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Author
Smith, Stefan G Llewellyn

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Desingularized propagating vortex equilibria

Stefan G Llewellyn Smith

Department of Mechanical and Aerospace Engineering, Jacobs School of Engineering, UCSD, 9500 Gilman Drive, La Jolla CA 92093-0411, USA

E-mail: `sgls@ucsd.edu`

Abstract. The correction to the propagation velocity of point vortex equilibria caused by allowing the vortices to have finite core size is calculated. A matched asymptotic expansion in the small parameter ϵ, given by the ratio of the core size to the dimension of the equilibrium configuration, is carried out. The resulting velocity correction is found to be of order ϵ^4 and arises from the interaction of second- and third-order terms in the inner expansion, which are themselves forced by the strain and strain derivatives of the outer field.

1. Introduction

Point vortices have been extensively studied since the original work of Helmholtz (1858) and Kirchhoff (1876), and many results are summarized in textbooks such as Lamb (1932) and Saffman (1992). The evolution of systems of point vortices may be viewed as a problem in the theory of dynamical systems, and hence the question of equilibrium states of point vortices is a natural and important one. Here, equilibrium refers to configurations that translate or rotate without change of shape. The simplest configurations of point vortices, the translating and co-rotating pair, are equilibrium states. Other configurations have been found. Many exploit symmetry, such as rotating polygonal arrays (Thomson 1883), as well as infinite rectilinear arrays and vortex streets. Many more complicated configurations, with and without symmetry, have also been obtained. A detailed discussion is given in Aref *et al* (2003).

Point vortices are singular solutions of the Euler equations. The justification of the dynamical equations governing their evolution is reviewed in Llewellyn Smith (2011). An interesting related question is the desingularization of point vortex solutions: constructing less singular solutions that reduce to point vortices in an appropriate limit. Two families of vortices have received particular attention: vortex patches and hollow vortices. The evolution of vortex patches can be obtained using contour dynamics. Some of the earliest work on vortex patches studied equilibrium configurations: Pierrehumbert (1980) found propagating dipole solutions, Wu *et al* (1984) found these and also co-rotating pairs, and Saffman and Schatzman (1981) and other authors obtained vortex street solutions. Dhanak (1992) examined the stability of polygonal vortex arrays. Hollow dipoles were originally so named because the fluid inside the vortex boundary is at rest. Pocklington (1895) found propagating hollow vortex dipoles and this work was revisited by Crowdy *et al* (2013), Baker *et al* (1876) found a linear array of hollow vortices and Crowdy and Green (2011) found streets of hollow vortices. The above equilibrium configurations are families that depend on a non-dimensional parameter measuring the size of the vortices, and there is a limit that approaches the point vortex configuration.

One can hence ask the following question: what is the general correction to the velocity of an equilibrium point vortex configuration and how does it depend on the desingularization used? We study this problem via the method of Matched Asymptotic Expansions, using a small parameter ϵ, the ratio of the core size to the size of the array. We obtain an asymptotic expansion in this parameter and compute the resulting velocity. For simplicity we limit ourselves to cases in which the configuration does not rotate, although the same approach should work in the case of rotating configurations. Neither do we consider generalizations to different geometries such as spheres or to domains with boundaries.

The plan of the paper is as follows. In section 2 we outline the mathematical problem. In section 3 we give properties of the radial Rayleigh equation that governs the behavior of the vortex cores. In section 4 we go through the matching procedure. This is a fairly lengthy section since we give all the details, but the underlying structure is straightforward. Finally we conclude in section 5.

2. Problem formulation

We consider the motion of an incompressible, inviscid fluid containing vortices in an equilibrium configuration. In a frame moving with the vortices, vortex m is at rest at $z = z_m$ and there is a uniform flow at infinity. The vortex cores are all taken to have the same structure, with compact vorticity support or exponentially decaying vorticity away from the core. Hence far from the cores, a complex potential $w = \phi + \mathrm{i}\psi$ exists. The complex potential corresponding to the uniform flow at infinity is $-Wz$.

