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Hermitian Sums of Squares Modulo Hermitian Ideals

A dissertation submitted in partial satisfaction
of the requirements for the degree

Doctor of Philosophy
in
Mathematics

by

Glen A. Frost

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December 2020

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December 2020

Hermitian Sums of Squares Modulo Hermitian Ideals

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Glen Frost

Dedicated to my parents.

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Abstract

Hermitian Sums of Squares Modulo Hermitian Ideals

by

Glen Frost

In this work we study the problem of writing a Hermitian polynomial as a Hermitian sum of squares modulo a Hermitian ideal. We investigate a novel idea of Putinar-Scheiderer to obtain necessary matrix positivity conditions for Hermitian polynomials to be Hermitian sums of squares modulo Hermitian ideals. We show that the conditions are sufficient for a class of examples making a connection to the operator-valued Riesz-Fejer theorem and block Toeplitz forms. The work fits into the larger themes of Hermitian versions of Hilbert's 17-th problem and characterizations of positivity.

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Chapter 0

Introduction

0.1 Problem Statement

In this work we study the problem of writing a Hermitian polynomial as a Hermitian sum of squares modulo a Hermitian ideal. We first introduce these concepts and state the problem.

Let $\mathbb{C}[z, \bar{z}]$ denote the polynomial algebra in the variables $z = (z_1, \dots, z_n)$ and $\bar{z} = (\bar{z}_1, \dots, \bar{z}_n)$ with coefficients in \mathbb{C} . We can write each element $f \in \mathbb{C}[z, \bar{z}]$ using standard multinomial notation:

$$f(z, \bar{z}) = \sum_{\alpha, \beta} a_{\alpha\beta} \bar{z}^\alpha z^\beta \quad (0.1)$$

Or in dimension one ($n = 1$) using matrix notation:

$$f(z, \bar{z}) = \begin{bmatrix} 1 \\ z \\ \vdots \\ z^d \end{bmatrix}^* \begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0d} \\ a_{10} & a_{11} & \cdots & a_{1d} \\ \vdots & \vdots & \ddots & \vdots \\ a_{d0} & a_{d1} & \cdots & a_{dd} \end{bmatrix} \begin{bmatrix} 1 \\ z \\ \vdots \\ z^d \end{bmatrix} = \psi_d(z)^* A \psi_d(z) \quad (0.2)$$

where

$$\psi_d(z) = \begin{bmatrix} 1 \\ z \\ \vdots \\ z^d \end{bmatrix} \quad (0.3)$$

denotes the *tautological monomial map* and $A = [a_{jk}]$ is an $(d+1) \times (d+1)$ matrix with complex entries. We often write $\psi(z)$ for $\psi_d(z)$ when d is understood.

Here we use standard matrix notation and operations. If A is an $n \times m$ matrix, then A^T denotes the transpose, \bar{A} denotes the conjugate, and A^* denotes the conjugate transpose. We think of elements $v \in \mathbb{C}^n$ as column vectors and $\langle v, w \rangle = v^*w$ denotes the standard inner product on complex Euclidean space, with the convention being conjugate linear in the first component.

We have an involution $f \mapsto f^*$ given by conjugation:

$$f^*(z, \bar{z}) := \overline{f(z, \bar{z})} = \psi(z)^* A^* \psi(z) \quad (0.4)$$

Say f is *Hermitian* if $f = f^*$. Hermitian polynomials are real-valued on \mathbb{C}^n . Let $\mathbb{C}_h[z, \bar{z}]$ denote the collection of Hermitian polynomials. An ideal I in $\mathbb{C}_h[z, \bar{z}]$ is called a *Hermitian ideal*.

If $h(z) \in \mathbb{C}[z]$ is a holomorphic polynomial, then

$$|h(z)|^2 = h(z)\overline{h(z)} \in \mathbb{C}_h[z, \bar{z}] \quad (0.5)$$

is a *Hermitian square*. A Hermitian polynomial of the form

$$|h_1(z)|^2 + \cdots + |h_\ell(z)|^2 \quad (0.6)$$

is called a *Hermitian sum of squares*. Let $\Sigma_h^2 \subset \mathbb{C}_h[z, \bar{z}]$ denote the collection of Hermitian sums of squares. We now state the main problem:

Problem 1 *Suppose f is a Hermitian polynomial and I is a Hermitian ideal. Under what conditions on f and I does there exist an identity:*

$$f(z, \bar{z}) = \sum_{j=1}^{\ell} |h_j(z)|^2 + g(z, \bar{z}) \quad (0.7)$$

where h_1, \dots, h_ℓ are holomorphic polynomials and $g \in I$.

We easily obtain the *trivial necessary condition*, point-wise positivity on the zero set of I :

$$f(p, \bar{p}) \geq 0 \quad \forall p \in \mathcal{Z}(I) \quad (0.8)$$

where $\mathcal{Z}(I)$ denotes the zero-set of I :

$$\mathcal{Z}(I) = \{p \in \mathbb{C}^n \mid g(p, \bar{p}) = 0 \ \forall g \in \mathcal{Z}(I)\} \quad (0.9)$$

The following example shows that this condition is not sufficient.

Example In dimension one ($n = 1$) consider

$$f(z, \bar{z}) = (z + \bar{z})^2 \quad I = (0) \quad (0.10)$$

Note that $\mathcal{Z}(I) = \mathbb{C}$ and $f(z, \bar{z}) \geq 0$ for all $z \in \mathbb{C}$ since $z + \bar{z} = 2\text{Re}(z)$, where $\text{Re}(z)$ denotes the real part of z .

Suppose for sake of contradiction that $f \in \Sigma_h^2$. Then

$$z^2 + 2z\bar{z} + \bar{z}^2 = |h_1(z)|^2 + \dots + |h_\ell(z)|^2 \quad (0.11)$$

where $h_1, \dots, h_\ell \in \mathbb{C}[z]$. We compare coefficients of the diagonal monomial $z^j \bar{z}^j$. Write $h_j(z) = h_{j0} + h_{j1}z + \dots + h_{jm}z^m$. Then by equating coefficients of diagonal monomials we obtain:

$$\begin{aligned} 0 &= |h_{10}|^2 + \dots + |h_{\ell 0}|^2 \\ 2 &= |h_{11}|^2 + \dots + |h_{\ell 1}|^2 \\ 0 &= |h_{12}|^2 + \dots + |h_{\ell 2}|^2 \\ &\vdots \end{aligned}$$

The equations with 0 on the left hand side imply that all $h_{jk} = 0$ for $k \neq 1$. Hence each $h_j(z)$ is of the form $h_j(z) = h_{1j}z$ with only the degree 1 term. Then

$$z^2 + 2z\bar{z} + \bar{z}^2 = (|h_{11}|^2 + \dots + |h_{\ell 1}|^2)z\bar{z} \quad (0.12)$$

is a contradiction from equating the off-diagonal coefficients. Hence $f \notin \Sigma_h^2$.

□

The following two examples show classical situations where the trivial necessary condition is sufficient.

Example: Riesz-Fejer Lemma Consider $I = (z\bar{z} - 1)$ in dimension one. Then $\mathcal{Z}(I) = \mathbb{T} \subset \mathbb{C}$ is the complex unit circle. In this case the trivial positivity condition is sufficient. The idea goes back to the Riesz-Fejer lemma:

Lemma 1 (Riesz-Fejer [15][7]) *Every non-negative trigonometric polynomial agrees with the modulus squared of a holomorphic polynomial on the unit circle.*

A trigonometric polynomial is a polynomial of the form $\sum_{j=-m}^m c_j e^{ij\theta}$ with $c_j \in \mathbb{C}$. Suppose $f(z) = \sum_{j=-m}^m c_j z^j \in \mathbb{C}[z, z^{-1}]$ is a Laurent polynomial. The Riesz-Fejer lemma

states that if $f(e^{i\theta}) \geq 0$ for all real θ , then there exists a holomorphic polynomial $p(z) = \sum_{j=0}^m a_j z^j$ such that

$$f(e^{i\theta}) = |p(e^{i\theta})|^2 \quad (0.13)$$

In our terminology:

Theorem 2 *Let f be a Hermitian polynomial. The following statements are equivalent:*

1. $f(\xi, \bar{\xi}) \geq 0$ for all $\xi \in \mathbb{T}$
2. $f \in \Sigma_h^2 + (z\bar{z} - 1)$

The Riesz-Fejer lemma is known to be equivalent to the spectral theorem for unitary operators.[16, p.281]

□

Example: Hermitian Linear Forms In arbitrary dimension n , consider the Hermitian ideal $I = (0)$ and take the Hermitian polynomial f to be homogeneous linear:

$$f(z, \bar{z}) = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}^* \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \quad (0.14)$$

where $A = [a_{jk}]$ is a Hermitian matrix. Note $\mathcal{Z}(I) = \mathbb{C}^n$ and $\Sigma_h^2 + I = \Sigma_h^2$. In this situation the trivial necessary condition is sufficient.

Theorem 3 *Suppose $f(z, \bar{z})$ is a linear Hermitian homogeneous form. The following statements are equivalent:*

1. $f(v, \bar{v}) \geq 0$ for all $v \in \mathbb{C}^n$
2. $f \in \Sigma_h^2$

The main idea is diagonalization of a Hermitian matrix with origins in Hermite (1854), Sylvester’s inertia (1853), and Sturm’s algorithm (1835).

