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Author
Ream, Robert

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by

Robert Orion Ream

Committee in Charge:

Professor John Douglas Moore, Chair

Professor Xianzhe Dai

Professor Guofang Wei

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The Dissertation of
Robert Orion Ream is approved:

Professor Xianzhe Dai

Professor Guofang Wei

Professor John Douglas Moore, Committee Chairperson

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by

Robert Orion Ream

For Jodie, Kalila, and Autumn.

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Curriculum Vitæ

Robert Orion Ream

Education

- Ph.D. Mathematics, University of California, Santa Barbara, December 2014 (expected).
- M.S. Mathematics, University State University, December 2008.
- B.S. Mathematics, University State University, May 2006.
- B.S. Physics, University State University, December 2005.

Employment

- Mathematics Lecturer, California State University, Channel Islands, 2014-Present.
- Graduate Student Teaching Associate, University of California, Santa Barbara, Summer 2010-2014.
- Graduate Student Teaching Assistant, University of California, Santa Barbara, 2008-2014.
- Graduate Student Teaching Associate, Utah State University, 2006-2008.
- Intern Physicist, ATK Launch Systems, 2005-2006.

Awards

- Raymond L. Wilder Award, University of California, Santa Barbara, Spring 2009.

Abstract

Index Estimates and Existence of Minimal Surfaces in Manifolds with Controlled Curvature

Robert Orion Ream

When the compact manifold M has a Riemannian metric satisfying a suitable curvature condition, we show that it has many minimal two-spheres of index between $n-2$ and $2n-5$, using Morse theory for the α-energy of Sacks and Uhlenbeck. The difficulty is controlling bad behavior of a sequence of α-energy critical points as α approaches one. The two bad behaviors which must be controlled are convergence toward a bubble tree and convergence to a branched cover of a minimal sphere of lower energy. We prevent these difficulties by making estimates on the index of bubble trees and branched covers.

These estimates require a new curvature condition, δ-controlled half-isotropic curvature. In order to better understand this new condition, we study the relationship between metrics with δ-controlled half-isotropic curvature and metrics satisfying the better studied conditions of pinched sectional curvature and pinched flag curvature. We are able to get a basically complete picture of the relationship between these three conditions.

If M is simply connected, then δ-controlled half-isotropic curvature implies that M is diffeomorphic to S^n. In this case the constant curvature metric on S^n can be used to compute the low degree $O(3)$-equivariant cohomology of $Map(S^2, S^n)$.

This then implies the existence of α-energy critical points of low index for generic metrics with δ-controlled half-isotropic curvature, when α is sufficiently close to one. Using index estimates to control the bad behavior of these critical points as α approaches 1 allows us to prove the existence of many minimal S^2 of low index.

Contents

Chapter 1

Introduction

The study of minimal surfaces has been of great interest to mathematicians since the 18^{th} century. Early mathematicians studied minimal surfaces in $\mathbb{R}^3$, and while this case is still of great interest, the theory has been generalized in many important ways. Of particular importance, has been the study of minimal hypersurfaces in a Riemannian manifold. As in the classical theory, the codimension in this case is 1. Geometric measure theory has played an important role in the study of this problem.

In this thesis we will study the case of dimension 2 minimal surfaces in a Riemannian manifold. Of course, when the dimension of the ambient manifold is 3 this coincides with the previously mentioned theory. Of particular interest will be minimal S^2 in a manifold satisfying some curvature condition. The approach that we use is to study the mapping space $\mathcal{M} = Map(S^2, M)$ via the α-energy functional of Sacks and Uhlenbeck.

$$E_\alpha(f) = \frac{1}{2}\int_{S^2}(1+|df|^2)^\alpha dA - \frac{1}{2}.$$

As $\alpha \to 1$ this approaches the usual energy functional, and critical points of E_α will approach a harmonic map except at a finite number of points. This approach was first used in [16] to prove the following existence theorems.

Theorem 1.0.1. *If M is compact, and $\pi_2(M) = 0$, then there exists a minimizing harmonic map in every homotopy class of maps in $C^0(\Sigma, M)$.*

Theorem 1.0.2. *Every conjugacy class of homomorphisms from $\pi_1(\Sigma)$ into $\pi_1(M)$ is induced by a minimizing harmonic map from Σ into M.*

Theorem 1.0.3. *If M is a compact Riemannian manifold. Then a generating set for $\pi_2(M)$ as a $\pi_1(M)$-module can be represented by minimal two-spheres, possibly with branch points.*

This approach was also used by Micallef and Moore in [9] to prove the following theorem.

Theorem 1.0.4. *Let M be a manifold of dimension at least 4 with positive isotropic curvature. Then $\pi_i(M) = 0$ for all $2 \leq i \leq \frac{n}{2}$.*

This theorem immediately implies that a simply connected manifold of dimension at least 4 with positive isotropic curvature must be homeomorphic to a sphere. In the paper [10] Moore again uses this approach to establish a Morse theory for the space of maps from S^2 to M with energy less than 8π when M has .733 pinched sectional curvature.

1.1 Results

The existence results presented in this thesis rely on index estimates for minimal surfaces in M. These estimates rely on a new condition on the curvature, δ-controlled half-isotropic curvature. In dimension four or higher, this new condition is equivalent the condition that for any 4-plane with orthonormal basis $\{e_i\}_{i=1}^4$ and any (a, b) in the unit circle,

$$\frac{1}{4}(a^2 R_{1331} + b^2 R_{1441} + a^2 R_{2332} + b^2 R_{2442} - 2abR_{1234}) > \delta R_{1221} > 0. \tag{1.1}$$

If M is a 3 manifold with Ricci tensor, Ric and scalar curvature S, then equation 1.1 only makes sense when $b = 0$. This results in the following condition which is equivalent to δ-controlled half-isotropic curvature in a 3-manifold,

$$Ric > \frac{\delta}{2\delta + 1} S > 0. \tag{1.2}$$

The index estimates proved in chapter 3 are as follows.

Theorem 1.1.1. *Suppose that M is a Riemannian manifold of positive half-isotropic curvature, and $n = dim(M)$. Then the Morse index of any minimal S^2 in M must be at least $n - 2$.*

Theorem 1.1.2. *Suppose that M is a Riemannian manifold with $\frac{1}{d}$-controlled half-isotropic curvature, for some integer, $d > 1$. If $n = dim(M)$, then the Morse* *index of any minimal S^2 which is a d-fold branched cover of another minimal S^2 in M must be at least $2(n-2)$.*

In chapter 4, the relationship of δ-controlled half-isotropic curvature to other conditions on the curvature is studied, and the following propositions are proved.

Proposition 1.1.3. *A manifold with pointwise λ-pinched sectional curvature has $\frac{4\lambda-1}{3}$-controlled half-isotropic curvature.*

Proposition 1.1.4. *If M has δ-controlled half-isotropic curvature, for some $\delta > \frac{1}{2}$, then M has $(2\delta-1)$ pinched flag curvature.*

Proposition 1.1.5. *If M has strictly λ-pinched flag curvature and $\delta > 0$, where*

$$\delta = \frac{16\lambda^2 + 13\lambda - 5}{12(\lambda+1)},$$

then M has δ-controlled half-isotropic curvature.

Since λ-pinched flag curvature implies λ^2 pinched sectional curvature, modulo possible improvements to the estimates, this gives a full picture of the relationship between δ-controlled half-isotropic curvature, pinched flag curvature, and pinched sectional curvature.

Finally in chapter 5, the estimates from chapter 2 are used to prevent a sequence of minimal surfaces of low index from converging to a bubble tree or a branched cover, establishing a Morse theory for these low-index surfaces. This is then used to calculate the low degree $O(3)$-equivariant cohomology of

$Map(S^2, S^n)$. For the round metric on S^n, the space of harmonic maps from $S^2 \to S^n$ is known and can be used to prove the following theorem for $\mathcal{M} = Map(S^2, S^n)$ and $\mathcal{M}_0$ the constant maps.

Theorem 1.1.6. *For $0 \leq k < n-2$,*

$$H^k_{O(3)}(\mathcal{M}, \mathcal{M}_0; \mathbb{Z}_2) = 0,$$

and for k satisfying $n-2 \leq k \leq 2n-5$ there are isomorphisms

$$H^k_{O(3)}(\mathcal{M}, \mathcal{M}_0; \mathbb{Z}_2) \cong H^{k-n+2}(Gr_3(\mathbb{R}^{n+1}); \mathbb{Z}_2).$$

Moreover the $H^*(BO(3); \mathbb{Z}_2)$ action on $H^*_{O(3)}(\mathcal{M}, \mathcal{M}_0; \mathbb{Z}_2)$ is trivial in degree less than $2n-4$.

The $O(3)$-equivariant Morse cohomology is then used to show the existence of minimal spheres in low degree when M has a generic metric with $\frac{1}{2}$-controlled half-isotropic curvature.

Theorem 1.1.7. *If S^n has a metric with $\frac{1}{2}$-controlled half-isotropic curvature, and if the prime minimal surfaces of M lie on nondegenerate critical submanifolds, each an orbit for the action of*

$$PSL(2, \mathbb{C}) \cup R \cdot PSL(2, \mathbb{C}),$$

then for each λ satisfying $n-2 \leq \lambda \leq 2n-5$, there are at least $p_3(\lambda - n + 2)$ minimal S^2 with Morse index λ.

