Submitted to Mathematische Zeitschrift

LBL-3883
Preprint

ON THE HEIGHT OF A CAPILLARY SURFACE

Paul Concus and Robert Finn

June 1975

Prepared for the U. S. Energy Research and
Development Administration under Contract W-7405-ENG-48

ON THE HEIGHT OF A CAPILLARY SURFACE

by

Paul Concus and Robert Finn

In the "capillary problem" one seeks to determine a surface $u(x)$ satisfying the equation

$$\text{div } Tu = \kappa u, \qquad Tu = \frac{1}{W}\nabla u, \qquad W = \sqrt{1+|\nabla u|^2}, \tag{1}$$

in a domain Ω, and the condition

$$\nu \cdot Tu = \cos\gamma \tag{2}$$

on $\Sigma = \partial\Omega$. Here γ is prescribed, $0 \le \gamma \le \pi$, $\kappa = \frac{\rho g}{\sigma}$ is the "capillarity constant", and ν is the exterior directed normal on Σ. The symbols ρ, g, σ denote density, gravitational acceleration and surface tension (all assumed constant). For background information see, e.g., [1,2]. In this note we assume the "contact angle" γ is constant[1], and that $\kappa > 0$, corresponding to the familiar physical situation of liquid rising in a vertical capillary tube of homogeneous composition.

M. Miranda has posed to us, informally, the question: let $\Omega_1 \subset \Omega$, and let u_1, u be solutions of (1,2) in, respectively, Ω_1 and Ω. Is $u_1 > u$ in Ω_1?

---

[1]We may assume $0 \le \gamma \le \pi/2$, as the complementary case reduces to this one by a formal transformation.

2. The question is physically suggestive, in the sense one expects the rise height of liquid in a capillary tube of small section to exceed that in a tube of large section.

If $\gamma = 0$ the question has an affirmative answer, under very general hypotheses, both on the domains Ω, Ω_1 and on assumption of boundary data; this follows from the particular forms of the maximum principle given by Gerhardt [3] and by these authors [4].

A heuristic reasoning, using the methods of [4], will convince the reader that the answer is again positive in the case Σ_1 and Σ are spheres (not necessarily concentric).

If $\gamma = \pi/2$ then the unique solution of (1,2) in any Ω is $u(x) \equiv 0$; thus an affirmative answer (in an extended sense) is obtained in this limiting case.

3. In the present note we show that in general the answer is negative, even for convex domains with smooth boundaries.

Consider the two dimensional region Ω indicated in Figure C. In §7 of [5] is proved the existence of a unique solution $u(x)$ of (1,2) in Ω. We note the data γ need not be prescribed at the corner, where the boundary angle is not defined. At all other boundary points, the data are achieved strictly and smoothly [6,7,8], at least if $\gamma \neq 0$.

We choose γ so that $\alpha + \gamma < \frac{\pi}{2}$. Introducing polar coordinates centered at V, we set

$$(3) \qquad v(r,\theta) = \frac{\cos\theta - \sqrt{k^2 - \sin^2\theta}}{\kappa\, kr}, \qquad k = \frac{\sin\alpha}{\cos\gamma}.$$