The solution for the equilibrium configuration of point vortices is

$$w_0 = \sum \frac{\Gamma_m}{2\pi\mathrm{i}} \log{(z - z_m)} - W_0 z. \tag{1}$$

A sum without subscripts runs over all the vortices. We view this as the leading-order term in an asymptotic expansion in ϵ, $w = w_0 + \epsilon w_1 + \cdots$, and the full propagation velocity W will be expanded in the same fashion. We will see that W_0 is obtained as part of the matching procedure. The outer solution at $O(\epsilon^p)$ is

$$w_p = \sum_{q=1}^{\infty} \sum a_{qm}^{(p)} (z - z_m)^{-q} - W_p z + C, \tag{2}$$

where C is a constant.

We now consider regions close to each vortex: near vortex n we define a new variable Z using $z = z_n + \epsilon Z$, with $Z = R\mathrm{e}^{\mathrm{i}\theta}$. The governing equation for steady solutions is

$$J(\psi, \nabla_R^2 \psi) = 0, \tag{3}$$

where ψ is the inner streamfunction, J is the Jacobian and

$$\nabla_R^2 = \frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} + \frac{1}{R^2}\frac{\partial^2}{\partial \theta^2}. \tag{4}$$

The leading-order inner streamfunction is determined by the model taken for the vortex cores. Any radially symmetric inner core structure $\psi_0(r)$ with circulation Γ_n is acceptable. Then, for large R,

$$\psi_0 = -\frac{\Gamma_n}{2\pi}\log R + O(R^{-\infty}), \tag{5}$$

where the order term denotes terms that vanish faster than any negative power of R.

3. The steady radial Rayleigh equation

At $O(\epsilon)$ and higher, the governing equation (3) becomes the steady version of the linear radial Rayleigh equation. This equation has been used to study the linear stability of two-dimensional inviscid vortices (Gent and McWilliams 1986). The homogeneous equation for azimuthal mode n of the streamfunction, f_n, is ($n > 0$ without loss of generality)

$$f_n'' + \frac{1}{R}f_n' - \left[\frac{n^2}{R^2} + \frac{Q'}{R\Omega}\right] f_n = 0, \tag{6}$$

where primes denote differentiation with respect to R, Ω is the background angular velocity and Q is the background vorticity. The origin is a regular singular point, so we take the solution that is bounded there with $f_n \sim R^n$. In general, the solution does not then decay at infinity, and one has $f_n \sim R^n + \beta_n R^{-n}$, where β_n usually has to be obtained numerically.

Mode 1 is special: there is a known steady solution due to Michalke and Timme (1967), $f_1 = R\Omega$. Llewellyn Smith (1995) and Llewellyn Smith (1997) discuss the approach to this solution in the initial-value problem and its relation to matching problems, respectively. The other, linearly independent, mode-1 solution is unbounded at the origin and infinity.

Some higher terms in the inner expansion will satisfy inhomogeneous equations. The interaction of $\psi_p = c_p f_p(R)\mathrm{e}^{\mathrm{i}p\theta} + c.c.$ and $\psi_q = c_q f_q(R)\mathrm{e}^{\mathrm{i}q\theta} + c.c.$ forces modes $\pm p \pm q$. The corresponding vorticity is $\zeta_p = c_p g_p(R)\mathrm{e}^{\mathrm{i}p\theta} + c.c.$, where $g_p = f_p'' + R^{-1}f_p' - p^2R^{-2}f_p$. The forcing terms are

$$\begin{aligned} -J(\psi_p,\zeta_q) - J(\psi_q,\zeta_p) = &- R^{-1}c_p c_q^* \mathrm{e}^{\mathrm{i}(p-q)\theta} \times \\ &[-\mathrm{i}q(f_p' g_q - g_p' f_q) + \mathrm{i}p(f_q' g_p - g_q' f_p)] + \cdots, \end{aligned} \tag{7}$$

where the dots refer to other modes. When $p = q$, mode 0 is not generated.

4. Expansion

The large-R expansion of the inner streamfunction is irrotational. For mode n of the inner solution at $O(\epsilon)$ or higher, we have, for large R for $n > 1$,

$$\psi = c_n f_n \mathrm{e}^{\mathrm{i}n\theta} + c.c. \sim c_n(R^n + \beta_n R^{-n})\mathrm{e}^{\mathrm{i}n\theta} + c.c. = c_n[Z^n + \beta_n (Z^*)^{-n}] + c.c. \tag{8}$$

for $n > 0$. Since β_n is real, this shows that in the matching region, ψ is the imaginary part of the meromorphic function $2\mathrm{i}(c_n Z^n + c_n^* \beta_n Z^{-n})$. We can hence carry out the matching using complex potentials, which we denote ϖ_n in the far field. We will not bother matching constant terms C or c in the potentials or streamfunctions, since they are dynamically irrelevant.