□

0.2 Context

A generalization of the Riesz-Fejer lemma to the odd-dimensional sphere $S^{2n-1} \subset \mathbb{C}^n$ appears in Quillen [14]:

Theorem 4 (Quillen-Catlin-D’Angelo) *Suppose $f(z, \bar{z}) \in \mathbb{C}_h[z, \bar{z}]$ is bihomogeneous. The following conditions are equivalent:*

1. $f(z, \bar{z}) > 0$ for all nonzero $z \in \mathbb{C}^n$
2. There exists positive integer N such that $\|z\|^{2N} f(z, \bar{z}) \in \Sigma_h^2$

The theorem is rediscovered by Catlin-D’Angelo [2]. Both proofs are analytic. A purely algebraic proof appears in [12] by applying the Archimedean Positivstellensatz of real algebra, allowing the following formulation in our terminology:

Theorem 5 *Suppose $f \in \mathbb{C}_h[z, \bar{z}]$ is a Hermitian polynomial and $I = (z_1 \bar{z}_1 + \cdots + z_n \bar{z}_n - 1)$. The following conditions are equivalent:*

1. $f(\xi, \bar{\xi}) > 0$ for all $\xi \in S^{2n-1} \subset \mathbb{C}^n$
2. $f \in \Sigma_h^2 + I$

For the story to here and other directions, see the survey [11].

The sphere is the prototypical example of a pseudoconvex hypersurface. At the 2006 AIM Conference “CR Complexity Theory” in Palo Alto, D’Angelo asks the following generalization of Quillen’s theorem:

D'Angelo's Question 1 *If f is a Hermitian polynomial on a pseudoconvex hypersurface, then does f agree with a Hermitian sum of squares along the hypersurface?*

The answer to this question lies in the paper [12]. Two main ideas of this paper:

1. An obstruction to the question is introduced and a counterexample is constructed. The obstruction provides necessary matrix positivity conditions for the Hermitian polynomial f beyond pointwise positivity given by the trivial necessary condition.
2. A characterization of the Hermitian ideals for which every positive Hermitian polynomial is a Hermitian sum of squares is obtained using the Archimedean Positivstellensatz.

The first point is explored further in the paper [4], and the second point is explored further in the paper [13].

Our goal is to continue the idea of the matrix positivity conditions for Hermitian polynomials, and obtain equivalent characterizations for Hermitian sums of squares modulo Hermitian ideals, thereby going beyond the Archimedean positivstellensatz and pointwise-positivity conditions.

0.3 Matrix Positivity Conditions for Hermitian Polynomials

In this section we describe the idea of matrix positivity conditions used in [12] to construct the counterexample to D'Angelo's question.

If A is an $n \times n$ Hermitian matrix, then we say A is *positive* (denoted $A \geq 0$) if

$$v^*Av \geq 0 \quad \forall v \in \mathbb{C}^n \tag{0.15}$$

Given points p_1, \dots, p_ℓ , we construct the *Gram matrix* of pairwise inner products:

$$\text{Gram}(p_1, \dots, p_\ell) := \begin{bmatrix} \langle p_1, p_1 \rangle & \cdots & \langle p_\ell, p_1 \rangle \\ \vdots & \ddots & \vdots \\ \langle p_1, p_\ell \rangle & \cdots & \langle p_\ell, p_\ell \rangle \end{bmatrix} \quad (0.16)$$

Let B be the matrix with columns p_1, \dots, p_ℓ . Then

$$\text{Gram}(p_1, \dots, p_\ell) = B^* B \quad (0.17)$$

and hence

$$v^* \text{Gram}(p_1, \dots, p_\ell) v = v^* B^* B v = \langle Bv, Bv \rangle \geq 0 \quad (0.18)$$

for all $v \in \mathbb{C}^\ell$. This demonstrates the classical idea: every Gram matrix is positive. In fact this provides a characterization of positivity (existence of a Cholesky factorization).

Theorem 6 *Let A be a Hermitian matrix. The following statements are equivalent:*

1. $A \geq 0$
2. $A = B^* B$ for some matrix B .

A *holomorphic polynomial map* is a map $h : \mathbb{C}^n \rightarrow \mathbb{C}^\ell$ whose components are holomorphic polynomials. Every Hermitian sum of squares may be written as the squared norm of a holomorphic polynomial map:

$$|h_1(z)|^2 + \cdots + |h_\ell(z)|^2 = \langle h(z), h(z) \rangle \quad (0.19)$$

where $h(z) = (h_1(z), \dots, h_\ell(z))$ is a holomorphic polynomial map.

Given a Hermitian polynomial f and a Hermitian ideal I , we now show how to obtain necessary matrix positivity conditions which must be satisfied by f .

For a Hermitian polynomial $f(z, \bar{z})$, we may “polarize” and treat z, \bar{z} as independent variables.

Suppose f is a Hermitian sum of squares modulo I :

$$f(z, \bar{z}) = \langle h(z), h(z) \rangle + g(z, \bar{z}) \quad (0.20)$$

where h is a holomorphic polynomial map and $g \in I$.

Suppose further that $p_1, \dots, p_\ell \in \mathbb{C}^n$ are points such that

$$g(p_j, \bar{p}_k) = 0 \quad \forall g \in I \quad \forall j, k = 1, \dots, \ell \quad (0.21)$$

Then

$$f(p_j, \bar{p}_k) = \langle h(p_j), h(p_k) \rangle \quad \forall j, k = 1, \dots, \ell \quad (0.22)$$

And since a Gram matrix is positive, we have

$$[f(p_j, \bar{p}_k)]_{j,k=1}^\ell \geq 0 \quad (0.23)$$

Thus we conclude:

Lemma 7 (Matrix Positivity Conditions) *Let f be a Hermitian polynomial and I a Hermitian ideal. If $f \in \Sigma_h^2 + I$, then f satisfies (0.23) for all collections of points p_1, \dots, p_ℓ satisfying (0.21)*

Using this idea we obtain necessary conditions for $f \in \Sigma_h^2 + I$. Our main question of interest: are these conditions sufficient?

Example Again consider $f = (z + \bar{z})^2$ and $I = (0)$. Choose $p_1 = 0$ and $p_2 = 1$. Then consider the Gram matrix of f on p_1, p_2 :

$$\text{Gram}(f)[p_1, p_2] = \begin{bmatrix} f(p_1, \bar{p}_1) & f(p_1, \bar{p}_2) \\ f(p_2, \bar{p}_1) & f(p_2, \bar{p}_2) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \quad (0.24)$$

The matrix has determinant -1 , hence is not positive. Therefore $f \notin \Sigma_h^2$.

Example We discuss the matrix positivity conditions for the ideal $I = (z^N \bar{z}^N - 1)$, where N is a positive integer. Let $P(z, \bar{z}) = z^N \bar{z}^N - 1$ and $\omega = e^{2\pi i/N}$. Then:

$$P(\omega^j \xi, \overline{\omega^k \xi}) = 0 \quad \forall \xi \in \mathbb{T} \quad \forall j, k = 0, \dots, N-1 \quad (0.25)$$

Therefore by the idea of matrix positivity conditions, if $f \in \Sigma_h^2 + (z^N \bar{z}^N - 1)$, then

$$[f(\omega^j \xi, \overline{\omega^k \xi})]_{j,k=0}^{N-1} \geq 0 \quad \forall \xi \in \mathbb{T} \quad (0.26)$$

Our main result (Chapter 3) is to show that these conditions are sufficient. The case $N = 1$ is the classical Riesz-Fejer lemma. We make a connection to the operator-valued RF lemma [5] by way of block Toeplitz forms and orthogonal polynomials on the circle.

0.4 Summary of Chapters

- **Chapter 1: Elementary Theory of Hermitian Polynomials** introduce matrix terminology and notation, concept of Hermitian matrix, concept of positive matrix, characterizations of positivity, Hermitian polynomials, polarization, holomorphic decomposition, Hermitian sums of squares, semirings, Hermitian modules, Archimedean positivstellensatz, matrix positivity conditions,

- **Chapter 2: Main Results** basic computations on the circle, Toeplitz forms, the functional \mathcal{F}_N , representing v^*Aw with \mathcal{F}_N , trigonometric representations, operator-valued RF theorem.

The main result is full characterization of Hermitian sums of squares modulo $(z^N \bar{z}^N - 1)$

- **Chapter 3: General Case** We discuss how the general case might be solved by analogy of development of the real spectrum solving Hilbert's 17-th problem.

Chapter 1

Elementary Theory of Hermitian Polynomials

1.1 Hermitian Matrices

For standard references on Hermitian matrix analysis, see [10, Horn, Johnson], [8, Gantmacher]. Let \mathcal{H} be the Hilbert space \mathbb{C}^n with inner product:

$$\langle x, y \rangle = x^* y \tag{1.1}$$

with the convention of conjugate linear in the first component. Regard elements of \mathcal{H} as column vectors.

$\mathbb{C}^{n \times m}$ denotes the collection of $n \times m$ matrices with entries from \mathbb{C} . Let M_n denote $\mathbb{C}^{n \times n}$ the collection of square matrices. Standard rules for matrix algebra. If $A \in \mathbb{C}^{n \times m}$ we wrote $A = [a_{jk}]$ to denote the components. Denote $\bar{A} = [\bar{a}_{jk}]$ the conjugate. Denote $A^T = [a_{kj}]$ the transpose.

Let $A^* := \bar{A}^T$ denote the conjugate transpose. This gives an involution $M_n \rightarrow M_n$

by $A \mapsto A^*$. We say $A \in M_n$ is *Hermitian* if $A = A^*$. Let $\text{Herm}_n \subseteq M_n$ denote the collection of Hermitian matrices.

We say $A \in M_n$ is *normal* if $A^*A = AA^*$. Say $A \in M_n$ is *unitary* if $U^*U = UU^* = I$. Hermitian matrices and unitary matrices are normal. Let $\mathcal{U}_n \subset M_n$ denote the group of unitary matrices.

Theorem 8 (Spectral Theorem) *Let $A \in M_n$. The following conditions are equivalent:*

1. A is normal
2. $A = U^*DU$ for $U \in \mathcal{U}_n$ and D diagonal.

Furthermore, D is real if and only if A is Hermitian.

