{2L}{\alpha\tau} \leq 1$ and $0 < \hat{q}_2 - \hat{q}_1 - 1 < (\frac{2}{\alpha} + \frac{1}{\beta})\frac{L}{\tau}$. Moreover, $\hat{x}^*(\cdot)$ is continuously differentiable on $[0, \hat{\ell}_1)$ and $(\hat{\ell}_1, \hat{p}]$.*

*Remark* 6.1. By Corollary 6.12 and by possibly taking $\tau^{(4)} \geq \tau_0$ larger, we can further assume that whenever $\tau > \tau^{(4)}$ and $x^*$ is a SOPS$^n$ of the DDER$^n$ with delay $\tau$ and $\hat{q}_0 = -1$, then

(i) $0 < \delta < \hat{q}_1 < \hat{\ell}_1 < \bar{q} + \delta < \bar{q} + 1 - \delta < \hat{q}_2 + \delta < \hat{q}_1 + 2$; and
(ii) $\hat{x}^*(t) > d$ for all $t \in [-1 + \delta, \bar{q} - \delta]$.

Here $\delta > 0$ and $d > 0$ are defined as in the proof of Lemma 6.14 below. The inequalities (i) and (ii) will be used in the proof of Lemma 6.24.

*Proof.* Define positive constants $\delta = \frac{1}{2}(\bar{q} \wedge 1)$ and $d = \frac{1}{2}\min(\bar{x}(-1+\delta), \bar{x}(\bar{q}-\delta))$. By Corollary 6.12 and (6.27), there exists $\tau^{(4)} \geq \tau_0$ such that whenever $\tau > \tau^{(4)}$ and $\hat{x}^*$ is a SOPS$^{\mathrm{n}}$ of the DDER$^{\mathrm{n}}$ with delay $\tau$ and $\hat{q}_0 = -1$, then

(i) $-1+\delta < \hat{q}_1 - 1 < \hat{q}_1 - 1 + \delta < \bar{q} - \delta < \hat{q}_2 < \hat{q}_1 + 2$;

(ii) $\hat{x}^*(t) > \tau d$ for all $t \in [\hat{q}_1 - 1, \hat{q}_1 - 1 + \delta]$;

(iii) $\hat{h}(s) < -\alpha/2$ for all $s \geq \tau d$; and

(iv) $\frac{2L}{\alpha\tau} < \delta$ and $\frac{L}{\beta\tau} < \delta$.

Combining (i)–(iii) yields $\hat{h}(\hat{x}^*(t-1)) < -\alpha/2$ for all $t \in [\hat{q}_1, \hat{q}_1+\delta]$. Then (6.22), (iv), and the fact that $\hat{z}^*(\hat{q}_1) = 0$ imply that $\hat{z}^*(\hat{q}_1 + \frac{2L}{\alpha\tau}) < -L$. Since $\hat{z}^*$ is continuous and decreasing on $[\hat{q}_1, \hat{q}_1+1)$, there exists $\hat{\ell}_1 \in (\hat{q}_1, \hat{q}_1 + \frac{2L}{\alpha\tau})$ such that $0 > \hat{z}^*(t) > -L$ for all $t \in (\hat{q}_1, \hat{\ell}_1)$ and $\hat{z}^*(t) < -L$ for all $t \in (\hat{\ell}_1, \hat{q}_1+1]$, so (6.39) and (6.40) hold. Equations (6.37) and (6.38) then follow from (6.4). Now consider $\hat{q}_2 - \hat{q}_1 - 1$. If $\hat{q}_2 \leq \hat{\ell}_1 + 1$, then $\hat{q}_2 - \hat{q}_1 - 1 \leq \hat{\ell}_1 - \hat{q}_1 < \frac{2L}{\alpha\tau}$. On the other hand, suppose $\hat{q}_2 > \hat{\ell}_1 + 1$. By (i) and (6.38), we have $\hat{x}^*(t) > -L$ and $\hat{h}(\hat{x}^*(t-1)) = \hat{h}(-L) = \beta$ for all $t \in [\hat{\ell}_1 + 1, \hat{q}_1 + 2]$. It then follows from Lemma 5.6 that $\hat{x}^*(\hat{\ell}_1 + 1 + \frac{L}{\beta\tau}) \geq 0$ and so $\hat{q}_2 - \hat{\ell}_1 - 1 \leq \frac{L}{\beta\tau}$. Combining this with $\hat{\ell}_1 - \hat{q}_1 < \frac{2L}{\alpha\tau}$ yields $q_2 - q_1 - 1 < (\frac{2}{\alpha} + \frac{1}{\beta})\frac{L}{\tau}$.

The fact that $\hat{x}^*(\cdot)$ is continuous differentiable on $[0, \hat{\ell}_1)$, $(\hat{\ell}_1, \hat{q}_1+1)$, and $(\hat{q}_1+1, \hat{p}]$ is due to Lemma 5.6, (5.12), (6.37), and (6.38). Now $\hat{x}^*(\cdot)$ is constant and equal to $-L$ in a left neighborhood of $\hat{q}_1+1$ and so is left differentiable with the left derivative equal to zero there. On the other hand, $\hat{x}^*(\cdot)$ is strictly above $-L$ in a right neighborhood of $\hat{q}_1 + 1$ (not including $\hat{q}_1 + 1$) and so is right differentiable with the right derivative equal to $\hat{h}(\hat{x}^*(\hat{q}_1)) = \hat{h}(0) = 0$ as well. Therefore $\hat{x}^*(\cdot)$ is continuously differentiable on $(\hat{\ell}_1, \hat{p}]$. ∎

**6.3. Solutions of the DDER near a SOPS.** In this section we consider solutions of the DDER whose initial conditions are in a small neighborhood of the initial condition of a SOPS. Throughout this section we fix $\tau > \tau^{(4)}$, where $\tau^{(4)}$ is as in Lemma 6.14, and let $x^*$ denote a SOPS of the DDER with delay $\tau$ and $q_0 = -\tau$. We let $z^*$ and $y^*$ be defined as in (2.3) and (2.1) with $f(\varphi) = h(\varphi(-\tau))$ and $x^*$, $y^*$, and $z^*$ in place of $x$, $y$, and $z$, respectively. In the following lemma, we state properties of $x^*$ and $z^*$ that are a consequence of (3.2), Lemmas 6.4, 6.14, and the one-to-one correspondence between SOPS and SOPS$^{\mathrm{n}}$.

LEMMA 6.15. *There exists $\ell_1 \in (q_1, q_1 + \tau)$ such that*

(i) *$x^*(t) > 0$ for all $t \in [0, \ell_1)$, $x^*(t) = 0$ for all $t \in [\ell_1, q_1 + \tau]$, and $x^*(t) > 0$ for all $t \in (q_1 + \tau, p]$;*

(ii) *$x^*(\cdot)$ is continuously differentiable on $[0, \ell_1)$ and $(\ell_1, p]$;*

(iii) *$x^*(\cdot)$ is decreasing on $(0, \ell_1)$ and increasing on $(q_1 + \tau, p)$;*

(iv) *$x^*(t) \leq L + \tau G$ for all $t \geq -\tau$;*

(v) *$z^*(t) > 0$ for all $t \in [0, \ell_1)$ and $z^*(t) < 0$ for all $t \in (\ell_1, q_1 + \tau]$.*

*Furthermore,*

$$0 < \ell_1 - q_1 < 2\alpha^{-1}L \qquad \textit{and} \qquad 0 < q_2 - q_1 - \tau < (2\alpha^{-1} + \beta^{-1})L. \tag{6.41}$$

*Proof.* Parts (i)–(iv) and (6.41) follow from Lemmas 6.4, 6.14, and 5.5 and the periodicity of $x^*$. Part (v) of the lemma is due to (6.39), (6.40), and the fact that $z^*(t) = \hat{z}^*(\tau^{-1}t) + L$ for every $t \geq 0$. ∎

To obtain estimates for solutions of the DDER, we have the following lemma on the continuity of solutions in their initial condition. Recall that $K_h < \infty$ is the Lipschitz constant for $h$.

LEMMA 6.16. *Given solutions $x$ and $x^\dagger$ of the DDER, define $z$ as in* (2.3) *and define $z^\dagger$ as in* (2.3) *but with $x^\dagger$ and $z^\dagger$ in place of $x$ and $z$, respectively. Then for all $t \geq 0$,*

$$\|x - x^\dagger\|_{[-\tau,t]} \leq 2\exp(2K_h t)\|x_0 - x_0^\dagger\|_{[-\tau,0]} \tag{6.42}$$

*and*

$$\|z - z^\dagger\|_{[0,t]} \leq (1 + K_h\tau)\exp(2K_h t)\|x_0 - x_0^\dagger\|_{[-\tau,0]}. \tag{6.43}$$

*Proof.* Fix $t \geq 0$. By (2.3) and (3.12), we obtain the following two bounds:

$$\|z - z^\dagger\|_{[0,t]} \leq \|x_0 - x_0^\dagger\|_{[-\tau,0]} + K_h \int_0^t \|x - x^\dagger\|_{[-\tau,s]} ds, \tag{6.44}$$

$$\|z - z^\dagger\|_{[0,t]} \leq (1 + K_h\tau)\|x_0 - x_0^\dagger\|_{[-\tau,0]} + K_h \int_0^t \|x - x^\dagger\|_{[0,s]} ds. \tag{6.45}$$

Applying the Lipschitz continuity of the Skorokhod map (see Proposition A.1) to the left-hand side of (6.44) and then extending the norm on the left to the interval $[-\tau, t]$, we obtain, for all $t \geq 0$,

$$\|x - x^\dagger\|_{[-\tau,t]} \leq 2\|x_0 - x_0^\dagger\|_{[-\tau,0]} + 2K_h \int_0^t \|x - x^\dagger\|_{[-\tau,s]} ds.$$

The bound (6.42) then follows from Gronwall's inequality. Next, applying the Lipschitz continuity of the Skorokhod map to the integrand in (6.45), we have, for all $t \geq 0$,

$$\|z - z^\dagger\|_{[0,t]} \leq (1 + K_h\tau)\|x_0 - x_0^\dagger\|_{[-\tau,0]} + 2K_h \int_0^t \|z - z^\dagger\|_{[0,s]} ds.$$

As above, the bound (6.43) follows from Gronwall's inequality. □

In the following lemma, we establish some properties for solutions of the DDER whose initial conditions are in a small neighborhood of $x_0^*$.

LEMMA 6.17. *For each $0 < \eta_0 < \frac{1}{2}\min(\tau, q_1, \ell_1 - q_1, q_1 + \tau - \ell_1, q_2 - q_1 - \tau)$ there exists $\varepsilon_0 \in (0, \eta_0)$ such that whenever $x$ is a solution of the DDER satisfying $\|x_0 - x_0^*\|_{[-\tau,0]} < \varepsilon_0$, then there exist $q_1^x \in (q_1 - \eta_0, q_1 + \eta_0)$, $\ell_1^x \in (\ell_1 - \eta_0, \ell_1 + \eta_0)$, and $q_2^x \in (q_2 - \eta_0, q_2 + \eta_0)$ such that*

(i) *$x(t) > 0$ for all $t \in [-\tau, -\tau + \eta_0]$, $x(t) > L$ for all $t \in [-\tau + \eta_0, q_1^x)$, $0 < x(t) < L$ for all $t \in (q_1^x, \ell_1^x)$, $x(t) = 0$ for all $t \in [\ell_1^x, q_1^x + \tau]$, $0 < x(t) < L$ for all $t \in (q_1^x + \tau, q_2^x)$, and $x(t) > L$ for all $t \in (q_2^x, p + \eta_0]$.*

(ii) *$x(\cdot)$ is continuously differentiable on $[0, \ell_1^x)$ and $(\ell_1^x, p + \eta_0]$;*

(iii) *$x(\cdot)$ is decreasing on $(\eta_0, \ell_1^x)$ and increasing on $(q_1^x + \tau, q_2^x + \tau)$.*

*Proof.* Fix $\eta_0 > 0$ as in the statement of the lemma. By (5.12), part (i) of Lemma 6.15, (6.42), and the continuity of $x$, there exists $\varepsilon_0 > 0$ such that whenever $x$ is a solution of the DDER with $\|x_0 - x_0^*\|_{[-\tau,0]} < \varepsilon_0$, then $x(t) > 0$ for all $t \in [-\tau, -\tau + \eta_0]$, $x(t) > L$ for all $t \in [-\tau + \eta_0, q_1 - \eta_0]$, $0 < x(t) < L$ for all $t \in [q_1 + \eta_0, \ell_1 - \eta_0]$, $0 \leq x(t) < L$ for all $t \in [\ell_1 - \eta_0, q_1 + \tau + \eta_0]$, $0 < x(t) < L$ for all $t \in [q_1 + \tau + \eta_0, q_2 - \eta_0]$, and $x(t) > L$ for all $t \in [q_2 + \eta_0, p + \eta_0]$. Then (2.5), with $f(\varphi) = h(\varphi(-\tau))$, and the negative feedback condition on $h$ imply that $x$ is decreasing on $[\eta_0, \ell_1 - \eta_0]$ and increasing on $[q_1 + \tau + \eta_0, q_2 + \tau - \eta_0]$. Consequently, there exist $q_1^x \in (q_1 - \eta_0, q_1 + \eta_0)$

and $q_2^x \in (q_2 - \eta_0, q_2 + \eta_0)$ such that $x(t) > L$ for all $t \in [-\tau + \eta_0, q_1^x)$, $0 \le x(t) < L$ for all $t \in (q_1^x, q_2^x)$, and $x(t) > L$ for all $t \in (q_2^x, p + \eta_0]$. To prove part (i) of the lemma, it remains to show that there exists $\ell_1^x \in (\ell_1 - \eta_0, \ell_1 + \eta_0)$ such that $x(t) > 0$ for all $t \in (q_1^x, \ell_1^x)$, $x(t) = 0$ for all $t \in [\ell_1^x, q_1^x + \tau]$ and $x(t) > 0$ for all $t \in (q_1^x + \tau, q_2^x)$.

Define $z \in \mathcal{C}_{[0,\infty)}$ as in (2.3). By part (v) of Lemma 6.15, (6.43), and the continuity of $z$, we can choose $\varepsilon_0 > 0$ possibly smaller so that $z$ is positive on $[0, \ell_1 - \eta_0]$ and negative on $[\ell_1 + \eta_0, q_1 + \tau - \eta_0]$. Furthermore, the negative feedback condition on $h$ and the previous paragraph imply that $z$ is decreasing on $(\eta_0, q_1^x + \tau)$ and increasing on $(q_1^x + \tau, q_2^x + \tau)$. Hence, there must exist $\ell_1^x \in (\ell_1 - \eta_0, \ell_1 + \eta_0)$ such that $z$ is positive on $[0, \ell_1^x)$ and negative on $(\ell_1^x, q_1^x + \tau]$. By (2.1), we see that $y(t) = 0$ for all $t \in [0, \ell_1^x]$, $y(t) = -z(t)$ for all $t \in [\ell_1^x, q_1^x + \tau]$, and $y(t) = -z(q_1^x + \tau)$ for all $t \in [q_1^x + \tau, q_2^x + \tau]$. Substituting into (2.2), we obtain that $x(t) > 0$ for all $t \in [0, \ell_1^x)$, $x(t) = 0$ for all $t \in [\ell_1^x, q_1^x + \tau]$, and $x(t) > 0$ for all $t \in (q_1^x + \tau, q_2^x)$. The fact that $x(\cdot)$ is continuous differentiable on $[0, \ell_1^x)$, $(\ell_1^x, q_1^x + \tau)$, and $(q_1^x + \tau, p]$ follows from part (i) of the lemma and Lemma 2.2. Now $x(\cdot)$ is identically zero in a left neighborhood of $q_1^x + \tau$ and so is left differentiable there with the left derivative equal to zero. On the other hand, $x(\cdot)$ is strictly positive in a right neighborhood of $q_1^x + \tau$ (not including $q_1^x + \tau$) and so is right differentiable with the right derivative equal to $h(x(q_1^x)) = h(L) = 0$ as well. Therefore $x(\cdot)$ is continuously differentiable on $(\ell_1^x, p]$. Part (iii) of the lemma follows from part (i) of the lemma, (2.5), and the negative feedback condition on $h$. ∎

Fix an $\eta_0 > 0$ and associated $\varepsilon_0 > 0$ as in Lemma 6.17. Given a solution $x$ of the DDER with initial condition satisfying $\|x_0 - x_0^*\|_{[-\tau,0]} < \varepsilon_0$, define

$$\varepsilon_1 = \varepsilon_1(x) = \varepsilon_0 - \|x_0 - x_0^*\|_{[-\tau,0]} > 0 \tag{6.46}$$

and

$$\eta_1 = \eta_1(x) = \eta_0 - \max\{|q_1^x - q_1|, |\ell_1^x - \ell_1|, |q_2^x - q_2|\} > 0. \tag{6.47}$$

As a consequence,

$$\begin{aligned}(q_1^x - \eta_1, q_1^x + \eta_1) &\subset (q_1 - \eta_0, q_1 + \eta_0),\\ (\ell_1^x - \eta_1, \ell_1^x + \eta_1) &\subset (\ell_1 - \eta_0, \ell_1 + \eta_0),\\ (q_2^x - \eta_1, q_2^x + \eta_1) &\subset (q_2 - \eta_0, q_2 + \eta_0).\end{aligned}$$

Suppose $x^\dagger$ is solution of the DDER with initial condition satisfying $\|x_0^\dagger - x_0\|_{[-\tau,0]} < \varepsilon_1$. Then by (6.46), we have

$$\|x^\dagger - x^*\|_{[-\tau,0]} \le \|x^\dagger - x\|_{[-\tau,0]} + \|x - x^*\|_{[-\tau,0]} < \varepsilon_0, \tag{6.48}$$

and so Lemma 6.17 holds with $x^\dagger$, $q_1^{x^\dagger}$, $\ell_1^{x^\dagger}$, and $q_2^{x^\dagger}$ in place of $x$, $q_1^x$, $\ell_1^x$, and $q_2^x$, respectively. The following lemma ensures that $q_1^x, \ell_1^x$, and $q_2^x$ are continuous in $x_0$ and is a consequence of the continuity of solutions in their initial conditions, as described in Lemma 6.16, the definitions of $\varepsilon_1$ and $\eta_1$, and Lemma 6.17.

LEMMA 6.18. *Given a solution $x$ of the DDER with initial condition satisfying $\|x_0 - x_0^*\|_{[-\tau,0]} < \varepsilon_0$, define $\varepsilon_1 > 0$ and $\eta_1 > 0$ as in* (6.46) *and* (6.47)*. Then for each $\eta \in (0, \eta_1)$, there exists $\varepsilon \in (0, \varepsilon_1)$ such that whenever $x^\dagger$ is a solution of the DDER with initial condition satisfying $\|x_0^\dagger - x_0\|_{[-\tau,0]} < \varepsilon$, properties* (i)–(iii) *of Lemma* 6.17 *hold with $x^\dagger$, $q_1^{x^\dagger}$, $\ell_1^{x^\dagger}$, and $q_2^{x^\dagger}$ in place of $x$, $q_1^x$, $\ell_1^x$, and $q_2^x$, respectively. Furthermore, $q_1^{x^\dagger} \in (q_1^x - \eta, q_1^x + \eta)$, $\ell_1^{x^\dagger} \in (\ell_1^x - \eta, \ell_1^x + \eta)$, and $q_2^{x^\dagger} \in (q_2^x - \eta, q_2^x + \eta)$.*

*Proof.* The fact that there exists $\varepsilon > 0$ such that $q_1^{x^\dagger} \in (q_1^x - \eta, q_1^x + \eta)$, $\ell_1^{x^\dagger} \in (\ell_1^x - \eta, \ell_1^x + \eta)$, and $q_2^{x^\dagger} \in (q_2^x - \eta, q_2^x + \eta)$ hold uses the continuity of solutions to the DDER in their initial condition, which is stated in Lemma 6.16. The argument is similar to the argument that $q_1^x \in (q_1 - \eta_0, q_1 + \eta_0)$, $\ell_1^x \in (\ell_1 - \eta_0, \ell_1 + \eta_0)$, and $q_2^x \in (q_2 - \eta_0, q_2 + \eta_0)$ found in the proof of Lemma 6.17, so we omit it here. ∎

**6.4. Variational equation.** In this section we introduce the notion of a variational equation (VE) along a solution of the DDER. Here we restrict ourselves to the case that $f$ satisfies the assumptions of Theorem 3.8 and that the solution of the DDER has an initial condition in a small neighborhood of the initial condition of a SOPS. A general definition and properties of a solution of the VE along a solution of the DDER for $f$ as in (1.1) are presented in Appendix B. The treatment in the appendix is more general than what we need for the proof of stability and uniqueness of SOPS, but we include it for independent interest.

Throughout this section we assume that $f$ is of the form (3.9) and satisfies Assumptions 3.3 and 3.4. We fix $\tau > \tau^{(4)}$, where $\tau^{(4)} \geq \tau_0$ is as in Lemma 6.14 and such that the properties in Remark 6.1 hold. Let $x^*$ denote a SOPS with delay $\tau$ and $q_0 = -\tau$ so that Lemmas 6.15–6.18 hold. Fix an $\eta_0 > 0$ and associated $\varepsilon_0 > 0$ as in Lemma 6.17 and let $x$ be a solution to the DDER with initial condition satisfying $\|x_0 - x_0^*\|_{[-\tau,0]} < \varepsilon_0$. We briefly summarize important definitions and properties from Appendix B for solutions of the VE along $x$. The following definition is a version of Definition B.2 specific to our current setting. Recall that $\mathcal{D}_{[-\tau,\infty)}$ denotes the set of functions from $[-\tau,\infty)$ to $\mathbb{R}$ with finite left and right limits at each $t > -\tau$ and a finite right limit at $-\tau$.

