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Essays on Robust Mechanism Design

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Wanchang Zhang

Committee in charge:

Professor Songzi Du, Co-Chair
Professor Joel Sobel, Co-Chair
Professor Snehal Banerjee
Professor Bradyn Breon-Drish
Professor Joel Watson

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TABLE OF CONTENTS

LIST OF FIGURES

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VITA

2015	B. A. in Economics, Xiamen University
2017	M. A. in Economics, Kyoto University
2023	Ph. D. in Economics, University of California San Diego

ABSTRACT OF THE DISSERTATION

Essays on Robust Mechanism Design

by

Wanchang Zhang

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Professor Songzi Du, Co-Chair
Professor Joel Sobel, Co-Chair

This dissertation studies the *robust* design of institutions when the mechanism designer does not fully know the environment.

In Chapter 1, I construct a novel *random double auction* as a robust bilateral trading mechanism for a profit-maximizing intermediary who facilitates trade between a buyer and a seller. It works as follows. The intermediary publicly commits to charging a *fixed commission fee* and *randomly* drawing a *spread* from a uniform distribution. Then the buyer submits a bid price and the seller submits an ask price simultaneously. If the difference between the bid price and the ask price is greater than the realized spread, then the asset is transacted at the *midpoint price*, and each pays the intermediary half of the fixed commission fee. Otherwise, no trade

takes place, and no one pays or receives anything. I show that the random double auction is a *dominant-strategy mechanism*, always guarantees a *positive* profit, and *maximizes* the profit guarantee across all dominant-strategy mechanisms.

In Chapter 2, I study the single-unit auction design when the seller is assumed to have information only about the marginal distribution of a generic bidder's valuation, but does not know the correlation structure of the joint distribution of bidders' valuations. For the two-bidder case, a *second-price auction with uniformly distributed random reserve* maximizes the worst-case expected revenue across *all* dominant-strategy mechanisms. For the N-bidder ($N \geq 3$) case, a *second-price auction with Beta-distributed random reserve* is a maxmin mechanism among *standard* (only a bidder with the highest bid could win the good) dominant-strategy mechanisms.

In Chapter 3, I study the auction design of selling *multiple* goods when the seller only knows the *upper bounds* of bidders' values for each good and has no additional distributional information. The designer takes a minimax regret approach. The expected regret from a mechanism given a joint distribution over value profiles and an equilibrium is the difference between the full surplus and the expected revenue. I find that a *separate second-price auction with random reserves* minimizes her worst-case expected regret across all participation-securing Bayesian mechanisms.

INTRODUCTION

The classic mechanism design theory assumes that the designer knows the agents' information structure. The design goal is to maximize some objective, such as profit, under the known information structure. While the classic theory is beautiful and influential, the optimal mechanism is sensitive to the detailed assumptions about the information structure. Robert Wilson criticized the classic theory for its heavy reliance on the strong common knowledge assumption of the environment. Instead, I assume that the designer only has *partial* knowledge of the information structure, and evaluates a mechanism by its worst-case performance under this partial knowledge. I construct a mechanism that maximizes the worst-case performance for the designer. This approach leads to the discovery of *novel* and attractive mechanisms along with new economic insights. This dissertation consists of three essays that study robust mechanism design problem in distinct contexts. The first essay studies the robust design in the context of bilateral trade. The remaining two essays studies the robust design in the context of auction.

In Chapter 1, I study the design of a trading platform for a profit-maximizing intermediary who facilitates trade between a buyer and a seller. The intermediary makes profit from the difference between what the buyer pays and what the seller receives. The intermediary can be a brokerage firm that typically gets compensation by means of commissions in the stock market, an automobile dealer who charges dealer fees in the car market, or a market maker who earns profit through the bid-ask spread in the over the counter (OTC) market. I consider the correlated private value environment. The intermediary only knows the *ex-ante gain from trade*, but does not know the joint distribution of the traders' private values[1]. The intermediary considers the class of all *dominant-strategy mechanisms*. The intermediary evaluates a trading mechanism by its worst-case expected profit (referred to as "profit guarantee") across all feasible value distributions consistent with the known ex-ante gain from trade. The intermediary seeks a trading mechanism that maximizes the profit guarantee across all dominant-strategy mechanisms.

[1]That is, the intermediary knows neither the marginal distributions nor the correlation structure except for the ex-ante gain from trade.

The main contribution is the construction of a *random double auction* as a robust two-sided trading mechanism, which runs as follows.

Step 0: *fixed commission fee*. The intermediary publicly commits to charging a fixed commission fee $r \in (0,1)$, where 1 is the normalized maximum value for each trader.

Step 1: *uniformly random spread*. The intermediary publicly commits to randomly drawing a spread s uniformly on $[r,1]$. Then a random spread is drawn whose realization is not observed by either the buyer or the seller. The buyer and the seller both know the fixed commission fee r and the uniform distribution on $[r,1]$ from which the random spread is drawn.

Step 2: *midpoint transaction price*. The buyer submits a bid price b, and the seller submits an ask price a, simultaneously. If the difference between the bid price and the ask price is greater than the realized spread, or $b-a>s$, then the seller sells the asset to the buyer at the midpoint price $\frac{b+a}{2}$, and each pays the intermediary half of the fixed commission fee $\frac{r}{2}$. Otherwise, no trade takes place, and no one pays or receives anything.

Conditional on trading, the random double auction reduces to a familiar double auction, as the transaction price is the midpoint of the bid price and the ask price. The main novelty is the uniformly random spread, which makes the trade take place randomly.

The random spread plays a *dual* role. First, the random spread decreases the traders' incentive to cheat. Without a random spread, the buyer has an incentive to submit a bid price lower than his true value, as he would lower the transaction price by doing so. However, with a random spread, if the buyer submits a lower bid price, then the trade will also take place with a lower probability, limiting the profit from deviating to a lower bid price. A similar argument can be made for the seller. A judiciously chosen random spread — uniformly random spread — eliminates the traders' incentive to cheat, and makes the mechanism strategy-proof.

Second, the random spread hedges against uncertainty about the information structure. The intermediary is *indifferent* to any feasible value distribution whose support is contained in the set of value profiles where the difference between values is higher than the fixed commission fee. This property holds because the ex-post profit from any value profile in the support of an aforementioned value distribution is linear, as uniformly random spread translates into a linear trading probability, and the profit conditional on trading is the fixed commission fee. The profit guarantee of the random double auction is always positive. In contrast, the profit guarantee of any deterministic dominant-strategy mechanism is zero if the known ex-ante gain from trade is weakly below one half. Furthermore, the random double auction (with a specific fixed commission fee) has the highest profit guarantee across all dominant-strategy mechanisms.

The remaining two chapters study other robust mechanism design problems, and offer variations of random double auction. These chapters study a *one-sided* auction market, whereas Chapter 1 studies a *two-sided* bilateral trade market . In addition, these studies differ in

A) what designer knows (known ex-ante gain from trade vs. known marginal distributions vs. known upper bounds),

B) objective function (maxmin profit vs. minimax regret),

C) solution concept (dominant strategy vs. Bayesian Nash Equilibrium),

D) dimensionality (single-unit good vs. multiple goods).

In Chapter 2, I study the single-unit auction design for a profit-maximizing seller. I consider the correlated private value environment. The seller only knows the marginal distribution of a generic bidder's value, but does not know the correlation structure among bidders' values. The seller considers the class of all dominant-strategy mechanisms. She evaluates a mechanism's performance by its worst-case expected profit across all possible joint distributions consistent with the know marginal distribution, and seeks a maxmin mechanism that maximizes the profit guarantee.

The main result is that *a second-price auction with uniformly random reserve price* is a maxmin mechanism for the two-bidder case, provided that the marginal distribution satisfies certain regularity conditions. The uniformly random reserve price makes the mechanism exhibit *a full-insurance property* : The expected profit is the same across all joint distributions consistent with the known marginal distribution, making it a good candidate for a maxmin mechanism. The regularity conditions capture a wide range of heavy-tailed distributions, which are observed in many real-world auctions. I partially extend the result to arbitrary number of bidders: A second-price auction with a Beta-distributed random reserve price is a maxmin mechanism among standard dominant-strategy mechanisms, whose defining property is that only a bidder with the highest bid could win the good.

In Chapter 3, I study the auction design of selling *multiple* goods when the seller only knows the *upper bounds* of bidders' values for each good and has no additional distributional information. Here the maxmin expected profit objective is uninteresting, as the worst case is simply that the bidders' values for all goods are zeros for sure. Instead, the seller takes a minimax regret approach. The seller considers all participation-securing mechanisms. The expected regret from a mechanism given a joint distribution over value profiles and an equilibrium is defined as the difference between the full surplus and the expected profit. The seller seeks a minimax regret mechanism that minimizes her worst-case expected regret across all possible joint distributions over value profiles and all equilibria.

The main result is that *a separate second-price auction with random reserve prices* is a minimax regret mechanism for general upper bounds. Under this mechanism, the seller holds a separate auction for each good; the formats of these auctions are second-price auctions with random reserve prices. To see the intuition behind separation, it is instructive to consider another mechanism of auctioning the *grand* bundle (only the bundle of all goods is auctioned). I argue that this mechanism may result in a high regret. Consider a three-good three-bidder example and an extremely *asymmetric* value profile in which each bidder values a different good (assuming that the upper bound on each bidder's values for each good is 1) : $(v_1^1, v_1^2, v_1^3) = (1,0,0), (v_2^1, v_2^2, v_2^3) = (0,1,0), (v_3^1, v_3^2, v_3^3) = (0,0,1)$.[2] The designer will *lose all but one good* if auctioning the grand bundle: She can at most obtain a profit of 1 from one of the goods but will suffer a regret of 2 from losing the other goods. In contrast, separation can guarantee a good regret performance for each good. Intuitively, auctioning the grand bundle performs just like selling one good at this value profile, while selling separately allows

[2]The superscript represents the good, and the subscript represents the bidder.

the designer to earn more. Furthermore, the same argument implies that *partial* bundling (a mechanism in which a bundle of some goods are auctioned) may perform worse than separate selling.

Chapter 1

Random Double Auction: A Robust Bilateral Trading Mechanism

1.1 Introduction

1.1.1 Background and Motivation

At every moment, a huge amount of trades are facilitated by intermediaries charging fees for their intermediary services in matching buyers with sellers. For example, stocks are sold through a trading platform that typically gets compensation by means of commissions; cars are sold through an automobile dealer who charges dealer fees; many bonds, commodities and derivatives are sold in the over-the-counter market (OTC) where a market maker earns profits through the bid-ask spread.

There are many situations in which the uncertainty about the value of the asset being traded is large, e.g., a newly public stock, and Tesla's new model. Intermediaries may then know little of the concerned parties' willingness to trade and only have an overall estimate about it. Given the large uncertainty towards the two-sided market, it is natural for the intermediary to seek for a trading mechanism that *guarantees* a good profit. How should a profit-maximizing intermediary design trading rules in such situations? Would the intermediary still be able to guarantee a positive profit and thus have strict incentives to offer intermediary services?

To answer these questions, I study the design of profit-maximizing trading mechanisms for the *two-sided* market when the intermediary has *limited* knowledge about the value distribution of the buyer and the seller. Specifically, I assume that the intermediary knows only the *ex-ante gain from trade*[1], denoted by GFT, but does not know the joint distribution of the traders' private values[2]. A joint distribution consistent with the known ex-ante gain from trade is referred to as a *feasible value distribution*. The intermediary considers the class of all *dominant-strategy mechanisms*[3]. Dominant-strategy mechanisms are attractive because the intermediary can predict trading behavior without making assumptions about the traders' beliefs. The intermediary evaluates a mechanism's performance by the expected profit under the dominant-strategy equilibrium in the worst case across all feasible value distributions, referred to as the *profit guarantee*, and seeks a mechanism that maximizes the profit guarantee across all dominant-strategy mechanisms, referred to as a *maxmin trading mechanism*.

Let me comment briefly on the maxmin modeling approach. At a high level, the maxmin modeling approach addresses an important issue of the classic mechanism design theory, in which the designer is assumed to know the agents' information structure and maximize some objective under her known information structure, e.g., Myerson (1981), Myerson and Satterthwaite (1983) and Crémer and McLean (1985, 1988). Although the classic theory is beautiful and influential, the optimal mechanism is sensitive to the detailed assumptions about the information structure. In contrast, the maxmin modeling approach leads to an answer that depends *less* on the details about the information structure.

Several motivations can be offered for the assumption about the intermediary's limited

[1]The ex-ante gain from trade is defined to be $E[\max\{\text{Buyer's value} - \text{Seller's value}, 0\}]$, where the expectation is taken with respect to the joint distribution of the traders' private values.

[2]That is, the intermediary knows neither the marginal distributions nor the correlation structure except for the ex-ante gain from trade, which is a summary statistics of the joint distribution.

[3]A trading mechanism is a dominant-strategy mechanism if each trader has a strategy that is optimal and yields a non-negative ex-post payoff, regardless of the other trader's strategy.

knowledge. First, the ex-ante gain from trade is a simple summary statistics, whereas the joint distribution is a high-dimensional object. Therefore, it is relatively easy to estimate the ex-ante gain from trade, while obtaining an accurate estimate of the whole joint distribution often requires unrealistically many data about the traders' joint value profiles. In addition, the knowledge of the ex-ante gain from trade is arguably the minimal amount of information under which, as I will show, one obtains a non-trivial answer. Therefore, this model can be viewed as a natural *benchmark*. More importantly, this assumption leads to the discovery of a *novel* trading mechanism with *appealing* properties along with new economic insights.

1.1.2 Results

The main contribution is the construction of a ***random double auction*** as a robust bilateral trading mechanism. It works as follows.

Step 0: ***Fixed commission fee***. The intermediary publicly commits to charging a fixed commission fee $r \in (0,1)$[4], where 1 is the normalized maximum value for each trader.

Step 1: ***Uniformly random spread***. The intermediary publicly commits to randomly drawing a spread s uniformly on $[r,1]$. Then a random spread is drawn whose realization is not observed by either the buyer or the seller. The buyer and the seller both know the fixed commission fee r and that the random spread is drawn from the uniform distribution on $[r,1]$.

Step 2: ***Midpoint transaction price***. The buyer submits a bid price b, and the seller submits an ask price a, simultaneously. If the difference between the bid price and the ask price is greater than the realized spread, or $b-a>s$, then the seller sells the asset to the buyer at the midpoint price $\frac{b+a}{2}$, and each pays the intermediary half of the fixed commission fee $\frac{r}{2}$. Otherwise, no trade takes place, and no one pays or receives anything.

Under this mechanism, the trade takes place randomly. Conditional on trading, the mechanism reduces to a *double auction*, as the transaction price is the midpoint of the bid price

[4] The optimal fixed commission fee r is determined by the known ex-ante gain from trade, details of which are given when deriving the profit guarantee of the random double auction.

and the ask price; in addition, the intermediary earns r as a fixed total commission from both parties. Although both traders have to pay half of the fixed commission fee to the intermediary conditional on trading, this mechanism is ex-post individually rational: Each trader's ex-post payoff is always non-negative by being honest, regardless of the other trader's submission. This is because the lower bound of the random spread is the fixed commission fee.

The random double auction is a trading mechanism that combines three features: A double auction, a fixed commission fee, and a random spread. Indeed, the first two features are *familiar* in the real world. First, a double auction is widely used in stock exchanges as well as in dark pools[5], e.g., the New York Stock Exchange (NYSE) and the Tokyo Stock Exchange (TSE) use a double auction to determine the opening prices; block-trading dark pools such as Liquidnet or POSIT typically match orders at the midpoint of the prevailing bid-ask prices (Duffie and Zhu, 2017). Second, brokerage firms often adopt the fixed-commission practice, e.g., Interactive Brokers offers fixed-commission plans for many financial assets[6]; E*TRADE charges a fixed commission per contract for futures contracts[7]. The main novelty of the random double auction comes from the third feature— a random spread[8]. Importantly, the random spread both *disciplines* the traders for cheating and *hedges* against uncertainty towards the traders' information structure. I next illustrate the key properties of the random double auction along with elaborating the *dual* role played by the random spread.

Strategy-proofness. The random double auction is strategy-proof (Proposition 1), i.e.,

[5]A dark pool is a privately organized financial forum or exchange for trading securities that are not accessible by the investing public. Dark pools came about primarily to facilitate block trading involving a huge number of securities.

[6]Interactive Brokers is a brokerage firm. From its official website (interactivebrokers.com), it offers a fixed-commission plan that charges $0.005 per share for stocks in US; it also offers a fixed-commission plan that charges $ 0.065 per contract for NANOS Options on CBOE.

[7]E*TRADE is also a brokerage firm. From its official website (us.etrade.com), it charges $1.5 per contract for futures contracts.

[8]The spread s in the random double auction is closely related to but different from the "bid-ask spread", also called "market-maker spread", which refers to the difference between the price at which a market-maker is willing to buy an asset and the price at which she is willing to sell the asset. Similar to the spread s, the bid-ask spread determines whether a trade takes place given a bid-ask pair. The bid-ask spread is an important source of profit for a market maker when she facilitates a trade successfully. In contrast, the spread s only determines whether a trade takes place, but does not affect the profit conditional on trading.

it is a dominant strategy for the buyer (resp, the seller) to submit a bid price (resp, an ask price) equal to his private value. This is a priori surprising, as conditional on trading, the mechanism reduces to a double auction, and a double auction per se is not strategy-proof (Chatterjee and Samuelson, 1983). This is because, the buyer (resp, the seller) has an incentive to submit a bid price (resp, an ask price) lower (resp, higher) than his true value to lower (resp, raise) the transaction price. A random spread makes it costly for the traders to cheat. This is because, with a random spread, if the buyer (resp, the seller) submits a lower bid price (resp, a higher ask price), then the trade will take place with a lower probability, which limits the buyer's (resp, the seller's) payoff from deviating to a lower bid price (resp, a higher ask price). Remarkably, a uniformly random spread eliminates the traders' incentive to cheat and makes the mechanism strategy-proof. To see this, note that the buyer's ex-post payoff from submitting a bid price b when his true value is v_B and the seller submits an ask price a (assuming trade takes place with a positive probability) is

$$\frac{b-a-r}{1-r} \cdot \left(v_B - \frac{b+a+r}{2} \right), \tag{1}$$

where the first term is the trading probability and the second term is the ex-post payoff of the buyer conditional on trading. Note that (1) is a quadratic function in the bid price b. It is straightforward to show that $b = v_B$ maximizes his ex-post payoff regardless of the seller's submitted ask price a. Similarly, truth-telling maximizes the ex-post payoff for the seller regardless of the buyer's submitted bid price b.

Positive profit guarantee. The profit guarantee of the random double auction is always positive (Proposition 2). In contrast, as I will show in Theorem 5, the profit guarantee of any deterministic dominant-strategy mechanism is *zero* if the known ex-ante gain from trade is weakly below one half. The key step to derive the profit guarantee of the random double auction is to show the convexity of the ex-post profit function in the ex-post gain from trade. Therefore, a point mass on the value profile $(GFT, 0)$ minimizes the expected profit across all feasible value distributions.

Furthermore, the random double auction exhibits a hedging property: The intermediary is *indifferent* to any feasible value distribution whose support is contained in the set of value profiles where the difference between values is higher than the fixed commission fee, which renders the random double auction a good candidate for a maxmin trading mechanism. This property holds because the ex-post profit from any value profile in the support of an aforementioned feasible value distribution is linear. Indeed, any aforementioned feasible value distribution minimizes the expected profit under the random double auction.

Optimal profit guarantee. The random double auction gives the optimal profit guarantee across all dominant-strategy mechanisms (Theorem 1). To show this, I construct a feasible value distribution, and show that the profit guarantee of the random double auction is the tight upper bound on the expected profit across all dominant-strategy mechanisms against the constructed value distribution. In addition, this upper bound is hit by the random double auction.

The constructed value distribution is a *symmetric triangular value distribution* that can be described as follows. The support is a symmetric triangular subset in the set of joint values, which is the same as the trading region[9] of the random double auction. The marginal distribution for the buyer is a combination of a uniform distribution on $(r, 1)$ and an atom on 1, while for the seller is a combination of a uniform distribution on $(0, 1-r)$ and an atom on 0. The conditional distribution is some truncated generalized Pareto distribution with an atom on 1 (resp, 0) for the buyer (resp, the seller).

There are many different ways to model the intermediary's limited knowledge about the value distribution, and the results can be extended to several other models of the limited knowledge. For the model where the intermediary knows only the difference between the expectations of the traders' values, I show that the random double auction remains a maxmin trading mechanism. For the model where the intermediary knows only the expectations of the traders' values, I show that the random double auction remains a maxmin trading mechanism for

[9] I refer to the set of value profiles in which trade takes place with a positive probability as the trading region.

the symmetric[10] informational environment. For the asymmetric informational environment, I show that a generalized random double auction is a maxmin trading mechanism. It generalizes the random double auction in that it approximates the random double auction as the asymmetric informational environment approximates the symmetric one.

Randomized trading is a salient property of the random double auction. This requires the intermediary to have *full commitment power*, which is a standard assumption in the mechanism design literature (e.g., Myerson (1981)). However, in practice, it is hard for the traders to check whether the randomization is done according to the specified trading rule. The traders then may not trust the specified randomization. This motivates the search for a trading mechanism that maximizes the profit guarantee across all deterministic dominant-strategy mechanisms. Such a trading mechanism is referred to as a *maxmin deterministic trading mechanism*. I characterize the class of maxmin deterministic trading mechanisms for any informational environment with a non-trivial profit guarantee (Theorem 5). Examples of maxmin deterministic trading mechanisms include a *linear trading mechanism*, in which trade takes place with probability one if and only if the difference between the bid price and the ask price exceeds a threshold, and a *double posted-price trading mechanism*, in which trade takes place with probability one if and only if the bid price exceeds a threshold and the ask price falls short of a threshold.

In addition, I extend my result to a more general model in which the intermediary can hold the asset. That is, the sum of the traders' allocations is only required to be weakly less than 1. I show that the random double auction remains a maxmin trading mechanism (Theorem 6). Finally, I apply my result to an information design problem in which a financial regulator can choose a probability distribution of the value profile of the buyer and the seller to maximize their welfare. The intermediary, after observing the choice of the distribution but not the realized joint values, designs a profit-maximizing trading mechanism across all dominant-strategy mechanisms. I show that the symmetric triangular value distribution is a solution to this financial regulator's

[10] Roughly speaking, the (a)symmetric information environment is one where the two-sided markets have (non-)identical willingness to trade on average.

information design problem (Theorem 7).

The remainder of the introduction discusses the related literature. Section 1.2 presents the model. Section 1.3 characterizes the main results. Section 1.4 characterizes the results for other models of limited knowledge. Section 1.5 characterizes the class of maxmin deterministic trading mechanisms. Section 1.6 extends and discusses the main results. Preliminary analysis and omitted proofs are in the Appendix.

1.1.3 Related Literature

This paper is related to the classic mechanism design literature. Myerson and Satterthwaite (1983) (henceforth MS) study the design of optimal bilateral trading mechanisms assuming the intermediary knows the distribution of the traders' private values and that these values are independently distributed. In contrast, the intermediary in my paper knows only the ex-ante gain from trade, but does not know the joint distribution of the traders' values. Importantly, I permit correlation between values. The intermediary in MS maximizes expected profit, whereas the intermediary in my paper maximizes the worst-case expected profit. The optimal trading mechanism in MS is deterministic, provided that some regularity conditions hold, whereas the maxmin trading mechanism in my paper involves randomized trade. Moreover, the optimal trading mechanism in MS is in general complicated. Under their mechanism, the trade takes place if and only if the buyer's virtual value is greater than the seller's one. These virtual values, however, depend on the fine details of the value distributions, and are non-linear functions of the traders' values in general[11]. In contrast, the maxmin trading mechanism in my paper is simple. Under the random double auction, the trade takes place if and only if the difference between the traders' values is greater than a uniformly random spread.

This paper contributes to the literature on robust mechanism design. One of the main differences is that I focus on a two-sided market, whereas most of the literature focuses on a

[11]Except for a special circumstance in which both traders' value are uniformly distributed.

one-sided market.

Carrasco et al. (2018) study the design of profit-maximizing selling mechanisms when a seller faced with a single buyer only knows the first n moments of the buyer's value distribution (n can be any positive integer), and solve the problem in which the seller only knows the expectation of the buyer's value as a special case. Indeed, their problem in the special case is equivalent to the intermediary's problem when she knows the ex-ante gain from trade and the seller's value is commonly known to be zero. This is because the ex-ante gain from trade is the same as the expectation of the buyer's value if the seller's value is zero. In contrast, my paper studies the intermediary's problem when she knows only the ex-ante gain from trade. Importantly, there is two-sided private information in my paper. This adds complications to the analysis in two ways. First, the mechanism in my paper has to respect the seller's incentive constraint, in addition to the buyer's one. Second, the intermediary is faced with a stronger "adversary" in my paper: The adversary can carefully choose the correlation structure between the traders' values to minimize the expected profit, in addition to choosing the distribution of the buyer's value. Indeed, the worst value distribution in my paper has a rather intricate correlation structure exhibiting a particular positive correlation.

Zhang (2022a) considers a model of one-sided auction design in which the designer (the auctioneer) knows the marginal distribution of each bidder's value but does not know the correlation structure. He finds that the second-price auction with the uniformly random reserve price is a maxmin auction across all dominant-strategy mechanisms under certain regularity conditions for the two-bidder case. In contrast, the designer (the intermediary) in this paper knows less: She does not know the marginal distribution of each trader's value, in addition to not knowing the correlation structure between the traders' values. Methodologically, both papers construct worst value distributions to proceed the analysis. However, the construction of the worst value distribution is more involved in this paper: It requires me to solve a partial integral equation in addition to ordinary differential equations.

There are other papers seeking robustness to value distributions in a one-side market, e.g., Auster (2018), Bergemann and Schlag (2011), Carroll (2017), Che and Zhong (2021). A separate strand of papers focuses on the case in which the designer does not have reliable information about the agents' hierarchies of beliefs about each other while assuming the knowledge of the payoff environment, e.g., Bergemann and Morris (2005), Chung and Ely (2007), Chen and Li (2018), Bergemann et al. (2016, 2017, 2019), Du (2018), Brooks and Du (2021), Libgober and Mu (2021), Yamashita and Zhu (2018).

This paper contributes to the double auction literature. Chatterjee and Samuelson (1983) analyze the simplest and most well-known double auction mechanism: If the bid price is higher than the ask price, then trade takes place, and the transaction price is the midpoint price; otherwise no trade takes place, and no one pays or receives anything. This mechanism has an undesirable property: Both traders have incentives to cheat under this mechanism. McAfee (1992) shows how to make the double auction mechanism strategy-proof when there are many buyers and sellers. However, McAfee's mechanism reduces to "no trade" when there are only one buyer and one seller. McAfee achieves strategy-proofness by making the price paid by any trader invariant to that trader's report conditional on trading.[12] In contrast, under the random double auction, a trader's report can still affect the price paid (midpoint price) conditional on trading. I achieve the strategy-proofness by introducing a random spread, which lowers the trading probability if the buyer (resp, the seller) underbids (resp, overbids) his value.

1.2 Model

1.2.1 Trading Environment

I consider an environment where an asset is traded between two risk-neutral traders through an intermediary. One of the traders is the seller (S), who holds the asset initially, while the other one is the buyer (B), who does not hold the asset initially. I denote by $I = \{S, B\}$ the set

[12] Under McAfee's mechanism, the only way a trader can affect the price paid is by eliminating himself from trading.

of the traders and $i \in I$ is a trader. Each trader i has private information about his value for the asset, which is modeled as a random variable v_i. I denote by V_i the set of possible values of trader i. Throughout, I assume $V_S = V_B$. I assume that V_i is bounded. As a normalization, I assume that $V_i = [0,1]$. The set of possible value profiles is denoted by $V = [0,1]^2$ with a typical value profile v. v_B and v_S may be correlated in an arbitrary way. I denote by π the joint distribution of the value profile. In addition, there is no technical assumption on π. That is, π can be continuous, discrete, or any mixtures. The set of all joint distributions on V is denoted by ΔV.

1.2.2 Knowledge

The intermediary only knows the ex-ante gain from trade GFT, but does not know the joint distribution π. Formally, I denote by

$$\Pi(GFT) = \left\{ \pi \in \Delta V : \int \max\{v_B - v_S, 0\} d\pi(v) = GFT \right\}$$

the collection of joint distributions that are consistent with the known ex-ante gain from trade. I refer to any $\pi \in \Pi(GFT)$ as a *feasible value distribution*. I assume $GFT \in (0,1)$ to rule out uninteresting cases.

1.2.3 Dominant-strategy Mechanisms

The intermediary seeks a dominant-strategy mechanism. The revelation principle holds, and it is without loss of generality to restrict attention to direct trading mechanisms. A direct trading mechanism (q, t_B, t_S) consists of a trading rule $q : V \to [0,1]$, a payment rule $t_B : V \to \mathbb{R}$ and a transfer rule $t_S : V \to \mathbb{R}$.[13] The buyer submits a bid price b and the seller submits an ask price a simultaneously to the intermediary. Upon receiving the bid-ask pair (b,a), the buyer obtains the asset with probability $q(b,a)$ and pays $t_B(b,a)$ to the intermediary, while the

[13] q is the probability that the buyer obtains the asset when the asset is indivisible. I allow randomization, which will play a crucial role in my analysis. q can be interpreted as the trading quantity when the asset is divisible.

seller holds the good with the remaining probability $1-q(b,a)$ and receives $t_S(b,a)$ from the intermediary. With slight abuse of notation, I sometimes use the true value profile $v=(v_B,v_S)$ to represent the submitted bid-ask pair because each trader truthfully reports his value in the dominant-strategy equilibrium.

A direct trading mechanism (q,t_B,t_S) is a dominant-strategy mechanism if

$$v_B q(v) - t_B(v) \geq v_B q(v'_B, v_S) - t_B(v'_B, v_S), \quad \forall v \in V, v'_B \in V_B; \tag{$DSIC_B$}$$

$$v_B q(v) - t_B(v) \geq 0, \quad \forall v \in V; \tag{$EPIR_B$}$$

$$v_S(1-q(v)) + t_S(v) \geq v_S(1-q(v_B, v'_S)) + t_S(v_B, v'_S), \quad \forall v \in V, v'_S \in V_S; \tag{$DSIC_S$}$$

$$v_S(1-q(v)) + t_S(v) \geq v_S, \quad \forall v \in V. \tag{$EPIR_S$}$$

The set of all dominant-strategy mechanisms is denoted by $\mathcal{D}$.

1.2.4 Objective

I am interested in the intermediary's *expected profit* in the dominant-strategy equilibrium in which each trader truthfully reports his value of the asset. The expected profit of a dominant-strategy mechanism (q,t_B,t_S) under the joint distribution π is $U((q,t_B,t_S),\pi) = \int_{v\in V} t(v) d\pi(v)$ where $t(v) = t_B(v) - t_S(v)$, referred to as the *ex-post profit*. The intermediary evaluates a trading mechanism by its worst-case expected profit over all feasible value distributions. Formally, the intermediary evaluates a trading mechanism (q,t_B,t_S) by its *profit guarantee* $PG((q,t_B,t_S))$, defined as

$$\inf_{\pi \in \Pi(GFT)} U((q,t_B,t_S),\pi). \tag{PG}$$

The intermediary aims to find a trading mechanism (q^*, t_B^*, t_S^*), referred to as a *maxmin trading mechanism*, that maximizes the profit guarantee. Formally, the intermediary solves

$$\sup_{(q,t_B,t_S)\in\mathcal{D}} PG((q,t_B,t_S)). \tag{MTM}$$

1.3 Main Results

Recall the random double auction: Given a submitted bid-ask pair (b,a), if $b-a>s$ in which s is a random spread drawn from the uniform distribution on $[r,1]$ where $r = 1-\sqrt{1-GFT} \in (0,1)$ is the fixed commission fee, then trade takes place at the midpoint price $p = \frac{b+a}{2}$, and each pays the intermediary $\frac{r}{2}$; otherwise, trade does not take place, and no one pays or receives anything.

It is straightforward to show that the random double auction can also be expressed as follows. If $b-a>r$,

$$q^*(b,a) = \frac{1}{1-r}\cdot(b-a-r),$$

$$t_B^*(b,a) = \frac{1}{2(1-r)}\cdot\left[b^2-(a+r)^2\right],$$

$$t_S^*(b,a) = \frac{1}{2(1-r)}\cdot\left[(b-r)^2-a^2\right].$$

If $b-a\le r$,

$$q^*(b,a) = t_B^*(b,a) = t_S^*(b,a) = 0.$$

The trading rule is a *linear* function; the payment rule and the transfer rule are both *quadratic* functions. In addition, this mechanism satisfies the standard weak budget balance property (as in Myerson and Satterthwaite (1983)), i.e., the intermediary never subsidizes the market.

1.3.1 Strategy-proofness

Proposition 1 (Strategy-proofness). *The random double auction is strategy-proof.*

The proof has been given in the introduction. The key idea is to use a random spread to decrease the traders' incentive to deviate in the double auction.

Remark 1 (Dropping the risk-neutral assumption). This idea extends to an environment where the traders' von Neumann-Morgenstern utility function is $u(x) = x^{\alpha}$ where $\alpha > 0$ and $\alpha \neq 1$. Note that the traders are risk-averse (resp, risk-loving) if $\alpha < 1$ (resp, $\alpha > 1$). Now I modify the random spread distribution so that the cumulative distribution function of the random spread s is $\left(\frac{s-r}{1-r}\right)^{\alpha}$ on the same support $[r, 1]$, then the random double auction is again strategy-proof. To see this, note that the non-risk-neutral buyer's ex-post utility from submitting a bid price b when his true value is v_B and the seller submits an ask price a (assuming trade takes place with a positive probability) becomes

$$\left(\frac{b-a-r}{1-r}\right)^{\alpha} \cdot \left(v_B - \frac{b+a+r}{2}\right)^{\alpha},$$

where the first term is the trading probability given the modified random spread distribution and the second term is the ex-post utility of the buyer conditional on trading. It is straightforward that $b = v_B$ maximizes his ex-post utility regardless of the seller's submitted ask price a, as a monotonic transformation preserves the optimal solution. Similarly, truthful-telling maximizes the ex-post utility for the seller regardless of the buyer's submitted bid price b.

1.3.2 Positive Profit Guarantee

Proposition 2 (Positive profit guarantee). *The random double auction has a positive profit guarantee for any non-trivial informational environment. The amount of the profit guarantee is* $\left(1-\sqrt{1-GFT}\right)^2$.