The diagonal entries of D are the eigenvalues of A .

We say $A \in \text{Herm}_n$ is *positive semidefinite* if

$$v^*Av \geq 0 \quad \forall v \in \mathbb{C}^n \quad (1.2)$$

We say $A \in \text{Herm}_n$ is *positive definite* if

$$v^*Av > 0 \quad \forall 0 \neq v \in \mathbb{C}^n \quad (1.3)$$

If A is positive semidefinite, then A is positive definite if and only if A is nonsingular. For convenience we use the term *(strictly) positive matrix* when A is positive (semi)definite and we write $(A > 0)A \geq 0$. Let $(\text{Herm}_n^>) \text{Herm}_n^{\geq}$ denote the collection of (strictly) positive matrices.

For integers $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, let $D(i_1, \dots, i_k)$ denote the determinant of the $k \times k$ submatrix of A formed by the i_j -th rows and columns of A . $D(i_1, \dots, i_k)$ is called a *principal minor of A* . $D(1, \dots, k)$ is called a *leading principal minor of A* .

The next theorem collects classical characterizations of matrix positivity: (see [1])

Theorem 9 *Let $A \in \text{Herm}_n$. The following conditions are equivalent:*

1. $A \geq 0$
2. all eigenvalues of A are ≥ 0 .
3. (Sylvester Criterion) All principal minors of $A \geq 0$.
4. (Gram Factorization) $A = B^*B$ for some B .
5. (Cholesky Factorization) $A = T^*T$ for some upper triangular T . T can be chosen to have non-negative diagonal entries.
6. $A = B^2$ for some unique positive matrix B , denoted $B = A^{1/2}$.
7. There exist $v_1, \dots, v_n \in \mathbb{C}^n$ such that $a_{jk} = \langle v_j, v_k \rangle$.
8. v^*Aw is an inner product
9. There exist $v_1, \dots, v_n \in \mathbb{C}^n$ such that $A = \sum v_j v_j^*$

Theorem 10 *Let $A \in \text{Herm}_n$. The following conditions are equivalent:*

1. $A > 0$
2. all eigenvalues of A are > 0 .
3. all principal minors of A are > 0 .
4. $A = B^*B$ for nonsingular matrix B .

5. $A = T^*T$ for a nonsingular upper triangular matrix T . T can be chosen to have strictly positive diagonal entries.
6. $A = B^2$ for some strictly positive matrix B .
7. There exist linearly independent $v_1, \dots, v_n \in \mathbb{C}^n$ such that $a_{jk} = \langle v_j, v_k \rangle$.
8. v^*Aw is a non-degenerate inner product
9. There exists linearly independent $v_1, \dots, v_n \in \mathbb{C}^n$ such that $A = \sum v_j v_j^*$

1.2 Hermitian Polynomials

Let $\mathbb{C}[z, \bar{z}]$ denote the algebra of polynomials in the variables $z = (z_1, \dots, z_n)$ and $\bar{z} = (\bar{z}_1, \dots, \bar{z}_n)$ with coefficients in \mathbb{C} . We can write each element $f \in \mathbb{C}[z, \bar{z}]$ using standard multinomial notation:

$$f(z, \bar{z}) = \sum_{\alpha, \beta} a_{\alpha\beta} \bar{z}^\alpha z^\beta \quad (1.4)$$

In dimension $n = 1$ we denote the variables as simply z and \bar{z} and we can write polynomials with matrix notation:

$$\begin{aligned} f(z, \bar{z}) &= \sum_{j,k=0}^d a_{jk} \bar{z}^j z^k \\ &= \begin{bmatrix} 1 \\ z \\ \vdots \\ z^d \end{bmatrix}^* \begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0d} \\ a_{10} & a_{11} & \cdots & a_{1d} \\ \vdots & \vdots & \ddots & \vdots \\ a_{d0} & a_{d1} & \cdots & a_{dd} \end{bmatrix} \begin{bmatrix} 1 \\ z \\ \vdots \\ z^d \end{bmatrix} \\ &= \psi_d(z)^* A \psi_d(z) \end{aligned}$$

where $A \in M_{d+1}$ and $\psi_d(z) = \begin{bmatrix} 1 & z & \dots & z^d \end{bmatrix}^T$ denotes the *tautological monomial map*. We often write $\psi(z)$ for $\psi_d(z)$ when d is understood.

We have an involution

$$\mathbb{C}[z, \bar{z}] \rightarrow \mathbb{C}[z, \bar{z}]$$

$$f \mapsto f^*$$

given by conjugation

$$f^*(z, \bar{z}) := \overline{f(z, \bar{z})} = \sum_{\alpha, \beta} \bar{a}_{\alpha\beta} \bar{z}^\beta z^\alpha \quad (1.5)$$

We say that f is a *Hermitian polynomial* if $f = f^*$ and let

$$\mathbb{C}_h[z, \bar{z}] := \{f \in \mathbb{C}[z, \bar{z}] \mid f = f^*\} \quad (1.6)$$

denote the collection of Hermitian polynomials. $\mathbb{C}_h[z, \bar{z}]$ is an \mathbb{R} -algebra isomorphic to $\mathbb{R}[x, y]$, where $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$, under the standard correspondence

$$z_j = x_j + iy_j \quad (1.7)$$

$$\bar{z}_j = x_j - iy_j \quad (1.8)$$

We have the following characterization of Hermitian polynomials:

Lemma 11 *Let $f \in \mathbb{C}[z, \bar{z}]$. The following statements are equivalent:*

1. $f(z, \bar{z}) = \overline{f(z, \bar{z})}$
2. $f(z, \bar{w}) = \overline{f(w, \bar{z})}$
3. *the coefficient matrix of f is Hermitian: $a_{\beta\alpha} = \bar{a}_{\alpha\beta}$*

1.3 Hermitian Sums of Squares

A polynomial $h \in \mathbb{C}[z, \bar{z}]$ is *holomorphic* if $\frac{\partial}{\partial \bar{z}}h = 0$. In other words, h depends only on z and we denote the dependence by $h(z)$. If $h(z)$ is a holomorphic polynomial, then

$$|h(z)|^2 := hh^* = h(z)\overline{h(z)} \quad (1.9)$$

is a Hermitian polynomial. Hermitian polynomials of this form are called *Hermitian squares*. A *sum of Hermitian squares* is a Hermitian polynomial of the form

$$|h_1(z)|^2 + \cdots + |h_\ell(z)|^2 \quad (1.10)$$

where $h_1(z), \dots, h_\ell(z)$ are holomorphic polynomials.

Let $\Sigma_h^2 \subset \mathbb{C}_h[z, \bar{z}]$ denote the collection of Hermitian sums of squares.

A *holomorphic polynomial map* is a map $h : \mathbb{C}^n \rightarrow \mathbb{C}^\ell$ whose components $h(z) = (h_1(z), \dots, h_\ell(z))$ are holomorphic polynomials. Every Hermitian sum of squares can be written as the squared norm of a holomorphic polynomial map

$$\langle h(z), h(z) \rangle = \sum h_j(z)\overline{h_j(z)} \quad (1.11)$$

Let $\Sigma^2 \subset \mathbb{C}_h[z, \bar{z}]$ be the collection of finite sums of squares of Hermitian polynomials. Then Σ^2 is the collection of sums of squares in $\mathbb{R}[x, y]$, a common object in real algebraic geometry.

We have the inclusion $\Sigma_h^2 \subset \Sigma^2$ from the identity:

$$|h(z)|^2 = \left(\frac{h + h^*}{2}\right)^2 + \left(\frac{h - h^*}{2i}\right)^2 \quad (1.12)$$

where $\frac{1}{2}(h + h^*), \frac{1}{2i}(h - h^*) \in \mathbb{C}_h[z, \bar{z}]$.

The inclusion is proper for a few reasons.

Lemma 12 *Let $f \in \Sigma_h^2$. Then $\mathcal{Z}(f) \subset \mathbb{C}^n$ is a complex algebraic set.*

Example Consider $f(z, \bar{z}) = (z + \bar{z})^2$. Then $\mathcal{Z}(f)$ is the imaginary axis in \mathbb{C} , not a complex algebraic set. Thus $f \notin \Sigma_h^2$.

□

An additional reason the inclusion is proper: elements of Σ_h^2 are plurisubharmonic. We say $f(z, \bar{z})$ is *plurisubharmonic (psh)* at p if the Levi matrix of f at p is positive:

$$\left[\frac{\partial^2}{\partial z_j \partial \bar{z}_k} f \right](p, \bar{p}) \geq 0 \quad (1.13)$$

And say f is *plurisubharmonic* on a domain if f is psh at all points of the domain. Note in particular that if f is psh at p , then by Sylvester's criterion we see $(\frac{\partial^2}{\partial z_k \partial \bar{z}_k} f)(p, \bar{p}) \geq 0$ for all k .

Lemma 13 *If $h(z)$ is a holomorphic polynomial, then $|h(z)|^2$ is plurisubharmonic on \mathbb{C}^n .*

Example Consider $f = (|z_1|^2 - |z_2|^2)^2$. Then f is not psh, hence $f \in \Sigma^2 \setminus \Sigma_h^2$.

□

We now describe a construction due to D'Angelo [3] known as the holomorphic decomposition.

Let $f \in \mathbb{C}_h[z, \bar{z}]$. Write $f = \psi_d(z)^* A \psi_d(z)$, where A is a Hermitian matrix. Apply the spectral theorem to A :

$$A = U^* D U$$

where $U \in \mathcal{U}(d+1)$ is unitary and D is a real diagonal matrix.