DEFINITION 6.19. *Given a solution $x$ of the DDER satisfying $\|x_0 - x_0^*\|_{[-\tau,0]} < \varepsilon_0$, a function $v \in \mathcal{D}_{[-\tau,\infty)}$ is a* solution of the VE along $x$ *if $v(t) \geq 0$ at all $t \geq -\tau$ such that $x(t) = 0$ and $v$ satisfies*

$$v(t) = \partial_w \Phi(z)(t), \qquad t \geq 0, \tag{6.49}$$

*where $\Phi$ denotes the Skorokhod map given by* (A.1) *and* (A.2)*, $z \in \mathcal{C}_{[0,\infty)}$ is defined via* (2.3)*, $w \in \mathcal{C}_{[0,\infty)}$ is defined by*

$$w(t) = v(0) + \int_0^t h'(x(s-\tau))v(s-\tau)ds, \qquad t \geq 0, \tag{6.50}$$

*and the directional derivative of $\Phi$ at $z$ in the direction $w$ is denoted by $\partial_w \Phi(z)$, is well defined as an element of $\mathcal{D}_{[0,\infty)}$ by Proposition* A.5 *and is given by*

$$\partial_w \Phi(z)(t) = w(t) + R(-z, -w)(t), \tag{6.51}$$

*where*

$$R(-z,-w)(t) = \begin{cases} \sup_{s \in \mathbb{S}_{-z}(t)}(-w(s)) & \text{if } \sup_{0\leq s\leq t}(-z(s)) > 0, \\ \sup_{s \in \mathbb{S}_{-z}(t)}(-w(s)) \vee 0 & \text{if } \sup_{0\leq s\leq t}(-z(s)) = 0, \\ 0 & \text{if } \sup_{0\leq s\leq t}(-z(s)) < 0, \end{cases} \tag{6.52}$$

*and*

$$\mathbb{S}_{-z}(t) = \left\{ s \in [0,t] : -z(s) = \sup_{0\leq u\leq t}(-z(u)) \vee 0 \right\}. \tag{6.53}$$

See Figure 7 for an example of a solution of the VE along a solution of the DDER.

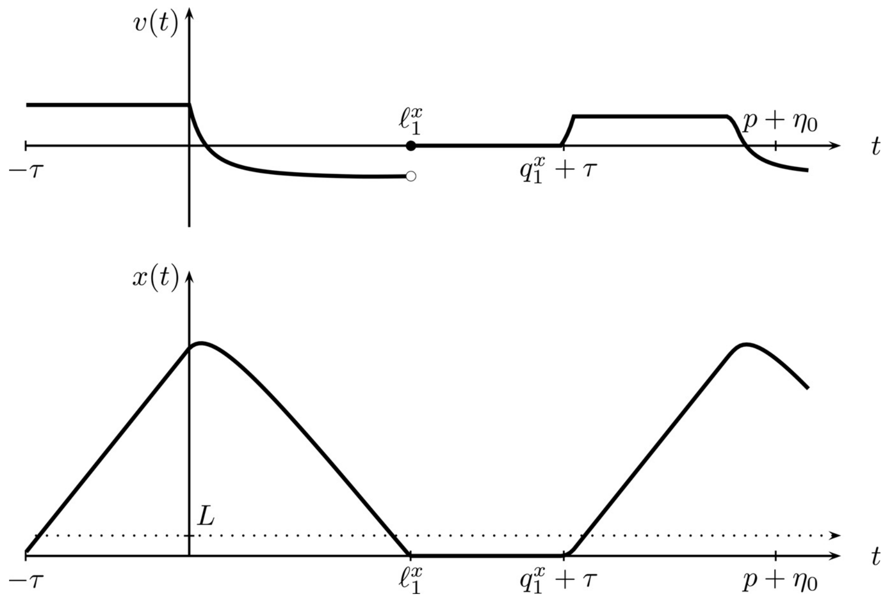


FIG. 7. *An example of a solution of the VE (on the top) along a solution of the DDER (on the bottom). Here* $h(x(t-\tau)) = \frac{\alpha C_0^2}{(C_0+x(t-\tau))^2} - \gamma$*, where* $\alpha > \gamma > 0$ *and* $C_0 > 0$ *as in Example* 4.2.

A solution $v$ of the VE along $x$ and with initial condition $v_0 \in \mathcal{C}_{[-\tau,0]}$ can be thought of as the direction that $x$ is perturbed in when its initial condition $x_0$ is perturbed in the direction $v_0$. In general, the element $v_0$ is constrained by the fact that the initial condition of a solution of the DDER cannot be perturbed in the negative direction when it is at the lower boundary. However, here we have $\|x_0 - x_0^*\|_{[-\tau,0]} < \varepsilon_0$, so by part (i) of Lemma 6.17, the initial condition $x_0(\cdot)$ is strictly positive on $[-\tau,0]$. Thus, we can take $v_0$ to be any element of $\mathcal{C}_{[-\tau,0]}$. Fix such a $v_0 \in \mathcal{C}_{[-\tau,0]}$. Then for all $\varepsilon > 0$ sufficiently small so that $x_0^\varepsilon = x_0 + \varepsilon v_0 \in \mathcal{C}^+_{[-\tau,0]}$, let $x^\varepsilon$ denote the unique solution of the DDER with initial condition $x_0^\varepsilon$ and define $v^\varepsilon \in \mathcal{C}_{[-\tau,\infty)}$ by

$$v^\varepsilon = \frac{x^\varepsilon - x}{\varepsilon}. \tag{6.54}$$

Let $z$ be as in (2.3) and let $z^\varepsilon$ be as in (2.3), but with $x^\varepsilon$ and $z^\varepsilon$ in place of $x$ and $z$, respectively. Define $w^\varepsilon \in \mathcal{C}_{[0,\infty)}$ by

$$w^\varepsilon = \frac{z^\varepsilon - z}{\varepsilon}. \tag{6.55}$$

The following proposition is a version of Theorem B.3 specific to this section. Recall that a family $\{u^\varepsilon : 0 \le \varepsilon \le \varepsilon^*\}$ in $\mathcal{C}_{[-\tau,\infty)}$ converges to $u \in \mathcal{D}_{[-\tau,\infty)}$ uniformly on compact intervals of continuity in $[-\tau,\infty)$ as $\varepsilon \downarrow 0$ provided that for each compact interval $I$ in $[-\tau,\infty)$ such that $u$ is continuous on $I$, $u^\varepsilon$ converges to $u$ uniformly on $I$ as $\varepsilon \downarrow 0$.

PROPOSITION 6.20. *Suppose $x$ is a solution of the DDER with initial condition satisfying $\|x_0 - x_0^*\|_{[-\tau,0]} < \varepsilon_0$ and $\psi$ is an element of $\mathcal{C}_{[-\tau,0]}$. Then there exists a unique solution $v$ of the VE along $x$ with $v_0 = \psi$. Furthermore, if $v^\varepsilon$ and $w^\varepsilon$ are defined as in* (6.54) *and* (6.55)*, respectively, then $v^\varepsilon$ converges to $v$ pointwise and* *uniformly on compact intervals of continuity in* $[-\tau,\infty)$ *as* $\varepsilon \downarrow 0$ *and* $w^\varepsilon$ *converges to* $w$ *uniformly on compact intervals in* $[0,\infty)$ *as* $\varepsilon \downarrow 0$.

In the following lemma we further describe solutions of the VE. The characterizations in (6.56) and (6.57) will be used later in this section to obtain bounds and to prove linearity properties for solutions of the VE.

LEMMA 6.21. *Given a solution* $x$ *of the DDER with initial condition satisfying* $\|x_0 - x_0^*\|_{[-\tau,0]} < \varepsilon_0$, *suppose* $v$ *is a solution of the VE along* $x$. *Then* $v$ *satisfies*

$$v(t) = \begin{cases} v(0) + \int_0^t h'(x(s-\tau))v(s-\tau)ds, & t \in [0,\ell_1^x), \\ \left(v(0) + \int_0^{\ell_1^x} h'(x(s-\tau))v(s-\tau)ds\right)^+, & t = \ell_1^x, \\ 0, & t \in (\ell_1^x, q_1^x+\tau], \\ \int_{q_1^x+\tau}^t h'(x(s-\tau))v(s-\tau)ds, & t \in (q_1^x+\tau, p+\eta_0]. \end{cases} \tag{6.56}$$

*If* $v^\dagger \in \mathcal{D}_{[-\tau,\infty)}$ *also satisfies* (6.56) *and* $v^\dagger(t) = v(t)$ *at almost every* $t \in [-\tau,0]$, *then* $v^\dagger(t) = v(t)$ *for all* $t \in [0,p+\eta_0]$. *Furthermore, if* $v$ *is a solution of the VE along the SOPS* $x^*$, *then* $v$ *satisfies*

$$v(kp+t) = \begin{cases} v(kp) + \int_{kp}^{kp+t} h'(x^*(s-\tau))v(s-\tau)ds, & t \in [0,\ell_1), \\ \left(v(kp) + \int_{kp}^{kp+\ell_1} h'(x^*(s-\tau))v(s-\tau)ds\right)^+, & t = \ell_1, \\ 0, & t \in (\ell_1, q_1+\tau], \\ \int_{kp+q_1+\tau}^{kp+t} h'(x^*(s-\tau))v(s-\tau)ds, & t \in (q_1+\tau, p], \end{cases} \tag{6.57}$$

*for each* $k = 0,1,\dots$. *If* $v^\dagger \in \mathcal{D}_{[-\tau,\infty)}$ *also satisfies* (6.57) *and* $v^\dagger(t) = v(t)$ *at almost every* $t \in [-\tau,0]$, *then* $v^\dagger(t) = v(t)$ *for all* $t \geq 0$.

*Proof.* Fix a solution $x$ of the DDER as in the statement of the lemma and let $v$ be a solution of the VE along $x$. Then (6.56) follows from Lemma 6.17 and parts (i), (ii), and (iv) of Lemma B.6, with $\partial_{v_s} f(x_s) = h'(x(s-\tau))v(s-\tau)$ by Example B.1. Suppose that $v^\dagger \in \mathcal{D}_{[-\tau,\infty)}$ satisfies (6.56) and $v^\dagger(t) = v(t)$ at almost every $t \in [-\tau,0]$. The continuity of $h'$ then implies that $v^\dagger(t) = v(t)$ at each $t \in [0,\tau \wedge \ell_1^x)$. By iterating this argument on intervals of length $\tau$, we see that $v^\dagger$ and $v$ are equal on $[0,\ell_1^x)$. The fact that $v^\dagger(\ell_1^x) = v(\ell_1^x)$ follows because $v^\dagger$ and $v$ are equal and continuous on $[0,\ell_1^x)$. Then, since $v^\dagger = v$ on $[q_1^x, q_1^x+\tau]$, we can again iterate on intervals of length $\tau$ to obtain that $v^\dagger$ and $v$ are equal on the interval $(q_1^x+\tau, p+\eta_0]$. The proof of the second part of the lemma follows from iterating the verification of (6.56) (with $\eta_0 = 0$) on intervals of length $p$. ∎

The following lemma is an immediate consequence of the above lemma and the linearity of integration with respect to the integral.

LEMMA 6.22. *Given a solution* $x$ *of the DDER with initial condition satisfying* $\|x_0 - x_0^*\|_{[-\tau,0]} < \varepsilon_0$, *suppose* $v$ *and* $v^\dagger$ *are solutions of the VE along* $x$. *For* $a,b \in \mathbb{R}$, *let* $v^\ddagger$ *denote the unique solution to the VE along* $x$ *with initial condition* $v_0^\ddagger = av_0 + bv_0^\dagger$. *Then*

$$v^\ddagger(t) = av(t) + bv^\dagger(t) \text{ for all } t \in [-\tau,p+\eta_0] \setminus \{\ell_1^x\}. \tag{6.58}$$

For the following lemma, let $\dot{x}^* \in \mathcal{D}_{[-\tau,\infty)}$ be the periodic function, with period $p$, defined for $t \geq 0$ by

$$\dot{x}^*(t) = \begin{cases} h(x^*(t-\tau)) & \text{if } x^*(t) > 0, \\ 0 & \text{if } x^*(t) = 0. \end{cases} \tag{6.59}$$

Then $\dot{x}^*$ is equal to the derivative of $x^*$ at almost every $t \in [-\tau, \infty)$ that $x^*$ is differentiable and (2.7) holds with $x^*$ and $\dot{x}^*$ in place of $x$ and $\dot{x}$, respectively.

LEMMA 6.23. *The function $\dot{x}^*$ is a solution of the VE along $x^*$.*

*Proof.* By Proposition 6.20, there exists a unique solution of the VE along $x^*$ with initial condition $\dot{x}^*_0$. In light of the last line in Lemma 6.21, it suffices to show that $\dot{x}^*$ satisfies (6.57) for all $t \geq 0$. Due to (6.59), the fact that $x^*$ is positive and continuously differentiable on $[0, \ell_1)$, and the fundamental theorem of calculus, it follows that $\dot{x}^*$ satisfies (6.57) on $[0, \ell_1)$. By (6.59), $\dot{x}^*(\ell_1) = 0$. On the other hand, by the fundamental theorem of calculus, the negative feedback condition on $h$, and the fact that $x^*(\ell_1 - \tau) > L$, we have

$$\left( \dot{x}^*(0) + \int_0^{\ell_1} h'(x^*(s-\tau))\dot{x}^*(s-\tau)ds \right)^+ = (h(x^*(\ell_1 - \tau)))^+ = 0,$$

so $\dot{x}^*$ satisfies (6.57) at $t = \ell_1$. By (6.59), $\dot{x}^*$ is zero on $(\ell_1, q_1 + \tau]$, so $\dot{x}^*$ satisfies (6.57) there. The fact that $\dot{x}^*$ satisfies (6.57) on $(q_1 + \tau, p]$ follows from the fundamental theorem of calculus and the fact that $x^*(\cdot)$ is continuously differentiable on $(\ell_1, p]$. Therefore $\dot{x}^*$ satisfies (6.57) on $[0, p]$. By iterating the above argument on intervals of length $p$, we obtain that $\dot{x}^*$ satisfies (6.57) for all $t \geq 0$, completing the proof. ∎

LEMMA 6.24. *There exists a positive constant $M < \infty$, which depends only on $h$ and $L$, such that whenever $v$ is a solution of the VE along $x^*$, the following bound holds:*

$$\|v\|_{[-\tau,p]} \leq M \|v\|_{[-\tau,0]}. \tag{6.60}$$

*Proof.* Recall that $\bar{q} = \alpha^{-1}\beta$ and the definition of $\bar{x}$ given in (6.27). Define positive constants $\delta = \frac{1}{2}(\bar{q} \wedge 1)$ and $d = \frac{1}{2}\min(\bar{x}(-1+\delta), \bar{x}(\bar{q} - \delta))$. By Remark 6.1 and Lemma 5.5,

(i) $0 < \tau\delta < q_1 < \ell_1 < \tau(\bar{q} + \delta) < \tau(\bar{q} + 1 - \delta) < q_2 + \tau\delta < q_1 + 2\tau$; and

(ii) $x^*(t) > L + \tau d$ for all $t \in [\tau(-1+\delta), \tau(\bar{q} - \delta)]$.

Let $v$ be a solution of the VE along $x^*$. We first bound $v(t)$ for $t \in [-\tau, \tau\delta]$. By (6.57) and the periodicity of $x^*$, we have, for $t \in [0, \tau\delta]$,

$$\begin{aligned} |v(t)| &\leq |v(0)| + \int_0^{\tau\delta} |h'(x^*(s-\tau))v(s-\tau)|ds \\ &\leq \|v\|_{[-\tau,0]} \left( 1 + \int_{q_2}^{q_2 \vee (\ell_1+\tau)} |h'(x^*(s))|ds + \int_{(\ell_1+\tau)\wedge(q_2+\tau\delta)}^{q_2+\tau\delta} |h'(x^*(s))|ds \right). \end{aligned}$$

If $\ell_1 + \tau < q_2 + \tau\delta$, then according to (i), we have $\ell_1 + \tau < q_2 + \tau\delta < q_1 + 2\tau$, so $x^*(t-\tau) = 0$ for all $t \in [\ell_1 + \tau, q_2 + \tau\delta]$. In this case, Lemma 2.3 and $h(0) = \beta$ imply that $x^*(t) = x^*(\ell_1 + \tau) + \beta(t - \ell_1 - \tau)$ for all $t \in [\ell_1 + \tau, q_2 + \tau\delta]$. By Lemma 3.6, the fact that $q_1 + \tau < q_2$, (6.41), and the $L^1$-integrability of $h'$ (see Assumption 3.4), it follows that in either case ($\ell_1 + \tau < q_2 + \tau\delta$ or $\ell_1 + \tau \geq q_2 + \tau\delta$) the following bound holds:

(6.61)

$$\begin{aligned}
&\|v\|_{[-\tau,\tau\delta]} \\
&\quad\le \|v\|_{[-\tau,0]}\left(1+K_h(\ell_1+\tau-q_2)^+ + \int_0^{(q_2+\tau\delta-\ell_1-\tau)^+} |h'(x^*(\ell_1+\tau)+\beta s)|ds\right)\\
&\quad\le \|v\|_{[-\tau,0]}\left(1+\frac{2LK_h}{\alpha}+\frac{1}{\beta}\|h'\|_{L^1(\mathbb{R}_+)}\right).
\end{aligned}$$

Proceeding, we consider the interval $[\tau\delta,\ell_1]$. By (6.57), the fact (gleaned from (i) and (ii)) that $x^*(t) > L+\tau d$ for all $t\in[-\tau+\tau\delta,\ell_1-\tau]\subset[\tau(-1+\delta),\tau(\bar{q}-\delta)]$, and because $m=\sup_{s\in\mathbb{R}_+}|sh'(s)|$ is finite (see Assumption 3.4), we have, for $t\in[\tau\delta,\ell_1]$,

$$\begin{aligned}
|v(t)| &\le |v(\tau\delta)| + \int_{\tau\delta}^t \frac{|h'(x^*(s-\tau))x^*(s-\tau)|}{|x^*(s-\tau)|}|v(s-\tau)|ds\\
&\le |v(\tau\delta)| + \frac{m}{L+\tau d}\left[\int_{-\tau+\tau\delta}^{(t-\tau)\wedge\tau\delta}|v(s)|ds + \int_{(t-\tau)\vee\tau\delta}^{t-\tau}|v(s)|ds\right]\\
&\le \|v\|_{[-\tau+\tau\delta,\tau\delta]}\left(1+\frac{\tau m}{L+\tau d}\right) + \frac{m}{L+\tau d}\int_{\tau\delta}^t |v(s)|ds.
\end{aligned}$$

An application of Gronwall's inequality on the interval $[\tau\delta,\ell_1]$ yields, for $t\in[\tau\delta,\ell_1]$,

$$|v(t)| \le \left(1+\frac{m}{d}\right)\exp\left(\frac{m(\ell_1-\tau\delta)}{\tau d}\right)\|v\|_{[-\tau,\tau\delta]}.$$

Taking the supremum over $t\in[-\tau,\ell_1]$ and using the facts that $\ell_1-\tau\delta\le\tau\bar{q}=\tau\alpha^{-1}\beta$ by (i) and $v$ is identically zero on $(\ell_1,q_1+\tau]$ by (6.57), we have

$$\|v\|_{[-\tau,q_1+\tau]} \le \left(1+\frac{m}{d}\right)\exp\left(\frac{m\beta}{d\alpha}\right)\|v\|_{[-\tau,\tau\delta]}. \tag{6.62}$$

Continuing, we bound $v(t)$ for $t\in[q_1+\tau,q_1+2\tau]$. By (6.57) and (6.41), we have, for $t\in[q_1+\tau,\ell_1+\tau]$,

$$|v(t)| \le \int_{q_1+\tau}^t |h'(x(s-\tau))v(s-\tau)|ds \le \frac{2LK_h}{\alpha}\|v\|_{[q_1,\ell_1]}.$$

Since $v(t-\tau)=0$ for all $t\in(\ell_1+\tau,q_1+2\tau]$, (6.57) implies that $v$ is constant on $[\ell_1+\tau,q_1+2\tau]$. Thus, taking supremums over $t\in[-\tau,q_1+2\tau]$, we have

$$\|v\|_{[-\tau,q_1+2\tau]} \le \|v\|_{[-\tau,q_1+\tau]}\vee\left(\frac{2LK_h}{\alpha}\|v\|_{[q_1,\ell_1]}\right) \le \left(\frac{2LK_h}{\alpha}\vee 1\right)\|v\|_{[-\tau,q_1+\tau]}. \tag{6.63}$$

Next, we bound $v(t)$ for $t\in[q_1+2\tau,(\ell_1+2\tau)\wedge p]$. By (6.57), (3.12), (6.41), and (6.63), we have, for $t\in[q_1+2\tau,(\ell_1+2\tau)\wedge p]$,

$$\begin{aligned}
|v(t)| &\le |v(q_1+2\tau)| + \int_{q_1+2\tau}^{(\ell_1+2\tau)\wedge p}|h'(x^*(s-\tau))v(s-\tau)|ds\\
&\le |v(q_1+2\tau)| + \frac{2LK_h}{\alpha}\|v\|_{[q_1+\tau,\ell_1+\tau]}.
\end{aligned}$$

Taking the supremum over $t \in [-\tau, (\ell_1 + 2\tau) \wedge p]$ yields

$$\|v\|_{[-\tau,(\ell_1+2\tau)\wedge p]} \leq \left(1 + \frac{2LK_h}{\alpha}\right) \|v\|_{[-\tau,q_1+2\tau]}. \tag{6.64}$$

Last, in the case that $\ell_1 + 2\tau < p$, because $q_2 < q_1 + 2\tau$ by (i), and $p = q_2 + \tau$, it follows that $x^*(t - \tau) = 0$ for all $t \in [\ell_1 + \tau, p - \tau]$ and $p < \ell_1 + 3\tau$. Thus, by Lemmas 2.3 and 6.15, $x^*(t) = x^*(\ell_1 + \tau) + \beta(t - \ell_1 - \tau)$ for all $t \in [\ell_1 + \tau, p - \tau]$. Substituting into (6.57) and using the $L^1$-integrability of $h'$ (see Assumption 3.4), we have, for $t \in [\ell_1 + 2\tau, p]$,

$$\begin{aligned}|v(t)| &\leq |v(\ell_1 + 2\tau)| + \|v\|_{[\ell_1+\tau,q_2]} \int_{\ell_1+2\tau}^{t} |h'(x^*(\ell_1 + \tau) + \beta(s - \ell_1 - \tau))| ds \\ &\leq |v(\ell_1 + 2\tau)| + \frac{1}{\beta}\|h'\|_{L^1(\mathbb{R}_+)}\|v\|_{[\ell_1+\tau,q_2]}.\end{aligned}$$

Taking the supremum over $t \in [\ell_1 + 2\tau, p]$ and using (6.61)–(6.64), we obtain

$$\|v\|_{[-\tau,p]} \leq \left(1 + \frac{1}{\beta}\|h'\|_{L^1(\mathbb{R}_+)}\right) \|v\|_{[-\tau,(\ell_1+2\tau)\wedge p]} \leq M\|v\|_{[-\tau,0]},$$

where

$$\begin{aligned}M = &\left(1 + \frac{1}{\beta}\|h\|_{L^1(\mathbb{R}_+)}\right)\left(1 + \frac{2LK_h}{\alpha}\right)\left(\frac{2LK_h}{\alpha} \vee 1\right) \\ &\times \left(1 + \frac{m}{d}\right) \exp\left(\frac{m\beta}{d\alpha}\right)\left(1 + \frac{2LK_h}{\alpha} + \frac{1}{\beta}\|h'\|_{L^1(\mathbb{R}_+)}\right). \qquad \square\end{aligned}$$

LEMMA 6.25. *Let $x$ be a solution of the DDER satisfying $\|x - x^*\|_{[-\tau,0]} < \varepsilon_0$. For each $\delta > 0$, there exists $\varepsilon > 0$ such that whenever $x^\dagger$ is a solution of the DDER such that $\|x_0^\dagger - x_0\|_{[-\tau,0]} < \varepsilon$, and $\psi \in \mathcal{C}_{[-\tau,0]}$ satisfies $\|\psi\|_{[-\tau,0]} \leq 1$, then*

$$\|v^x - v^{x^\dagger}\|_{[p-\tau-\eta_0,p+\eta_0]} < \delta, \tag{6.65}$$

*where $v^x$ and $v^{x^\dagger}$ denote the unique solutions of the VE along $x$ and $x^\dagger$, respectively, with $v_0^x = v_0^{x^\dagger} = \psi$.*

*Proof.* Fix $\delta > 0$. Define $\varepsilon_1 = \varepsilon_1(x) > 0$ and $\eta_1 = \eta_1(x) > 0$ as in (6.46) and (6.47). For a solution $x^\dagger$ of the DDER and $t \geq 0$, define

$$d_{h,x}(x^\dagger, t) = \sup_{s\in[-\tau,t]} |h'(x(s)) - h'(x^\dagger(s))|. \tag{6.66}$$

Then $d_{h,x}(x^\dagger, \cdot)$ is a nondecreasing function and by (6.42) and the continuity of $h'$, for fixed $t \geq -\tau$, we have $d_{h,x}(x^\dagger, t) \to 0$ as $x_0^\dagger \to x_0$ in $\mathcal{C}^+_{[-\tau,0]}$. Choose $\eta \in (0, \eta_1)$ such that

$$\eta < \frac{\delta}{12K_h \exp(3K_h(p + \eta_0))}. \tag{6.67}$$

Given $\eta$, we can choose $\varepsilon \in (0, \varepsilon_1)$ such that the conclusion of Lemma 6.18 holds and

$$d_{h,x}(x^\dagger, q_2 + \eta_0) < \frac{\delta}{4(q_2 - q_1 + 2\eta_0)(1 + K_h(\ell_1 + \eta_0)) \exp(4K_h(p + \eta_0))} \tag{6.68}$$

holds whenever $\|x_0^\dagger - x_0\|_{[-\tau,0]} < \varepsilon$.