Chapter 2

Preliminaries

2.1 Energy

Let $(M, \langle\ ,\ \rangle)$ be a Riemannian manifold, and (Σ, ω) a Riemann surface, then choosing a metric, η, in the conformal class, ω, the energy of a map, $f : \Sigma \to M$ is

$$E(f) = \frac{1}{2}\int_{\Sigma} |df|^2 dA.$$

Where dA is the area form for η, and $|\cdot|$ is the norm induced on $\mathrm{Hom}(T\Sigma, f^*TM)$. This integral is unchanged by conformal scalings of the metric, η, and thus only depends on the conformal class, ω. Using holomorphic coordinates $z = x + iy$, so that $\eta = \lambda^2(dx^2 + dy^2)$, this can be written as

$$E(f) = \frac{1}{2}\int_{\Sigma} (|f_x|^2 + |f_y|^2) dxdy.$$

The critical points for this functional are called harmonic maps. A map is called *weakly conformal* if it satisfies

$$\langle f_z, f_z \rangle = \frac{1}{4}(|f_x|^2 - |f_y|^2 - 2i\langle f_x, f_y \rangle) = 0.$$

Harmonic maps which are also weakly conformal parametrize minimal surfaces. This can be seen by comparing $E(f)$ to the area integral

$$A(f) = \int_\Sigma |f_x \wedge f_y| dxdy.$$

The integrands of both integrals can be compared, as

$$|f_x \wedge f_y| = \sqrt{|f_x|^2|f_y|^2 - \langle f_x, f_y \rangle^2} \leq |f_x||f_y| \leq \frac{1}{2}(|f_x|^2 + |f_y|^2).$$

In the first inequality, equality holds precisely when $\langle f_x, f_y \rangle = 0$, and in the second inequality equaliy holds if and only if $|f_x| = |f_y|$. Therefore $A(f) \leq E(f)$, with equality holding precisely when f is weakly conformal. Now, if $f : \Sigma \to M$ is a weakly conformal harmonic map, and $\sigma : (-\varepsilon, \varepsilon) \times \Sigma \to M$ is a variation of f, then

$$f(p) = \sigma(0, p) \qquad \text{and} \qquad \frac{d}{dt}E(\sigma(t, \cdot)) = 0.$$

Let $a(t) = A(\sigma(t, \cdot))$ and $e(t) = E(\sigma(t, \cdot))$, then $a(t) \leq e(t)$ for all $t \in (-\varepsilon, \varepsilon)$, $a(0) = A(f) = E(f) = e(0)$, and $e'(0) = 0$. This implies that $a'(0) = 0$, as

$$a'(0) = \lim_{h \to 0} \frac{a(h) - a(0)}{h} \leq \lim_{h \to 0} \frac{e(h) - e(0)}{h} = e'(0) = 0,$$

and

$$a'(0) = \lim_{h \to 0} \frac{a(0) - a(-h)}{h} \geq \lim_{h \to 0} \frac{e(0) - e(-h)}{h} = e'(0) = 0.$$

Since this holds for all variations of f, this shows that f must parametrize a minimal surface.

2.1.1 First Variation

Suppose that f is critical point for the energy functional. Let $\sigma : (-\varepsilon, \varepsilon) \times \Sigma \to M$ be a variation of f with variation field $X \in \Gamma(f^*TM)$. Then $\sigma(0, \cdot) = f$, and $\sigma_t(0, \cdot) = X$. Since f is a local extremum,

$$\frac{d}{dt} E(\sigma(t, \cdot))\Big|_{t=0} = 0.$$

This is then used to find the Euler-Lagrange equations,

$$\begin{aligned}
\frac{d}{dt} E(\sigma(t, \cdot))\Big|_{t=0} &= \frac{1}{2} \int_\Sigma \frac{d}{dt}(|\sigma_x|^2 + |\sigma_y|^2) dxdy\Big|_{t=0}, \\
&= \int_\Sigma \left(\langle \nabla_{\partial_t} \sigma_x, \sigma_x \rangle + \langle \nabla_{\partial_t} \sigma_y, \sigma_y \rangle\right) dxdy\Big|_{t=0}, \\
&= \int_\Sigma \big(\langle \nabla_{\partial_x} \sigma_t, \sigma_x \rangle + \langle \nabla_{\partial_y} \sigma_t, \sigma_y \rangle\big)\big|_{t=0} dxdy, \\
&= -\int_\Sigma \big(\langle \sigma_t, \nabla_{\partial_x} \sigma_x + \nabla_{\partial_y} \sigma_y \rangle\big)\big|_{t=0} dxdy, \\
&= -\int_\Sigma \big(\langle X, \nabla_{\partial_x} f_x + \nabla_{\partial_y} f_y \rangle\big) dxdy, \\
&= -\int_\Sigma \left(\langle X, \Delta_\eta f \rangle\right) dA.
\end{aligned}$$

Since this is true for any variation field, X, f must satisfy

$$\Delta_\eta f = \frac{1}{\lambda^2}(\nabla_{\partial_x} f_x + \nabla_{\partial_y} f_y) = \frac{4}{\lambda^2} \nabla_{\partial_{\bar{z}}} f_z = 0.$$

When f is harmonic, $\langle f_z, f_z \rangle dz^2$ is a holomorphic section of the square of the canonical bundle, K^2. In the genus zero case, the degree of K^2 is -4, and this section must be trivial, that is $\langle f_z, f_z\rangle = 0$. This shows that when Σ is a sphere every harmonic map must parametrize a minimal surface.

2.1.2 Second Variation

To check the stability of a critical point we take the second derivative using a variation, $\sigma : (-\varepsilon,\varepsilon)^2 \times \Sigma \to M$, with $\sigma(0,0,\cdot) = f$, $\sigma_s(0,0,\cdot) = X$, and $\sigma_t(0,0,\cdot) = Y$. Then

$$\begin{aligned}
d^2E(f)(X,Y) &= \frac{\partial^2}{\partial s\partial t}E(\sigma(s,t,\cdot))\Big|_{s=0,t=0} \\
&= \frac{1}{2}\int_\Sigma \frac{\partial^2}{\partial s\partial t}(|\sigma_x|^2 + |\sigma_y|^2)dxdy\Big|_{s=0,t=0}, \\
&= \int_\Sigma \frac{\partial}{\partial s}\left(\langle\nabla_{\partial_t}\sigma_x,\sigma_x\rangle + \langle\nabla_{\partial_t}\sigma_y,\sigma_y\rangle\right)dxdy\Big|_{s=0,t=0}, \\
&= \int_\Sigma (\langle\nabla_{\partial_s}\nabla_{\partial_t}\sigma_x,\sigma_x\rangle + \langle\nabla_{\partial_t}\sigma_x,\nabla_{\partial_s}\sigma_x\rangle \\
&\qquad + \langle\nabla_{\partial_s}\nabla_{\partial_t}\sigma_y,\sigma_y\rangle + \langle\nabla_{\partial_t}\sigma_y,\nabla_{\partial_s}\sigma_y\rangle)\,dxdy|_{s=0,t=0}, \\
&= \int_\Sigma (\langle\nabla_{\partial_s}\nabla_{\partial_x}\sigma_t,\sigma_x\rangle + \langle\nabla_{\partial_x}\sigma_t,\nabla_{\partial_x}\sigma_s\rangle \\
&\qquad + \langle\nabla_{\partial_s}\nabla_{\partial_y}\sigma_t,\sigma_y\rangle + \langle\nabla_{\partial_y}\sigma_t,\nabla_{\partial_y}\sigma_s\rangle\big)\,dxdy|_{s=0,t=0}, \\
&= \int_\Sigma (\langle R(\sigma_s,\sigma_x)\sigma_t + \nabla_{\partial_x}\nabla_{\partial_s}\sigma_t,\sigma_x\rangle \\
&\qquad + \langle R(\sigma_s,\sigma_y)\sigma_t + \nabla_{\partial_y}\nabla_{\partial_s}\sigma_t,\sigma_y\rangle \\
&\qquad\quad + \langle\nabla_{\partial_x}\sigma_t,\nabla_{\partial_x}\sigma_s\rangle + \langle\nabla_{\partial_y}\sigma_t,\nabla_{\partial_y}\sigma_s\rangle\big)\,dxdy|_{s=0,t=0}.
\end{aligned}$$

Integrating by parts, evaluating, and using the fact that f is harmonic gives the formula

$$\begin{aligned} d^2E(f)(X,Y) &= \int_\Sigma \big(\langle \nabla_{\partial_x} X, \nabla_{\partial_x} Y\rangle + \langle \nabla_{\partial_y} X, \nabla_{\partial_y} Y\rangle \\ &\qquad -\langle R(X,f_x)f_x, Y\rangle - \langle R(X,f_y)f_y, Y\rangle\big)\, dxdy, \qquad (2.1) \\ &= \int_\Sigma \left(\langle \nabla X, \nabla Y\rangle - \langle \mathcal{K}(X), Y\rangle\right) dA, \end{aligned}$$

where $\mathcal{K}$ is given by

$$\mathcal{K}(X) = \frac{1}{\lambda^2}\left(R(X,f_x)f_x + R(X,f_y)f_y\right).$$

Definition 2.1.1. A harmonic map, f, has *Morse index* equal to the dimension of largest subspace of $\Gamma(f^*TM)$ on which $d^2E(f)$ is negative definite. i.e.

$$\text{Morse Index}(f) = \sup\{\dim V \mid \forall X \in V, d^2E(f)(X,X) < 0\}.$$

This is equal to the number of negative eigenvalues of the Schrödinger operator $\mathcal{J} = \nabla^*\nabla - \mathcal{K}$. Extending the metric to a complex bilinear form on $\mathbf{E} = f^*TM \otimes \mathbb{C}$ allows one to define a Hermitian metric by

$$\langle\langle V, W\rangle\rangle = \langle V, \bar{W}\rangle.$$

By a theorem of Koszul and Malgrange, there is a unique holomorphic structure so that $\bar\partial : \Omega^{(0,0)}(\mathbf{E}) \to \Omega^{(0,1)}(\mathbf{E})$ satisfies

$$\bar\partial V = \nabla_{\partial_{\bar z}} V \otimes d\bar z.$$

This holomorphic structure will be the one used on $\mathbf{E}$ unless stated otherwise. The adjoint operator $\bar{\partial}^* : \Omega^{(0,1)}(\mathbf{E}) \to \Omega^{(0,0)}(\mathbf{E})$ is given by

$$\bar{\partial}^*(V \otimes d\bar{z}) = -\frac{1}{\lambda^2}\nabla_{\partial_z} V.$$

Using this, gives

$$\begin{aligned}
\Delta'' V &= 2\bar{\partial}^*\bar{\partial} V, \\
&= -\frac{2}{\lambda^2}\nabla_{\partial_z}\nabla_{\partial_{\bar{z}}} V, \\
&= -\frac{1}{2\lambda^2}(\nabla_{\partial_x} - i\nabla_{\partial_y})(\nabla_{\partial_x} V + i\nabla_{\partial_y} V), \\
&= -\frac{1}{2\lambda^2}(\nabla_{\partial_x}\nabla_{\partial_x} V + \nabla_{\partial_y}\nabla_{\partial_y} V + i(\nabla_{\partial_x}\nabla_{\partial_y} V - \nabla_{\partial_y}\nabla_{\partial_x} V)), \\
&= \frac{1}{2}\nabla^*\nabla V - \frac{i}{2\lambda^2} R(f_x, f_y) V.
\end{aligned}$$

This shows that after a $\mathbb{C}$-linear extension, $\mathcal{J} = 2\Delta'' - 2\mathcal{K}''$, where

$$\begin{aligned}
\mathcal{K}''(V) &= \frac{1}{2}\mathcal{K}(V) - \frac{i}{2\lambda^2} R(f_x, f_y) V, \\
&= \frac{1}{2\lambda^2}(R(V, f_x) f_x + R(V, f_y) f_y - iR(f_x, f_y) V), \\
&= \frac{1}{2\lambda^2}(R(V, f_x) f_x + R(V, f_y) f_y + iR(V, f_x) f_y - iR(V, f_y) f_x), \\
&= \frac{1}{2\lambda^2}(R(V, f_x)(f_x + i f_y) - iR(V, f_y)(f_x + i f_y)), \\
&= \frac{1}{2\lambda^2} R(V, f_x - i f_y)(f_x + i f_y), \\
&= \frac{2}{\lambda^2} R(V, f_z) f_{\bar{z}}.
\end{aligned}$$

Since $\mathcal{J}$ is formally self-adjoint with respect to the real L^2 metric, its complex linear extension is formally self-adjoint with respect to the corresponding Hermitian L^2 metric. Thus $\mathcal{J}$ has real eigenvalues and the complex eigenspaces are just the complexification of the real eigenspaces. Therefore the Morse index is also equal to the complex dimension of the largest subspace, $\mathcal{N} \subset \Gamma(\mathbf{E})$ such that for every vector, $V \in \mathcal{N}$,

$$d^2E(f)(V, \bar{V}) = \int_\Sigma \langle\langle \mathcal{J}V, V \rangle\rangle dA < 0.$$

This is the complex form of the second variation which can be written as

$$d^2E(f)(V, \bar{V}) = 4 \int_\Sigma (\|\nabla_{\partial_{\bar{z}}} V\|^2 - \langle\langle \mathcal{R}(f_z \wedge V), f_z \wedge V \rangle\rangle) dxdy, \qquad (2.2)$$

and was used in [9].