The outer solution w_p becomes, in terms of the inner variable,

$$w_p = \sum_{q=1}^{\infty} \left\{ a_{Mn}^{(p)} (\epsilon Z)^{-q} + {\sum}' a_{qm}^{(p)} (z_n - z_m)^{-q} \left(1 + \frac{\epsilon Z}{z_n - z_m}\right)^{-q} \right\} - W_p(z_n + \epsilon Z) + C, \tag{9}$$

where the prime indicates that the sum over m does not include the term in n. To carry out the matching, we use van Dyke's rule in the form $\varpi^{(m,n)} = w^{(n,m)}$ where the notation is as follows (Crighton *et al* 1991): the first superscript indicates the order of truncation of the expansion considered; then the inner and outer expansions are rewritten in terms of the outer and inner variable, respectively, and truncated at the order given by the second superscript. When the second superscript is absent, the solution is not truncated in the 'wrong' variable.

The $O(1)$ matching is almost automatic. From (1) we find

$$w^{(0,.)} = \frac{\Gamma_n}{2\pi\mathrm{i}} \log \epsilon Z + {\sum}' \frac{\Gamma_m}{2\pi\mathrm{i}} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{(\epsilon Z)^k}{k(z_n - z_m)^k} - \epsilon W_0 Z + C. \tag{10}$$

From (5)

$$\varpi^{(0,.)} = \frac{\Gamma_n}{2\pi\mathrm{i}} \log (z - z_n) + c, \tag{11}$$

neglecting terms that decay faster than any power. There is also an $O(\log \epsilon)$ term in the inner expansion, but it is dynamically irrelevant and hence suppressed. One finds $\varpi^{(0,0)} = w^{(0,0)}$, as expected.

The outer solution to $O(\epsilon)$ gives

$$w^{(1,.)} = \frac{\Gamma_n}{2\pi\mathrm{i}} \log \epsilon Z + {\sum}' \frac{\Gamma_m}{2\pi\mathrm{i}} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{(\epsilon Z)^k}{k(z_n - z_m)^k} - \epsilon W_0 Z + C + \epsilon w_1. \tag{12}$$

The $O(\epsilon)$ inner problem is homogeneous, so

$$\varpi^{(1,.)} = \frac{\Gamma_n}{2\pi\mathrm{i}} \log (z - z_n) + c + \epsilon \left[\frac{\epsilon b_1^{(1)}}{z - z_n} + \sum_{m=2}^{\infty} \left\{ b_m^{(1)} \left(\frac{z - z_n}{\epsilon}\right)^m + b_m^{(1)*} \beta_m \left(\frac{z - z_n}{\epsilon}\right)^{-m} \right\} \right] \tag{13}$$

and

$$\varpi^{(1,1)} = \cdots + \epsilon \sum_{m=2}^{\infty} b_m^{(1)} \left(\frac{z - z_n}{\epsilon}\right)^m, \tag{14}$$

where the ellipsis corresponds to terms that have already been matched and constants have been included in it. By the matching principle, $\varpi^{(1,1)}$ is equal to

$$w^{(1,1)} = \cdots + \epsilon \left[{\sum}' \frac{\Gamma_m}{2\pi \mathrm{i}} \frac{Z}{z_n - z_m} - W_0 Z + \sum_{q=1}^{\infty} a_{qn}^{(1)} (\epsilon Z)^{-q} \right]. \tag{15}$$

The matching gives $a_{qm}^{(1)} = 0$ for $q > 0$ (this is true for vortex n and hence for all vortices m), $b_m^{(1)} = 0$ for $m > 1$ and

$$W_0 = {\sum}' \frac{\Gamma_m}{2\pi \mathrm{i}} \frac{1}{z_n - z_m}, \tag{16}$$

the expected leading-order velocity of the point vortex limit. Note that $b_1^{(1)}$ is unspecified at this point. We take it to be zero for a number of reasons. First this gives the correct matching subsequently. Second there is no outer term for it to match at this or higher orders; hence it cannot be dynamically important and must be set to zero. Finally if one considers the linearized stability problem for a general circular vortex, Llewellyn Smith (1995) shows that the mode-1 response tends to the $r\Omega$ solution mentioned above with amplitude proportional to the initial vorticity in mode 1. There is an arbitrary degree of freedom, so we take it to be zero until $O(\epsilon^4)$ when the mode-1 problem is no longer homogeneous. As a result, ψ_1 is just a constant and dynamically irrelevant.