Let $x^\dagger$ be a solution of the DDER such that $\|x_0^\dagger - x_0\|_{[-\tau,0]} < \varepsilon$. Define $z$ as in (2.3) and define $z^\dagger$ as in (2.3), but with $x^\dagger$ and $z^\dagger$ in place of $x$ and $z$, respectively. Suppose $\psi \in \mathcal{C}_{[-\tau,0]}$ satisfies $\|\psi\|_{[-\tau,0]} \le 1$ and let $v^x$ and $v^{x^\dagger}$ denote the unique solutions of the VE along $x$ and $x^\dagger$, respectively, with $v_0^x = v_0^{x^\dagger} = \psi$. Define $w^x$ as in (6.50), but with $v^x$ and $w^x$ in place of $v$ and $w$, respectively, and define $w^{x^\dagger}$ as in (6.50), but with $x^\dagger$, $v^{x^\dagger}$, and $w^{x^\dagger}$ in place of $x$, $v$, and $w$, respectively. By (B.11), with $v = v^x$, $v^\dagger \equiv 0$ and with $v = v^{x^\dagger}$, $v^\dagger \equiv 0$, we have, for $t \ge 0$,

$$\|v^x\|_{[-\tau,t]} \le 2\exp(2K_h t) \qquad \text{and} \qquad \|v^{x^\dagger}\|_{[-\tau,t]} \le 2\exp(2K_h t), \tag{6.69}$$

respectively, where we have used the fact that $\|\psi\|_{[-\tau,0]} \le 1$. Last, define the points $q_1^x, q_1^{x^\dagger} \in (q_1 - \eta, q_1 + \eta)$, $\ell_1^x, \ell_1^{x^\dagger} \in (\ell_1 - \eta, \ell_1 + \eta)$, and $q_2^x, q_2^{x^\dagger} \in (q_2 - \eta, q_2 + \eta)$ as in Lemmas 6.17 and 6.18.

Here we assume that $q_1^x \le q_1^{x^\dagger}$. The proof for the case that $q_1^x \ge q_1^{x^\dagger}$ is similar and is omitted. First consider the interval $[0, \ell_1^x \wedge \ell_1^{x^\dagger}]$. By (6.56), (6.66), (6.69), (3.12), and the fact that $v_0^x = v_0^{x^\dagger} = \psi$, we have, for $t \in [0, \ell_1^x \wedge \ell_1^{x^\dagger}]$,

$$\begin{aligned} |v^x(t) - v^{x^\dagger}(t)| &\le \int_{-\tau}^{t-\tau} |h'(x(s)) - h'(x^\dagger(s))||v^x(s)|ds \\ &\quad + \int_0^{t-\tau} |h'(x^\dagger(s))||v^x(s) - v^{x^\dagger}(s)|ds \\ &\le d_{h,x}(x^\dagger, t) 2t \exp(2K_h t) + K_h \int_0^t |v^x(s) - v^{x^\dagger}(s)|ds. \end{aligned}$$

An application of Gronwall's inequality yields

$$|v^x(t) - v^{x^\dagger}(t)| \le d_{h,x}(x^\dagger, t) 2t \exp(3K_h t), \qquad t \in [0, \ell_1^x \wedge \ell_1^{x^\dagger}]. \tag{6.70}$$

Next, it follows from (6.56) that either $v^x$ or $v^{x^\dagger}$ is zero on the interval $(\ell_1^x \wedge \ell_1^{x^\dagger}, \ell_1^x \vee \ell_1^{x^\dagger}]$. Therefore, by (6.69), for $t \in (\ell_1^x \wedge \ell_1^{x^\dagger}, \ell_1^x \vee \ell_1^{x^\dagger}]$,

$$|v^x(t) - v^{x^\dagger}(t)| = |v^x(t)| \vee |v^{x^\dagger}(t)| \le 2\exp(2K_h(\ell_1 + \eta_0)). \tag{6.71}$$

By (6.56) and the fact that $q_1^x \le q_1^{x^\dagger}$, we have $v^x$ and $v^{x^\dagger}$ are both zero on $(\ell_1^x \vee \ell_1^{x^\dagger}, q_1^x + \tau]$ and $v^{x^\dagger}$ is zero on the interval $[q_1^x + \tau, q_1^{x^\dagger} + \tau]$. Thus, by (6.69), for $t \in [q_1^x + \tau, q_1^{x^\dagger} + \tau]$,

$$|v^x(t) - v^{x^\dagger}(t)| = |v^x(t)| \le 2\exp(2K_h(q_1 + \eta_0 + \tau)). \tag{6.72}$$

Consider the interval $(q_1^{x^\dagger} + \tau, p + \eta_0]$. By (6.56) and (6.69), we have, for $t \in (q_1^{x^\dagger} + \tau, p + \eta_0]$,

$$\begin{aligned} |v^x(t) - v^{x^\dagger}(t)| &\le |v^x(q_1^{x^\dagger} + \tau)| + \int_{q_1^{x^\dagger}}^{t-\tau} |h'(x(s)) - h'(x^\dagger(s))||v^x(s)|ds \\ &\quad + \int_{q_1^{x^\dagger}}^{t-\tau} |h'(x^\dagger(s))||v^x(s) - v^{x^\dagger}(s)|ds. \end{aligned} \tag{6.73}$$

Since $v^x(q_1^x+\tau)=0$ and $q_1^{x^\dagger}\in(q_1^x-\eta,q_1^x+\eta)\subset(q_1-\eta_0,q_1+\eta_0)$, the first term on the right-hand side of the inequality (6.73) satisfies

$$|v^x(q_1^{x^\dagger}+\tau)|\le K_h\int_{q_1^x}^{q_1^{x^\dagger}}|v^x(s)|ds\le 2\eta K_h\exp(2K_h(q_1+\eta_0)). \tag{6.74}$$

For the second term on the right-hand side of the inequality (6.73), we have, for $t\in[q_1^{x^\dagger}+\tau,p+\eta_0]$,

$$\begin{aligned}&\int_{q_1^{x^\dagger}}^{t-\tau}|h'(x(s))-h'(x^\dagger(s))||v^x(s)|ds\\ &\qquad\le 2(q_2-q_1+2\eta_0)d_{h,x}(x^\dagger,q_2+\eta_0)\exp(2K_h(q_2+\eta_0)),\end{aligned} \tag{6.75}$$

where we have used (6.66), (6.69), and the facts that $p=q_2+\tau$, $|p-q_1^{x^\dagger}-\tau+\eta_0|\le|q_2-q_1+2\eta_0|$. For the last term on the right-hand side of the inequality (6.73), we have, for $t\in[q_1^{x^\dagger}+\tau,p+\eta_0]$,

$$\begin{aligned}&\int_{q_1^{x^\dagger}}^{t-\tau}|h'(x^\dagger(s))||v^x(s)-v^{x^\dagger}(s)|ds\\ &\quad\le K_h\int_{q_1^{x^\dagger}}^{\ell_1^x\wedge\ell_1^{x^\dagger}}|v^x(s)-v^{x^\dagger}(s)|ds+K_h\int_{\ell_1^x\wedge\ell_1^{x^\dagger}}^{\ell_1^x\vee\ell_1^{x^\dagger}}|v^x(s)-v^{x^\dagger}(s)|ds\\ &\qquad+K_h\int_{q_1^x+\tau}^{q_1^{x^\dagger}+\tau}|v^x(s)|ds+K_h\int_{q_1^{x^\dagger}+\tau}^{t}|v^x(s)-v^{x^\dagger}(s)|ds\\ &\quad\le 2K_h(\ell_1-q_1+2\eta_0)d_{h,x}(x^\dagger,\ell_1+\eta_0)(\ell_1+\eta_0)\exp(3K_h(\ell_1+\eta_0))\\ &\qquad+4\eta K_h\exp(2K_h(q_1+\eta_0+\tau))+K_h\int_{q_1^{x^\dagger}+\tau}^{t}|v^x(s)-v^{x^\dagger}(s)|ds,\end{aligned} \tag{6.76}$$

where we have used (6.70)–(6.72). Combining (6.73)–(6.76) yields, for $t\in[q_1^{x^\dagger}+\tau,p+\eta_0]$,

$$\begin{aligned}|v^x(t)-v^{x^\dagger}(t)|&\le 6\eta K_h\exp(2K_h(q_1+\tau+\eta_0))\\ &\quad+2(q_2-q_1+2\eta_0)d_{h,x}(x^\dagger,q_2+\eta_0)\exp(2K_h(q_2+\eta_0))\\ &\quad+2K_h(\ell_1-q_1+2\eta_0)(\ell_1+\eta_0)d_{h,x}(x^\dagger,\ell_1+\eta_0)\exp(3K_h(\ell_1+\eta_0))\\ &\quad+K_h\int_{q_1^{x^\dagger}+\tau}^{t}|v^x(s)-v^{x^\dagger}(s)|ds.\end{aligned}$$

Applying Gronwall's inequality and substituting from (6.67)–(6.68) results in the desired bound:

$$|v^x(t)-v^{x^\dagger}(t)|<\delta,\qquad t\in[q_1^{x^\dagger}+\tau,p+\eta_0].$$

The lemma then follows from the fact that $[p-\tau-\eta_0,p+\eta_0]\subset[q_1^{x^\dagger}+\tau,p+\eta_0]$. ∎

**6.5. Semiflow and Poincaré-type map.** In this section we define a semiflow and a variant of a Poincaré map that will be used in the next section to prove the exponential stability of a SOPS. Throughout this section we fix $\tau \geq \tau^{(4)}$, where $\tau^{(4)} \geq \tau_0$ is as in Lemma 6.14 and the properties in Remark 6.1 hold.

Define the semiflow $\Sigma : \mathbb{R}_+ \times \mathcal{C}^+_{[-\tau,0]} \to \mathcal{C}^+_{[-\tau,0]}$ by

$$\Sigma(t,\varphi) = x_t, \tag{6.77}$$

where $x$ denotes the unique solution of the DDER with $x_0 = \varphi$. Since the DDER is autonomous and solutions to the DDER are unique, we have, for $s, t \in \mathbb{R}_+$ and $\varphi \in \mathcal{C}^+_{[-\tau,0]}$,

$$\Sigma(t, \Sigma(s,\varphi)) = \Sigma(t, x_s) = x_{s+t} = \Sigma(s+t, \varphi). \tag{6.78}$$

LEMMA 6.26. *The semiflow* $\Sigma : \mathbb{R}_+ \times \mathcal{C}^+_{[-\tau,0]} \to \mathcal{C}^+_{[-\tau,0]}$ *is continuous.*

*Proof.* The lemma follows from (6.42) and because solutions of the DDER are continuous. ∎

Let $x^*$ denote a SOPS with delay $\tau$ and $q_0 = -\tau$. Fix an $\eta_0 > 0$ and associated $\varepsilon_0 \in (0, \eta_0)$ as in Lemma 6.17. Define the neighborhood $\mathcal{U}$ of $(p, x_0^*)$ in the Banach space $\mathbb{R} \times \mathcal{C}_{[-\tau,0]}$ (with norm $\|(t,\varphi)\| = |t| \vee \|\varphi\|_{[-\tau,0]}$) by

$$\mathcal{U} = \left\{(t,\varphi) \in \mathbb{R} \times \mathcal{C}_{[-\tau,0]} : \|(t,\varphi) - (p, x_0^*)\| < \varepsilon_0\right\}. \tag{6.79}$$

Note that Lemma 6.17 implies $\mathcal{U} \subset \mathbb{R}_+ \times \mathcal{C}^+_{[-\tau,0]}$ and, moreover, if $(t,\varphi) \in \mathcal{U}$, then $\varphi(s) > 0$ for all $s \in [-\tau, 0]$ and $x_t(s) = \Sigma(t,\varphi)(s) > 0$ for all $s \in [-\tau, 0]$. Hence, by (2.6), $\dot{x}_t \in \mathcal{C}_{[-\tau,0]}$, where $\dot{x} \in \mathcal{D}_{[-\tau,\infty)}$ is defined as in (2.6). For the following lemma, recall that given vector spaces $X$ and $Y$, $\mathcal{L}(X, Y)$ denotes the vector space of bounded linear operators from $X$ into $Y$.

LEMMA 6.27. *The semiflow* $\Sigma$ *is continuously Fréchet differentiable on* $\mathcal{U}$ *and for each* $(t,\varphi) \in \mathcal{U}$*, the derivative* $D\Sigma(t,\varphi) \in \mathcal{L}(\mathbb{R} \times \mathcal{C}_{[-\tau,0]}, \mathcal{C}_{[-\tau,0]})$ *is given by*

$$D\Sigma(t,\varphi)(s,\psi) = s\dot{x}_t + v_t \qquad \text{for all } (s,\psi) \in \mathbb{R} \times \mathcal{C}_{[-\tau,0]}, \tag{6.80}$$

*where* $x$ *denotes the unique solution of the DDER with* $x_0 = \varphi$*,* $\dot{x}$ *is defined as in* (2.6)*, and* $v$ *denotes the unique solution of the VE along* $x$ *with* $v_0 = \psi$*.*

*Proof.* For each $(t,\varphi) \in \mathcal{U}$, define the operator $F(t,\varphi) : \mathbb{R} \times \mathcal{C}_{[-\tau,0]} \to \mathcal{C}_{[-\tau,0]}$ by

$$F(t,\varphi)(s,\psi) = s\dot{x}_t + v_t, \qquad (s,\psi) \in \mathbb{R} \times \mathcal{C}_{[-\tau,0]}. \tag{6.81}$$

It suffices to show (see, e.g., Theorem 1.3 in [9]) that (i) the function $(t,\varphi) \to F(t,\varphi)$ is continuous as a mapping from $\mathcal{U}$ into $\mathcal{L}(\mathbb{R} \times \mathcal{C}_{[-\tau,0]}, \mathcal{C}_{[-\tau,0]})$ and (ii) for each $(t,\varphi) \in \mathcal{U}$ and $(s,\psi) \in \mathbb{R} \times \mathcal{C}_{[-\tau,0]}$,

$$F(t,\varphi)(s,\psi) = \lim_{\varepsilon \downarrow 0} \varepsilon^{-1}\{\Sigma(t + \varepsilon s, \varphi + \varepsilon\psi) - \Sigma(t,\varphi)\},$$

where the limit is taken in $\mathcal{C}_{[-\tau,0]}$.

We first show that (i) holds. Fix $(t,\varphi) \in \mathcal{U}$ and let $x$ be the unique solution of the DDER with $x_0 = \varphi$. The linearity of $F(t,\varphi)$ follows from Lemma 6.22 and the fact that $t - \tau > p - \eta_0 - \tau > \ell_1 + \eta_0 > \ell_1^x$. This, along with (3.10) and (B.11) (with $v^\dagger \equiv 0$), establishes that $F(t,\varphi)$ is a bounded linear operator. We now show that $(t,\varphi) \to F(t,\varphi)$ is continuous. Let $\delta > 0$. By Lemma 6.25, we can choose $\varepsilon^\dagger > 0$ such that if $x^\dagger$ is another solution of the DDER satisfying $\|x - x^\dagger\|_{[-\tau,0]} < \varepsilon^\dagger$ and $\psi \in \mathcal{C}_{[-\tau,0]}$ satisfies $\|\psi\|_{[-\tau,0]} \leq 1$, then

$$\|v^x - v^{x^\dagger}\|_{[p-\tau-\eta_0, p+\eta_0]} < \delta/3, \tag{6.82}$$

where $v^x$ and $v^{x^\dagger}$ denote the unique solutions of the VE along $x$ and $x^\dagger$, respectively, with $v_0^x = v_0^{x^\dagger} = \psi$. By choosing a possibly smaller $\varepsilon^\dagger > 0$, we can assume that

$$\varepsilon^\dagger < \frac{\delta}{3K_h[H + 2\exp(2K_h(q_2 + \eta_0))]}. \tag{6.83}$$

Suppose $(t^\dagger, \varphi^\dagger) \in \mathcal{U}$ satisfies $\|(t,\varphi) - (t^\dagger, \varphi^\dagger)\| < \varepsilon^\dagger$ and $(s,\psi) \in \mathbb{R} \times \mathcal{C}_{[-\tau,0]}$ satisfies $\|(s,\psi)\| = 1$. Let $x^\dagger$ denote the unique solution of the DDER with $x_0^\dagger = \varphi^\dagger$. Define $\dot{x}$ as in (2.6) and define $\dot{x}^\dagger$ as in (2.6), but with $\dot{x}^\dagger$ and $x^\dagger$ in place of $\dot{x}$ and $x$, respectively. By (6.81),

$$\|F(t,\varphi)(s,\psi) - F(t^\dagger,\varphi^\dagger)(s,\psi)\|_{[-\tau,0]} \le \|\dot{x}_t - \dot{x}^\dagger_{t^\dagger}\|_{[-\tau,0]} + \|v_t^x - v_{t^\dagger}^{x^\dagger}\|_{[-\tau,0]}. \tag{6.84}$$

Consider the first term on the right-hand side of (6.84). Using (2.6), with $f(x_t) = h(x(t-\tau))$, (3.12), (3.10), (6.42), and (6.83), we have

$$\begin{aligned}
\|\dot{x}_t - \dot{x}^\dagger_{t^\dagger}\|_{[-\tau,0]} &\le \sup_{u\in[-\tau,0]} |h(x(t-\tau+u)) - h(x^\dagger(t^\dagger - \tau + u))| \\
&\le K_h \left[ \|x_{t-\tau} - x_{t^\dagger - \tau}\|_{[-\tau,0]} + \|x_{t^\dagger-\tau} - x^\dagger_{t^\dagger - \tau}\|_{[-\tau,0]} \right] \\
&\le K_h \left[ H|t - t^\dagger| + 2\exp(2K_h(t^\dagger - \tau))\|\varphi - \varphi^\dagger\|_{[-\tau,0]} \right] \\
&\le \delta/3.
\end{aligned} \tag{6.85}$$

Now consider the second term on the right-hand side of (6.84). By (6.82), (6.56), (3.12), (B.11) (with $v = v^{x^\dagger}$, $v^\dagger = 0$, and $K_f = K_h$), and (6.83), we have

$$\begin{aligned}
\|v_t^x - v_{t^\dagger}^{x^\dagger}\|_{[-\tau,0]} &\le \|v_t^x - v_t^{x^\dagger}\|_{[-\tau,0]} + \|v_t^{x^\dagger} - v_{t^\dagger}^{x^\dagger}\|_{[-\tau,0]} \\
&\le \delta/3 + \sup_{u\in[-\tau,0]} \int_{t+u}^{t^\dagger + u} |h'(x^\dagger(s-\tau)) v^{x^\dagger}(s-\tau)| ds \\
&\le \delta/3 + 2K_h \exp(2K_h(q_2 + \eta_0))|t - t^\dagger| \\
&\le 2\delta/3.
\end{aligned} \tag{6.86}$$

Combining (6.84)–(6.86) and recalling that $(s,\psi)$ was only subject to the constraint $\|(s,\psi)\| = 1$, we see that $\|F(t,\varphi) - F(t^\dagger,\varphi^\dagger)\| < \delta$, so $F$ is continuous at $(t,\varphi)$. Since $(t,\varphi) \in \mathcal{U}$ was arbitrary, $(t,\varphi) \to F(t,\varphi)$ is continuous on $\mathcal{U}$ and so (i) holds.