For a Hermitian matrix A , denote by $\text{sign}(A) = (a, b, c)$ the signature of A , where a is the number of positive eigenvalues of A , b is the number of negative eigenvalues of A , and c is the number of zero eigenvalues of A .

Write $D = \text{diag}(\rho_1, \dots, \rho_a, \eta_1, \dots, \eta_b, \zeta_1, \dots, \zeta_c)$, where the ρ_j are the positive eigenvalues, the η_j are the negative eigenvalues, and the ζ_j are zero. Then:

$$f(z, \bar{z}) = \psi_d(z)^* U^* D U \psi_d(z) \quad (1.14)$$

The entries of the column vector $U\psi_d(z)$ are holomorphic polynomials. Write:

$$U\psi_d(z) = \begin{bmatrix} F_1(z) & \cdots & F_a(z) & G_1(z) & \cdots & G_b(z) & H_1(z) & \cdots & H_c(z) \end{bmatrix}^T \quad (1.15)$$

Then we have:

$$f(z, \bar{z}) = \sum_{j=1}^a |\sqrt{\lambda_j} F_j(z)|^2 - \sum_{j=1}^b |\sqrt{\lambda_j} G_j(z)|^2 \quad (1.16)$$

From this decomposition we have the following basic test for a Hermitian polynomial to be a Hermitian sum of squares:

Lemma 14 (Basic Test) *Let $f \in \mathbb{C}_h[z, \bar{z}]$ be a Hermitian polynomial. Write $f(z, \bar{z}) = \psi(z)^* A \psi(z)$ where $\psi(z)$ is a vector of monomials and A is a Hermitian matrix. The following statements are equivalent:*

1. $f \in \Sigma_h^2$
2. $A \geq 0$

Example Consider $f = (z + \bar{z})^2$. Write f using matrix notation:

$$f = \begin{bmatrix} 1 \\ z \\ z^2 \end{bmatrix}^* \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ z \\ z^2 \end{bmatrix} \quad (1.17)$$

The Hermitian coefficient matrix not positive (by Sylvester's criterion) since the determinant is -2 . Thus $f \notin \Sigma_h^2$.

□

1.4 Hermitian Modules

Let A be an \mathbb{R} -algebra. A *semiring* in A is a subset $S \subset A$ such that

$$S + S \subset S \quad SS \subset S \quad \mathbb{R}_{\geq 0} \subset S \quad (1.18)$$

Two examples of semirings in $\mathbb{C}_h[z, \bar{z}]$ are Σ_h^2 and Σ^2 .

Let $A = [a_{jk}], B = [b_{jk}] \in \mathbb{C}^{n \times m}$. The *Hadamard product* of A and B is obtained by entrywise multiplication and denoted $A \circ B$:

$$A \circ B = [a_{jk}b_{jk}] \quad (1.19)$$

The *Schur product theorem* states that if $A \geq 0$ and $B \geq 0$, then $A \circ B \geq 0$. A simple proof follows from Sylvester's criterion and the inequality $\det(A \circ B) \geq \det(A) \det(B)$.

Example Herm_n has the structure of a ring with matrix addition and Hadamard product. By the Schur product theorem, we see that Herm_n^{\geq} is a semiring in Herm_n .

□

Let S be a semiring in A . An S -module is a subset $M \subset A$ such that

$$M + M \subset M \quad SM \subset M \quad 1 \in M \quad (1.20)$$

We refer to Σ_h^2 -modules as *Hermitian modules*. Σ^2 -modules are known in real algebraic geometry as quadratic modules.

Let M be an S -module in A . We say M is *Archimedean* if $A = \mathbb{R} + M$. This condition encodes a notion of compactness: regular functions are bounded.

The following lemma provides a basic characterization of the Archimedean property:

Lemma 15 ([12] [13]) *Let M be a Hermitian module. The following statements are equivalent:*

1. M is Archimedean.
2. $C - \|z\|^2 \in M$ for some $C \in \mathbb{R}$

Let I be a Hermitian ideal. Then $S = \Sigma_h^2 + I$ is a semiring in $\mathbb{C}_h[z, \bar{z}]$. We say I is *Archimedean* if $C - \|z\|^2 \in \Sigma_h^2 + I$ for some $C \in \mathbb{R}$. We say I is *Quillen* if $f(z, \bar{z}) > 0$ for all $z \in \mathcal{Z}(I)$ implies $f \in \Sigma_h^2 + I$.

The following result appears in [12] and [13] as an application of the Archimedean positivstellensatz:

Theorem 16 (Archimedean Positivstellensatz [12] [13]) *Let I be a Hermitian ideal. The following statements are equivalent:*

1. I is Archimedean

2. I is Quillen and $\mathcal{Z}(I)$ is compact

As a corollary we obtain Quillen's theorem since the ideal of the sphere is Archimedean. Note that if I is Archimedean, then I is Quillen. However the converse is false. For example, consider $I = (\text{Im}(z))$. Then I is Quillen, not Archimedean, and $\mathcal{Z}(I)$ is the noncompact real axis.

1.5 Matrix Positivity Conditions for Hermitian Polynomials

This is the main idea of Putinar and Scheiderer to construct the counterexample to the question of D'Angelo. It provides necessary conditions for $f \in \Sigma_h^2 + I$.

Let f be a Hermitian polynomial and I a Hermitian polynomial.

Suppose $p_1, \dots, p_\ell \in \mathbb{C}^n$ are points such that

$$g(p_j, \bar{p}_k) = 0 \quad \forall g \in I \quad \forall j, k = 1, \dots, \ell \quad (1.21)$$

We say such a collection of points p_1, \dots, p_ℓ is *orthogonal with respect to I* .

Suppose $f \in \Sigma_h^2 + I$ and p_1, \dots, p_ℓ are orthogonal with respect to I . Write:

$$f(z, \bar{z}) = \langle h(z), h(z) \rangle + g(z, \bar{z}) \quad (1.22)$$

where $h(z)$ is a holomorphic polynomial map and $g \in I$. Then

$$f(p_j, \bar{p}_k) = \langle h(p_j), h(p_k) \rangle \quad \forall j, k = 1, \dots, \ell \quad (1.23)$$

Since a Gram matrix is always positive, we have:

$$\text{Gram}(f)[p_1, \dots, p_\ell] := [f(p_j, \bar{p}_k)]_{j,k=1}^\ell \geq 0 \quad (1.24)$$

We summarize the discussion:

Lemma 17 *Let f be a Hermitian polynomial and I a Hermitian ideal. If $f \in \Sigma_h^2 + I$, then $\text{Gram}(f)[p_1, \dots, p_\ell] \geq 0$ for all collections of points p_1, \dots, p_ℓ orthogonal with respect to I .*

Our main problem is the following:

Problem 2 *Are the necessary Gram matrix positivity conditions for $f \in \Sigma_h^2 + I$ also sufficient?*

The matrix positivity conditions lead to non-Archimedean semirings as follows.

Let I be a Hermitian ideal. Suppose p_1, p_2 are distinct points orthogonal with respect to I . Then by the method of [12] we can construct a Hermitian polynomial f such that $f > 0$ on $\mathcal{Z}(I)$ and $f \notin \Sigma_h^2 + I$. Let $S = \Sigma_h^2 + I$ be the semiring. By the Archimedean positivstellensatz, S is not Quillen, hence S is non-Archimedean.

For instance, Σ_h^2 and $\Sigma_h^2 + (z^N \bar{z}^N - 1)$ for $N > 1$ are non-Archimedean semirings.

Chapter 2

Main Results

Our main result is to characterize Hermitian sums of squares modulo the Hermitian ideal $I = (z^N \bar{z}^N - 1)$, where N is a positive integer. The case $N = 1$ is the classical Riesz-Fejer lemma. For $N > 1$, matrix positivity conditions are necessary and pointwise-positivity on the circle is not sufficient. Matrix positivity conditions are obtained as follows. Let $\omega = e^{2\pi i/N}$ and $P(z, \bar{z}) = z^N \bar{z}^N - 1$. For every $\xi \in \mathbb{T}$ and for every $j, k \in \{0, \dots, N-1\}$ we have

$$P(\omega^j \xi, \overline{\omega^k \xi}) = 0 \tag{2.1}$$

Thus, for a Hermitian polynomial f , if $f \in \Sigma_h^2 + I$, then it is necessary that

$$\text{Gram}(f)[\xi, \omega \xi, \dots, \omega^{N-1} \xi] := [f(\omega^j \xi, \overline{\omega^k \xi})]_{j,k=0}^{N-1} \geq 0 \quad \forall \xi \in \mathbb{T} \tag{2.2}$$

We want to show that these conditions are sufficient. The next example supplies evidence to the investigation.

Example 18 ($N = 2$) *We demonstrate an example of a Hermitian polynomial f such that:*

$$(i) f \notin \Sigma_h^2$$

$$(ii) \text{Gram}(f)[e^{i\theta}, -e^{i\theta}] \geq 0 \text{ for all } \theta$$

$$(iii) f \in \Sigma_h^2 + (z^2 \bar{z}^2 - 1)$$

Consider:

$$f(z, \bar{z}) = 10 + 2z + 2\bar{z} + 10z\bar{z} - 2z^2\bar{z} - 2z\bar{z}^2 \quad (2.3)$$

$$= \begin{bmatrix} 1 \\ z \\ z^2 \end{bmatrix}^* \begin{bmatrix} 10 & 2 & 0 \\ 2 & 10 & -2 \\ 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ z \\ z^2 \end{bmatrix} \quad (2.4)$$

(i) By Sylvester's criterion, the Hermitian coefficient matrix is not positive, hence $f \notin \Sigma_h^2$.

