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Publication Date
1986-12-23

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Working Paper 8621

CORE ALLOCATIONS AND SMALL INCOME TRANSFERS

Robert M. Anderson

December 23, 1986

Key words: Core, Walrasian equilibrium, approximate equilibrium, non-convex preferences, Shapley-Folkman theorem, income transfers, gap-minimizing prices.

Abstract

We show that, given any allocation f in the core of an exchange economy, we can find small income transfers and a Walrasian allocation $\tilde{f}$ relative to the transfers such that most agents are indifferent between f and $\tilde{f}$. In addition, we can find small income transfers and an approximate Walrasian allocation $\hat{f}$ relative to the transfers such that all agents are indifferent between $\hat{f}$ and f.

JEL Classification: 021, 024

ACKNOWLEDGEMENTS

This research was supported in part by a grant from the National Science Foundation. The author is grateful to Will Geller, Wan-Jin Kim, Alejandro Manelli, Hugo Sonnenschein and Pinghua Young for helpful comments; and to Richard Steele for the artwork.

1. Introduction

Most decentralization results on core allocations in exchange economies with nonconvex preferences have focussed on the so-called "competitive gap," (Hildenbrand (17), Dierker (13), Anderson (1)) which measures the distance from the agents' consumption to the budget frontier plus the distance that the cheapest vector preferred to their consumption lies below the budget frontier. Anderson (6) gives a formulation showing that the utility levels achieved converge on average to the utility levels generated by the agents' demands. However, it need not be the case that core allocations are close to Walrasian allocations. Indeed, Anderson and Mas-Colell (10) give an example of a sequence of exchange economies with a sequence of core allocations in which every agent is a fixed distance from his/her demand set for every price.[2] See Anderson (5) for a survey of the results known with nonconvex preferences.

One might hope that, given a core allocation, one could find a Walrasian allocation under which each individual achieves approximately the same utility level as at the core allocation. Note first, however, that there is no guarantee that there will be any price q that clears the markets, since preferences are nonconvex; we are forced to consider prices that approximately clear the markets. As shown in Figure 1, it is possible to have a core allocation f with a local supporting price p that makes f lie in the budget set of each agent, but so that p is *not* an equilibrium price: agent I's demand will be at the point x, while II's demand will be at the point y, so that the excess demand is not small in per capita terms. Worse still, there is a unique equilibrium price q which yields an allocation g far from f and such that the utility levels of agents at g are very different from the levels at f. Furthermore, there is no way to transfer income (in the spirit of the Second Welfare Theorem) so that there is a post-transfer Walrasian equilibrium which yields utility levels close to those of f to both agents; see Anderson (8) for a fuller discussion of this point.

The purpose of this paper is to give a closer link between core allocations and Walrasian allocations by showing that in large finite economies there are small income transfers and (post transfer) Walrasian allocations which gives most agents the same utility they experience at core allocations. We choose a decentralizing price vector which, essentially,

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2 There are results showing that core allocations are close to demand sets in two generic formulations: see Mas-Colell and Neuefeind (21), in combination with Proposition 4 on page 200 and condition (*) on page 201 of Hildenbrand (17), for a topological formulation; and Anderson (4) for a probabilistic formulation.

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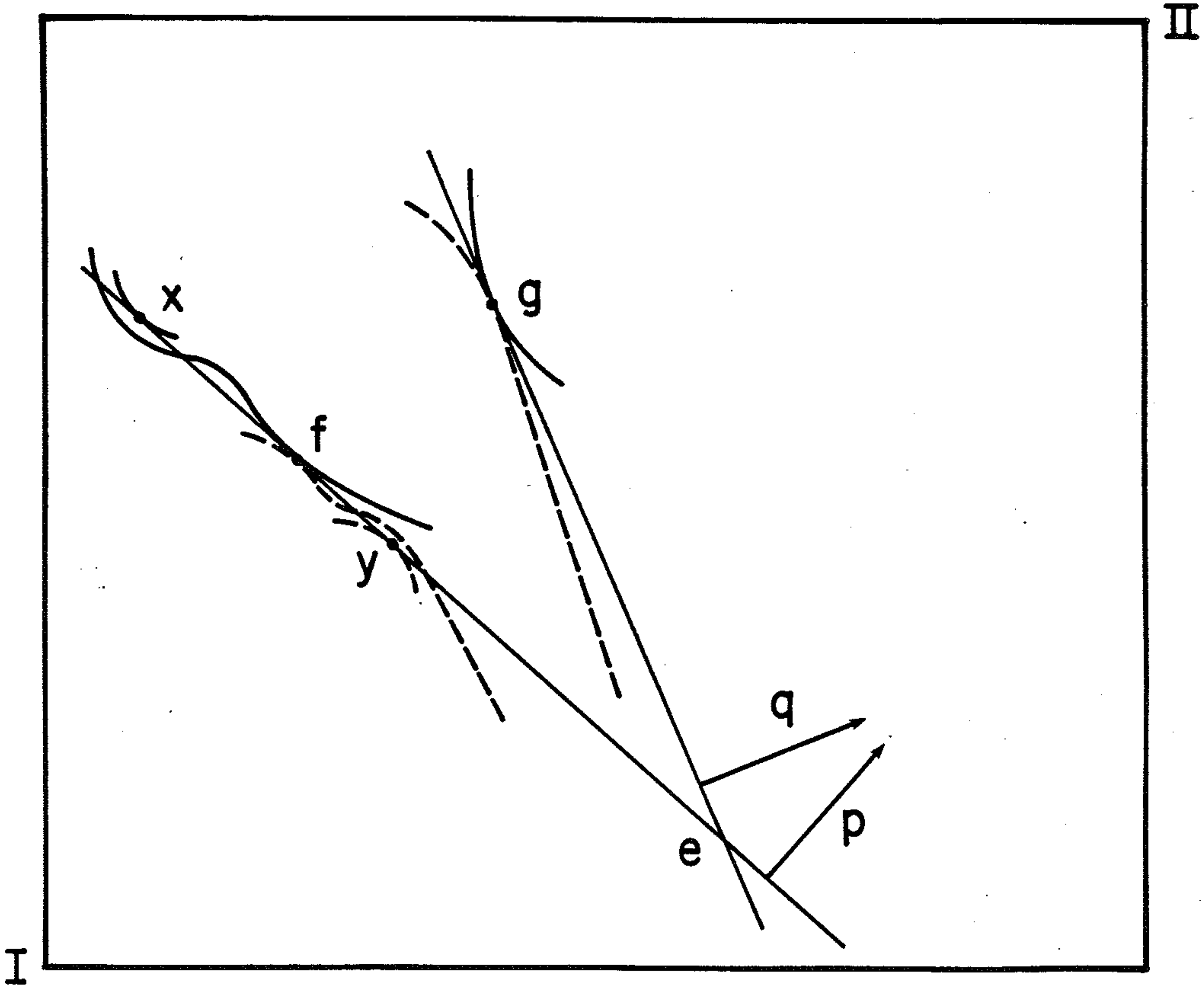