We now show that (ii) holds. Fix $(t,\varphi) \in \mathcal{U}$ and $(s,\psi) \in \mathbb{R} \times \mathcal{C}_{[-\tau,0]}$. Let $x$ be the unique solution of the DDER with $x_0 = \varphi$ and let $v$ be the unique solution of the VE along $x$ with $v_0 = \psi$. Clearly, $(t + \varepsilon s, \varphi + \varepsilon\psi) \in \mathcal{U}$ for $\varepsilon$ sufficiently small. For such $\varepsilon > 0$, let $x^\varepsilon$ denote the unique solution of the DDER with $x_0^\varepsilon = \varphi + \varepsilon\psi$ and define $v^\varepsilon = \varepsilon^{-1}(x^\varepsilon - x) \in \mathcal{C}_{[-\tau,\infty)}$. Then, by the triangle inequality,

$$\begin{aligned}
&\left\| \varepsilon^{-1} \{ \Sigma(t + \varepsilon s, \varphi + \varepsilon\psi) - \Sigma(t,\varphi) \} - F(t,\varphi)(s,\psi) \right\|_{[-\tau,0]} \\
&\quad \le \varepsilon^{-1} \left\| x^\varepsilon_{t+\varepsilon s} - x_t^\varepsilon - \varepsilon s \dot{x}_t \right\|_{[-\tau,0]} + \|v_t^\varepsilon - v_t\|_{[-\tau,0]}.
\end{aligned} \tag{6.87}$$

For the first term on the right-hand side of the above inequality, we have

$$\begin{aligned}\left\|x^\varepsilon_{t+\varepsilon s} - x^\varepsilon_t - \varepsilon s\dot{x}_t\right\|_{[-\tau,0]} &= \sup_{u\in[-\tau,0]}\left|\int_0^{\varepsilon s}(h(x^\varepsilon(t+u-\tau+r)) - h(x(t+u-\tau)))dr\right| \\ &\le K_h \sup_{u\in[-\tau,0]}\int_0^{\varepsilon s}|x^\varepsilon(t+u-\tau+r) - x^\varepsilon(t+u-\tau)|dr \\ &\quad + K_h \sup_{u\in[-\tau,0]}\int_0^{\varepsilon s}|x^\varepsilon(t+u-\tau) - x(t+u-\tau)|dr \\ &\le K_h H\int_0^{\varepsilon s} r dr \\ &\quad + K_h\varepsilon s \sup_{u\in[-\tau,0]}|x^\varepsilon(t+u-\tau) - x(t+u-\tau)| \\ &\le \frac{1}{2}K_h H\varepsilon^2 s^2 + 2K_h\|\psi\|_{[-\tau,0]}\varepsilon^2 s\exp(2K_h t).\end{aligned}$$

The first equality follows from (2.6) and the fact that $x$ and $x^\varepsilon$ are positive in the $|\varepsilon s|$ neighborhood of $[t-\tau,t]$. The first and second inequalities follow from (3.12) and (3.10), respectively. The final inequality follows from (6.42), but with $x^\varepsilon$ in place of $x^\dagger$. Therefore, the first term on the right-hand side of the inequality in (6.87) converges to zero as $\varepsilon\downarrow 0$. For the second term on the right-hand side of the inequality in (6.87), we first note that by (6.56) and the inclusion $[t-\tau,t]\subset[p-\tau-\eta_0,p+\eta_0]\subset[q_1^x+\tau,p+\eta_0]$, $v$ is continuous on $[t-\tau,t]$. Thus, by Proposition 6.20, $\|v^\varepsilon_t - v_t\|_{[-\tau,0]}$ converges to zero as $\varepsilon\downarrow 0$. Since the convergence holds for each $(t,\varphi)\in\mathcal{U}$ and $(s,\psi)\in\mathbb{R}\times\mathcal{C}_{[-\tau,0]}$, (ii) holds and the proof is complete. ◻

Define the function $Z:\mathbb{R}_+\times\mathcal{C}^+_{[-\tau,0]}\to\mathbb{R}$ by

$$Z(t,\varphi) = z(t), \tag{6.88}$$

where $z$ is defined as in (2.3) with $x$ being the unique solution of the DDER such that $x_0=\varphi$. Let $\mathcal{V}$ be the neighborhood of $(\ell_1,x_0^*)$ in $\mathbb{R}\times\mathcal{C}_{[-\tau,0]}$ given by

$$\mathcal{V} = \left\{(t,\varphi)\in\mathbb{R}\times\mathcal{C}_{[-\tau,0]} : \|(t,\varphi)-(\ell_1,x_0^*)\| < \varepsilon_0\right\}.$$

Note that by Lemma 6.17 and because $\varepsilon_0\in(0,\eta_0)$, $\mathcal{V}\subset\mathbb{R}_+\times\mathcal{C}^+_{[-\tau,0]}$. In the following lemma we prove that $Z$ is continuously Fréchet differentiable on $\mathcal{V}$. This combined with the implicit function theorem will allow us to define a continuously Fréchet differentiable function $\Delta$ that maps $\varphi$ in a small neighborhood of $x_0^*$ to the first time that the associated solution of the DDER hits the lower boundary (see Lemma 6.29).

LEMMA 6.28. *The function $Z$ is continuously Fréchet differentiable on $\mathcal{V}$ and for each $(t,\varphi)\in\mathcal{V}$, the derivative $DZ(t,\varphi)\in\mathcal{L}(\mathbb{R}\times\mathcal{C}_{[-\tau,0]},\mathbb{R})$ is given by*

$$DZ(t,\varphi)(s,\psi) = s\dot{z}(t) + w(t) \text{ for all } (s,\psi)\in\mathbb{R}\times\mathcal{C}_{[-\tau,0]}. \tag{6.89}$$

*Here $\dot{z}\in\mathcal{C}_{[0,\infty)}$ is defined by $\dot{z}(t)=h(x(t-\tau))$ for all $t\ge 0$ with $x$ being the unique solution of the DDER with $x_0=\varphi$, and if $v$ is the unique solution of the VE along $x$ with $v_0=\psi$, then $w$ is defined by* (6.50).

*Proof.* For each $(t,\varphi)\in\mathcal{V}$, define the operator $G(t,\varphi):\mathbb{R}\times\mathcal{C}_{[-\tau,0]}\to\mathbb{R}$ by

$$G(t,\varphi)(s,\psi) = s\dot{z}(t) + w(t), \qquad (s,\psi)\in\mathbb{R}\times\mathcal{C}_{[-\tau,0]}. \tag{6.90}$$

As in the proof of Lemma 6.27, it suffices to show that (i) the function $(t,\varphi)\to G(t,\varphi)$ is a continuous function from $\mathcal{V}$ into $\mathcal{L}(\mathbb{R}\times\mathcal{C}_{[-\tau,0]},\mathbb{R})$ and (ii) for each $(t,\varphi)\in\mathcal{V}$ and $(s,\psi)\in\mathbb{R}\times\mathcal{C}_{[-\tau,0]}$,

$$G(t,\varphi)(s,\psi)=\lim_{\varepsilon\downarrow 0}\varepsilon^{-1}\{Z(t+\varepsilon s,\varphi+\varepsilon\psi)-Z(t,\varphi)\}.$$

We first show that (i) holds. Fix $(t,\varphi)\in\mathcal{V}$, let $x$ denote the unique solution of the DDER with $x_0=\varphi$ and define $z$ as in (2.3). The linearity of $G(t,\varphi)$ follows from (6.50), Lemma 6.22, and the fact that $\ell_1+\varepsilon_0-\tau<\ell_1^x$. Using Lemma 3.5, (6.50), (6.42), and (B.11), it is simple to check that $G(t,\varphi)$ is indeed a bounded linear operator. Now fix $\delta>0$. By (6.42) and the continuity of $h'$, we can choose $\varepsilon^\dagger>0$ such that if $x^\dagger$ is another solution of the DDER satisfying $\|x_0-x_0^\dagger\|_{[-\tau,0]}<\varepsilon^\dagger$, then given $t^\dagger\in[0,t+\varepsilon^\dagger)$,

$$\sup_{u\in[-\tau,t^\dagger-\tau]}|h'(x(u))-h'(x^\dagger(u))|<\frac{\delta}{8\exp(3K_h(\ell_1+\eta_0))(\ell_1+\eta_0)}. \tag{6.91}$$

By choosing a possibly smaller $\varepsilon^\dagger>0$, we can ensure that $\varepsilon^\dagger$ satisfies

$$\varepsilon^\dagger\le\frac{\delta}{4K_h(H+2\exp(3K_h(\ell_1+\eta_0))}. \tag{6.92}$$

Suppose $(t^\dagger,\varphi^\dagger)\in\mathcal{V}$ satisfies $\|(t,\varphi)-(t^\dagger,\varphi^\dagger)\|<\varepsilon^\dagger$ and let $x^\dagger$ denote the unique solution of the DDER with $x_0^\dagger=\varphi^\dagger$. Define $z^\dagger$ as in (2.3), but with $x^\dagger$ and $z^\dagger$ in place of $x$ and $z$, respectively. Consider $(s,\psi)\in\mathbb{R}\times\mathcal{C}_{[-\tau,0]}$ such that $\|(s,\psi)\|=1$. Let $v^x$ and $v^{x^\dagger}$ be the unique solutions of the VE along $x$ and $x^\dagger$, respectively, and both with initial condition $\psi$. Define $w^x$ as in (6.50), but with $v^x$ and $w^x$ in place of $v$ and $w$, respectively, and define $w^{x^\dagger}$ as in (6.50), but with $v^{x^\dagger}$ and $w^{x^\dagger}$ in place of $v$ and $w$, respectively. Note that by (6.50) and (6.56), we have

$$v^x(t)=w^x(t),\qquad 0\le t<\ell_1^x. \tag{6.93}$$

Similarly, (6.93) holds with $v^{x^\dagger},w^{x^\dagger}$, and $\ell_1^{x^\dagger}$ in place of $v^x,w^x$, and $\ell_1^x$, respectively. Applying the triangle inequality in (6.90), we have

$$|G(t,\varphi)(s,\psi)-G(t^\dagger,\varphi^\dagger)(s,\psi)|\le|\dot z(t)-\dot z^\dagger(t^\dagger)|+|w^x(t)-w^{x^\dagger}(t^\dagger)|. \tag{6.94}$$

Using (2.3), (3.10), (3.12), (6.42), and (6.92), we can bound the first term on the right-hand side of (6.94) by

$$|\dot z(t)-\dot z^\dagger(t^\dagger)|\le K_h\left[|x(t-\tau)-x(t^\dagger-\tau)|+|x(t^\dagger-\tau)-x^\dagger(t^\dagger-\tau)|\right]\le\delta/4. \tag{6.95}$$

By (6.50), (3.12), (6.93), and (B.11), the second term on the right-hand side of (6.94) satisfies

$$\begin{aligned}|w^x(t)-w^{x^\dagger}(t^\dagger)|&\le|w^x(t)-w^x(t^\dagger)|+|w^x(t^\dagger)-w^{x^\dagger}(t^\dagger)|\\&\le 2K_h\exp(2K_h(\ell_1-\tau+\eta_0))|t-t^\dagger|\\&\quad+\int_0^{t^\dagger}|h'(x(s-\tau))-h'(x^\dagger(s-\tau))||v^x(s-\tau)|ds\\&\quad+\int_0^{t^\dagger}|h'(x^\dagger(s-\tau))||v^x(s-\tau)-v^{x^\dagger}(s-\tau)|ds\\&\le\frac{3\delta}{4\exp(K_h(\ell_1+\eta_0))}+K_h\int_0^{t^\dagger}|w^x(s)-w^{x^\dagger}(s)|ds.\end{aligned}$$

An application of Gronwall's inequality to the above inequality yields

$$|w^x(t) - w^{x^\dagger}(t)| \le 3\delta/4. \tag{6.96}$$

Combining (6.94)–(6.96), we obtain $|G(t,\varphi)(s,\psi) - G(t^\dagger,\varphi^\dagger)(s,\psi)| \le \delta$. Since $(s,\psi) \in \mathbb{R}\times\mathcal{C}_{[-\tau,0]}$ was only subject to the constraint $\|(s,\psi)\| = 1$, $G$ is continuous at $(t,\varphi)$. Noting that our choice of $(t,\varphi)\in\mathcal{V}$ was arbitrary, we have $G$ is continuous on $\mathcal{V}$, and so (i) holds.

We now show that (ii) holds. Fix $(t,\varphi)\in\mathcal{V}$ and $(s,\psi)\in\mathbb{R}\times\mathcal{C}_{[-\tau,0]}$. Let $x$ be the unique solution of the DDER with $x_0 = \varphi$, $z$ be defined as in (2.3), $v$ be the unique solution of the VE along $x$, and $w$ be defined as in (6.50). For $\varepsilon > 0$ sufficiently small such that $(t+\varepsilon s, \varphi + \varepsilon\psi)\in\mathcal{V}$, let $x^\varepsilon$ denote the unique solution of the DDER with initial condition $\varphi+\varepsilon\psi$, define $z^\varepsilon$ as in (2.3), but with $x^\varepsilon$ and $z^\varepsilon$ in place of $x$ and $z$, respectively, and define $w^\varepsilon = \varepsilon^{-1}(z^\varepsilon - z)\in\mathcal{C}_{[-\tau,0]}$. By the triangle inequality,

$$\begin{aligned} &\left|\varepsilon^{-1}\left(Z(t+\varepsilon s,\varphi+\varepsilon\psi) - Z(t,\varphi)\right) - s\dot z(t) - w(t)\right| \\ &\quad\le \varepsilon^{-1}|z^\varepsilon(t+\varepsilon s) - z^\varepsilon(t) - \varepsilon s\dot z(t)| + |w^\varepsilon(t) - w(t)|. \end{aligned} \tag{6.97}$$

For the first term on the right-hand side of the inequality in (6.97), we have

$$\begin{aligned} |z^\varepsilon(t+\varepsilon s) - z^\varepsilon(t) - \varepsilon s\dot z(t)| &\le K_h\int_0^{\varepsilon s}|x^\varepsilon(t+u-\tau) - x^\varepsilon(t-\tau)|du \\ &\quad + K_h\varepsilon s|x^\varepsilon(t-\tau) - x(t-\tau)| \\ &\le K_h H\int_0^{\varepsilon s} u du + K_h\varepsilon s\|x^\varepsilon - x\|_{[-\tau,\ell_1+\eta_0]} \\ &\le \frac{1}{2}K_h H\varepsilon^2 s^2 + 2K_h\|\psi\|_{[-\tau,0]}\varepsilon^2 s\exp(2K_h(\ell_1+\eta_0)). \end{aligned}$$

The first inequality follows from (3.12) and the triangle inequality. The second inequality follows from (3.10). The last inequality follows from (6.42) with $x^\varepsilon$ in place of $x^\dagger$. Thus, the first term on the right-hand side of the inequality in (6.97) converges to zero as $\varepsilon\downarrow 0$. By Proposition 6.20, the second term on the right-hand side of the inequality in (6.97) converges to zero as $\varepsilon\downarrow 0$. Since the limit holds for each $(t,\varphi)\in\mathcal{V}$ and $(s,\psi)\in\mathbb{R}\times\mathcal{C}_{[-\tau,0]}$, (ii) holds and the proof is complete. ∎

Lemma 6.29. *There is a neighborhood $\mathcal{T}\times\mathcal{W}$ of $(0,x_0^*)$ in $(-\varepsilon_0,\varepsilon_0)\times\mathcal{C}_{[-\tau,0]}$ and a continuously Fréchet differentiable function $\Delta:\mathcal{W}\to\mathcal{T}$ such that $\Delta(x_0^*) = 0$, $(\ell_1+\Delta(\varphi),\varphi)\in\mathcal{V}$ for all $\varphi\in\mathcal{W}$ and for all $(t,\varphi)\in\mathcal{T}\times\mathcal{W}$, $Z(\ell_1+t,\varphi) = 0$ if and only if $t = \Delta(\varphi)$. Furthermore, $\Delta(\varphi) = \ell_1^x - \ell_1$, where $\ell_1^x$ is as in Lemma* 6.17 *with $x$ the unique solution of the DDER satisfying $x_0 = \varphi$, and the derivative of $\Delta$ at $x_0^*$, $D\Delta(x_0^*)\in\mathcal{L}(\mathcal{C}_{[-\tau,0]},\mathbb{R})$, is given by*

$$D\Delta(x_0^*)\psi = -\frac{w(\ell_1)}{\dot z^*(\ell_1)} \qquad \text{for all } \psi\in\mathcal{C}_{[-\tau,0]}, \tag{6.98}$$

*where $w$ is given by* (6.50) *with $x$ replaced by $x^*$ and $v$ is the solution of the VE along $x^*$ with $v_0 = \psi$.*

*Proof.* Note that $Z$ is continuously Fréchet differentiable on $\mathcal{V}$ and $Z(\ell_1, x_0^*) = 0$. Additionally, due to the fact that $\ell_1 - \tau\in(-\tau, q_1)$, we have $h(x^*(\ell_1-\tau)) < 0$ and so $s\to DZ(\ell_1,x_0^*)(s,0) = s\dot z^*(\ell_1) = sh(x^*(\ell_1-\tau))$ is an isomorphism on $\mathbb{R}$. It then follows from the implicit function theorem on Banach spaces (see, e.g., Theorem 2.3

in [9]) that there exists a neighborhood $\mathcal{T} \times \mathcal{W}$ of $(0, x_0^*)$ in $(-\varepsilon_0, \varepsilon_0) \times \mathcal{C}_{[-\tau,0]}$ and a continuously Fréchet differentiable function $\Delta^\ell : \mathcal{W} \to \mathbb{R}$ such that $\Delta^\ell(x_0^*) = \ell_1$, $(\Delta^\ell(\varphi), \varphi) \in \mathcal{V}$ for all $\varphi \in \mathcal{W}$, and for all $(t, \varphi) \in \mathcal{T} \times \mathcal{W}$, $Z(\ell_1 + t, \varphi) = 0$ if and only if $\ell_1 + t = \Delta^\ell(\varphi)$. Moreover, the derivative of $\Delta^\ell$ at $x_0^*$ is given by

$$D\Delta^\ell(x_0^*)\psi = -[D_1 Z(\ell_1, x_0^*)]^{-1} D_2 Z(\ell_1, x_0^*)\psi = -\frac{w(\ell_1)}{\dot{z}^*(\ell_1)} \text{ for all } \psi \in \mathcal{C}_{[-\tau,0]}.$$

where $D_i Z$ denotes the derivative of $Z$ with respect to its $i$th argument. Define $\Delta : \mathcal{W} \to \mathbb{R}$ by $\Delta(\varphi) = \Delta^\ell(\varphi) - \ell_1$ for each $\varphi \in \mathcal{W}$. Then $\Delta$ is continuously Fréchet differentiable on $\mathcal{W}$, $\Delta(x_0^*) = 0$ and $(\ell_1 + \Delta(\varphi), \varphi) \in \mathcal{V}$. Thus, by the definition of $\mathcal{V}$, $|\Delta(\varphi)| < \varepsilon_0 \le \eta_0$. Given $\varphi \in \mathcal{W}$, let $x$ denote the unique solution of the DDER with $x_0 = \varphi$, define $z \in \mathcal{C}_{[-\tau,\infty)}$ as in (2.3), and let $\ell_1^x$ be as in Lemma 6.17. Then $\ell_1^x$ is the unique point in $(\ell_1 - \eta_0, \ell_1 + \eta_0)$ such that $z(\ell_1^x) = 0$. Since $\ell_1 + \Delta(\varphi) \in (\ell_1 - \eta_0, \ell_1 + \eta_0)$ and $z(\ell_1 + \Delta(\varphi)) = Z(\ell_1 + \Delta(\varphi), \varphi) = 0$, we must have that $\ell_1 + \Delta(\varphi) = \ell_1^x$. ∎

We can now define our variant of a Poincaré map for $x^*$, $\Gamma : \mathcal{W} \to \mathcal{C}_{[-\tau,0]}^+$, as follows:

$$\Gamma(\varphi) = \Sigma(p + \Delta(\varphi), \varphi), \qquad \varphi \in \mathcal{W}. \tag{6.99}$$

THEOREM 6.30. *The function $\Gamma$ is continuously Fréchet differentiable, $x_0^*$ is a fixed point of $\Gamma$, and for all $\psi \in \mathcal{C}_{[-\tau,0]}$, the derivative of $\Gamma$ in the direction $\psi$ and evaluated at $x_0^*$ is given by*

$$D\Gamma(x_0^*)\psi = -\frac{w(\ell_1)}{\dot{z}^*(\ell_1)} \dot{x}_p^* + v_p. \tag{6.100}$$

*Here $v$ denotes the unique solution of the VE along $x^*$ with $v_0 = \psi$ and $w$ is defined as in* (6.50)*, but with $x^*$ in place of $x$.*

*Proof.* The fact that $\Gamma$ is continuously Fréchet differentiable on $\mathcal{U}$ is due to the facts that $\Sigma$ is continuously Fréchet differentiable on $\mathcal{U}$, $\Delta$ is continuously Fréchet differentiable on $\mathcal{W}$, and $(p + \Delta(\varphi), \varphi) \in \mathcal{U}$ for all $\varphi \in \mathcal{W}$. Since $\Delta(x_0^*) = 0$, the chain rule implies that

$$D\Gamma(x_0^*)\psi = D_1\Sigma(p, x_0^*) D\Delta(x_0^*)\psi + D_2\Sigma(p, x_0^*)\psi. \tag{6.101}$$

By (6.98), we have $D\Delta(x_0^*)\psi = -w(\ell_1)/\dot{z}^*(\ell_1)$. Then by (6.80), with $x_0^*$ in place of $\varphi$ and $D\Delta(x_0^*)\psi$ in place of $s$, we have

$$D_1\Sigma(p, x_0^*) D\Delta(x_0^*)\psi = -\frac{w(\ell_1)}{\dot{z}^*(\ell_1)} \dot{x}_p^*,$$

where $\dot{z}^*$ is defined by $\dot{z}^*(t) = h(x^*(t - \tau))$ for all $t \ge 0$ and $w$ is defined as in (6.50), but with $x^*$ in place of $x$, and also

$$D_2\Sigma(p, x_0^*)\psi = v_p,$$

where $v$ denotes the unique solution of the VE along $x^*$ with $v_0 = \psi$. ∎

**6.6. Proof of stability.** In this section we prove that if the delay $\tau$ is sufficiently large, then any SOPS is exponentially stable. In section 6.7, we prove that such a SOPS is unique, which will complete the proof of Theorem 3.8. Let $\tau^{(4)} \ge \tau_0$ be such that Lemma 6.14 and the properties in Remark 6.1 hold. Recall the positive constants

$\alpha$, $\beta$, $m$, $H$, $K_h$, and $M$ introduced in Assumptions 3.3 and 3.4, and Lemmas 3.5, 3.6, and 6.24.