(ii) We compute $\text{Gram}(f)[e^{i\theta}, -e^{i\theta}]$:

$$\text{Gram}(f)[e^{i\theta}, -e^{i\theta}] = \begin{bmatrix} 20 & 8i \sin(\theta) \\ -8i \sin(\theta) & 20 \end{bmatrix} \quad (2.5)$$

We have:

$$\det \text{Gram}(f)[e^{i\theta}, -e^{i\theta}] = 400 - 64 \sin^2(\theta) > 0 \quad (2.6)$$

Thus by Sylvester's criterion:

$$\text{Gram}(f)[e^{i\theta}, -e^{i\theta}] \geq 0 \quad \forall \theta \in [0, 2\pi] \quad (2.7)$$

(iii) Now

$$f(z, \bar{z}) + 5(z^2 \bar{z}^2 - 1) = \begin{bmatrix} 1 \\ z \\ z^2 \end{bmatrix}^* \begin{bmatrix} 5 & 2 & 0 \\ 2 & 10 & -2 \\ 0 & -2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ z \\ z^2 \end{bmatrix} \quad (2.8)$$

By Sylvester's criterion, the Hermitian coefficient matrix is positive. Thus:

$$f \in \Sigma_h^2 + (z^2 \bar{z}^2 - 1) \quad (2.9)$$

This chapter is organized as follows:

In section 1 we prove some basic lemmas and computations involving orthogonal polynomials on the circle inspired by the $N = 1$ case.

In section 2 we define a functional \mathcal{F}_N , then discuss basic properties and how to use the functional to represent matrix inner products.

In section 3 we discuss trace parametrization of trigonometric polynomials and the operator-valued Riesz-Fejer lemma.

In section 4 we utilize the developed tools to prove the full characterization of Hermitian sums of squares modulo the Hermitian ideal $I = (z^N \bar{z}^N - 1)$.

2.1 Basic Computations on the Circle

To begin, let $h_j(z) = z^j$ for $j \in \mathbb{Z}$. We have the classical formulae:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} h_j(e^{i\theta}) d\theta = \delta_{j0} \quad (2.10)$$

and

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} h_j(e^{i\theta}) \overline{h_k(e^{i\theta})} d\theta = \delta_{jk} \quad (2.11)$$

where δ_{jk} is the standard Kronecker symbol:

$$\delta_{jk} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

We also note the identities:

$$\overline{h_j(e^{i\theta})} = h_{-j}(e^{i\theta}) \quad (2.12)$$

$$h_j(z)h_k(z) = h_{j+k}(z) \quad (2.13)$$

Given $f \in \mathbb{C}[z, \bar{z}]$, we can integrate around the circle to obtain the trace of the coefficient matrix:

Lemma 19 *Let $f(z, \bar{z}) = \sum_{j,k=0}^m a_{jk} \bar{z}^j z^k \in \mathbb{C}[z, \bar{z}]$. Then:*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}, \overline{e^{i\theta}}) d\theta = \sum_{\ell=0}^m a_{\ell\ell} \quad (2.14)$$

Proof:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}, \overline{e^{i\theta}}) d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{j,k=0}^m a_{jk} e^{-ij\theta} e^{ik\theta} \right) d\theta \quad (2.15)$$

$$= \sum_{j,k=0}^m a_{jk} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\theta(k-j)} d\theta \right) \quad (2.16)$$

$$= \sum_{j,k=0}^m a_{jk} \delta_{jk} \quad (2.17)$$

$$= \sum_{\ell=0}^m a_{\ell\ell} \quad (2.18)$$



Consider a Hermitian polynomial $g(z, \bar{z})$ written in matrix notation:

$$g(z, \bar{z}) = \begin{bmatrix} 1 \\ z \\ \vdots \\ z^m \end{bmatrix}^* \begin{bmatrix} b_{00} & b_{01} & \cdots & b_{0m} \\ b_{10} & b_{11} & \cdots & b_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m0} & b_{m1} & \cdots & b_{mm} \end{bmatrix} \begin{bmatrix} 1 \\ z \\ \vdots \\ z^m \end{bmatrix} \quad (2.19)$$

By reducing the coefficients modulo $(z\bar{z} - 1)$, we see that g is congruent modulo $(z\bar{z} - 1)$ to a unique polynomial of the form:

$$f(z, \bar{z}) = \begin{bmatrix} 1 \\ z \\ \vdots \\ z^m \end{bmatrix}^* \begin{bmatrix} a_0 & a_1 & \cdots & a_m \\ a_{-1} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{-m} & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} 1 \\ z \\ \vdots \\ z^m \end{bmatrix} \quad (2.20)$$

where $a_{-j} = \bar{a}_j$. A Hermitian polynomial of this form is called a *trigonometric polynomial* with data (a_0, \dots, a_m) .

Since $g(z, \bar{z}) = f(z, \bar{z}) + q(z, \bar{z})(z\bar{z} - 1)$ for some Hermitian polynomial $q(z, \bar{z})$, we have the following observations:

- (i) $f(e^{i\theta}, \overline{e^{i\theta}}) \geq 0$ for all $\theta \in [0, 2\pi]$ if and only if $g(e^{i\theta}, \overline{e^{i\theta}}) \geq 0$ for all $\theta \in [0, 2\pi]$
- (ii) $f \in \Sigma_h^2 + (z\bar{z} - 1)$ if and only if $g \in \Sigma_h^2 + (z\bar{z} - 1)$

It is known [9, pg. 17] that the condition

$$f(e^{i\theta}, \overline{e^{i\theta}}) \geq 0 \quad \forall \theta \in [0, 2\pi] \quad (2.21)$$

is equivalent to positivity of the associated Toeplitz matrix

$$\text{Toep}(a_0, \dots, a_m) := \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_m \\ a_{-1} & a_0 & a_1 & \ddots & \vdots \\ a_{-2} & a_{-1} & a_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & a_1 \\ a_{-m} & \cdots & \cdots & a_{-1} & a_0 \end{bmatrix} \quad (2.22)$$

A matrix is *Toeplitz* if the entries are constant along the diagonals. We collect the information in the following lemma:

Lemma 20 [9, pg. 17] *Suppose f is a trigonometric polynomial with data (a_0, \dots, a_m) . Let $w = \begin{bmatrix} w_0 & \cdots & w_m \end{bmatrix}^T \in \mathbb{C}^{m+1}$. Define $w(z) = w_0 + w_1z + \cdots + w_mz^m \in \mathbb{C}[z]$. Then:*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |w(e^{i\theta})|^2 f(e^{i\theta}, \overline{e^{i\theta}}) d\theta = w^* \text{Toep}(a_0, \dots, a_m) w \quad (2.23)$$

In particular, if $f(e^{i\theta}, \overline{e^{i\theta}}) \geq 0$ for all $\theta \in [0, 2\pi]$, then $\text{Toep}(a_0, \dots, a_m) \geq 0$.

Now consider a holomorphic polynomial $h(z) = h_0 + h_1z + \cdots + h_mz^m \in \mathbb{C}[z]$. Then

$$|h(z)|^2 = \begin{bmatrix} 1 \\ z \\ \vdots \\ z^m \end{bmatrix}^* \begin{bmatrix} h_0\bar{h}_0 & h_1\bar{h}_0 & \cdots & h_m\bar{h}_0 \\ h_0\bar{h}_1 & h_1\bar{h}_1 & \cdots & h_m\bar{h}_1 \\ \vdots & \vdots & \ddots & \vdots \\ h_0\bar{h}_m & h_1\bar{h}_m & \cdots & h_m\bar{h}_m \end{bmatrix} \begin{bmatrix} 1 \\ z \\ \vdots \\ z^m \end{bmatrix} \quad (2.24)$$

Suppose further that f is a trigonometric polynomial with data (a_0, \dots, a_m) . Then, after reducing coefficients modulo $(z\bar{z} - 1)$, we see that the following conditions are equivalent:

- (i) $f(z, \bar{z}) \equiv |h(z)|^2 \pmod{(z\bar{z} - 1)}$

$$(ii) \quad a_k = \sum_{j=k}^m h_j \bar{h}_{j-k} \text{ for } k = 0, \dots, m$$

From which we can conclude that the following conditions are equivalent:

$$(i) \quad f \in \Sigma_h^2 + (z\bar{z} - 1)$$

$$(ii) \quad \text{There exist } h_0, \dots, h_m \in \mathbb{C} \text{ such that } a_k = \sum_{j=k}^m h_j \bar{h}_{j-k} \text{ for } k = 0, \dots, m$$

These equations characterize the coefficients of the holomorphic polynomial $h(z)$ (see [9, pg. 22]), and can be solved by the method of spectral factorization.

Our plan for the case $N > 1$ is to develop block analogues of these ideas and invoke the operator-valued Riesz-Fejer lemma.

2.2 Block Trace Parametrization

In this section we discuss a block analog of the trace parametrization technique utilized throughout [6].

Now consider a Hermitian polynomial $g(z, \bar{z})$. We may write g in block matrix form:

$$g(z, \bar{z}) = \begin{bmatrix} \psi(z) \\ z^N \psi(z) \\ \vdots \\ z^{Nm} \psi(z) \end{bmatrix}^* \begin{bmatrix} B_{00} & B_{01} & \cdots & B_{0m} \\ B_{10} & B_{11} & \cdots & B_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ B_{m0} & B_{m1} & \cdots & B_{mm} \end{bmatrix} \begin{bmatrix} \psi(z) \\ z^N \psi(z) \\ \vdots \\ z^{Nm} \psi(z) \end{bmatrix} \quad (2.25)$$

where $B_{jk} \in \mathbb{C}^{N \times N}$ and $B_{kj} = B_{jk}^*$. By reducing the coefficients modulo $(z^N \bar{z}^N - 1)$ we

see that g is congruent modulo $(z^N \bar{z}^N - 1)$ to a unique polynomial of the form

$$f(z, \bar{z}) = \begin{bmatrix} \psi(z) \\ z^N \psi(z) \\ \vdots \\ z^{Nm} \psi(z) \end{bmatrix}^* \begin{bmatrix} A_0 & A_1 & \cdots & A_m \\ A_{-1} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{-m} & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \psi(z) \\ z^N \psi(z) \\ \vdots \\ z^{Nm} \psi(z) \end{bmatrix} \quad (2.26)$$

where $A_j \in \mathbb{C}^{N \times N}$, $A_{-j} = A_j^*$ and $\psi(z) = \begin{bmatrix} 1 & z & \cdots & z^{N-1} \end{bmatrix}^T$. If a Hermitian polynomial f has the above form, then we say f is *trigonometric modulo $(z^N \bar{z}^N - 1)$ of degree m with data (A_0, \dots, A_m)* .