2.2 α-Energy

The tool we will use to study critical points of the energy functional is the α-energy functional introduced in [16],

$$E_\alpha(f) = \frac{1}{2} \int_\Sigma (1 + |df|^2)^\alpha dA - \frac{1}{2}.$$

When $\alpha = 1$, this is just the usual energy functional, but when $\alpha > 1$, unlike E, it satisfies condition C of Palais and Smale, and can be perturbed to a Morse function on $L_1^{2\alpha}(S^2, M)$. For $\alpha > 1$ this functional is only invariant under isometries of the metric, η, on Σ.

2.2.1 α-Harmonic Maps

Using holomorphic coordinates $z = x + iy$, so that $\eta = \lambda^2(dx^2 + dy^2)$, the α-energy can be written as

$$E_\alpha(f) = \frac{1}{2}\int_\Sigma \left(1 + \frac{1}{\lambda^2}(|f_x|^2 + |f_y|^2)\right)^\alpha \lambda^2 dxdy - \frac{1}{2}.$$

Suppose that f is critical point for the α-energy functional. Let $\sigma : (-\varepsilon, \varepsilon) \times \Sigma \to M$ be a variation of f with variation field $X \in \Gamma(f^*TM)$. Then $\sigma(0, \cdot) = f$, and $\sigma_t(0, \cdot) = X$. Since f is a critical point,

$$\frac{d}{dt}E_\alpha(\sigma(t, \cdot))\bigg|_{t=0} = 0.$$

Again, this is used to find the Euler-Lagrange equations,

$$\begin{aligned}
\frac{d}{dt}E_\alpha(\sigma(t, \cdot))\bigg|_{t=0} &= \frac{1}{2}\int_\Sigma \frac{d}{dt}\left(1 + \frac{1}{\lambda^2}(|\sigma_x|^2 + |\sigma_y|^2)\right)^\alpha \lambda^2 dxdy\bigg|_{t=0}, \\
&= \alpha\int_\Sigma (1 + |d\sigma|^2)^{\alpha-1}\left(\langle\nabla_{\partial_t}\sigma_x, \sigma_x\rangle + \langle\nabla_{\partial_t}\sigma_y, \sigma_y\rangle\right)\lambda^2 dxdy\bigg|_{t=0}, \\
&= \alpha\int_\Sigma (1 + |d\sigma|^2)^{\alpha-1}\left(\langle\nabla_{\partial_x}\sigma_t, \sigma_x\rangle + \langle\nabla_{\partial_y}\sigma_t, \sigma_y\rangle\right)\big|_{t=0} dA, \\
&= -\alpha\int_\Sigma \left(\langle X, \nabla_{\partial_x}((1 + |df|^2)^{\alpha-1}f_x) + \nabla_{\partial_y}((1 + |df|^2)^{\alpha-1}f_y)\rangle\right) dA.
\end{aligned}$$

Since this is true for any variation field, X, f must satisfy

$$\nabla_{\partial_x}((1 + |df|^2)^{\alpha-1}f_x) + \nabla_{\partial_y}((1 + |df|^2)^{\alpha-1}f_y) = 0,$$

which can also be written as

$$\nabla_{\partial_x}f_x + \nabla_{\partial_y}f_y + \frac{\partial(1 + |df|^2)^{\alpha-1}}{\partial x}f_x + \frac{\partial(1 + |df|^2)^{\alpha-1}}{\partial y}f_y = 0.$$

A calculation similar to the one carried out in section 2.1.2 gives the second variation formula for E_α.

$$d^2E_\alpha(f)(X,Y) = \alpha \int_\Sigma (1+|df|^2)^{\alpha-1}(\langle \nabla X, \nabla Y\rangle - \langle \mathcal{K}(X), Y\rangle)dA$$
$$+ 2\alpha(\alpha-1)\int_\Sigma (1+|df|^2)^{\alpha-2}\langle df, \nabla X\rangle\langle df, \nabla Y\rangle dA$$

Unlike the usual energy, E_α is not conformally invariant, but is only invariant under isometries of Σ. Thus, in the case of $\Sigma = S^2$, E_α is invariant under the action of the group $O(3)$. On the other hand, E is invariant under the action of the group of conformal and anticonformal automorphsims,

$$PSL(2,\mathbb{C}) \cup R \cdot PSL(2,\mathbb{C}),$$

where R is a reflection across a great circle. This difference in the symmetry group is accounted for by the fact that α-harmonic maps satisfy a center-of-mass condition. It is shown in [12, 11] that for $\alpha > 1$ and

$$\psi_\alpha(t) = \alpha \int_0^t \frac{\tau}{(1+\tau)^{2-\alpha}} d\tau,$$

the following theorem holds.

Theorem 2.2.1. *Let* $\mathbf{X} : S^2 \to \mathbb{R}^3$ *be the standard inclusion,* $\mathbf{0}$ *be the origin in* $\mathbb{R}^3$*, and* $f : S^2 \to M$ *be* α*-harmonic. Then*

$$\int_{S^2} \mathbf{X}\psi_\alpha(|df|^2)dA = \mathbf{0}. \tag{2.3}$$

A function $f : S^2 \to M$ satisfying (2.3) will be said to have α *center of mass zero*. When $\alpha = 1$ the α will be dropped, so for

$$\psi_1(t) = \int_0^t \frac{\tau}{1+\tau} d\tau = t - \log(t+1)$$

a map satisfying (2.3) has *center of mass zero*. Imposing this condition reduces the symmetry group of E to $O(3)$.

It is also important to understand the isotropy of a critical point $f \in L_1^{2\alpha}(S^2, M)$. Since the isotropy group must be a Lie subgroup of $O(3)$, it will either be a discrete subgroup, a circle, or all of $O(3)$. The only maps that have isotropy $O(3)$ are the constant maps, and it is shown in [11] that there are no E_α critical points with finite α-energy and isotropy S^1. Thus nonconstant α-harmonic maps below a given α-energy level will have discrete isotropy.

Using the implicit function theorem, the theorem below is proved in [12].

Theorem 2.2.2. *Suppose that M is a compact connected Riemannian manifold and that $f : S^2 \to M$ is a prime oriented parametrized minimal immersion lying on a nondegenerate critical submanifold N for E of dimension six, an orbit for the $PSL(2, \mathbb{C})$ action. If $N_0 \subset N$ is the set of zero center of mass critical points, then for some $\alpha_0 \in (1, \infty)$, there is a unique smooth map*

$$\eta : N_0 \times [1, \alpha_0) \to \left\{ (f, \alpha) \in L_k^2(S^2, M) \times [1, \alpha_0) \,\middle|\, \int_{S^2} \mathbf{X}\psi_\alpha(|df|^2) dA = 0 \right\}$$

satisfying

1. *$\eta(f, \alpha)$ is a critical point for E_α and*

2. η is $O(3)$-equivariant.

Furthermore, there is an $\varepsilon_0 > 0$ such that if f' is a critical point for E_α such that $\|f' - f\|_{C^2} < \varepsilon_0$ for some $f \in N_0$, then f is in the image of η.

2.3 Bubble Trees

In the paper [16], it is shown that for $\alpha_i \to 1$, a sequence of E_{α_i} critical points will converge to a harmonic map away from a finite number of points.

Theorem 2.3.1. *Let M be a compact Riemannian manifold, and let Σ be a compact Riemann surface. If $f_i : \Sigma \to M$ is a sequence of E_{α_i} critical points with $\alpha_i \to 1$ and $E(f_i) < E_0$ for some $E_0 > 0$ then there is a finite set, $P = \{p_1, p_2, \ldots, p_l\} \subset \Sigma$ such that a subsequence of f_i converges uniformly in C^k on compact subsets of $\Sigma \setminus P$ for all $k \geq 0$ to a harmonic map $f_\infty : \Sigma \setminus P \to M$.*

Furthermore, there is a removable singularity theorem

Theorem 2.3.2. *Let D be the unit disk. If $f : D \setminus \{0\} \to M$ is a harmonic map then f extends to a harmonic map on all of D.*

This allows the limiting map f_∞ to be extended to all of Σ. Sacks and Uhlenbeck were also able to show that if f_i does not converge to f_∞ in C^1, then at one of the points, p_j, a minimal sphere bubbles off. That is there is a harmonic map $g : S^2 \to M$ satisfying $E(f_\infty) + E(g) \leq \limsup_{i\to\infty} E_{\alpha_i}(f_i)$ and

$$g(S^2) \subset \bigcap_{\alpha\to 1} \overline{\bigcup_{\beta<\alpha} s_\beta(\Sigma)}.$$

The papers [14, 15] further clarified the behavior of the limiting maps f_i with their introduction of a *bubble tree*.

Definition 2.3.3. A *bubble tree* based on Σ is a collection of maps indexed by a rooted tree, T. If T has vertex set T_0, edge set T_1 and root $\hat{v}$, then The root corresponds to a harmonic map $f_{\hat{v}} : \Sigma \to M$ for the given conformal structure on Σ. The other vertices $v \in T_0 \backslash \{\hat{v}\}$, correspond to harmonic maps $f_v : S^2 \to M$, and this map cannot be constant if v does not have any descendants. An edge, $e \in T_1$, joining a vertex, v_1, to its parent, v_0, corresponds to a geodesic, $\gamma_e : [0,1] \to M$, satisfying $\gamma_e(0) \in \mathrm{im}(f_{v_0})$ and $\gamma_e(1) \in \mathrm{im}(f_{v_1})$, and which may degenerate to a point.