FIGURE 1

minimizes the competitive gap. This so-called "gap-minimizing" price has been used by Mas-Colell (20) (Proposition 7.4.1) in a variation of the proof of Anderson (1); he also used a related construction for approximate decentralization of Pareto optima (Mas-Colell (20), Proposition 4.5.1). This construction has since been used in Anderson (7) to obtain a quadratic rate of convergence for the competitive gap under smoothness assumptions. The construction has also been used in Anderson (8) to obtain a version of the Second Welfare Theorem in exchange economies with nonconvex preferences. Specifically, (8) shows that, given any Pareto optimum f, there exist income transfers and a Walrasian allocation $\tilde{f}$ relative to those transfers such that all but k agents are indifferent between f and $\tilde{f}$, where k is the number of commodities.

Our present result is most closely related to (8). It asserts that, given a core allocation f, there exist income transfers and a Walrasian allocation $\tilde{f}$ such that all but roughly $\sqrt{kn}$ agents are indifferent between f and $\tilde{f}$, where k is the number of commodities and n is the number of agents. Moreover, the income transfers are small: the average absolute value is bounded by a term of order the norm of the largest endowment times $\sqrt{k/n}$. The bounds on the size of the transfers and on the number of agents who are not indifferent between f and $\tilde{f}$ are expressed in terms of the competitive gap. Under smoothness hypotheses, the average competitive gap can be shown to be $O(1/n^2)$ (Anderson (7), Kim (18)). It then follows that the number of agents who are not indifferent between f and $\tilde{f}$ is at most $k + 2$, and the average absolute value of the income transfers is $O(1/n)$. We also introduce an income transfer $\hat{t}$ and an approximate Walrasian allocation $\hat{f}$ (relative to $\hat{t}$) such that $\hat{f}(a) \sim_a f(a)$ for *all* a; the per capita excess demand at $\hat{f}$ is of the order $1/\sqrt{n}$ times the norm of the largest endowment and the average income transfer is of the order $\sqrt{k/n}$. With smoothness, these rates improve to $O(1/n)$.

The rate we obtain for the convergence of the transfers to 0 in the smooth case is the best possible for replica sequences with two types of agents. A faster rate of convergence to 0 of the income transfers would yield a rate of convergence of the distance in the commodity space to Walrasian equilibrium faster than the $O(1/n)$ rate obtained in Debreu (12), but Debreu's rate of convergence is best possible in the case of two types of agents. However, it is likely that the rate of convergence of the competitive gap becomes faster as the number of types increases, and faster still if one considers non-replica economies; furthermore, this acceleration should occur even without smoothness; see Anderson (7) and Geller (15) for a discussion of these points. If a $O(n^{-2})$ rate holds for the average competitive gap without smoothness, the rate of convergence on average for $\tilde{f}$, $\tilde{t}$, $\hat{f}$ and $\hat{t}$ would automatically improve to $O(1/n)$. If still faster convergence rates hold for the competitive gap, a rate faster than $O(1/n)$ would automatically hold for $\hat{t}$, but not necessarily for $\tilde{f}$, $\tilde{t}$ and $\hat{f}$, since their bounds involve additional terms which do not shrink along with the competitive gap; we conjecture that the $O(1/n)$ rate is the best possible for $\tilde{f}$, $\tilde{t}$ and $\hat{f}$.