At $O(\epsilon^2)$, the outer solution becomes

$$w^{(2,.)} = \frac{\Gamma_n}{2\pi \mathrm{i}} \log \epsilon Z + {\sum}' \frac{\Gamma_m}{2\pi \mathrm{i}} \sum_{k=2}^{\infty} (-1)^{k-1} \frac{(\epsilon Z)^k}{k (z_n - z_m)^k} + C - \epsilon^2 W_1 Z + \epsilon^2 w_2. \tag{17}$$

The problem for the inner streamfunction is again homogeneous, so

$$\varpi^{(2,.)} = \frac{\Gamma_n}{2\pi \mathrm{i}} \log (z - z_n) + c + \epsilon^2 \sum_{m=2}^{\infty} \left\{ b_m^{(2)} \left(\frac{z - z_n}{\epsilon} \right)^m + b_m^{(2)*} \beta_m \left(\frac{z - z_n}{\epsilon} \right)^{-m} \right\}. \tag{18}$$

Now truncate:

$$\varpi^{(2,2)} = \cdots + \epsilon^2 \sum_{m=2}^{\infty} b_m^{(2)} \left(\frac{z - z_n}{\epsilon} \right)^m, \tag{19}$$

and match to

$$w^{(2,2)} = \cdots + \epsilon^2 \left[- {\sum}' \frac{\Gamma_m}{2\pi \mathrm{i}} \frac{Z^2}{2 (z_n - z_m)^2} - W_1 Z + \sum_{q=1}^{\infty} a_{qn}^{(2)} (\epsilon Z)^{-q} \right]. \tag{20}$$

This gives almost the same matching problem as before, with $a_{qm}^{(2)} = 0$ for $q > 0$ and all m, $b_m^{(2)} = 0$ for $m > 2$, $W_1 = 0$ and

$$b_2^{(2)} = a_2 = - {\sum}' \frac{\Gamma_m}{2\pi \mathrm{i}} \frac{1}{2 (z_n - z_m)^2}. \tag{21}$$

This is the strain induced by the other vortices at the location of point vortex n. There is no $O(\epsilon)$ correction to the propagation velocity since $W_1 = 0$. The inner solution is hence, up to a constant,

$$\psi_2 = -\frac{\mathrm{i}}{2} a_2 f_2(R) \mathrm{e}^{2\mathrm{i}\theta} + c.c. \tag{22}$$

The $O(\epsilon^3)$ matching will follow exactly the same pattern as before, since the inner problem is homogeneous. The result is $a^{(3)}_{qm} = 0$ for $q > 0$ and all m, $W_2 = 0$,

$$b^{(3)}_3 = a_3 = \sum{}' \frac{\Gamma_m}{2\pi\mathrm{i}} \frac{1}{3(z_n - z_m)^3} \tag{23}$$

and $b^{(3)}_m = 0$ otherwise. As above, we find

$$\psi_3 = -\frac{\mathrm{i}}{2} a_3 f_3(R) \mathrm{e}^{3\mathrm{i}\theta} + c.c. \tag{24}$$

The difference at $O(\epsilon^4)$ is that the inner problem is no longer homogeneous. It is forced by the self-interaction of the $O(\epsilon^2)$ solution. The interaction terms, which are quadratic, give modes 0 and 4, but, as shown previously, the forcing terms for mode 0 vanish. For mode 4, we generalize our earlier notation to allow β_4 to include the self-interaction term. The other difference is the presence of the $b^{(2)}_2 Z^{-2}$ term. We have

$$w^{(4,.)} = \frac{\Gamma_n}{2\pi\mathrm{i}} \log \epsilon Z + \sum{}' \frac{\Gamma_m}{2\pi\mathrm{i}} \sum_{k=2}^{\infty} (-1)^{k-1} \frac{(\epsilon Z)^k}{k(z_n - z_m)^k} + C - \epsilon^4 W_3 Z + \epsilon^4 w_4 \tag{25}$$

for the outer problem. For the inner problem

$$\begin{aligned} \varpi^{(4,.)} = & \frac{\Gamma_n}{2\pi\mathrm{i}} \log(z - z_n) + c \\ & + \sum_{p=2}^{3} \epsilon^p \left[a_p \left(\frac{z - z_n}{\epsilon} \right)^p + a_p^* \beta_p \left(\frac{z - z_n}{\epsilon} \right)^{-p} \right] \\ & + \epsilon^4 \sum_{m=2}^{\infty} \left\{ b^{(4)}_m \left(\frac{z - z_n}{\epsilon} \right)^m + b^{(4)*}_m \beta_4 \left(\frac{z - z_n}{\epsilon} \right)^{-m} \right\}. \end{aligned} \tag{26}$$