At each point p_j one can construct a sequence of disks $D_{j,m}$ and $B_{j,m}$ of radius less than $\frac{C}{m}$ and $\frac{C}{m^3}$, respectively, for some constant C. Then for a conformal contraction $T_{j,m} : D_m(0) \to B_{j,m}$ from the disk of radius m in $\mathbb{C}$, a new sequence of maps

$$g_{j,m} = f_m \circ T_{j,m} : D_m(0) \to M,$$

for which a subsequence convergers uniformly in C^k, for all k, on compact subsets of

$$\mathbb{C} \setminus \{p_{j,1}, p_{j,2}, \cdots, p_{j,l_j}\} = S^2 \setminus \{p_{j,1}, p_{j,2}, \cdots, p_{j,l_j}, \infty\}$$

to a harmonic map $g_j : S^2 \to M$. Then the map f_∞ is the base of the bubble tree for the sequence f_i, its descendants are the maps g_j, and this process can be repeated for the points $p_{j,1}, \cdots, p_{j,l_j}$ to get the descendants of g_j and so on.

The geodesics corresponding to the edges connecting f_∞ to g_j are obtained by a conformal scaling of the annular region $D_{j,m} \setminus B_{j,m}$. The resulting sequence will converge to a radial function $h_j(r)$ which parametrizes a geodesic joining a point in the image of f_∞ to a point in the image of g_j.

If the sequence f_i satisfies $E_{\alpha_i}(f_i) < E_0$ and the sectional curvature of M is bounded above by 1, then the number of bubbles in the resulting tree must be less than $\frac{E_0}{4\pi}$. This is guaranteed because total energy of all the maps in the bubble tree must also be bounded by E_0. The Gauss equation shows that for a minimal surface $f : S^2 \to M$ the Gauss curvature of the pulled back metric, K_f, must also be less or equal to 1. If dA_f is the area form for the pulled back metric, then

$$E(f) = A(f) = \int_{S^2} dA_f \geq \int_{S^2} K_f dA_f \geq 4\pi.$$

In [6] it is shown that for sequences which minimize energy for a given constraint, no energy is lost in the necks, and the sequence $E_{\alpha_i}(f_i)$ converges to the total energy of the bubble tree. In [11] it is shown that this result can be extended to minimax sequences provided $\pi_1(M)$ is finite. Some of the index may be lost, but if f_m is converging to a bubble tree of harmonic maps indexed by T, then there is an $M > 0$ such that for every $m > M$,

$$\sum_{v \in T} \big(\text{Morse index}(f_v)\big) \leq \text{Morse index}(f_m). \tag{2.4}$$

Chapter 3

Index Estimates

Since the curvature appears in the second variation formula, some curvature estimate is needed to get an estimate on the index. Stronger curvature conditions allow better estimates. The strongest curvature condition considered, δ-controlled half-isotropic curvature, will be studied in greater detail in the following chapter. This condition seems to be the weakest possible condition which guarantees that the index of a branched covering of a minimal sphere is at least $2(n-2)$, where n is the dimension of the ambient manifold.

3.1 Positive Scalar Curvature

For a Riemannian manifold of dimension 3 the scalar curvature controls the stability of minimal surfaces. This can be seen from the well-known second variation formula

$$d^2A(f)(u\nu, u\nu) = \int_\Sigma \left[|\nabla' u|^2 - u^2(|\alpha|^2 + \mathrm{Ric}(\nu))\right] dA',$$

where ∇' and dA' are the gradient and area form for the pulled back metric, and α is the second fundamental form. This can be written in terms of the scalar curvature of M, s, and the Gauss curvature of the pulled back metric, K, as

$$d^2A(f)(u\nu, u\nu) = \int_\Sigma \left[|\nabla' u|^2 - u^2\left(\frac{1}{2}|\alpha|^2 + s - K\right)\right] dA'. \tag{3.1}$$

When $u = 1$ this becomes

$$d^2A(f)(u\nu, u\nu) = -\int_\Sigma \left(\frac{1}{2}|\alpha|^2 + s\right) dA' + 2\pi\chi(\Sigma).$$

This is always negative provided that $s > 0$ and Σ is not a sphere. This was used by Schoen and Yau[17] to show that a 3-manifold of positive scalar curvature does not contain any incompressible surfaces. The following is a special case of the theorem proved in [8], and allows us to give an index estimate under the same hypothesis.

Theorem 3.1.1. *Let Σ be a compact surface with metric g and $V \in L^\infty(\Sigma)$. If $\mathcal{N}(V)$ denotes the number of negative eigenvalues of the operator $(-\Delta_g - V)$, then*

$$\mathcal{N}(V) \geq C \int_\Sigma V dA_g,$$

where C is a constant depending only on the genus of Σ.

Using the second variation formula 3.1 gives the following corollary.

Corollary 3.1.2. *Let M be a Riemannian manifold satisfying $s > s_0 > 0$. If Σ is a minimal surface immersed in M then*

$$\textit{index}(f) \geq C_0 A(f) + C_1.$$

Where the constant C_0 depends only on the genus of Σ and s_0 and C_1 depends only on the genus of Σ and is nonnegative for positive genus.

Proof. Let $f : \Sigma \to M$ be a minimal immersion, then

$$d^2A(f)(u\nu, u\nu) = \int_\Sigma (-\Delta' u - Vu) u dA',$$

for $V = \frac{1}{2}|\alpha|^2 + s - K$. By theorem 3.1.1, this implies

$$\begin{aligned} \text{index}(f) &\geq C \int_\Sigma \left(\frac{1}{2}|\alpha|^2 + s - K \right) dA' \\ &\geq C(\int_\Sigma s dA' - 2\pi\chi(\Sigma)) \\ &\geq C(s_0 A(f) - 2\pi\chi(\Sigma)). \end{aligned}$$

□

3.2 Positive Half-Isotropic Curvature

A complex two-plane, $\sigma \subset T_pM \otimes \mathbb{C}$ is *half-isotropic* if it has a Hermitian basis $\{w_1, w_2\}$ satisfying

$$\langle w_1, w_1 \rangle = \langle w_1, w_2 \rangle = 0.$$

Such a basis will be called *adapted*. The manifold, M, has *positive half-isotropic curvature* if for every such two-plane,

$$\langle\langle \mathcal{R}(w_1 \wedge w_2), w_1 \wedge w_2 \rangle\rangle > 0.$$

Note that if M is 3-dimensional, then this reduces to positive Ricci curvature.

Theorem 3.2.1. *Let M be an n-dimensional Riemannian manifold with positive half-isotropic curvature. If $f : S^2 \to M$ is a paramatrized minimal sphere, then the Morse index of f is at least $(n-2)$.*

Proof. Let $\mathbf{L} \subset \mathbf{E}$ be a line bundle given locally by scalar multiples of f_z. If p is a zero of f_z, then $f_z = (z-p)^k Z$ for some k and some local holomorphic section, $Z \in \Gamma(\mathbf{E})$ with $Z(p) \neq 0$. At such a point $\mathbf{L}$ is given by scalar multiples of Z. The bundle $\mathbf{L}^\perp = \{Z | \langle f_z, Z \rangle = 0\}$, can be defined similarly. Noting that since f is weakly conformal, $\mathbf{L} \subset \mathbf{L}^\perp$, the holomorphic normal bundle is given by $\mathbf{N} = \mathbf{L}^\perp / \mathbf{L}$. Since $\mathbf{N}$ is rank $n-2$, the Riemann-Roch theorem implies that

$$\dim \mathcal{O}(\mathbf{N}) \geq n-2.$$

If $W \in \mathcal{O}(\mathbf{N})$ then $\sigma =$span(f_z, W) is half-isotropic and from the second variation formula (2.2),

$$d^2E(f)(W, \bar{W}) = -4 \int_\Sigma \|f_z\|^2 \|W\|^2 K_{\mathbb{C}}(\sigma) dxdy < 0.$$

Therefore

$$\text{Morse Index}(f) \geq \dim \mathcal{O}(\mathbf{N}) \geq n-2.$$

□

3.3 δ-controlled Half-Isotropic Curvature

The manifold, M, has *δ-controlled half-isotropic curvature* if for every half-isotropic two-plane, σ, and any adapted basis $\{w_1 = \frac{1}{\sqrt{2}}(e_1 + ie_2), w_2\}$

$$\langle\langle \mathcal{R}(w_1 \wedge w_2), w_1 \wedge w_2 \rangle\rangle > \delta \langle \mathcal{R}(e_1 \wedge e_2), e_1 \wedge e_2 \rangle.$$

Theorem 3.3.1. *Let M be an n-dimensional Riemannian manifold with $\frac{1}{d}$-controlled half-isotropic curvature. If $g : S^2 \to M$ is a parametrized minimal surface, and $h : S^2 \to S^2$ is a degree d holomorphic map, then $f = g \circ h$ is a parametrized minimal surface and the Morse index of f is at least $2(n-2)$.*

Proof. Suppose that $f : S^2 \to M$ is a weakly conformal harmonic map, and define $\mathbf{N}$ as above. Using the Kozul-Malgrange holomorphic structure, $\mathbf{N}$ is a holomorphic vector bundle over S^2. For W a section of $\mathbf{N}$, the complex form of the second variation of energy is given by

$$d^2E(f)(W, \bar{W}) = 2\int_{S^2} \left(\|\bar{\partial} W\|^2 - K_{\mathbb{C}}(\sigma)e(f)\|W\|^2\right) dA,$$

where σ is the space spanned by f_z and W.