Since

$$g(z, \bar{z}) = f(z, \bar{z}) + q(z, \bar{z})(z^N \bar{z}^N - 1) \quad (2.27)$$

for some Hermitian polynomial q , we see that

$$\text{Gram}(f)[\xi, \omega\xi, \dots, \omega^{N-1}\xi] = \text{Gram}(g)[\xi, \omega\xi, \dots, \omega^{N-1}\xi] \quad (2.28)$$

Hence the Gram matrices are simultaneously positive.

Furthermore, we have that $f \in \Sigma_h^2 + (z^N \bar{z}^N - 1)$ if and only if $g \in \Sigma_h^2 + (z^N \bar{z}^N - 1)$.

If f is trigonometric modulo $(z^N \bar{z}^N - 1)$ with data (A_0, \dots, A_m) , then our goal is to show that the condition

$$\text{Gram}(f)[\xi, \omega\xi, \dots, \omega^{N-1}\xi] \geq 0 \quad \forall \xi \in \mathbb{T} \quad (2.29)$$

implies positivity of the associated block Toeplitz matrix

$$\text{Toep}(A_0, \dots, A_m) := \begin{bmatrix} A_0 & A_1 & A_2 & \cdots & A_m \\ A_{-1} & A_0 & A_1 & \ddots & \vdots \\ A_{-2} & A_{-1} & A_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & A_1 \\ A_{-m} & \cdots & \cdots & A_{-1} & A_0 \end{bmatrix} \quad (2.30)$$

which allows us to apply the operator-valued RF theorem to obtain $f \in \Sigma_h^2 + (z^N \bar{z}^N - 1)$.

The idea of the trace parametrization depends on the following observation: Let T_k denote the elementary block Toeplitz matrix with I on the k -th diagonal and 0 elsewhere, where the main diagonal is counted as $k = 0$ and positive diagonals count to the right. Let $\text{trace}[Q]$ denote the sum of the diagonal blocks of $Q = [Q_{jk}]_{j,k=0}^m$. Then

$$\text{trace}[T_{-k}Q] = \sum_{j=k}^m Q_{j-k,j} \quad (2.31)$$

is the sum of the k -th diagonal of Q .

Suppose $h(z)$ is a holomorphic polynomial of degree $N(m+1)$. We can write $|h(z)|^2$ in block matrix form:

$$|h(z)|^2 = \begin{bmatrix} \psi(z) \\ z^N \psi(z) \\ \vdots \\ z^{Nm} \psi(z) \end{bmatrix}^* \begin{bmatrix} Q_{00} & Q_{01} & \cdots & Q_{0m} \\ Q_{10} & Q_{11} & \cdots & Q_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ Q_{m0} & Q_{m1} & \cdots & Q_{mm} \end{bmatrix} \begin{bmatrix} \psi(z) \\ z^N \psi(z) \\ \vdots \\ z^{Nm} \psi(z) \end{bmatrix} \quad (2.32)$$

where $Q = [Q_{jk}]_{j,k=0}^m$ is a positive Hermitian block matrix with $Q_{jk} \in \mathbb{C}^{N \times N}$ and $Q_{kj} = Q_{jk}^*$.

Suppose further that f is trigonometric modulo $(z^N \bar{z}^N - 1)$ with data (A_0, \dots, A_m) . Then, after by considering the reduction of coefficients modulo $(z^N \bar{z}^N - 1)$, we see that the following conditions are equivalent:

- (i) $f(z, \bar{z}) \equiv |h(z)|^2 \pmod{(z^N \bar{z}^N - 1)}$
- (ii) $A_k = \text{trace}[T_{-k}Q]$

From which we conclude that the following conditions are equivalent:

- (i) $f \in \Sigma_h^2 + (z^N \bar{z}^N - 1)$
- (ii) There exists a positive block matrix $Q = [Q_{jk}]_{j,k=0}^m$ such that $A_k = \text{trace}[T_{-k}Q]$, $k = 0, \dots, m$.

2.3 Operator-Valued RF Theorem

We recall the operator-valued Riesz-Fejer theorem:

Theorem 21 [5, Theorem 2.1] *Let $A(z) = \sum_{k=-m}^m A_k z^k$ be a Laurent polynomial with matrix coefficients $A_k \in \mathbb{C}^{N \times N}$. The following conditions are equivalent:*

- (i) $A(\xi) \geq 0$ for all $\xi \in \mathbb{T}$
- (ii) $\text{Toep}(A_0, \dots, A_m) \geq 0$
- (iii) *There exists $P(z) = P_0 + P_1 z + \dots + P_m z^M$ with matrix coefficients $P_k \in \mathbb{C}^{N \times N}$ such that $A(z) = P(z)^* P(z)$*

Let $Q_{jk} = P_j^* P_k$ and let $Q = [Q_{jk}]_{j,k=0}^m$. If $A(z) = P(z)^* P(z)$, then we can equate coefficients to get $A_k = \text{Trace}[T_{-k}Q]$. Thus we get:

Corollary 22 *Let $A_0, \dots, A_m \in \mathbb{C}^{N \times N}$. The following conditions are equivalent:*

(i) $\text{Toep}(A_0, \dots, A_m) \geq 0$

(ii) There exists a positive block matrix $Q = [Q_{jk}]_{j,k=0}^m$ such that $A_k = \text{trace}[T_{-k}Q]$.

With these considerations in mind, starting with f trigonometric modulo $(z^N \bar{z}^N - 1)$ with data (A_0, \dots, A_m) , we will assume

$$\text{Gram}(f)[\xi, \omega\xi, \dots, \omega^{N-1}\xi] \geq 0 \quad \forall \xi \in \mathbb{T} \quad (2.33)$$

and show that this implies

$$\text{Toep}(A_0, \dots, A_m) \geq 0 \quad (2.34)$$

from which we can obtain $f \in \Sigma_h^2 + (z^N \bar{z}^N - 1)$.

2.4 The Functional \mathcal{F}_N

In this section we define a functional $\mathcal{F}_N : \mathbb{C}[z, \bar{z}] \rightarrow \mathbb{C}$ and use it to represent matrix products. The goal is to transfer the matrix positivity conditions into block Toeplitz positivity conditions.

2.4.1 Definition and Basic Properties

Definition 23 For $f(z, \bar{z}) \in \mathbb{C}[z, \bar{z}]$, define:

$$\mathcal{F}_N(f) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\frac{1}{N^2} \sum_{j,k=0}^{N-1} f(\omega^j e^{i\theta}, \overline{\omega^k e^{i\theta}}) \right] d\theta \quad (2.35)$$

Observe that the integrand is the average of the entries of $\text{Gram}(f)[e^{i\theta}, \dots, \omega^{N-1}e^{i\theta}]$.

We require the following computation for the next proposition.

Lemma 24

$$\frac{1}{N^2} \sum_{j,k=0}^{N-1} \omega^{\ell(j-k)} = \begin{cases} 1 & \text{if } \ell \mid N \\ 0 & \text{if } \ell \nmid N \end{cases} \quad (2.36)$$

Proof:

Define the symbol:

$$\mu(\ell) := \begin{cases} N & \text{if } N \mid \ell \\ 0 & \text{if } N \nmid \ell \end{cases} \quad (2.37)$$

Then

$$\sum_{j=0}^{N-1} \omega^{\ell j} = 1 + \omega^\ell + \omega^{2\ell} + \dots + \omega^{\ell(N-1)} \quad (2.38)$$

$$= \mu(\ell) \quad (2.39)$$

which gives

$$\sum_{j,k=0}^{N-1} \omega^{\ell(j-k)} = \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \omega^{\ell j} \omega^{-\ell k} \quad (2.40)$$

$$= \sum_{j=0}^{N-1} \omega^{\ell j} \left(\sum_{k=0}^{N-1} \omega^{-\ell k} \right) \quad (2.41)$$

$$= \sum_{j=0}^{N-1} \omega^{\ell j} \mu(\ell) \quad (2.42)$$

$$= \mu(\ell)^2 \quad (2.43)$$

■

The key idea is that $\mathcal{F}_N(f)$ computes the sum of the diagonal entries $a_{\ell\ell}$ of the coefficient matrix of f such that ℓ is a multiple of N .