LEMMA 6.31. *For each $\theta > 0$, there exists $\tau_\theta \geq \tau_0$ such that whenever $\tau > \tau_\theta$, $x^*$ is a SOPS of the DDER with delay $\tau$ and $q_0 = -\tau$ and $\Gamma$ is the associated Poincaré type map defined in* (6.99)*, then $\|D\Gamma(x_0^*)\| \leq \theta$.*

*Proof.* Fix $\theta > 0$. By (6.27), Corollary 6.12, Lemma 6.14, Remark 6.1, and Lemma 5.5, there exist constants $d > 0$ and $\tau_\theta \geq \tau^{(4)}$ such that if $\tau > \tau_0$ and $x^*$ is a SOPS of the DDER with delay $\tau$, then (i) $q_2 < q_1 + 2\tau$; (ii) $x^*(t) \geq L + \tau d$ for all $t \in [q_1 - \tau, \ell_1 - \tau]$; and (iii) $h(s) \leq -\alpha/2$ for all $s \geq L + \tau d$. By choosing a possibly larger $\tau_\theta$ we can ensure that

$$\tau_\theta \geq \frac{4K_h(1+K_h)L^3 Mm(2MH+\alpha)(2\beta+\alpha)}{\theta\alpha^4\beta d}. \tag{6.102}$$

Fix $\tau > \tau_\theta$ and a SOPS $x^*$ with delay $\tau$ and $q_0 = -\tau$. Suppose $\psi \in \mathcal{C}_{[-\tau,0]}$ satisfies $\|\psi\|_{[-\tau,0]} = 1$. Let $\xi \in \mathcal{D}_{[-\tau,\infty)}$ denote the unique solution of the VE along $x^*$ with

$$\xi_0 = -\frac{w(\ell_1)}{\dot{z}^*(\ell_1)}\dot{x}_0^* + \psi \in \mathcal{C}_{[-\tau,0]},$$

where if $v$ is the unique solution to the VE along $x^*$ with $v_0 = \psi$, then $w$ is defined as in (6.50), but with $x^*$ in place of $x$. Note that by Lemma 6.24, $\|v\|_{[0,\ell_1]} \leq \|v\|_{[-\tau,p]} \leq M\|\psi\|_{[-\tau,0]}$. Since $w$ and $v$ are equal on $[0,\ell_1)$ and $w$ is continuous, we have $\|w\|_{[0,\ell_1]} \leq M\|\psi\|_{[-\tau,0]}$ as well. This along with (ii), (iii) above, the fact that $\dot{z}^*(t) = h(x^*(t-\tau))$ for all $t \geq 0$, and (3.10) together imply that $\|\xi_0\|_{[-\tau,0]} \leq 2\alpha^{-1}MH + 1$. From Lemmas 6.22 (with $a = 1, b = -w(\ell_1)/\dot{z}^*(\ell_1)$ and $v^\dagger = \dot{x}^*$) and 6.23, we obtain

$$\xi(t) = -\frac{w(\ell_1)}{\dot{z}^*(\ell_1)}\dot{x}^*(t) + v(t), \qquad t \in [-\tau,p] \setminus \{\ell_1\}. \tag{6.103}$$

Note that $\ell_1 \notin [p-\tau,p]$, so by (6.100),

$$\xi_p = -\frac{w(\ell_1)}{\dot{z}^*(\ell_1)}\dot{x}_0^* + v_p = D\Gamma(x_0^*)\psi.$$

Since $\dot{x}^*(t) = \dot{z}^*(t)$ and $v(t) = w(t)$ for all $t \in [0,\ell_1)$, and $\dot{z}^*$ and $w$ are continuous, we have

$$\lim_{t\uparrow\ell_1}\xi(t) = \lim_{t\uparrow\ell_1}\left\{-\frac{w(\ell_1)}{\dot{z}^*(\ell_1)}\dot{z}^*(t) + w(t)\right\} = 0. \tag{6.104}$$

By (6.57) and (6.104), we have, for $q_1 \leq s < \ell_1$,

$$\begin{aligned} |\xi(s)| = \lim_{t\uparrow\ell_1}|\xi(t)-\xi(s)| &\leq \int_s^{\ell_1} |h'(x^*(u-\tau))\xi(u-\tau)|du \\ &\leq \|\xi\|_{[-\tau,\ell_1]}\int_s^{\ell_1}\frac{|h'(x^*(u-\tau))x^*(u-\tau)|}{|x^*(u-\tau)|}du \\ &\leq \frac{M|\ell_1-q_1|m}{L+\tau d}\|\xi_0\|_{[-\tau,0]}. \end{aligned} \tag{6.105}$$

The final inequality above follows from Lemma 6.24, Assumption 3.4, and the fact that $x^*(u) \geq L + \tau d$ for all $u \in [q_1-\tau, \ell_1-\tau]$.

By (6.57) and (6.104), $\xi$ is continuous at $\ell_1$ and equal to zero on $[\ell_1, q_1+1]$, so

$$\|\xi\|_{[q_1,q_1+\tau]} \le \frac{M|\ell_1 - q_1|m}{L+\tau d}\|\xi_0\|_{[-\tau,0]}. \tag{6.106}$$

By (6.57) and (3.12), for $t \in [q_1+\tau, \ell_1+\tau]$, $|\xi(t)| \le K_h|\ell_1 - q_1|\|\xi\|_{[q_1,q_1+\tau]}$. For $t \in [\ell_1+\tau, q_1+2\tau]$, $\xi(t-\tau) = 0$, so $\xi$ is constant on $[\ell_1+\tau, q_1+2\tau]$ and thus

$$\|\xi\|_{[q_1+\tau,q_1+2\tau]} \le K_h|\ell_1 - q_1|\|\xi\|_{[q_1,q_1+\tau]}. \tag{6.107}$$

By (i) above, $q_2 < q_1 + 2\tau$. Then by (6.57), for $t \in [q_1+2\tau, p] = [q_1+2\tau, q_2+\tau]$,

$$\begin{aligned}|\xi(t)| &\le |\xi(q_1+2\tau)| + \int_{q_1+2\tau}^{p} |h'(x^*(s-\tau))\xi(s-\tau)|ds \\ &\le (1+K_h)|q_2 - q_1 - \tau|\|\xi\|_{[q_1+\tau,q_1+2\tau]}.\end{aligned}$$

Then by (6.102), (6.106), (6.107), and (6.41),

$$\|D\Gamma(x_0^*)\psi\| = \|\xi_p\|_{[-\tau,0]} \le \theta. \tag{6.108}$$

Since (6.108) holds for all $\psi \in \mathcal{C}_{[-\tau,0]}$ satisfying $\|\psi\|_{[-\tau,0]} = 1$, the conclusion of the lemma follows. ∎

We now establish that for $\tau$ sufficiently large, any SOPS to the DDER is exponentially stable. The statement and proof of Theorem 6.32 below are similar to the statement and proof of Theorem 1.1 in [40], which establishes the exponential stability of periodic solutions to an unconstrained state dependent delay differential equation. We include the proof here for completeness.

THEOREM 6.32. *Fix $\theta \in (0,1)$ and $\tau > \tau_\theta$. Suppose $x^*$ is a SOPS of the DDER with delay $\tau$ and $q_0 = -\tau$. Then there exist constants $\varepsilon > 0$ and $K_\rho > 0$ such that given any*

$$0 < \gamma < \frac{|\log\theta|}{p}, \tag{6.109}$$

*there exists $K_\gamma > 0$ such that if $\varphi \in \mathcal{C}^+_{[-\tau,0]}$ satisfies $\|\varphi - x^*_\sigma\|_{[-\tau,0]} \le \varepsilon$ for some $\sigma \in [0,p)$, then there exists $\rho \in (-p,p)$ satisfying*

$$|\rho| \le K_\rho\|\varphi - x^*_\sigma\|_{[-\tau,0]} \tag{6.110}$$

*and such that*

$$\|x_t - x^*_{t+\sigma+\rho+p}\|_{[-\tau,0]} \le K_\gamma e^{-\gamma t}\|\varphi - x^*_\sigma\|_{[-\tau,0]} \text{ for all } t \ge 0, \tag{6.111}$$

*where $x$ is the unique solution of the DDER with delay $\tau$ and $x_0 = \varphi$.*

*Remark* 6.2. Note that since $\sigma$ is nonnegative and $|\rho| < p$, it follows that $t+\sigma+\rho+p \ge 0$ for all $t \ge 0$ and so $x^*_{t+\sigma+\rho+p}$ in (6.111) is well defined for all $t \ge 0$. When $\sigma+\rho \ge 0$, $x^*_{t+\sigma+\rho}$ is well defined for all $t \ge 0$, so by the periodicity of $x^*$, we can replace $x^*_{t+\sigma+\rho+p}$ with $x^*_{t+\sigma+\rho}$.

*Proof.* Let $x^*$ be a SOPS of the DDER. First note that by (3.10) and the periodicity of $x^*$,

$$\|x^*_t - x^*_s\|_{[-\tau,0]} \le H|t-s|, \qquad 0 \le s,t < \infty. \tag{6.112}$$

Our proof proceeds by sequentially proving that the following statements hold:

(i) Given any $\gamma$ satisfying (6.109), there exist positive constants $\varepsilon_1(\gamma)$, $\widetilde{K}_\rho(\gamma)$, and $K_1(\gamma)$ such that if $\varphi \in \mathcal{C}^+_{[-\tau,0]}$ satisfies $\|\varphi - x^*_0\|_{[-\tau,0]} \le \varepsilon_1(\gamma)$, then there exists $\rho \in (-p, p)$ satisfying (6.110), with $\sigma = 0$ and $\widetilde{K}_\rho(\gamma)$ in place of $K_\rho$, and such that (6.111) holds with $\sigma = 0$ and $K_1(\gamma)$ in place of $K_\gamma$.

(ii) There exist positive constants $\varepsilon_2$ and $\overline{K}_\rho$ such that given any $\gamma$ satisfying (6.109), there exists $K_5(\gamma) > 0$ such that if $\varphi \in \mathcal{C}^+_{[-\tau,0]}$ satisfies $\|\varphi - x^*_0\|_{[-\tau,0]} \le \varepsilon_2$, then there exists $\rho \in (-p, p)$ satisfying (6.110), with $\sigma = 0$ and $\overline{K}_\rho$ in place of $K_\rho$, and such that (6.111) holds with $\sigma = 0$ and $K_5(\gamma)$ in place of $K_\gamma$.

(iii) The statement of the theorem.

*Proof of* (i). Since $\Gamma$ is continuously Fréchet differentiable, it follows that

$$\Gamma(\varphi) - x^*_0 = D\Gamma(x^*_0)(\varphi - x^*_0) + o(\|\varphi - x^*_0\|_{[-\tau,0]})$$

as $\|\varphi - x^*_0\|_{[-\tau,0]} \to 0$. Suppose $\gamma > 0$ satisfies (6.109). Then $e^{-\gamma p} \in (\theta, 1)$. By Lemma 6.31, $\|D\Gamma(x^*_0)\| \le \theta$, so by taking a sufficiently small neighborhood $\mathcal{W}$ of $x^*_0$ in $\mathcal{C}^+_{[-\tau,0]}$, we can ensure that

$$\|\Gamma(\varphi) - x^*_0\|_{[-\tau,0]} \le e^{-\gamma p}\|\varphi - x^*_0\|_{[-\tau,0]} \text{ for all } \varphi \in \mathcal{W}. \tag{6.113}$$

By Lemma 6.29, we can choose $\varepsilon_1(\gamma), K_2(\gamma) > 0$ such that

$$\mathcal{W}(\varepsilon_1(\gamma)) = \left\{\varphi \in \mathcal{C}^+_{[-\tau,0]} : \|\varphi - x^*_0\|_{[-\tau,0]} < \varepsilon_1(\gamma)\right\} \subseteq \mathcal{W}$$

and

$$|\Delta(\varphi)| \le K_2(\gamma)\|\varphi - x^*_0\|_{[-\tau,0]} \text{ for all } \varphi \in \mathcal{W}(\varepsilon_1(\gamma)). \tag{6.114}$$

By choosing a possibly smaller $\varepsilon_1(\gamma) > 0$ such that $\varepsilon_1(\gamma)K_2(\gamma) < \tau$, (6.114) ensures that

$$|\Delta(\varphi)| < \tau \text{ for all } \varphi \in \mathcal{W}(\varepsilon_1(\gamma)). \tag{6.115}$$

Given $\varphi \in \mathcal{W}(\varepsilon_1(\gamma))$, we can iterate (6.113) to obtain that $\Gamma^k(\varphi) \in \mathcal{W}(\varepsilon_1(\gamma))$ for each $k = 0, 1, \dots$ and

$$\|\Gamma^k(\varphi) - x^*_0\|_{[-\tau,0]} \le e^{-\gamma kp}\|\varphi - x^*_0\|_{[-\tau,0]}. \tag{6.116}$$

Define $t_0 = 0$ and $\Gamma^0(\varphi) = \varphi$. For $k = 1, 2, \dots$, recursively define

$$\Delta(\varphi, k) = \sum_{j=0}^{k-1} \Delta\left(\Gamma^j(\varphi)\right), \tag{6.117}$$

$$t_k = kp + \Delta(\varphi, k), \tag{6.118}$$

Let $x$ denote the unique solution of the DDER with $x_0 = \varphi$. We will use induction to show that $\Gamma^k(\varphi) = x_{t_k}$ for all $k = 0, 1, \dots$. By definition, $\Gamma^0(\varphi) = \varphi$. Now suppose that $\Gamma^k(\varphi) = x_{t_k}$ for some $k \in \{0, 1, \dots\}$. Then using the semiflow property of $\Sigma$ we have

$$\begin{aligned}\Gamma^{k+1}(\varphi) = \Gamma\left(\Gamma^k(\varphi)\right) &= \Sigma\left(p + \Delta(\Gamma^k(\varphi)), \Sigma(t_k, \varphi)\right)\\ &= \Sigma\left(t_k + p + \Delta(\Gamma^k(\varphi)), \varphi\right) = x_{t_{k+1}}.\end{aligned}$$

Therefore, by the induction principle, $\Gamma^k(\varphi) = x_{t_k}$ for each $k = 0, 1, \dots$. Note that (6.115) implies $t_k < t_{k+1}$ for each $k = 0, 1, \dots$. Using (6.114) and (6.116), we have for all $k = 0, 1, \dots,$

$$\left|\Delta\left(\Gamma^k(\varphi)\right)\right| \leq K_2(\gamma)\|\Gamma^k(\varphi) - x_0^*\|_{[-\tau,0]} \leq K_2(\gamma)e^{-\gamma kp}\|\varphi - x_0^*\|_{[-\tau,0]}. \tag{6.119}$$

Define

$$\rho = -\lim_{k\to\infty} \Delta(\varphi, k), \tag{6.120}$$

where the existence of $\rho$ as a finite limit follows from (6.117) and (6.119). By (6.117), (6.119), and (6.120), we have for each $k = 1, 2, \dots,$

$$|\Delta(\varphi, k)| \leq \sum_{j=0}^{\infty} |\Delta(\Gamma^j(\varphi))| \leq \widetilde{K}_\rho(\gamma)\|\varphi - x_0^*\|_{[-\tau,0]}, \tag{6.121}$$

$$|\rho| \leq \sum_{j=0}^{\infty} |\Delta(\Gamma^j(\varphi))| \leq \widetilde{K}_\rho(\gamma)\|\varphi - x_0^*\|_{[-\tau,0]}, \tag{6.122}$$

and

$$|\rho + \Delta(\varphi, k)| \leq \sum_{j=k}^{\infty} |\Delta(\Gamma^j(\varphi))| \leq \widetilde{K}_\rho(\gamma)e^{-\gamma kp}\|\varphi - x_0^*\|_{[-\tau,0]}, \tag{6.123}$$

where

$$\widetilde{K}_\rho(\gamma) = K_2(\gamma)\sum_{j=0}^{\infty} e^{-\gamma jp} = \frac{K_2(\gamma)}{1 - e^{-\gamma p}}. \tag{6.124}$$

By choosing $\varepsilon_1(\gamma) > 0$ possibly smaller so that $\varepsilon_1(\gamma)\widetilde{K}_\rho(\gamma) < p$, it follows from (6.122) that $\rho \in (-p, p)$.

Now let $I_k = [t_k, t_{k+1}]$ for each $k \in \mathbb{N}_0$. By (6.118), (6.121), and the fact that $\varepsilon_1(\gamma)\widetilde{K}_\rho(\gamma) < p$, $t_k \to \infty$ as $n \to \infty$, so $\bigcup_{k=0}^{\infty} I_k$ covers $[0, \infty)$. From (6.117)–(6.119) and (6.121), we have for $\varphi \in \mathcal{W}(\varepsilon_1(\gamma))$ and $k = 0, 1, \dots,$

$$t_{k+1} - t_k = p + \Delta(\Gamma^k(\varphi)) \leq p + \widetilde{K}_\rho(\gamma)\varepsilon_1(\gamma). \tag{6.125}$$

By (6.42), for $K_3(\gamma) = 2\exp[2K_h(p + \widetilde{K}_\rho(\gamma)\varepsilon_1(\gamma))]$, we have, for all $\varphi \in \mathcal{W}(\varepsilon_1(\gamma))$,

$$\|x_t - x_t^*\|_{[-\tau,0]} \leq K_3(\gamma)\|\varphi - x_0^*\|_{[-\tau,0]}, \qquad 0 \leq t \leq p + \widetilde{K}_\rho(\gamma)\varepsilon_1(\gamma). \tag{6.126}$$

It follows from (6.116), (6.125), (6.126), the relation $\Gamma^k(\varphi) = x_{t_k}$, and the semiflow property of $\Sigma$ that for $t \in I_k$

$$\begin{aligned}\|x_t - x_{t-t_k}^*\|_{[-\tau,0]} &= \|\Sigma(t - t_k, x_{t_k}) - \Sigma(t - t_k, x_0^*)\|_{[-\tau,0]} \\ &\leq K_3(\gamma)\|\Gamma^k(\varphi) - x_0^*\|_{[-\tau,0]} \\ &\leq K_3(\gamma)e^{-\gamma kp}\|\varphi - x_0^*\|_{[-\tau,0]}.\end{aligned}$$

Also, by (6.112), (6.118), and (6.123), for all $t \in I_k$,

$$\begin{aligned}\|x_{t-t_k}^* - x_{t+p+\rho}^*\|_{[-\tau,0]} &= \|x_{t-t_k+(k+1)p}^* - x_{t+p+\rho}^*\|_{[-\tau,0]} \\ &\leq H|\rho - kp + t_k| \\ &\leq H|\rho + \Delta(\varphi, k)| \\ &\leq H\widetilde{K}_\rho(\gamma)e^{-\gamma kp}\|\varphi - x_0^*\|_{[-\tau,0]}.\end{aligned}$$

Combining the previous two inequalities yields

$$\|x_t - x^*_{t+p+\rho}\|_{[-\tau,0]} \le K_4(\gamma)e^{-\gamma kp}\|\varphi - x^*_0\|_{[-\tau,0]} \text{ for all } t \in I_k, \tag{6.127}$$

where

$$K_4(\gamma) = K_3(\gamma) + H\widetilde{K}_\rho(\gamma). \tag{6.128}$$

Furthermore, by (6.118) and (6.121), we have

$$t_{k+1} - kp = p + \Delta(\varphi, k+1) \le p + \widetilde{K}_\rho(\gamma)\varepsilon_1(\gamma). \tag{6.129}$$

Therefore, by (6.127) and (6.129), for all $t \in I_k$,

$$\begin{aligned} \|x_t - x^*_{t+p+\rho}\|_{[-\tau,0]} &\le K_4(\gamma)e^{-\gamma kp}e^{\gamma(t_{k+1}-t)}\|\varphi - x^*_0\|_{[-\tau,0]} \\ &\le K_1(\gamma)e^{-\gamma t}\|\varphi - x^*_0\|_{[-\tau,0]}, \end{aligned} \tag{6.130}$$

where $K_1(\gamma) = K_4(\gamma)e^{\gamma(p+\widetilde{K}_\rho(\gamma)\varepsilon_1(\gamma))}$. From (6.122) and (6.130), we see that (i) holds.

*Proof of* (ii). Fix $\bar{\gamma} > 0$ satisfying (6.109) and set

$$\varepsilon_2 = \varepsilon_1(\bar{\gamma}), \qquad \overline{K}_\rho = \widetilde{K}_\rho(\bar{\gamma}).$$

Now given $\varphi \in \mathcal{C}^+_{[-\tau,0]}$ satisfying $\|\varphi - x^*_0\|_{[-\tau,0]} \le \varepsilon_2$, let $x$ denote the unique solution of the DDER with $x_0 = \varphi$. By (i), there is a $\rho \in (-p, p)$ satisfying

$$|\rho| \le \overline{K}_\rho\|\varphi - x^*_0\|_{[-\tau,0]} \tag{6.131}$$

and such that

$$\|x_t - x^*_{t+p+\rho}\|_{[-\tau,0]} \le K_1(\bar{\gamma})e^{-\bar{\gamma}t}\|\varphi - x^*_0\|_{[-\tau,0]} \qquad \text{for all } t \ge 0. \tag{6.132}$$

Suppose $\gamma > 0$ also satisfies (6.109). Then set

$$T(\gamma) = \max\left\{0, \frac{1}{\bar{\gamma}}\log\left(\frac{K_1(\bar{\gamma})\varepsilon_2}{\varepsilon_1(\gamma)}\right)\right\}, \tag{6.133}$$

where $\varepsilon_1(\gamma) > 0$ is as in (i). Due to (6.132), (6.133), and the fact that $\|\varphi - x^*_0\|_{[-\tau,0]} \le \varepsilon_2$, we have, for all $t \ge T(\gamma)$,

$$\|x_t - x^*_{t+p+\rho}\|_{[-\tau,0]} \le K_1(\bar{\gamma})e^{-\bar{\gamma}t}\|\varphi - x^*_0\|_{[-\tau,0]} \le \varepsilon_1(\gamma). \tag{6.134}$$

Let $n_\gamma = \min\{k \in \mathbb{N} : kp \ge T(\gamma) + \varepsilon_2\overline{K}_\rho\}$ and define

$$t_\gamma = n_\gamma p - \rho. \tag{6.135}$$

By (6.131) and the definition of $n_\gamma$, we have $t_\gamma \ge n_\gamma p - \varepsilon_2\overline{K}_\rho \ge T(\gamma)$ and so (6.134) holds for all $t \ge t_\gamma$. Then (6.134), (6.135), and the periodicity of $x^*$ imply that

$$\|x_{t_\gamma} - x^*_0\|_{[-\tau,0]} = \|x_{t_\gamma} - x^*_{t_\gamma+p+\rho}\|_{[-\tau,0]} \le \varepsilon_1(\gamma). \tag{6.136}$$

Additionally, by (6.133) and the fact that $\rho \in (-p, p)$, we have the following bound on $t_\gamma$:

$$t_\gamma \le T(\gamma) + \varepsilon_2\overline{K}_\rho + p + |\rho| \le T(\gamma) + \varepsilon_2\overline{K}_\rho + 2p. \tag{6.137}$$

Define $x^\gamma \in \mathcal{C}^+_{[-\tau,\infty)}$ by $x^\gamma(t) = x(t_\gamma + t)$ for all $t \geq -\tau$, so that $x^\gamma$ is a solution of the DDER with $x^\gamma_0 = x_{t_\gamma}$. By (6.136) and (i), there exists $\tilde\rho \in (-p, p)$ such that for all $t \geq 0$,

$$\|x^\gamma_t - x^*_{t+p+\tilde\rho}\|_{[-\tau,0]} \leq K_1(\gamma)e^{-\gamma t}\|x^\gamma_0 - x^*_0\|_{[-\tau,0]}. \tag{6.138}$$