By the Grothendieck splitting theorem,

$$\mathbf{N} \cong \mathbf{L}_1 \oplus \cdots \oplus \mathbf{L}_{n-2}.$$

The Riemannian metric restricted to $\mathbf{N}$ is then a holomorphic bilinear form, and the induced map, $X \mapsto \langle X, \cdot \rangle$, is a holomorphic isomorphism. Therefore $\mathbf{N} \cong \mathbf{N}^* \cong \bar{\mathbf{N}}$, and

$$\mathbf{N} \cong \bar{\mathbf{L}}_1 \oplus \cdots \oplus \bar{\mathbf{L}}_{n-2},$$

but this decomposition is unique up to isomorphism of the line bundles. Thus the line bundles can be arranged so that $\mathbf{L}_i \cong \bar{\mathbf{L}}_{n-1-i}$, and $c_1(\mathbf{L}_i) = -c_1(\mathbf{L}_{n-1-i})$. Further, they can be arranged so that $c_1(\mathbf{L}_1) \geq \cdots \geq c_1(\mathbf{L}_{n-2})$. Now, choosing $m = \max\{i | c_1(\mathbf{L}_i) > 0\}$, let $W \in \mathcal{O}(\mathbf{L}_i)$, for some $1 \leq i \leq m$. Then $\langle W, W \rangle$ is a holomorphic function, and must be constant. Since $c_1(\mathbf{L}_i) > 0$ is the number of zeroes of W, $\langle W, W \rangle$ must be identically zero. Showing that the line bundles $\mathbf{L_i}$ for $1 \leq i \leq m$ are all isotropic. This implies that $\mathbf{L}_i$ considered as a real $SO(2)$ bundle, E_i, has an orthogonal complex structure, J_i. Now $E_i \otimes \mathbb{C} = \mathbf{L}_i \oplus \bar{\mathbf{L}}_i$ and $\mathbf{L}_i$ is the i-eigenspace of J_i. Recall that the dimension of the largest subspace of $\Gamma(f^*TM)$ on which the index form $d^2E(f) = \int_{S^2} \langle\langle \mathcal{J}(\cdot), \cdot \rangle\rangle dA$ is negative definite determines the index of f. The operator $\mathcal{J}_i = \pi_{E_i} \circ \mathcal{J}_{|E_i}$ is formally self-adjoint, thus it has real eigenvalues. Let $\mathcal{N}_i$ be the space spanned by eigensections of $\mathcal{J}_i$ with negative eigenvalue. Extending $\mathcal{J}_i$ to be complex linear, the second variation formula (2.2) shows that $\mathcal{O}(\mathbf{L}_i) \subset \mathcal{N}_i \otimes \mathbb{C}$. This implies that both the real and imaginary parts of a holomorphic section of $\mathbf{L}_i$, which are non-zero and orthogonal, must be contained in $\mathcal{N}_i$ and thus $\dim_{\mathbb{R}}(\mathcal{N}_i) \geq 2\dim_{\mathbb{C}}(\mathcal{O}(\mathbf{L}_i))$. Since $d^2E(f)$ is negative definite on each $\mathcal{N}_i$ this implies that the Morse index of f is at least $4m$.

In order to complete the proof, it must be shown that each line bundle summand with degree 0 contributes at least 2 to the Morse index. Since these summands only have one holomorphic section and need not be isotropic, an eigenvalue estimate is needed. Let $l = \max\{i | c_1(\mathbf{L}_{m+i}) = 0\}$, then $\dim \mathcal{O}(\mathbf{L}_i) = 1$ for

$m+1 \leq i \leq m+l$. Thus we can let Z be a nowhere zero holomophic section of L_i, and since $\sigma = \mathrm{span}(f_z, Z)$ is half-isotropic, $d^2E(f)(Z, \bar{Z}) < 0$.

Now for $w : S^2 \to \mathbb{C}$, let $W = wZ$. Since M satisfies the curvature estimate (4.1), σ is half isotropic, and $T_f S^2$ is an associated real 2-plane of σ,

$$K_{\mathbb{C}}(\sigma) > \delta K_{\mathbb{R}}(T_f S^2) > 0.$$

$$d^2E(f)(W, \bar{W}) < 2 \int_{S^2} \left(\|\bar{\partial} W\|^2 - \delta K_{\mathbb{R}}(T_f S^2) e(f) \|W\|^2 \right) dA$$

Which can be rewritten as

$$d^2E(f)(W, \bar{W}) < C \left(\frac{2 \int_{S^2} \|\bar{\partial} W\|^2 dA}{\int_{S^2} \delta K_{\mathbb{R}}(T_f S^2) e(f) \|W\|^2 dA} - 2\delta \right),$$

where $C = \int_{S^2} \delta K_{\mathbb{R}}(T_f S^2) e(f) \|W\|^2 dA > 0$. Since $W = wZ$, and the eigenvalues of $\bar{\partial}$ do not depend on the Hermitian structure on $\mathbf{L}_i$, a hermitian structure can be chosen so that $\|Z\| = 1$, giving

$$d^2E(f)(W, \bar{W}) < C \left(\frac{2 \int_{S^2} |\bar{\partial} w|^2 dA}{\int_{S^2} K_{\mathbb{R}}(T_f S^2) e(f) |w|^2 dA} - 2\delta \right).$$

Now, since the Dirichlet integral is conformally invariant in dimension 2,

$$\frac{2 \int_{S^2} |\bar{\partial} w|^2 dA}{\int_{S^2} K_{\mathbb{R}}(T_f S^2) e(f) |w|^2 dA} = \frac{\int_{S^2} |\nabla w|^2 dA}{\int_{S^2} K_{\mathbb{R}}(T_f S^2) e(f) |w|^2 dA}$$

is just the Rayleigh quotient for the metric $K_{\mathbb{R}}(T_f S^2) e(f) \eta = K_{\mathbb{R}}(T_f S^2) f^* g$, where η is the reference metric on S^2 of constant curvature and total area 1.

This yields the estimate

$$d^2E(f)(W, \bar{W}) < C\left(\lambda_1 - 2\delta\right),$$

where λ_1 is the first eigenvalue of the metric, $K_{\mathbb{R}}(T_f S^2) f^* g$. By the eigenvalue estimate of Yang and Yau[19],

$$\lambda_1 \leq \frac{8\pi}{\int_{S^2} K_{\mathbb{R}}(T_f S^2) e(f) dA},$$

By the Gauss equation, for a local orthonormal frame e_1, e_2 of $T_f S^2$

$$\begin{aligned} R_f(e_1, e_2, e_1, e_2) &= R(e_1, e_2, e_1, e_2) + \langle \alpha(e_1, e_1), \alpha(e_2, e_2) \rangle - \|\alpha(e_1, e_2)\|^2, \\ &= R(e_1, e_2, e_1, e_2) - \|\alpha(e_1, e_1)\|^2 - \|\alpha(e_1, e_2)\|^2, \end{aligned}$$

where α is the second fundamental form. Thus the curvature of $f(S^2)$ is

$$K_f = K_{\mathbb{R}}(T_f S^2) - \frac{1}{2}\|\alpha\|^2,$$

Therefore

$$\lambda_1 \leq \frac{8\pi}{\int_{S^2} \left(K_f + \frac{1}{2}\|\alpha\|^2\right) e(f) dA} \leq \frac{8\pi}{\int_{S^2} K_f e(f) dA},$$

but the denominator depends only on the topology,

$$\int_{S^2} K_f e(f) dA = 2\pi \chi(T_f S^2) = 2\pi(2 + \nu),$$

where ν is the total branching order of f. Using this gives the final estimate,

$$d^2 E(f)(W, \bar{W}) < 2C \left(\frac{2}{2 + \nu} - \delta \right) < 0,$$

as the Riemann-Hurwitz formula implies f must have branching order at least $2(d - 1)$.

The above shows that the Morse index of f must be at least $4m + 2l = 2(2m + l)$. Since $\text{rank}(\mathbf{N}) = 2m + l = n - 2$, this completes the proof of the theorem. □

Chapter 4

Curvature Conditions

For a Riemannian manifold, M, with Levi-Civita connection, ∇, the curvature tensor is given by

$$R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z.$$

This gives rise to the curvature operator, $\mathcal{R} : \bigwedge^2 T_pM \to \bigwedge^2 T_pM$, given by

$$\langle \mathcal{R}(X \wedge Y), Z \wedge W \rangle = \langle R(X,Y)W, Z \rangle,$$

where the metric on $\bigwedge^2 T_pM$ is determined by

$$\langle X \wedge Y, Z \wedge W \rangle = \langle X, Z \rangle \langle Y, W \rangle - \langle X, W \rangle \langle Y, Z \rangle.$$

The sectional curvature of a real 2-plane, $\sigma \subset T_pM$, with orthonormal basis $\{e_1, e_2\}$ is given by

$$K_{\mathbb{R}}(\sigma) = \langle \mathcal{R}(e_1 \wedge e_2), e_1 \wedge e_2 \rangle.$$

These tensors can all be extended to be complex linear, then there is a Hermitian inner product on T_pM given by

$$\langle\langle V, W\rangle\rangle = \langle V, \bar{W}\rangle.$$

The complex sectional curvature of a complex 2-plane, σ with Hermitian basis $\{w_1, w_2\}$ is then

$$K_{\mathbb{C}}(\sigma) = \langle\langle \mathcal{R}(w_1 \wedge w_2), w_1 \wedge w_2\rangle\rangle.$$

4.1 Half-Isotropic Curvature

A vector $V \in T_pM \otimes \mathbb{C}$ is *isotropic* $V \in \sigma$, $\langle V, V\rangle = 0$. A subspace $\sigma \subset T_pM \otimes \mathbb{C}$, is *isotropic* if all vectors $V \in \sigma$, are isotropic. Since $\langle\langle V, \bar{V}\rangle\rangle = \langle V, V\rangle$, a vector is isotropic precisely when it is Hermitian orthogonal to its complex conjugate. This is also true for a subspace, thus σ is isotropic precisely when it Hermitian orthogonal to $\bar{\sigma}$. In contrast, σ, is the complexification of a real subspace of T_pM if $\sigma = \bar{\sigma}$. To get conditions between these two extremes, consider the $\mathbb{C}$-linear map $q_\sigma : \sigma \to \bar{\sigma}$, which conjugates a vector, projects it orthogonally back onto σ, then conjugates again. Then σ is real if and only if $\sigma = \bar{\sigma}$ and $q_\sigma = \mathrm{id}_\sigma$, and in this case $\ker q_\sigma = \{0\}$. On the other hand, σ is isotropic if and only if $\ker q_\sigma = \sigma$. The dimension of $\ker q_\sigma$ could thus be used as a measure of how close σ is to being isotropic.

For 2-planes, σ, the only new possibility is dim ker $q_\sigma = 1$. Assume this is the case and let $\{w_1, w_2\}$ be a Hermitian basis for σ such that $w_1 \in \ker q_\sigma$. Then

$$q_\sigma(w_1) = \langle\langle w_1, \bar{w}_1 \rangle\rangle \bar{w}_1 + \langle\langle w_1, \bar{w}_2 \rangle\rangle \bar{w}_2 = \langle w_1, w_1 \rangle w_1 + \langle w_1, w_2 \rangle w_2 = 0,$$

implies that $\langle w_1, w_1 \rangle = \langle w_1, w_2 \rangle = 0$. We call any two plane with dim ker $q_\sigma \geq 1$, *half-isotropic*, and any such basis *adapted*.

There is a one-to-one correspondence between isotropic lines in $T_pM \otimes \mathbb{C}$ and real 2-planes in T_pM. To see this note that if V is isotropic, $\|V\| = 1$, and $V = \frac{1}{\sqrt{2}}(X + iY)$ for real vectors X and Y, then $\langle X, Y \rangle = 0$ and $\|X\| = \|Y\| = 1$. If σ is half-isotropic, then it contains an isotropic line. Any real two plane corresponding to such an isotropic line we call an *associated real two-plane* of the half-isotropic plane, σ.