The representation of a core allocation as a Walrasian equilibrium with small income transfers is implicit in Debreu (12), Grodal (16), and Cheng (11); and almost explicit in Proposition 7.4.12 in Mas-Colell (20). Each of the four results just cited deals with exchange economies in which agents have smooth, strongly convex preferences. The proofs (and, in the case of Mas-Colell (20), the statement of the result) establish that $\max_{a\in A}|p \cdot (f(a) - e(a))| = O(1/|A|)$, where f is a core allocation and p is the *supporting* price for the core allocation. Then defining $t(a) = p \cdot (f(a) - e(a))$, we see immediately that (f, p) is a Walrasian equilibrium with respect to t, that $\sum_{a\in A} t(a) = 0$, and that $\max_{a\in A}|t(a)| = O(1/|A|)$. On the other hand, the rate of convergence results in Debreu (12) and Grodal (16) can be deduced from the results in this paper (except that, in Grodal's case, we obtain the rate of convergence on average, not uniformly). Thus, our notion of decentralization has important roots in the literature on cores with smooth, strongly convex preferences. The principal contribution of this paper is to show that this strong form of decentralization can also be extended to the nonsmooth and, more importantly, the nonconvex case.

The proof is elementary. The key observation is that, with respect to the gap-minimizing price, we can find a point in the negative orthant which is in the convex hull of points which minimize expenditure, subject to the constraint of having utility at least as high as the core allocation, for some coalition S; such points are, of course, in the quasidemand sets after income transfers. Using the Shapley-Folkman theorem, we can move all but k agents in S into their quasi-demand sets. However, this does not tell us anything about individuals *not* in S. These individuals will not be indifferent between f and $\widetilde{f}$; moreover, we will have to make relatively large income transfers to these individuals. Therefore, we seek to choose S relatively large so as to limit the number of such individuals. On the other hand, we cannot force S to be too large. Our ability to control the average absolute value of the transfers is based on knowledge of the average (signed) value of the transfers in S. If S is required to be too large, then it will be forced to contain individuals with positive as well as negative transfers, thereby limiting our ability to infer anything about the average absolute transfer. The trade-off between these two considerations leads us to choose the coalition S to have roughly $n - \sqrt{n}$ individuals in general ($n - (k + 2)$ in the smooth case).

We believe that the usefulness of gap-minimizing price construction in this wide variety of problems demonstrates that it is not just a convenient technical device, but an important construction in its own right.

With nonconvex preferences, the core may be empty. However, various notions of approximate cores are not empty. The analysis of this paper extends easily to these approximate core notions; see Anderson (4).

2. Preliminaries

We begin with some notation and definitions which will be used throughout. Suppose $x, y \in \mathbb{R}^k$, $B \subset \mathbb{R}^k$. x^i denotes the i^{th} component of x; $x \geq y$ means $x^i \geq y^i$ for all i; $x > y$ means $x \geq y$ and $x \neq y$; $x >> y$ means $x^i > y^i$ for all i; $(x_+)^i = \max\{x^i, 0\}$; $(x_-)^i = -\min\{x^i, 0\}$; $\|x\|_1 = \sum_{i=1}^{k} |x^i|$; $\|x\|_\infty = \max_i |x^i|$; $\mathbb{R}^k_+ = \{x \in \mathbb{R}^k: x \geq 0\}$; $\mathbb{R}^k_{++} = \{x \in \mathbb{R}^k: x >> 0\}$; and $\overline{B}$ denotes the closure of B.

A preference is a binary relation $\blacktriangleright$ on $\mathbb{R}^k_+$ satisfying the following conditions: (i) monotonicity: $x >> y \Rightarrow x \blacktriangleright y$; (ii) free disposal: $x >> y,\ y \blacktriangleright z \Rightarrow x \blacktriangleright z$; (iii) continuity: $\{(x,y) \in \mathbb{R}^k_+ \times \mathbb{R}^k_+ : x \blacktriangleright y\}$ is relatively open in $\mathbb{R}^k_+ \times \mathbb{R}^k_+$; (iv) transitivity: if $x \blacktriangleright y$ and $y \blacktriangleright z$, then $x \blacktriangleright z$; and (v) irreflexivity: $x \not\blacktriangleright x$. Let $\mathcal{P}$ denote the set of preferences. Let $\mathcal{P}'$ denote the set of preferences in $\mathcal{P}$ satisfying the following additional conditions: (vi) strong monotonicity: if $x > y$, then $x \blacktriangleright y$; and (vii) for all $x \in \mathbb{R}^k_+$, $0 \not\blacktriangleright x$. If $\blacktriangleright \in \mathcal{P}$, we *define* $x \sim y \iff (x \not\blacktriangleright y \text{ and } y \not\blacktriangleright x)$.

An exchange economy is a map $\varepsilon: A \to \mathcal{P} \times \mathbb{R}^k_+$, where A is a finite set. For $a \in A$, let $\blacktriangleright_a$ denote the preference of a (i.e. the projection of $\varepsilon(a)$ onto $\mathcal{P}$) and $e(a)$ the initial endowment of a (i.e. the projection of $\varepsilon(a)$ onto $\mathbb{R}^k_+$). An allocation is a map $f: A \to \mathbb{R}^k_+$ such that $\sum_{a \in A} f(a) = \sum_{a \in A} e(a)$. A coalition is a non-empty subset of A. A coalition S can improve on an allocation f if there exists $g: S \to \mathbb{R}^k_+$, $g(a) \blacktriangleright_a f(a)$ for all $a \in S$, and $\sum_{a \in S} g(a) = \sum_{a \in S} e(a)$. The core of ε, $\mathcal{C}(\varepsilon)$, is the set of all allocations which cannot be improved on by any coalition. Let $M^m_\varepsilon = \max\{\|\sum_{a \in S} e(a)\|_\infty : S \subset A,\ |S| \leq m\}$.