To derive the profit guarantee of a random double auction with a general fixed commission fee, I first show that the ex-post profit earned from an arbitrary value profile (v_B, v_S) is

$\max\left\{\frac{v_B-v_S-r}{1-r}\cdot r,0\right\}$. To see this, note that the profit collected from a bid-ask pair (b,a) if $b-a>r$ is

$$\frac{b-a-r}{1-r}\cdot r, \tag{2}$$

where the first term is the trading probability and the second term is the profit conditional on trading. Importantly, (2) is linear in the difference between the bid and the ask, as uniformly random spread translates into a linear trading probability, and the profit conditional on trading is the fixed commission fee. If $b-a\leq r$, then the trade will not take place and the profit is trivially zero. Recall that the bid price (resp, the ask price) is equal to the true value of the buyer (resp, the seller) because the mechanism is strategy-proof. Next, I show that a lower bound on the expected profit is $\max\left\{\frac{GFT-r}{1-r}\cdot r,0\right\}$. To see this, note that the expected profit [14]

$$E\left[\max\left\{\frac{\max\{v_B-v_S,0\}-r}{1-r}\cdot r,0\right\}\right]\geq\max\left\{E\left[\frac{\max\{v_B-v_S,0\}-r}{1-r}\cdot r\right],0\right\}=\max\left\{\frac{GFT-r}{1-r}\cdot r,0\right\},$$

where the inequality follows from Jensen's inequality, and the equality follows from the linearity of the ex-post profit when it is positive. Finally, I show that the lower bound is tight, i.e., the profit guarantee is $\max\left\{\frac{GFT-r}{1-r}\cdot r,0\right\}$. To see this, note that a degenerate distribution— a point mass on the value profile $(GFT,0)$— hits the lower bound. Indeed, a random double auction with any positive fixed commission fee below the ex-ante gain from trade has a positive profit guarantee. A high fixed commission fee translates into a high profit conditional on trading, but also leads to a low trading probability. Optimal fixed commission fee $r=1-\sqrt{1-GFT}$ balances these two effects, resulting in the profit guarantee of $\left(1-\sqrt{1-GFT}\right)^2$.

Remark 2 (Positive welfare guarantee)**.** In terms of the traders' welfare, how does the random double auction perform? Define the *ex-post welfare* for a value profile (v_B,v_S) as the sum of the traders' ex-post payoffs, or $q(v)(v_B-v_S)-(t_B(v)-t_S(v))$. The *expected welfare* and the *welfare guarantee* can then be similarly defined. I will show below that the random double auction has a

[14] Observe that $\max\{v_B-v_S,0\}=v_B-v_S$ when $v_B-v_S>r$.

positive welfare guarantee.

To derive the welfare guarantee of the random double auction, I first show that the ex-post welfare given an arbitrary value profile (v_B, v_S) is $\frac{(v_B - v_S - r)^2}{1-r}\mathbb{1}_{v_B - v_S > r}$. To see this, note that the welfare from a bid-ask pair (b, a) if $b - a > r$ is

$$\frac{b-a-r}{1-r} \cdot (v_B - v_S - r),$$

where the first term is the trading probability and the second term is the realized welfare conditional on trading. If $b - a \leq r$, then the trade will not take place and the realized welfare is trivially zero. Recall that the bid price (resp, the ask price) is equal to the true value of the buyer (resp, the seller) because the mechanism is strategy-proof. Next, I show that a lower bound on the expected welfare is $\frac{(GFT-r)^2}{1-r}$. To see this, note that the expected welfare

$$\begin{aligned} E\left[\frac{(v_B - v_S - r)^2}{1-r}\mathbb{1}_{v_B - v_S > r}\right] &= E\left[\frac{((v_B - v_S - r)\mathbb{1}_{v_B - v_S > r})^2}{1-r}\right] \\ &\geq \frac{(E\left[(v_B - v_S - r)\mathbb{1}_{v_B - v_S > r}\right])^2}{1-r} \\ &= \frac{(E\left[\max\{v_B - v_S - r, 0\}\right])^2}{1-r} \\ &\geq \frac{(E\left[\max\{v_B - v_S, 0\} - r\right])^2}{1-r} \\ &= \frac{(GFT - r)^2}{1-r}, \end{aligned}$$

where the first line follows from $\mathbb{1}_{v_B - v_S > r} = \mathbb{1}^2_{v_B - v_S > r}$, the second line follows from Jensen's inequality, the third line follows from $(v_B - v_S - r)\mathbb{1}_{v_B - v_S > r} = \max\{v_B - v_S - r, 0\}$, the fourth line follows from $\max\{v_B - v_S - r, 0\} \geq \max\{v_B - v_S, 0\} - r$, and the last line follows from the definition of GFT. Finally, I show that the lower bound is tight, i.e., the gain from trade guarantee is $\frac{(GFT-r)^2}{1-r}$. To see this, note that a degenerate distribution— a point mass on the value profile $(GFT, 0)$— hits the lower bound. Clearly, any fixed commission fee below the difference between the ex-ante gain from trade leads to a positive welfare guarantee. Raising fixed commission fee leads to both a low welfare conditional on trading and a low trading probability. Therefore, optimal fixed commission fee (for welfare) is zero, resulting in the welfare guarantee of GFT^2.

1.3.3 Optimal Profit Guarantee

In this section, I will show that the random double auction is a maxmin trading mechanism (Theorem 1) by constructing a feasible value distribution, referred to as a *worst value distribution*, and showing that $\left(1-\sqrt{1-GFT}\right)^2$ is the tight upper bound on expected profit across all dominant-strategy mechanisms against the worst value distribution. In addition, the random double auction is an optimal mechanism against the worst value distribution. Essentially, the random double auction and the worst value distribution form a "saddle point": The random double auction maximizes the expected profit given the worst value distribution, and the worst value distribution minimizes the expected profit under the random double auction. The properties of a saddle point imply that the random double auction is maxmin optimal. More details about the saddle point approach are given in Appendix 1.7.1. Subsection 1.3.3 gives details about the construction of the worst value distribution.

Let me first specify the symmetric triangular value distribution, which is the worst value distribution that I construct. The support is a symmetric triangular subset of joint values $ST := \{v \in V | v_B - v_S > r\}$. The marginal distribution for the buyer is a combination of a uniform distribution on $(r,1)$ and an atom of size r on 1: $\pi^*_B(v_B) = 1$ for $v_B \in (r,1)$ and $Pr^*_B(1) = r$. The marginal distribution for the seller is a combination of a uniform distribution on $(0,1-r)$ and an atom of size r on 0: $\pi^*_S(v_S) = 1$ for $v_S \in (0,1-r)$ and $Pr^*_S(0) = r$. The conditional distribution for the buyer is a combination of some generalized Pareto distribution on $(v_S + r, 1)$ and an atom on 1: When $v_S \in (0,1-r)$, $\pi^*_B(v_B|v_S) = \frac{2r^2}{(v_B-v_S)^3}$ for $v_B \in (v_S + r, 1)$ and $Pr^*_B(v_B = 1|v_S) = \frac{r^2}{(1-v_S)^2}$; when $v_S = 0$, $\pi^*_B(v_B|v_S = 0) = \frac{r}{(v_B)^2}$ for $v_B \in (r,1)$ and $Pr^*_B(v_B = 1|v_S = 0) = r$. The conditional distribution for the seller is a

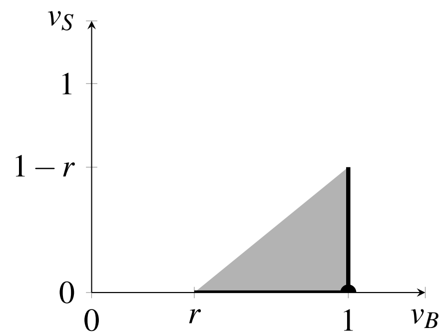

Figure 1.1: Symmetric Triangular Value Distribution

combination of some generalized Pareto distribution on $(0, v_B - r)$ and an atom on 0: When $v_B \in (r,1)$, $\pi^*_S(v_S|v_B) = \frac{2r^2}{(v_B-v_S)^3}$ for $v_S \in (0, v_B - r)$ and $Pr^*_S(v_S = 0|v_B) = \frac{r^2}{(v_B)^2}$; when $v_B = 1$, $\pi^*_S(v_S|v_B = 1) = \frac{r}{(1-v_S)^2}$ for $v_S \in (0, 1-r)$ and $Pr^*_S(v_S = 0|v_B = 1) = r$.

Equivalently, the symmetric triangular value distribution can be described as a combination of a joint density function on $ST \backslash \{(1,0)\}$ and an atom of size r^2 on the value profile $(1,0)$ as follows (See Figure 1.1).

$$\pi^*(v_B, v_S) = \begin{cases} \frac{2r^2}{(v_B-v_S)^3} & \text{if } v_B - v_S > r,\ v_B \neq 1 \text{ and } v_S \neq 0, \\ \frac{r^2}{(1-v_S)^2} & \text{if } v_B = 1 \text{ and } 0 < v_S < 1-r, \\ \frac{r^2}{(v_B)^2} & \text{if } r < v_B < 1 \text{ and } v_S = 0. \end{cases}$$

$$Pr^*(1,0) = r^2.$$

To construct the symmetric triangular value distribution, it is useful to define a "virtual value".

Definition 1 (Virtual value)**.** Fix any value distribution π[15], the expected profit of an optimal

[15] For exposition, I assume that π is differentiable everywhere when deriving the virtual values. It can be easily extended to joint distributions which admits an atom on the value profile $(1,0)$.

trading mechanism (q,t_B,t_S) admits a "virtual" representation[16]:

$$E[t(v)] = \int q(v)\phi(v)d\pi(v),$$

where $\phi(v) := (v_B - v_S) - \left(\frac{1-\Pi_B(v_B|v_S)}{\pi_B(v_B|v_S)} + \frac{\Pi_S(v_S|v_B)}{\pi_S(v_S|v_B)}\right)$ is defined to be the "virtual value"[17] of the value profile (v_B, v_S), where the first term is the maximum possible profit the intermediary could have earned if she knew the value profile (v_B, v_S), and the second term is the sum of the traders' information rents, which are pinned down by dominant-strategy incentive compatibility and the binding ex-post participation constraints of zero-value buyer and one-value seller. Here $\pi_B(\cdot|\cdot)$ and $\Pi_B(\cdot|\cdot)$ (resp, $\pi_S(\cdot|\cdot)$ and $\Pi_S(\cdot|\cdot)$) are conditional PDF and conditional CDF for the buyer (resp, the seller).

Using the virtual value, the problem of maximizing the expected profit across all dominant-strategy mechanisms is equivalent to the problem of maximizing the expected virtual value of the value profile in which trade takes places, subject to that the trading rule is *monotone*[18] (a monotonicity constraint associated with dominant-strategy incentive compatibility). This simplifies the problem, as one can now point-wise maximize the objective, ignoring the monotonicity constriant[19]. The symmetric triangular value distribution is constructed by solving a *zero virtual value condition* requiring the virtual value be zero for any value profile in the support except for the highest joint type. The intuition behind this condition is that the intermediary is *indifferent* between trading and no trading for any those value profiles under the random double auction.

Lemma 1. *The symmetric triangular value distribution satisfies a zero virtual value condition*

[16]The details are given in Appendix 1.7.1.

[17]This is a straightforward adaptation of the virtual value in Myerson and Satterthwaite (1983) to dominant-strategy mechanisms and the correlated private value environment.

[18]A trading rule q is monotone if q is non-decreasing in v_B and non-increasing in v_S. This is analogous to a monotone allocation rule in the auction design. Details are given in Appendix 1.7.1.

[19]Of course, one need to check that the monotonicity constraint holds in the end.

for any value profile in the support except for the highest joint type. Formally,

$$\phi(v) = 0, \quad \forall v \in ST \setminus \{(1,0)\}. \tag{ZVV}$$

Indeed, this condition guarantees that the intermediary is indifferent to any dominant-strategy mechanism in which 1) trade does not take place if the value profile lies outside the support and trade takes place with probability one when the value profile is $(1,0)$, and 2) ex-post participation constraints are binding for zero-value buyer and one-value seller. In addition, such a trading mechanism is an optimal trading mechanism given the symmetric triangular value distribution. Using the virtual representation, the optimal expected profit given the symmetric triangular value distribution is

$$Pr^*(1,0) \times 1 = \left(1 - \sqrt{1 - GFT}\right)^2.$$

This is because $(1,0)$ is the only value profile with a positive virtual value, and its virtual value is 1 as it is the highest joint type.

To understand why the symmetric triangular value distribution is a worst value distribution, it is useful to observe that it exhibits a *positive correlation*: If the buyer's value is higher, then the seller's value is more likely to be higher as well. Intuitively, positive correlation levels the maximal gain from trade across value profiles and therefore limits the intermediary's incentive to discriminate across value profiles. Indeed, the symmetric triangular value distribution exhibits "extreme" positive correlation in the following sense: It renders the intermediary *indifferent* across all value profiles in the support but the highest joint type $(1,0)$.

Definition 2 (Positive correlation for bivariate distributions)**.** Let $Z = (X,Y)$ be a bivariate random vector whose distribution is F. I say that Z exhibits positive correlation for D_X and D_Y if $F(X|Y=y)$ *first order stochastically dominates* $F(X|Y=y')$ for any $y > y', y, y' \in D_Y$ and $F(Y|X=x)$ *first order stochastically dominates* $F(Y|X=x')$ for any $x > x', x, x' \in D_X$.

Lemma 2. *The symmetric triangular value distribution exhibits a positive correlation for* $r < v_B < 1$ *and* $0 < v_S < 1 - r$.[20]

Theorem 1. *The random double auction is a maxmin trading mechanism with a profit guarantee of* $\left(1 - \sqrt{1 - GFT}\right)^2$, *and the symmetric triangular value distribution is a worst value distribution.*

Remark 3. It is useful to compare the profit guarantee of the random double auction and the optimal profit across dominant-strategy mechanisms if the value distribution were known to the intermediary. One case could be the following value distribution: The buyer's value follows a uniform distribution on $[GFT, 1]$ and the seller's value follows a uniform distribution on $[0, 1 - GFT]$; their values are independent. By a straightforward adaptation of the revenue equivalence theorem, the profit achievable by the optimal dominant-strategy mechanism can be computed. For example, When $GFT = \frac{3}{4}$, the optimal profit is $\frac{1}{2}$, whereas the profit guarantee of the random double auction is $\frac{1}{4}$, so the ratio between the profit guarantee and the optimal profit is $\frac{1}{2}$. In addition, this ratio is large when GFT is large and converges to 1 as $GFT \to 1$. Another case could be that the value distribution is a point mass on $(GFT, 0)$. Then the optimal profit is GFT. When $GFT = \frac{3}{4}$, the ratio between the profit guarantee and the optimal profit is $\frac{1}{3}$. In addition, this ratio is increasing in GFT and converges to 1 as $GFT \to 1$.

Construction of Symmetric Triangular Value Distribution

In this subsection, I illustrate how I construct a feasible value distribution such that (ZVV) holds. I start from value profiles in which either $v_B = 1$ or $v_S = 0$. Assume that $Pr^*(1,0) = \alpha$. Consider value profiles $(v_B, 0)$ in which $v_B \in (r, 1)$. Let $S^*(v_B, 0) := \int_{(v_B,1)} \pi^*(x,0)dx + Pr^*(1,0)$ for $v_B \in (r,1)$ and $S^*(1,0) := Pr^*(1,0)$. Note that $\pi^*(v_B, 0) = -\frac{\partial S^*(v_B,0)}{\partial v_B}$ for $v_B \in (r,1)$. By

[20] To see this, note that $\Pi^*_S(v_S|v_B) = \frac{r^2}{(v_B - v_S)^2}$ is decreasing w.r.t. v_B for $v_B \in (r,1)$. When $v_B = 1$, $\Pi^*_S(v_S|v_B = 1) = \frac{r}{1-v_S} \geq \frac{r^2}{(1-v_S)^2}$, so the positive correlation breaks when $v_B = 1$. Similarly, $\Pi^*_B(v_B|v_S) = 1 - \frac{r^2}{(v_B - v_S)^2}$ is decreasing w.r.t. v_S for $v_S \in (0, 1-r)$. When $v_S = 0$, $\Pi^*_B(v_B|v_S = 0) = 1 - \frac{r}{v_B} \leq 1 - \frac{r^2}{(v_B)^2}$, so the positive correlation breaks when $v_S = 0$.

(ZVV), I have that for any $(v_B, 0)$ in which $v_B \in (r, 1)$,

$$\pi^*(v_B, 0)(v_B - 0) - S^*(v_B, 0) = 0.$$

Note that this is a simple ordinary differential equation, to which the solution is

$$S^*(v_B, 0) = \frac{\alpha}{v_B}, \quad \pi^*(v_B, 0) = \frac{\alpha}{v_B^2}, \quad \forall v_B \in (r, 1).$$

Then consider value profiles $(1, v_S)$ in which $v_S \in (0, 1-r)$. Similarly, let $S^*(1, v_S) := \int_{(0,v_S)} \pi^*(1, x)dx + Pr^*(1, 0)$ for $v_S \in (0, 1-r)$. Note that $\pi^*(1, v_S) = \frac{\partial S^*(1, v_S)}{\partial v_S}$ for $v_S \in (0, 1-r)$. By (ZVV), I have that for any $(1, v_S)$ in which $v_S \in (0, 1-r)$,

$$\pi^*(1, v_S)(1 - v_S) - S^*(1, v_S) = 0.$$

Note that this is also a simple ordinary differential equation, to which the solution is

$$S^*(1, v_S) = \frac{\alpha}{1 - v_S}, \quad \pi^*(1, v_S) = \frac{\alpha}{(1 - v_S)^2}, \quad \forall v_S \in (0, 1-r).$$

Finally consider any value profile (v_B, v_S) in which $v_B - v_S > r$, $v_B \neq 1$ and $v_S \neq 0$. Let $S^*(v_B, v_S) := \int_{(v_B,1)} \pi^*(b, v_S)db + \pi^*(1, v_S)$ if $v_B - v_S > r$, $v_B \neq 1$ and $v_S \neq 0$. Note that $\pi^*(v_B, v_S) = -\frac{\partial S^*(v_B, v_S)}{\partial v_B}$ if $v_B - v_S > r$, $v_B \neq 1$ and $v_S \neq 0$. By (ZVV), I have that if $v_B - v_S > r$, $v_B \neq 1$ and $v_S \neq 0$,

$$\pi^*(v_B, v_S)(v_B - v_S) - S^*(v_B, v_S) - \int_{(0,v_S)} \pi^*(v_B, s)ds - \pi^*(v_B, 0) = 0. \tag{PIE}$$

Note that (PIE) is a (second order) partial integral equation. It is straightforward to see that $S^*(v_B, v_S)$ is not separable by taking the cross partial derivative. I take the guess-and-verify approach to solve (PIE). I guess that if $v_B - v_S > r$, $v_B \neq 1$ and $v_S \neq 0$,

$$S^*(v_B, v_S) = \frac{\alpha}{(v_B - v_S)^2}.$$

Under this guess, the L.H.S. of (PIE) equals $\frac{2\alpha}{(v_B-v_S)^3}(v_B - v_S) - \frac{\alpha}{(v_B-v_S)^2} - \int_{(0,v_S)} \frac{2\alpha}{(v_B-s)^3} ds - \frac{\alpha}{v_B^2}$, which can be shown to be 0 with simple algebra. Thus, I verified the guess.

To solve for α, I use the requirement that $\pi^*(v)$ is a distribution. Note that the marginal distribution for S is $\pi_S^*(v_S) = S^*(v_S + r, v_S) = \frac{\alpha}{(v_S+r-v_S)^2} = \frac{\alpha}{r^2}$ for $0 < v_S < 1 - r$ and $Pr_S^*(v_S = 0) = S^*(r, 0) = \frac{\alpha}{r}$. Since the integration is 1, I obtain that

$$\frac{\alpha}{r} + \frac{\alpha}{r^2} \cdot (1 - r) = 1.$$

Thus, $\alpha = r^2$.

The final step is to show that the constructed joint distribution is a feasible value distribution. To see this, note that

$$\begin{aligned}
\int \max\{v_B - v_S, 0\}\, d\pi^* &= \int (v_B - v_S)\, d\pi^* \\
&= \left(r \cdot 1 + \int_r^1 v_B dv_B\right) - \left(r \cdot 0 + \int_0^{1-r} v_S dv_S\right) \\
&= GFT,
\end{aligned}$$

where the first line follows from $v_B > v_S$ for any value profile in the support of π^*, the second line uses the marginal distributions of π^*, and the third line uses $r = 1 - \sqrt{1 - GFT}$.

1.4 Other Models of Limited Knowledge

1.4.1 Known Difference In Expectations

In this section, I consider a model in which the intermediary only knows the difference between the expectations of the traders' values, denoted by DE, but does not know the joint distribution π. Formally, I denote by

$$\Pi(DE) = \left\{ \pi \in \Delta V : \int (v_B - v_S) d\pi(v) = DE \right\} \tag{KDE}$$

the collection of joint distributions that are consistent with the known difference in expectations. If $DE \leq 0$, then the maxmin profit is zero, as no trading mechanism can generate a positive profit against the point mass on the value profile $(0, -DE)$. Therefore, I focus on *non-trivial* informational environments in which $DE > 0$.

Theorem 2. *Under the model* (KDE)*, The random double auction is a maxmin trading mechanism with a profit guarantee of* $\left(1 - \sqrt{1 - DE}\right)^2$*, and the symmetric triangular value distribution is a worst value distribution.*

Knowing DE is different from knowing GFT. That is, the sets of feasible value distributions are different under these two assumptions. Indeed, for any value distribution in which the seller's value is greater than the buyer's one with a positive probability, GFT is strictly higher than DE. $GFT = DE$ if and only if the seller's value is always weakly lower than the buyer's one. Nonetheless, the results are the same under these two different assumptions. This is because the ex-post profit under the random double auction is convex in either the ex-post gain from trade or the difference between the values[21]. Therefore, any value distribution in which the seller's value is greater than the buyer's one with a positive probability is not a "worst case" for the random double auction under either assumption. In other words, the differences in

[21]The ex-post profit $\max\left\{\frac{v_B - v_S - r}{1-r} \cdot r, 0\right\} = \max\left\{\frac{\max\{v_B - v_S, 0\} - r}{1-r} \cdot r, 0\right\}$.

the sets of feasible value distributions do not matter.

1.4.2 Known Expectations

In this section, I consider a model in which the intermediary only knows the expectations of the buyer's value and the seller's value respectively, denoted by M_B and M_S, but does not know the joint distribution π. Formally, I denote by

$$\Pi(M_B, M_S) = \left\{\pi \in \Delta V : \int v_B d\pi(v) = M_B, \int v_S d\pi(v) = M_S\right\} \tag{KE}$$

the collection of joint distributions that are consistent with the known expectations. If $M_B \leq M_S$, then the maxmin profit is zero, as no trading mechanism can generate a positive profit against the point mass on the value profile (M_B, M_S). Therefore, we focus on *non-trivial* informational environments in which $M_B > M_S$.

Symmetric Informational Environment: $M_B + M_S = 1$

The higher the seller's value, the lower his willingness to trade. Thus, it is plausible to regard the highest-value seller as the lowest-type seller. When the known expectations sum up to 1, the expectation of the buyer's value and the expectation of the seller's value have the same distance from the lowest-type buyer and the lowest-type seller respectively, i.e., $M_B - 0 = 1 - M_S$. Therefore I refer to this case as the symmetric informational environment. The symmetric informational environment captures situations in which both parties have similar willingness to trade. Likewise, I refer to the case in which $M_B + M_S \neq 1$ as the asymmetric informational environment.

Theorem 3. *Under the model* (KE)*, for the symmetric informational environment, the random double auction is a maxmin trading mechanism with a profit guarantee of* $\left(1 - \sqrt{1-(M_B - M_S)}\right)^2$*, and the symmetric triangular value distribution is a worst value* *distribution.*

The derivation of the profit guarantee under the model (KE) is the same as that under the model (KDE). The construction of a worst value distribution is the same. Observe that the symmetric triangular value distribution satisfies $M_B + M_S = 1$, because

$$\begin{aligned}\int (v_B + v_S)\, d\pi^* &= \left(r \cdot 1 + \int_r^1 v_B dv_B\right) + \left(r \cdot 0 + \int_0^{1-r} v_S dv_S\right) \\ &= 1,\end{aligned}$$

where the first line uses the marginal distributions of π^*, and the second line holds for any $r \in (0,1)$.

Knowing the expectations and knowing the difference in expectations are comparable. Indeed, $\Pi(DE)$ is a larger set: It contains both the symmetric informational environment and the asymmetric ones. For example, if DE= 0.2, then it is possible that $M_B = 0.6$ and $M_S = 0.4$ (the symmetric one), and it is possible that $M_B = 0.8$ and $M_S = 0.6$ (an asymmetric one). Therefore, although the random double auction is maxmin optimal under the model (KDE), it is maxmin optimal only for the symmetric informational environment under the model (KE). For the asymmetric one, as I show in the next section, a variation of random double auction does strictly better.

Asymmetric Informational Environment: $M_B + M_S \neq 1$

I extend the analysis to construct a maxmin trading mechanism for the asymmetric informational environment. I will propose a *generalized random double auction* and an *asymmetric triangular value distribution*, and then show that they form a saddle point. The illustration of the result is relegated to Appendix 1.7.2. This section generalizes the results for the symmetric informational environment, as the generalized random double auction (resp, the asymmetric triangular value distribution) converges to the random double auction (resp, the symmetric triangular value distribution) when the asymmetric informational environment converges to the symmetric informational environment (See Remark 7).

Let (r_1, r_2) in which $r_1 \in (0,1)$, $r_2 \in (0,1)$ and $r_1 + r_2 \neq 1$ be a solution to the following system of equations

$$M_B = \int_{r_1}^{1} \frac{r_1(1-r_2)}{(\frac{1-r_1-r_2}{1-r_1} v_B + \frac{r_1 r_2}{1-r_1})^2} v_B dv_B + r_1 := H_1(r_1, r_2), \tag{KE-B}$$

$$M_S = \int_{0}^{r_2} \frac{r_1(1-r_2)}{(\frac{1-r_1-r_2}{r_2} v_S + r_1)^2} v_S dv_S := H_2(r_1, r_2). \tag{KE-S}$$

Lemma 3. *For the asymmetric informational environment, there exists a solution* $(r_1, r_2) \in (0,1)^2$ *to the system of equations* (KE-B) *and* (KE-S). *In addition,* $r_1 + r_2 \neq 1$.

Let $\gamma := \frac{1-r_2}{r_1}$, $\delta := \frac{2(1-r_1-r_2)}{1-r_1+r_2}$, $\tau := \frac{2r_1 r_2}{1-r_1+r_2}$. The generalized random double auction is described as follows.

Step 0: ***Transformed bid and ask***. The intermediary publicly commits to transforming a bid price b and an ask price a as follows: $b' = \frac{1}{\ln\gamma} \cdot \left[\ln\left(\frac{1-r_1-r_2}{1-r_1} b + \frac{r_1 r_2}{1-r_1}\right)\right], a' = \frac{1}{\ln\gamma} \cdot \left[\ln\left(\frac{1-r_1-r_2}{r_2} a + r_1\right)\right]$. The buyer and the seller both know r_1 and r_2 as well as the transformations.

Step 1: ***Uniformly random spread***. The intermediary publicly commits to randomly drawing a spread s' uniformly on $[0,1]$. Then a random spread is drawn whose realization is not observed by either the buyer or the seller. The buyer and the seller both know the uniform distribution on $[0,1]$ from which the random spread is drawn.

Step 2: ***Exponential transaction price and floating commission fee***. The buyer submits a bid price b, and the seller submits an ask price a, simultaneously. If the difference between the transformed bid price and the transformed ask price is greater than the realized spread, or $b' - a' > s'$, then the seller sells the asset to the buyer at the price $p' = \frac{\gamma^{b'} - \gamma^{a'}}{\delta(\ln\gamma)(b'-a')} - \frac{\tau}{\delta}$, and each pays the intermediary half of the commission fee $\frac{r'}{2} = \frac{\delta p' + \tau}{2}$. Otherwise, no trade takes place, and no one pays or receives anything.

Remark 4. The transaction price p' is no-longer midpoint of the bid price and the ask price. The floating commission fee r', however, has a fixed commission fee component τ, plus an price-adjusted component $\delta p'$, which is linear in the transaction price.

It is straightforward to show that the generalized random double auction can also be expressed as follows. If $r_2 b-(1-r_1)a > r_1 r_2$,

$$q^{**}(b,a)=\frac{1}{\ln\frac{1-r_2}{r_1}}\cdot\left[\ln\left(\frac{1-r_1-r_2}{1-r_1}b+\frac{r_1 r_2}{1-r_1}\right)-\ln\left(\frac{1-r_1-r_2}{r_2}a+r_1\right)\right],$$

$$\begin{aligned}t_B^{**}(b,a)=&-\frac{r_1 r_2}{(1-r_1-r_2)\ln\frac{1-r_2}{r_1}}\cdot\left[\ln\left(\frac{1-r_1-r_2}{1-r_1}b+\frac{r_1 r_2}{1-r_1}\right)-\ln\left(\frac{1-r_1-r_2}{r_2}a+r_1\right)\right]\\&+\frac{1}{\ln\frac{1-r_2}{r_1}}\cdot\left(b-\frac{1-r_1}{r_2}a-r_1\right),\end{aligned}$$

$$\begin{aligned}t_S^{**}(b,a)=&-\frac{r_1 r_2}{(1-r_1-r_2)\ln\frac{1-r_2}{r_1}}\cdot\left[\ln\left(\frac{1-r_1-r_2}{1-r_1}b+\frac{r_1 r_2}{1-r_1}\right)-\ln\left(\frac{1-r_1-r_2}{r_2}a+r_1\right)\right]\\&+\frac{1}{\ln\frac{1-r_2}{r_1}}\cdot\left(\frac{r_2}{1-r_1}b-a-\frac{r_1 r_2}{1-r_1}\right).\end{aligned}$$

If $r_2 b-(1-r_1)a\le r_1 r_2$,

$$q^{**}(b,a)=t_B^{**}(b,a)=t_S^{**}(b,a)=0.$$

Remark 5. The generalized random double auction also satisfies the standard weak budget balance property.

Now let me specify the asymmetric triangular value distribution. The support is an asymmetric triangular subset of joint values $AT:=\{v|r_2 v_B-(1-r_1)v_S>r_1 r_2\}$. The marginal distribution for the buyer is a combination of some generalized Pareto distribution on $(r_1,1)$ and an atom of size r_1 on 1: $\pi_B^{**}(v_B)=\frac{r_1(1-r_2)}{\left(\frac{1-r_1-r_2}{1-r_1}v_B+\frac{r_1 r_2}{1-r_1}\right)^2}$ for $v_B\in(r_1,1)$ and $Pr_B^{**}(1)=r_1$. The marginal distribution for the buyer is a combination of some generalized Pareto distribution on $(0,r_2)$ and an atom of size $1-r_2$ on 0: $\pi_S^{**}(v_S)=\frac{r_1(1-r_2)}{\left(\frac{1-r_1-r_2}{r_2}v_S+r_1\right)^2}$ for $v_S\in(0,r_2)$ and $Pr_S^{**}(0)=1-r_2$. The conditional distribution for the buyer is a combination of some generalized

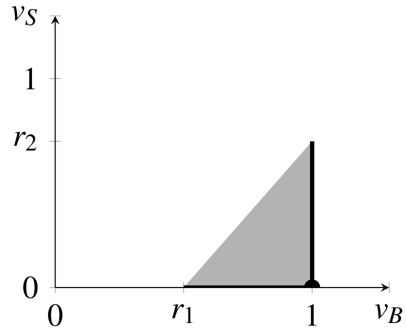

Figure 1.2: Asymmetric Triangular Value Distribution

Pareto distribution on $\left(r_1+\frac{1-r_1}{r_2}v_S,1\right)$ and an atom on 1: When $v_S\in(0,r_2)$, $\pi_B^{**}(v_B|v_S)=\frac{2\left(\frac{1-r_1-r_2}{r_2}v_S+r_1\right)^2}{(v_B-v_S)^3}$ for $v_B\in\left(r_1+\frac{1-r_1}{r_2}v_S,1\right)$ and $Pr_B^{**}(v_B=1|v_S)=\frac{\left(\frac{1-r_1-r_2}{r_2}v_S+r_1\right)^2}{(1-v_S)^2}$; when $v_S=0$, $\pi_B^{**}(v_B|v_S=0)=\frac{r_1}{(v_B)^2}$ for $v_B\in(r_1,1)$ and $Pr_B^{**}(v_B=1|v_S=0)=r_1$. The conditional distribution for the seller is a combination of some generalized Pareto distribution on $(0,\frac{r_2(v_B-r_1)}{1-r_1})$ and an atom on 0: When $v_B\in(r_1,1)$, $\pi_S^{**}(v_S|v_B)=\frac{2\left(\frac{1-r_1-r_2}{1-r_1}v_B+\frac{r_1r_2}{1-r_1}\right)^2}{(v_B-v_S)^3}$ for $v_S\in\left(0,\frac{r_2(v_B-r_1)}{1-r_1}\right)$ and $Pr_B^{**}(v_S=0|v_B)=\frac{\left(\frac{1-r_1-r_2}{1-r_1}v_B+\frac{r_1r_2}{1-r_1}\right)^2}{(v_B)^2}$; when $v_B=1$, $\pi_S^{**}(v_S|v_B=1)=\frac{1-r_2}{(1-v_S)^2}$ for $v_S\in(0,r_2)$ and $Pr_S^{**}(v_S=0|v_B=1)=1-r_2$.

Equivalently, the asymmetric triangular value distribution can be described as a combination of a joint density function on $AT\backslash\{(1,0)\}$ and an atom of size $r_1(1-r_2)$ on the value profile $(1,0)$ as follows (See Figure 1.2).

$$\pi^{**}(v_B,v_S)=\begin{cases}\frac{2r_1(1-r_2)}{(v_B-v_S)^3} & \text{if } r_2v_B-(1-r_1)v_S\geq r_1r_2,\ v_B\neq 1 \text{ and } v_S\neq 0,\\ \frac{r_1(1-r_2)}{(1-v_S)^2} & \text{if } v_B=1 \text{ and } 0<v_S<r_2,\\ \frac{r_1(1-r_2)}{(v_B)^2} & \text{if } r_1<v_B<1 \text{ and } v_S=0.\end{cases}$$

$$Pr^{**}(1,0)=r_1(1-r_2).$$

Lemma 4. *The asymmetric triangular value distribution exhibits a positive correlation for*

$r_1 < v_B < 1$ and $0 < v_S < r_2$.[22]

Remark 6. If $M_S = 0$, then it is common knowledge that the seller's value $v_S = 0$. Note that $q^{**}(b,0) = \frac{1}{\ln \frac{1-r_2}{r_1}} \cdot \ln\left(\frac{1-r_1-r_2}{r_1(1-r_1)} b + \frac{r_2}{1-r_1}\right)$. If $r_2 = 0$, it is straightforward that $q^{**}(b,0)$ (resp, π^{**}) reduces to the mechanism (resp, the worst-case distribution) found by Carrasco et al. (2018) when the monopolistic seller only knows the expectation of the buyer's value.

Theorem 4. *Under the model* (KE)*, for the asymmetric informational environment, the generalized random double auction is a maxmin trading mechanism with a profit guarantee of $r_1(1-r_2)$, and the asymmetric triangular value distribution is a worst value distribution.*

Remark 7 (Convergence)**.** If $M_B + M_S \to 1$, it is straightforward to show that there is a solution in which $r_1 + r_2 \to 1$. Then by L'Hôpital's rule, $q^{**} \to q^*$, $p' \to p$, $r' \to r$, $t_B^{**} \to t_B^*$, $t_S^{**} \to t_S^*$. In addition, $\pi^{**} \to \pi^*$.