We say that M has δ-controlled half-isotropic curvature if for any half-isotropic two-plane, σ, with associated real two-plane, $\widehat{\sigma}$,

$$K_{\mathbb{C}}(\sigma) > \delta K_{\mathbb{R}}(\widehat{\sigma}) > 0. \tag{4.1}$$

If M satisfies (4.1) with non-strict inequalities then it is said to have weakly δ-controlled half-isotropic curvature.

In the next section we compare the above condition to more commonly used pinching conditions.

4.2 Relationship to Other Curvature Conditions

A Riemannian manifold, M, has λ-pinched sectional curvature if for each point, $p \in M$, and two-plane $\sigma \subset T_pM$

$$\lambda < K_{\mathbb{R}}(\sigma) \leq 1.$$

If there is a positive function $f : M \to \mathbb{R}$ so that

$$\lambda f(p) < K_{\mathbb{R}}(\sigma) \leq f(p),$$

then M has pointwise λ-pinched sectional curvature.

Proposition 4.2.1. *A manifold with pointwise λ-pinched sectional curvature has $\frac{4\lambda-1}{3}$-controlled half-isotropic curvature.*

Proof. Working at a point, $p \in M$ we can scale the metric so that $f(p) = 1$. Let σ be a half isotropic two-plane with adapted basis $\{w_1, w_2\}$ then there is an orthonormal basis for T_pM , $\{e_i\}$, such that $w_1 = \frac{1}{\sqrt{2}}(e_1 + ie_2)$ and $w_2 = ae_3 + bie_4$ for some $a, b \in \mathbb{R}$ satisfying $a^2 + b^2 = 1$. Then

$$K_{\mathbb{C}}(\sigma) = \frac{1}{2}[a^2(R_{1331} + R_{2332}) + b^2(R_{1441} + R_{2442}) + 2abR_{1234}].$$

By a well-known estimate of Berger[3], λ-pinched sectional curvature implies $|R_{1234}| \leq \frac{2}{3}(1 - \lambda)$. Therefore

$$K_{\mathbb{C}}(\sigma) > \lambda - \frac{2}{3}ab(1 - \lambda).$$

Since $2ab \geq a^2 + b^2 = 1 \geq K_{\mathbb{R}}(\widehat{\sigma})$,

$$K_{\mathbb{C}}(\sigma) > \lambda - \frac{1}{3}(1-\lambda) = \frac{4\lambda - 1}{3} \geq \frac{4\lambda - 1}{3} K_{\mathbb{R}}(\widehat{\sigma}).$$

□

When M has δ-controlled half-isotropic curvature, and $\delta > 1/2$, the sectional curvature will be $(2\delta - 1)^2$-pinched. To show this we first get an estimate on the flag curvature.

A Riemannian manifold, M, of dimension at least 3, has strictly λ-pinched *flag curvature* if for each point, $p \in M$, and unit vector, $E \in T_pM$, the symmetric form R_E given by

$$R_E(X, Y) = R(X, E, E, Y)$$

satisfies

$$R_E(X, X) > \lambda R_E(Y, Y),$$

for every X and Y perpendicular to E with $\|X\| = \|Y\|$. This condition was studied in [1] and [13]. It is equivalent the eigenvalues of R_E being contained in the interval $(\lambda k, k]$, where k depends on the point p and the unit vector E. It is also equivalent to requiring $K_{\mathbb{R}}(\sigma_1) > \lambda K_{\mathbb{R}}(\sigma_2)$ whenever $\sigma_1 \cap \sigma_2 \neq \{0\}$. Thus when dim $M = 3$, strictly λ pinched sectional curvature is equivalent to λ-pinched sectional curvature. If dim $M \geq 3$ strictly λ-pinched flag curvature immediately implies pointwise λ^2-pinched sectional curvature. To see this let $\sigma_1, \sigma_2 \subset T_pM$ be two-planes at $p \in M$ then there is another two-plane, $\sigma_3 \subset T_pM$ which intersects both σ_1 and σ_2 nontrivially. Then $K_{\mathbb{R}}(\sigma_1) > \lambda K_{\mathbb{R}}(\sigma_3) > \lambda^2 K_{\mathbb{R}}(\sigma_2)$.

A four dimensional algebraic curvature tensor is given in section 4 of [13] which has λ-pinched flag curvature, but only λ^2-pinched sectional curvature, showing that this is the best possible estimate in dimension greater than or equal to four.

Proposition 4.2.2. *If M has δ-controlled half-isotropic curvature, for some $\delta > \frac{1}{2}$, then M has strictly $(2\delta - 1)$-pinched flag curvature.*

Proof. Let $p \in M$ and $E \in T_pM$ be a unit vector. Let $\{e_i\}$ be an orthonormal basis for T_pM satisfying $e_1 = E$, e_i is an eignevector of R_E with eigenvalue λ_i for all $2 \leq i \leq n$, and $\lambda_2 \leq \lambda_i \leq \lambda_3$ for all $4 \leq i \leq n$. Then $\sigma_1 =$span$(e_1, \frac{1}{\sqrt{2}}(e_2 + ie_3))$ is half-isotropic with associated real two-plane $\widehat{\sigma}_1 =$span(e_2, e_3). Therefore

$$K_{\mathbb{C}}(\sigma_1) = \frac{1}{2}(R_{1221} + R_{1331}) > \delta R_{2332}.$$

Similarly,

$$\frac{1}{2}(R_{2332} + R_{1221}) > \delta R_{1331}.$$

Therefore

$$R_{1221} + R_{1331} > 2\delta R_{2332} > 4\delta^2 R_{1331} - 2\delta R_{1221},$$

which implies that

$$(2\delta + 1)R_{1221} > (4\delta^2 - 1)R_{1331}.$$

Since $R_{1221} = \lambda_2$ and $R_{1331} = \lambda_3$, this implies that

$$\lambda_2 > (2\delta - 1)\lambda_3,$$

showing that the eigenvalues of R_E are all contained in the interval $((2\delta - 1)\lambda_3, \lambda_3]$.

□

Corollary 4.2.3. *If M has δ-controlled half-isotropic curvature, for some $\delta > \frac{1}{2}$, then M has strictly $(2\delta - 1)^2$-pinched sectional curvature.*

For $1 > \delta > \frac{1}{2}$, consider the algebraic curvature operator $\mathcal{R} : \bigwedge^2 \mathbb{R}^3 \to \bigwedge^2 \mathbb{R}^3$ written in the basis $\{e_1 \wedge e_2, e_2 \wedge e_3, e_3 \wedge e_1\}$ with its Ricci tensor, $Ric : \mathbb{R}^3 \to \mathbb{R}^3$, written in the basis $\{e_1, e_2, e_3\}$

$$\mathcal{R} = \begin{pmatrix} 2\delta - 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Ric = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2\delta & 0 \\ 0 & 0 & 2\delta \end{pmatrix}.$$

The scalar curvature is then $S = 4\delta + 2$, and by equation (1.2), $\mathcal{R}$ has weakly δ-controlled half-isotropic curvature and $(2\delta - 1)$-pinched sectional curvature. On the other hand, if $0 < \delta < \frac{1}{2}$ using the same basis as before,

$$\mathcal{R} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2\delta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Ric = \begin{pmatrix} 2\delta + 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2\delta \end{pmatrix}$$

has weakly δ-controlled half-isotropic curvature, but does not have λ-pinched sectional curvature for any λ.

For λ sufficiently large, λ-pinched flag curvature will imply δ-controlled half-isotropic curvature.

Proposition 4.2.4. *If M has strictly λ-pinched flag curvature and $\delta > 0$, where*

$$\delta = \frac{16\lambda^2 + 13\lambda - 5}{12(\lambda + 1)},$$

then M has δ-controlled half-isotropic curvature.

Proof. Let σ be a half isotropic two-plane with adapted basis $\{w_1, w_2\}$ then there is an orthonormal basis for T_pM , $\{e_i\}$, such that $w_1 = \frac{1}{\sqrt{2}}(e_1 + ie_2)$ and $w_2 = ae_3 + bie_4$ for some $a, b \in \mathbb{R}$ satisfying $a^2 + b^2 = 1$. Then

$$K_{\mathbb{C}}(\sigma) = \frac{1}{2}[a^2(R_{1331} + R_{2332}) + b^2(R_{1441} + R_{2442}) + 2abR_{1234}].$$

From corollary 2.2 in [13] it follows that

$$\begin{aligned}3(1+\lambda)K_{\mathbb{C}}(\sigma) \geq (1+2\lambda)(a^2(R_{1331} + R_{2332}) + b^2(R_{1441} + R_{2442}))\\ - (1-\lambda)(R_{1221} + a^2b^2R_{3443}).\end{aligned}$$

Now for any $A, B \geq 0$ satisfying $A + B = 2\lambda + 1$ one can use λ-pinched flag curvature to get

$$\begin{aligned}3(1+\lambda)K_{\mathbb{C}}(\sigma) &\geq (A+B)(a^2(R_{1331} + R_{2332}) + b^2(R_{1441} + R_{2442}))\\ &\quad - (1-\lambda)(R_{1221} + a^2b^2R_{3443}),\\ &> A\lambda(a^2(R_{1221} + R_{2112}) + b^2(R_{1221} + R_{2112}))\\ &\quad B\lambda(a^2(R_{4334} + R_{4334}) + b^2(R_{3443} + R_{3443}))\\ &\quad - (1-\lambda)(R_{1221} + a^2b^2R_{3443}),\\ &= (2A\lambda - 1 + \lambda)R_{1221} + (2B\lambda - a^2b^2(1-\lambda))R_{3443},\\ &\geq (2A\lambda - 1 + \lambda)R_{1221} + (2B\lambda - (1-\lambda)/4)R_{3443}.\end{aligned}$$

Using $A = (16\lambda^2 + 9\lambda - 1)/(8\lambda)$ and $B = (1-\lambda)/(8\lambda)$ then gives the estimate

$$K_{\mathbb{C}}(\sigma) > \frac{16\lambda^2 + 13\lambda - 5}{12(1+\lambda)} R_{1221}.$$

□

Chapter 5

Existence Theorems

In this concluding chapter, Morse theory for the α-energy and the index estimates from chapter 3 will be used to prove the existence of minimal surfaces with index between $n-2$ and $2n-5$. The first step is to calculate the equivariant cohomology groups for $\mathcal{M} = L_1^{2\alpha}(S^2, S^n)$. This is done using the Morse-Bott theory for the functional E_α when S^n is given the round metric. This calculation is then used to show the existence of critical points for a perturbation of E_α to an equivariant Morse function for any metric on S^n. As the perturbation approaches zero and α approaches 1, these critical points will converge to bubble trees of minimal surfaces[15]. When the metric on S^n has positive half-isotropic curvature, the critical points of index lower than $2n-4$ cannot bubble. If the stronger condition of $\frac{1}{2}$-controlled half-isotropic curvature is imposed critical points of index lower than $2n-4$ cannot converge to branched covers of other minimal S^2 or minimal $\mathbb{R}P^2$. The module structure of the equivariant cohomology can then be used to rule out double covers of minimal $\mathbb{R}P^2$. Thus, these critical points must converge to unique minimal spheres.