A price p is an element of $\mathbb{R}^k_+$ with $\|p\|_1 = 1$. Δ_+ denotes the set of prices, $\Delta_{++} = \{p \in \Delta_+ : p >> 0\}$. The demand set for $(\blacktriangleright, e)$, with income augmented by $r \in \mathbb{R}$ is $D(p, (\blacktriangleright, e), r) = \{x \in \mathbb{R}^k_+ : p \cdot x \leq p \cdot e + r,\ y \blacktriangleright x \Rightarrow p \cdot y > p \cdot e + r\}$. $D(p, (\blacktriangleright, e), r)$ could be empty under the hypotheses we have placed on preferences. An income transfer is a function $t: A \to \mathbb{R}$ with $\sum_{t \in A} t(a) \leq 0$. By abuse of notation, we let $D(p, a, t) = D(p, (\blacktriangleright_a, e(a)), t(a))$ if $a \in A$.

The quasidemand set for $(\blacktriangleright, e)$, with income augmented by $r \in \mathbb{R}$ is $Q(p, (\blacktriangleright, e), r) = \{x \in \mathbb{R}^k_+ : p \cdot x \leq p \cdot e + r,\ y \blacktriangleright x \Rightarrow p \cdot y \geq p \cdot e + r\}$. $Q(p, (\blacktriangleright, e), r)$ could be empty under the hypotheses we have placed on preferences. By abuse of notation, we let $Q(p, a, t) = Q(p, (\blacktriangleright_a, e(a)), t(a))$ if $a \in A$.

A Walrasian equilibrium for ε, relative to the income transfer t, is a pair (f, p), where $\sum_{a \in A} f(a) \leq \sum_{a \in A} e(a)$, $p \in \Delta$, and $f(a) \in D(p, a, t)$ for all $a \in A$. Let $\mathcal{W}(\varepsilon, t)$ denote the set of Walrasian equilibria for ε, relative to the income transfer t. A Walrasian quasiequilibrium for ε, relative to the income transfer t, is a pair (f, p), where $\sum_{a \in A} f(a) \leq \sum_{a \in A} e(a)$, $p \in \Delta$, and $f(a) \in Q(p, a, t)$ for all $a \in A$. Let $Q(\varepsilon, t)$ denote the set of Walrasian quasiequilibria for ε, relative to the income transfer t.

3.Results

Definition 3.1: Given an exchange economy ε: $A \to P \times \mathbb{R}^k_+$, and $f \in C(\varepsilon)$, define $\gamma(a) = \{x - e(a): x \blacktriangleright_a f(a)\}$, $\pi(a) = \gamma(a) \cup \{0\}$, and $\Pi = \sum_{a \in A} \pi(a)$. Let $\bar{p}$ be chosen to maximize $\inf p \cdot \Pi$ over $p \in \Delta_+$. $\bar{p}$ is called the *gap-minimizing price* for f; this choice of price vector first appears in Mas-Colell (20), Proposition 7.4.1. Define $\alpha = \inf \bar{p} \cdot \Pi$. Suppose $\zeta \in (0, 1]$. Let $\Gamma = \{\sum_{a \in S} h(a): h(a) \in \gamma(a),\ |S| \geq (1 - \zeta)|A|\}$, $\bar{q}$ maximizes $\inf q \cdot \Gamma$ over $q \in \Delta_+$, $\beta = \inf \bar{q} \cdot \Gamma$, $z = \beta \bar{q} / |\bar{q}|^2$. For $a \in A$, define $g(a) = \operatorname{argmin} \bar{q} \cdot \overline{\gamma(a)}$; g may be a correspondence, and it might be empty-valued. For any $S \subset A$, define $g(S) = \sum_{a \in S} g(a)$ and $\beta(S) = \bar{q} \cdot g(S)$. The existence of $\bar{p}$ and $\bar{q}$ is shown in the proof of Theorem 3.4 of Anderson (7).

Theorem 3.2: Suppose ε: $A \to P \times \mathbb{R}^k_+$ is an exchange economy and $f \in C(\varepsilon)$. There exists $p \in \Delta_+$ such that $\inf p \cdot \Pi \geq -M^k_\varepsilon$.

Proof: This follows immediately from the proof of Theorem 1 of Anderson (1).

Lemma 3.3: (i) $-M^k_\varepsilon \leq \alpha \leq \beta \leq \zeta\alpha$; and (ii) $\inf \bar{q} \cdot \Pi \geq \zeta^{-1}\beta \geq \zeta^{-1}\alpha \geq -\zeta^{-1}M^k_\varepsilon$.

Proof: This follows from the proof of Lemma 3.3 of Anderson (7), taking $\zeta = 1 - \xi$.

Lemma 3.4: Let ε: $A \to P \times \mathbb{R}^k_+$ be an exchange economy and $f \in C(\varepsilon)$. Then $z \in \operatorname{con} \overline{\Gamma}$.

Proof: This is a portion of the statement of Theorem 3.4 of Anderson (7).