From the definition of $x^\gamma$, (6.138), (6.135), the periodicity of $x^*$, and (6.132), it follows that for $t \geq 0$,

$$\begin{aligned}\|x_{t+t_\gamma} - x^*_{t+p+\tilde\rho}\|_{[-\tau,0]} &= \|x^\gamma_t - x^*_{t+p+\tilde\rho}\|_{[-\tau,0]}\\ &\leq K_1(\gamma)e^{-\gamma t}\|x_{t_\gamma} - x^*_{t_\gamma+p+\rho}\|_{[-\tau,0]}\\ &\leq K_1(\gamma)e^{-\gamma t}K_1(\bar\gamma)e^{-\bar\gamma t_\gamma}\|\varphi - x^*_0\|_{[-\tau,0]}.\end{aligned} \tag{6.139}$$

By (6.135), the periodicity of $x^*$, (6.132), (6.139), and the fact that $\|\varphi - x^*_0\|_{[-\tau,0]} < \varepsilon_2$, we have, for $t \geq 0$,

$$\begin{aligned}\|x^*_t - x^*_{t+p+\tilde\rho}\|_{[-\tau,0]} &= \|x^*_{t+t_\gamma+\rho} - x^*_{t+p+\tilde\rho}\|\\ &\leq \|x^*_{t+t_\gamma+p+\rho} - x_{t+t_\gamma}\|_{[-\tau,0]}\\ &\quad + \|x_{t+t_\gamma} - x^*_{t+p+\tilde\rho}\|_{[-\tau,0]}\\ &\leq K_1(\bar\gamma)e^{-\bar\gamma t}\|\varphi - x^*_0\|_{[-\tau,0]}\\ &\quad + K_1(\gamma)K_1(\bar\gamma)e^{-\bar\gamma t_\gamma}e^{-\gamma t}\|\varphi - x^*_0\|_{[-\tau,0]}\\ &\to 0 \text{ as } t \to \infty.\end{aligned} \tag{6.140}$$

Thus, by periodicity, $x^*(t) = x^*(t + p + \tilde\rho)$ for all $t \geq 0$. Since $p$ is the minimal period of $x^*$, $\tilde\rho$ must be an integer multiple of $p$. Then $|\tilde\rho| < p$ implies that $\tilde\rho = 0$. Now by (6.139), with $\tilde\rho = 0$ and $t - t_\gamma$ in place of $t$, we have, for $t \geq t_\gamma$,

$$\begin{aligned}\|x_t - x^*_{t+p-t_\gamma}\|_{[-\tau,0]} &\leq K_1(\gamma)K_1(\bar\gamma)e^{-\gamma(t-t_\gamma)-\bar\gamma t_\gamma}\|\varphi - x^*_0\|_{[-\tau,0]}\\ &\leq K_6(\gamma)e^{-\gamma t}\|\varphi - x^*_0\|_{[-\tau,0]},\end{aligned} \tag{6.141}$$

where $K_6(\gamma) = K_1(\gamma)K_1(\bar\gamma)e^{t_\gamma(\gamma-\bar\gamma)}$. For $0 \leq t \leq t_\gamma$, by (6.132) and (6.137), we have

$$\begin{aligned}\|x_t - x^*_{t+p+\rho}\|_{[-\tau,0]} &\leq K_1(\bar\gamma)e^{\gamma t_\gamma}e^{-\gamma t}\|\varphi - x^*_0\|_{[-\tau,0]}\\ &\leq K_1(\bar\gamma)\exp\left(\gamma\left(T(\gamma) + \varepsilon_2\overline{K}_\rho + 2p\right)\right)e^{-\gamma t}\|\varphi - x^*_0\|_{[-\tau,0]}\\ &\leq K_7(\gamma)e^{-\gamma t}\|\varphi - x^*_0\|_{[-\tau,0]},\end{aligned} \tag{6.142}$$

where $K_7(\gamma) = K_1(\bar\gamma)\exp\left(\gamma\left(T(\gamma) + \varepsilon_2\overline{K}_\rho + 2p\right)\right)$. Combining (6.141) and (6.142) and letting $K_5(\gamma) = \max(K_6(\gamma), K_7(\gamma))$ we have, for $t \geq 0$,

$$\|x_t - x^*_{t+p+\rho}\|_{[-\tau,0]} \leq K_5(\gamma)e^{-\gamma t}\|\varphi - x^*_0\|_{[-\tau,0]}. \tag{6.143}$$

It follows from (6.131) and (6.143) that (ii) holds.

*Proof of* (iii). Suppose $\sigma \in [0, p)$ and $\varphi \in \mathcal{C}^+_{[-\tau,0]}$. Let $x$ denote the unique solution of the DDER with $x_0 = \varphi$. By (6.42), with $x^*(\sigma + \cdot)$ in place of $x^\dagger(\cdot)$, for $\sigma \in [0, p)$ and $0 \leq t \leq p$,

$$\|x_t - x^*_{t+\sigma}\|_{[-\tau,0]} \leq 2\exp(2K_h p)\|\varphi - x^*_\sigma\|_{[-\tau,0]}. \tag{6.144}$$

Choose $\varepsilon > 0$ satisfying

$$\varepsilon \leq \frac{\varepsilon_2}{2\exp(2K_h p)}. \tag{6.145}$$

Fix $\gamma > 0$ satisfying (6.109) and suppose $\varphi \in \mathcal{C}^+_{[-\tau,0]}$ is such that $\|\varphi - x^*_\sigma\|_{[-\tau,0]} \leq \varepsilon$ for some $\sigma \in [0, p)$. By (6.144), for $0 \leq t \leq p - \sigma$,

$$\|x_t - x^*_{t+\sigma}\|_{[-\tau,0]} \leq 2\exp(2K_h p)e^{\gamma p}e^{-\gamma t}\|\varphi - x^*_\sigma\|_{[-\tau,0]}. \tag{6.146}$$

Now (6.144), (6.145), and the periodicity of $x^*$ imply that

$$\|x_{p-\sigma} - x^*_0\|_{[-\tau,0]} \leq 2\exp(2K_h p)\|\varphi - x^*_\sigma\|_{[-\tau,0]} \leq \varepsilon_2. \tag{6.147}$$

Then by (ii) and (6.147), there exists $K_5(\gamma) > 0$ and $\rho \in (-p, p)$ such that for $t \geq 0$,

$$\begin{aligned}\|x_{t+p-\sigma} - x^*_{t+p+\rho}\|_{[-\tau,0]} &\leq K_5(\gamma)e^{-\gamma t}\|x_{p-\sigma} - x^*_0\|_{[-\tau,0]} \\ &\leq 2K_5(\gamma)\exp(2K_h p)e^{-\gamma t}\|\varphi - x^*_\sigma\|_{[-\tau,0]}\end{aligned} \tag{6.148}$$

and

$$|\rho| \leq \overline{K}_\rho\|x_{p-\sigma} - x^*_0\|_{[-\tau,0]} \leq 2\overline{K}_\rho\exp(2K_h p)\|\varphi - x^*_\sigma\|_{[-\tau,0]}. \tag{6.149}$$

Set $K_\rho = 2\overline{K}_\rho\exp(2K_h p)$ so that (6.110) holds. By (6.148) and the periodicity of $x^*$, for $t \geq p - \sigma$,

$$\|x_t - x^*_{t+p+\sigma+\rho}\|_{[-\tau,0]} \leq 2K_5(\gamma)\exp(2K_h p)e^{-\gamma t}\|\varphi - x^*_\sigma\|_{[-\tau,0]}. \tag{6.150}$$

On the other hand, by (6.146), (6.112), and (6.149), we have, for $0 \leq t \leq p - \sigma$,

$$\begin{aligned}\|x_t - x^*_{t+p+\sigma+\rho}\|_{[-\tau,0]} &\leq \|x_t - x^*_{t+\sigma}\|_{[-\tau,0]} + \|x^*_{t+p+\sigma} - x^*_{t+p+\sigma+\rho}\|_{[-\tau,0]} \\ &\leq 2(1 + H\overline{K}_\rho)\exp(2K_h p)e^{\gamma p}e^{-\gamma t}\|\varphi - x^*_\sigma\|_{[-\tau,0]}.\end{aligned} \tag{6.151}$$

Upon setting $K_\gamma = 2\exp(2K_h p)\max(K_5(\gamma), e^{\gamma p}(1 + H\overline{K}_\rho))$, the combination of (6.150) and (6.151) implies that (6.111) holds, which completes the proof. ∎

**6.7. Proof of uniqueness.** In this section we show that if the delay $\tau$ is sufficiently large, then any SOPS $x^*$ of the DDER with delay $\tau$ is unique up to time translation, which will complete the proof of Theorem 3.8. The proof of uniqueness is similar to the proof of Corollary 19 in [80] and to the proof of Part 1 of Theorem A in [41]. The main tool which we use to prove the uniqueness of a SOPS is the *fixed point index*. For an in-depth discussion of the fixed point index and its properties, see [54]. We briefly introduce a special case of the fixed point index used here and remark on some of its relevant properties.

Suppose that $K$ is a closed, bounded, convex, infinite-dimensional subset of a Banach space and $f : K \to K$ is a continuous, compact function. For each relatively open subset $U$ of $K$, for which the set of fixed points of $f$ in $U$, $S_U = \{x \in U : f(x) = x\}$, is compact (possibly empty), there is an integer $\iota_K(f, U)$, called the fixed point index of $f$ on $U$. This index $\iota_K(f, \cdot)$ is uniquely characterized by the fact that it satisfies the following properties: additivity, homotopy, and normalization (see [54] for details of these properties). Here we state the additivity property since it will be used in the proof of uniqueness.

- *Additivity.* If $U_1$ and $U_2$ are disjoint subsets of $U$, where $U_1$, $U_2$, and $U$ are relatively open subsets of $K$, and if $S_U$ is compact and satisfies $S_U \subset U_1 \cup U_2$, then $\iota_K(f, U_j)$ is defined for $j = 1, 2$ and

$$\iota_K(f, U) = \iota_K(f, U_1) + \iota_K(f, U_2).$$

In addition, $\iota_K(f, \cdot)$ has the following useful properties:

(i) The fixed point index $\iota_K(f, K)$ is defined and equal to 1.

(ii) If $x \in K$ is an ejective fixed point of $f$ (see Definition 5.1), then there exists a relatively open subset $U$ of $K$ such that $x \in U$, $x$ is the only fixed point of $f$ in $U$ and $\iota_K(f, U) = 0$ (see, e.g., Corollary 1.1 in [50]).

(iii) If $x \in K$ is an attractive fixed point of $f$ (see Definition 6.33 below), then there exists a relatively open subset $V$ of $K$ such that $x \in V$, $x$ is the only fixed point of $f$ in $V$ and $\iota_K(f, V) = 1$ (see, e.g., Theorem 3.5 in [54]).

DEFINITION 6.33. *Suppose $x_0 \in K$ is a fixed point of $f$. Then $x_0$ is an* attractive fixed point *if there exists a relatively open neighborhood $U$ of $x_0$ in $K$ such that if $V$ is any relatively open neighborhood of $x_0$ in $K$, there exists a positive integer $n_0 = n_0(V)$ such that $f^n(x) \in V$ for all $n \geq n_0$ and $x \in U$.*

THEOREM 6.34. *Fix $\theta \in (0, 1)$ and let $\tau_\theta \geq \tau_0$ be as in Lemma* 6.31. *For each $\tau > \tau_\theta$ there exists a unique SOPS of the DDER with delay $\tau$ and $q_0 = -\tau$.*

*Proof.* Define $\widetilde{\mathcal{K}}$ and $\Lambda$ as in (5.18)–(5.20) and (5.26). Then $\widetilde{\mathcal{K}}$ is a closed, bounded, convex, infinite-dimensional subset of the Banach space $\mathcal{C}_{[-\tau,0]}$, and $\Lambda$ is continuous and compact. Thus, by property (i) above, the fixed point index of $\Lambda$ on $\widetilde{\mathcal{K}}$ is defined and $\iota_{\widetilde{\mathcal{K}}}(\Lambda, \widetilde{\mathcal{K}}) = 1$. Recall that $\hat{\varphi} \equiv 0$ is the unique constant fixed point of $\Lambda$. By Lemma 5.12, $\varphi \equiv 0$ is an ejective fixed point of $\Lambda$, so property (ii) above implies that there is a relatively open subset $\mathcal{U}$ of $\widetilde{\mathcal{K}}$ containing $\hat{\varphi} \equiv 0$ that does not contain any other fixed points of $\Lambda$ and such that $\iota_{\widetilde{\mathcal{K}}}(\Lambda, \mathcal{U}) = 0$. If $\hat{\varphi}$ is a nonconstant fixed point of $\Lambda$, it follows from Lemma 5.8 that the associated solution $\hat{x}^*$ of the DDER$^{\rm n}$ is a SOPS$^{\rm n}$. Moreover, by Lemma 5.5 and Theorem 6.32, the corresponding SOPS $x^*$ is exponentially stable. Conversely, if $x^*$ is an exponentially stable SOPS with $q_0 = -\tau$, then by Lemmas 5.5 and 6.15, the initial condition of the corresponding SOPS$^{\rm n}$, $\hat{x}_0^*$, is in $\widetilde{\mathcal{K}}$ and is a nonconstant fixed point of $\Lambda$. Thus, there is a one-to-one correspondence between nonconstant fixed points of $\Lambda$ and exponentially stable SOPS with $q_0 = -\tau$. Furthermore, it is straightforward to check that the exponential stability property of $x^*$ implies that $\hat{x}_0^*$ is an attractive fixed point of $\Lambda$. Therefore, by property (iii) above, for each nonconstant fixed point $\hat{\varphi} \not\equiv 0$ of $\Lambda$, there is a neighborhood $\mathcal{V}_{\hat{\varphi}}$ of $\hat{\varphi}$ that does not contain any other fixed points of $\Lambda$ and such that $\iota_{\widetilde{\mathcal{K}}}(\Lambda, \mathcal{V}_{\hat{\varphi}}) = 1$.

Let $\mathcal{S} = \{\hat{\varphi} \in \widetilde{\mathcal{K}} : \Lambda(\hat{\varphi}) = \hat{\varphi}\}$ denote the set of fixed points of $\Lambda$ in $\widetilde{\mathcal{K}}$. Since $\Lambda$ is continuous and compact, $\mathcal{S}$ is compact. From the above paragraph, each fixed point of $\Lambda$ is contained in a neighborhood that does not contain another fixed point of $\Lambda$, so by compactness, $\mathcal{S}$ is a finite set. Then by the additivity property of the fixed point index, and the fact that $\mathcal{S} \subset \mathcal{U} \cup (\bigcup_{\hat{\varphi} \in \mathcal{S} \setminus \{0\}} \mathcal{V}_{\hat{\varphi}}) \subset \widetilde{\mathcal{K}}$ and properties (ii) and (iii) above, we have

$$1 = \iota_{\widetilde{\mathcal{K}}}(\Lambda, \widetilde{\mathcal{K}}) = \iota_{\widetilde{\mathcal{K}}}(\Lambda, \mathcal{U}) + \sum_{\hat{\varphi} \in \mathcal{S} \setminus \{0\}} \iota_{\widetilde{\mathcal{K}}}(\Lambda, \mathcal{V}_{\hat{\varphi}}) = |\mathcal{S} \setminus \{0\}|,$$

where $|\mathcal{S} \setminus \{0\}|$ denotes the cardinality of the set $\mathcal{S} \setminus \{0\}$. Therefore $\mathcal{S}$ contains exactly one point besides $\hat{\varphi} \equiv 0$ and so $\Lambda$ has exactly one nonconstant fixed point. By the one-to-one correspondence between nonconstant fixed points of $\Lambda$ and SOPS$^{\rm n}$

with $\hat{q}_0 = -1$, there is a unique SOPS$^{\rm n}$ of the DDER$^{\rm n}$ with $\hat{q}_0 = -1$ and hence by Lemma 5.5, there is a unique SOPS of the DDER with $q_0 = -\tau$. ◻

*Proof of Theorem* 3.8. Fix $\theta \in (0,1)$ and set $\tau^* = \tau_\theta$, where $\tau_\theta \geq \tau_0$ is as in Lemma 6.31. Suppose $\tau > \tau^*$. By Theorem 6.34 there exists a unique SOPS $x^*$ of the DDER with delay $\tau$ and $q_0 = -\tau$. Thus, if $x^\dagger$ is another SOPS of the DDER such that $q_0^\dagger \neq -\tau$, then the time translated function $\tilde{x}^\dagger \in \mathcal{C}^+_{[-\tau,\infty)}$, defined by $\tilde{x}^\dagger(t) = x^\dagger(q_0^\dagger + \tau + t)$ for all $t \geq -\tau$, satisfies $\tilde{x}^\dagger = x^*$, which proves that $x^*$ is unique up to time translation. From Theorem 6.32, we see that $x^*$ is exponentially stable and thus any time translation of $x^*$ is also exponentially stable, which completes the proof. ◻

**Appendix A. One-dimensional Skorokhod problem.** Solutions of the DDER (1.1) can be thought of as solutions of the well-known (one-dimensional) Skorokhod problem. This problem was first introduced by Skorokhod [62] to construct solutions of one-dimensional stochastic differential equations with nonnegativity constraints. The one-dimensional Skorokhod problem and its multidimensional generalization are frequently used in the study of stochastic differential equations with state constraints (see, e.g., [16, 17] and the references therein). Before defining the problem, we first introduce the (one-dimensional) Skorokhod map and note a couple of its properties.

Define the *(one-dimensional) Skorokhod map* $(\Phi,\Psi) : \mathcal{C}_{[0,\infty)} \to \mathcal{C}^+_{[0,\infty)} \times \mathcal{C}^+_{[0,\infty)}$ by

$$\Phi(z)(t) = z(t) + \Psi(z)(t), \qquad t \geq 0, \tag{A.1}$$

$$\Psi(z)(t) = \sup_{0\leq s\leq t} (z(s))^-, \qquad t \geq 0. \tag{A.2}$$

The following are well-known properties of the Skorokhod map that follow from (A.1) and (A.2). For more details, see [12, 77].

PROPOSITION A.1. *For $z, z^\dagger \in \mathcal{C}_{[0,\infty)}$ and $t \geq 0$,*

$$\|\Phi(z) - \Phi(z^\dagger)\|_{[0,t]} \leq 2\|z - z^\dagger\|_{[0,t]},$$
$$\|\Psi(z) - \Psi(z^\dagger)\|_{[0,t]} \leq \|z - z^\dagger\|_{[0,t]}.$$

*It follows that the map $(\Phi,\Psi) : \mathcal{C}_{[0,\infty)} \to \mathcal{C}^+_{[0,\infty)} \times \mathcal{C}^+_{[0,\infty)}$ is continuous. (Recall that $\mathcal{C}_{[0,\infty)}$ is endowed with the topology of uniform convergence on compact time intervals.)*

PROPOSITION A.2. *For $z \in \mathcal{C}_{[0,\infty)}$,*

$$Osc(\Phi(z), [t_1,t_2]) \leq Osc(z, [t_1,t_2]),$$
$$Osc(\Psi(z), [t_1,t_2]) \leq Osc(z, [t_1,t_2]),$$

*for each $0 \leq t_1 \leq t_2 < \infty$, where for any $u \in \mathcal{C}_{[0,\infty)}$,*

$$Osc(u, [t_1,t_2]) = \sup_{t_1\leq s<t\leq t_2} |u(t) - u(s)|.$$

We now define the Skorokhod problem and give its solution in terms of the Skorokhod map.

DEFINITION A.3. *Let $z \in \mathcal{C}_{[0,\infty)}$ satisfy $z(0) \geq 0$. A pair $(x,y) \in \mathcal{C}^+_{[0,\infty)} \times \mathcal{C}^+_{[0,\infty)}$ is a solution of the* (one-dimensional) Skorokhod problem for *$z$ if the following hold:*

(i) *$x(t) = z(t) + y(t)$ for each $t \geq 0$,*
(ii) *$y(0) = 0$ and $y(\cdot)$ is nondecreasing, and*
(iii) *$\int_0^t x(s)dy(s) = 0$ for all $t \geq 0$.*

PROPOSITION A.4. *Given $z \in \mathcal{C}_{[0,\infty)}$, suppose that $z(0) \geq 0$. Then there exists a unique solution $(x, y) \in \mathcal{C}^+_{[0,\infty)} \times \mathcal{C}^+_{[0,\infty)}$ of the Skorokhod problem for $z$, given by*

$$(x, y) = (\Phi, \Psi)(z).$$

*Proof.* See, e.g., section 8.2 of [12]. ∎

To define the VE along solutions of the DDER, we have employed the following results on "directional derivatives" of the (one-dimensional) Skorokhod map, which are described in [44, 45] and Chapter 9 of [76]. For $z, w \in \mathcal{C}_{[0,\infty)}$ and $\varepsilon > 0$, define approximate derivatives $\partial^\varepsilon_w \Phi(z), \partial^\varepsilon_w \Psi(z) \in \mathcal{C}_{[0,\infty)}$ by

$$\partial^\varepsilon_w \Phi(z) = \frac{\Phi(z + \varepsilon w) - \Phi(z)}{\varepsilon} = w + \partial^\varepsilon_w \Psi(z), \tag{A.3}$$

$$\partial^\varepsilon_w \Psi(z) = \frac{\Psi(z + \varepsilon w) - \Psi(z)}{\varepsilon}. \tag{A.4}$$

For $z, w \in \mathcal{C}_{[0,\infty)}$ and $t \geq 0$, define

$$R(z, w)(t) = \begin{cases} 0 & \text{if } \sup_{0 \leq s \leq t} z(s) < 0, \\ \sup_{s \in \mathbb{S}_z(t)} w(s) \vee 0 & \text{if } \sup_{0 \leq s \leq t} z(s) = 0, \\ \sup_{s \in \mathbb{S}_z(t)} w(s) & \text{if } \sup_{0 \leq s \leq t} z(s) > 0, \end{cases} \tag{A.5}$$

where

$$\mathbb{S}_z(t) = \left\{ s \in [0, t] : z(s) = \sup_{0 \leq u \leq t} z(u) \vee 0 \right\}. \tag{A.6}$$

Note that we have used a slightly different definition for $\mathbb{S}_z(t)$ than the one used in [44, 45]. There the authors use $\tilde{\mathbb{S}}_z(t) = \{s \in [0, t] : z(s) = \sup_{0 \leq u \leq t} z(u)\}$ in place of $\mathbb{S}_z(t)$. However, the functional $R(z, w)(t)$ is defined to be zero at times $t$ where $\mathbb{S}_z(t) \neq \tilde{\mathbb{S}}_z(t)$ and so it remains unchanged. Here we have modified the definition to simplify subsequent proofs. Finally, for $z, w \in \mathcal{C}_{[0,\infty)}$, define the functions $\partial_w \Phi(z) : [0, \infty) \to \mathbb{R}$ and $\partial_w \Psi(z) : [0, \infty) \to \mathbb{R}$, for $t \geq 0$, by

$$\partial_w \Phi(z)(t) = w(t) + \partial_w \Psi(z)(t), \tag{A.7}$$

$$\partial_w \Psi(z)(t) = R(-z, -w)(t). \tag{A.8}$$

Now, given $z, w$ and $\{w^\varepsilon : 0 < \varepsilon \leq \varepsilon^*\}$, $\varepsilon^* > 0$, in $\mathcal{C}_{[0,\infty)}$ such that $w^\varepsilon \to w$ uniformly on compact intervals in $[0, \infty)$ as $\varepsilon \downarrow 0$, we are interested in the pointwise limits of $\partial^\varepsilon_{w^\varepsilon} \Phi(z)$ and $\partial^\varepsilon_{w^\varepsilon} \Psi(z)$ as $\varepsilon \downarrow 0$. In [44], Mandelbaum and Massey considered this problem under the restriction that $z(0) = 0$, $w^\varepsilon = w$ for all $\varepsilon > 0$, and $\partial_w \Phi(z)$ has a finite number of discontinuities in any compact interval. In Theorem 9.5.3 of [76, Chapter 9], Whitt extended their results by relaxing the condition that $z(0) = 0$ and that $\partial_w \Phi(z)$ has a finite number of discontinuities in any compact interval. Furthermore, that theorem allows $w^\varepsilon$ to be $\varepsilon$ dependent but requires that $z$ is Lipschitz continuous. More recently, in [45], Mandelbaum and Ramanan proved that $\partial^\varepsilon_{w^\varepsilon} \Phi(z)$ converges to $\partial_w \Phi(z)$ pointwise and uniformly on compact sets of continuity points of $\partial_w \Phi(z)$ when $w^\varepsilon$ is monotonically decreasing and converges pointwise to $w$. Our specific case is not covered by any of the aforementioned papers, but it is a simple extension of Theorem 1.1 in [45]. Indeed, we have the following proposition.