Hence

$$\varpi^{(4,4)} = \cdots + \epsilon^4 \left[a_2^* \beta_2 (z - z_n)^{-2} + \sum_{m=2}^{\infty} b^{(4)}_m \left(\frac{z - z_n}{\epsilon} \right)^m \right]. \tag{27}$$

Match to

$$w^{(4,4)} = \cdots + \epsilon^4 \left[-\sum{}' \frac{\Gamma_m}{2\pi\mathrm{i}} \frac{Z^4}{4(z_n - z_m)^4} - W_3 Z + \sum_{q=1}^{\infty} a^{(4)}_{qn} (\epsilon Z)^{-q} \right]. \tag{28}$$

This gives $a^{(4)}_{qm} = 0$ for $q = 1$, $q > 2$ and all m, $W_3 = 0$,

$$a^{(4)}_{2n} = a_2^* \beta_2, \qquad b^{(4)}_4 = a_4 = -\sum{}' \frac{\Gamma_m}{2\pi\mathrm{i}} \frac{1}{4(z_n - z_m)^4}, \tag{29}$$

and $b^{(4)}_m = 0$ otherwise.

At $O(\epsilon^5)$ the inner problem is not homogeneous and is forced by interactions between modes 2 and 3. We care about the resulting mode 1 response. The coefficient β_m will contain contributions from the inhomogeneous term, leading in particular to a β_1 term which did not exist for previous orders. As usual,

$$w^{(5,.)} = \frac{\Gamma_n}{2\pi\mathrm{i}}\log\epsilon Z + \sum{}' \frac{\Gamma_m}{2\pi\mathrm{i}}\sum_{k=2}^{\infty}(-1)^{k-1}\frac{(\epsilon Z)^k}{k(z_n-z_m)^k} + C - \epsilon^5 W_4 Z + \epsilon^5 w_5. \quad (30)$$

For the inner problem

$$\begin{aligned}\varpi^{(5,.)} &= \frac{\Gamma_n}{2\pi\mathrm{i}}\log(z-z_n) + c \\ &+ \sum_{p=2}^{4}\epsilon^p\left[a_p\left(\frac{z-z_n}{\epsilon}\right)^p + a_p^*\beta_p\left(\frac{z-z_n}{\epsilon}\right)^{-p}\right] \\ &+ \epsilon^5\sum_{m=1}^{\infty}\left\{b_m^{(5)}\left(\frac{z-z_n}{\epsilon}\right)^m + b_m^{(5)*}\beta_m\left(\frac{z-z_n}{\epsilon}\right)^{-m}\right\}. \end{aligned} \quad (31)$$

Hence

$$\varpi^{(5,5)} = \cdots + \epsilon^5\sum_{m=1}^{\infty} b_m^{(5)}\left(\frac{z-z_n}{\epsilon}\right)^m, \quad (32)$$

which, by matching, is equal to

$$w^{(5,5)} = \cdots + \epsilon^5\left[\sum{}'\frac{\Gamma_m}{2\pi\mathrm{i}}\frac{Z^5}{5(z_n-z_m)^5} - W_4 Z + \sum_{q=1}^{\infty} a_{qn}^{(5)}(\epsilon Z)^{-q}\right]. \quad (33)$$

The matching gives $a_{qm}^{(5)} = 0$ for $q > 1$ for all m,

$$b_5^{(5)} = a_5 = \sum{}'\frac{\Gamma_m}{2\pi\mathrm{i}}\frac{1}{5(z_n-z_m)^5}, \qquad W_4 = -b_1^{(5)} \quad (34)$$

and $b_m^{(5)} = 0$ otherwise.