Proposition 3.5: Suppose ε: $A \to \mathcal{P} \times \mathbb{R}^k_+$ is an exchange economy. If $f \in \mathcal{C}(\varepsilon)$, and $0 < \zeta \leq 1$, then there exists an income transfer $\tilde{t}$ with

$$\sum_{a \in A} |\tilde{t}(a)| \leq \frac{2|\alpha|}{\zeta} + M_\varepsilon^{\zeta|A|} + M_\varepsilon^{k+1}$$

and $(\tilde{f}, \bar{q}) \in \mathcal{Q}(\varepsilon, \tilde{t})$ such that there is a set of agents S, $|S| \geq (1 - \zeta)|A| - (k + 1)$ with $f(a) \sim_a \tilde{f}(a)$ for all $a \in S$. Alternatively, we may find an income transfer $\hat{t}$ with

$$\sum_{a \in A} |\hat{t}(a)| \leq \frac{2|\alpha|}{\zeta}$$

and $\hat{f}(a) \in Q(\bar{q}, a, \hat{t})$ such that, for *all* $a \in A$, $f(a) \sim_a \hat{f}(a)$ and

$$\bar{q} \cdot \left[\left(\sum_{a \in A} (\hat{f}(a) - e(a)) \right)_+ + \left(\sum_{a \in A} (\hat{f}(a) - e(a)) \right)_- \right] \leq 6M_\varepsilon^{k+1} + 2M_\varepsilon^{\zeta|A|}.$$

If in addition we assume that $\blacktriangleright_a \in \mathcal{P}'$ for all $a \in A$ and $\bar{q} \cdot \hat{f}(a) > 0$ for some $a \in A$, then we can take $\bar{q} >> 0$, $(\tilde{f}, \bar{q}) \in \mathcal{W}(\varepsilon, \tilde{t})$, and $\hat{f}(a) \in D(\bar{q}, a, \hat{t})$ for all $a \in A$.

Remark 3.6: Since preferences may be nonconvex, $\mathcal{W}(\varepsilon, t)$ may be empty. The conclusion that it is not empty is less surprising that it might at first appear. Since we only require that $\sum_{a \in A} \tilde{f}(a) \leq \sum_{a \in A} e(a)$, and we may have $\sum_{a \in A} \tilde{t}(a) < 0$, some quantity of goods is left over. It is as if we added an agent with a linear preference relation with indifference curves perpendicular to p who receives the residual income; this provides the necessary freedom to obtain a Walrasian equilibrium. The alternative formulation involving $\hat{f}$ is a notion of approximate Walrasian equilibrium. The theorem indicates that the market value of the absolute value (taken componentwise) of the excess demand is bounded. This result is obtained essentially by combining the formulation involving $\tilde{f}$ with the argument in Anderson (3). Corollary 3.7 is obtained by optimizing the choice of ζ, following Anderson, Khan and Rashid (9) and Geller (14).

Corollary 3.7: Suppose $\varepsilon: A \to \mathcal{P} \times \mathbb{R}^k_+$ is an exchange economy, $|A| = n$. If $f \in \mathcal{C}(\varepsilon)$, then there exists an income transfer $\tilde{t}$ with

$$\sum_{a \in A} |\tilde{t}(a)| \quad \leq \quad \left(2\sqrt{2n(k+1)} + (k+1)\right) \max_{a \in A} \|e(a)\|_\infty .$$

and $(\tilde{f}, \bar{q}) \in \mathcal{Q}(\varepsilon, \tilde{t})$ such that there is a set of agents S, $|S| \geq n - \sqrt{2(k+1)n} - (k+1)$ with $f(a) \sim_a \tilde{f}(a)$ for all $a \in S$. Alternatively, we may find an income transfer $\hat{t}$ with

$$\sum_{a \in A} |\hat{t}(a)| \quad \leq \quad \sqrt{2n(k+1)} \max_{a \in A} \|e(a)\|_\infty .$$

and $\hat{f}(a) \in Q(\bar{q}, a, \hat{t})$ such that, for *all* $a \in A$, $f(a) \sim_a \hat{f}(a)$ and

$$\bar{q} \cdot \left[\left(\sum_{a \in A} (\hat{f}(a) - e(a)) \right)_+ + \left(\sum_{a \in A} (\hat{f}(a) - e(a)) \right)_- \right]$$

$$\leq \left(6(k+1) + 2\sqrt{2n(k+1)}\right) \max_{a \in A} \|e(a)\|_\infty$$

If in addition we assume that $\blacktriangleright_a \in \mathcal{P}'$ for all $a \in A$ and

$$\min_j \left(\sum_{a \in A} e(a) \right)^j \quad > \quad \sqrt{2n(k+1)} \max_{a \in A} \|e(a)\|_\infty ,$$

then we can take $\bar{q} >> 0$, $(\tilde{f}, \bar{q}) \in \mathcal{W}(\varepsilon, t)$, and $\hat{f}(a) \in D(\bar{p}, a, \hat{t})$ for all $a \in A$.

Proof: Take

$$\zeta = \sqrt{\frac{2|\alpha|}{nM_\varepsilon^1}}.$$

$$\sum_{a \in A} |\widetilde{t}(a)| \leq 2\zeta^{-1}|\alpha| + \zeta n M_\varepsilon^1 + M_\varepsilon^{k+1}$$

$$\leq 2\sqrt{\frac{nM_\varepsilon^1}{2|\alpha|}}|\alpha| + \sqrt{\frac{2|\alpha|}{nM_\varepsilon^1}} nM_\varepsilon^1 + M_\varepsilon^{k+1} \leq \sqrt{2nM_\varepsilon^1|\alpha|} + \sqrt{2nM_\varepsilon^1|\alpha|} + M_\varepsilon^{k+1}$$

$$= 2\sqrt{2nM_\varepsilon^1|\alpha|} + M_\varepsilon^{k+1} \leq 2\sqrt{2n(k+1)}M_\varepsilon^1 + (k+1)M_\varepsilon^1$$

$$\leq \left(2\sqrt{2n(k+1)} + (k+1)\right) \max_{a \in A} ||e(a)||_\infty.$$

The other bounds are established similarly. To show that at least one agent a has $\bar{q} \cdot \hat{f}(a) > 0$, note that

$$\sum_{a \in A} \bar{q} \cdot \hat{f}(a) = \bar{q} \cdot \left(\sum_{a \in A} e(a)\right) - \sum_{a \in A} \hat{t}(a)$$

$$\geq \min_j \left(\sum_{a \in A} e(a)\right)^j - \sqrt{2n(k+1)} \max_{a \in A} ||e(a)||_\infty > 0.$$