There hold [4],

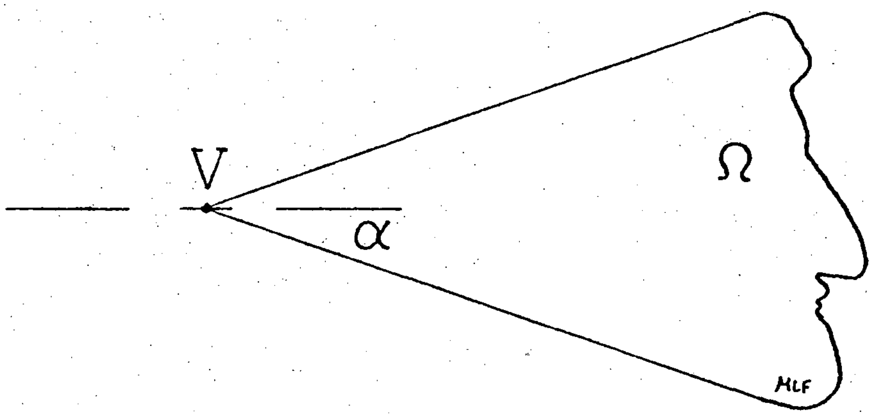

FIGURE C

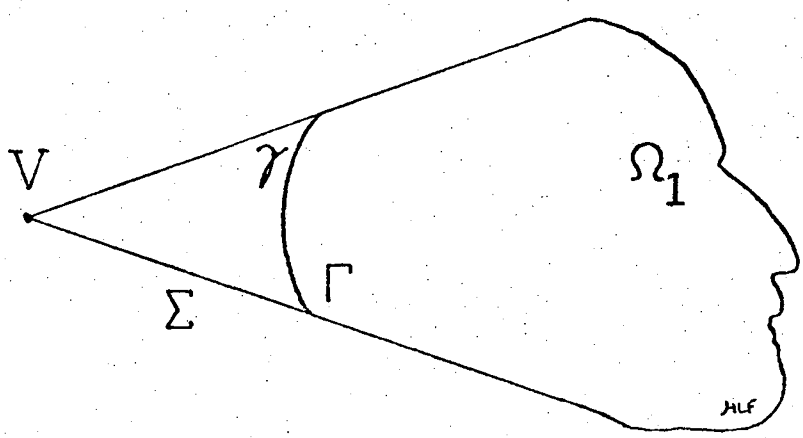

FIGURE F

(4) $$\operatorname{div} Tv = \kappa v + O(r^3)$$

in Ω, and

(5) $$\nu \cdot Tv = \cos\gamma + O(r^4)$$

on the two straight segments. Further, there exists a constant C such that

(6) $$|u - v| < C$$

uniformly in a neighborhood of V.

Let Ω_1 be as indicated in Figure F. We note the circular arc Γ is a level curve for v; since $|\nabla v| > |v_r| = \frac{1}{r} v$ becomes infinite as $r \to 0$, there follows

(7) $$\nu \cdot Tv]_\Gamma \to 1$$

as $r \to 0$. This relation is uniform on Γ.

4. We apply the divergence theorem to v in $\Omega \setminus \Omega_1$, obtaining

(8) $$|\Sigma| \cos\gamma - \int_\Gamma \nu \cdot Tv \, d\sigma = \kappa \int_{\Omega\setminus\Omega_1} v\, dx + O(|\Sigma|^5)$$

by (4,5). No contribution appears from the singularity at V, since $|Tf| < 1$ for any function f.

Repeating the procedure with v replaced by u and taking the difference of the two expressions yields

(9) $$\left|\int_\Gamma \nu\cdot Tu\, d\sigma - \int_\Gamma \nu\cdot Tv\, d\sigma\right| \le \kappa \int_{\Omega\setminus\Omega_1} |u-v|dx + O(|\Sigma|^5) .$$

Hence, by (6), if $\gamma \neq 0$,

(10) $$\int_\Gamma \nu\cdot Tu\, d\sigma > |\Gamma|\cos\gamma$$

if Γ is chosen sufficiently close to V.

5. Let $u_1(x)$ be the solution of (1,2) in Ω_1. The divergence theorem now yields

(11) $$\int_\Gamma \nu\cdot Tu\, d\sigma - |\Gamma|\cos\gamma = \kappa\int_{\Omega_1}(u-u_1)dx$$

and we conclude, by (10), that there exists an open subset of Ω_1 in which $u > u_1$. This completes the proof of our assertion, for the particular domains considered.

6. An objection may possibly be raised, that the boundaries appearing in the above construction have singular points. However, an examination of our procedure, and of the method of construction of the function $u(x)$ in [5], shows that all corners--including the one at V--can be smoothed without affecting the qualitative result. The construction can also be effected so that $\Omega_1 \subset\subset \Omega$, and it can be repeated without essential change in any number of dimensions.

References

1. Bakker, G.: Kapillarität und Oberflächenspannung, Wien-Harms Handbuch der Experimentalphysik, vol. 6 (1928).

2. Finn, R.: Capillarity phenomena, Uspehi Mat. Nauk 29 (1974), 131-152.

3. Gerhardt, C.: Existence and regularity of capillary surfaces, Boll. Un. Mat. Ital., to appear.

4. Concus, P., and R. Finn: On capillary free surfaces in a gravitational field, Acta Math. 132 (1974), 207-224.

5. Finn, R., and C. Gerhardt: The internal sphere condition and the capillary problem, to appear.

6. Uraltseva, N.N.: Solution of the capillary problem, Vestnik Univ. Leningrad 19 (1973), 54-64.

7. Gerhardt, C.: Global regularity of the solutions to the capillary problem, to appear.

8. Simon, L., and J. Spruck: Existence and regularity of a capillary surface with prescribed contact angle, to appear.

Lawrence Berkeley Laboratory
Berkeley, CA 94720

Stanford University
Stanford, CA 94305

This work was supported in part by the Energy Research and Development Administration, by the National Aeronautics and Space Administration, and by the National Science Foundation.

TECHNICAL INFORMATION DIVISION
LAWRENCE BERKELEY LABORATORY
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA 94720