1.5 Deterministic Mechanisms

In this section, I restrict attention to the class of deterministic dominant-strategy mechanisms, i.e., the trading rule has an additional property: $q(v)$ [23] is either 0 or 1 for any $v \in V$. I characterize maxmin deterministic trading mechanisms across mechanisms in this class.

Definition 3. The *trade boundary* of a given deterministic dominant-strategy mechanism (q, t_B, t_S) is a set of value profiles $\mathcal{B} := \{\bar{v} = (\bar{v}_B, \bar{v}_S) \in V | q(\bar{v}) = 0$[24] and for any small $\varepsilon > 0$, $q(\bar{v}_B + \varepsilon, \bar{v}_S) = 1$ or $q(\bar{v}_B, \bar{v}_S - \varepsilon) = 1\}$.

[22]To see this, note that $\Pi_S^{**}(v_S|v_B) = \frac{\left(\frac{1-r_1-r_2}{1-r_1} v_B + \frac{r_1 r_2}{1-r_1}\right)^2}{(v_B - v_S)^2}$ is decreasing w.r.t. v_B for $v_B \in (r_1, 1)$. When $v_B = 1$, $\Pi_S^{**}(v_S|v_B) = \frac{1-r_2}{1-v_S} \geq \frac{(1-r_2)^2}{(1-v_S)^2}$, so the positive correlation breaks when $v_B = 1$. Similarly, $\Pi_B^{**}(v_B|v_S) = 1 - \frac{\left(\frac{1-r_1-r_2}{r_2} v_S + r_1\right)^2}{(v_B - v_S)^2}$ is decreasing w.r.t. v_S for $v_S \in (0, r_2)$. When $v_S = 0$, $\Pi_B^{**}(v_B|v_S = 0) = 1 - \frac{r_1}{v_B} \leq 1 - \frac{(r_1)^2}{(v_B)^2}$, so the positive correlation breaks when $v_S = 0$.

[23]I define $q(v)$ to be 0 if $v \notin V$.

[24]For exposition, I assume that trade does not take place on the trade boundary. As will be clear, this is to guarantee that a best response for adversarial Nature exists. This assumption does not affect the solution and the value of the problem. Similar assumption is also made in Kos and Messner (2015).

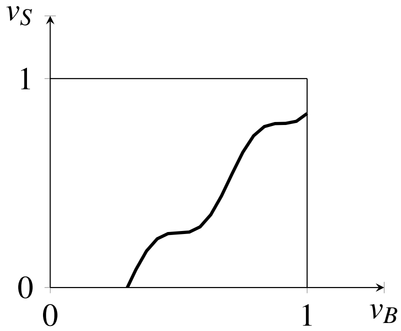

Figure 1.3: Trade Boundary

I observe that the trade boundary of a deterministic dominant-strategy mechanism is non-decreasing (See Figure 1.3[25]).

Remark 8 (Non-decreasing trade boundary)**.** If $\bar{v} = (\bar{v}_B, \bar{v}_S) \in \mathcal{B}, \bar{v}' = (\bar{v}'_B, \bar{v}'_S) \in \mathcal{B}$ and $\bar{v}_B > \bar{v}'_B$, then $\bar{v}_S \geq \bar{v}'_S$. [26]

The main idea of searching for a maxmin deterministic trading mechanism is as follows. I divide all possible deterministic dominant-strategy mechanisms into four classes according to the trade boundary. By *strong duality*[27], I can work on the dual program. I propose a relaxation of the dual program by ignoring a lot of constraints. The merit of doing so is to have a finite-dimensional linear programming problem. Then I derive an upper bound of the value of the relaxation and show that it can be attained by constructing deterministic dominant-strategy mechanisms as well as a feasible value distribution.

Theorem 5. *When $GFT > \frac{1}{2}$, any deterministic dominant-strategy mechanism satisfying the following properties is a maxmin deterministic trading mechanism (See Figure 1.4[28]):*
(i). $\left(1-\sqrt{\frac{1-GFT}{2}}, 0\right) \in \mathcal{B}, \left(1, \sqrt{\frac{1-GFT}{2}}\right) \in \mathcal{B}$.

[25]The thick black curve is a trade boundary $\mathcal{B}$ that is non-decreasing.

[26]To see this, note that by the definition of the trade boundary, I have that $q(\bar{v}_B, \bar{v}'_S) = 1$ because $\bar{v}' \in \mathcal{B}$ and $\bar{v}_B > \bar{v}'_B$. Then, again by the definition of the trade boundary, I have that $\bar{v}_S \geq \bar{v}'_S$ because $\bar{v} \in \mathcal{B}$.

[27]That is, given a dominant-strategy mechanism, the value of the primal minimization problem equals that of its dual maximization problem, details of which are in Appendix 1.7.4.

[28]$B_1 = \left(1-\sqrt{\frac{1-GFT}{2}}, 0\right), B_2 = \left(1, \sqrt{\frac{1-GFT}{2}}\right)$. If $GFT > \frac{1}{2}$, then $B_1 \in \mathcal{B}, B_2 \in \mathcal{B}$, and $\mathcal{B}$ lies in the black region for a maxmin deterministic trading mechanism.

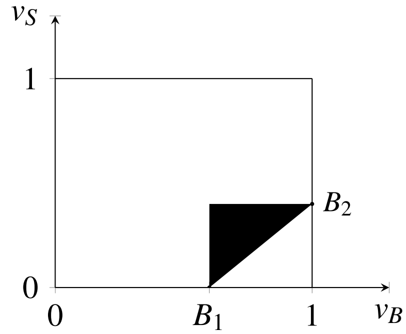

Figure 1.4: Maxmin Deterministic Trading Mechanisms

(ii). $\mathcal{B}$ is above (including) the line $v_B - v_S = \sqrt{\frac{1-GFT}{2}}$.

(iii). The payment rule and the transfer rule are characterized by Lemma 5.

The profit guarantee is $\left(1-\sqrt{2(1-GFT)}\right)^2$. The worst value distribution puts probability masses of $\sqrt{\frac{1-GFT}{2}}, \sqrt{\frac{1-GFT}{2}}$ and $1-2\sqrt{\frac{1-GFT}{2}}$ on the value profiles $\left(1-\sqrt{\frac{1-GFT}{2}}, 0\right), \left(1, \sqrt{\frac{1-GFT}{2}}\right)$ and $(1,0)$ respectively.

When $GFT \leq \frac{1}{2}$, the Never Trading Mechanism[29] is a maxmin deterministic trading mechanism with a profit guarantee of 0.

That is, I characterize the class of maxmin deterministic trading mechanisms for any informational environment with a non-trivial profit guarantee (i.e., $GFT > \frac{1}{2}$). The worst value distribution is discrete, and is the same for the mechanisms in this class. Now I provide examples of some maxmin deterministic trading mechanisms.

Linear Trading Mechanism: Trade takes place with probability one if $v_B - v_S > \sqrt{\frac{1-GFT}{2}}$, and conditional on trading, the buyer pays $1-\sqrt{\frac{1-GFT}{2}} + v_S$ and the seller receives $v_B - \left(1-\sqrt{\frac{1-GFT}{2}}\right)$; otherwise, no trade takes place, and no one pays or receives anything.

Double Posted-Price Trading Mechanism: Trade takes place with probability one if $v_B > 1-\sqrt{\frac{1-GFT}{2}}$ and $v_S < \sqrt{\frac{1-GFT}{2}}$, and conditional on trading, the buyer pays $1-\sqrt{\frac{1-GFT}{2}}$ and the seller receives $\sqrt{\frac{1-GFT}{2}}$; otherwise, no trade takes place, and no one pays or receives anything.

[29]Trade never takes place, and no one pays or receives anything.

1.6 Extension and Discussion

1.6.1 Can-hold Case

Consider a more general model in which the intermediary can hold the asset. To wit, this only requires that the sum of the buyer's allocation (denoted by q_B) and the seller's allocation (denoted by q_S) do not exceed 1. Recall that this sum is required to be 1 for the main results. Formally, the intermediary seeks a trading mechanism (q_B, q_S, t_B, t_S) such that the following constraints hold:

$$v_B q_B(v) - t_B(v) \geq v_B q_B(v'_B, v_S) - t_B(v'_B, v_S), \quad \forall v \in V, v'_B \in V_B; \qquad (DSIC_B)$$

$$v_B q_B(v) - t_B(v) \geq 0, \quad \forall v \in V; \qquad (EPIR_B)$$

$$v_S q_S(v) + t_S(v) \geq v_S q_S(v_B, v'_S) + t_S(v_B, v'_S), \quad \forall v \in V, v'_S \in V_S; \qquad (DSIC'_S)$$

$$v_S q_S(v) + t_S(v) \geq v_S, \quad \forall v \in V; \qquad (EPIR'_S)$$

$$q_B(v) + q_S(v) \leq 1, \quad \forall v \in V. \qquad (CH)$$

I denote the set of such trading mechanisms as $\mathcal{D}'$[30]. The intermediary's problem is to seek for a trading mechanism that solves

$$\sup_{(q_B, q_S, t_B, t_S) \in \mathcal{D}'} \inf_{\pi \in \Pi(GFT)} \int t(v) d\pi(v). \qquad \text{(MTM')}$$

Theorem 6. *The random double auction is a solution to* (MTM').

That is, the solution to the more general problem (MTM') coincides with the solution to the problem (MTM). To see this, first note that the value of (MTM') is weakly higher than the value of (MTM) because $\mathcal{D} \subset \mathcal{D}'$. I will show that the value of (MTM) is weakly higher than the

[30] Note that here the monotonicity constraints are that $q_B(v_B, v_S)$ is non-decreasing w.r.t. v_B for any v_S and $q_S(v_B, v_S)$ is non-decreasing w.r.t. v_S for any v_B.

value of (MTM'). Indeed, given the symmetric triangular value distribution, the random double auction is an optimal mechanism even among this wider class of trading mechanism $\mathcal{D}'$. To show this, first note that a simple adaptation of Lemma 5 yields an analogous virtual representation of the expected profit for this more general model:

$$E[t(v)] = \int_v [q_B(v)\phi_B(v) + q_S(v)\phi_S(v)]d\pi(v) - 1,$$

where $\phi_B(v) = v_B - \frac{1-\Pi_B(v_B|v_S)}{\pi_B(v_B|v_S)}$ and $\phi_S(v) = v_S + \frac{\Pi_S(v_S|v_B)}{\pi_S(v_S|v_B)}$. Here $\phi_B(v)$ (resp, $\phi_S(v)$) is the buyer's (resp, the seller's) virtual value when the value profile is $v = (v_B, v_S)$. Given the symmetric triangular value distribution, $\phi_B = \phi_S > 0$ for any value profile in the support except for the highest joint type (1,0), in which $\phi_B(1,0) > \phi_S(1,0) = 0$; in addition, $\phi_B \leq 0$ and $\phi_S \geq 0$ for any value profile outside the support. Then any trading mechanism in $\mathcal{D}'$ will be optimal if 1) the ex-post participation constraints are binding for zero-value buyer and one-value seller, and 2) $q_B = 0$ and $q_S = 1$ for any value profile outside the support, $q_B + q_S = 1$ for any value profile inside the support and $q_B(1,0) = 1, q_S(1,0) = 0$. It is straightforward to see that the random double auction[31] is such a mechanism and therefore remains optimal to the symmetric triangular value distribution in this general model. Indeed, given the property that the buyer's virtual value is equal to the seller's virtual value for any value profile in the support except for (1,0), the intermediary does not have an incentive to hold the asset in an optimal trading mechanism.

1.6.2 Information Design Problem

A well-known result in models of private information is that the distribution of agents' private information is a key determinant of their welfare. For example, in the environment of bilateral trade, Myerson and Satterthwaite (1983) consider the independent private value model and show that the two trading parties' welfare is not the full surplus for general distributions and the amount of their welfare depends on their distributions of private values. Indeed, most

[31] In this more general model, $q_B = q^*$ and $q_S = 1 - q^*$.

of the existing models of private information in the environment of bilateral trade assume that the distribution of the two trading parties' private information is exogenous. However, it is conceivable that a *financial regulator*, e.g., the Security and the Exchange Commission (SEC), may optimally design the nature of the private information held by the two trading parties to maximize their welfare, given the fact that their welfare is affected by the distribution of their private information.

In this section, I consider an information design problem of a *financial regulator* whose objective is to maximize the expected welfare. Recall that the expected welfare is defined as the sum of the traders' expected profits. I assume that the financial regulator can carefully design the private information of the traders by choosing a value distribution subject to the constraint that the ex-ante gain from trade is GFT, i.e., $\pi \in \Pi(GFT)$. The intermediary, after observing the choice of the distribution but not the realized joint values, designs a profit-maximizing trading mechanism across dominant-strategy mechanisms. Formally, the financial regulator solves [32]

$$\sup_{\pi \in \Pi(GFT)} \int [q^*(v)(v_B - v_S) - t^*(v)] d\pi(v) \tag{MW}$$

subject to

$$(q^*, t_B^*, t_S^*) \in \arg \sup_{(q, t_B, t_S) \in \mathcal{D}} \int_v t(v) d\pi(v).$$

Theorem 7. *The symmetric triangular value distribution is a solution to* (MW).

That is, the symmetric triangular value distribution is an optimal information structure for the financial regulator's information design problem (MW).

Recall that a symmetric triangular value distribution has the property that the virtual value is zero for any value profile in the support except for the value profile $(1,0)$. This property has

[32] If the intermediary has multiple optimal trading mechanisms, I break ties in favor of the financial regulator by selecting one that maximizes the gain from trade for the traders. This is a standard tie-breaking rule in the information design literature (e.g., Kamenica and Gentzkow (2011), Roesler and Szentes (2017) and Condorelli and Szentes (2020)).

two implications. First, it implies that an *efficient* [33] trading mechanism is a best response for the intermediary. Second, it implies that in a best response, the intermediary does not discriminate across all value profiles in the support but the value profile $(1,0)$. These two implications render a symmetric triangular value distribution a good candidate as a solution.

Under the symmetric triangular value distribution, the expected welfare is the difference between the ex-ante gain from trade GFT and the expected profit of the intermediary $\left(1-\sqrt{1-GFT}\right)^2$. Indeed, the symmetric triangular value distribution minimizes the expected profit of the intermediary. This is because the expected profit under the random double auction is weakly higher than $\left(1-\sqrt{1-GFT}\right)^2$ for any feasible value distribution (Recall Proposition 2). Therefore, the symmetric triangular value distribution solves (MW). In addition, the expected welfare is equally shared by the traders: Each trader obtains an expected profit of $\sqrt{1-GFT}(1-\sqrt{1-GFT})$.

The information design problem (MW) is closely related to Condorelli and Szentes (2020) who consider a *buyer-optimal* information design problem: The buyer can choose the probability distribution of her valuation for the good to maximize her profit. The seller, after observing the buyer's choice of the distribution but not the realized valuation, designs a revenue-maximizing selling mechanism. The problem (MW) may be interpreted as a *traders-optimal* information design problem. Critically, trade is efficient under the solution in either problem.

[33] Precisely, trade takes place with probability one for any value profile in the support of the symmetric triangular value distribution, and 0 otherwise.

1.7 Appendix

1.7.1 Preliminaries

Zero-Sum Game

The intermediary's maxmin optimization problem (MTM) can be interpreted as a two-player sequential *zero-sum game*. The two players are the intermediary and adversarial Nature. The intermediary first chooses a trading mechanism $(q, t_B, t_S) \in \mathcal{D}$. After observing the intermediary's choice of the trading mechanism, adversarial Nature chooses a feasible value distribution $\pi \in \Pi(M_B, M_S)$. The intermediary's payoff is $U((q, t_B, t_S), \pi)$, and adversarial Nature's payoff is $-U((q, t_B, t_S), \pi)$. Instead of solving directly for such a subgame perfect equilibrium, I can solve for a Nash equilibrium $((q^*, t_B^*, t_S^*), \pi^*)$ of the simultaneous-move version of this zero-sum game, which corresponds to a saddle point of the payoff functional U. Formally, for any $(q, t_B, t_S) \in \mathcal{D}$ and any $\pi \in \Pi(M_B, M_S)$,

$$U((q^*, t_B^*, t_S^*), \pi) \geq U((q^*, t_B^*, t_S^*), \pi^*) \geq U((q, t_B, t_S), \pi^*). \tag{SP}$$

Indeed, the first inequality implies that the mechanism (q^*, t_B^*, t_S^*)'s profit guarantee is the expected profit when adversarial Nature chooses the value distribution π^*, and the second inequality implies that no other dominant-strategy mechanism can yield a strictly higher expected profit under the value distribution π^*. Hence, the two inequalities together imply the mechanism (q^*, t_B^*, t_S^*) is a maxmin trading mechanism.

Revenue Equivalence

When searching an optimal dominant-strategy mechanism given a value distribution, it will be useful to simplify the problem. I will use the following proposition: Its proof is standard but included in Appendix 1.7.1 for completeness.

Lemma 5 (Revenue Equivalence)**.** *When searching an optimal dominant-strategy mechanism, it is without loss to restrict attention to trading mechanisms satisfying the following properties:*

(i). $q(v)$ *is non-decreasing in* v_B *and non-increasing in* v_S*.*

(ii). $t_B(v) = v_B q(v) - \int_0^{v_B} q(x, v_S)dx.$

(iii). $t_S(v) = v_S q(v) + \int_{v_S}^1 q(v_B, x)dx.$

(iv). $t(v) = (v_B - v_S)q(v) - \int_0^{v_B} q(x, v_S)dx - \int_{v_S}^1 q(v_B, x)dx.$

That is, the trading rule $q(v)$ is monotone; the payment rule t_B and the transfer rule t_S admit an envelope representation. In addition, the ex-post participation constraints for zero-value buyer and one-value seller are binding. Lemma 5 is standard in the mechanism design literature. The envelope representation of the ex-post profit (property (iv)) implies $E[t(v)] = \int q(v)\phi(v)d\pi(v)$, using integration by parts.

Proof of Lemma 5

(i). Dominant-strategy incentive compatibility (DSIC) for a type v_B of B requires that for any v_S and $v'_B \neq v_B$:

$$v_B q(v_B, v_S) - t_B(v_B, v_S) \geq v_B q(v'_B, v_S) - t_B(v'_B, v_S).$$

DSIC for a type v'_B of B requires that for any v_S and $v_B \neq v'_B$:

$$v'_B q(v'_B, v_S) - t_B(v'_B, v_S) \geq v'_B q(v_B, v_S) - t_B(v_B, v_S).$$

Adding the two inequalities, I have that:

$$(v_B - v'_B)(q(v_B, v_S) - q(v'_B, v_S)) \geq 0.$$

It follows that $q(v_B, v_S) \geq q(v'_B, v_S)$ whenever $v_B > v'_B$.

Similarly, DSIC for a type v_S of S requires that for any v_B and $v'_S \neq v_S$:

$$v_S(1 - q(v_B, v_S)) + t_B(v_B, v_S) \geq v_S(1 - q(v_B, v'_S)) + t_S(v_B, v'_S).$$

DSIC for a type v'_S of S requires that for any v_B and $v_S \neq v'_S$:

$$v'_S(1 - q(v_B, v'_S)) + t_S(v_B, v'_S) \geq v'_S(1 - q(v_B, v_S)) + t_S(v_B, v_S).$$

Adding the two inequalities, I have that:

$$(v_S - v'_S)(q(v_B, v'_S) - q(v_B, v_S)) \geq 0.$$

It follows that $q(v_B, v_S) \leq q(v_B, v'_S)$ whenever $v_S > v'_S$.

(ii). Fix v_S, and define

$$U_B(v_B) := v_B q(v_B, v_S) - t_B(v_B, v_S).$$

By the first two inequalities in (i), I obtain that

$$(v'_B - v_B)q(v_B, v_S) \leq U_B(v'_B) - U_B(v_B) \leq (v'_B - v_B)q(v'_B, v_S).$$

Therefore $U_B(v_B)$ is Lipschitz, hence absolutely continuous w.r.t. v_B and therefore differentiable w.r.t. v_B almost everywhere. Then applying the envelope theorem to the above inequality at each point of differentiability, I obtain that

$$\frac{dU_B(v_B)}{dv_B} = q(v_B, v_S).$$

Then I have that

$$t_B(v_B, v_S) = v_B q(v_B, v_S) - \int_0^{v_B} q(x, v_S)dx - U_B(0).$$

Note that $U_B(0) \geq 0$ by the ex-post individually rational constraint. If $U_B(0) > 0$, then I can reduce it to 0 so that I can increase the payment from B for any value profile in which the seller's value is v_S. And the profit will be weakly greater. Thus, when searching for an optimal dominant-strategy mechanism, it is without loss of generality to let $U_B(0) = 0$. Then I obtain that $t_B(v_B, v_S) = v_B q(v_B, v_S) - \int_0^{v_B} q(x, v_S)dx$.

(iii). Similarly, fix v_B, and define

$$U_S(v_S) := v_S(1 - q(v_B, v_S)) + t_S(v_B, v_S).$$

By the fourth and fifth inequalities in (i), I obtain that

$$(v'_S - v_S)(1 - q(v_B, v_S)) \leq U_S(v'_S) - U_S(v_S) \leq (v'_S - v_S)(1 - q(v_B, v'_S)).$$

Therefore $U_S(v_S)$ is Lipschitz, hence absolutely continuous w.r.t. v_S and therefore differentiable w.r.t. v_S almost everywhere. Then applying the envelope theorem to the above inequality at each point of differentiability, I obtain that

$$\frac{dU_S(v_S)}{dv_S} = 1 - q(v_B, v_S).$$

Then I have that

$$t_B(v_B, v_S) = U_S(1) - v_S(1 - q(v_B, v_S)) - \int_{v_S}^1 q(v_B, x)dx.$$

Note that $U_S(1) \geq 1$ by the ex-post individually rational constraint. If $U_S(1) > 1$, then I can reduce it to 1 so that I can decrease the payment to S for any value profile in which the buyer's

value is v_B. And the profit will be weakly greater. Thus, when searching for an optimal dominant-strategy mechanism, it is without loss of generality to let $U_S(1) = 1$. Then I obtain that $t_S(v) = 1-(1-q(v))v_S - \int_{v_S}^1 (1-q(v_B,x)dx = q(v)v_S + \int_{v_S}^1 q(v_B,x)dx$.
(iv). This is implied by (ii) and (iii).

1.7.2 Illustration of Theorem 4

Construction of Generalized Random Double Auction

Lemma 6. *Given a trading mechanism* $(q,t_B,t_S) \in \mathcal{D}$*, if* π *minimizes the expected profit over* $\Pi(M_B,M_S)$*, then there exist real numbers* λ_B*,* λ_S *and* μ *such that*

$$\lambda_B v_B + \lambda_S v_S + \mu = t(v), \quad \forall v \in supp(\pi). \tag{CS}$$

(CS) is a *complementary slackness condition*, stating that the ex-post profit is a linear function of the true values for any value profile in the support of a worst value distribution. The complementary slackness condition is a result of strong duality. The proof is similar to the one for the main model (See Appendix 1.7.4 for details). The complementary slackness condition is useful in the construction of a maxmin trading mechanism for the asymmetric informational environment.

For the asymmetric informational environment, it is natural to attach different weights to the submitted bid price and the submitted ask price. I thus form an educated guess of the trading region in a maxmin trading mechanism: Trade takes place with positive probability if and only if the difference between a *weighted* bid (true value of the buyer) $r_2 \cdot v_B$ and a (different) *weighted* ask (true value of the seller) $(1-r_1)\cdot v_S$ exceeds a threshold $r_1 r_2 > 0$, or $r_2 v_B - (1-r_1)v_S > r_1 r_2$. In addition, again, the support of a worst value distribution coincides with the trading region (including the boundary). Together with (iv) of Lemma 5, the complementary slackness condition

(CS) can be expressed as follows: For any $(v_B, v_S) \in AT$,

$$\lambda_B^{**} v_B + \lambda_S^{**} v_S + \mu^{**} = (v_B - v_S) \cdot q^{**}(v_B, v_S) - \int_{\frac{1-r_1}{r_2} v_S + r_1}^{v_B} q^{**}(x, v_S) dx - \int_{v_S}^{\frac{r_2}{1-r_1}(v_B - r_1)} q^{**}(v_B, x) dx. \tag{CS-A}$$

Now I solve for the trading rule q^{**}. Similarly, I first take the first order derivatives with respect to v_B and v_S respectively, and I obtain that for any $(v_B, v_S) \in AT$,

$$(v_B - v_S) \cdot \frac{\partial q^{**}(v_B, v_S)}{\partial v_B} - \frac{\partial \int_{v_S}^{\frac{r_2}{1-r_1}(v_B - r_1)} q^{**}(v_B, x) dx}{\partial v_B} = \lambda_B^{**}, \tag{FOC-B}$$

$$(v_B - v_S) \cdot \frac{\partial q^{**}(v_B, v_S)}{\partial v_S} - \frac{\partial \int_{\frac{1-r_1}{r_2} v_S + r_1}^{v_B} q^{**}(x, v_S) dx}{\partial v_S} = \lambda_S^{**}. \tag{FOC-S}$$

Then, I take the cross partial derivative, with some algebra, I obtain that

$$(v_B - v_S) \cdot \frac{\partial q^{**}(v_B, v_S)}{\partial v_B \partial v_S} = 0.$$

Thus, $q^{**}(v_B, v_S)$ is separable, which can be expressed as the sum of two functions f^{**} and g^{**}: For any $(v_B, v_S) \in AT$,

$$q^{**}(v_B, v_S) = f^{**}(v_B) + g^{**}(v_S). \tag{B.1.1}$$

Again, the separable nature is crucial for solving (CS-A). Plugging (B.1.1) into (FOC-B) and (FOC-S), I obtain that for any $(v_B, v_S) \in AT$,

$$\left[\left(1 - \frac{r_2}{1-r_1}\right) v_B + \frac{r_1 r_2}{1-r_1}\right] \cdot (f^{**})'(v_B) - \frac{r_2}{1-r_1} \cdot \left[f^{**}(v_B) + g^{**}\left(\frac{r_2}{1-r_1}(v_B - r_1)\right)\right] = \lambda_B^{**}, \tag{B.1.2}$$

$$\left[\left(\frac{1-r_1}{r_2} - 1\right) v_S + r_1\right] \cdot (g^{**})'(v_S) + \frac{1-r_1}{r_2} \cdot \left[f^{**}\left(\frac{1-r_1}{r_2} v_S + r_1\right) + g^{**}(v_S)\right] = \lambda_S^{**}. \tag{B.1.3}$$

Note that $f^{**}(v_B) + g^{**}\left(\frac{r_2}{1-r_1}(v_B - r_1)\right) = 0$ and that $f^{**}\left(\frac{1-r_1}{r_2} v_S + r_1\right) + g^{**}(v_S) = 0$ because trade does not take place in the boundary of the trading region, i.e., $q^{**}(v_B, v_S) = 0$ for

$r_2 v_B - (1-r_1)v_S = r_1 r_2$. Then it is straightforward to solve for $f^{**}(v_B)$ and $g^{**}(v_S)$, and I obtain that

$$f^{**}(v_B) = \frac{(1-r_1)\lambda_B^{**}}{1-r_1-r_2} \cdot \ln\left[\left(1-\frac{r_2}{1-r_1}\right)v_B + \frac{r_1 r_2}{1-r_1}\right] + c_B^{**}, \tag{B.1.4}$$

$$g^{**}(v_S) = \frac{r_2\lambda_S^{**}}{1-r_1-r_2} \cdot \ln\left[\left(\frac{1-r_1}{r_2}-1\right)v_S + r_1\right] + c_S^{**}, \tag{B.1.5}$$

where c_B^{**} and c_S^{**} are some constants. Observe that

$$g^{**}\left(\frac{r_2(v_B-r_1)}{1-r_1}\right) = \frac{r_2\lambda_S^{**}}{1-r_1-r_2} \cdot \ln\left[\left(1-\frac{r_2}{1-r_1}\right)v_B + \frac{r_1 r_2}{1-r_1}\right] + c_S^{**}.$$

Then, again, using that $q^{**}(v_B, v_S) = 0$ for $r_2 v_B - (1-r_1)v_S = r_1 r_2$, I have that

$$(1-r_1)\lambda_B^{**} + r_2\lambda_S^{**} = c_B^{**} + c_S^{**} = 0. \tag{B.1.6}$$

Now plugging (B.1.4),(B.1.5) and (B.1.6) into (B.1.1), I obtain that for any $(v_B, v_S) \in AT$,

$$q^{**}(v_B, v_S) = \frac{(1-r_1)\lambda_B^{**}}{1-r_1-r_2} \cdot \left[\ln\left(\frac{1-r_1-r_2}{1-r_1}v_B + \frac{r_1 r_2}{1-r_1}\right) - \ln\left(\frac{1-r_1-r_2}{r_2}v_S + r_1\right)\right].$$

Likewise, to solve for λ_B^{**}, I let $q^{**}(1,0)$ be 1 and obtain that $\lambda_B^{**} = \frac{1-r_1-r_2}{(1-r_1)\ln\frac{1-r_2}{r_1}}$. So far I have obtained the trading rule q^{**}[34]. The payment rule t_B^{**} (resp, the transfer rule t_S^{**}) is then characterized by (ii) (resp, (iii)) of Lemma 5.

Construction of Asymmetric Triangular Value Distribution

Similar to the symmetric informational environment, I impose a *zero virtual value condition* on the joint distribution, stating that virtual value is 0 for any value profile in the

[34]Plugging the trading rule q^{**} into (CS-A), it is straightforward that $\mu^{**} = -\frac{r_1(1-r_1-r_2)}{(1-r_1)\ln\frac{1-r_2}{r_1}}$.

support except for the highest joint type. Formally,

$$\phi(v) = 0, \quad \forall v \in AT \setminus \{(1,0)\}. \tag{ZVV-A}$$

The construction procedure for the joint distribution is exactly the same. Therefore I omit it. Note that the marginal distribution no longer has a uniform distribution part since $v_B - v_S$ is no longer a constant on the boundary of the trading region due to different weights for the buyer and the seller. The final step is to make sure that the constructed joint distribution has the known expectations. Given the marginal distributions for the buyer and the seller, I have a system of two equations (KE-B) and (KE-S). Lemma 3 states that a solution exists for the asymmetric informational environment, details of which are given in Appendix 1.7.3.

1.7.3 Proofs for Section 1.3

Proof of Lemma 3

I start from establishing the following four claims regarding some properties of the functions $H_1(r_1,r_2)$ and $H_2(r_1,r_2)$, which play a crucial role in establishing Lemma 3. First, by simple calculation, I have that for $(r_1,r_2) \in (0,1)^2$,

$$H_1(r_1,r_2) = \frac{r_1(1-r_2)(1-r_1)^2}{(1-r_1-r_2)^2} \cdot \ln\frac{1-r_2}{r_1} - \frac{r_1 r_2(1-r_1)}{1-r_1-r_2} + r_1, \tag{C.1.1}$$

$$H_2(r_1,r_2) = \frac{r_1(1-r_2)r_2^2}{(1-r_1-r_2)^2} \cdot \ln\frac{1-r_2}{r_1} - \frac{r_1 r_2^2}{1-r_1-r_2}. \tag{C.1.2}$$

First note that H_1 and H_2 are not well-defined when $0 < r_1 = 1-r_2 < 1$. Using L'Hôpital's rule, it is straightforward to show that $\lim_{1-r_2 \to r_1} H_1(r_1,r_2) = \frac{1-r_1^2+2r_1}{2}$ and $\lim_{1-r_2 \to r_1} H_2(r_1,r_2) = \frac{(1-r_1)^2}{2}$. I thus define $H_1(r_1,r_2) := \lim_{1-r_2 \to r_1} H_1(r_1,r_2)$ and $H_2(r_1,r_2) := \lim_{1-r_2 \to r_1} H_2(r_1,r_2)$ when $0 < r_1 = 1-r_2 < 1$. This makes H_1 and H_2 continuous on $(0,1)^2$. In addition, using L'Hôpital's rule, it is straightforward to show that $\lim_{r_1 \to 0} H_1(r_1,r_2) = 0$

for $r_2 \in (0,1)$, $\lim_{r_1 \to 1} H_1(r_1,r_2) = 1$ for $r_2 \in (0,1)$, $\lim_{r_2 \to 0} H_1(r_1,r_2) = r_1 - r_1 \ln r_1$ for $r_1 \in (0,1)$, $\lim_{r_2 \to 1} H_1(r_1,r_2) = 1$ for $r_1 \in (0,1)$, $\lim_{r_1 \to 0} H_2(r_1,r_2) = 0$ for $r_2 \in (0,1)$, $\lim_{r_1 \to 1} H_2(r_1,r_2) = (1-r_2)\ln(1-r_2) + r_2$ for $r_2 \in (0,1)$, $\lim_{r_2 \to 0} H_2(r_1,r_2) = 0$ for $r_1 \in (0,1)$ and $\lim_{r_2 \to 1} H_1(r_1,r_2) = 1$ for $r_1 \in (0,1)$. Therefore I define $H_1(r_1,r_2)$ and $H_2(r_1,r_2)$ as follows.

$$H_1(r_1,r_2) = \begin{cases} \frac{(1-r_2)r_1(1-r_1)^2}{(1-r_1-r_2)^2} \cdot \ln \frac{1-r_2}{r_1} - \frac{r_1 r_2 (1-r_1)}{1-r_1-r_2} + r_1 & \text{if } (r_1,r_2) \in (0,1)^2 \text{ and } r_1 + r_2 \neq 1, \\ \frac{1-r_1^2+2r_1}{2} & \text{if } \quad 0 < r_1 = 1 - r_2 < 1, \\ 0 & \text{if } r_1 = 0 \text{ and } r_2 \in (0,1), \\ 1 & \text{if } r_1 = 1 \text{ and } r_2 \in (0,1), \\ r_1 - r_1 \ln r_1 & \text{if } r_2 = 0 \text{ and } r_1 \in (0,1), \\ 1 & \text{if } r_2 = 1 \text{ and } r_1 \in (0,1). \end{cases}$$

$$H_2(r_1,r_2) = \begin{cases} \frac{r_1(1-r_2)r_2^2}{(1-r_1-r_2)^2} \cdot \ln \frac{1-r_2}{r_1} - \frac{r_1 r_2^2}{1-r_1-r_2} & \text{if } (r_1,r_2) \in (0,1)^2 \text{ and } r_1 + r_2 \neq 1, \\ \frac{(1-r_1)^2}{2} & \text{if } \quad 0 < r_1 = 1 - r_2 < 1, \\ 0 & \text{if } r_1 = 0 \text{ and } r_2 \in (0,1), \\ (1-r_2)\ln(1-r_2) + r_2 & \text{if } r_1 = 1 \text{ and } r_2 \in (0,1), \\ 0 & \text{if } r_2 = 0 \text{ and } r_1 \in (0,1), \\ 1 & \text{if } r_2 = 1 \text{ and } r_1 \in (0,1). \end{cases}$$

Claim 1. *Fix any $r_2 \in [0,1)$, $H_1(r_1,r_2)$ is strictly increasing in r_1. Moreover, for any $r_2 \in (0,1)$, $\lim_{r_1 \to 1-r_2} \frac{\partial H_1(r_1,r_2)}{\partial r_1}$ exists and is positive. In addition, for any $r_2 \in [0,1)$, as $r_1 \to 1$, $H_1(r_1,r_2) \to 1$.*

Proof of Claim 1. When $r_2 = 0$, $H_1(r_1,r_2) = r_1 - r_1 \ln r_1$. This is an strictly increasing function because $\frac{\partial H_1(r_1,r_2)}{\partial r_1} = -\ln r_1 > 0$. In addition, by L'Hôpital's rule, $\lim_{r_1 \to 1} H_1(r_1,r_2) = 1$. Thus, Claim 1 holds when $r_2 = 0$. When $0 < r_2 < 1$, I already have that $\lim_{r_1 \to 1} H_1(r_1,r_2) = 1$. Now taking the first order derivative w.r.t. r_1 to (C.1.1), I obtain that

$$\frac{\partial H_1(r_1,r_2)}{\partial r_1} = \frac{(1-r_1)(1-r_2)}{(1-r_1-r_2)^2} \cdot \left[\left(1-3r_1+\frac{2r_1(1-r_1)}{1-r_1-r_2}\right)\cdot \ln\frac{1-r_2}{r_1} - 2r_2\right]. \qquad \text{(C.1.3)}$$

Then to show the first part of Claim 1, it suffices to show that if $(r_1,r_2)\in(0,1)^2$ and $r_1+r_2\neq 1$,

$$\left(1-3r_1+\frac{2r_1(1-r_1)}{1-r_1-r_2}\right)\cdot \ln\frac{1-r_2}{r_1} - 2r_2 > 0. \qquad \text{(C.1.4)}$$

Let $\beta := \frac{1-r_2}{r_1}$, then $\beta \in (0,1)\cup(1,\infty)$. Plugging $r_2 = 1-\beta r_1$ into (C.1.4), it suffices to show that for any $\beta\in(0,1)\cup(1,\infty)$,

$$\left(1-3r_1+\frac{2(1-r_1)}{\beta-1}\right)\cdot \ln\beta - 2\cdot(1-\beta r_1) > 0. \qquad \text{(C.1.5)}$$

Slightly rewriting (C.1.5), it suffices to show that for any $\beta\in(0,1)\cup(1,\infty)$,

$$\frac{\beta+1}{\beta-1}\cdot\ln\beta - 2 + \left(-\frac{3\beta-1}{\beta-1}\cdot\ln\beta+2\beta\right)\cdot r_1 > 0. \qquad \text{(C.1.6)}$$

Then, it suffices to show that for any $\beta\in(0,1)\cup(1,\infty)$, the following two inequalities hold:

$$\frac{\beta+1}{\beta-1}\cdot\ln\beta - 2 > 0, \qquad \text{(C.1.7)}$$

$$-\frac{3\beta-1}{\beta-1}\cdot\ln\beta+2\beta > 0. \qquad \text{(C.1.8)}$$

Now to prove (C.1.7), it suffices to show that $f(\beta) := \ln\beta - \frac{2(\beta-1)}{\beta+1} > 0$ for $\beta\in(1,\infty)$ and $f(\beta)<0$ for $\beta\in(0,1)$. Taking the first order derivative to $f(\beta)$, I obtain that

$$f'(\beta) = \frac{(\beta-1)^2}{\beta(\beta+1)^2}. \qquad \text{(C.1.9)}$$

Therefore, $f(\beta)$ is strictly increasing. Note that $f(1)=0$. Thus, I proved (C.1.7). To prove

(C.1.8), it suffices to show that $h(\beta) := (1-3\beta)\ln\beta + 2\beta(\beta-1) > 0$ for $\beta \in (1,\infty)$ and $h(\beta) < 0$ for $\beta \in (0,1)$. Taking the first order derivative to $h(\beta)$, I obtain that

$$h'(\beta) = 4\beta - 3\ln\beta + \frac{1}{\beta} - 5. \tag{C.1.10}$$

Now taking the second order derivative to $h(\beta)$, I obtain that

$$h''(\beta) = \frac{(4\beta+1)(\beta-1)}{\beta^2}. \tag{C.1.11}$$

Note that $h''(\beta) > 0$ when $\beta > 1$, $h''(\beta) < 0$ when $\beta < 1$ and $h''(1) = 0$. This implies that $h'(\beta)$ is minimized at $\beta = 1$. Note that $h'(1) = 0$. This implies that $h(\beta)$ is strictly increasing. Finally, note that $h(1) = 0$. This implies that (C.1.8) holds.