Corollary 3.8: Suppose ε_n: $A_n \to P \times I\!R_+^k$ satisfies the hypotheses of Theorem 3, p. 202 of Hildenbrand (17). If $f_n \in C(\varepsilon)$, then there exist income transfers $\widetilde{t}_n$ with

$$\frac{1}{|A_n|}\sum_{a\in A_n}|\tilde{t}_n(a)| \to 0$$

and $(\tilde{f}_n, \bar{q}_n) \in \mathcal{W}(\varepsilon_n, \tilde{t}_n)$ for n sufficiently large such that there is a set of agents S_n, $|S_n| / |A_n| \to 1$ with $f_n(a) \sim_a \tilde{f}_n(a)$ for all $a \in S_n$. Alternatively, we may find an income transfer $\hat{t}_n$ with

$$\frac{1}{|A_n|}\sum_{a\in A_n}|\hat{t}_n(a)| \to 0$$

and $\hat{f}_n(a) \in D(\bar{q}_n, a, \hat{t}_n)$ for n sufficiently large such that, for *all* $a \in A$, $f_n(a) \sim_a \hat{f}_n(a)$ and

$$\frac{1}{|A_n|}\sum_{a\in A}(\hat{f}_n(a) - e_n(a)) \;\to\; 0.$$

Remark 3.9: Since demand is continuous as a function of income when preferences are strongly convex, it is possible to deduce Theorem 1 on page 179 of Hildenbrand (17) from Corollary 3.8.

Proof: Since ε_n is purely competitive, the sequence of endowments is uniformly integrable, i.e.

$$E_n \subset A_n,\ \frac{|E_n|}{|A_n|} \to 0 \quad \Rightarrow \quad \frac{1}{|A_n|}\sum_{a\in E_n} e_n(a) \to 0$$

(Hildenbrand (17), page 137). In ε_n, take $\zeta_n = \sqrt{\frac{M^1_{\varepsilon_n}}{|A_n|}} \to 0$ as $n \to \infty$; we thus have

$$\frac{2M_{\varepsilon_n}^k}{\zeta_n|A_n|} \leq \sqrt{\frac{2kM_{\varepsilon_n}^1}{|A_n|}} \to 0, \quad \frac{M_{\varepsilon_n}^{\zeta_n|A_n|}}{|A_n|} \to 0, \text{ and } \frac{M_{\varepsilon_n}^{k+1}}{|A_n|} \to 0.$$

Hence, the average convergence of $|t_n(a)|$ and $|\hat{t}_n(a)|$ to 0 follow from the corresponding bounds in Proposition 3.5. The hypotheses of Theorem 3 of Hildenbrand (17) imply that $\hat{\blacktriangleright}_a \in P'$ for all $a \in A_n$. To see that for n sufficiently large, there is $a \in A_n$ with $\bar{q}_n \cdot \hat{f}_n(a) > 0$, observe

$$\frac{1}{|A_n|}\sum_{a\in A_n} \bar{q}_n \cdot \hat{f}_n(a) \geq \frac{1}{|A_n|}\bar{q}_n \cdot \left(\sum_{a\in A_n} e_n(a)\right) - \frac{1}{|A_n|}\sum_{a\in A_n} |\hat{t}_n(a)|$$

$$\geq \min_j\left(\frac{1}{|A_n|}\sum_{a\in A_n} e_n(a)\right)^j - \frac{1}{|A_n|}\sum_{a\in A_n} |\hat{t}_n(a)| \to \lim_{n\to\infty} \min_j\left(\frac{1}{|A_n|}\sum_{a\in A_n} e_n(a)\right)^j > 0.$$

The argument of Lemma 4 of Anderson (2) is easily adapted to show that there is a compact set $K \subset \Delta_{++}$ such that $\bar{q}_n \in K$ for all n; the equi-convexity assumption assumed there is only used to establish an equi-monotonocity condition, which follows directly from the hypotheses here. Hence

$$\frac{\|\sum_{a\in A_n} (\hat{f}_n(a) - e_n(a))\|_\infty}{|A_n|} \leq \frac{\bar{q}_n \cdot \left[\left(\sum_{a\in A_n} (\hat{f}_n(a) - e_n(a))\right)_+ + \left(\sum_{a\in A_n} (\hat{f}_n(a) - e_n(a))\right)_-\right]}{|A_n| \min_j (\bar{q}_n)^j}$$

$$\leq \frac{6M_{\varepsilon_n}^{k+1} + 2M_{\varepsilon_n}^{\zeta_n|A_n|}}{|A_n| \inf\{q^j \colon q \in K,\ 1 \leq j \leq k\}} \to 0.$$