Definition 5.0.5. Let $\mathcal{M}$ be a Banach manifold. If $F : \mathcal{M} \to \mathbb{R}$ is C^2 then

1. F is *Morse* if its critical locus consists of isolated, nondegenerate critical points,

2. F is *Morse-Bott* if its critical locus consists of isolated, nondegenerate critical submanifolds,

3. F is *equivariant Morse* if it is invariant under the action a Lie group, G on $\mathcal{M}$, and its critical locus consists of isolated, nondegenerate critical submanifolds, each an orbit for the action of G.

If $F : \mathcal{M} \to \mathbb{R}$ is a Morse function, and $\mathcal{M}^a = F^{-1}((-\infty, a])$ then the following theorems hold [4]

Theorem 5.0.6. *If the interval $[a, b]$ contains no critical values for F, then $\mathcal{M}^a$ is a strong deformation retract of $\mathcal{M}^b$.*

Theorem 5.0.7. *Suppose that $[a, b]$ contains a single critical value c and there is exactly one Morse nondegenerate critical point p such that $F(p) = c$, and*

$$\textit{Morse index}(p) = \lambda.$$

Then $\mathcal{M}^b$ is homotopy equivalent to $\mathcal{M}^a$ with a λ-handle attached.

If $F : \mathcal{M} \to \mathbb{R}$ is Morse-Bott, then a critical value of F, c, will correspond to a critical submanifold, $\mathcal{M}_c = F^{-1}(c)$. To define a similar theorem for a Morse-Bott function requires a definition of the normal space for a critical submanifold. The following definition is used in [12] for $\mathcal{M} = L_1^{2\alpha}(S^2, S^n)$,

$$N_f\mathcal{M}_c = \left\{ X \in T_f\mathcal{M} | \, \forall Y \in T_f\mathcal{M}_c, \langle X, Y \rangle_{L_1^2(\mu)} = 0 \right\},$$

where $\langle \cdot, \cdot \rangle_{L_1^2(\mu)}$ is the weighted L_1^2 metric with measure $\mu = (1 + |df|^2)^{\alpha-1} dA$. Explicitly,

$$\langle X, Y \rangle_{L_1^2(\mu)} = \int_{S^2} \left(\langle X, Y \rangle + \langle \nabla X, \nabla Y \rangle \right) \left(1 + |df|^2\right)^{\alpha-1} dA.$$

Let $\mathcal{A}$ be the operator given by

$$d^2F(f)(X, Y) = \langle \mathcal{A}X, Y \rangle_{L_1^2(\mu)},$$

Then the negative normal space, $N_f^-\mathcal{M}_c$ is the negative eigenspace of $\mathcal{A}$. If λ is the rank of $N^-\mathcal{M}_c$, then $\mathcal{M}_c$ has Morse index λ.

Theorem 5.0.8. *Suppose that $[a, b]$ contains a single critical value c and the critical points with value c form a Morse nondegenerate critical submanifold, $\mathcal{M}_c$. Then $\mathcal{M}^b$ is homotopy equivalent to $\mathcal{M}^a$ with the disk bundle $D(N^-\mathcal{M}_c)$ attached along $S(N^-\mathcal{M}_c)$.*

Therefore there is an isomorphism on cohomology,

$$H^k(\mathcal{M}^b, \mathcal{M}^a; R) \cong H^k(D(N^-\mathcal{M}_c), S(N^-\mathcal{M}_c); R),$$

for any coefficient ring R. If $N^-\mathcal{M}_c$ is orientable over $\mathcal{M}_c$, then the Thom isomorphism theorem implies

$$H^k(D(N^-\mathcal{M}_c), S(N^-\mathcal{M}_c); R) \cong H^{k-\lambda}(\mathcal{M}_c; R).$$

If F is G-invariant and the action of G on $\mathcal{M}_c$ preserves the orientation of $N^-\mathcal{M}_c$ there are isomorphisms on equivariant cohomology,

$$H^k_G(\mathcal{M}^b, \mathcal{M}^a; R) \cong H^{k-\lambda}_G(\mathcal{M}_c; R),$$

for any coefficient ring R. The equivariant cohomology is constructed using the universal principal bundle $EG \to BG$. The product space $\mathcal{M} \times EG$ has a free diagonal action by G. For $\mathcal{M}_G = (\mathcal{M} \times EG)/G$, the equivariant cohomology is then

$$H^k_G(\mathcal{M}; R) = H^k(\mathcal{M}_G; R).$$

5.1 Topology of the Mapping Space

When S^n is given the round metric, the minimal surfaces can be described explicitly[2, 5]. In particular, the energy of any minmal surface is an integer multiple of 4π. The minimal surfaces of 0 energy are just the constant maps. Of course these are also α-harmonic, and form a critical submanifold $E_\alpha^{-1}(0) \cong S^n$. The energy 4π minimal surfaces are totally geodesic S^2 given by intersecting a 3-plane in $\mathbb{R}^{n+1}$ with S^n.

Proposition 5.1.1. *If $f : S^2 \to M$ is a harmonic isometric immersion then it is α-harmonic for $\alpha > 1$.*

Proof. if η is the metric on S^2 and g is the metric on M, then as f is an isometric immersion, $f^*g = \eta$, and $\frac{1}{2}|df|^2 = e(f) = \frac{1}{2}tr_\eta f^*g = 1$. Using isothermal coordinates, $\eta = \lambda^2(dx^2 + dy^2)$ and the equation for an α-harmonic map is

$$\nabla_{\partial_x}((1 + |df|^2)^{\alpha-1} f_x) + \nabla_{\partial_y}((1 + |df|^2)^{\alpha-1} f_y) = 0.$$

Since $|df|^2 = 2$, this becomes

$$3^{\alpha-1}(\nabla_{\partial_x} f_x + \nabla_{\partial_y} f_y) = 0.$$

Now, since f is harmonic, this implies that it is also α-harmonic. □

Note that this also follows if $f^*g = c\eta$ for any constant $c \in \mathbb{R}^+$.

Corollary 5.1.2. *Let S^2 and S^n be given round metrics, η and g, respectively. If $f : S^2 \to S^n$ is a center-of-mass zero harmonic map with energy 4π, then f is α-harmonic for $\alpha > 1$.*

Proof. Since f is harmonic and has energy 4π, it can be written as $f = A \circ \iota \circ \zeta$ where $A \in O(n+1)$ is an isometry of S^n, ι is the standard inclusion of S^2 into S^n, and ζ is a conformal or anti-conformal automorphism of S^2. The center-of-mass zero condition, (2.3), implies that $\zeta \in O(3)$ is an isometry of S^2. Thus $f^*g = c\eta$, and f must be α-harmonic. □

If η is the constant curvature metric of total area 1 on S^2, and $f : S^2 \to S^n$ is a center-of-mass zero harmonic map of energy 4π then $f^*g = 4\pi\eta$. Thus $|df|^2 = 8\pi$ and

$$E_\alpha(f) = \frac{1}{2}(1 + 8\pi)^\alpha - \frac{1}{2} = k_\alpha.$$

Theorem 5.1.3. *There is an $\varepsilon_0 > 0$ such that if $E_\alpha(f) < \varepsilon_0$ then f is a constant function. For all $\varepsilon \in (0, \varepsilon_0)$, there exists $\alpha_0 > 1$ such that for any $\alpha \in (1, \alpha_0)$,*

1. *if f is a critical point of E_α and $E_\alpha(f) < 4\pi - \varepsilon$, then f is a constant map,*
2. *if f is a critical point of E_α and $\varepsilon < E_\alpha(f) < 8\pi - \varepsilon$, then f is totally geodesic.*

Proof. The first statement is proved in [16]. To prove part 1 of the second statement, assume to the contrary. Then there is an $\varepsilon \in (0, \varepsilon_0)$ and a sequence $\alpha_i \to 1$ with f_i non-constant critical points for E_{α_i} satisfying $E_{\alpha_i}(f_i) < 4\pi - \varepsilon$. A subsequence of f_i must converge to a bubble tree of harmonic maps. Since this sequence does not have enough energy to bubble, it must be converging to a harmonic map of energy less than 4π. Since the only harmonic maps from S^2 with energy less than 4π are constant maps, f_i must be converging to a constant map, and $E_{\alpha_i}(f_i) \to 0$. By the first statement, this implies that there is an I such that for $i > I$, f_i is a constant map, a contradiction.

To prove part 2 of the second statment, again assume to the contrary. Then there is an $\varepsilon \in (0, \varepsilon_0)$ and a sequence $\alpha_i \to 1$ with $f_i \notin \mathcal{M}_{k_\alpha}$ critical points for E_{α_i} satisfying $4\pi - \varepsilon < E_{\alpha_i}(f_i) < 8\pi - \varepsilon$. A subsequence of f_i must converge to a bubble tree of harmonic maps. Since the maps have energy less than 8π, the bubble tree cannot have more than two vertices. If it has two bubbles, one must have energy at least 4π, the other must have energy less than $4\pi - \varepsilon$. This shows that one of the bubbles must be constant, showing that there is no bubbling. Therefore f_i must converge in C^k, for all k, to a totally geodesic S^2 with energy 4π.

By part 3 of theorem 2.2.2, for large enough i, f_i is in the image of a map η satisfying properties 1 and 2 from the theorem. By corollary 5.1.2, the inclusion map $\mathcal{M}_{k_\alpha} \times [1, \alpha_0) \to L^2_k(S^2, M) \times [1, \alpha_0)$ also satisfies properties 1 and 2 of theorem 2.2.2. Since the map η in theorem 2.2.2 is unique, it must just be the inclusion map, thus f_i is a totally geodesic embedding of S^2 in S^n. □

Proposition 5.1.4. *There is an $\alpha_0 > 1$ such that for $1 < \alpha < \alpha_0$, $\mathcal{M}_{k_\alpha} = E_\alpha^{-1}(k_\alpha)$ is a nondegenerate critical submanifold of index $n-2$ on which the symmetry group $O(3)$ acts freely, and*

$$\mathcal{M}_{k_\alpha}/O(3) \cong Gr_3(\mathbb{R}^{n+1}).$$

Proof. For α sufficiently close to 1, $4\pi < k_\alpha < 8\pi$, therefore by theorem 5.1.3 $\mathcal{M}_{k_\alpha} = E_\alpha^{-1}(k_\alpha)$ is just the totally geodesic embeddings of S^2. Any such embedding is given by composing the inclusion $S^2 \to \mathbb{R}^3$ with an isometric embedding $\mathbb{R}^3 \to \mathbb{R}^{n+1}$. Since an isometric embedding $\mathbb{R}^3 \to \mathbb{R}^{n+1}$ defines a 3-frame of $\mathbb{R}^{n+1}$, $\mathcal{M}_{k_\alpha}$ is just the Stiefel manifold $V_3(\mathbb{R}^{n+1})$. A rotation of S^2 is then equivalent to rotating the 3-frame, so $O(3)$ acts freely on $\mathcal{M}_{k_\alpha}$ and $\mathcal{M}_{k_\alpha}/O(3) \cong Gr_3(\mathbb{R}^{n+1})$.