Using L'Hôpital's rule, I have that $\lim_{r_1 \to 1-r_2} \frac{\partial H_1(r_1,r_2)}{\partial r_1} = \frac{r_2(6-5r_2)}{1-r_2} > 0$ for $r_2 \in (0,1)$. □

Claim 2. *Fix any $r_1 \in (0,1)$, $H_1(r_1,r_2)$ is strictly increasing in r_2. Moreover, for any $r_1 \in (0,1)$, $\lim_{r_2 \to 1-r_1} \frac{\partial H_1(r_1,r_2)}{\partial r_2}$ exists and is positive. In addition, for any $r_1 \in (0,1)$, as $r_2 \to 1$, $H_1(r_1,r_2) \to 1$.*

Proof of Claim 2. When $0 < r_1 < 1$, I already have that $\lim_{r_2 \to 1} H_1(r_1,r_2) = 1$. Now taking the first order derivative w.r.t. r_2 to (C.1.1), with some algebra, I obtain that

$$\frac{\partial H_1(r_1,r_2)}{\partial r_2} = \frac{(1-r_1)^2 r_1}{(1-r_1-r_2)^2} \cdot \left[\left(-1 + \frac{2(1-r_2)}{1-r_1-r_2}\right) \cdot \ln\frac{1-r_2}{r_1} - 2\right]. \tag{C.1.12}$$

Then it suffices to show that if $(r_1,r_2) \in (0,1)^2$ and $r_1 + r_2 \neq 1$,

$$\left(-1 + \frac{2(1-r_2)}{1-r_1-r_2}\right) \cdot \ln\frac{1-r_2}{r_1} - 2 > 0. \tag{C.1.13}$$

Plugging $r_2 = 1 - \beta r_1$ into (C.1.13), it suffices to show that for any $\beta \in (0,1) \cup (1,\infty)$,

$$\frac{\beta+1}{\beta-1} \cdot \ln\beta - 2 > 0. \tag{C.1.14}$$

This is exactly (C.1.7) and has been shown in the Proof of Claim 1.

Using L'Hôpital's rule, I have that $\lim_{r_2 \to 1-r_1} \frac{\partial H_1(r_1,r_2)}{\partial r_2} = \frac{(1-r_1)^2}{6r_1} > 0$ for $r_1 \in (0,1)$. □

Claim 3. *Fix any $r_2 \in (0,1)$, $H_2(r_1,r_2)$ is strictly increasing in r_1. Moreover, for $r_2 \in (0,1)$, $\lim_{r_1 \to 1-r_2} \frac{\partial H_2(r_1,r_2)}{\partial r_1}$ exists and is positive.*

Proof of Claim 3. Taking the first order derivative w.r.t. r_1 to (C.1.2), I obtain that

$$\frac{\partial H_2(r_1,r_2)}{\partial r_1} = \frac{(1-r_2)r_2^2}{(1-r_1-r_2)^2} \cdot \left[\left(1 + \frac{2r_1}{1-r_1-r_2}\right) \cdot \ln\frac{1-r_2}{r_1} - 2\right]. \tag{C.1.15}$$

Then it suffices to show that if $(r_1,r_2) \in (0,1)^2$ and $r_1 + r_2 \neq 1$,

$$\left(1 + \frac{2r_1}{1-r_1-r_2}\right) \cdot \ln\frac{1-r_2}{r_1} - 2 > 0. \tag{C.1.16}$$

Plugging $r_2 = 1 - \beta r_1$ into (C.1.16), it suffices to show that for any $\beta \in (0,1) \cup (1,\infty)$, $\frac{\beta+1}{\beta-1} \ln\beta - 2 > 0$, which is exactly (C.1.7) and has been shown in the Proof of Claim 1.

Using L'Hôpital's rule, I have that $\lim_{r_1 \to 1-r_2} \frac{\partial H_2(r_1,r_2)}{\partial r_1} = \frac{(r_2)^2}{6(1-r_2)} > 0$ for $r_2 \in (0,1)$. □

Claim 4. *Fix any $r_1 \in (0,1]$, $H_2(r_1,r_2)$ is strictly increasing in r_2. Moreover, for any $r_1 \in (0,1)$, $\lim_{r_2 \to 1-r_1} \frac{\partial H_2(r_1,r_2)}{\partial r_2}$ exists and is positive. In addition, for any $r_1 \in (0,1]$, as $r_2 \to 1$, $H_2(r_1,r_2) \to 1$.*

Proof of Claim 4. When $r_1 = 1$, $H_2(r_1,r_2) = (1-r_2) \cdot \ln(1-r_2) + r_2$. This is an strictly increasing function because $\frac{\partial H_2(r_1,r_2)}{\partial r_2} = -\ln(1-r_2) > 0$. In addition, by L'Hôpital's rule, $\lim_{r_2 \to 1} H_2(r_1,r_2) = 1$. Thus, Claim 4 holds when $r_1 = 1$. When $0 < r_1 < 1$, I already have that

$\lim_{r_2 \to 1} H_1(r_1, r_2) = 1$. Now taking the first order derivative w.r.t. r_2 to (C.1.2), I obtain that

$$\frac{\partial H_2(r_1,r_2)}{\partial r_2} = \frac{r_1 r_2}{(1-r_1-r_2)^2} \cdot \left[\left(2 - 3r_2 + \frac{2r_2(1-r_2)}{1-r_1-r_2} \right) \cdot \ln \frac{1-r_2}{r_1} - 2 \cdot (1-r_1) \right]. \quad \text{(C.1.17)}$$

Then to show the first part of Claim 4, it suffices to show that if $(r_1, r_2) \in (0,1)^2$ and $r_1 + r_2 \neq 1$,

$$\left(2 - 3r_2 + \frac{2r_2(1-r_2)}{1-r_1-r_2} \right) \cdot \ln \frac{1-r_2}{r_1} - 2 \cdot (1-r_1) > 0. \quad \text{(C.1.18)}$$

Plugging $r_2 = 1 - \beta r_1$ into (C.1.18), it suffices to show that for any $\beta \in (0,1) \cup (1, \infty)$,

$$\left(3\beta r_1 - 1 + \frac{2\beta(1-\beta r_1)}{\beta - 1} \right) \cdot \ln \beta - 2 \cdot (1-r_1) > 0. \quad \text{(C.1.19)}$$

Slightly rewriting (C.1.19), it suffices to show that for any $\beta \in (0,1) \cup (1, \infty)$,

$$\frac{\beta+1}{\beta-1} \cdot \ln \beta - 2 + \left(\frac{\beta^2 - 3\beta}{\beta - 1} \ln \beta + 2 \right) \cdot r_1 > 0. \quad \text{(C.1.20)}$$

Then, it suffices to show that for any $\beta \in (0,1) \cup (1, \infty)$, the following two inequalities hold:

$$\frac{\beta+1}{\beta-1} \cdot \ln \beta - 2 > 0, \quad \text{(C.1.21)}$$

$$\frac{\beta^2 - 3\beta}{\beta - 1} \cdot \ln \beta + 2 > 0. \quad \text{(C.1.22)}$$

Note that (C.1.21) is exactly (C.1.7), which has been shown in the Proof of Claim 1. To prove (C.1.22), it suffices to show that $g(\beta) := (\beta^2 - 3\beta) \ln \beta + 2(\beta - 1) > 0$ for $\beta \in (1, \infty)$ and $g(\beta) < 0$ for $\beta \in (0,1)$. Taking the first order derivative to $g(\beta)$, I obtain that

$$g'(\beta) = (2\beta - 3) \cdot \ln \beta + \beta - 1. \quad \text{(C.1.23)}$$

Now taking the second order derivative to $g(\beta)$, I obtain that

$$g''(\beta) = 2\ln\beta - \frac{3}{\beta} + 3. \tag{C.1.24}$$

Note that $g''(\beta)$ is strictly increasing and $g''(1) = 0$. This implies that $g'(\beta)$ is minimized at $\beta = 1$. Note that $g'(1) = 0$. This implies that $g(\beta)$ is strictly increasing. Finally, note that $g(1) = 0$. This implies that (C.1.22) holds.

Using L'Hôpital's rule, I have that $\lim_{r_2 \to 1-r_1} \frac{\partial H_2(r_1,r_2)}{\partial r_2} = \frac{(1-r_1)(5r_1+1)}{6r_1} > 0$ for $r_1 \in (0,1)$. □

I am now ready to prove Lemma 3. Fix any (M_B, M_S) in which $0 < M_S < M_B < 1$ and $M_B + M_S \neq 1$. By Claim 1, 2 and the Intermediate Value Theorem, I have that for any $r_2 \in [0,1)$, there exists a unique $I(r_2) \in (0,1)$ such that $r_1 = I(r_2)$ is a solution to $H_1(r_1,r_2) = M_B$. In addition, $I(r_2)$ is a strictly decreasing function. Moreover, by the Implicit Function Theorem[35], $I(r_2)$ is continuous at each $r_2 \in [0,1)$. By Claim 3, 4 and the Intermediate Value Theorem, I have that for any $r_1 \in (0,1]$, there exists a unique $J(r_1) \in (0,1)$ such that $r_2 = J(r_1)$ is a solution to $H_2(r_1,r_2) = M_S$. In addition, $J(r_1)$ is a strictly decreasing function. Moreover, by the Implicit Function Theorem[36], $J(r_1)$ is continuous at each $r_1 \in (0,1]$. Thus it suffices to prove that there exists $r_2 \in (0,1)$ such that

$$J(I(r_2)) = r_2. \tag{C.1.25}$$

Note that $J(I(r_2))$ is a continuous and strictly increasing function for $r_2 \in [0,1)$. Also note that $J(I(0)) \in (0,1)$ because $I(0) \in (0,1)$ and $J(r_1) \in (0,1)$ when $r_1 \in (0,1)$. Now, by the

[35]The Implicit Function Theorem applies for any $r_2 \in [0,1)$ because by Claim 1 and 2, $\frac{\partial H_1(I(r_2),r_2)}{\partial r_1} > 0$ and $\frac{\partial H_1(I(r_2),r_2)}{\partial r_2} > 0$ for any $r_2 \in [0,1)$.

[36]The Implicit Function Theorem applies for any $r_1 \in (0,1]$ because by Claim 3 and 4, $\frac{\partial H_2(r_1,J(r_1))}{\partial r_1} > 0$ and $\frac{\partial H_2(r_1,J(r_1))}{\partial r_2} > 0$ for any $r_1 \in (0,1]$.

Intermediate Value Theorem, it suffices to show that there exists some $r_2 \in (0,1)$ such that

$$J(I(r_2)) \leq r_2. \tag{C.1.26}$$

Because J is strictly decreasing, it is equivalent to showing that there exists some $r_2 \in (0,1)$ such that

$$I(r_2) \geq J^{-1}(r_2). \tag{C.1.27}$$

By Claim 1, this is equivalent to showing that there exists some $r_2 \in (0,1)$ such that

$$H_1(J^{-1}(r_2), r_2) \leq M_B. \tag{C.1.28}$$

Let $\varepsilon := M_B - M_S > 0$. I observe a relationship between the two functions H_1 and H_2 when $(r_1, r_2) \in (0,1)^2$:

$$H_1(r_1,r_2) - H_2(r_1,r_2) = \left(\frac{(1-r_1)^2}{r_2^2} - 1 \right) \cdot H_2(r_1,r_2) + r_1 \cdot (2 - r_1). \tag{C.1.29}$$

Note that when $r_2 \to 1$, $J^{-1}(r_2) \to 0$. To see this, suppose not, then by Claim 4, $H_2(J^{-1}(r_2), r_2) \to 1$ when $r_2 \to 1$, a contradiction to $H_2(J^{-1}(r_2), r_2) = M_S < 1$. Then by (C.1.29), as $r_2 \to 1$,

$$\begin{aligned}
H_1(J^{-1}(r_2), r_2) - M_S &= H_1(J^{-1}(r_2), r_2) - H_2(J^{-1}(r_2), r_2) \\
&= \left(\frac{(1-J^{-1}(r_2))^2}{(r_2)^2} - 1 \right) \cdot H_2(J^{-1}(r_2), r_2) + J^{-1}(r_2) \cdot (2 - J^{-1}(r_2)) \\
&= \left(\frac{(1-J^{-1}(r_2))^2}{(r_2)^2} - 1 \right) \cdot M_S + J^{-1}(r_2) \cdot \big(2 - J^{-1}(r_2)\big) \\
&\to \left(\frac{(1-0)^2}{1^2} - 1 \right) \cdot M_S + 0 \cdot (2-0) \\
&= 0.
\end{aligned}$$

This implies that there exists some $r_2 \in (0,1)$ such that

$$\left|H_1(J^{-1}(r_2), r_2) - M_S\right| \leq \frac{\varepsilon}{2}. \tag{C.1.30}$$

Note that (C.1.30) implies (C.1.28) because $H_1(J^{-1}(r_2), r_2) \leq M_S + \frac{\varepsilon}{2} < M_S + \varepsilon = M_B$.

Finally, suppose that $r_1 + r_2 = 1$ for the solution (r_1, r_2), then $M_B + M_S = H_1(r_1, r_2) + H_2(r_1, r_2) = 1$ by the definition of $H_1(r_1, r_2)$ and $H_2(r_1, r_2)$, a contradiction to the assumption that $M_B + M_S \neq 1$. Therefore, I have that $r_1 + r_2 \neq 1$ for the solution (r_1, r_2).

Proof of Theorem 4

Step 1: The generalized random double auction maximizes the expected profit under the asymmetric triangular value distribution. To show this, first note that (ZVV-A) holds by construction. In addition, the virtual value is non-positive for any value profile outside the support and positive for the value profile $(1,0)$. Then any dominant-strategy mechanism in which 1) ex-post participation constraints are binding for zero-value buyer and one-value seller, and 2) trade does not take place when $r_2 v_B - (1-r_1) v_S < r_1 r_2$ and trade takes place with probability one when $(v_B, v_S) = (1,0)$ is an optimal trading mechanism. It is straightforward to see that the generalized random double auction is such a mechanism.

Step 2: The asymmetric triangular value distribution minimizes the expected profit under the generalized random double auction. I use the duality theory to show this. Note that the asymmetric triangular value distribution is a feasible value distribution by construction. By the virtual representation, the expected profit (the value of the primal problem) given the random double auction and the symmetric triangular value distribution is $Pr(1,0) \times (1-0) = r_1(1-r_2)$. Second, the constraints in the dual problem hold for all value profiles. To see this, note that for any value profile $v = (v_B, v_S)$ inside the support (or $v \in AT$), $\lambda_B^{**} v_B + \lambda_S^{**} v_S + \mu^{**} = t^{**}(v)$ by (CS-A). Also, for any value profile $v = (v_B, v_S)$ in which $r_2 v_B - (1-r_1) v_S = r_1 r_2$, $\lambda_B^{**} v_B + \lambda_S^{**} v_S + \mu^{**} =$

$0 = t^{**}(v)$ because $\lambda_B^{**} = \frac{1-r_1-r_2}{(1-r_1)\ln\frac{1-r_2}{r_1}}$, $\lambda_S^{**} = -\frac{1-r_1-r_2}{r_2\ln\frac{1-r_2}{r_1}}$ and $\mu^{**} = -\frac{r_1(1-r_1-r_2)}{(1-r_1)\ln\frac{1-r_2}{r_1}}$. Then, for any value profile $v = (v_B, v_S)$ in which $r_2 v_B - (1-r_1)v_S < r_1 r_2$, $\lambda_B^{**} v_B + \lambda_S^{**} v_S + \mu^{**} < 0 = t^{**}(v)$ because $\lambda_B^{**} > 0$ and $\lambda_S^{**} < 0$. Finally, the value of the dual problem given the constructed dual variables is equal to $r_1(1-r_2)$ by simple calculation. The details of the constructed dual variables as well as the characterization are given in Appendix 1.7.2.

1.7.4 Proof of Theorem 5

To facilitate the analysis, I first establish a strong duality result. Given a dominant-strategy mechanism (q, t_B, t_S), the primal minimization problem of adversarial Nature is as follows (with dual variables in the bracket):

$$\inf_{\pi} \int t(v) d\pi(v) \tag{P}$$

subject to

$$\int \max\{v_B - v_S, 0\} d\pi(v) = GFT, \quad (\lambda)$$

$$\int d\pi(v) = 1. \quad (\mu)$$

It has the following dual maximization problem:

$$\sup_{\lambda \in \mathcal{R}, \mu \in \mathcal{R}} \lambda GFT + \mu \tag{D}$$

subject to

$$\lambda \max\{v_B - v_S, 0\} + \mu \leq t(v), \quad \forall v \in V.$$

Note that the value of (P) is bounded by 1 as $t(v) \leq 1$. In addition, the joint distribution that puts all probability masses on the value profile $(\frac{1+GFT}{2}, \frac{1-GFT}{2})$ is in the interior of the primal cone. Then, by Theorem 3.12 in Anderson and Nash (1987), strong duality holds. Theorem 5 is established by the following three steps.

Step 1: Narrow down the search to a class of mechanisms.

I divide all deterministic dominant-strategy mechanisms that satisfy the properties stated in Lemma 5 into the following four classes:

Class 1: The trade boundary touches on the value profiles $(c_1, 1)$ and $(0, c_2)$ for some $0 \leq c_1 \leq 1, 0 \leq c_2 \leq 1$.

Class 2: The trade boundary touches on the value profiles $(0, c_1)$ and $(1, c_2)$ for some $0 \leq c_1 \leq 1, 0 \leq c_2 \leq 1$.

Class 3: The trade boundary touches on the value profiles $(c_1, 0)$ and $(c_2, 1)$ for some $0 \leq c_1 \leq 1, 0 \leq c_2 \leq 1$.

Class 4: The trade boundary touches on the value profiles $(c_1, 0)$ and $(1, c_2)$ for some $0 < c_1 \leq 1, 0 \leq c_2 < 1$[37].

By (iv) of Lemma 5, I can show that the ex-post profit from the value profile $(0,0)$ or $(0,1)$ will never be positive for any mechanism in *Class 1*, *Class 2* or *Class 3*. To see this, note that for any mechanism in *Class 1*: $t(0,0) = 0 - c_2 = -c_2 \leq 0$, $t(1,0) = (1-0) \cdot 1 - 1 - 1 = -1 < 0$; for any mechanism in *Class 2*: $t(0,0) = 0 - c_1 = -c_1 \leq 0$, $t(1,0) = (1-0) \cdot 1 - 1 - c_2 = -c_2 \leq 0$; for any mechanism in *Class 3*: $t(0,0) = 0$, $t(1,0) = (1-0) \cdot 1 - (1-c_1) - 1 = -(1-c_1) \leq 0$. Consider the joint distribution that puts probability masses GFT and $1 - GFT$ on the value profiles (1,0) and (0,0) respectively. It is straightforward to verify that this is a feasible value distribution; moreover, the profit under this joint distribution cannot be positive. Therefore, I can restrict attention to *Class 4* only.

Step 2: Identify an upper bound of the profit guarantee.

I propose a relaxation of (D) by ignoring many constraints. Specifically, the only remaining ones are the constraints for the following four value profiles: (0,0), (1,0), $(c_1, 0)$ and $(1, c_2)$. Formally, I have the following relaxed problem:

$$\max_{\lambda \in \mathcal{R}, \mu \in \mathcal{R}} \lambda GFT + \mu \tag{D'}$$

[37] The cases where $c_1 = 0$ are included in *Class 2*, and the cases where $c_2 = 1$ are included in *Class 3*.

subject to

$$\mu \leq 0, \tag{D.1.1}$$

$$\lambda + \mu \leq c_1 - c_2. \tag{D.1.2}$$

$$\lambda c_1 + \mu \leq 0, \tag{D.1.3}$$

$$\lambda(1 - c_2) + \mu \leq 0. \tag{D.1.4}$$

Note that the value of (D'), denoted by val(D'), is weakly greater than the value of (D). Now I will find an upper bound of the value of (D') across $0 < c_1 \leq 1$ and $0 \leq c_2 < 1$, and show that it is attainable by constructing deterministic dominant-strategy mechanisms and a feasible value distribution.

If $\lambda \leq 0$, then by (D.1.1), val(D')≤ 0 for any $0 < c_1 \leq 1$ and $0 \leq c_2 < 1$. Henceforth, I restrict attention to $\lambda > 0$. If $c_1 \geq GFT$, then $\lambda GFT + \mu \leq \lambda c_1 + \mu \leq 0$, where the first inequality follows from $\lambda > 0$ and the second inequality follows from (D.1.3). If $c_2 \leq 1 - GFT$, then $\lambda GFT + \mu \leq \lambda(1 - c_2) + \mu \leq 0$, where the first inequality follows from $\lambda > 0$ and the second inequality follows from (D.1.4). If $c_1 \leq c_2$, then $\lambda GFT + \mu \leq \lambda + \mu \leq 0$, where the first inequality follows from $\lambda > 0$ and the second inequality follows from (D.1.2). Therefore, I can restrict attention to $1 - GFT < c_2 < c_1 < GFT$, because otherwise the profit guarantee cannot be positive. This also implies that I can restrict attention to informational environments in which $GFT > \frac{1}{2}$, because otherwise the profit guarantee cannot be positive. Now I am left with (D.1.2), (D.1.3) and (D.1.4) as they imply (D.1.1).

If $c_1 \geq 1 - c_2$, then I am left with (D.1.2) and (D.1.3), as (D.1.4) is not binding. It is straightforward that the solution occurs when both (D.1.2) and (D.1.3) are binding, because $c_1 < GFT < 1$. The solution is $\lambda = \frac{c_1 - c_2}{1 - c_1}$, $\mu = -\frac{c_1(c_1 - c_2)}{1 - c_1}$. val(D')$= \frac{(c_1 - c_2)(GFT - c_1)}{1 - c_1} := K(c_1, c_2)$. Now I maximize $K(c_1, c_2)$ subject to the constraints that $1 - GFT < c_2 < c_1 < GFT$ and $c_1 \geq 1 - c_2$. Observe that $c_2 = 1 - c_1$ in the optimal solution as $K(c_1, c_2)$ is decreasing in c_2.

Now, with some algebra,

$$K(c_1, 1-c_1) = 1 + 2(1-GFT) - 2(1-c_1) - \frac{1-GFT}{1-c_1}.$$

Then it is straightforward that the optimal solution is $c_1 = 1 - \sqrt{\frac{1-GFT}{2}}$, $c_2 = 1 - c_1 = \sqrt{\frac{1-GFT}{2}}$, and the maximized value of $K(c_1, c_2)$ is $\left(1 - \sqrt{2(1-GFT)}\right)^2$.

If $c_1 \leq 1 - c_2$, then I am left with (D.1.2) and (D.1.4), as (D.1.3) is not binding. It is straightforward that the solution occurs when both (D.1.2) and (D.1.4) are binding, because $1 - c_2 < GFT < 1$. The solution is $\lambda = \frac{c_1 - c_2}{c_2}$, $\mu = -\frac{(1-c_2)(c_1-c_2)}{c_2}$. val(D')= $\frac{(c_1-c_2)(GFT-1+c_2)}{c_2} :=$ $L(c_1, c_2)$. Now I maximize $L(c_1, c_2)$ subject to the constraints that $1 - GFT < c_2 < c_1 < GFT$ and $c_1 \leq 1 - c_2$. Observe that $c_1 = 1 - c_2$ in the optimal solution as $L(c_1, c_2)$ is increasing in c_1. Now, with some algebra,

$$L(1-c_2, c_2) = 1 + 2(1-GFT) - 2c_2 - \frac{1-GFT}{c_2}.$$

Then it is straightforward that the optimal solution is $c_2 = \sqrt{\frac{1-GFT}{2}}$, $c_1 = 1 - c_2 = 1 - \sqrt{\frac{1-GFT}{2}}$, and the maximized value of $L(c_1, c_2)$ is $\left(1 - \sqrt{2(1-GFT)}\right)^2$.

Step 3: Show that the upper bound is attainable.

The last step is to construct deterministic trading mechanisms whose profit guarantee is $\left(1 - \sqrt{2(1-GFT)}\right)^2$ when $GFT > \frac{1}{2}$. Consider any deterministic trading mechanism satisfying (*i*), (*ii*) and (*iii*) of Theorem 5. Let $\lambda = \frac{1-\sqrt{2(1-GFT)}}{\sqrt{\frac{1-GFT}{2}}}$ and $\mu = -\frac{(1-\sqrt{2(1-GFT)})(1-\sqrt{\frac{1-GFT}{2}})}{\sqrt{\frac{1-GFT}{2}}}$. I will show that they are feasible for the original dual problem (D).

Note that the constraint for the value profile (1,0) holds with equality by construction. Then the constraint holds for any *interior* value profile[38]. Indeed, the constraint is the most stringent for the value profile (1,0) because the trade boundary is non-decreasing. To see this,

[38] A value profile in which trade takes place with probability one is referred to as an interior value profile.

note that the constraint for any interior value profile (v_B, v_S) is equivalent to that

$$\lambda \max\{v_B - v_S, 0\} + b_1(v_B) - b_2(v_S) + \mu \leq 0, \tag{D.1.5}$$

where $(v_B, b_1(v_B))$ and $(b_2(v_S), v_S)$ are in the trade boundary. Then the L.H.S. of (D.1.5) is maximized at (1,0) because that $\lambda > 0$ and that b_1 as well as b_2 are non-decreasing. For any *exterior* value profile[39], the constraint also holds if (ii) of Theorem 5 holds. To see this, note that given the constructed λ and μ, $\lambda \max\{v_B - v_S, 0\} + \mu = 0$ for the value profiles $\left(1 - \sqrt{\frac{1-GFT}{2}}, 0\right)$ and $\left(1, \sqrt{\frac{1-GFT}{2}}\right)$. Then, by linearity, $\lambda(v_B - v_S) + \mu = 0$ for any value profile on the line linking $\left(1 - \sqrt{\frac{1-GFT}{2}}, 0\right)$ and $\left(1, \sqrt{\frac{1-GFT}{2}}\right)$. Therefore, if (ii) of Theorem 5 holds, the constraint also holds for any exterior value profile because that $\lambda > 0$ and $\mu < 0$. Finally, the value of (D) under the constructed dual variables is $\left(1 - \sqrt{2(1-GFT)}\right)^2$ by simple calculation.

Now consider the joint distribution described in Theorem 5. First, it is straightforward to verify that it is a feasible value distribution. Second, given any trading mechanism satisfying (i), (ii) and (iii) of Theorem 5, the value of (P) is $\left(1 - \sqrt{2(1-GFT)}\right)^2$ under the joint distribution by simple algebra. This finishes the proof.

[39]A value profile in which trade does not take place is referred to as an exterior value profile. Note that by the definition of the trade boundary, a value profile on the trade boundary is also an exterior value profile.

Chapter 2

Correlation-Robust Optimal Auctions

2.1 Introduction

The mechanism design literature assumes that bidders' valuation profile follows a *commonly known* joint distribution. For example, Myerson (1981) assumes that the auctioneer knows the marginal distribution of each bidder's valuation, and also knows that bidders' valuations are independently distributed. While the independent private value model is widely acknowledged as a useful benchmark, little is known about how the optimal mechanism would perform once the model is misspecified. In addition, it is not clear how the auctioneer should determine which model is the correct one to use.

In this paper, I study the single-object auction problem in the correlated private value environment in which bidders' valuation profile is drawn from a general joint distribution. I assume that the auctioneer knows the marginal distribution of a generic bidder's valuation, but does not have any knowledge about the correlation structure of different bidders' valuations[1]. A joint distribution of bidders' valuation profile is said to be *feasible* if it is consistent with the known marginal distribution of a generic bidder's valuation. The auctioneer seeks a dominant-

[1]The framework is originally proposed by Carroll (2017) for the multi-dimensional screening problem. His solution is simple and conveys a clear and intuitive message: if you do not know how to bundle, then do not. It is natural to adapt his framework to an environment with multiple bidders whose private valuations may be correlated.

strategy mechanism. A mechanism is evaluated according to the auctioneer's expected revenue in the dominant-strategy equilibrium derived in the worst case, referred to as the *revenue guarantee*, over all feasible joint distributions. The objective of the auctioneer is to design a mechanism that maximizes the revenue guarantee across some general class of dominant-strategy mechanisms. I call such a mechanism a *maxmin mechanism*.

This framework is in the same spirit as the robust mechanism literature in that it assumes away detailed knowledge of the auctioneer (Wilson (1987)). It is motivated by the observation that the joint distribution is a much higher-dimensional object than the marginal distribution of a generic bidder. Therefore it is more difficult to estimate the joint distribution. Practically, it fits into the situations where the bidder pool changes constantly and then there is no data for estimating the correlation structure. Another situation where the auctioneer may only know the marginal distribution for each bidder is the one where the identities of the participating bidders cannot be observed.

The first main result (Theorem 8) is that, under certain regularity conditions on the marginal distribution, *the second-price auction with the uniformly distributed random reserve* is a maxmin mechanism across all dominant-strategy mechanisms for the two-bidder case. Under this mechanism, a random reserve is drawn from a uniform distribution on $[0,\bar{v}]$ where $\bar{v}$ is the maximum valuation.

The randomness in this mechanism hedges against uncertainty over correlation structures. Indeed, the specific random device in this mechanism exhibits a *full-insurance property*: the expected revenue is the same across all joint distributions consistent with the marginal distribution. To see this, consider a valuation profile (v_1,v_2) in which $v_1 > v_2$. Under this mechanism, if the random reserve r is lower than v_2, occurring with a probability of $\frac{v_2}{\bar{v}}$, then bidder 1 pays v_2; if the random reserve r is between v_2 and v_1, then bidder 1 pays r. Therefore the revenue from the valuation profile (v_1,v_2) is $v_2 \cdot \frac{v_2}{\bar{v}} + \int_{v_1}^{v_2} r \cdot \frac{1}{\bar{v}} dr = \frac{v_1^2+v_2^2}{2\bar{v}}$, which is separable in v_1 and v_2. This implies the full-insurance property. In addition, the expected revenue is the second moment of the marginal distribution over the maximum valuation.

I show that this mechanism is a maxmin mechanism across all dominant-strategy mechanisms under certain regularity conditions by constructing a feasible joint distribution such that (i) the joint distribution minimizes the expected revenue under the mechanism across all feasible joint distributions and (ii) the mechanism maximizes the expected revenue under the joint distribution across all dominant-strategy mechanisms. I call such a joint distribution a *worst-case correlation structure*. It is straightforward that (i) and (ii) imply that the mechanism is a maxmin mechanism.