Corollary 3.10: Let $\varepsilon_n\colon A_n \to P' \times \mathbb{R}_+^k$ be a sequence of exchange economies which satisfies the hypotheses of Theorem 3.8 of Anderson (7) or Theorem 1 of Kim (18). If $f_n \in C(\varepsilon_n)$, then there exist income transfers $\tilde{t}_n$ with

$$\frac{1}{|A_n|}\sum_{a\in A_n}|\tilde{t}_n(a)| = O\left(\frac{1}{|A_n|}\right)$$

and $(\tilde{f}_n, \bar{q}_n) \in \mathcal{W}(\varepsilon_n, \tilde{t}_n)$ for n sufficiently large such that there is a set of agents S_n, $|A_n - S_n| \le k+2$, with $f_n(a) \sim_a \tilde{f}_n(a)$ for all $a \in S_n$. Alternatively, we may find an income transfer $\hat{t}_n$ with

$$\frac{1}{|A_n|}\sum_{a\in A_n}|\hat{t}_n(a)| = O\left(\frac{1}{|A_n|}\right)$$

and $\hat{f}_n(a) \in D(\bar{q}_n, a, \hat{t}_n)$ for n sufficiently large such that, for *all* $a \in A$, $f_n(a) \sim_a \hat{f}_n(a)$ and

$$\frac{1}{|A_n|}\sum_{a\in A}(\hat{f}_n(a) - e_n(a)) = O\left(\frac{1}{|A_n|}\right).$$

Proof: Under the hypotheses of Theorem 3.8 of Anderson (7) or Theorem 1 of Kim (18), $\alpha_n = O(1/n)$. Take $\zeta_n = 1/|A_n|$, and apply Proposition 3.5. The existence of a with $\bar{q}_n \cdot \hat{f}_n(a) > 0$ and the bound on the per capita excess demand at $\hat{f}$ are established by the same arguments as in the proof of Corollary 3.8.

Proof of Proposition 3.5: Let $n = |A|$. By Lemma 3.4, $z \in \text{con } \bar{\Gamma}$. From the proof of Theorem 3.4 of Anderson (7), there is some $t \ge (1-\zeta)n$ and $a_1, \dots, a_m$ with $m \le k+1$ and $h(a_i) \in \text{con } \{(0,0),(1,g(a_i))\}$, $h(a) \in \{(0,0),(1,g(a_i))\}$ for $a \notin \{a_1, \dots, a_m\}$, such that $(t,z) = \sum_{a\in A} h(a)$. Let $S = \{a \in A\colon h^1(a) = 1\}$. Note that $|S| \ge (1-\zeta)n - (k+1)$. Let $\tilde{h}(a) = (h^2(a), \dots, h^{k+1}(a))$. Let $f'(a) = \tilde{h}(a) + e(a)$ if $a \in S$.

Next, we adjust $f'(a)$ to ensure that it lies on the relative boundary of $\gamma(a)$ with respect to $I\!R^k_+$. For $a \in S$, let $r(a) = \inf \{r\colon rf'(a) - e(a) \in \bar{\gamma}(a)\}$. Note that $r(a) \leq 1$. Define $\tilde{f}(a) = r(a)f'(a)$ if $a \in S$; note that $\bar{q} \cdot \tilde{f}(a) = \inf \bar{q} \cdot (\gamma(a) + e(a))$. Let $\tilde{f}(a) = 0$ if $a \notin S$.

$$\sum_{a\in A} \tilde{f}(a) \leq \sum_{a\in A} \left(\tilde{h}(a) + e(a)\right) \leq z + \sum_{a\in A} e(a) \leq \sum_{a\in A} e(a).$$

Define $\tilde{t}(a) = \bar{q} \cdot (\tilde{f}(a) - e(a))$. We will show that $\tilde{f}(a) \in Q(\bar{q}, \tilde{a}, t)$ for all a. If $a \notin S$, $\bar{q} \cdot \tilde{f}(a) = \bar{q} \cdot e(a) + \tilde{t}(a) = 0$, so it is trivial that $\tilde{f}(a) \in Q(\bar{q}, a, t)$. Suppose now that $a \in S$. Suppose $x \blacktriangleright_a \tilde{f}(a)$. By continuity there exists $y \in \gamma(a)$ such that $x \blacktriangleright_a (y + e(a))$. By the definition of $\gamma(a)$, $(y + e(a)) \blacktriangleright_a f(a)$; by transitivity, $x \blacktriangleright_a f(a)$. Hence, $\bar{q} \cdot x \geq \inf \bar{q} \cdot \gamma(a) + \bar{q} \cdot e(a) = \bar{q} \cdot \tilde{f}(a) = \bar{q} \cdot e(a) + \tilde{t}(a)$. Thus, $\tilde{f}(a) \in Q(\bar{q}, a, \tilde{t})$. Hence, $(\tilde{f}, \bar{q}) \in Q(\varepsilon, t)$.

Next, we compute the bound on the average absolute transfer.

$$\sum_{a\in A} |\tilde{t}(a)| \leq \sum_{a\in S} |\inf \bar{q} \cdot \gamma(a)| + \sum_{a\notin S} \bar{q} \cdot e(a) \quad \leq \quad \sum_{a\in A} |\inf \bar{q} \cdot \gamma(a)| \quad + \quad M_\varepsilon^{\zeta n + k + 1}$$

$$\leq \quad 2|\inf \bar{q} \cdot \Pi| \quad + \quad M_\varepsilon^{\zeta n} \quad + \quad M_\varepsilon^{k+1} \quad \leq \quad \frac{2|\alpha|}{\zeta} \quad + \quad M_\varepsilon^{\zeta n} \quad + \quad M_\varepsilon^{k+1}.$$