PROPOSITION A.5. *Let $z, w$ and $\{w^\varepsilon : 0 < \varepsilon \leq \varepsilon^*\}$ be in $\mathcal{C}_{[0,\infty)}$ such that $w^\varepsilon \to w$ uniformly on compact intervals in $[0, \infty)$ as $\varepsilon \downarrow 0$. Then as $\varepsilon \downarrow 0$,*

$$\partial^\varepsilon_{w^\varepsilon} \Phi(z) \to \partial_w \Phi(z) \qquad \textit{and} \qquad \partial^\varepsilon_{w^\varepsilon} \Psi(z) \to \partial_w \Psi(z),$$

*where the convergence is pointwise and uniform on compact sets of continuity points of $\partial_w\Psi(z)$ and $\partial_w\Phi(z)$, respectively. Furthermore, both $\partial_w\Phi(z)$ and $\partial_w\Psi(z)$ are in $\mathcal{D}_{[0,\infty)}$.*

*Proof.* By Theorem 1.1(i) in [45], we have $\partial_w\Psi(z) \in \mathcal{D}_{[0,\infty)}$ and $\partial_w^\varepsilon\Psi(z)$ converges to $\partial_w\Psi(z)$ pointwise and uniformly on compact sets of continuity points of $\partial_w\Phi(z)$ as $\varepsilon \downarrow 0$. (Note that in [45], the authors use $\mathcal{D}_{\lim}$ to denote the space of functions on $[0,\infty)$ with finite left and right limits, which we denote by $\mathcal{D}_{[0,\infty)}$.) To prove that $\partial_{w^\varepsilon}^\varepsilon\Psi(z)$ also converges to $\partial_w\Psi(z)$ pointwise and uniformly on compact sets of continuity points of $\partial_w\Phi(z)$, it suffices, by the triangle inequality, to show that $\lim_{\varepsilon\downarrow 0}\|\partial_{w^\varepsilon}^\varepsilon\Psi(z) - \partial_w^\varepsilon\Psi(z)\|_{[0,t]} = 0$ for each $t \geq 0$. To show this, we use the Lipschitz continuity of $\Psi$ and the uniform convergence of $w^\varepsilon$ to $w$ as follows: for each $t \geq 0$, we have

$$\begin{aligned}\|\partial_w^\varepsilon\Psi(z) - \partial_{w^\varepsilon}^\varepsilon\Psi(z)\|_{[0,t]} &= \varepsilon^{-1}\|\Psi(z+\varepsilon w) - \Psi(z+\varepsilon w^\varepsilon)\|_{[0,t]} \\ &\leq \|w^\varepsilon - w\|_{[0,t]} \to 0 \text{ as } \varepsilon \downarrow 0.\end{aligned}$$

Last, the fact that $\partial_{w^\varepsilon}^\varepsilon\Phi(z)$ converges to $\partial_w\Phi(z)$ pointwise and uniformly on compact sets of continuity points of $\partial_w\Phi(z)$ follows from the above convergence results and the definition of $\partial_{w^\varepsilon}^\varepsilon\Phi(z)$ given in (A.3), but with $w^\varepsilon$ in place of $w$. ∎

PROPOSITION A.6. *For $z, w, w^\dagger \in \mathcal{C}_{[0,\infty)}$, and $t \geq 0$,*

$$\begin{aligned}\|\partial_w\Phi(z) - \partial_{w^\dagger}\Phi(z)\|_{[0,t]} &\leq 2\|w - w^\dagger\|_{[0,t]}, \\ \|\partial_w\Psi(z) - \partial_{w^\dagger}\Psi(z)\|_{[0,t]} &\leq \|w - w^\dagger\|_{[0,t]}.\end{aligned}$$

*Proof.* For $z, w, w^\dagger \in \mathcal{C}_{[0,\infty)}$, we have

$$\|R(z,w) - R(z,w^\dagger)\|_{[0,t]} \leq \Big\|\sup_{s\in\mathbb{S}_z(\cdot)} w(s) - \sup_{s\in\mathbb{S}_z(\cdot)} w^\dagger(s)\Big\|_{[0,t]} \leq \|w - w^\dagger\|_{[0,t]}.$$

The proposition then follows from (A.7)–(A.8). ∎

**Appendix B. Variational equation.** In this section, we introduce the notion of a VE along a solution $x$ of the DDER. As we will see, solutions of our VE differ considerably from solutions to the VE in the unconstrained setting. In particular, with the lower boundary constraint in the DDER, the VE is no longer linear and its solutions can be discontinuous.

Recall that $\mathcal{D}_{[-\tau,0]}$ is the space of functions from the interval $[-\tau,0]$ to $\mathbb{R}$ that have finite left and right limits at all $t \in (-\tau,0)$ and finite one-sided limits at the endpoints of $[-\tau,0]$. Given $\varphi \in \mathcal{C}^+_{[-\tau,0]}$, define

$$\text{(B.1)} \qquad \mathcal{D}^\varphi = \{\psi \in \mathcal{D}_{[-\tau,0]} : 1_{\{\varphi(s)=0\}}\psi(s) \geq 0 \text{ for all } s \in [-\tau,0]\}.$$

Here $1_{\{\varphi(s)=0\}}$ is the indicator that is one if $\varphi(s) = 0$ and zero otherwise. If we consider $\varphi$ as the initial condition of a solution to the DDER, then $\mathcal{D}^\varphi$ denotes the directions in $\mathcal{D}_{[-\tau,0]}$ that we allow $\varphi$ to be perturbed. For $\varphi \in \mathcal{C}^+_{[-\tau,0]}$, let $\mathcal{C}^\varphi = \{\psi \in \mathcal{C}_{[-\tau,0]} : \varphi + \varepsilon\psi \in \mathcal{C}^+_{[-\tau,0]}$ for all sufficiently small $\varepsilon > 0\}$. To ensure that the VE is well defined, we assume that the function $f : \mathcal{C}^+_{[-\tau,0]} \to \mathbb{R}$ in (1.1) satisfies the following regularity properties.

*Assumption* B.1. There exists $K_f < \infty$ such that for all $\varphi, \varphi^\dagger \in \mathcal{C}^+_{[-\tau,0]}$,

$$\text{(B.2)} \qquad |f(\varphi) - f(\varphi^\dagger)| \leq K_f\|\varphi - \varphi^\dagger\|_{[-\tau,0]}.$$

*Assumption* B.2. Given $\varphi \in \mathcal{C}^+_{[-\tau,0]}$, then for each $\psi \in \mathcal{D}^\varphi$ there is a unique *derivative of $f$ in the direction $\psi$*, denoted $\partial_\psi f(\varphi)$, that satisfies the following three properties:

(i) Whenever $\{\psi_n\}_{n=1}^\infty$ is a uniformly bounded sequence in $\mathcal{C}_{[-\tau,0]}$ that converges *pointwise* to $\psi \in \mathcal{D}^\varphi$ as $n \to \infty$ and $\{\varepsilon_n\}_{n=1}^\infty$ is a sequence of positive real numbers such that $\varepsilon_n \downarrow 0$ as $n \to \infty$ and $\varphi + \varepsilon_n \psi_n \in \mathcal{C}^+_{[-\tau,0]}$ for each $n$, we have

$$\partial_\psi f(\varphi) = \lim_{n\to\infty} \frac{f(\varphi + \varepsilon_n \psi_n) - f(\varphi)}{\varepsilon_n}. \tag{B.3}$$

(ii) If $a, b \in \mathbb{R}$ and $\psi, \psi^\dagger \in \mathcal{D}^\varphi$ such that $a\psi + b\psi^\dagger \in \mathcal{D}^\varphi$, then

$$\partial_{a\psi+b\psi^\dagger} f(\varphi) = a\partial_\psi f(\varphi) + b\partial_{\psi^\dagger} f(\varphi).$$

(iii) For all $\psi, \psi^\dagger \in \mathcal{D}^\varphi$

$$|\partial_\psi f(\varphi) - \partial_{\psi^\dagger} f(\varphi)| \le K_f \|\psi - \psi^\dagger\|_{[-\tau,0]}. \tag{B.4}$$

LEMMA B.1. *Let $f : \mathcal{C}^+_{[-\tau,0]} \to \mathbb{R}$ be given by*

$$f(\varphi) = \int_{[-\tau,0]} \zeta(\varphi(s)) d\mu(s) \text{ for all } \varphi \in \mathcal{C}^+_{[-\tau,0]},$$

*where $\zeta : \mathbb{R}_+ \to \mathbb{R}$ is a function that is Lipschitz continuous (with Lipschitz constant $K_\zeta$) and continuously differentiable on $\mathbb{R}_+$, and $\mu$ is a finite measure on the interval $[-\tau, 0]$. Then $f$ satisfies Assumptions* B.1 *and* B.2 *with $K_f = K_\zeta \mu([-\tau, 0])$ and*

$$\partial_\psi f(\varphi) = \int_{[-\tau,0]} \zeta'(\varphi(s))\psi(s) d\mu(s) \tag{B.5}$$

*for all $\psi \in \mathcal{D}^\varphi$, where $\zeta' : \mathbb{R}_+ \to \mathbb{R}$ denotes the first derivative of $\zeta$.*

*Proof.* The Lipschitz continuity of $\zeta$ implies Assumption B.1 holds with $K_f = K_\zeta \mu([-\tau, 0])$. We now prove that Assumption B.2 holds. Fix $\varphi \in \mathcal{C}^+_{[-\tau,0]}$, $\psi \in \mathcal{D}^\varphi$ and let $\{\psi_n\}_{n=1}^\infty$ and $\{\varepsilon_n\}_{n=1}^\infty$ be sequences as in part (i) of Assumption B.2. Since $\zeta$ is continuously differentiable, we have, for each $s \in [-\tau, 0]$,

$$\lim_{n\to\infty} \frac{\zeta(\varphi(s) + \varepsilon_n \psi_n(s)) - \zeta(\varphi(s))}{\varepsilon_n} = \zeta'(\varphi(s))\psi(s).$$

Then, due to the Lipschitz continuity of $\zeta$, we can use bounded convergence to obtain that (B.5) holds. Part (ii) of Assumption B.2 follows because the integral in (B.5) is linear in $\psi$. Part (iii) is immediate from the Lipschitz continuity of $\zeta$. ∎

*Example* B.1. Let $f : \mathcal{C}^+_{[-\tau,0]} \to \mathbb{R}$ be given by

$$f(\varphi) = h(\varphi(-\tau)) \text{ for all } \varphi \in \mathcal{C}^+_{[-\tau,0]},$$

where $h : \mathbb{R}_+ \to \mathbb{R}$ is continuously differentiable with bounded derivative $h' : \mathbb{R}_+ \to \mathbb{R}$. Then Lemma B.1, with $\zeta = h$ and $\mu$ equal to the point mass at $s = -\tau$, implies that $f$ satisfies Assumptions B.1 and B.2 with

$$\partial_\psi f(\varphi) = h'(\varphi(-\tau))\psi(-\tau) \text{ for all } \psi \in \mathcal{D}^\varphi, \qquad \varphi \in \mathcal{C}^+_{[-\tau,0]}.$$

Throughout the remainder of this section, we assume that $f$ satisfies Assumptions B.1 and B.2 and we fix a solution $x$ of the DDER and define $z$ as in (2.3).

Definition B.2. *A function $v \in \mathcal{D}_{[-\tau,\infty)}$ is a solution of the VE along $x$ if for each $s \geq 0$, $v_s \in \mathcal{D}^{x_s}$, the function $s \to \partial_{v_s} f(x_s)$ is Lebesgue integrable on each compact set in $[0,\infty)$, and $v$ satisfies*

$$v(t) = \partial_w \Phi(z)(t), \qquad t \geq 0, \tag{B.6}$$

*where $\Phi$ denotes the Skorokhod map given by* (A.1) *and* (A.2), *$w \in \mathcal{C}_{[0,\infty)}$ is defined by*

$$w(t) = v(0) + \int_0^t \partial_{v_s} f(x_s) ds, \qquad t \geq 0, \tag{B.7}$$

*and the directional derivative of $\Phi$ at $z$ in the direction $w$ is denoted by $\partial_w \Phi(z)$ and is well defined as an element of $\mathcal{D}_{[0,\infty)}$ by Proposition* A.5.

Suppose $\psi \in \mathcal{C}^{x_0}$. Then there exists $\varepsilon^* > 0$ such that $x_0 + \varepsilon\psi \in \mathcal{C}^+_{[-\tau,0]}$ for all $\varepsilon \in (0,\varepsilon^*]$. For each $\varepsilon \in (0,\varepsilon^*]$ let $x^\varepsilon$ denote the unique solution of the DDER with $x_0^\varepsilon = x_0 + \varepsilon\psi$ and define $v^\varepsilon \in \mathcal{C}_{[-\tau,\infty)}$ by

$$v^\varepsilon = \frac{x^\varepsilon - x}{\varepsilon}. \tag{B.8}$$

Additionally, define $z^\varepsilon \in \mathcal{C}_{[0,\infty)}$ as in (2.3) but with $x$ and $z$ replaced with $x^\varepsilon$ and $z^\varepsilon$, respectively, and define $w^\varepsilon \in \mathcal{C}_{[0,\infty)}$ by

$$w^\varepsilon(t) = \frac{z^\varepsilon(t) - z(t)}{\varepsilon} = \psi(0) + \int_0^t \frac{f(x_s + \varepsilon v_s^\varepsilon) - f(x_s)}{\varepsilon} ds, \qquad t \geq 0. \tag{B.9}$$

Recall that a family $\{u^\varepsilon : 0 < \varepsilon \leq \varepsilon^*\}$ in $\mathcal{D}_{[0,\infty)}$ converges to $u \in \mathcal{D}_{[0,\infty)}$ uniformly on compact intervals of continuity (u.o.c.c.), as $\varepsilon \downarrow 0$ if for each compact interval $I$ contained in $[0,\infty)$ on which $u$ is continuous, $u^\varepsilon$ converges to $u$ uniformly on $I$ as $\varepsilon \downarrow 0$. We have the following theorem on the existence and uniqueness of a solution of the VE given an appropriate initial condition as well as the pointwise and u.o.c.c. convergence of $v^\varepsilon$ to $v$ as $\varepsilon \downarrow 0$.

Theorem B.3. *Given $\psi \in \mathcal{C}^{x_0}$, there exists a unique solution $v$ of the VE along $x$ with $v_0 = \psi$. Furthermore, as $\varepsilon \downarrow 0$, $v^\varepsilon \to v$ pointwise as well as uniformly on compact intervals of continuity of $v$ in $[-\tau,\infty)$, and $w^\varepsilon \to w$ uniformly on compact intervals in $[0,\infty)$. Here $w$ is defined by* (B.7) *and for each $\varepsilon \in (0,\varepsilon^*]$, $v^\varepsilon$ and $w^\varepsilon$ are defined by* (B.8) *and* (B.9), *respectively.*

In preparation for proving Theorem B.3, we prove the following lemmas.

Lemma B.4. *For each $\varepsilon \in (0,\varepsilon^*]$,*

$$\|v^\varepsilon\|_{[-\tau,t]} \leq 2\|\psi\|_{[-\tau,0]} \exp(2K_f t), \qquad t \geq 0. \tag{B.10}$$

*Proof.* Fix $t \geq 0$. By (B.9) and (B.2), for each $\varepsilon \in (0,\varepsilon^*]$ and all $s \in [0,t]$ we have

$$|w^\varepsilon(s)| \leq \|\psi\|_{[-\tau,0]} + K_f \int_0^s \|v^\varepsilon\|_{[-\tau,r]} dr.$$

By taking the supremum over $s$ in the interval $[0,t]$, using (2.4) and applying the Lipschitz continuity of the Skorokhod map (see Proposition A.1), we have

$$\|v^\varepsilon\|_{[0,t]} \leq 2\|w^\varepsilon\|_{[0,t]} \leq 2\|\psi\|_{[-\tau,0]} + 2K_f \int_0^t \|v^\varepsilon\|_{[-\tau,s]} ds.$$

We can extend the supremum norm on the left to the interval $[-\tau, t]$ and apply Gronwall's inequality to obtain (B.10). ◻

LEMMA B.5. *Suppose $v, v^\dagger$ are solutions of the VE along $x$. Then*

$$\|v - v^\dagger\|_{[-\tau,t]} \le 2\exp(2K_f t)\|v - v^\dagger\|_{[-\tau,0]}, \qquad t \ge 0. \tag{B.11}$$

*Proof.* Suppose $v$ and $v^\dagger$ are solutions of the VE along $x$. Define $w \in \mathcal{C}_{[0,\infty)}$ as in (B.7) and define $w^\dagger \in \mathcal{C}_{[0,\infty)}$ as in (B.7), but with $v^\dagger$ in place of $v$. By definition,

$$v(t) = \partial_w \Phi(z)(t), \qquad v^\dagger(t) = \partial_{w^\dagger}\Phi(z)(t), \qquad t \ge 0,$$

where $z$ is defined as in (2.3). For $t \ge 0$, by (B.7) and (B.4), we have for all $s \in [0, t]$,

$$|w(s) - w^\dagger(s)| \le |v(0) - v^\dagger(0)| + K_f \int_0^s \|v - v^\dagger\|_{[-\tau,u]} du.$$

By taking the supremum over $s \in [0, t]$ and applying Proposition A.6, we have

$$\|v - v^\dagger\|_{[0,t]} \le 2\|v - v^\dagger\|_{[-\tau,0]} + 2K_f \int_0^t \|v - v^\dagger\|_{[-\tau,u]} du.$$

The supremum norm on the left can be extended to the interval $[-\tau, t]$, after which a simple application of Gronwall's inequality yields (B.11). ◻

*Proof of Theorem* B.3. Uniqueness follows from (B.11). We now establish existence. Given $\varepsilon \in (0, \varepsilon^*]$ and $t \ge 0$, it follows from (B.9), (B.2), and Lemma B.4 that, for all $0 \le t_1 \le t_2 \le t$,

$$|w^\varepsilon(t_2) - w^\varepsilon(t_1)| \le 2K_f \|\psi\|_{[-\tau,0]} \exp(2K_f t)|t_2 - t_1|.$$

Thus, $\{w^\varepsilon : 0 < \varepsilon \le \varepsilon^*\}$ is uniformly bounded and uniformly Lipschitz continuous on each interval $[0, t]$ and therefore, by the Arzelà–Ascoli theorem, is relatively compact in $\mathcal{C}_{[0,t]}$. Since $t \ge 0$ was arbitrary, using a diagonal sequence argument, we have for any sequence $\{\varepsilon_n\}_{n=1}^\infty$ in $(0, \varepsilon^*]$ with $\varepsilon_n \downarrow 0$ as $n \to \infty$, there exists a subsequence, also denoted $\{\varepsilon_n\}_{n=1}^\infty$, and $w \in \mathcal{C}_{[0,\infty)}$ such that $w^{\varepsilon_n}$ converges to $w$ uniformly on compact intervals in $[0, \infty)$ as $n \to \infty$. Applying Proposition A.5, we have

$$\lim_{n\to\infty} v^{\varepsilon_n} = \lim_{n\to\infty} \frac{\Phi(z + \varepsilon_n w^{\varepsilon_n}) - \Phi(z)}{\varepsilon_n} = \partial_w \Phi(z),$$

where the convergence is pointwise and u.o.c.c. in $[0, \infty)$. Define $v \in \mathcal{D}_{[-\tau,\infty)}$ by $v(t) = \psi(t)$ for $t \in [-\tau, 0]$ and $v(t) = \partial_w \Phi(z)(t)$ for $t \ge 0$. Note that this is a proper definition because $\partial_w \Phi(z)(0) = \psi(0)$ since $\psi \in \mathcal{C}^{x_0}$. Then $v^{\varepsilon_n} \to v$ pointwise and u.o.c.c. on $[-\tau, \infty)$ as $n \to \infty$.