To construct such a joint distribution, I first reformulate the problem of maximizing the expected revenue across all dominant-strategy mechanisms as the problem of maximizing the expected "virtual value" of the bidder who is allocated the object subject to that the allocation rule is monotone (a monotonicity constraint associated with dominant-strategy incentive compatibility), where the "virtual value" is that the bidder's valuation less information rents that are pinned down by dominant-strategy incentive compatibility and the binding ex-post participation constraints of zero-valuation bidders. This is a straightforward adaption of the well-known revenue equivalence result of Myerson (1981). Importantly, this simplifies the problem in that one can now point-wise maximizes the objective, ignoring the monotonicity constriant[2]. Then such a joint distribution is obtained by letting the virtual value of the high bidder (the bidder with a higher valuation than that of her opponent) be zero except when the high bidder's valuation is $\bar{v}$. The intuition behind this property is that the auctioneer is *indifferent* between allocating and not allocating the object to the high bidder as long as her valuation is below $\bar{v}$ under the second-price auction with the uniformly distributed random reserve.

To illustrate, consider a special case in which the marginal distribution is an *equal-revenue* distribution[3], defined by the property of a unit-elastic demand: in the monopoly pricing problem, the monopoly's revenue from charging any price in the support of this distribution is the same. Notably, there is a probability mass on the maximum valuation in an equal-revenue distribution.

[2]Of course, one need to check that the monotonicity constraint holds in the end.

[3]This distribution is identified as a buyer-optimal signal distribution in a monopoly selling problem by Roesler and Szentes (2017).

Under the joint distribution where the high bidder's virtual value is 0 except when her valuation is $\bar{v}$, the two bidders' valuations turn out to be *independent*. Notably, the low bidder's virtual value also *equals* 0. The proposed regularity conditions generalize the special case: they guarantee that under the constructed joint distribution, the high bidder's virtual value is 0 and the low bidder's virtual value is *weakly negative*. Then if the proposed regularity conditions hold, the second-price auction with the uniformly distributed random reserve maximizes the expected revenue across all dominant-strategy mechanisms under the constructed joint distribution. Indeed, if the proposed regularity conditions hold, any dominant-strategy mechanism, in which 1) the ex-post participation constraints are binding for zero-valuation bidders and 2) the object is allocated with probability one to the high bidder with the valuation of $\bar{v}$ and the object is never allocated to the low bidder, maximizes the expected revenue across all dominant-strategy mechanisms under the constructed joint distribution.

Notably, the proposed regularity conditions contain a probability mass condition on the maximum valuation. That is, the result requires that the marginal distribution have an atom on the maximum valuation and that the size of the atom be bounded from below. Indeed, these conditions capture many heavy-tailed distributions[4], which are observed in many real-world auctions. For example, according to Arnosti et al. (2016), it has been observed that a huge fraction of the total value comes from a small number of very valuable impressions in online advertising. In addition, according to Ibragimov and Walden (2010), very diverse private valuations have been observed in markets for cultural and sport events as well as in those for antiques and collectibles and online auctions and marketplaces such as eBay and StubHub. Many papers have studied mechanism design problems when the distribution of values exhibits a heavy tail (e.g., Arnosti et al. (2016) and Ibragimov and Walden (2010)).

I extend the analysis to the case of general number of bidders $(N \geq 3)$. For tractability, I restrict attention to a subclass of dominant-strategy mechanisms in which a bidder whose bid is not the highest is never allocated. A mechanism in this subclass is referred to as a

[4] I present a detailed discussion of these conditions in Section 2.4.1.

standard[5] dominant-strategy mechanism. The second main result (Theorem 9) is that, under certain regularity conditions on the marginal distribution, *the second-price auction with the* $\bar{v}-$*scaled* $Beta(\frac{1}{N-1},1)$ *distributed random reserve* is a maxmin mechanism across standard dominant-strategy mechanisms. Under this mechanism, the cumulative distribution function of the random reserve r is $G(r)=(\frac{r}{\bar{v}})^{\frac{1}{N-1}}$ for $r\in[0,\bar{v}]$. Following a methodology similar to the two-bidder case, I show this result by constructing a worst-case correlation structure.

This mechanism embodies, albeit not the full-insurance property, a good hedging property: it yields the same expected revenue across a range of correlation structures. Precisely, as I will show, the expected revenue is the same for any feasible joint distribution whose support lies in the set of valuation profiles in which either all bidders have the same valuations or there is a unique highest bidder and the other bidders have the same valuations. Indeed, the constructed worst-case correlation structure has such a support. Intuitively, given the restriction to standard mechanisms, only the highest bidders are possible to generate positive revenue to the auctioneer. Thus, the other bidders' valuations except for the highest one are "wasted". To reduce the expected revenue as much as possible while maintaining the consistency with the marginal distribution, a worst-case correlation structure maximizes the waste by increasing the other bidders' valuations as much as possible until all the other bidders' valuations are the same. Then similar to the two-bidder case, the worst-case correlation structure is obtained by requiring the highest bidder's virtual value be zero except when the highest bidder's valuation is $\bar{v}$. Here, the proposed regularity conditions on the marginal distribution guarantee that the construction is feasible. As I focus on standard dominant-strategy mechanisms, the bidders whose valuations are not the highest do not contribute to the expected revenue and therefore their virtual values do not matter. Then it is straightforward that the second-price auction with the $\bar{v}-$scaled $Beta(\frac{1}{N-1},1)$ distributed random reserve maximizes the expected revenue across standard dominant-strategy mechanisms under the constructed joint distribution.

[5]The terminology of "standard" comes from Bergemann et al. (2019) who define standard mechanisms in a similar manner. He and Li (2022) also adopts this terminology.

Moreover, I show that the second-price auction with the $\bar{v}-$scaled $Beta(\frac{1}{N-1},1)$ distributed random reserve is asymptotically optimal across all dominant-strategy mechanisms as the number of the bidders goes to infinity, regardless of the marginal distribution (Remark 13). To establish this, I show that the revenue guarantee of this mechanism converges to the expectation of a generic bidder's valuation. Indeed, the expectation of a generic bidder's valuation is an upper bound of the revenue guarantee for any dominant-strategy mechanism, as it is possible that the correlation structure is the maximally positively correlated distribution (that is, all bidders have the same valuations for any valuation profile in the support), under which the expectation of a generic bidder's valuation is the most surplus that the auctioneer can extract.

The first two main results both require the probability mass on the maximum valuation to be bounded from below. The third main result (Theorem 10) characterizes *the second-price auction with the* s^{*}[6]*−scaled* $Beta(\frac{1}{N-1},1)$ *distributed random reserve* as a maxmin mechanism across standard dominant-strategy mechanisms if the probability mass condition does not hold. Under this mechanism, the cumulative distribution function of the random reserve r is $G_{s^*}(r)=(\frac{r}{s^*})^{\frac{1}{N-1}}$ with support $[0,s^*]$ where $s^*\in(0,\bar{v})$. Notably, the highest bidder will be fully allocated the object provided that her valuation is higher than s^*.

A second-price auction (albeit with some random reserve), which is simple and widely adopted in practice, arises as a robustly optimal mechanism across some general class of mechanisms. Importantly, this is true for a wide range of marginal distributions. Therefore, the main results provide a positive explanation why the second-price auction is prevalent in the real world: it maximizes the worst-case expected revenue for a wide range of marginal distributions. Furthermore, the explanation is particularly convincing for the two-bidder case: the robust optimality is established across *all* dominant-strategy mechanisms.

In addition to the main results, I propose a family of second-price auctions with $t-$scaled $Beta(\frac{1}{N-1},1)$ distributed random reserves where $t\in(0,\bar{v})$. I identify a non-trivial lower bound of the revenue guarantee for each auction in this family. For any given marginal distribution, I

[6] s^* is characterized by the known marginal distribution, details of which are given in Section 2.4.3.

am able to find an auction from this family that has a strictly higher revenue guarantee than that of any posted-price mechanism and any second-price auction with a non-negative deterministic reserve (Theorem 11 and Theorem 12).

The remainder of the introduction discusses related literature. Section 2.2 presents the model. Section 2.3 illustrates the methodology and conducts preliminary analysis. Section 2.4 characterizes the main results. Section 2.5 proposes a family of auctions and studies their performance in terms of the revenue guarantee. Section 2.6 is a conclusion. Omitted proofs are in Appendix 2.7.1, 2.7.2 and 2.7.3. Appendix 2.7.4 contains the result for the essential necessity of the regularity conditions.

2.1.1 Related Literature

The closest related paper is He and Li (2022), who study the design of auctions within the correlation-robust framework. They show, among others, that a second-price auction with a random reserve is a maxmin mechanism across standard dominant-strategy mechanisms under certain conditions on the marginal distribution. Methodologically, both papers use duality theory to proceed the analysis. The main differences can be summarized as follows. My setting is more general than theirs in that I allow the marginal distribution to have a probability mass on the maximum valuation[7]. More importantly, I obtain a new and strong result for the two-bidder case: I establish that, under certain regularity conditions (different from theirs), my proposed mechanism is a maxmin mechanism across *all* dominant-strategy mechanisms. In addition, I establish the main results under weaker conditions on the marginal distribution[8].

This paper is closely related to Che (2020), Koçyiğit et al. (2020a) and Zhang (2021). The first two papers both consider a model of auction design in which the auctioneer only knows the expectation of each bidder's valuation. Specifically, Che (2020) shows that a second-price

[7]They assume continuous distributions and do not allow for a mass point on the maximum valuation.

[8]For valuations below the maximum valuation, both papers assume that the marginal distribution admits a density f. For the main results, I only require that $x^2 f(x)$ be non-decreasing in x instead of that $xf(x)$ be non-decreasing in x, required in their paper.

auction with an optimal random reserve is a maxmin mechanism within a class of competitive mechanisms[9]; Koçyiğit et al. (2020a) characterize a maxmin mechanism across highest-bidder lotteries[10] for the case where the known expectations are the same across bidders. Similarly, my paper also considers some general class of mechanisms. The main difference is that I assume that the auctioneer knows exactly the marginal distribution. That is, I assume that the auctioneer knows more and therefore the revenue guarantee in my setting is an upper bound of theirs. Zhang (2021) considers a model of bilateral trade in which the profit-maximizing intermediary only knows the expectations of each trader's valuation. He characterizes maxmin trading mechanisms across all dominant-strategy mechanisms. The maxmin trading mechanism features fixed-commission fee, uniformly random spread and midpoint transaction price in the symmetric case. In contrast, this paper considers a model of auction design and assumes that the auctioneer knows exactly the marginal distribution. Like that paper, I consider all dominant-strategy mechanisms for the two-bidder case. In addition, this paper employs a similar methodology to proceed the analysis. More specifically, both papers use properties of "virtual value" to construct worst-case distributions.

This paper is also closely related to Bei et al. (2019), who study the design of auctions within the correlation-robust framework with a focus on *simple* mechanisms. They show, among others, that the revenue guarantee of the sequential posted-price mechanism is at least $\frac{1}{2\ln 4+2}$ times the revenue guarantee of the optimal dominant-strategy mechanism.

This paper is related to Bose et al. (2006) who study the design of auctions assuming that the bidders' valuations are independently distributed but there may be ambiguity about the marginal distribution of a generic bidder's valuation. In contrast, my paper assumes that the marginal distribution of a generic bidder's valuation is known but the correlation structure of bidders' valuations is unknown. Because of the different framework, the methodology of this paper differs significantly from that one. They show, among others, that an auction that "fully

[9]The class of dominant-strategy mechanisms is not a subset of the class of competitive mechanisms, and vice versa.

[10]This is the same as standard dominant-strategy mechanisms.

insures" the auctioneer is a maxmin mechanism when the auctioneer does not know the marginal distribution but the bidders know it. Similarly, in my framework, the first main result (Theorem 8) characterizes a maxmin mechanism exhibiting a full-insurance property. However, the notion of full-insurance is different from theirs. While full-insurance requires that the same *expected revenue* be obtained across all feasible distributions in this paper, it requires that the same *ex-post revenue* be obtained across all valuation profiles in that one.

Broadly, this paper joins the robust mechanism design literature (Bergemann and Morris (2005)). There are other papers searching optimal solutions in the worst case over the space of parameters (e.g., Carroll and Meng (2016), Garrett (2014), Bergemann and Schlag (2011), Carroll (2017),Giannakopoulos et al. (2020),Chen et al. (2019)). Bergemann et al. (2016), Du (2018) and Brooks and Du (2020) consider a model of auction design with common values. They assume that bidders' valuations for the object are drawn from a commonly known prior, but they may have arbitrary information (high-order beliefs) about the prior distribution unknown to the seller. An auction's performance is measured by the worst expected revenue across a class of incomplete information correlated equilibria termed Bayes correlated equilibria (BCE) in Bergemann and Morris (2013). In this paper, I completely ignore the beliefs of bidders by focusing on the dominant-strategy mechanisms. This assumption is more appropriate for situations in which one not say much about bidders' beliefs, as dominant-strategy mechanisms are robust to misspecification of bidders' beliefs.

2.2 Preliminaries

2.2.1 Notation

I introduce the following technical notations. First, all spaces considered are polish spaces; I endow them with their Borel $\sigma-$algebra. Second, product spaces are endowed with product $\sigma-$algebra. Third, I use $\Delta(X)$ to denote the set of all probability measures over X.

2.2.2 Environment

I consider an environment where a single indivisible object is sold to $N \geq 2$ risk-neutral bidders. I denote by $I = \{1, 2, ..., N\}$ the set of bidders. Each bidder i has private information about her valuation for the object, which is modeled as a random variable v_i with cumulative distribution function F_i[11]. Throughout the paper, I focus on symmetric environment, i.e., $F_i = F_j = F$[12] for any $i, j \in I$. I denote the support of F_i by V_i. I assume that $V_i = [0, \bar{v}]$ for some $\bar{v} > 0$. The joint support of all F_i is $V = \times_{i=1}^{N} V_i = [0, \bar{v}]^N$ with a typical valuation profile v. I denote bidder i's opponents' valuation profiles by v_{-i}, i.e., $v_{-i} \in V_{-i} = \times_{j \neq i} V_j$.

The valuation profile v is drawn from a joint distribution $\mathcal{P}$, which may have an arbitrary correlation structure. The auctioneer only knows the marginal distribution F of each bidder's valuation but does not know how these bidders' valuations are correlated. To the auctioneer, any joint distribution is *feasible* as long as the joint distribution is consistent with the marginal distribution. I denote by

$$\Pi(F) = \{\pi^{13} \in \Delta V : \forall i \in I, \forall \text{measurable } A_i \subset V_i, \pi(A_i \times V_{-i}) = F(A_i)\}$$

the collection of feasible joint distributions.

2.2.3 Marginal Distribution

I assume that the marginal distribution F admits a probability density function $f(x)$ for any $x \in [0, \bar{v})$. I allow F to have a probability mass on $\bar{v}$, the size of which is denoted by $Pr(\bar{v})$. [14] Importantly, as will be seen, the first two main results (Theorem 8 and 9) require that the marginal

[11] As will be discussed later, I allow distributions to have a probability mass on the maximum valuation. Furthermore, all results (with slight modifications) hold in discrete environments.

[12] With slight abuse of notation, I also use F to denote the probability measure consistent with the distribution F.

[13] With slight abuse of notation, I also use π to denote the probability density of the probability measure π if the probability density exists.

[14] He and Li (2022) restrict attention to marginal distributions that admits a positive density function everywhere, whereas this paper allows a probability mass on the maximum valuation. That is, the assumption in this paper is more general.

distribution have an atom on $\bar{v}$ and that the size of the atom be bounded from below. Indeed, distributions with a probability mass on the maximum valuation are familiar in many information design and robust mechanism design environments. Roesler and Szentes (2017) analyze a model where the buyer has a given value distribution and designs a signal structure to learn about her valuation. After observing the signal structure, the seller makes a take-it-or-leave-it offer. They show that an equal-revenue distribution is the buyer-optimal signal distribution. In a closely related work, Condorelli and Szentes (2020) analyze a model where the buyer can choose the probability distribution of her valuation for the good. After observing the buyer's choice of the distribution, the seller makes a take-it-or-leave-it offer. They show that an equal-revenue distribution is the buyer-optimal value distribution. Besides, Bergemann and Schlag (2008) consider a minimax regret design problem of selling a single object to a single buyer and find that an equal-revenue distribution is a worst-case distribution; Zhang (2022b) considers a robust public good mechanism design problem and finds that the worst-case marginal distribution has a probability mass on the maximum valuation.

Moreover, distributions with a probability mass on the maximum valuation admits an ”approximating” interpretation as follows. Consider a marginal distribution $\hat{F}$ with a non-negative density function $\hat{f}$ everywhere on $[0,\infty)$. In the real world, it is reasonable to assume that bidders' valuations are bounded as the total wealth, which is physically impossible to be infinite, is an upper bound of bidders' valuations. Then the marginal distribution F with bounded support $[0,\bar{v}]$ is generated by truncating $\hat{F}$ on $\bar{v}$ as follows: $f(x) = \hat{f}(x)$ for $x < \bar{v}$ and $Pr(\bar{v}) = 1 - \hat{F}(\bar{v})$. When $\bar{v}$ is large, F is a natural approximation of $\hat{F}$.

2.2.4 (Standard) Dominant-strategy Mechanisms

I focus on dominant-strategy mechanisms. The revelation principle holds and it is without loss of generality to restrict attention to direct mechanisms. A direct mechanism (q,t) is defined as an allocation rule $q : V \to [0,1]^N$ and a payment function $t : V \to \mathbb{R}^N$. With slight abuse of notation, each bidder submits a sealed bid $v_i \in V_i$ to the auctioneer. Upon receiving the bids profile $v = (v_1, v_2, \cdots, v_N)$, the allocation probabilities are $q(v) = (q_1(v), q_2(v), \cdots, q_N(v))$ and the payments are $t(v) = (t_1(v), t_2(v), \cdots, t_N(v))$ where $q(v) \geq 0$ and $\sum_i q_i(v) \leq 1$ for all $v \in V$. A direct mechanism is a dominant-strategy mechanism if for all $i \in I$, all $v \in V$, and all $v_i' \in V_i$,

$$v_i q_i(v) - t_i(v) \geq v_i q_i(v_i', v_{-i}) - t_i(v_i', v_{-i}),$$

$$v_i q_i(v) - t_i(v) \geq 0.$$

The set of all such mechanisms is denoted by $\mathcal{D}$. I say a direct mechanism (q,t) is *standard* if for any $v \in V$ and $i \in I$ such that $v_i < \max_{j \in I} v_j$, the allocation to the bidder i is $q_i(v) = 0$. That is, only the highest bidders are possible to be allocated in a standard mechanism. The set of all standard dominant-strategy mechanisms is denoted by $\mathcal{E}$.

2.2.5 Objective Function

I am interested in the auctioneer's expected revenue in the dominant-strategy equilibrium in which each bidder truthfully reports her valuation of the object. Then the expected revenue of a dominant-strategy mechanism (q,t) when the joint distribution is π is $U((q,t),\pi) = \int_{v \in V} \sum_{i=1}^{N} t_i(v) d\pi(v)$. The auctioneer evaluates a mechanism (q,t) by its worst-case expected revenue, referred to as the *revenue guarantee* of the mechanism (q,t), over all feasible joint distributions. Formally, the mechanism (q,t)'s revenue guarantee is $REG((q,t)) = \inf_{\pi \in \Pi(F)} U((q,t),\pi)$. The auctioneer's goal is to find a *maxmin mechanism* from either $\mathcal{D}$ or $\mathcal{E}$ with the maximal revenue guarantee[15]. Formally, the auctioneer solves

$$\sup_{(q,t) \in \mathcal{D} (\text{or} \mathcal{E})} REG((q,t)). \tag{MRG}$$

[15]As will be seen later, the first main result (Theorem 8) characterizes a maxmin mechanism from $\mathcal{D}$, and either of the other main results (Theorem 9 and Theorem 10) characterizes a maxmin mechanism from $\mathcal{E}$.

2.3 Methodology and Preliminary Analysis

The maxmin optimization problem (MRG) can be interpreted as a two-player sequential zero-sum game. The two players are the auctioneer and adversarial nature. The auctioneer first chooses a mechanism $(q,t) \in \mathcal{D}$ (or $(q,t) \in \mathcal{E}$). After observing the auctioneer's choice of the mechanism, adversarial nature chooses a feasible joint distribution $\pi \in \Pi(F)$. The auctioneer's payoff is $U((q,t),\pi)$, and adversarial nature's payoff is $-U((q,t),\pi)$. One can also consider the simultaneous-move version of this zero-sum game, whose Nash equilibrium is indeed a *saddle point* of the payoff functional U, i.e., for any $(q,t) \in \mathcal{D}$ (or $(q,t) \in \mathcal{E}$) and any $\pi \in \Pi(F)$,

$$U((q^*,t^*),\pi) \geq U((q^*,t^*),\pi^*) \geq U((q,t),\pi^*).$$

The first inequality says the joint distribution π^* minimizes the expected revenue under the mechanism (q^*,t^*), and the second inequality implies that, under the joint distribution π^*, the other dominant-strategy mechanisms cannot attain a strictly higher expected revenue. Hence, the auctioneer's equilibrium strategy in the simultaneous-move version of this zero-sum game, (q^*,t^*), is a maxmin mechanism. π^* is referred to as a *worst-case correlation structure*. I will construct a saddle point for each of the main results.

Proposition 3 (Revenue Equivalence)**.** *When searching for a maxmin mechanism, it is without loss to restrict attention to mechanisms satisfying the following properties: 1) $q_i(\cdot,v_{-i})$ is non-decreasing in v_i for all v_{-i} and 2) $t_i(v_i,v_{-i}) = v_i q_i(v_i,v_{-i}) - \int_0^{v_i} q_i(x,v_{-i})dx$.*

Proof. The proof is in Appendix 2.7.1. □

Proposition 3 simplifies the analysis by establishing two properties of a maxmin mechanism. The first property says the allocation rule is *monotone*, and the second property says that the payment rule can be characterized by the allocation rule and that the ex-post participation constraints are binding for zero-valuation bidders. This is standard in the mechanism design literature (e.g., Myerson (1981)).

Moreover, Proposition 3 allows me to obtain a virtual representation of the expected revenue, which is essential for my analysis. Precisely, consider the problem that fixing an arbitrary joint distribution π, the auctioneer designs an optimal mechanism (q,t). For exposition, I assume that π admits a positive density function[16]. The density function of v_i conditional on v_{-i} is denoted by $\pi_i(v_i|v_{-i})$, and the cumulative distribution function of v_i conditional on v_{-i} is denoted by $\Pi_i(v_i|v_{-i})$. Then an direct implication of Proposition 3 is that the expected revenue of (q,t) under the joint distribution π is

$$E[\sum_{i=1}^{N} t_i(v)] = E[\sum_{i=1}^{N} q_i(v)\phi_i(v)],$$

where $\phi_i(v) = v_i - \frac{1-\Pi_i(v_i|v_{-i})}{\pi_i(v_i|v_{-i})}$ is the *virtual value* of bidder i when the valuation profile is v. Thus the problem of designing an optimal mechanism given a joint distribution is equivalent to maximizing the *expected total virtual surplus*, which refers to the expected sum of the allocation times the virtual value, subject to that the allocation rule is monotone.

2.4 Main Results

In Section 2.4.1, I characterize a maxmin mechanism across all dominant-strategy mechanisms under certain regularity conditions for the two-bidder case. In Section 2.4.2, I characterize a maxmin mechanism across standard dominant-strategy mechanisms under certain regularity conditions for the $N-$bidder ($N \geq 3$) case. In Section 2.4.3, I characterize a maxmin mechanism across standard dominant-strategy mechanisms condition for the $N-$bidder ($N \geq 2$) case when the probability mass condition (part of the regularity conditions) fails.

[16]The virtual representation can be similarly derived for joint distributions in which there is a probability mass on $\underbrace{(1,\cdots,1)}_{N}$.

2.4.1 Two Bidders

I first define a mechanism and a joint distribution. Then I define regularity conditions under which the mechanism and the joint distribution form a saddle point (Theorem 8). Then I illustrate Theorem 8. Finally, I give a discussion of the regularity conditions.

The second-price auction with the uniformly distributed random reserve is defined as follows. The auctioneer first draws a random reserve r from the uniform distribution with support $[0,\bar{v}]$. Then the two bidders bid simultaneously. The high bidder (the bidder with a higher bid than that of her opponent) wins the object if her bid is also higher than r, and she pays the maximum of r and her opponent's bid; the low bidder loses the auction and pays nothing. In case of ties, each bidder wins the object with a half probability if the bid is higher than r, and the winner pays the bid.

Equivalently, it can be defined by (q^*,t^*) as follows. If $v_1 > v_2$, then $q_1^*(v_1,v_2) = \frac{v_1}{\bar{v}}, q_2^*(v_1,v_2) = 0$ and $t_1^*(v_1,v_2) = \frac{v_1^2+v_2^2}{2\bar{v}}, t_2^*(v_1,v_2) = 0$; if $v_1 < v_2$, then $q_1^*(v_1,v_2) = 0, q_2^*(v_1,v_2) = \frac{v_2}{\bar{v}}$ and $t_1^*(v_1,v_2) = 0, t_2^*(v_1,v_2) = \frac{v_1^2+v_2^2}{2\bar{v}}$; if $v_1 = v_2 = x$, then $q_1^*(v_1,v_2) = q_2^*(v_1,v_2) = \frac{x}{2\bar{v}}$ and $t_1^*(v_1,v_2) = t_2^*(v_1,v_2) = \frac{x^2}{2\bar{v}}$.

The joint distribution π^* is defined as follows[17].

$$\pi^*(v_1,v_2)=\pi^*(v_2,v_1)=\begin{cases} f(0) & \text{if } v_1=v_2=0; \\ 0 & \text{if } v_1>v_2=0; \\ \frac{1}{v_1^2}\left(v_2 f(v_2)-\frac{\int_0^{v_2} x^2 f(x)dx}{v_2^2}\right) & \text{if } \bar{v}>v_1\geq v_2>0; \\ \frac{1}{\bar{v}}\left(v_2 f(v_2)-\frac{\int_0^{v_2} x^2 f(x)dx}{v_2^2}\right) & \text{if } \bar{v}=v_1>v_2>0. \end{cases}$$

$$Pr^*(\bar{v},\bar{v})=Pr(\bar{v})-\frac{\int_{x\in(0,\bar{v})} x^2 f(x)dx}{\bar{v}^2}.$$

The two-bidder robust regularity conditions are defined as follows: $x^2 f(x)$ is non-decreasing for $x\in(0,\bar{v})$ and $Pr(\bar{v})\geq\frac{\int_{x\in(0,\bar{v})} x^2 f(x)dx}{\bar{v}^2}$. I refer to the second part of the conditions as the probability mass condition.

Remark 9. Note that the probability mass condition becomes non-restrictive as $\bar{v}\to\infty$ if $\int_{x\in(0,\bar{v})} x^2 f(x)dx$ is of order $\bar{v}^\gamma$ with $\gamma<2$.

Theorem 8. *For the two-bidder case, the second-price auction with the uniformly distributed random reserve is a maxmin mechanism across all dominant-strategy mechanisms if the two-bidder robust regularity conditions hold. The revenue guarantee is $\frac{E[X^2]}{\bar{v}}$.[18] The joint distribution π^* is a worst-case correlation structure.*

Now I illustrate Theorem 8. I start with the illustration of the mechanism.

[17] Here $\pi^*(v_1,v_2)$ denotes the density of the valuation profile (v_1,v_2) whenever the density exists and $Pr^*(v_1,v_2)$ denotes the probability mass of the valuation profile (v_1,v_2) whenever there is some probability mass on (v_1,v_2). The marginal distributions that the result covers have a probability mass on the maximum valuation $\bar{v}$. In the joint distribution π^*, there is (non-negative) probability mass on the point $(\bar{v},\bar{v})$.

[18] The distribution of X is F.

Definition 4. I say a dominant-strategy mechanism (q,t) exhibits the *full-insurance property* if the expected revenue of (q,t) is the same across all feasible joint distributions.

Proposition 4. *For the two-bidder case, the second-price auction with the uniformly distributed random reserve exhibits the full-insurance property.*

Proof. Note that under the second-price auction with the uniformly distributed random reserve, the total revenue from a valuation profile (v_1,v_2) is $t^*(v_1,v_2) = t_1^*(v_1,v_2) + t_2^*(v_1,v_2) = \frac{v_1^2+v_2^2}{2\bar{v}}$. Then fix any feasible joint distribution π, the expected revenue is

$$\begin{aligned}
\int_{[0,\bar{v}]^2} t^*(v_1,v_2) d\pi(v_1,v_2) &= \int_{[0,\bar{v}]^2} \frac{v_1^2+v_2^2}{2\bar{v}} d\pi(v_1,v_2) \\
&= \int_{[0,\bar{v}]^2} \frac{v_1^2}{2\bar{v}} d\pi(v_1,v_2) + \int_{[0,\bar{v}]^2} \frac{v_2^2}{2\bar{v}} d\pi(v_1,v_2) \\
&= \int_{[0,\bar{v}]} \frac{v_1^2}{2\bar{v}} dF(v_1) + \int_{[0,\bar{v}]} \frac{v_2^2}{2\bar{v}} dF(v_2) \\
&= \frac{E[X^2]}{\bar{v}}.
\end{aligned}$$

□

The joint distribution π^* is obtained by a condition requiring that the high bidder's virtual value be 0 except when her valuation is $\bar{v}$. Formally,

$$\phi_i^*(v_i,v_j) = 0 \quad \text{if} \quad v_j \leq v_i < \bar{v}. \tag{1}$$

The property (1) is motivated by a property of the second-price auction with the uniformly distributed random reserve: the auctioneer is indifferent between allocating and not allocating the object to the high bidder as long as her valuation is not $\bar{v}$.

Furthermore, I impose a condition on the constructed joint distribution π^* that the virtual value of the low bidder is weakly smaller than that of the high bidder. Formally,

$$\phi_j^*(v_i,v_j) \leq \phi_i^*(v_i,v_j) \quad \text{if} \quad v_j \leq v_i. \tag{2}$$

The property (2) is motivated by another property of the second-price auction with the uniformly distributed random reserve: the low bidder is never allocated the object.

Indeed, if the property (1) and the property (2) hold for a joint distribution, it is straightforward that the second-price auction with the uniformly distributed random reserve maximizes the expected revenue across all dominant-strategy mechanisms given this joint distribution. The two-bidder robust regularity conditions, as I will show, guarantee that the property (1) and the property (2) hold for the constructed joint distribution π^*. In summary, I obtain a virtual value matrix for π^* as follows if the two-bidder robust regularity conditions hold.

$$\begin{pmatrix}
(0,0)_{0,0} & (0,0)_{0,>0} & \cdots & \cdots & (0,0)_{0,<\bar{v}} & (0,0)_{0,\bar{v}} \\
(0,0)_{>0,0} & (0,0)_{v_1=v_2>0} & (-,0)_{v_1<v_2} & \cdots & (-,0)_{v_1<v_2<1} & (\leq,+)_{<\bar{v},\bar{v}} \\
\vdots & (0,-)_{v_1>v_2} & \ddots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \cdots & \ddots & \vdots & \vdots \\
(0,0)_{<\bar{v},0} & (0,-)_{<\bar{v},>0} & \cdots & \cdots & (0,0)_{v_1=v_2<\bar{v}} & (\leq,+)_{<\bar{v},\bar{v}} \\
(0,0)_{\bar{v},0} & (+,\leq)_{\bar{v},>0} & \cdots & \cdots & (+,\leq)_{\bar{v},<\bar{v}} & (+,+)_{\bar{v},\bar{v}}
\end{pmatrix}$$

Here “0” in the bracket means zero virtual value, “$-$” means a non-positive virtual value, “+” means a non-negative virtual value, “$\leq$” means the virtual value of the bidder is weakly smaller than that of her opponent. The subscript denotes the corresponding valuation profile.

Remark 10. The probability mass condition arises because the property (1) requires that the conditional distribution of the high bidder’s valuation be an equal-revenue distribution, which has an atom on the maximum valuation $\bar{v}$.

Proposition 5. *If the two-bidder robust regularity conditions hold, the second-price auction with the uniformly distributed random reserve maximizes the expected revenue across all dominant-strategy mechanism under the joint distribution* π^*.

Proof. The proof is in Appendix 2.7.2 which presents details about the construction of π^*. □

Theorem 8 follows immediately from Proposition 4 and Proposition 5.

There is an important special case where Theorem 8 applies: the equal-revenue distribution. Recall that the equal-revenue distribution is familiar in the information design literature and robust mechanism design literature (e.g., Roesler and Szentes (2017), Bergemann and Schlag (2008), Du (2018), etc).

Corollary 1. *For the two-bidder case, if the marginal distribution is an equal-revenue distribution with* $\alpha \in (0,\bar{v})$*:*

$$F(x) = \begin{cases} 1-\frac{\alpha}{x} & \text{if} \quad \alpha \le x < \bar{v}; \\ 1 & \text{if} \quad x = \bar{v}, \end{cases}$$

then the second-price auction with the uniformly distributed random reserve is a maxmin mechanism across all dominant-strategy mechanism. The revenue guarantee is $2\alpha - \frac{\alpha^2}{\bar{v}}$*. The independent equal-revenue distribution*[19] *is a worst-case correlation structure.*

Proof. It is straightforward to show that an equal-revenue distribution satisfies the two-bidder robust regularity conditions. Then Theorem 8 implies Corollary 1. □

There are many other distributions satisfying the two-bidder robust regularity conditions. I now provide some examples.

Example 1. Any (truncated) Pareto distribution with $\alpha \in (0,\bar{v}), \beta \in (0,1)$:

$$F(x) = \begin{cases} 1-\frac{\alpha^\beta}{x^\beta} & \text{if} \quad \alpha \le x < \bar{v}; \\ 1 & \text{if} \quad x = \bar{v}. \end{cases}$$

To see this, note that $x^2 f(x) = \alpha^\beta \beta x^{1-\beta}$ is non-decreasing when $\beta \in (0,1)$. For the probability mass condition, note that $Pr(\bar{v}) = (\frac{\alpha}{\bar{v}})^\beta \ge (\frac{\alpha}{\bar{v}})^\beta \frac{\beta}{2-\beta}[1-(\frac{\alpha}{\bar{v}})^{2-\beta}] = \frac{\int_{(0,\bar{v})} x^2 f(x)dx}{\bar{v}^2}$ when $\beta \in (0,1)$.

[19] That is, the marginal distribution of each bidder's valuation is the known equal-revenue distribution; bidders' valuations are independently distributed.

Example 2. A combination of an uniform distribution on $[0,\bar{v})$ and a probability mass on $\bar{v}$ with $Pr(\bar{v}) \geq \frac{1}{4}$.

To see this, note that the first part of the conditions holds trivially because it is uniformly distributed on $[0,\bar{v})$. For the probability mass condition, note that $Pr(\bar{v}) \geq \frac{\int_{(0,\bar{v})} x^2 \cdot \frac{1-Pr(\bar{v})}{\bar{v}} dx}{\bar{v}^2} = \frac{\int_{(0,\bar{v})} x^2 f(x) dx}{\bar{v}^2}$ when $Pr(\bar{v}) \geq \frac{1}{4}$.