Proposition 25 Let $f(z, \bar{z}) = \sum a_{jk} \bar{z}^j z^k \in \mathbb{C}[z, \bar{z}]$. Then:

$$\mathcal{F}_N(f) = a_{0,0} + a_{N,N} + a_{2N,2N} + \cdots = \sum_{\substack{\ell=0 \\ N|\ell}}^m a_{\ell\ell} \quad (2.44)$$

Proof:

$$\mathcal{F}_N(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\frac{1}{N^2} \sum_{j,k=0}^{N-1} f(\omega^j e^{i\theta}, \overline{\omega^k e^{i\theta}}) \right] d\theta \quad (2.45)$$

$$= \frac{1}{N^2} \sum_{j,k=0}^{N-1} \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} f(\omega^j e^{i\theta}, \overline{\omega^k e^{i\theta}}) d\theta \right] \quad (2.46)$$

$$= \frac{1}{N^2} \sum_{j,k=0}^{N-1} \left[\sum_{\ell=0}^m a_{\ell\ell} \omega^{\ell(j-k)} \right] \quad (2.47)$$

$$= \sum_{\ell=0}^m a_{\ell\ell} \left[\frac{1}{N^2} \sum_{j,k=0}^{N-1} \omega^{\ell(j-k)} \right] \quad (2.48)$$

$$= \sum_{\substack{\ell=0 \\ N|\ell}}^m a_{\ell\ell} \quad (2.49)$$

where we use Lemma 19 and Lemma 24. ■

Corollary 26 \mathcal{F}_N satisfies the following properties:

(i) $\mathcal{F}_N(z^N \bar{z}^N f(z, \bar{z})) = \mathcal{F}_N(f(z, \bar{z}))$ for all $f \in \mathbb{C}_h[z, \bar{z}]$

(ii) \mathcal{F}_N is a \mathbb{C} -linear map $\mathbb{C}[z, \bar{z}] \rightarrow \mathbb{C}$

(iii) \mathcal{F}_N is an \mathbb{R} -linear map $\mathbb{C}_h[z, \bar{z}] \rightarrow \mathbb{R}$

Proof: Follows from Proposition 25. ■

Proposition 27 *Let $f \in \mathbb{C}_h[z, \bar{z}]$. Suppose $\text{Gram}(f)[e^{i\theta}, \omega e^{i\theta}, \dots, \omega^{N-1}e^{i\theta}] \geq 0$ for all $\theta \in [0, 2\pi]$. Then:*

$$\mathcal{F}_N(f(z, \bar{z})) \geq 0 \quad (2.50)$$

Furthermore:

$$\mathcal{F}_N(|h(z)|^2 f(z, \bar{z})) \geq 0 \quad (2.51)$$

for all holomorphic polynomials $h(z) \in \mathbb{C}[z]$.

Proof:

The average of the entries of a positive matrix is positive. Thus:

$$\frac{1}{N^2} \sum_{j,k=0}^{N-1} f(\omega^j e^{i\theta}, \overline{\omega^k e^{i\theta}}) \geq 0 \quad \text{for all } \theta \in [0, 2\pi] \quad (2.52)$$

The average of positive numbers is positive. Thus:

$$\mathcal{F}_N(f) := \int_{-\pi}^{\pi} \left[\frac{1}{N^2} \sum_{j,k=0}^{N-1} f(\omega^j e^{i\theta}, \overline{\omega^k e^{i\theta}}) \right] \frac{d\theta}{2\pi} \geq 0 \quad (2.53)$$

For a Hermitian polynomial $g \in \mathbb{C}_h[z, \bar{z}]$, for ease of notation, let $\text{Gram}(g)$ denote the matrix $\text{Gram}(g)[e^{i\theta}, \omega e^{i\theta}, \dots, \omega^{N-1}e^{i\theta}]$. We know $\text{Gram}(|h(z)|^2) \geq 0$ for all holomorphic polynomials $h(z) \in \mathbb{C}[z]$. Then

$$\text{Gram}(|h(z)|^2 f(z, \bar{z})) = \text{Gram}(|h(z)|^2) \circ \text{Gram}(f(z, \bar{z})) \geq 0 \quad (2.54)$$

by the Schur product theorem (where \circ denotes the Hadamard product).

Then $\mathcal{F}_N(|h(z)|^2 f(z, \bar{z})) \geq 0$ follows from the first part of the proof. ■

2.4.2 Representing Matrix Products with \mathcal{F}_N

Given $v, w \in \mathbb{C}^N$ and $A \in M_N$, our goal is to construct a polynomial f such that $\mathcal{F}_N(f) = v^*Aw$. Let us expand the expression v^*Aw so that we may recognize its appearance later. Write

$$\begin{aligned} v &= \begin{bmatrix} v_0 & \cdots & v_{N-1} \end{bmatrix}^T \\ w &= \begin{bmatrix} w_0 & \cdots & w_{N-1} \end{bmatrix}^T \\ A &= [a_{jk}]_{j,k=0}^{N-1} \end{aligned}$$

Then

$$v^*Aw = \sum_{j,k=0}^{N-1} a_{jk} \bar{v}_j w_k \quad (2.55)$$

The following example demonstrates the construction for the case $N = 2$.

Example 28 Let $v, w \in \mathbb{C}^2$ and $A \in \mathbb{C}^{2 \times 2}$. Write:

$$v = \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} \quad w = \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \quad A = \begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix}$$

Define

$$f(z, \bar{z}) = \sum_{j,k=0}^1 a_{jk} \bar{z}^j z^k$$

$$\tilde{v}(z) = v_0 z^2 + v_1 z$$

$$\tilde{w}(z) = w_0 z^2 + w_1 z$$

Then

$$\overline{\tilde{v}(z)} \tilde{w}(z) f(z, \bar{z}) = a_{00} \bar{v}_0 w_0 z^2 \bar{z}^2 + z a_{01} \bar{v}_0 w_1 z \bar{z}^2 + \bar{z} a_{10} \bar{v}_1 w_0 z^2 \bar{z} + z \bar{z} a_{11} \bar{v}_1 w_1 z \bar{z}$$

Then

$$\mathcal{F}_2(\overline{\tilde{v}(z)}\tilde{w}(z)f(z, \bar{z})) = a_{00}\bar{v}_0w_0 + a_{01}\bar{v}_0w_1 + a_{10}\bar{v}_1w_0 + a_{11}\bar{v}_1w_1 = v^*Aw$$

by Proposition 25.

Written in matrix notation: (consider the coefficient of $z^2\bar{z}^2$ before applying \mathcal{F}_2)

$$\mathcal{F}_2\left(\begin{bmatrix} z^2 \\ z \end{bmatrix}^* \begin{bmatrix} \bar{v}_0 \\ \bar{v}_1 \end{bmatrix} \begin{bmatrix} \bar{w}_0 \\ \bar{w}_1 \end{bmatrix}^* \begin{bmatrix} z^2 \\ z \end{bmatrix} \begin{bmatrix} 1 \\ z \end{bmatrix}^* \begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} \begin{bmatrix} 1 \\ z \end{bmatrix}\right) = \begin{bmatrix} v_0 \\ v_1 \end{bmatrix}^* \begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \quad (2.56)$$

We generalize the previous example to obtain the desired construction:

Proposition 29 *Let:*

$$v = \begin{bmatrix} v_0 & \cdots & v_{N-1} \end{bmatrix}^T \in \mathbb{C}^N$$

$$w = \begin{bmatrix} w_0 & \cdots & w_{N-1} \end{bmatrix}^T \in \mathbb{C}^N$$

$$A = [a_{jk}]_{j,k=0}^{N-1} \in \mathbb{C}^{N \times N}$$

Define:

$$\tilde{v}(z) = \sum_{j=0}^{N-1} v_j z^{N-j} \in \mathbb{C}[z] \quad (2.57)$$

$$\tilde{w}(z) = \sum_{j=0}^{N-1} w_j z^{N-j} \in \mathbb{C}[z] \quad (2.58)$$

$$f(z, \bar{z}) = \sum_{j,k=0}^{N-1} a_{jk} \bar{z}^j z^k \in \mathbb{C}[z, \bar{z}] \quad (2.59)$$

Then:

$$\mathcal{F}_N(\overline{\tilde{v}(z)}\tilde{w}(z)f(z, \bar{z})) = v^*Aw \quad (2.60)$$

Furthermore, for integers $s, t \geq 0$ we have:

$$\mathcal{F}_N(\bar{z}^{Ns} z^{Nt} \overline{\tilde{v}(z)} \tilde{w}(z) f(z, \bar{z})) = \delta_{st} v^* A w \quad (2.61)$$

Proof: We have

$$\overline{\tilde{v}(z)} \tilde{w}(z) = \sum_{j,k=0}^{N-1} \bar{v}_j w_k \bar{z}^{N-j} z^{N-k}$$

and

$$f(z, \bar{z}) = \sum_{c,d=0}^{N-1} a_{cd} \bar{z}^c z^d$$

Thus

$$\overline{\tilde{v}(z)} \tilde{w}(z) f(z, \bar{z}) = \sum_{j,k,c,d=0}^{N-1} a_{cd} \bar{v}_j w_k \bar{z}^{N-j+c} z^{N-k+d}$$

Use the \mathbb{C} -linearity of \mathcal{F}_N :

$$\mathcal{F}_N(\overline{\tilde{v}(z)} \tilde{w}(z) f(z, \bar{z})) = \sum_{j,k,c,d=0}^{N-1} a_{cd} \bar{v}_j w_k \mathcal{F}_N(\bar{z}^{N-j+c} z^{N-k+d})$$

Since $0 \leq j, k, c, d \leq N-1$, applying Proposition we get

$$\mathcal{F}_N(\bar{z}^{N-j+c} z^{N-k+d}) = \delta_{jc} \delta_{kd}$$

Therefore only terms with $j = c$ and $k = d$ will survive \mathcal{F}_N :

$$\mathcal{F}_N(\overline{\tilde{v}(z)} \tilde{w}(z) f(z, \bar{z})) = \sum_{j,k=0}^{N-1} a_{jk} \bar{v}_j w_k = v^* A w$$

In order to prove the second part, for a Hermitian polynomial f , let $\text{Mon}(f) \subset \mathbb{N} \times \mathbb{N}$ denote the monomial support (the set of monomials corresponding to nonzero coefficients of f). Note $\text{Mon}(fg) = \text{Mon}(f) + \text{Mon}(g)$. Say a monomial in $\mathbb{N} \times \mathbb{N}$ is N -divisible if

both components are divisible by N and say the monomial is *diagonal* if both entries are equal. We know that $\mathcal{F}_N(f)$ is the sum of the coefficients corresponding to monomials of f which are both diagonal and N -divisible.