Now, we show that $\tilde{f}(a) \sim_a f(a)$ for all $a \in S$. If $f(a) \blacktriangleright_a \tilde{f}(a)$, then by continuity and the fact that $\tilde{f}(a) \in \bar{\gamma}(a) + e(a)$, we may find $y \in \gamma(a)$ such that $f(a) \blacktriangleright_a (y + e(a))$. By the definition of $\gamma(a)$, $(y + e(a)) \blacktriangleright_a f(a)$; by transitivity, $f(a) \blacktriangleright_a f(a)$, contradicting irreflexivity. Hence, for all $a \in S$, $f(a) \not\blacktriangleright_a \tilde{f}(a)$. On the other hand, suppose $\tilde{f}(a) \blacktriangleright_a f(a)$. If $r(a) = 0$, then $\tilde{f}(a) = 0$, contradicting assumption (vii) on preferences. If $r(a) > 0$, then we may find a point $x \in I\!R^k_+$ arbitrarily close to $\tilde{f}(a)$ such that $x \notin \gamma(a) + e(a)$, and hence $x \not\succ_a f(a)$; but this would contradict continuity. Thus, for $a \in S$, $\tilde{f}(a) \sim_a f(a)$. $|S| \geq \zeta n - (k+1)$.

Define $\hat{f}(a) = \tilde{f}(a)$ if $a \in S$ and let $\hat{f}(a)$ be chosen arbitrarily from

$$\{x: x - e(a) \in \bar{\gamma}(a),\ \bar{q} \cdot x = \bar{q} \cdot e(a) + \inf \bar{q} \cdot \gamma(a),\ rx \notin e(a) + \gamma(a) \text{ if } r < 1 \text{ and } x \neq 0\}.$$

Let $\hat{t}(a) = \bar{q} \cdot (\hat{f}(a) - e(a))$. $\hat{f}(a) \in Q(\bar{q}, a, \hat{t})$ and $\hat{f}(a) \sim_a f(a)$ for all $a \in A$, by the same arguments that worked for $\tilde{f}$.

Next we compute the bound on the average absolute income transfer represented by $\hat{t}$.

$$\sum_{a \in A} |\hat{t}(a)| \quad = \quad \sum_{a \in A} |\inf \bar{q} \cdot \gamma(a)| \quad \leq \quad 2|\inf \bar{q} \cdot \Pi| \quad \leq \quad 2\zeta^{-1}|\alpha|.$$

Next, we compute the bound on the excess demand given the price $\bar{q}$ and the income transfer $\hat{t}$.

$$\sum_{a \in S} \left(\hat{f}(a) - e(a)\right) \quad = \quad z - \sum_{a \in S} \left(\tilde{h}(a) + e(a) - \hat{f}(a)\right) \; - \; \sum_{i=1}^{m} \tilde{h}(a_i)$$

$$= z - \sum_{a \in S} \left(f'(a) - \hat{f}(a)\right) \; - \; \sum_{i=1}^{m} \tilde{h}(a_i).$$

Note that $a \in S, f'(a) \neq \hat{f}(a) \;\Rightarrow \bar{q} \cdot f'(a) = \bar{q} \cdot \hat{f}(a) = 0 \qquad \Rightarrow \bar{q} \cdot (\tilde{h}(a) - \hat{f}(a)) +$

$= \bar{q} \cdot (f'(a) - \hat{f}(a))_- = 0.$ Therefore

$$\bar{q} \cdot \left[\left(\sum_{a \in A} \hat{f}(a) - e(a)\right)_+ + \left(\sum_{a \in A} \hat{f}(a) - e(a)\right)_-\right]$$

$$\leq \bar{q} \cdot z + \bar{q} \cdot \sum_{i=1}^{m} \tilde{h}(a_i) + \bar{q} \cdot \left[\sum_{a \notin S} \left(\hat{f}(a) - e(a)\right)_+ + \sum_{a \notin S} \left(\hat{f}(a) - e(a)\right)_-\right]$$

$$\leq M_\varepsilon^{k+1} + 2M_\varepsilon^{k+1} + \sum_{a \notin S} \left(\bar{q} \cdot \hat{f}(a) + \bar{q} \cdot e(a)\right)$$

$$\leq 3M_\varepsilon^{k+1} + \sum_{a \notin S} (\bar{q} \cdot f(a) + \bar{q} \cdot e(a))$$

$$\leq 3M_\varepsilon^{k+1} + 2\sum_{a \notin S} \bar{q} \cdot e(a) + |\inf \bar{q} \cdot \Pi| \leq 3M_\varepsilon^{k+1} + 2M_\varepsilon^{\zeta n + k + 1} + 2M_\varepsilon^{k+1}$$

$$= 6M_\varepsilon^{k+1} + 2M_\varepsilon^{\zeta n}.$$

Now suppose that preferences satisfy the strong monotonicity assumption (vi) and $\bar{q} \cdot \hat{f}(a) > 0$ for some $a \in A$. We will show that $\bar{q} > > 0$. Suppose $\bar{q}^i = 0$ for some i. By monotonicity and continuity, there exists $y \blacktriangleright_a \hat{f}(a)$, $\bar{q} \cdot y < \bar{q} \cdot \hat{f}(a)$. By continuity we may find x such that $y \blacktriangleright_a x$ and $x - e(a) \in \gamma(a)$. By the definition of $\gamma(a)$, $x \blacktriangleright_a f(a)$, and so $y \blacktriangleright_a f(a)$ by transitivity, a contradiction. Therefore, $\bar{q} > > 0$. But then it follows by standard arguments that the demand and quasidemand sets coincide, completing the proof.

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