As a final topic of this section, I discuss the two-bidder robust regularity conditions. Using the approximation interpretation in Section 2.2.3, F is obtained via a truncation of $\hat{F}$ on $\bar{v}$. Now consider the following regularity conditions for $\hat{F}$: $x^2 \hat{f}(x)$ is non-decreasing on $[0,\infty)$ and $\frac{\int_0^s x^2 \hat{f}(x) dx}{s^2} \to 0$ as $s \to \infty$. These conditions imply that the two-bidder robust regularity conditions hold for any $\bar{v} > 0$.[20] It is straightforward to verify that these conditions hold for many heavy-tailed distributions including a family of power law distributions[21], Cauchy distributions[22], log-Cauchy distributions[23], Lévy distributions[24], etc. Thus, when $\bar{v}$ is large, the two-bidder robust regularity conditions hold for a distribution that is an approximation of some heavy-tailed distribution.

[20]Indeed, that $x^2 \hat{f}(x)$ is non-decreasing implies that the function $K(s) := \hat{F}(s) + \frac{\int_0^s x^2 \hat{f}(x) dx}{s^2}$ is non-decreasing. To see this, note that $K'(s) = 2[\hat{f}(s) - \frac{\int_0^s x^2 \hat{f}(x) dx}{s^3}] \geq 0$, shown in (B.9). Then $\hat{F}(\bar{v}) + \frac{\int_0^{\bar{v}} x^2 \hat{f}(x) dx}{\bar{v}^2} \leq 1$ for any $\bar{v} > 0$ if $\frac{\int_0^s x^2 \hat{f}(x) dx}{s^2} \to 0$ as $s \to \infty$.

[21]The power law distribution is given by $F(x) = 1 - \frac{\alpha^\beta}{x^\beta}$ with $\alpha > 0$ for $x \in [\alpha, \infty)$. The parameter β determines the weight of the tail. These conditions hold for any power law distribution with $\beta \in (0,1]$.

[22]The density function of the Cauchy distribution is given by $f(x) = \frac{2b}{\pi(b^2+x^2)}$ with $b > 0$ for $x \in [0,\infty)$. It is straightforward to show that these conditions hold for Cauchy distributions with any $b > 0$.

[23]The density function of the log-Cauchy distribution is given by $f(x) = \frac{1}{\pi x}[\frac{\sigma}{(\ln x - \mu)^2 + \sigma^2}]$ with $\sigma > 0$ for $x \in (0,\infty)$. It is straightforward to show that these conditions hold for log-Cauchy distributions with any $\sigma \geq 1$ and any real number μ.

[24]The density function of the Lévy distribution is given by $f(x) = \sqrt{\frac{c}{2\pi}} \frac{e^{-\frac{c}{2x}}}{x^{\frac{3}{2}}}$ with $c > 0$ for $x \in (0,\infty)$. It is straightforward to show that these conditions hold for Lévy distributions with any $c > 0$.

2.4.2 *N* Bidders

The structure of this section is similar to Section 2.4.2: I first define a mechanism and a joint distribution. Then I define regularity conditions under which the mechanism and the joint distribution form a saddle point (Theorem 9). Then I illustrate Theorem 9.

The second-price auction with $\bar{v}$-scaled $Beta(\frac{1}{N-1},1)$ distributed random reserve is defined as follows. The auctioneer first draws a random reserve r from the $\bar{v}-$scaled $Beta(\frac{1}{N-1},1)$ distribution. That is, the cumulative distribution function of the random reserve r is $G(r)=(\frac{r}{\bar{v}})^{\frac{1}{N-1}}$ with support $[0,\bar{v}]$. Then the N bidders bid simultaneously. The highest bidder wins the object if her bid is also higher than r, and she pays the maximum of r and the second highest bid; a bidder whose bid is not the highest loses the auction and pays nothing. In case of ties, each bidder wins the object with an equal probability if the bid is higher than r, and the winner pays the bid.

Equivalently, it can be defined by (q^*,t^*) as follows. I denote the highest valuation in a valuation profile v by $v(1)$, and the second highest valuation (if any) by $v(2)$. If $\#\{k:v_k=v(1)\}=1$, then $q_i^{**}(v)=(\frac{v(1)}{\bar{v}})^{\frac{1}{N-1}}, q_j^{**}(v)=0$ and $t_i^{**}(v)=\frac{v(1)^{\frac{N}{N-1}}+(N-1)v(2)^{\frac{N}{N-1}}}{N\bar{v}^{\frac{1}{N-1}}}, t_j^{**}(v)=0$ for $i\in\{k:v_k=v(1)\}$ and $j\notin\{k:v_k=v(1)\}$; if $\#\{k:v_k=v(1)\}=K\geq 2$, then $q_i^{**}(v)=\frac{1}{K}(\frac{v(1)}{\bar{v}})^{\frac{1}{N-1}}, q_j^{**}(v)=0$ and $t_i^{**}(v)=\frac{v(1)^{\frac{N}{N-1}}}{K\bar{v}^{\frac{1}{N-1}}}, t_j^{**}(v)=0$ for $i\in\{k:v_k=v(1)\}$ and $j\notin\{k:v_k=v(1)\}$.

The joint distribution π^{**} is symmetric and is defined as follows[25]. The support of π^{**} is $V^+:=\{v\in V|v_i=v(1)$ for any i or $\exists i$ s.t. $v_i=v(1)>v_j=v(2)$ for any $j\neq i\}$. That is, $v\in V^+$ if either all bidders have the same valuations or there is a unique highest bidder and all of the

[25]Here $\pi^{**}(v)$ denote the density of the valuation profile v whenever the density exists and $Pr^{**}(v)$ denote the probability mass of the valuation profile v whenever there is some probability mass on v. The marginal distributions that this result covers have a probability mass on the maximum valuation $\bar{v}$. In the joint distribution π^{**}, there is a (non-negative) probability mass on the point $(\underbrace{\bar{v},\cdots,\bar{v}}_{N})$.

remaining bidders have the same valuations. If $v \notin V^+$, then $\pi^{**}(v) = 0$. If $v \in V^+$, then

$$\pi^{**}(v_i, v_{-i}) = \begin{cases} f(0) & \text{if } v = (0, \cdots, 0); \\ 0 & \text{if } 0 = v_j < v_i, \forall j \neq i; \\ \frac{1}{(N-1)v(1)^2}\left(v(2)f(v(2)) - \frac{v(2)^{-\frac{N}{N-1}}}{N-1}\int_0^{v(2)} x^{\frac{N}{N-1}} f(x)dx\right) & \text{if } 0 < v(2) = v_j \leq v_i = v(1) < \bar{v}, \forall j \neq i; \\ \frac{1}{(N-1)\bar{v}}\left(v(2)f(v(2)) - \frac{v(2)^{-\frac{N}{N-1}}}{N-1}\int_0^{v(2)} x^{\frac{N}{N-1}} f(x)dx\right) & \text{if } 0 < v(2) = v_j < v_i = \bar{v}, \forall j \neq i. \end{cases}$$

$$Pr^{**}(\underbrace{\bar{v}, \cdots, \bar{v}}_{N}) = Pr(\bar{v}) - \frac{\int_{(0,\bar{v})} x^{\frac{N}{N-1}} f(x)dx}{(N-1)\bar{v}^{\frac{N}{N-1}}}.$$

The N-bidder robust regularity conditions (I) are defined as follows: $f(x) \geq \frac{x^{-\frac{2N-1}{N-1}}}{N-1}\int_0^x s^{\frac{N}{N-1}} f(s)ds$ for $x \in (0, \bar{v})$ and $Pr(\bar{v}) \geq \frac{\int_{(0,\bar{v})} x^{\frac{N}{N-1}} f(x)dx}{(N-1)\bar{v}^{\frac{N}{N-1}}}$[26]. The *N-bidder robust regularity conditions (II)* are defined as follows: $x^2 f(x)$ is non-decreasing for $x \in (0, \bar{v})$ and $Pr(\bar{v}) \geq \frac{\int_{(0,\bar{v})} x^{\frac{N}{N-1}} f(x)dx}{(N-1)\bar{v}^{\frac{N}{N-1}}}$.

Remark 11. Here the probability mass condition will vanish as the number of the bidders goes to infinity. To see this, note that $\frac{\int_{(0,\bar{v})} x^{\frac{N}{N-1}} f(x)dx}{(N-1)\bar{v}^{\frac{N}{N-1}}} \leq \frac{\int_{(0,\bar{v})} \bar{v}^{\frac{N}{N-1}} f(x)dx}{(N-1)\bar{v}^{\frac{N}{N-1}}} \leq \frac{1}{N-1} \to 0$ as $N \to \infty$. Therefore, the probability mass condition is non-restrictive when the number of bidders is large.

Theorem 9. *For the N-bidder case ($N \geq 3$), the second-price auction with the $\bar{v}-$scaled $Beta(\frac{1}{N-1}, 1)$ distributed random reserve is a maxmin mechanism across standard dominant-strategy mechanisms if the $N-$bidder robust regularity conditions (I) hold. The revenue guarantee is $\frac{E[X^{\frac{N}{N-1}}]}{\bar{v}^{\frac{1}{N-1}}}$. The joint distribution π^{**} is a worst-case correlation structure. In addition, the $N-$bidder robust regularity conditions (II) imply the $N-$bidder robust regularity conditions (I).*

[26]With slight abuse of notation, here the condition for $Pr(\bar{v})$ is also referred to as the probability mass condition.

Remark 12. It is straightforward that the second-price auction with the $\bar{v}-$scaled $Beta(\frac{1}{N-1},1)$ distributed random reserve converges to the second-price auction without a reserve as the number of bidders goes to infinity. The asymptotic behaviour of the random reserve is consistent with the empirical finding that reserve prices are substantially lower than the optimal ones under the estimated distribution of values (e.g., Paarsch (1997), McAfee et al. (2002), Haile and Tamer (2003) and Bajari and Hortaçsu (2003)).

Definition 5. Given a marginal distribution F, suppose the revenue guarantee of a maxmin mechanism across all dominant-strategy mechanisms is $Opt_N(F)$ for each $N-$bidder case. Consider a dominant-strategy mechanism M_N for each $N-$bidder case. Suppose the revenue guarantee of M_N is $Reg_N(F)$. I say M_N is *asymptotically optimal* across all dominant-strategy mechanisms given the marginal distribution F if $Opt_N(F) - Reg_N(F) \to 0$ as $N \to \infty$.

Remark 13. The second-price auction with the $\bar{v}-$scaled $Beta(\frac{1}{N-1},1)$ distributed random reserve is asymptotically optimal across all dominant-strategy mechanisms, regardless of the marginal distribution. Furthermore, the rate of convergence is $O(\frac{1}{N})$[27]. To see these, recall that $E[X]$ is an upper bound of the revenue guarantee for any dominant-strategy mechanism. This is because it is always possible that adversarial nature chooses the maximally positively correlated distribution. But by the Dominated Convergence Theorem, I have that

$$\frac{E[X^{\frac{N}{N-1}}]}{\bar{v}^{\frac{1}{N-1}}} \to E[X]$$

as $N \to \infty$. Furthermore, let $j(x) := x - \frac{x^{\frac{N}{N-1}}}{\bar{v}^{\frac{1}{N-1}}}$. Because $j'(x) = 1 - \frac{Nx^{\frac{1}{N-1}}}{(N-1)\bar{v}^{\frac{1}{N-1}}}$ and $j''(x) = -\frac{Nx^{\frac{2-N}{N-1}}}{(N-1)^2\bar{v}^{\frac{1}{N-1}}} \leq 0$, $j(x)$ is maximized at $x = (\frac{N-1}{N})^{N-1}\bar{v}$ and the maximized value is $(\frac{N-1}{N})^{N-1} \cdot \frac{\bar{v}}{N}$ by simple calculation. Then I have that $E[X] - \frac{E[X^{\frac{N}{N-1}}]}{\bar{v}^{\frac{1}{N-1}}} \leq (\frac{N-1}{N})^{N-1} \cdot \frac{\bar{v}}{N} \leq \frac{\bar{v}}{N}$. Therefore the rate of convergence is $O(\frac{1}{N})$.

[27] In addition, this rate of convergence is the fastest across all standard dominant-strategy mechanisms, as is shown in He and Li (2020).

Now I illustrate Theorem 9. I start with the illustration of the mechanism.

This mechanism exhibits a robust property in the following sense: for any feasible joint distribution whose support is V^+, the expected revenue under this mechanism is the same. To see this, note that the total revenue from a valuation $v \in V^+$ is $t^{**}(v) = \sum_{i=1}^{N} t_i^{**}(v) = \sum_{i=1}^{N} \frac{v_i^{\frac{N}{N-1}}}{N\bar{v}^{\frac{1}{N-1}}}$. Therefore, the expected revenue is $\frac{E[X^{\frac{N}{N-1}}]}{\bar{v}^{\frac{1}{N-1}}}$ for any feasible joint distribution whose support is V^+. I will use the linear programming duality theorem to show that such an joint distribution indeed minimizes the expected revenue across all feasible joint distributions.

Proposition 6. *For the N-bidder case ($N \geq 3$), under the second-price auction with the $\bar{v}-$scaled $Beta(\frac{1}{N-1}, 1)$ distributed random reserve, any feasible joint distribution whose support is V^+ minimizes the expected revenue across all feasible joint distributions. The minimized expected revenue is* $\frac{E[X^{\frac{N}{N-1}}]}{\bar{v}^{\frac{1}{N-1}}}$.

Proof. The proof is in Appendix 2.7.2 which presents the details about the construction of the mechanism. □

Proposition 6 implies that the constructed joint distribution π^{**} minimizes the expected revenue across all feasible joint distributions under the mechanism, as the support of the constructed joint distribution π^{**} is V^+.

The joint distribution π^{**} is then obtained by a condition requiring that the highest bidder's virtual value be 0 except when her valuation is $\bar{v}$. Formally,

$$\phi_i^{**}(v_i, v_{-i}) = 0 \quad \text{if} \quad v \in V^+, \max_{j \neq i} v_j \leq v_i < \bar{v}. \tag{1'}$$

If the property (1') holds for a joint distribution, then it is straightforward that the second-price auction with the $\bar{v}-$scaled $Beta(\frac{1}{N-1}, 1)$ distributed random reserve maximizes the expected revenue across all standard dominant-strategy mechanisms given this joint distribution. Note that I do not impose any condition on lower bidders' virtual values. This is because I restrict attention to standard dominant-strategy mechanisms, and then a bidder whose bid is not the highest is not allocated the object and pays nothing. The $N-$bidder robust regularity conditions (I) guarantee that the property (1') holds for the constructed joint distribution π^{**}. The $N-$bidder robust regularity conditions (II) are simpler conditions, and, as I will show, they imply the $N-$bidder robust regularity conditions (I).

Proposition 7. *For the N-bidder case ($N \geq 3$), if the $N-$bidder robust regularity conditions (I) hold, then the second-price auction with the $\bar{v}$-scaled $Beta(\frac{1}{N-1}, 1)$ distributed random reserve maximizes the expected revenue across all standard dominant-strategy mechanisms under the joint distribution π^{**}. In addition, the $N-$bidder robust regularity conditions (II) imply the $N-$bidder robust regularity conditions (I).*

Proof. The proof is in Appendix 2.7.2 which presents details about the construction of the joint distribution π^{**}. □

Theorem 9 follows immediately from Proposition 6 and Proposition 7.

Now I present the result for the special case: the equal-revenue distribution. In contrast to the two-bidder case, bidders' valuations are not independently distributed in the worst-case correlation structure for the $N-$bidders case ($N \geq 3$).

Corollary 2. *For the $N-$bidders case ($N \geq 3$), if the marginal distribution is an equal-revenue distribution ($\alpha \in (0, \bar{v})$):*

$$F(v) = \begin{cases} 1 - \frac{\alpha}{v} & \text{if} \quad \alpha \leq v < \bar{v}; \\ 1 & \text{if} \quad v = \bar{v}, \end{cases}$$

then the second-price auction with the $\bar{v}-$scaled $Beta(\frac{1}{N-1}, 1)$ distributed random reserve is a maxmin mechanism across standard dominant-strategy mechanisms. The revenue guarantee is

$N\alpha - \frac{(N-1)\alpha^{\frac{N}{N-1}}}{\bar{v}^{\frac{1}{N-1}}}$. *The worst-case correlation structure is symmetric and is defined as follows.*

$$\pi(v_i, v_{-i}) = \begin{cases} \frac{(\frac{v(2)}{\alpha})^{-\frac{N}{N-1}}}{(N-1)v(1)^2} & \text{if } \alpha \leq v(2) = v_j \leq v_i = v(1) < \bar{v}, \forall j \neq i; \\ \frac{(\frac{v(2)}{\alpha})^{-\frac{N}{N-1}}}{(N-1)\bar{v}} & \text{if } \alpha \leq v_j = v(2) < v_i = \bar{v}, \forall j \neq i; \\ 0 & \text{if otherwise.} \end{cases}$$

$$Pr(\underbrace{\bar{v}, \cdots, \bar{v}}_{N}) = (\frac{\alpha}{\bar{v}})^{\frac{N}{N-1}}.$$

Proof. It is straightforward to show that an equal-revenue distribution satisfies the N-bidder robust regularity conditions (I) (and (II)). Then Theorem 9 implies Corollary 2. □

2.4.3 When Probability Mass Condition Fails

When the probability mass condition fails, I characterize a maxmin mechanism across standard dominant-strategy mechanisms.

Theorem 10. *For the N-bidder case ($N \geq 2$), if $x^2 f(x)$ is non-decreasing for $x \in (0, \infty)$ and $Pr(\bar{v}) < \frac{\int_{(0,\bar{v})} x^{\frac{N}{N-1}} f(x) dx}{(N-1)\bar{v}^{\frac{N}{N-1}}}$, then the second-price auction with the s^*-scaled $Beta(\frac{1}{N-1}, 1)$ distributed random reserve is a maxmin mechanism across standard dominant-strategy mechanisms where $s^* \in (0, \bar{v})$ is a solution to*

$$\frac{\int_{(0,s)} x^{\frac{N}{N-1}} dF(x)}{s^{\frac{N}{N-1}}} = (N-1)(1-F(s)). \tag{SD}$$

In addition, the revenue guarantee is $Ns^*(1-F(s^*))$.

Proof. The proof is in Appendix 2.7.2. □

Remark 14. This result generalizes and strengthens Theorem 3 in He and Li (2022). First, the conditions in this result allow for a probability mass on the maximum valuation. Second, the condition for $x < \bar{v}$ is weaker than that $xf(x)$ is non-decreasing[28], required in their result.

Under this mechanism, the cumulative distribution function of the random reserve r is $G_{s*}(r) = (\frac{r}{s^*})^{\frac{1}{N-1}}$ with support $[0, s^*]$. Therefore, the highest bidder will be allocated the object with a probability less than one if her valuation is less than s^*, and will be allocated the object with probability one if her valuation is weakly higher than s^*. I follow the saddle point approach to show Theorem 10. Notably, the constructed worst-case correlation structure exhibits the property that the highest bidder's virtual value is 0 when her valuation is less than s^*, and is weakly positive when her valuation is weakly higher than s^*. I relegate the details to Appendix 2.7.2.

2.5 Robust Dominance

I have shown that the second-price auction with the $\bar{v}-$scaled $Beta(\frac{1}{N-1}, 1)$ distributed random reserve is a maxmin mechanism under certain regularity conditions. Then what if the regularity conditions do not hold? How does this mechanism perform? As a first topic of this section, I compare the performance of this mechanism with that of the posted-price mechanism, which is a dominant-strategy mechanism commonly used in practice. For exposition, I assume that the marginal distribution admits a positive density function everywhere on $[0, \bar{v}]$ in this section. I denote the set of all such distributions by $\Delta^c[0, \bar{v}]$.

Definition 6. *I say a mechanism M dominates a family of mechanisms $\mathcal{M}$ for a set of marginal distributions if for any marginal distribution in this set, the revenue guarantee of M is strictly greater than that of any mechanism in the family $\mathcal{M}$.*

Definition 7. I say a family of mechanisms $\mathcal{M}_1$ *universally dominates* another family of

[28] It is straightforward that $xf(x)$ is non-decreasing implies that $x^2 f(x)$ is non-decreasing, but not vice versa.

mechanisms $\mathcal{M}_2$ if for any $F \in \Delta^c[0,\bar{v}]$, there exists a mechanism in $\mathcal{M}_1$ with a revenue guarantee strictly greater than that of any mechanism in $\mathcal{M}_2$.

Proposition 8. *For any $N \geq 2$, the second-price auction with the $\bar{v}-$scaled $Beta(\frac{1}{N-1},1)$ distributed random reserve dominates the family of posted-price mechanisms for the set $\mathcal{H} = \{F \in \Delta^c[0,\bar{v}] | \text{the revenue function } x \cdot (1-F(x)) \text{ is strictly concave}\}$.*

Proof. The proof is in Appendix 2.7.3. □

Motivated by the main idea embedded in the construction, I propose a family of second-price auctions with $t-$scaled $Beta(\frac{1}{N-1},1)$ distributed random reserves where $t \in (0,\bar{v})$, denoted by $\mathcal{M}_{SP-\beta}$. As a second topic of this section, I study the performance of the proposed family of auctions. Formally, the cumulative distribution function of the random reserve r in this family is $G_t(r) = (\frac{r}{t})^{\frac{1}{N-1}}$ with support $[0,t]$ for some $t \in (0,\bar{v})$. Under such a random reserve, if the valuation of the highest bidder is above the threshold t, the object will be fully allocated to her. For each t, I am able to identify a non-trivial lower bound of the revenue guarantee by constructing a set of feasible dual variables.

Lemma 7. *A lower bound of the revenue guarantee of the second-price auction with $t-$scaled $Beta(\frac{1}{N-1},1)$ distributed random reserve where $t \in (0,\bar{v})$ is*

$$\int_{(0,t)} \frac{x^{\frac{N}{N-1}}}{t^{\frac{1}{N-1}}} dF(x) + t(1-F(t)).$$

Proof. The proof is in Appendix 2.7.3. □

Lemma 7 suggests a potential criterion under which the auctioneer selects an auction from this family. Although the revenue guarantee of an auction in this family may depend on the details of the marginal distribution and thus may be hard to be identified, it has a non-trivial lower bound. Then I can compare this lower bound with the revenue guarantees of some dominant-strategy mechanisms commonly used in practice.

Theorem 11. *For any $N \geq 2$, $\mathcal{M}_{SP-\beta}$ universally dominates the family of posted-price mechanisms.*

Proof. The proof is in Appendix 2.7.3. □

Theorem 12. *For any $N \geq 2$, $\mathcal{M}_{SP-\beta}$ universally dominates the family of second-price auctions with non-negative deterministic reserves.*

Proof. The proof is in Appendix 2.7.3. □

I do not require any distributional assumption for these two theorems to hold. Moreover, Theorem 11 and 12 imply that for a given marginal distribution, the auctioneer can find a second-price auction with a $t-$scaled $Beta(\frac{1}{N-1}, 1)$ distributed random reserve whose revenue guarantee is strictly higher than that of any posted-price mechanism and any second-price auction with a non-negative deterministic reserve. In addition, Theorem 11 can be interpreted as that the competition effect dominates the adversarial correlation effect. To see this, note that Theorem 11 implies that there exists an auction for the two-bidder case that generates strictly higher revenue guarantee than the monopoly revenue from one bidder, regardless of the marginal distribution. Thus, even if nature picks the worst-case correlation structure, it is always *strictly* more desirable for the auctioneer to have just one more bidder.

2.6 Concluding Remarks

In this paper, I consider the correlation-robust framework and show, among others, that the second-price auction with the uniformly distributed random reserve maximizes the revenue guarantee across all dominant-strategy mechanisms for the two-bidder case, and that the second-price auction with the $Beta(\frac{1}{N-1}, 1)$ distributed random reserve maximizes the revenue guarantee across standard dominant-strategy mechanisms for the N-bidder ($N \geq 3$) case. These auctions have familiar formats and admit simple descriptions which do not require the information of the marginal distribution except for the maximum valuation of of a generic bidder. Thus, these auctions are more practical compared with, for example, Myerson's auction, which often requires the full information of the marginal distribution to calculate the optimal reserve. To my knowledge, this paper is the first to characterize a maxmin mechanism across *all* dominant-strategy mechanism in the correlation-robust framework. It remains an open question what the maxmin mechanisms across all dominant-strategy mechanisms are for general number of bidders. The constructive method may shed light on other robust design problems and general robust optimization problems.

2.7 Appendix

2.7.1 Proof for Section 2.3: Proposition 3

1) Dominant-strategy incentive compatibility (DSIC) for a type v_i requires that for any $v_i' \neq v_i$:

$$v_i q_i(v_i, v_{-i}) - t_i(v_i, v_{-i}) \geq v_i q_i(v_i', v_{-i}) - t_i(v_i', v_{-i}).$$

DSIC also requires that:

$$v_i' q_i(v_i', v_{-i}) - t_i(v_i', v_{-i}) \geq v_i' q_i(v_i, v_{-i}) - t_i(v_i, v_{-i}).$$

Adding the two inequalities, I obtain that

$$(v_i - v_i')(q_i(v_i, v_{-i}) - q_i(v_i', v_{-i})) \geq 0.$$

It follows that $q_i(v_i, v_{-i}) \geq q_i(v_i', v_{-i})$ whenever $v_i > v_i'$.

2) Fix any v_{-i}, and define

$$U_i(v_i) = v_i q_i(v_i, v_{-i}) - t_i(v_i, v_{-i}).$$

By the two inequalities in 1), I obtain that

$$(v_i' - v_i)q_i(v_i, v_{-i}) \leq U_i(v_i') - U_i(v_i) \leq (v_i' - v_i)q_i(v_i', v_{-i}).$$

Dividing throughout by $v_i' - v_i$, I obtain that

$$q_i(v_i, v_{-i}) \leq \frac{U_i(v_i') - U_i(v_i)}{(v_i' - v_i)} \leq q_i(v_i', v_{-i}).$$

As $v \uparrow v'$, I have that

$$\frac{dU_i(v_i)}{dv_i} = q_i(v_i, v_{-i}).$$

Then I obtain that

$$t_i(v_i, v_{-i}) = v_i q_i(v_i, v_{-i}) - \int_0^{v_i} q_i(s, v_{-i}) ds - U_i(0).$$

Note that $U_i(0) \geq 0$ by the ex-post individually rational constraint. If $U_i(0) > 0$, then I can reduce it to 0 so that I can increase the revenue from any valuation profile in which the others' valuation profile is v_{-i}. And the revenue guarantee will be weakly greater. Thus, when searching for a maxmin mechanism, it is without loss to let $U_i(0) = 0$. Then I obtain that $t_i(v_i, v_{-i}) = v_i q_i(v_i, v_{-i}) - \int_0^{v_i} q_i(s, v_{-i})$.

2.7.2 Proofs for Section 2.4

Proof of Proposition 5

First, I illustrate the details about the construction of π^*. Note that by allocating all marginal density $f(0)$ to valuation profile $(0,0)$, I have that $\phi_i^*(v_i, 0) = \phi_j^*(0, v_j) = 0$ for any v_i and v_j. Thus, the property (1) trivially holds for any one of these valuation profiles. Now let $A_{kj} := \{v | k \leq v_1 \leq j, v_2 = k\}$, and define $c(0) := f(0)$ and $c(k) := \int_{A_{k\bar{v}}} d\pi^*$ for $k > 0$. Consider the valuation profile (v_1,v_2) where $0<v_2\le v_1<\bar{v}$. In order for the virtual value to satisfy the property (1), I have that

$$\phi_1^*(v_1,v_2)=v_1-\frac{c(v_2)-\int_{v_2}^{v_1}\pi^*(x,v_2)dx}{\pi^*(v_1,v_2)}=0,\quad \forall 0<v_2\le v_1<\bar{v}.$$

These equations are essentially a system of ordinary differential equations, whose solution is well known[29]:

$$\pi^*(v_1,v_2)=\frac{v_2c(v_2)}{v_1^2},\quad \forall 0<v_2\le v_1<\bar{v}, \tag{B.1}$$

$$\pi^*(\bar{v},v_2)=\frac{v_2c(v_2)}{\bar{v}},\quad \forall 0<v_2<\bar{v}. \tag{B.2}$$

By symmetry, I also obtain $\pi^*(v_2,v_1)=\pi^*(v_1,v_2)$ for $0<v_2\le v_1<\bar{v}$ and $\pi^*(v_2,\bar{v})=\pi^*(\bar{v},v_2)$ for $0<v_2<\bar{v}$. Finally,

$$Pr^*(\bar{v},\bar{v})=Pr(\bar{v})-\frac{\int_{j\in(0,\bar{v})}jc(j)dj}{\bar{v}}. \tag{B.3}$$

Now I solve for $c(k)$. Note since the marginal distribution is the same across the two bidders, given the above derivation, $c(k)$ must satisfy the following condition:

$$c(k)=f(k)-\frac{\int_0^k jc(j)dj}{k^2},\quad \forall 0<k<\bar{v}. \tag{B.4}$$

To see this, note $f(k)=\int_{\{0\le v_1\le\bar{v},v_2=k\}}d\pi^*=\int_{A_{k\bar{v}}\cup\{0\le v_1<k,v_2=k\}}d\pi^*=\int_{A_{k\bar{v}}}d\pi^*+\int_{\{0\le v_1<k,v_2=k\}}d\pi^*=\int_{A_{k\bar{v}}}d\pi^*+\int_{\{v_1=k,0\le v_2<k\}}d\pi^*$ where the last equality follows from symmetry. Multiplying both sides of (B.4) by k, I obtain that

$$kc(k)=kf(k)-\frac{\int_0^k jc(j)dj}{k},\quad \forall 0<k<\bar{v}.$$

[29] The solution is reminiscent of the equal-revenue distribution.

Define $g(k) := \int_0^k jc(j)dj$ for $0 < k < \bar{v}$. Then I have that

$$g'(k) = kf(k) - \frac{g(k)}{k}, \quad \forall 0 < k < \bar{v}.$$

Note this is an ordinary differential equation, and I solve for $g(k)$:

$$g(k) = \frac{1}{k}\int_0^k j^2 f(j)dj, \quad \forall 0 < k < \bar{v}. \tag{B.5}$$

From this I compute $c(k)$:

$$c(k) = f(k) - \frac{\int_0^k j^2 f(j)dj}{k^3}, \quad \forall 0 < k < \bar{v}. \tag{B.6}$$

Plugging (B.6) to (B.1), (B.2) and (B.3), I obtain the joint distribution π^*.

To guarantee that it is possible to construct π^*, it has to be a feasible joint distribution in that the density (or probability mass) has to be non-negative for all valuation profiles, i.e., $\pi^*(v_1, v_2) \geq 0$ for $0 \leq v_1, v_2 < \bar{v}$ and $Pr^*(\bar{v}, \bar{v}) \geq 0$. Therefore, I have that

$$f(k) - \frac{\int_0^k j^2 f(j)dj}{k^3} \geq 0, \quad \forall 0 < k < \bar{v}, \tag{B.7}$$

$$Pr(\bar{v}) \geq \frac{\int_{x\in(0,1)} x^2 f(x)dx}{\bar{v}^2}. \tag{B.8}$$

Now I show that the first part of the two-bidder robust regularity conditions implies (B.7). To see this, note that if $x^2 f(x)$ is non-decreasing for $x \in (0, \bar{v})$, then for any $0 < k < \bar{v}$, I have that

$$f(k) - \frac{\int_0^k j^2 f(j)dj}{k^3} \geq f(k) - \frac{\int_0^k k^2 f(k)dj}{k^3} = 0, \tag{B.9}$$

where the inequality follows from that $j^2 f(j) \leq k^2 f(k)$ if $j \leq k$.

Now given that the construction is feasible, I argue that the two-bidder robust regularity conditions guarantee that the property (2) holds. Given that the property (1) holds for π^*, it suffices to show

$$\phi_2(v_1, v_2) \leq 0$$

if $0 < v_2 \leq v_1 < \bar{v}$. I now calculate $\phi_2(v_1, v_2)$ for $0 < v_2 \leq v_1 < \bar{v}$:

$$\begin{aligned}
\phi_2(v_1, v_2) &= v_2 - \frac{f(v_1) - \int_0^{v_2} \pi^*(v_1, t)dt}{\pi^*(v_1, v_2)} \\
&= v_2 - \frac{f(v_1) - \int_0^{v_2} c(t)\frac{t}{v_1^2}dt}{c(v_2)\frac{v_2}{v_1^2}} \\
&= v_2 - \frac{f(v_1) - \frac{1}{v_1^2 v_2}\int_0^{v_2} t^2 f(t)dt}{(f(v_2) - \frac{\int_0^{v_2} s^2 f(s)ds}{v_2^3})\frac{v_2}{v_1^2}},
\end{aligned}$$

where the second equality follows from (B.1) and the third equality follows from (B.5) and (B.6). Now it is straightforward that for any $0 < v_2 \leq v_1 < \bar{v}$:

$$\phi_2(v_1, v_2) \leq 0 \Longleftrightarrow v_2^2 f(v_2) \leq v_1^2 f(v_1).$$

Proof of Proposition 6

First, I illustrate the details about the construction of the mechanism. I first write down the primal minimization problem for adversarial nature given a mechanism (q, t) and derive its dual maximization problem. Formally, let $\{\lambda_i(v_i)\}_{i \in \{1,2,\cdots,N\}, v_i \in [0,\bar{v}]}$ be dual variables.

$$(P) \qquad \inf_{\pi \in \Pi(F)} \int_{v \in [0,\bar{v}]^N} \sum_{i=1}^{N} t_i(v) d\pi(v)$$

subject to

$$\int_{[0,\bar{v}]^{N-1}} d\pi(v_i, v_{-i}) = f(v_i), \quad \forall v_i \in [0, \bar{v}),$$

$$\int_{[0,\bar{v}]^{N-1}} d\pi(v_i = \bar{v}, v_{-i}) = Pr(v_i = \bar{v}).$$

$$(D) \quad \sup_{\{\lambda_i(v_i)\}} \sum_{i=1}^{N} \int_0^{\bar{v}} \lambda_i(v_i) dF(v_i)$$

subject to

$$\sum_{i=1}^{N} \lambda_i(v_i) \leq \sum_{i=1}^{N} t_i(v), \quad \forall v \in [0, \bar{v}]^N. \tag{B.10}$$

It is straightforward to show that weak duality holds [30]. The mechanism is constructed by a complementary slackness condition as follows.

$$\sum_{i=1}^{N} \lambda_i(v_i) = \sum_{i=1}^{N} t_i(v), \quad \forall v \in V^+. \tag{B.11}$$

I assume that $\lambda_i = \lambda$ for all $i \in I$, and that the mechanism is a second-price auction with a random reserve whose cumulative distribution function is G, then (B.11) implies

$$N\lambda(v_i) = v_i G(v_i), \quad \forall v_i \in [0, \bar{v}], \tag{B.12}$$

$$\lambda(v(1)) + (N-1)\lambda(v(2)) = v(1)G(v(1)) - \int_{v(2)}^{v(1)} G(s)ds, \quad \forall 0 \leq v(2) < v(1) \leq 1. \tag{B.13}$$

Note by (B.12), I have that for $v_i \in [0, \bar{v}]$,

$$\lambda(v_i) = \frac{v_i G(v_i)}{N}. \tag{B.14}$$

Plugging (B.14) to (B.13), I obtain that for $0 \leq v(2) < v(1) \leq \bar{v}$,

$$\frac{v(1)G(v(1)) + (N-1)v(2)G(v(2))}{N} = v(1)G(v(1)) - \int_{v(2)}^{v(1)} G(s)ds. \tag{B.15}$$

Taking first order derivatives with respect to $v(1)$ and $v(2)$, I obtain the same ordinary differential equation that for $x \in [0, \bar{v}]$,

$$(N-1)xG'(x) = G(x). \tag{B.16}$$

[30] See, for example, He and Li (2022).