For integers $a, b \in \mathbb{Z}$ let $[a, b] := \{j \in \mathbb{Z} | a \leq j \leq b\}$. Then:

$$\text{Mon}(\overline{\tilde{v}(z)}\tilde{w}(z)) = [1, N] \times [1, N] \quad (2.62)$$

$$\text{Mon}(f(z, \bar{z})) = [0, N - 1] \times [0, N - 1] \quad (2.63)$$

Hence

$$\text{Mon}(\overline{\tilde{v}(z)}\tilde{w}(z)f(z, \bar{z})) = [1, 2N - 1] \times [1, 2N - 1] \quad (2.64)$$

whose only diagonal N -divisible element is (N, N) .

We have

$$\text{Mon}(\overline{\tilde{v}(z)}\tilde{w}(z)f(z, \bar{z})\bar{z}^{Ns}z^{Nt}) = [1 + Ns, 2N - 1 + Ns] \times [1 + Nt, 2N - 1 + Nt] \quad (2.65)$$

For convenience, denote $Q(s, t) = Q(s, t)(z, \bar{z}) = \overline{\tilde{v}(z)}\tilde{w}(z)f(z, \bar{z})\bar{z}^{Ns}z^{Nt}$.

As we vary $(s, t) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ we obtain a partition of $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ by squares $\text{Mon}(Q(s, t))$. Only the bottom right monomial of each tile is N -divisible. If $s = t$, then the bottom right element of $\text{Mon}(Q(s, t))$ is both N -divisible and diagonal. If $s \neq t$, then no elements of $\text{Mon}(Q(s, t))$ are both N -divisible and diagonal. From these observations the result follows. ■

2.5 Proof of Main Results

We are now in position to utilize all the tools developed so far and show that the matrix positivity conditions imply positivity of the associated block Toeplitz matrix. We then prove the theorem characterizing Hermitian sums of squares modulo $(z^N \bar{z}^N - 1)$.

Proposition 30 *Let $\omega = e^{2\pi i/N}$.*

Let $f \in \mathbb{C}_h[z, \bar{z}]$ be trigonometric mod $(z^N \bar{z}^N - 1)$ with data (A_0, \dots, A_m) .

Suppose $\text{Gram}(f)[e^{i\theta}, \omega e^{i\theta}, \dots, \omega^{N-1} e^{i\theta}] \geq 0$ for all $\theta \in [0, 2\pi]$.

Then $\text{Toep}(A_0, \dots, A_m) \geq 0$.

Proof:

Let $v^{(0)}, \dots, v^{(m)} \in \mathbb{C}^N$ be arbitrary.

Write $v^{(j)} = \begin{bmatrix} v_0^{(j)} & \dots & v_{N-1}^{(j)} \end{bmatrix}^T$ with $v_k^{(j)} \in \mathbb{C}$.

Let $v = \begin{bmatrix} v^{(0)} & \dots & v^{(m)} \end{bmatrix}^T \in (\mathbb{C}^N)^{m+1}$.

It suffices to show $v^* \text{Toep}(A_0, \dots, A_m) v \geq 0$.

Observe that

$$v^* \text{Toep}(A_0, \dots, A_m) v = \sum_{j,k=0}^m v^{(j)*} A_{k-j} v^{(k)}$$

For $j = 0, \dots, m$ define $\tilde{v}^{(j)}(z) = \sum_{k=0}^{N-1} v_k^{(j)} z^{N-k} \in \mathbb{C}[z]$.

Define $v(z) = \sum_{j=0}^m z^{Nj} \tilde{v}^{(j)}(z) \in \mathbb{C}[z]$.

Let $\psi(z) = \phi_N(z) = \begin{bmatrix} 1 & z & \dots & z^{N-1} \end{bmatrix}^T$.

Claim: $\mathcal{F}_N(|v(z)|^2 f(z, \bar{z})) = v^* \text{Toep}(A_0, \dots, A_m) v \geq 0$.

Introduce double-index notation for matrix coefficients of f in order to simplify com-

putations:

$$f(z, \bar{z}) = \begin{bmatrix} \psi(z) \\ z^N \psi(z) \\ \vdots \\ z^{Nm} \psi(z) \end{bmatrix}^* \begin{bmatrix} A_{00} & A_{01} & \cdots & A_{0m} \\ A_{10} & A_{11} & \cdots & A_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m0} & A_{m1} & \cdots & A_{mm} \end{bmatrix} \begin{bmatrix} \psi(z) \\ z^N \psi(z) \\ \vdots \\ z^{Nm} \psi(z) \end{bmatrix}$$

where $A_{jk} = 0$ if $\min(k, j) \neq 0$ and $A_{jk} = A_{k-j}$ otherwise.

Then:

$$f(z, \bar{z}) = \sum_{j,k=0}^m \bar{z}^j z^{kN} \psi(z)^* A_{jk} \psi(z)$$

We have:

$$|v(z)|^2 = \sum_{j,k=0}^m \bar{z}^j z^{Nk} \overline{\tilde{v}^{(j)}(z)} \tilde{v}^{(k)}(z)$$

Hence:

$$|v(z)|^2 f(z, \bar{z}) = \sum_{j,k,c,d=0}^m \bar{z}^{N(c+j)} z^{N(d+k)} \overline{\tilde{v}^{(j)}(z)} \tilde{v}^{(k)}(z) \psi(z)^* A_{cd} \psi(z)$$

Then using Proposition 25 and passing between the double and single index coefficients

for f :

$$\mathcal{F}_N(|v(z)|^2 f(z, \bar{z})) = \sum_{j,k,c,d=0}^m \mathcal{F}_N(\bar{z}^{N(c+j)} z^{N(d+k)} \overline{\tilde{v}^{(j)}(z)} \tilde{v}^{(k)}(z) \psi(z)^* A_{cd} \psi(z)) \quad (2.66)$$

$$= \sum_{j,k,c,d=0}^m \delta_{c+j,d+k} v^{(j)*} A_{cd} v^{(k)} \quad (2.67)$$

$$= \sum_{\substack{j,k,c,d=0 \\ c+j=d+k}}^m v^{(j)*} A_{cd} v^{(k)} \quad (2.68)$$

$$= \sum_{\substack{j,k,c,d=0 \\ j-k=d-c}}^m v^{(j)*} A_{d-c} v^{(k)} \quad (2.69)$$

$$= \sum_{j,k=0}^m v^{(j)*} A_{j-k} v^{(k)} \quad (2.70)$$

$$= v^* \text{Toep}(A_0, \dots, A_m)^T v \quad (2.71)$$

By Proposition 27 we have

$$v^* \text{Toep}(A_0, \dots, A_m)^T v \geq 0 \quad \forall v \in (\mathbb{C}^N)^{m+1}$$

Therefore $\text{Toep}(A_0, \dots, A_m)^T \geq 0$, and hence $\text{Toep}(A_0, \dots, A_m) \geq 0$. ■

Theorem 31 *Let $\omega = e^{2\pi i/N}$.*

Let $f \in \mathbb{C}_h[z, \bar{z}]$ be trigonometric mod $(z^N \bar{z}^N - 1)$ with data (A_0, \dots, A_m) .

The following conditions are equivalent:

(i) $f \in \Sigma_h^2 + (z^N \bar{z}^N - 1)$

(ii) $\text{Gram}(f)[e^{i\theta}, \omega e^{i\theta}, \dots, \omega^{N-1} e^{i\theta}] \geq 0$ for all $\theta \in [0, 2\pi]$

(iii) $\text{Toep}(A_0, \dots, A_m) \geq 0$

Proof:

(i) \implies (ii)) By Lemma 17, the idea of matrix positivity conditions.

(ii) \implies (iii)) By Proposition 30.

(iii) \implies (i)) By Corollary 22 and the conclusion of Section 2.2.

■

Chapter 3

Conclusion

The original contribution of this thesis is to show that the matrix positivity conditions for the Hermitian ideal $I = (z^N \bar{z}^N - 1)$. Some reasons why this is interesting:

- (i) The characterization of $\Sigma_h^2 + I$ was not covered by the application of the Archimedean positivstellensatz. Pointwise positivity is never sufficient and $\Sigma_h^2 + I$ is a non-Archimedean semiring. The ideas of real algebra are not enough.
- (ii) The matrix positivity conditions are parametrized by a continuum. This is in contrast to the following example: let $p_1, p_2 \in \mathbb{C}$ and let $I = ((z - p_1)(z - p_2), \overline{(z - p_1)(z - p_2)})$. Then $f \in \Sigma_h^2 + I$ if and only if $\text{Gram}(f)[p_1, p_2] \geq 0$. This is the basic idea of the Kolmogorov factorization of a positive kernel. Only the one positivity condition is required, and $\mathcal{Z}(I) = \{p_1, p_2\} \subset \mathbb{C}$.

In the case of $(z^N \bar{z}^N - 1)$, we have an N -th order matrix positivity condition for each point on the unit circle.

The question remains: are the matrix positivity conditions sufficient in general?

Let f be a Hermitian polynomial and I a Hermitian ideal.

Suppose

$$[f(p_j, \bar{p}_k)]_{j,k=1}^{\ell} \geq 0$$

for all collections of points $p_1, \dots, p_{\ell} \in \mathbb{C}^n$ such that

$$g(p_j, \bar{p}_k) = 0 \quad \forall j, k = 1, \dots, \ell \quad \forall g \in I$$

Then do we have $f \in \Sigma_h^2 + I$?

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