We have completed the matching. If W_4 is not the same for all vortices, we have not found an equilibrium correction and this approach fails. To compute the correction to the propagation velocity, we calculate w_0, a_2 and a_3 from the equilibrium configuration. The first term gives the propagation velocity of the point vortex equilibrium. We then solve the homogeneous inner radial Rayleigh problem for modes 2 and 3, giving $f_2(R)$ and $f_3(R)$. The inner solution at these orders is $\psi_2 = -\frac{1}{2}\mathrm{i}a_2 f_2(R)\mathrm{e}^{2\mathrm{i}\theta} + c.c.$ and $\psi_3 = -\frac{1}{2}\mathrm{i}a_3 f_3(R)\mathrm{e}^{3\mathrm{i}\theta} + c.c.$ We then solve the mode-1 inner problem at $O(\epsilon^5)$, for which the forcing term F is, from (7) with $p = 3$ and $q = 2$,

$$\psi_1'' + \frac{1}{R}\psi_1' - \left[\frac{Q'}{R\Omega} + \frac{1}{R^2}\right]\psi_1 = \frac{\mathrm{i}a_3 a_2^*}{4R\Omega}[-2\mathrm{i}(f_3' g_2 - g_3' f_2) + 3\mathrm{i}(f_2' g_3 - g_2' f_3)]. \quad (35)$$

The solution to this equation that is bounded at the origin can be found in closed form. The operator on the left-hand side has the solution $f_1 = R\Omega$ that is bounded at 0 and ∞, as mentioned before, and also the unbounded solution

$$g_1 = R\Omega\int\frac{\mathrm{d}u}{u^3\Omega(u)^2} \sim \frac{\pi R}{\Gamma} \quad \text{as } R\to\infty. \quad (36)$$

We can find a Green's function bounded at the origin in the form

$$G(R,\xi) = \begin{cases} a(\xi) f_1(R) & \text{for } R < \xi, \\ b(\xi) f_1(R) + c(\xi) g_1(R) & \text{for } R > \xi. \end{cases} \tag{37}$$

The continuity and jump conditions on G give

$$a f_1(\xi) = b f_1(\xi) + c g_1(\xi), \qquad b f_1'(\xi) + c g_1'(\xi) - a f_1'(\xi) = 1, \tag{38}$$

leading to $c(\xi) = f_1(\xi) W(\xi)^{-1} = \xi f_1(\xi) = \xi^2 \Omega(\xi)$, where $W(\xi) = f_1(\xi) g_1'(\xi) - f_1'(\xi) g_1(\xi) = \xi^{-1}$ is the Wronskian. The solution is

$$f_1(R) = \int_R^\infty a(\xi) F(\xi)\,\mathrm{d}\xi + \int_0^R [b(\xi) f_1(R) + c(\xi) g_1(R)] F(\xi)\,\mathrm{d}\xi. \tag{39}$$

where $F(\xi)$ is the right-hand side of (35). The coefficient of Z in the far field is then

$$W_4 = -\frac{2\pi\mathrm{i}}{\Gamma} \int_0^\infty R^2 \Omega(R) F(R)\,\mathrm{d}R. \tag{40}$$

5. Conclusions

We have found the correction to the velocity of steadily translating vortex arrays. It enters at $O(\epsilon^4)$, where ϵ is the ratio of vortex core size to array dimension. The correction is proportional to the second and third derivatives of the regularized complex potential at the vortex cores. The case of configurations that rotate as well as (or instead of) translating should be amenable to the same approach, although one will no longer have the luxury of working in an inertial frame moving with the array. If the vortices have different interior core structures or sizes, each will feel a different strain and strain derivative from the others. These must be related in just such a way as to give the same correction W_4 for all vortices, yielding a steadily propagating array.

The analysis given here is reminiscent of that of Ting and Klein (1991), who showed that the motion of a point vortex can be found by ignoring the divergent part of the velocity field that it induces, by matching the evolution of a Rankine vortex core to a far field. In our notation, this corresponds to obtaining W_0 in the time-dependent case, while our goal was to compute the higher-order corrections to the velocity of steadily propagating equilibria for arbitrary core profiles. The results here also show that the MAE approach works for arbitrary core structures for obtaining the steady propagation velocity at $O(1)$, and that the details of the core structure do not matter, as one might have expected.

More work remains to be done, in particular relating the results to numerical solutions and investigating rotating cases, starting with the co-rotating vortex pair for which vortex patch solutions have previously been obtained but for which no hollow vortices have been found.

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