Given that G is a distribution, the solution to (B.16) is

$$G(x) = (\frac{x}{\bar{v}})^{\frac{1}{N-1}}, \quad \forall x \in [0, \bar{v}].$$

This is the $\bar{v}-$scaled $Beta(\frac{1}{N-1}, 1)$ distribution.

Now I show that under the second-price auction with the $\bar{v}-$scaled $Beta(\frac{1}{N-1}, 1)$ distributed random reserve, any feasible joint distribution with the support V^+ minimizes the expected revenue across all feasible joint distributions. As I have argued in Section 2.4.2, the expected revenue given such a joint distribution is $\frac{E[X^{\frac{N}{N-1}}]}{\bar{v}^{\frac{1}{N-1}}}$, then it suffices to show the value of (D) is also $\frac{E[X^{\frac{N}{N-1}}]}{\bar{v}^{\frac{1}{N-1}}}$. To this end, I construct the dual variables as follows. For all $i \in \{1, 2, \cdots, N\}$ and $x \in [0, 1]$,

$$\lambda_i(x) = \frac{x^{\frac{N}{N-1}}}{N\bar{v}^{\frac{1}{N-1}}}.$$

Note that $\sum_{i=1}^{N} \int_0^{\bar{v}} \lambda_i(v_i) dF(v_i) = \frac{E[X^{\frac{N}{N-1}}]}{\bar{v}^{\frac{1}{N-1}}}$ under the constructed dual variables. Then, it suffices to show that (B.10) holds under the constructed dual variables. To see this, I divide valuation profiles into two cases.

Case 1: $\#\{k : v_k = v(1)\} = 1$.

In this case, $t(v) = \frac{v(1)^{\frac{N}{N-1}} + (N-1)v(2)^{\frac{N}{N-1}}}{N\bar{v}^{\frac{1}{N-1}}}$. The L.H.S. of (B.10) is maximized when all bidders except the highest bidder have the same valuations, and (B.10) holds with equality when the L.H.S. of (B.10) is maximized. Therefore (B.10) holds.

Case 2: $\#\{k : v_k = v(1)\} \geq 2$.

In this case, $t(v) = \frac{v(1)^{\frac{N}{N-1}}}{\bar{v}^{\frac{1}{N-1}}}$. The L.H.S. of (B.10) is maximized when all bidders have the same valuations, and (B.10) holds with equality when the L.H.S. of (B.10) is maximized. Therefore (B.10) holds.

Proof of Proposition 7

First, I illustrate the details about the construction of π^{**}. Note that by allocating all marginal density $f(0)$ to the valuation profile $\underbrace{(0,\cdots,0)}_{N}$, I have that $\phi_i^{**}(v)=0$ for any i and $v_i\geq 0, v_j=0, \forall j\neq i$. Thus, the property (1') trivially holds for any one of these valuation profiles. Now let $B_{kj}:=\{v|k\leq v_1\leq j, v_i=k, \quad \forall i\neq 1\}$, and define $d(0):=f(0)$ and $d(k):=\int_{B_{k\bar{v}}}d\pi^*$ for $k>0$. Consider the valuation profile $(v_1,\underbrace{v_2,\cdots,v_2}_{N-1})$ where $0<v_2\leq v_1<\bar{v}$. In order for the virtual value of bidder 1 to satisfy the property (1'), I have that

$$\phi_1^{**}(v_1,\underbrace{v_2,\cdots,v_2}_{N-1})=v_1-\frac{d(v_2)-\int_{v_2}^{v_1}\pi^{**}(s,\underbrace{v_2,\cdots,v_2}_{N-1})ds}{\pi^{**}(v_1,\underbrace{v_2,\cdots,v_2}_{N-1})}=0,\quad \forall 0<v_2\leq v_1<\bar{v}.$$

These equations are essentially a system of ordinary differential equations, whose solution is well known:

$$\pi^{**}(v_1,\underbrace{v_2,\cdots,v_2}_{N-1})=\frac{v_2 d(v_2)}{v_1^2},\quad \forall 0<v_2\leq v_1<\bar{v}, \tag{B.17}$$

$$\pi^{**}(1,\underbrace{v_2,\cdots,v_2}_{N-1})=\frac{v_2 d(v_2)}{\bar{v}},\forall \quad 0<v_2<\bar{v}. \tag{B.18}$$

By symmetry, I also obtain that $\pi^{**}(v)=\pi^{**}(v_1,\underbrace{v_2,\cdots,v_2}_{N-1})$ for $0<v_j=v_2\leq v_i=v_1<\bar{v},\forall j\neq i,\forall i$ and $\pi^{**}(v)=\pi^{*}(1,\underbrace{v_2,\cdots,v_2}_{N-1})$ for $0<v_j=v_2<v_i=\bar{v},\forall j\neq i,\forall i$. Finally,

$$Pr^{**}(\underbrace{\bar{v},\cdots,\bar{v}}_{N})=Pr(\bar{v})-\frac{\int_{j\in(0,\bar{v})}jd(j)dj}{\bar{v}}. \tag{B.19}$$

Now I solve for $d(k)$. Note that $d(k)$ must satisfy the following condition:

$$f(k)=(N-1)d(k)+\frac{\int_0^k jd(j)dj}{k^2},\quad \forall 0<k<\bar{v}.$$

To see this, suppose that the bidder 1's valuation is k. Then either k is the highest valuation and other bidders all have a valuation of $j \in [0,k]$ (with a probability of $\frac{\int_0^k jd(j)dj}{k^2}$) or k is the second highest valuation and one of the other bidders has the highest valuation (with a probability of $(N-1)d(k)$). Multiplying both sides of (2.7.2) by k, I obtain that

$$kf(k) = (N-1)kd(k) + \frac{\int_0^k jd(j)dj}{k}, \quad \forall 0 < k < \bar{v}.$$

Define $h(k) := \int_0^k jd(j)dj$ for $0 < k < \bar{v}$. Then I have that

$$kf(k) = (N-1)h'(k) + \frac{h(k)}{k}, \quad \forall 0 < k < \bar{v}.$$

Note that this is an ordinary differential equation, and I solve for $h(k)$:

$$h(k) = \frac{\int_0^k j^{\frac{N}{N-1}} f(j)dj}{(N-1)k^{\frac{1}{N-1}}}, \quad \forall 0 < k < \bar{v}. \tag{B.20}$$

From this I compute $d(k)$:

$$d(k) = \frac{1}{N-1}(f(k) - \frac{\int_0^k j^{\frac{N}{N-1}} f(j)dj}{(N-1)k^{1+\frac{N}{N-1}}}), \quad \forall 0 < k < \bar{v}. \tag{B.21}$$

Plugging (B.21) to (B.17),(B.18) and (B.19), I obtain the joint distribution π^{**}.

To guarantee that π^* is a feasible joint distribution in that the density (or probability mass) has to be non-negative for all valuation profiles, it is straightforward that the $N-$bidder robust regularity conditions (I) have to hold. Now I show that the $N-$bidder robust regularity conditions (II) imply the $N-$bidder robust regularity conditions (I). To see this, note that if $x^2 f(x)$ is non-decreasing for $x \in (0,\bar{v})$, then for any $0 < k < \bar{v}$, I have that

$$f(k) - \frac{\int_0^k j^{\frac{N}{N-1}} f(j)dj}{(N-1)k^{1+\frac{N}{N-1}}} \geq f(k) - \frac{\int_0^k j^{\frac{N}{N-1}-2} k^2 f(k)dj}{(N-1)k^{1+\frac{N}{N-1}}} = 0, \tag{B.22}$$

where the inequality follows from that $j^2 f(j) \leq k^2 f(k)$ if $j \leq k$.

Proof of Theorem 10

Lemma 8. *If $x^2 f(x)$ is non-decreasing for $x \in (0,\bar{v})$ and $Pr(\bar{v}) < \frac{\int_{(0,\bar{v})} x^{\frac{N}{N-1}} f(x)dx}{(N-1)\bar{v}^{\frac{N}{N-1}}}$, then there exists $s^* \in (0,\bar{v})$ that is a solution to* (SD).

Proof. First, note that if $s \uparrow \bar{v}$, the R.H.S of (SD) converges to $(N-1)Pr(\bar{v})$, thus the L.H.S. of (SD)¿ the R.H.S of (SD).

Next, take a monotone sequence $\{s_n\}_{n\in\mathbb{N}}$ where $s_n \downarrow 0$ as $n \to \infty$, $s_1 \in (0,\bar{v})$ and $\frac{s_{n+1}}{s_n} \leq \frac{1}{2}$ for any n. [31] I will prove that $\limsup_{n\to\infty} s_n f(s_n) = 0$ by contradiction. Suppose that $\limsup_{n\to\infty} s_n f(s_n) = c > 0$, then for any $\varepsilon > 0$, there exists a subsequence $\{s_{n_k}\}$ such that $s_{n_k} f(s_{n_k}) - c \geq \varepsilon$ for any k. So $f(s_{n_k}) \geq \frac{c-\varepsilon}{s_{n_k}}$ for any k. Let ε be $\frac{c}{2}$. That $x^2 f(x)$ is non-decreasing implies that for any $x \in (s_{n_{k+1}}, s_{n_k})$, $f(x) \geq \frac{s_{n_{k+1}}^2 f(s_{n_{k+1}})}{x^2} \geq \frac{s_{n_{k+1}}(c-\varepsilon)}{x^2} = \frac{cs_{n_{k+1}}}{2x^2}$ for any k. Therefore $\int_{s_{n_{k+1}}}^{s_{n_k}} f(x) \geq \frac{c\cdot s_{n_{k+1}}}{2}(\frac{1}{s_{n_{k+1}}} - \frac{1}{s_{n_k}}) \geq \frac{c}{4}$. Thus, $\int_0^{\bar{v}} dF(x) \geq \sum_{k=1}^{K} \int_{s_{n_{k+1}}}^{s_{n_k}} f(x) \geq \frac{cK}{4} \to \infty$ as $K \to \infty$, a contradiction to the fact that F is a probability measure. Therefore $\limsup_{n\to\infty} s_n f(s_n) = 0$. This implies that $\lim_{n\to\infty} s_n f(s_n) = 0$. Now, by L'Hôpital's rule,

$$\begin{aligned}
\lim_{n\to\infty} \frac{\int_{(0,s_n)} x^{\frac{N}{N-1}} dF(x)}{s_n^{\frac{N}{N-1}}} &= \lim_{n\to\infty} \frac{s_n^{\frac{N}{N-1}} f(s_n)}{\frac{N}{N-1} s_n^{\frac{1}{N-1}}} \\
&= \lim_{n\to\infty} \frac{(N-1)s_n f(s_n)}{N} \\
&= 0.
\end{aligned}$$

Then, if $s_n \downarrow 0$, the L.H.S. of (SD) ¡ the R.H.S of (SD). By the Intermediate Value Theorem, there exists $s^* \in (0,\bar{v})$ that is a solution to (SD). □

Proposition 9. *If $x^2 f(x)$ is non-decreasing for $x \in (0,\bar{v})$ and $Pr(\bar{v}) < \frac{\int_{(0,\bar{v})} x^{\frac{N}{N-1}} f(x)dx}{(N-1)\bar{v}^{\frac{N}{N-1}}}$, then the revenue guarantee of the second-price auction with the s^*-scaled $Beta(\frac{1}{N-1}, 1)$ distributed random reserve is at least $(N-1)s^*(1-F(s^*))$.*

[31] For example, $s_n = \frac{\bar{v}}{2^n}$ for $n \in \mathbb{N}$.

Proof. This follows immediately from Lemma 7 and Lemma 8. □

Proposition 10. *If $x^2 f(x)$ is non-decreasing for $x \in (0,\bar{v})$ and $Pr(\bar{v}) < \frac{\int_{(0,\bar{v})} x^{\frac{N}{N-1}} f(x)dx}{(N-1)\bar{v}^{\frac{N}{N-1}}}$, then there exists a joint distribution $\pi^{***} \in \pi(F)$ under which the second-price auction with the s^*-scaled $Beta(\frac{1}{N-1},1)$ distributed random reserve maximizes the expected revenue across standard dominant-strategy mechanisms. In addition, the maximized expected revenue is $(N-1)s^*(1-F(s^*))$.*

Proof. The joint distribution π^{***} is symmetric and is defined as follows. The support of π^{***} is V^+. If $v \notin V^+$, then $\pi^{***}(v) = 0$. If $v \in V^+$, then

$$\pi^{***}(v_i, v_{-i})^{32} = \begin{cases} f(0) & \text{if } v = (0,\cdots,0); \\ 0 & \text{if } 0 = v_j < v_i, \forall j \neq i; \\ \frac{1}{(N-1)v(1)^2}\left(v(2)f(v(2)) - \frac{v(2)^{-\frac{N}{N-1}}}{N-1}\int_0^{v(2)} x^{\frac{N}{N-1}} f(x)dx\right) & \text{if } 0 < v(2) = v_j \leq v_i = v(1) \leq s^*, \forall j \neq i; \\ \frac{f(v(1))}{(N-1)s^*(1-F(s^*))}\left(v(2)f(v(2)) - \frac{v(2)^{-\frac{N}{N-1}}}{N-1}\int_0^{v(2)} x^{\frac{N}{N-1}} f(x)dx\right) & \text{if } 0 < v(2) = v_j \leq s^* < v_i < \bar{v}, \forall j \neq i; \\ \frac{Pr(\bar{v})}{(N-1)s^*(1-F(s^*))}\left(v(2)f(v(2)) - \frac{v(2)^{-\frac{N}{N-1}}}{N-1}\int_0^{v(2)} x^{\frac{N}{N-1}} f(x)dx\right) & \text{if } 0 < v(2) = v_j \leq s^* < v_i = \bar{v}, \forall j \neq i. \end{cases}$$

It is straightforward to verify that $\pi^{***} \in \Pi(F)$. When $v_i = v(1) \leq s^*$, the density function coincides with π^{**}. Therefore by the proof of Proposition 7,

$$\phi_i^{***}(v) = 0 \quad \text{for } v \in V^+, v_i = v(1) \leq s^*.$$

Note that under π^{***}, when $v_i = v(1) > s^*$, v_i and v_{-i} are independent. Therefore $\phi_i^{***}(v) =$

[32] The density function π^{***} in the region $(0,\bar{v})^N$ is similar to the density function η_F^* in the region $(0,1)^N$ in He and Li (2022).

$v_i - \frac{1-F(v_i)}{f(v_i)}$ for $s^* < v_i = v(1) < \bar{v}$ and $\phi_i^{***}(v) = \bar{v}$ for $v_i = v(1) = \bar{v}$.

Now I show that $\phi_i^{***}(v) \geq 0$ for $s^* < v_i = v(1) < \bar{v}$. First I show that $1 - F(x) - xf(x)$ is non-increasing if $x^2 f(x)$ is non-decreasing. To see this, note that for any $0 < x_1 \leq x_2 < \bar{v}$,

$$\begin{aligned}
1 - F(x_2) - x_2 f(x_2) - [1 - F(x_1) - x_1 f(x_1)] &= x_1 f(x_1) - x_2 f(x_2) - \int_{x_1}^{x_2} f(x) dx \\
&\leq x_1 f(x_1) - x_2 f(x_2) - \int_{x_1}^{x_2} \frac{x_1^2 f(x_1)}{x^2} dx \\
&= \frac{x_1^2 f(x_1)}{x_2} - x_2 f(x_2) \\
&\leq 0,
\end{aligned}$$

where the first inequality follows from that $x^2 f(x) \geq x_1^2 f(x_1)$ for $x_1 \leq x \leq x_2$ and the second inequality follows from that $x_2^2 f(x_2) \geq x_1^2 f(x_1)$.

Recall that

$$\frac{\int_{(0,s^*)} x^{\frac{N}{N-1}} dF(x)}{(s^*)^{\frac{N}{N-1}}} = (N-1)(1 - F(s^*)).$$

Subtracting $(N-1)s^* f(s^*)$ from both sides, I obtain that

$$\frac{\int_{(0,s^*)} x^{\frac{N}{N-1}} dF(x)}{(s^*)^{\frac{N}{N-1}}} - (N-1)s^* f(s^*) = (N-1)[1 - F(s^*) - s^* f(s^*)].$$

The L.H.S. of the above equation is weakly negative, shown in (B.22). Together with that $1 - F(x) - xf(x)$ is non-increasing, I have that $1 - F(x) - xf(x) \leq 0$ for any $x \geq s^*$. Hence,

$$\phi_i^{***}(v) = v_i - \frac{1 - F(v_i)}{f(v_i)} \geq 0 \quad \text{for } v \in V^+, s^* < v_i = v(1) < \bar{v}.$$

.

Then, any standard dominant-strategy mechanism, in which 1) the ex-post participation constraints are binding for zero-valuation bidders and 2) the highest bidder with a valuation higher than s^* is allocated with probability one, maximizes the expected revenue across standard dominant-strategy mechanisms under π^{***}. And the second-price auction with the s^*-scaled $Beta(\frac{1}{N-1},1)$ distributed random reserve is such a mechanism. Finally, it is straightforward to show that the maximized expected revenue is $Ns^*(1-F(s^*))$ by calculating the expected total virtual surplus. Indeed, under this mechanism and the joint distribution π^{***} , the expected total virtual surplus is

$$
\begin{aligned}
& N\{\int_{(0,\bar{v})}\int_{(0,s^*)}\frac{f(v_1)}{(N-1)s^*(1-F(s^*))}[v_2f(v_2)-\frac{v_2^{-\frac{N}{N-1}}}{N-1}\int_0^{v_2}x^{\frac{N}{N-1}}f(x)dx)]\cdot(v_1-\frac{1-F(v_1)}{f(v_1)})dv_2dv_1+\\
& \int_{(0,s^*)}\frac{Pr(\bar{v})}{(N-1)s^*(1-F(s^*))}[v_2f(v_2)-\frac{v_2^{-\frac{N}{N-1}}}{N-1}\int_0^{v_2}x^{\frac{N}{N-1}}f(x)dx)]\cdot\bar{v}dv_2\}\\
& =\frac{N}{(N-1)s^*(1-F(s^*))}\cdot[\int_{(0,\bar{v})}(v_1f(v_1)-1+F(v_1))dv_1+Pr(\bar{v})\bar{v}]\cdot\frac{\int_{(0,s^*)}s^{\frac{N}{N-1}}f(s)ds}{(s^*)^{\frac{1}{N-1}}}\\
& =\frac{N}{(N-1)s^*(1-F(s^*))}\cdot[\int_{(0,\bar{v})}(v_1f(v_1)-1+F(v_1))dv_1+Pr(\bar{v})\bar{v}]\cdot(N-1)s^*(1-F(s^*))\\
& =N\cdot[\int_{(0,\bar{v})}(v_1f(v_1)-1+F(v_1))dv_1+Pr(\bar{v})\bar{v}]\\
& =Ns^*(1-F(s^*)),
\end{aligned}
$$

where the first equality follows from (B.20), the second equality follows from (SD) and the last equality uses integration by parts. □

Theorem 10 follows immediately from Proposition 9 and Proposition 10.

2.7.3 Proofs for Section 2.5

Proof of Proposition 8

Note that under a posted-price mechanism, the maximally positively correlated distribution (the valuations of the bidders are always the same) is a worst-case correlation structure, and the revenue guarantee of any posted-price mechanism is thus at most $\max_{x\in[0,\bar{v}]}x(1-F(x))$, which is the monopoly profit when there is only one bidder. It is

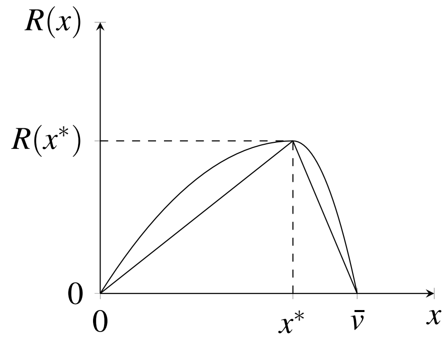

Figure 2.1: Revenue Function

straightforward to show that $\frac{x^{\frac{N}{N-1}}}{\bar{v}^{\frac{1}{N-1}}} \geq \frac{x^2}{\bar{v}}$ for any $x \in [0, \bar{v}]$ and $N \geq 2$. Thus, it suffices to compare $\frac{E[X^2]}{\bar{v}}$ with $\max_{x\in[0,\bar{v}]} x \cdot (1 - F(x))$ if the revenue function $R(x) = x \cdot (1 - F(x))$ is strictly concave. Using integration by parts, I obtain that

$$E[X^2] = 2\int_0^{\bar{v}} x(1-F(x))dx.$$

Let x^* denote the unique solution to $\max_{x\in[0,\bar{v}]} R(x)$. Then using graph (see Figure 2.1[33]) it is straightforward that

$$\int_0^{\bar{v}} R(x)dx > \frac{1}{2} \cdot \bar{v} \cdot R(x^*). \tag{D.1}$$

(D.1) is equivalent to that

$$\frac{E[X^2]}{\bar{v}} = \frac{2\int_0^{\bar{v}} x(1-F(x))dx}{\bar{v}} > \max_{x\in[0,\bar{v}]} x(1-F(x)).$$

[33]The curve is a strictly concave revenue function. The L.H.S. of (D.1) is the area under the curve. The R.H.S. of (D.1) is the area of the triangle.

Proof of Lemma 7

For each t, I construct the dual variables for the second-price auction with the random reserve whose cumulative distribution function is $G_t(r) = (\frac{r}{t})^{\frac{1}{N-1}}$ as follows:

$$\lambda_i(x) = \frac{x^{\frac{N}{N-1}}}{Nt^{\frac{1}{N-1}}} \quad \text{if} \quad 0 \leq x \leq t, \forall i \in I,$$

$$\lambda_i(x) = \frac{t}{N} \quad \text{if} \quad t < x \leq \bar{v}, \forall i \in I.$$

Given the constructed dual variables above, the value of (D) is

$$\int_0^t \frac{x^{\frac{N}{N-1}}}{t^{\frac{1}{N-1}}} dF(x) + t(1 - F(t)).$$

Then it suffices to show that the constructed dual variables are feasible, or (B.10) holds. I divide the valuation profiles into three cases.

Case 1: $v(1) \leq t$.

(B.10) holds by a similar argument with that in the proof of Proposition 5.

Case 2: $v(1) > t, \#\{k : v_k = v(1)\} = 1$.

When $v(2) > t$, then $t(v) = v(2)$. The L.H.S. of (B.10) is maximized when $v_i \geq t$ for all i, and the maximized value is $N \cdot \frac{t}{N} = t < t(v)$. When $v(2) \leq t$, then $t(v) = v(1) \cdot 1 - \int_t^{v(1)} dx - \int_{v(2)}^t (\frac{x}{t})^{\frac{1}{N-1}} dx = \frac{t}{N} + \frac{(N-1)v(2)^{\frac{N}{N-1}}}{Nt^{\frac{1}{N-1}}}$. The L.H.S. of (B.10) is maximized when $v_i = v(2)$ for all $i \notin \{k : v_k = v(1)\}$, and the maximized value is $\frac{t}{N} + \frac{(N-1)v(2)^{\frac{N}{N-1}}}{Nt^{\frac{1}{N-1}}} = t(v)$. Therefore (B.10) holds.

Case 3: $v(1) > t, \#\{k : v_k = v(1)\} \geq 2$.

Now $t(v) = v(1)$. The L.H.S. of (B.10) is maximized when $v_i \geq t$ for all i, and the maximized value is $N \cdot \frac{t}{N} = t < t(v)$. Therefore (B.10) holds.

Proof of Theorem 11

Recall that the revenue guarantee of a posted-price mechanism is at most $\max_{x\in[0,\bar{v}]} x\cdot(1-F(x))$ for a given $F\in\Delta^c[0,\bar{v}]$. Denote a solution to $\max_{x\in[0,\bar{v}]} x\cdot(1-F(x))$ as x^*. For a given $F\in\Delta^c[0,\bar{v}]$, consider the second-price auction with the x^*-scaled $Beta(\frac{1}{N-1},1)$ distributed random reserve, and I have that

$$\int_0^{x^*}\frac{x^{\frac{N}{N-1}}}{(x^*)^{\frac{1}{N-1}}}dF(x)+x^*(1-F(x^*))>x^*(1-F(x^*)),$$

where the inequality follows from that $x^*>0$.

Proof of Theorem 12

By He and Li (2022), given a $F\in\Delta^c[0,\bar{v}]$, the revenue guarantee of the second-price auction with the optimal deterministic reserve for the N-bidder case is as follows:

$$\frac{N}{N-1}\int_{r^*}^{c(r^*)}xdF(x),$$

where r^* satisfies $F(Nr^*)=F(\frac{N-1+F(r^*)}{N})$, and $c(r^*)=F^{-1}(\frac{N-1+F(r^*)}{N})$.

Define $J(x):=\frac{N}{N-1}x-\frac{x^{\frac{N}{N-1}}}{c(r^*)^{\frac{1}{N-1}}}$. Because $J'(x)=\frac{N}{N-1}(1-(\frac{x}{c(r^*)})^{\frac{1}{N-1}})$ and $J''(x)=-\frac{Nx^{\frac{1}{N-1}-1}}{(N-1)^2c(r^*)^{\frac{1}{N-1}}}\leq 0$, $J(x)$ is maximized at $x=c(r^*)$ and the maximized value is $\frac{1}{N-1}c(r^*)$ by simple calculation. For a given $F\in\Delta^c[0,\bar{v}]$, consider the second-price auction with the $c(r^*)-$scaled $Beta(\frac{1}{N-1},1)$ distributed random reserve, and I have that

$$\begin{aligned}\int_0^{c(r^*)}\frac{x^{\frac{N}{N-1}}}{[c(r^*)]^{\frac{1}{N-1}}}dF(x)+c(r^*)[1-F(c(r^*))] &> \int_{r^*}^{c(r^*)}\frac{x^{\frac{N}{N-1}}}{[c(r^*)]^{\frac{1}{N-1}}}dF(x)+c(r^*)[1-F(c(r^*))]\\ &\geq \int_{r^*}^{c(r^*)}[\frac{N}{N-1}x-\frac{1}{N-1}c(r^*)]dF(x)+c(r^*)[1-F(c(r^*))]\\ &= \frac{N}{N-1}\int_{r^*}^{c(r^*)}xdF(x),\end{aligned}$$

where the first inequality follows from that $r^* > 0$, the second inequality follows from that $J(x) \leq \frac{1}{N-1}c(r^*)$ and the equality follows from that

$$\begin{aligned}\int_{r^*}^{c(r^*)} \frac{1}{N-1}c(r^*)dF(x) &= \frac{1}{N-1}c(r^*)[F(c(r^*)) - F(r^*)] \\ &= \frac{1}{N-1}[\frac{N-1+F(r^*)}{N} - F(r^*)]F^{-1}(\frac{N-1+F(r^*)}{N}) \\ &= [\frac{1-F(r^*)}{N}]F^{-1}(\frac{N-1+F(r^*)}{N}) \\ &= c(r^*)[1-F(c(r^*))].\end{aligned}$$

2.7.4 "Necessity" of Robust Regularity Conditions

Definition 8. I say the allocation rule q is *strictly monotone* if for any i, any v_{-i} and any pair of valuation v_i and v'_i in which $q_i(v_i, v_{-i}) > 0$ and $q_i(v'_i, v_{-i}) > 0$, I have that $q_i(v_i, v_{-i}) < q_i(v'_i, v_{-i})$ whenever $v_i < v'_i$.

Proposition 11. *For the two-bidder case, if the second-price auction with the uniformly distributed random reserve is a maxmin mechanism across dominant-strategy mechanisms, then the two-bidder robust regularity conditions hold almost surely.*

Proof. The intuition behind this is the observation that under the second-price auction with the uniformly distributed random reserve, the allocation rule is strictly monotone. In addition, the high bidder's allocation is positive but less than 1 when her valuation is positive but less than $\bar{v}$. Thus in a Nash equilibrium, the high bidder's virtual value has to be 0 for these valuations under the joint distribution, otherwise Myerson's ironing argument implies that allocation rule in equilibrium should exhibit "flatness" across some range. Formally, I will establish Lemma 9 and Lemma 10 below.

Lemma 9. *For the two-bidder case, for the second-price auction with the uniformly distributed random reserve to be part of a Nash equilibrium across dominant-strategy mechanisms, the equilibrium joint distribution has to be π^* almost surely.*

Proof. let π be a best response of adversarial nature to the second-price auction with the uniformly distributed random reserve. Suppose (1) does not hold for a set of (v_1, v_2) where $\bar{v} > v_1 \geq v_2$ with some positive measure. If virtual values of bidder 1 for these valuation profiles are all positive, then consider a modified allocation exhibiting the property that the allocation to bidder 1 is one from the valuation profile in which the virtual value of bidder 1 becomes positive for the first time. Formally, let $\overline{v_1(v_2)} = \inf\{v_1 : \phi_1(v_1, v_2) > 0, v_1 \geq v_2\}$. Let $\tilde{q}_1(v_1, v_2) = 1$ for $v_1 > \overline{v_1(v_2)}$ and $\tilde{q}(v) := q^*(v)$ otherwise. Such modification is feasible since bidder 2 gets zero allocation for any one of these valuation profiles in the second-price auction with the uniformly distributed random reserve. Thus I have a profitable and feasible deviation. If virtual values of bidder 1 for these valuation profiles are all negative, by a similar argument, I rule out the possibility that the second-price auction with the uniformly distributed random reserve is a best response of the auctioneer to π. Now If virtual values of bidder 1 for these valuation profiles are not all positive and not all negative, I have to discuss two cases. The first case is that the virtual value is still (weakly) monotone. Then by a similar argument, the second-price auction with the uniformly distributed random reserve can not a best response to π. The second case is that the virtual value is not monotone, then a best response has to exhibit flatness across a range of valuation profiles, which can be done by Myerson's ironing procedure. Recall that the allocation rule of the second-price auction with the uniformly distributed random reserve is strictly monotone. Thus, it cannot be a best response. To illustrate this, suppose $\phi_1(\cdot, v_2)$ is decreasing in v_1 for $v_1 \in (a(v_2), b(v_2))$ and $\phi_1(a(v_2), v_2) > \phi_1(\hat{v}_1(v_2), v_2) = 0 > \phi_1(b(v_2), v_2)$ for some $\hat{v}_1(v_2) \in (a(v_2), b(v_2))$. Then let $\tilde{q}_1(v_1, v_2) = q^*(\hat{v}_1(v_2), v_2)$ for $v_1 \in [a(v_2), b(v_2)]$ and $\tilde{q}(v) = q^*(v)$ otherwise. Since this is a feasible and profitable deviation, I conclude that the second-price auction with the uniformly distributed random reserve can not be a best response.

Together with the proof of Proposition 5, the equilibrium joint distribution is π^* almost surely. □

Lemma 10. *For the two-bidder case, for the second-price auction with the uniformly distributed* *random reserve to be part of a Nash equilibrium across dominant-strategy mechanisms,* π^* *exhibits* (2) *almost surely.*

Proof. Suppose not. Then, there exists a set of (v_1, v_2) where $0 < v_2 < v_1 < \bar{v}$ but $\phi_2(v_1, v_2) > 0$ with some positive measure. Then by increasing the allocation to bidder 2 by a small amount ε when the valuation profile lies in this set, I have a feasible and profitable deviation. Thus, the second-price auction with the uniformly distributed random reserve is not a best response. □

Proposition 11 follows immediately from Lemma 9, Lemma 10 and the proof of Proposition 5. □

Proposition 12. *For the $N-$bidder ($N \geq 3$) case, If the second-price auction with the $\bar{v}-$scaled $Beta(\frac{1}{N-1}, 1)$ distributed random reserve is a maxmin mechanism across standard dominant-strategy mechanisms, then the $N-$bidder robust regularity conditions (I) hold almost surely.*

Proof. First, I establish Lemma 11 below.

Lemma 11. *For the $N-$bidder ($N \geq 3$) case, for the second-price auction with the $Beta(\frac{1}{N-1}, 1)$ distributed random reserve to be part of a Nash equilibrium across standard dominant-strategy mechanisms, the equilibrium joint distribution has to be* π^{**} *almost surely.*

Proof. As shown in the proof of Proposition 6, (B.10) holds with equality if and only if $v \in V^+$. This implies the equilibrium joint distribution has the support V^+. Then this lemma follows from a similar argument to the proof of Lemma 9. □

Proposition 12 follows immediately from Lemma 11 and the proof of Proposition 7. □

Chapter 3

Auctioning Multiple Goods without Priors

3.1 Introduction

The standard auction literature focuses on the single-good environment and assumes that bidders' value profile follows a commonly known joint distribution. It is assumed that the designer seeks a mechanism that maximizes the expected revenue. Myerson (1981) characterizes optimal mechanisms for selling a single good when bidders' values are independent; Crémer and McLean (1988) characterize optimal mechanisms for selling a single good given generic correlation structures of bidders' values. However, optimal mechanisms vary widely with the model of correlation structure and relatively little is known about how optimal mechanisms would perform once the correlation structure is misspecified. In addition, it is not clear how the designer should form a prior in the first place.

In this paper, I am going to extend the analysis in two ways. First, I consider the multiple-good environment, in which little is known about the optimal mechanisms. Even for the special case where there is only one bidder, the optimal mechanism is hard to characterize or to describe (Daskalakis et al. (2014) and Manelli and Vincent (2007)). Second, I consider a robust version of the analysis. Specifically, I consider a (correlated) private value model where the designer auctioning multiple different goods knows no distributional information except for the *upper* *bounds* of bidders' values for each good. In contrast, bidders agree on a joint distribution over their value profiles[1]. The designer considers any joint distribution consistent with the known upper bounds to be possible. I consider general mechanisms with the only requirement that the mechanism "secures" bidders' participation: there exists a message for each payoff type of each bidder that guarantees a non-negative ex-post payoff regardless of the other bidders' messages. I assume that the designer takes the *minimax regret approach*. Precisely, the *expected regret* from a mechanism given a joint distribution over value profiles and a Bayesian equilibrium is defined as the difference between the full surplus[2] and the expected revenue. The designer evaluates a mechanism by its highest expected regret across all possible joint distributions and all Bayesian equilibria, which is referred to as its *regret cap*. The designer aims to find a mechanism, referred to as a *minimax regret mechanism*, that minimizes the regret cap.

The assumption that the designer only knows the upper bounds is appropriate for situations where little information is known about the bidders and it is costly and time-consuming to collect information. For example, in an auction of initial public offerings, there is no distributional information about the bidders' values. Note that in this example, bidders' budgets, which can be viewed as (reasonable approximations of) the upper bounds of bidders' values, are typically known by the designer, as bidders for initial public offerings are often institutional investors, whose financial resources are publicly known, or can be estimated fairly precisely from their financial reports. On the contrary, the assumption may be too conservative for situations in which data about bidders are abundant, e.g., online advertising in which auctions are held repeatedly and frequently. In addition, the assumption is formally necessary to obtain non-trivial results because if there is no known upper bound, then minimax regret will be infinite [3]. Thus, this model can be viewed as a theoretical benchmark that provides a first step toward a broad study of robust auction design problems in the multiple-good environment.