Now let $f \in \mathcal{M}_{k_\alpha}$ since $|df|^2 = 8\pi$ is constant, $d^2E_\alpha(f) = \alpha(1+8\pi)^{\alpha-1}d^2E(f)$. Since S^n has the constant curvature 1 metric, the second variation formula (2.2) is just

$$d^2E(f)(X,X) = \int_{S^2} \left(|\nabla X|^2 - 2|X|^2 e(f)\right) dA$$

for X a normal variation field, and

$$d^2E(f)(X,X) = \int_{S^2} |\nabla X|^2 dA$$

for X a tangential variation field. Thus tangential fields cannot contribute to the index, but contribute $\dim_{\mathbb{C}}\mathcal{O}(\bar{K}) = 3$ to the nullity. The pulled back metric has area form $e(f)dA$ and constant curvature 1, therefore the eigenvalues of S^2 with this metric are 0 with multiplicity 1, 2 with multiplicity three, and so on. Thus, as the normal bundle is trivial with rank $n-2$ it must contribute $3(n-2) = \dim Gr_3(\mathbb{R}^{n+1})$ to the nullity and $n-2$ to the index. Therefore nullity$(f) = 3(n-1) = \dim(\mathcal{M}_{k_\alpha})$ and index$(f) = n-2$. □

This shows that below the 8π energy level, the only critical submanifolds for E_α are the constant maps, $\mathcal{M}_0$ and the totally geodesic maps, $\mathcal{M}_{k_\alpha}$.

Theorem 5.1.5. *For $0 \leq k < n-2$,*

$$H^k_{O(3)}(\mathcal{M}, \mathcal{M}_0; \mathbb{Z}_2) = 0,$$

and for k satisfying $n-2 \leq k \leq 2n-5$ there are isomorphisms

$$H^k_{O(3)}(\mathcal{M}, \mathcal{M}_0; \mathbb{Z}_2) \cong H^{k-n+2}(Gr_3(\mathbb{R}^{n+1}); \mathbb{Z}_2).$$

Moreover the $H^*(BO(3);\mathbb{Z}_2)$ action on $H^*_{O(3)}(\mathcal{M},\mathcal{M}_0;\mathbb{Z}_2)$ is trivial in degree less than $2n-4$.

Proof. By theorem 5.1.3, for α close to 1, there is an ε so that the only critical points with values between ε and $8\pi-\varepsilon$ are in the nondegenerate critical submanifold $\mathcal{M}_{k_\alpha}$. Since the bundle $N^-\mathcal{M}_{k_\alpha}$ is trivial, theorem 5.0.8 then implies that

$$H^k_{O(3)}(\mathcal{M}^{8\pi-\varepsilon},\mathcal{M}^{\varepsilon};\mathbb{Z}_2) \cong H^{k-n+2}_{O(3)}(\mathcal{M}_{k_\alpha};\mathbb{Z}_2).$$

Since the action of $O(3)$ on $\mathcal{M}_{k_\alpha}$ is free, and $\mathcal{M}_{k_\alpha}/O(3) \cong Gr_3(\mathbb{R}^{n+1})$,

$$H^{k-n+2}_{O(3)}(\mathcal{M}_{k_\alpha};\mathbb{Z}_2) \cong H^{k-n+2}(Gr_3(\mathbb{R}^{n+1});\mathbb{Z}_2).$$

For ε sufficiently small, $\mathcal{M}_0$ is a strong deformation retraction of $\mathcal{M}^{\varepsilon}$, thus

$$H^k_{O(3)}(\mathcal{M}^{8\pi-\varepsilon},\mathcal{M}^{\varepsilon};\mathbb{Z}_2) \cong H^k_{O(3)}(\mathcal{M}^{8\pi-\varepsilon},\mathcal{M}_0;\mathbb{Z}_2).$$

The index estimates of Ejiri [7] show that harmonic maps of energy 8π or higher have Morse index at least $2n-4$. This implies that for α sufficiently close to 1 the index of α-harmonic maps with α energy 8π or higher must also be at least $2n-4$. Therefore for $k \leq 2n-5$,

$$H^k_{O(3)}(\mathcal{M},\mathcal{M}_0;\mathbb{Z}_2) \cong H^k_{O(3)}(\mathcal{M}^{8\pi-\varepsilon},\mathcal{M}_0;\mathbb{Z}_2).$$

The action $H^*(BO(3);\mathbb{Z}_2)$ must also be the same, and since the $O(3)$ acts freely on $\mathcal{M}_{k_\alpha}$, the action of $H^*(BO(3);\mathbb{Z}_2)$ on $H^*_{O(3)}(\mathcal{M}_{k_\alpha};\mathbb{Z}_2)$ must be trivial. □

5.2 Low Index Minimal S^2

The existence of low index minimal surfaces in S^n with metrics that are not round, but have $\frac{1}{2}$-controlled half-isotropic curvature can now be established. By perturbing E_α to an equivariant Morse function and using the $O(3)$ equivariant cohomology of the mapping space calculated in the previous section.

Considering M as isometrically embedded in $\mathbb{R}^N$ for some large N, which is always possible by the Nash embedding theorem, we define a perturbed functional,

$$E_{\alpha,\psi}(f) = E_\alpha(f) + \int_{S^2} f \cdot \psi dA.$$

Proposition 5.2.1. *For α sufficiently close to 1 and a generic map $\psi : S^2 \to \mathbb{R}^N$, the perturbed α-energy, $E_{\alpha,\psi}$ is a Morse function and satisfies condition C.*

This is proved in [11], and in [12] this idea is combined with the arguments found in [18] to show it is possible to perturb E_α to an equivariant Morse function, F. Since the equivariant cohomology of $\mathcal{M}$ does not depend on the metric, the equivariant Morse cohomology for the functional must agree with the one computed for the round metric.

Theorem 5.2.2. *For $n-2 \leq k \leq 2n-5$, F has at least $p_3(\lambda - n + 2)$ critical submanifolds of Morse index λ, each an orbit for the action of $O(3)$.*

Proof. This follows from theorem 5.0.8, the computation of the low degree equivariant cohomology in theorem 5.1.5, and the $\mathbb{Z}_2$ cohomology of $Gr_3(\mathbb{R}^{n+1})$. $\square$

Theorem 5.2.3. *If S^n has a metric with $\frac{1}{2}$-controlled half-isotropic curvature, and if the prime minimal surfaces of M lie on nondegenerate critical submanifolds, each an orbit for the action of*

$$PSL(2, \mathbb{C}) \cup R \cdot PSL(2, \mathbb{C}),$$

then for each λ satisfying $n-2 \leq \lambda \leq 2n-5$, there are at least $p_3(\lambda - n + 2)$ minimal S^2 with Morse index λ.

Proof. Let $\mathcal{C}$ be the collection of critical submanifolds for E of index between $n-2$ and $2n-5$. Let $\bar{\mathcal{M}}$ be the maps with center of mass zero, and set $\bar{\mathcal{C}} = \{K \cap \bar{\mathcal{M}} | K \in \mathcal{C}\}$. For every $K \in \bar{\mathcal{C}}$ use the map η from theorem 2.2.2 to form the collection $\bar{\mathcal{C}}_\alpha = \{\eta(K, \alpha) | K \in \bar{\mathcal{C}}\}$.

For α sufficiently close to 1, $\bar{\mathcal{C}}_\alpha$ must contain every critical point of E_α with index between $n-2$ and $2n-5$. If this were not the case, one could construct a sequence, f_i, of E_{α_i} critical points with $\alpha_i \to 1$ such that $f_i \notin \bigcup \bar{\mathcal{C}}_\alpha$. Since M has positive half-isotropic curvature, the sequence f_i cannot bubble and must converge in C^k, for all k, to a harmonic map f with

$$\text{Morse index}(f) \leq \liminf_{i \to \infty} \text{Morse index}(f_i).$$

In fact, since the sequence cannot bubble, it will not have any necks in whcih index can be lost. Therefore Morse index$(f) = \lim_{i \to \infty}$ Morse index(f_i). For large i the maps f_i are C^2 close to a center of mass zero harmonic map, f. Thus they must be in the image of η for the critical set K containing f, giving a contradiction.

By a similar argument, the critical orbits in $\bar{\mathcal{C}}_\alpha$ can be perturbed to give all the critical orbits of index between $n-2$ and $2n-5$ for a perturbation of E_α to an equivariant Morse function F. By the previous theorem, there are at least $p_3(\lambda-n+2)$ critical orbits for F of index λ for each $n-2 \leq \lambda \leq 2n-5$. The critical orbits can now be split into two classes. In one class, $\mathcal{A}$ are the critical orbits on which $O(3)$ acts freely, these orbits are homeomorphic to $O(3)$. In the other class, $\mathcal{B}$, are the critical orbits with $\mathbb{Z}_2$ isotropy, these orbits are homeomorphic to $SO(3)$. The orbits in both classes represent cohomology classes in the Morse cohomology. The action of $H^*(BO(3);\mathbb{Z}_2) = P[w_1, w_2, w_3]$ on the orbits in $\mathcal{A}$ is trivial, but for the orbits in B, w_1 does not act trivially [12]. Thus the class $\mathcal{B}$ does not contribute to the cohomology of $H^*_{O(3)}(\mathcal{M}, \mathcal{M}_0; \mathbb{Z}_2)$, as the action of $H^*(BO(3);\mathbb{Z}_2)$ on $H^*_{O(3)}(\mathcal{M}, \mathcal{M}_0; \mathbb{Z}_2)$ is trivial in degree less than $2n-4$. $\square$

In [12], it is also shown that branched double covers of prime minimal S^2 must be acted on nontrivially by w_2, and this case can be ruled out as well. Thus theorem 5.2.3 will also hold for metrics with only $\frac{1}{3}$-controlled half-isotropic curvature.

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