For each $s \ge 0$, $x_s^{\varepsilon_n} = x_s + \varepsilon_n v_s^{\varepsilon_n} \in \mathcal{C}^+_{[-\tau,0]}$ for all $n$. Suppose that $x_s(u) = 0$ for some $s \ge 0$ and some $u \in [-\tau, 0]$. Then $v_s^{\varepsilon_n}(u) = \varepsilon_n^{-1} x_s^{\varepsilon_n}(u) \ge 0$ for all $n$ and taking limits as $n \to \infty$, we must have $v_s(u) \ge 0$. Since this holds for all $s \ge 0$ and $u \in [-\tau, 0]$, it follows that $v_s \in \mathcal{D}^{x_s}$ for all $s \ge 0$, and so by part (i) of Assumption B.2 and Lemma B.4,

$$\lim_{n\to\infty} \frac{f(x_s + \varepsilon_n v_s^{\varepsilon_n}) - f(x_s)}{\varepsilon_n} = \partial_{v_s} f(x_s) \qquad \text{for all } s \ge 0. \tag{B.12}$$

By (B.2) and Lemma B.4, for each $n$ we have, for all $s \geq 0$,

$$\left|\frac{f(x_s + \varepsilon_n v_s^{\varepsilon_n}) - f(x_s)}{\varepsilon_n}\right| \leq K_f \|v^{\varepsilon_n}\|_{[-\tau,s]} \leq 2K_f\|\psi\|_{[-\tau,0]}\exp(2K_f s). \tag{B.13}$$

Since the function $s \to \partial_{v_s} f(x_s)$ is the pointwise limit of a sequence of Borel measurable functions, it is also Borel measurable. Furthermore, by (B.12) and (B.13), on each compact set in $[0,\infty)$, the function is bounded and hence integrable. Then by dominated convergence, implied by (B.13), we have $w$ satisfies, for $t \geq 0$,

$$\begin{aligned} w(t) = \lim_{n\to\infty} w^{\varepsilon_n}(t) &= \psi(0) + \lim_{n\to\infty}\int_0^t \frac{f(x_s + \varepsilon_n v_s^{\varepsilon_n}) - f(x_s)}{\varepsilon_n} ds \\ &= \psi(0) + \int_0^t \partial_{v_s} f(x_s) ds. \end{aligned}$$

This establishes the existence of a unique solution $v$ to the VE along $x$ with $v_0 = \psi$. Furthermore, given a sequence $\{\varepsilon_n\}_{n=1}^\infty$ in $(0,\varepsilon^*]$ with $\varepsilon_n \downarrow 0$ as $n \to \infty$, there is a subsequence along which $v^{\varepsilon_n}$ converges pointwise and u.o.c.c. on $[0,\infty)$ to $v$ and along which $w^{\varepsilon_n}$ converges uniformly on compact time intervals in $[0,\infty)$ to $w$, defined as in (B.7). Since this is true for every sequence $\{\varepsilon_n\}_{n=1}^\infty$ with $\varepsilon_n \downarrow 0$ as $n \to \infty$, it follows from a standard real analysis argument that the family $\{v^\varepsilon : 0 < \varepsilon \leq \varepsilon^*\}$ converges to $v$ pointwise and u.o.c.c. on $[0,\infty)$ as $\varepsilon \downarrow 0$ and that the family $\{w^\varepsilon : 0 < \varepsilon \leq \varepsilon^*\}$ converges to $w$ uniformly on compact time intervals in $[0,\infty)$ as $\varepsilon \downarrow 0$. ☐

In the following lemma we further describe solutions of the VE along a solution $x$ of the DDER.

LEMMA B.6. *Suppose that $x$ is a solution of the DDER and $v$ is a solution of the VE along $x$. Then for $0 \leq t_1 < t_2 < \infty$,*

(i) *if $x(t) > 0$ for all $t \in (t_1, t_2)$, then*

$$v(t) = v(t_1) + \int_{t_1}^t \partial_{v_s} f(x_s) ds, \qquad t \in [t_1, t_2);$$

(ii) *if $x(t) > 0$ for all $t \in (t_1, t_2)$ and $x(t_2) = 0$, then*

$$v(t_2) = \left(v(t_1) + \int_{t_1}^{t_2} \partial_{v_s} f(x_s) ds\right)^+;$$

(iii) *if $x(t) = 0$ for all $t \in [t_1, t_2]$ and $f(x_t) = 0$ for all $t \in [t_1, t_2]$, then*

$$v(t) = v(t_1) + \int_{t_1}^t \partial_{v_s} f(x_s) ds + \sup_{s\in[t_1,t]}\left(-v(t_1) - \int_{t_1}^s \partial_{v_u} f(x_u) du\right) \vee 0, \qquad t \in [t_1, t_2];$$

(iv) *if $x(t) = 0$ for all $t \in (t_1, t_2]$ and $f(x_t) < 0$ for all $t \in (t_1, t_2)$, then*

$$v(t) = 0, \qquad t \in (t_1, t_2].$$

*Proof.* Define $z$ and $w$ as in (2.3) and (B.7), respectively. By (B.6), (A.7), and (A.8), we have, for $0 \leq t_1 \leq t$,

$$v(t) = v(t_1) + \int_{t_1}^t \partial_{v_s} f(x_s) ds - R(-z,-w)(t_1) + R(-z,-w)(t). \tag{B.14}$$

Throughout this proof, given $x \in \mathcal{C}_{[0,\infty)}$, we let $\overline{x}(t) = \sup_{0\leq s\leq t} x(s)$ for all $t \geq 0$.

*Proof of* (i). By (2.2) and (2.1), we have $-z(t) < \overline{-z}(t) \vee 0$ for each $t \in (t_1, t_2)$. Fix such a $t$. If $\overline{-z}(t) < 0$, then $\overline{-z}(t_1) < 0$ and (i) follows from (B.14) and (A.5). If $\overline{-z}(t) \geq 0$, then $-z(s) < \overline{-z}(t) = \overline{-z}(t_1)$ for all $s \in (t_1, t]$ and so $\mathbb{S}_{-z}(t) = \mathbb{S}_{-z}(t_1)$. Thus $R(-z,-w)(t) = R(-z,-w)(t_1)$ and (i) follows from (B.14).

*Proof of* (ii). By (2.2)–(2.1), we have $-z(t) < \overline{-z}(t) \vee 0$ for each $t \in (t_1, t_2)$ and also that $-z(t_2) = \overline{-z}(t_2) \vee 0 = \overline{-z}(t_1) \vee 0$. Therefore (A.6) implies that $\mathbb{S}_{-z}(t_2) = \mathbb{S}_{-z}(t_1) \cup \{t_2\}$. Then by (A.5), we have

$$\begin{aligned} R(-z,-w)(t_2) &= \begin{cases} (-w(t_2)) \vee 0 & \text{if } \overline{-z}(t_1) < 0,\ \overline{-z}(t_2) = 0, \\ \sup_{s \in \mathbb{S}_{-z}(t_1)} (-w(s)) \vee (-w(t_2)) \vee 0 & \text{if } \overline{-z}(t_1) = \overline{-z}(t_2) = 0, \\ \sup_{s \in \mathbb{S}_{-z}(t_1)} (-w(s)) \vee (-w(t_2)) & \text{if } \overline{-z}(t_1) = \overline{-z}(t_2) > 0, \end{cases} \\ &= R(-z,-w)(t_1) \vee (-w(t_2)). \end{aligned}$$

By (B.6) and (A.3)–(A.6),

$$\begin{aligned} v(t_2) &= w(t_2) + R(-z,-w)(t_1) \vee (-w(t_2)) \\ &= \left( w(t_1) + R(-z,-w)(t_1) + \int_{t_1}^{t_2} \partial_{v_s} f(x_s) ds \right)^+ \\ &= \left( v(t_1) + \int_{t_1}^{t_2} \partial_{v_s} f(x_s) ds \right)^+. \end{aligned}$$

*Proof of* (iii). By (2.2)–(2.1), we have $-z(t) = \overline{-z}(t) \geq 0$ for each $t \in [t_1, t_2]$ and, since $f$ is zero there, $z$ is constant there. Thus, for $t \in [t_1, t_2]$, $\mathbb{S}_{-z}(t) = \mathbb{S}_{-z}(t_1) \cup [t_1, t]$ and by (B.14),

$$\begin{aligned} v(t) &= v(t_1) + \int_{t_1}^{t} \partial_{v_s} f(x_s) ds + \left( \sup_{s \in [t_1, t]} (-w(s)) - R(-z,-w)(t_1) \right) \vee 0 \\ &= v(t_1) + \int_{t_1}^{t} \partial_{v_s} f(x_s) ds + \sup_{s \in [t_1, t]} \left( -v(t_1) - \int_{t_1}^{s} \partial_{v_u} f(x_u) du \right) \vee 0. \end{aligned}$$

Proof of (iv): By (2.2)–(2.1), for each $t \in (t_1, t_2]$, we have $-z(t) = \overline{-z}(t) > 0$ and $\mathbb{S}_{-z}(t) = \{t\}$. Thus $v(t) = w(t) + (-w(t)) = 0$ for all such $t$. ◻

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## REFERENCES


[1] D. F. Anderson, *A modified next reaction method for simulating chemical systems with time dependent propensities and delays*, J. Chem. Phys., 127 (2007), 214107.

[2] O. Arino, K. P. Hadeler, and M. L. Hbid, *Existence of periodic solutions for delay differential equations with state-dependent delay*, J. Differential Equations, 144 (1998), pp. 263–301.

[3] O. Arino, M. Hbid, and E. Ait Dads, eds., *Delay Differential Equations with Applications*, Springer, New York, 2002.

[4] F. M. Atay, *Periodic solutions of delay-differential equations with a restorative condition*, Nonlinear Anal., 45 (2001), pp. 555–576.

[5] R. D. Bliss, P. R. Painter, and A. G. Marr, *Role of feedback inhibition in stabilizing the classical operon*, J. Theoret. Biol., 97 (1982), pp. 177–193.

[6] D. Bratsun, D. Volfson, L. S. Tsimring, and J. Hasty, *Delay-induced stochastic oscillations in gene regulation*, Proc. Natl. Acad. Sci. USA, 102 (2005), pp. 14593–14598.

[7] F. Brauer and C. Castillo-Chávez, *Mathematical Models in Population Biology and Epidemiology*, Springer-Verlag, New York, 2012.

[8] F. E. Browder, *A further generalization of the Schauder fixed point theorem*, Duke Math. J., 32 (1965), pp. 575–578.

[9] S.-N. Chow and J. K. Hale, *Methods of Bifurcation Theory*, Springer-Verlag, New York, 1982.

[10] S.-N. Chow and H.-O. Walther, *Characteristic multipliers and stability of symmetric periodic solutions of* $\dot{x}(t) = g(x(t-1))$, Trans. Amer. Math. Soc., 307 (1988), pp. 127–142.

[11] S.-N. Chow, *Existence of periodic solutions of autonomous functional differential equations*, J. Differential Equations, 15 (1974), pp. 350–378.

[12] K. L. Chung and R. J. Williams, *Introduction to Stochastic Integration*, Birkhäuser, Boston, 1990.

[13] S. Deb and R. Srikant, *Global stability of congestion controllers for the Internet*, IEEE Trans. Automat. Control, 48 (2003), pp. 1055–1060.

[14] O. Diekmann, S. A. van Gils, S. M. Verduyn Lunel, and H.-O. Walther, *Delay Equations: Functional-, Complex-, and Nonlinear Analysis*, Springer-Verlag, New York, 1995.

[15] G. Dunkel, *Single species model for population growth depending on past history*, in Seminar on Differential Equations and Dynamical Systems, G. S. Jones, ed., Springer-Verlag, New York, 1968.

[16] P. Dupuis and K. Ramanan, *Convex duality and the Skorokhod problem.* I, Probab. Theory Related Fields, 115 (1999), pp. 153–195.

[17] P. Dupuis and K. Ramanan, *Convex duality and the Skorokhod problem.* II, Probab. Theory Related Fields, 115 (1999), pp. 197–236.

[18] T. Erneux, *Applied Delay Differential Equations*, Springer, New York, 2009.

[19] K. Gopalsamy, *Stability and Oscillations in Delay Differential Equations of Population Dynamics*, Kluwer Academic, Dordrecht, the Netherlands, 1992.

[20] J. R. Graef and J. K. Hale, eds., *Oscillation and Dynamics in Delay Equations*, AMS, Providence, RI, 1992.

[21] R. B. Grafton, *A periodicity theorem for autonomous functional differential equations*, J. Differential Equations, 6 (1969), pp. 87–109.

[22] I. Györi and G. Ladas, *Oscillation Theory of Delay Differential Equations*, Clarendon Press, Oxford, UK, 1991.

[23] K. P. Hadeler and J. Tomiuk, *Periodic solutions of difference-differential equations*, Arch. Ration. Mech. Anal., 65 (1977), pp. 87–95.

[24] J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Springer-Verlag, New York, 1993.

[25] G. S. Jones, *The existence of periodic solutions of* $f'(x) = -\alpha f(x-1)\{1 + f(x)\}$, J. Math. Anal. Appl., 5 (1962), pp. 435–450.

[26] J. L. Kaplan and J. A. Yorke, *Ordinary differential equations which yield periodic solutions of differential delay equations*, J. Math. Anal. Appl., 48 (1974), pp. 317–324.

[27] J. L. Kaplan and J. A. Yorke, *On the stability of a periodic solution of a differential-delay equation*, SIAM J. Math. Anal., 6 (1975), pp. 268–282.

[28] J. L. Kaplan and J. A. Yorke, *On the nonlinear differential delay equation* $x'(t) = -f(x(t), x(t-1))$, J. Differential Equations, 23 (1977), pp. 293–314.

[29] T. Krisztin and O. Arino, *The 2-dimensional attractor of a differential equation with state-dependent delay*, J. Dynam. Differential Equations, 13 (2001), pp. 453–522.

[30] T. Krisztin, *Global dynamics of delay differential equations*, Period. Math. Hungar., 56 (2008), pp. 83–95.

[31] Y. Kuang and H. L. Smith, *Periodic solutions of differential delay equations with threshold-type delays*, in Oscillation and Dynamics in Delay Equations, J. R. Graef and J. K. Hale, eds., AMS, Providence, RI, 1992, pp. 153–176.

[32] Y. Kuang and H. L. Smith, *Slowly oscillating periodic solutions of autonomous state-dependent delay differential equations*, Nonlinear Anal., 19 (1992), pp. 855–872.

[33] D. Lipshutz, *Existence, Uniqueness and Stability of Slowly Oscillating Periodic Solutions for Delay Differential Equations with Non-negativity Constraints*, Ph.D. thesis, University of California, San Diego, 2013.

[34] N. MacDonald, *Time lag in a model of a biochemical reaction sequence with end product inhibition*, J. Theoret. Biol., 67 (1977), pp. 549–556.

[35] P. Magal and O. Arino, *Existence of periodic solutions for a state-dependent delay differential equation*, J. Differential Equations, 165 (2000), pp. 61–95.

[36] J. Mallet-Paret, R. D. Nussbaum, and P. Paraskevopoulos, *Periodic solutions for functional differential equations with multiple state-dependent time lags*, Topol. Methods Nonlinear Anal., 3 (1994), pp. 101–162.

[37] J. Mallet-Paret and R. D. Nussbaum, *Boundary layer phenomena for differential-delay equations with state-dependent time-lags:* I, Arch. Ration. Mech. Anal., 120 (1992), pp. 99–146.

[38] J. Mallet-Paret and R. D. Nussbaum, *Boundary layer phenomena for differential-delay equations with state-dependent time-lags:* II, J. Reine Angew. Math., 477 (1996), pp. 129–197.

[39] J. Mallet-Paret and R. D. Nussbaum, *Boundary layer phenomena for differential-delay equations with state-dependent time-lags:* III, J. Differential Equations, 189 (2003), pp. 640–692.

[40] J. Mallet-Paret and R. D. Nussbaum, *Stability of periodic solutions of state-dependent delay-differential equations*, J. Differential Equations, 250 (2011), pp. 4085–4103.

[41] J. Mallet-Paret and R. D. Nussbaum, *Superstability and rigorous asymptotics in singularly perturbed state-dependent delay-differential equations*, J. Differential Equations, 250 (2011), pp. 4037–4084.

[42] J. Mallet-Paret and G. Sell, *The Poincaré-Bendixson theorem for monotone cyclic feedback systems with delay*, J. Differential Equations, 125 (1996), pp. 441–489.

[43] J. Mallet-Paret and G. Sell, *Systems of differential delay equations: Floquet multipliers and discrete Lyapunov functions*, J. Differential Equations, 125 (1996), pp. 385–440.

[44] A. Mandelbaum and W. A. Massey, *Strong approximations for time-dependent queues*, Math. Oper. Res., 20 (1995), pp. 33–64.

[45] A. Mandelbaum and K. Ramanan, *Directional derivatives of oblique reflection maps*, Math. Oper. Res., 35 (2010), pp. 527–558.

[46] W. Mather, M. R. Bennett, J. Hasty, and L. S. Tsimring, *Delay-induced degrade-and-fire oscillations in small genetic circuits*, Phys. Rev. Lett., 102 (2009), pp. 1–4.

[47] V. Misra, W. Gong, and D. Towsley, *Fluid-based analysis of a network of AQM routers supporting TCP flows with an application to RED*, in Proceedings of the Conference on Applications, Technologies, Architectures, and Protocols for Computer Communication, 2000, pp. 151–160.

[48] N. A. M. Monk, *Oscillatory expression of Hes*1*, p*53*, and NF-$\kappa B$ driven by transcriptional time delays*, Current Biology, 13 (2003), pp. 1409–1413.

[49] R. D. Nussbaum, *Periodic solutions of some nonlinear autonomous functional differential equations* II, J. Differential Equations, 14 (1973), pp. 368–394.

[50] R. D. Nussbaum, *Periodic solutions of some nonlinear autonomous functional differential equations*, Ann. Mat. Pura Appl. (4), 10 (1974), pp. 263–306.

[51] R. D. Nussbaum, *A global bifurcation theorem with applications to functional differential equations*, J. Funct. Anal., 19 (1975), pp. 319–339.

[52] R. D. Nussbaum, *The range of periods of periodic solutions of* $x'(t) = -\alpha f(x(t-1))$, J. Math. Anal. Appl., 58 (1977), pp. 280–292.

[53] R. D. Nussbaum, *Uniqueness and nonuniqueness for periodic solutions of* $x'(t) = -g(x(t-1))$, J. Differential Equations, 34 (1979), pp. 25–54.

[54] R. D. Nussbaum, *Fixed Point Index and Some Applications*, Presses de l'Université de Montréal, Montréal, 1985.

[55] F. Paganini, J. Doyle, and S. Low, *Scalable laws for stable network congestion control*, in Proceedings of the Conference on Decision and Control, 2001, pp. 185–190.

[56] F. Paganini and Z. Wang, *Global stability with time-delay in network congestion control*, in Proceedings of the 41st IEEE Conference on Decision and Control, 2002, 2002, pp. 3632–3637.

[57] A. Papachristodoulou, J. C. Doyle, and S. H. Low, *Analysis of nonlinear delay differentiable equation models of TCP/AQM protocols using sums of squares*, in Proceedings of the 43rd IEEE Conference on Decision and Control, 2004, pp. 4684–4689.

[58] A. Papachristodoulou, *Global stability analysis of TCP/AQM protocol for arbitrary networks with delay*, in Proceedings of the 43rd IEEE Conference on Decision and Control, 2004, pp. 1029–1034.

[59] M. Peet and S. Lall, *Global stability analysis of a nonlinear model of Internet congestion control with delay*, IEEE Trans. Automat. Control, 52 (2007), pp. 553–559.

[60] Y. B. Pesin, *On the behavior of a strongly nonlinear differential equation with retarded argument*, Differ. Uravn., 10 (1974), pp. 1025–1036.

[61] L. Pujo-Menjouet, S. Bernard, and M. C. Mackey, *Long period oscillations in a* $g_0$ *model of hematopoietic stem cells*, SIAM J. Appl. Dyn. Syst., 4 (2005), pp. 312–332.

[62] A. Skorokhod, *Stochastic differential equations for a bounded region*, Theoret. Probab. Appl., 6 (1961), pp. 264–274.

[63] H. Smith, *An Introduction to Delay Differential Equations with Applications to the Life Sciences*, Springer, New York, 2011.

[64] R. A. Smith, *Existence of periodic solutions of autonomous retarded functional differential equations*, Math. Proc. Cambridge Philos. Soc., 88 (1980), pp. 89–109.
[65] R. A. Smith, *Poincaré-Bendixson theory for certain retarted functional differential equations*, Differential Integral Equations, 5 (1992), pp. 213–240.
[66] P. Smolen, D. A. Baxter, and J. H. Byrne, *A reduced model clarifies the role of feedback loops and time delays in the Drosphila circadian oscillator*, Biophys. J., 83 (2002), pp. 2349–2359.
[67] R. Srikant, *The Mathematics of Internet Congestion Control*, Birkhäuser, Boston, 2004.
[68] E. Stumpf, *On a differential equation with state-dependent delay: A global center-unstable manifold connecting an equilibrium and a periodic orbit*, J. Dynam. Differential Equations, 24 (2012), pp. 197–248.
[69] J. J. Tyson, K. C. Chen, and B. Novak, *Sniffers, buzzers, toggles and blinkers: Dynamics of regulatory and signaling pathways in the cell*, Current Opinion in Cell Biology, 15 (2003), pp. 221–231.
[70] H.-O. Walther, *On instability, $\omega$-limit sets and periodic solutions of nonlinear autonomous differential delay equations*, in Functional Differential Equations and Approximation of Fixed Points, H.-O. Peitgen and H.-O. Walther, eds., Springer, Berlin, 1979.
[71] H.-O. Walther, *Contracting returns maps for some delay differential equations*, in Topics in Functional Differential and Difference Equations, T. Faria and P. Freitas, eds., Fields Inst. Commun. 29, AMS, Providence, RI, 2001, pp. 349–360.
[72] H.-O. Walther, *Contracting return maps for monotone delayed feedback*, Discrete Contin. Dynam. Syst., 7 (2001), pp. 259–274.
[73] H.-O. Walther, *Stable periodic motion of a delayed spring*, Topol. Methods Nonlinear Anal., 21 (2003), pp. 1–28.
[74] H.-O. Walther, *Stable periodic motion of a system using echo for position control*, J. Dynam. Differential Equations, 15 (2003), pp. 143–223.
[75] H.-O. Walther, *Topics in delay differential equations*, Jahresber. Dtsch. Math.-Ver., 116 (2014), pp. 87–114.
[76] W. Whitt, *An Introduction to Stochastic-Process Limits and Their Applications to Queues*, Internet supplement, http://www.columbia.edu/∼ww2040/supplement.html (2002).
[77] R. J. Williams, *An invariance principle for semimartingale reflecting Brownian motions in an orthant*, Queueing Syst., 30 (1998), pp. 5–25.
[78] E. M. Wright, *A non-linear difference-differential equation*, J. Reine Angew. Math., 194 (1955), pp. 66–87.
[79] J. Wu, *Stable phase-locked periodic solutions in a delay differential system*, J. Differential Equations, 194 (2003), pp. 237–286.
[80] X. Xie, *Uniqueness and stability of slowly oscillating periodic solutions of delay equations with bounded nonlinearity*, J. Dynam. Differential Equations, 3 (1991), pp. 515–540.
[81] X. Xie, *The multiplier equation and its applications to $S$-solutions of a differential delay equation*, J. Differential Equations, 95 (1992), pp. 259–280.
[82] X. Xie, *Uniqueness and stability of slowly oscillating periodic solutions of differential delay equations with unbounded nonlinearity*, J. Differential Equations, 103 (1993), pp. 350–374.
[83] L. Ying, G. Dullerud, and R. Srikant, *Global stability of Internet congestion controllers with heterogeneous delays*, IEEE/ACM Trans. Networking, 14 (2006), pp. 579–591.