[1] In the Appendix, I show that the main result still holds when bidders can acquire additional information.

[2] The full surplus under a joint distribution over value profiles is the expected revenue attainable were the designer able to sell the goods with full information about bidders' value profile.

[3] See Remark 19 for the formal proof.

The minimax regret approach can be traced back to Wald (1950) and Savage (1951). It captures the idea that a decision maker is concerned about missing out opportunities. A decision theoretical axiomatization of regret can be found in Milnor (1951) and Stoye (2011). It is adapted to multiple priors by Hayashi (2008) and Stoye (2011). Another leading approach is the maxmin utility approach, which is adopted by most of the robust mechanism design literature. However, in the setting of this paper, under the maxmin utility approach, it is optimal for the designer to keep all goods to herself because it is possible that all bidders have zero values towards all goods. Note that in this extreme case, there is no surplus to extract even under complete information. Thus, the maxmin utility approach is too conservative to be useful, whereas the minimax regret approach protects the surplus when there is some surplus to extract and will be shown to lead to a non-trivial answer.

The main result is that a *separate second-price auction with random reserves* is a minimax regret mechanism. This mechanism can be described as follows. For each good, the designer holds a separate auction; the formats of these auctions are second-price auctions with bidder-specific random reserves that depend on the upper bounds of values[4]. It is remarkable that a simple mechanism arises as a robustly optimal mechanism for auctioning multiple goods, across all participation-securing mechanisms that include highly complicated mechanisms, e.g., combinatorial auctions[5].

Importantly, I allow for general upper bounds of values for the main result. In particular, the upper bounds of the values for a given good can be different across bidders. This captures the widely observed *asymmetries* in many real-life auction environments. For example, in art auctions, there are obvious asymmetries associated with differing budget constraints across bidders. Asymmetric auctions have been studied in Maskin and Riley (2000), Hafalir and Krishna (2008),Güth et al. (2005) and Athey et al. (2013) among others.

The main result provides a possible explanation why separate second-price auctions -

[4]The distributions of these random reserves are given in the Section 3.4.1.

[5]In a combinatorial auction, bidders can place bids on combinations of discrete heterogeneous goods.

or their more practical equivalents in the private value environment, separate English auctions - are widely used in practice for auctioning multiple goods. For example, at the popular online auction site eBay, each good is typically auctioned separately via an English auction (Krishna (2009), Anwar et al. (2006) and Feldman et al. (2020)); Sotheby's and Christie's (two major auction houses of art) typically sell works of art separately via an English auction (Ashenfelter and Graddy (2003)).[6] The main result justifies this empirical rule of thumb by an optimal performance guarantee: a separate second-price auction (albeit with random reserves) minimizes the worst-case expected regret. This may be one reason why complicated mechanisms that require the full information of the joint distribution over bidders' value profiles, to my knowledge, are not used in practice for auctioning multiple goods.

The role of randomized reserves can be seen considering the one-good one-bidder case, in which they are reduced to randomized pricing. The designer suffers from a large regret if she charges a high price when the value of the bidder is low or if she charges a low price when the value of the bidder is high. She can lower her regret by randomizing. Bergemann and Schlag (2008) characterize the solution for the one-good one-bidder case. Indeed, the well-crafted distribution of the randomized pricing renders the designer indifferent across values over a range. The second-price auction with random reserves extends the robust property to the one-good multiple-bidder case. In this case, the regret from a value profile is the difference between the highest value and the collected revenue. When the second highest value is low enough (e.g., 0), it boils down to the one-good one-bidder case and the regret remains the same; when the second highest value is high enough (e.g., above the lower bound of the random reserve for the highest bidder), then the revenue is even higher and the regret is thus lower. To see the intuition

[6] In practice, there are other mechanisms used for auctioning multiple goods. While most of spectrum auctions in US do not allow for "package bidding" due to its complexities (Cramton (2002) and Filiz-Ozbay et al. (2015)), the Federal Communications Commission (FCC) has used a "package bidding auction" to sell spectrum licenses in rare cases. In such an auction, a bidder is allowed to select a group of licenses to bid on as a package. In an early 2008 FCC auction (Auction 73), AT&T and Verizon both bought geographically diverse packages of 700MHz spectrum. The principal rationale to consider a package bidding auction is that there may be complementarity between different licenses (Goeree and Holt (2005)). Complementarity is ruled out in my model. It is an interesting open question what the minimax regret mechanism would be like when there is complementarity between goods.

behind separation, it is instructive to consider another mechanism of auctioning the *grand* bundle (only the bundle of all goods is auctioned). I argue that this mechanism may result in a high regret. Consider a three-good three-bidder example and an extremely *asymmetric* value profile in which each bidder values a different good (assuming that the upper bound on each bidder's values for each good is 1) : $(v_1^1, v_1^2, v_1^3) = (1,0,0), (v_2^1, v_2^2, v_2^3) = (0,1,0), (v_3^1, v_3^2, v_3^3) = (0,0,1)$.[7] The designer will *lose all but one good* if auctioning the grand bundle: she can at most obtain a revenue of 1 from one of the goods but will suffer a regret of 2 from losing the other goods. In contrast, separation can guarantee a good regret performance for each good. It can be shown that for this example, the separate second-price auction with random reserves yields a regret cap[8] lower than 2. Intuitively, auctioning the grand bundle performs just like selling one good at this value profile, while selling separately allows the designer to earn more. Furthermore, the same argument implies that *partial* bundling (a mechanism in which a bundle of some goods are auctioned) may perform worse than separate selling.

I show that the separate second-price auction with random reserves is a minimax regret mechanism by constructing a joint distribution over value profiles, referred to as a *worst-case distribution*, such that the lower bound of the expected regret for any mechanism and any equilibrium under this joint distribution is *equal* to the upper bound of the expected regret for any joint distribution and any equilibrium under the separate second-price auction with random reserves. One can imagine that adversarial nature is constructing a worst-case distribution to let the designer suffer from a high expected regret.

The worst-case distribution admits a description as follows[9]. For each good, adversarial nature *selects* one bidder whose upper bound of the values of the good is the highest among the bidders (breaking ties arbitrarily). For each bidder, the marginal distributions of the values of the goods for which the bidder is selected are *equal-revenue distributions*, defined by the property of a unit-elastic demand: in the monopoly pricing problem, the monopoly's revenue

[7] The superscript represents the good, and the subscript represents the bidder.

[8] Indeed, it can be shown that the regret cap is $\frac{3}{e}$ by simple calculation.

[9] Its formal definition is given in Section 3.4.2.

from charging any price in the support is the same; the values for these goods are *comonotonic* (maximal positive correlation); the values for the other goods are all *zeros*[10]. For the goods across the bidders, the values are *independent*.

Under this distribution, each selected bidder values a totally different set of goods; for each good, exactly one bidder values it; each selected bidder's values for the goods he values are comonotonic; the values of the goods across the selected bidders are independent.

Now I illustrate the idea behind this distribution. First, to understand the part of equal-revenue distributions, consider the one-good one-bidder case in which the mechanism collapses to randomized pricing over a range. As the designer is indifferent between these prices, the marginal revenue must be zero over these prices, from which an equal-revenue distribution arises[11]. Second, the intuition for the part of selection can be summarized by a *scale effect*: because the minimax regret in the one-good one-bidder case is proportional to the upper bound of the values, by selecting a bidder whose upper bound of the values is the highest for each good, the potential regret is made the highest for each good. Third, the intuition for the part of comonotonicity can be summarized by a *screening effect*: consider the multiple-good one-bidder case, the comonotonicity between goods limits the ability of the designer to screen different goods by reducing the multi-dimensional screening to the single-dimensional screening. Fourth, the intuition for the part of zeros can be summarized by a *competition effect*: it eliminates the competition among bidders for each good by letting only one bidder have a positive value for each good. Fifth, the intuition for the part of independence can be summarized by an *information effect*: one bidder's values for goods do not provide any information about any other bidder's values for other goods, which prevents the designer from extracting surplus from one bidder based on information about other bidders.

The main result incorporates multi-dimensional screening and single-good auction as two special cases. For the multi-dimensional screening, a *separate randomized posted-price*

[10]Note that if a bidder is not selected for any good, then his values for all goods are zeros.
[11]See Bergemann and Schlag (2008) for details about the derivation.

mechanism is a minimax regret mechanism, and a distribution in which the values across the goods are comonotonic is a worst-case distribution.[12] For the single-good auction, a *second-price auction with random reserves* is a minimax regret mechanism, and a distribution in which only one bidder has a positive value for the good is a worst-case distribution.

The remainder of the introduction discusses related work. Section 3.2 presents the model. Section 3.3 illustrates the methodology. Section 3.4 characterizes the main result. Section 3.5 presents the solutions to two special cases. Section 3.6 is a discussion. Section 3.7 is a conclusion. The Appendix extends the result to an environment where bidders can acquire any additional information.

3.1.1 Related Work

The closest related work is Koçyiğit et al. (2020b), who consider the same environment and find a (different) separate second-price auction with random reserves has good robust properties. There are, however, several critical differences. First, they restrict attention to dominant-strategy mechanisms[13], whereas I allow for general mechanisms with essentially the only requirement that there is a message that secures bidders' participation. That is, I search for a minimax regret mechanism from a much wider class of mechanisms. Second, they show that their proposed mechanism is a minimax regret mechanism for the symmetric case where the upper bounds of the values for a given good are the same across bidders, whereas I establish that my proposed mechanism is a minimax regret mechanism for general upper bounds. That is, I place no restrictions on the upper bounds of the values for a given good across bidders. It is important and interesting to understand the minimax regret mechanism in asymmetric environments considered in this paper, as the symmetric case is a knife-edge case. In this sense, this paper complements their work. The key factor that drives these differences is that

[12]The solution for the multi-dimensional screening has been found by Koçyiğit et al. (2021). I offer an alternative proof using a quantile-version of virtual values. Carroll (2017) also uses quantiles to parameterize the single buyer's values for the multi-dimensional screening.

[13]See Definition 9 for the formal definition of dominant-strategy mechanisms.

I construct a different joint distribution over value profiles that yields a higher lower bound of the expected regret in general. In addition, there is a Pareto ranking between my proposed mechanism and theirs: in the truth-telling equilibrium, the ex-post regret is always weakly lower and sometimes strictly lower under my proposed mechanism than that under theirs (Remark 16). Besides, technically, they take the duality approach for their result, whereas I adopt an adaptation of the classic Myerson's approach to identify a lower bound of the expected regret under my constructed joint distribution.

Bergemann and Schlag (2008, 2011) consider the problem of monopoly pricing where the monopolist is faced with uncertainty about the demand curve and characterize randomized posted price mechanisms as minimax regret mechanisms. My result reduces to that of Bergemann and Schlag (2008) in the one-good one-bidder case. Koçyiğit et al. (2021) consider the problem of multi-dimensional screening without priors and characterize randomized separate posted price mechanisms as minimax regret mechanisms. My result reduces to theirs in the multiple-good one-bidder case. Moreover, I offer another minimax regret mechanism: a randomized grand bundling (Remark 21).

More broadly, this paper is related to the robust mechanism design literature and the information design literature. Carrasco et al. (2018) characterize maxmin selling mechanisms when the seller faced with a single buyer only knows the first N moments of distribution (N is an arbitrary positive integer). Che (2019), He and Li (2022) and Zhang (2022a) study the robust auction design problem when the designer has limited distributional information. Zhang (2021) studies the profit-maximizing bilateral trade problem and characterize maxmin trade mechanisms when the designer knows only the expectations of the values. Similar to mine, these papers all assume that the values are private and all characterize some randomized mechanism as a maxmin mechanism. Carroll (2017) studies the multi-dimensional screening problem when the designer only knows the marginal distributions. Similar to the multiple-good one-bidder case in my paper, a separate selling mechanism turns out to be a maxmin solution. Different from the multiple-good one-bidder case in my paper, his maxmin solution does not require randomization. Chung and Ely (2007) and Chen and Li (2018) study maxmin foundations for dominant-strategy mechanisms. Similar to my model, they study the private value environment. In contrast to my model, beliefs are not required to be consistent with a common prior; in addition, they select for the designer's most preferred equilibrium. Roesler and Szentes (2017) and Condorelli and Szentes (2020) derive optimal information structures for maximizing buyer's surplus. My worst-case distribution reduces to theirs in the one-good one-bidder case. Du (2018) derives the optimal informationally robust mechanism for the one-good one-bidder case and constructs a mechanism that asymptotically extracts full surplus for the single-good auction. Brooks and Du (2021) derive the optimal informationally robust mechanism for the single-good auction. In contrast to my model, they study the common value environment. However, our solution concepts are similar. My solution is indeed a *strong minimax solution*: holding the joint distribution fixed, the mechanism and equilibrium minimize regret, and holding the mechanism fixed, the joint distribution and equilibrium maximize regret; in addition, there is an equilibrium (the truth-telling equilibrium) under which the regret cap is hit.

3.2 Model

I consider a (correlated) private value environment where a designer sells J different indivisible goods to I risk-neutral bidders. I denote by $\mathcal{I} = \{1, 2, \cdots, I\}$ the set of bidders and by $\mathcal{J} = \{1, 2, \cdots, J\}$ the set of goods. Bidder i's value of the good j is denoted by v_i^j, and bidder i's value vector for all goods is denoted by $\mathbf{v_i} = (v_i^1, v_i^2, \cdots, v_i^J)$. The value profile across bidders is denoted by $\mathbf{v} = (\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_I})$. Each bidder's value vector is his private information, which the designer perceives as uncertain. I assume that the designer only knows an upper bound $\bar{v}_i{}^j$ on the value v_i^j for all $i \in \mathcal{I}$ and all $j \in \mathcal{J}$. Then I denote by $V_i = \times_{j \in \mathcal{J}} [0, \bar{v}_i{}^j]$ the set of all possible value vectors of bidder i, by $V = \times_{i \in \mathcal{I}} V_i$ the set of all possible value profiles across bidders and by ΔV the set of all possible joint distributions on V. In contrast, bidders share a common prior

$\pi \in \Delta(V)$. For exposition, I assume that the supply cost for each good is zero[14].

A *mechanism* $\mathcal{M}$ consists of measurable sets of messages M_i for each i and measurable allocation rules $\mathbf{q_i} = (q_i^j)_{j \in \mathcal{J}} : M \to [0,1]^J$ and measurable payment rules: $t_i : M \to \mathbb{R}$ for each i, where $M = \times_{i=1}^{I} M_i$ is the set of message profiles, such that $\sum_{i=1}^{I} q_i^j(\mathbf{m}) \leq 1$ for each j. Given a mechanism $\mathcal{M}$ and a simultaneously submitted message profile $\mathbf{m}$, bidder i with a value vector of $\mathbf{v_i}$ has an ex-post payoff

$$U_i(\mathbf{v_i}, \mathbf{m}) = \mathbf{v_i} \cdot \mathbf{q_i}(\mathbf{m}) - t_i(\mathbf{m}). \tag{1}$$

Bidders' preferences are quasilinear and additively separable across the goods. I require the mechanism to satisfy a *participation security* constraint: For each i, there exists $\mathbf{0} \in M_i$ such that for each $\mathbf{v_i} \in V_i$ and each $\mathbf{m_{-i}} \in M_{-i}$,

$$U_i(\mathbf{v_i}, (\mathbf{0}, \mathbf{m_{-i}})) \geq 0. \tag{PS}$$

Bidder i with a value vector $\mathbf{v_i}$ can guarantee a nonnegative ex-post payoff by sending this message, regardless of messages sent by the other bidders.

Given a mechanism $\mathcal{M}$ and a joint distribution (common prior among bidders) π, I have a game of incomplete information. A *Bayes Nash Equilibrium* (BNE) of the game is a strategy profile $\sigma = (\sigma_i)$, $\sigma_i : V_i \to \Delta(M_i)$, such that σ_i is best response to σ_{-i}: Let $U_i(\mathbf{v_i}, \mathcal{M}, \pi, \sigma) = \int_{\mathbf{v_{-i}}} U_i(\mathbf{v_i}, (\sigma_i(\mathbf{v_i}), \sigma_{-i}(\mathbf{v_{-i}}))) d\pi(\mathbf{v_{-i}} | \mathbf{v_i})$ where $U_i(\mathbf{v_i}, (\sigma_i(\mathbf{v_i}), \sigma_{-i}(\mathbf{v_{-i}})))$ is the multilinear extension of U_i in Equation (1), then for any $i, \mathbf{v_i}, \sigma_i'$,

$$U_i(\mathbf{v_i}, \mathcal{M}, \pi, \sigma) \geq U_i(\mathbf{v_i}, \mathcal{M}, \pi, (\sigma_i', \sigma_{-i})). \tag{BR}$$

The set of all Bayes Nash Equilibria for a given mechanism $\mathcal{M}$ and a given joint distribution π is denoted by $\Sigma(\mathcal{M}, \pi)$.

[14] All results can be easily extended to the case where the supply cost can be positive and different for each good. Formal statement and proofs are omitted but available upon request.

The designer's *expected regret* is defined as the difference between the full surplus given a joint distribution and the expected revenue under a mechanism and an equilibrium. Given a joint distribution π, the full surplus is $\int_{\mathbf{v}}\{\sum_{j=1}^{J}\max_{i\in I} v_i^j\}d\pi(\mathbf{v})$, and the expected revenue given a mechanism $\mathcal{M}$ and an equilibrium σ is $\int_{\mathbf{v}}\{\sum_{i=1}^{I} t_i(\sigma(\mathbf{v}))\}d\pi(\mathbf{v})$. The expected regret thus is $ER(\mathcal{M},\pi,\sigma) = \int_{\mathbf{v}}\{\sum_{j=1}^{J}\max_{i\in I} v_i^j - \sum_{i=1}^{I} t_i(\sigma(\mathbf{v}))\}d\pi(\mathbf{v})$. The integrand is defined as the *ex-post regret* from $\mathbf{v}$ under the equilibrium σ. The designer evaluates a mechanism by its worst-case expected regret across all possible joint distributions and equilibria. Formally, the designer evaluates a mechanism $\mathcal{M}$ by $GER(\mathcal{M}) = \sup_{\pi\in\Delta(V)}\sup_{\sigma\in\Sigma(\mathcal{M},\pi)} ER(\mathcal{M},\pi,\sigma)$, referred to as the *regret cap*. I say $\bar{R}$ is an *upper bound* of the expected regret under a mechanism $\mathcal{M}$ if $GER(\mathcal{M}) \leq \bar{R}$. I say $\underline{R}$ is a *lower bound* of the expected regret given a joint distribution π if $\inf_{\mathcal{M}}\inf_{\sigma\in\Sigma(\mathcal{M},\pi)} ER(\mathcal{M},\pi,\sigma) \geq \underline{R}$. The designer's goal is to find a mechanism with the minimal regret cap. I refer to the minimal regret cap as the *minimax regret*. Formally, the designer aims to find a mechanism $\mathcal{M}^*$, referred to as a *minimax regret mechanism*, that solves the following problem:

$$\inf_{\mathcal{M}} GER(\mathcal{M}). \tag{MRM}$$

3.3 Methodology

The problem (MRM) can be interpreted as a two-player sequential game. The two players are the designer and adversarial nature. The designer first chooses a mechanism $\mathcal{M}$. After observing the designer's choice of the mechanism, adversarial nature chooses a joint distribution over value profiles $\pi \in \Delta(V)$ as well as an equilibrium $\sigma \in \Sigma(\mathcal{M},\pi)$ to maximize the expected regret. The designer's payoff is $-ER(\mathcal{M},\pi,\sigma)$, and nature's payoff is $ER(\mathcal{M},\pi,\sigma)$ if $\Sigma(\mathcal{M},\pi) \neq \emptyset$; otherwise, both players' payoffs are minus infinity. One can also consider a game in which nature moves first by choosing a joint distribution and the designer moves next by choosing a mechanism and an equilibrium. Although it is not obvious that these two problems are payoff equivalent because the equilibrium correspondence is not lower-hemicontinuous, I will construct a mechanism $\mathcal{M}^*$ and a joint distribution π^* that form (a version of) a saddle point and therefore (a version of) the minimax theorem holds. Precisely, I will (i) find an upper bound R^* of the expected regret under the mechanism $\mathcal{M}^*$ and (ii) show that R^* is also a lower bound of the expected regret given the joint distribution π^*. Note that (ii) implies that no mechanism can achieve a regret cap strictly lower than R^*, and (i) says that the regret cap of $\mathcal{M}^*$ is weakly lower than R^*. Therefore, (i) and (ii) together imply that $\mathcal{M}^*$ is a minimax regret mechanism. I refer to π^* as a *worst-case distribution*.

3.4 Main Result

In this section, I first formally define the separate second-price auction with random reserves $(M^*, \mathbf{q}^*, \mathbf{t}^*)$ (Section 3.4.1) and the joint distribution over value profiles π^* (Section 3.4.2), then I present the formal statement of the result (Section 3.4.3) that the proposed mechanism (resp, the proposed distribution) is a minimax regret mechanism (resp, a worst-case distribution). Finally, I prove the formal statement (Section 3.4.4).

3.4.1 Separate Second-price Auction with Random Reserves

The separate second-price auction with random reserves, $(M^*, \mathbf{q}^*, \mathbf{t}^*)$, is defined as follows. First, it is a direct mechanism, i.e., $M^* = V$. With slight abuse of notations, I use $\mathbf{v} = (v_i^j)_{i\in I, j\in \mathcal{J}}$ to also denote the reported message. Let $v_{(2)}^j$ be the second highest reported value for the good j (the second highest value is the same as the highest one if there are ties).

If there are no ties, $\mathbf{q}^*(\mathbf{v}) = (q_i^{j*}(\mathbf{v}))_{i\in I, j\in \mathcal{J}}$ where

$$q_i^{j*}(\mathbf{v}) = \begin{cases} 1 + \ln \frac{v_i^j}{\bar{v}_i{}^j} & \text{if } \forall k \neq i, v_i^j > v_k^j \text{ and } \frac{\bar{v}_i{}^j}{e} \leq v_i^j \leq \bar{v}_i{}^j; \\ 0 & \text{if } \exists k \neq i \text{ s.t. } v_i^j < v_k^j \text{ or } 0 \leq v_i^j < \frac{\bar{v}_i{}^j}{e}. \end{cases}$$

$\mathbf{t}^*(\mathbf{v}) = (t_i^*(\mathbf{v}))_{i\in I}$ in which $t_i^*(\mathbf{v}) = \sum_{j\in\mathcal{J}} t_i^{j*}(\mathbf{v})$[15] where

$$t_i^{j*}(\mathbf{v}) = \begin{cases} v_i^j - \frac{\bar{v}_i{}^j}{e} & \text{if } \forall k \neq i,\, v_i^j > v_k^j \text{ and } v_{(2)}^j < \frac{\bar{v}_i{}^j}{e} \leq v_i^j \leq \bar{v}_i{}^j; \\ v_i^j + v_{(2)}^j \ln \frac{v_{(2)}^j}{\bar{v}_i{}^j} & \text{if } \forall k \neq i,\, v_i^j > v_k^j \text{ and } \frac{\bar{v}_i{}^j}{e} \leq v_{(2)}^j < v_i^j \leq \bar{v}_i{}^j; \\ 0 & \text{if } \exists k \neq i \text{ s.t. } v_i^j < v_k^j \text{ or } 0 \leq v_i^j < \frac{\bar{v}_i{}^j}{e}. \end{cases}$$

Now I specify the tie breaking rule when there are ties. Given a value profile $\mathbf{v}$ in which there are ties in the auction of good j, let $I(\mathbf{v^j}) := \{s \in I | v_s^j \geq v_i^j \quad \forall i \in I \quad \text{and} \quad v_s \geq \frac{\bar{v}_s{}^j}{e}\}$. If $I(\mathbf{v^j})$ is empty, then $q_i^{j*}(\mathbf{v}) = t_i^{j*}(\mathbf{v}) = 0$ for any $i \in I$; otherwise, pick a bidder $i \in \arg\min_{s\in I(\mathbf{v^j})} \bar{v}_s{}^j$, let $q_i^{j*}(\mathbf{v}) = 1 + \ln \frac{v_i^j}{\bar{v}_i{}^j}, t_i^{j*}(\mathbf{v}) = v_i^j + v_i^j \ln \frac{v_i^j}{\bar{v}_i{}^j}$, and $q_s^{j*}(\mathbf{v}) = t_s^{j*}(\mathbf{v}) = 0$ for any $s \neq i$. In words, among the bidders whose values are weakly higher than their lower bounds of the random reserves respectively, pick a bidder whose upper bound of the values is the lowest and allocate good j to this bidder when his bid is higher than the random reserve.

Note that the allocation probabilities for each good j are independent of the bidders' values for the other goods. The payment rule is characterized by the envelope theorem. Then clearly the above mechanism is equivalent to holding a separate second-price auction with bidder-specific random reserves for each good.

3.4.2 Joint Distribution

The joint distribution over value profiles π^* is defined via the following five steps.

Step 1: Selection. For each good j, pick (breaking ties arbitrarily) a bidder $i \in \arg\max_{s\in I} \bar{v}_s{}^j$. Let $\mathcal{J}(i)$ denote the set of goods for which i is picked. Note that $\mathcal{J}(i)$ (could be empty) is disjoint and $\cup_{i\in I}\mathcal{J}(i) = \mathcal{J}$. If a bidder is not picked for any good, then his values for all goods are zeros.

Step 2: Equal-revenue Distributions. For each bidder i and $j \in \mathcal{J}(i)$ (this part is irrelevant if $\mathcal{J}(i)$ is empty), the marginal distribution of v_i^j is an equal-revenue distribution whose cumulative

[15] t_i^{j*} can be interpreted as the payment from bidder i for good j under the mechanism $(M^*, \mathbf{q}^*, \mathbf{t}^*)$.

distribution function is

$$\pi_i^{j*}(v_i^j) = \begin{cases} 1 - \frac{\bar{v}_i{}^j}{e v_i^j} & \text{if } \frac{\bar{v}_i{}^j}{e} \le v_i^j < \bar{v}_i{}^j; \\ 1 & \text{if } v_i^j = \bar{v}_i{}^j. \end{cases}$$

Step 3: Comonotonicity. For each bidder i and cross $j \in \mathcal{J}(i)$ (this part is irrelevant if $\mathcal{J}(i)$ is empty), the dependence structure is comonotonic. Formally, for good $j \in \mathcal{J}(i)$, define the inverse quantile function

$$v_i^j(z_i) = \min\{\tilde{v}_i{}^j | \pi_i^{j*}(\tilde{v}_i{}^j) \ge z_i\} = \begin{cases} \frac{\bar{v}_i{}^j}{e(1-z_i)} & \text{if } 0 \le z_i < 1 - \frac{1}{e}; \\ \bar{v}_i{}^j & \text{if } z_i \ge 1 - \frac{1}{e}. \end{cases}$$

Then I define the joint distribution across $j \in \mathcal{J}(i)$ by randomly drawing $z_i \sim U[0,1]$ and taking $v_i^j = v_i^j(z_i)$ for each $j \in \mathcal{J}(i)$.

Step 4: Zeros. For each bidder i and $j \notin \mathcal{J}(i)$ (this part is irrelevant if $\mathcal{J}(i) = \mathcal{J}$), $v_i^j = 0$.

Step 5: Independence. For goods across bidders (this part is irrelevant if $\mathcal{J}(i) = \mathcal{J}$ for some $i \in I$), the values are independently distributed. Formally, z_i's are independently distributed uniform distributions.

3.4.3 Formal Statement: Theorem 13

Theorem 13. *The mechanism $(M^*, \mathbf{q}^*, \mathbf{t}^*)$ is a minimax regret mechanism with the regret cap of $\sum_{j \in \mathcal{J}} \max_{i \in I} \frac{\bar{v}_i{}^j}{e}$. The joint distribution π^* is a worst-case distribution.*

3.4.4 Proof of Theorem 13

Upper Bound on Regret for $(M^*, \mathbf{q}^*, \mathbf{t}^*)$

Proposition 13. *An upper bound of the expected regret under the mechanism $(M^*, \mathbf{q}^*, \mathbf{t}^*)$ is $\sum_{j \in \mathcal{J}} \max_{i \in I} \frac{\bar{v}_i{}^j}{e}$.*

Proof. Consider the auction of good j under the mechanism $(M^*, \mathbf{q}^*, \mathbf{t}^*)$. Fix an arbitrary joint distribution $\pi \in \Delta V$. First consider the truth-telling equilibrium. Given any value profile $\mathbf{v} \in supp(\pi)$, suppose i is the unique highest bidder for good j . If $v^j_{(2)} < \frac{\bar{v_i}^j}{e} \le v^j_i \le \bar{v_i}^j$, then the ex-post regret is $v^1_i - t^{j*}_i(\mathbf{v}) = v^j_i - (v^j_i - \frac{\bar{v_i}^j}{e}) = \frac{\bar{v_i}^j}{e}$; if $\frac{\bar{v_i}^j}{e} \le v^j_{(2)} < v^j_i \le \bar{v_i}^j$, then the ex-post regret is $v^j_i - t^{j*}_i(\mathbf{v}) = v^j_i - (v^j_i + v^j_{(2)} \ln \frac{v^j_{(2)}}{\bar{v_i}^j}) = -v^j_{(2)} \ln \frac{v^j_{(2)}}{\bar{v_i}^j}$, which is maximized at $v^j_{(2)} = \frac{\bar{v_i}^j}{e}$, yielding an ex-post regret of $\frac{\bar{v_i}^j}{e}$; finally, if $0 \le v^j_i < \frac{\bar{v_i}^j}{e}$, then the ex-post regret is less than $\frac{\bar{v_i}^j}{e}$ because the maximal revenue is less than $\frac{\bar{v_i}^j}{e}$. Suppose now there are ties. Then under the specified tie breaking rule, if $I(\mathbf{v^j})$ is empty, then $v^j_i < \max_{i\in I} \frac{\bar{v_i}^j}{e}$ for any $i \in I$, and so the ex-post regret is less than $\max_{i\in I} \frac{\bar{v_i}^j}{e}$; if $I(\mathbf{v^j})$ is not empty, then the ex-post regret is $v^j_i \ln \frac{v^j_i}{\bar{v_i}^j}$ where $i \in \arg\min_{s\in I(\mathbf{v^j})} \bar{v_s}^j$, which is maximized at $v^j_i = \frac{\bar{v_i}^j}{e}$, yielding an ex-post regret of $\frac{\bar{v_i}^j}{e} \le \max_{i\in I} \frac{\bar{v_i}^j}{e}$. Thus, in the truth-telling equilibrium, the ex-post regret from good j is at most $\max_{i\in I} \frac{\bar{v_i}^j}{e}$ for any value profile.

Next consider any equilibrium σ. Given any value profile $\mathbf{v} \in supp(\pi)$, pick a bidder i whose value for good j is the highest among the bidders, or $i \in \arg\max_{i\in I} v^j_i$. If $v^j_i > \frac{\bar{v_i}^j}{e}$ and there is a positive measure of the others' reports under the conditional equilibrium report distribution $\sigma_{-i}(\mathbf{v_{-i}})$ such that bidder i wins the good by truthfully reporting v^j_i provided that v^j_i is higher than the random reserve, then bidder i has a strict incentive to truthfully report his value for good j, and thus the argument in the previous paragraph implies that the ex-post regret must not exceed $\max_{i\in I} \frac{\bar{v_i}^j}{e}$. Otherwise, there are two cases to consider. 1) If $v^j_i \le \frac{\bar{v_i}^j}{e}$, then the most to lose does not exceed $\frac{\bar{v_i}^j}{e}$ and thus the ex-post regret must not exceed $\max_{i\in I} \frac{\bar{v_i}^j}{e}$. 2) If $v^j_i > \frac{\bar{v_i}^j}{e}$ and there is a zero measure of the others' reports under the conditional equilibrium report distribution $\sigma_{-i}(\mathbf{v_{-i}})$ such that bidder i wins the good by truthfully reporting v^j_i provided that v^j_i is higher than the random reserve, then (almost surely) the highest report among the other bidders is (weakly) higher than bidder i's value for good j. In this case, the ex-post regret must not exceed $\max_{i\in I} \frac{\bar{v_i}^j}{e}$ as the difference between the highest report and the ex-post revenue is weakly less than $\max_{i\in I} \frac{\bar{v_i}^j}{e}$ by the argument in the previous paragraph. Thus, in any equilibrium, the ex-post regret from good j is at most $\max_{i \in I} \frac{\bar{v_i}^j}{e}$ for any value profile. This implies that in any equilibrium, the expected regret from good j is at most $\max_{i \in I} \frac{\bar{v_i}^j}{e}$ given the arbitrary joint distribution π.

Finally, because of the separable nature of the mechanism $(M^*, \mathbf{q}^*, \mathbf{t}^*)$, an upper bound of the expected regret is $\sum_{j \in \mathcal{J}} \max_{i \in I} \frac{\bar{v_i}^j}{e}$. □

Remark 15. This upper bound is hit given the joint distribution π^* and the truth-telling equilibrium. To see this, fix any $\mathbf{v} \in \pi^*$ and consider good j. By the definition of π^*, there is only one bidder, denoted by i, whose value for good j is positive. In addition, $v_i^j \geq \frac{\bar{v_i}^j}{e}$. Then by the proof of Proposition 13, the ex-post regret from good j is $\frac{\bar{v_i}^j}{e}$ in the truth-telling equilibrium. Note that by the definition of π^*, $i \in \arg\max_{i \in I} \bar{v_i}^j$. Thus the ex-post regret from good j is equal to $\max_{i \in I} \frac{\bar{v_i}^j}{e}$ in the truth-telling equilibrium. Because this is true for any $\mathbf{v} \in \pi^*$, the expected regret from good j is equal to $\max_{i \in I} \frac{\bar{v_i}^j}{e}$ given the joint distribution π^* and the truth-telling equilibrium. Summing up across goods, the expected regret is $\sum_{j \in \mathcal{J}} \max_{i \in I} \frac{\bar{v_i}^j}{e}$ given the joint distribution π^* and the truth-telling equilibrium.

Remark 16. Koçyiğit et al. (2020b) present a separate second-price auction with anonymous random reserves in which

$$q_i^j(\mathbf{v}) = \begin{cases} 1 + \ln \frac{v_i^j}{\max_{i \in I} \bar{v_i}^j} & \text{if } \forall k \neq i,\ v_i^j > v_k^j \text{ and } \frac{\max_{i \in I} \bar{v_i}^j}{e} \leq v_i^j \leq \bar{v_i}^j; \\ 0 & \text{if } \exists k \neq i \text{ s.t. } v_i^j < v_k^j \text{ or } 0 \leq v_i^j < \frac{\max_{i \in I} \bar{v_i}^j}{e}. \end{cases}$$

And the payment rule is characterized by the envelope theorem. They show that given this mechanism, $\sum_{j \in \mathcal{J}} \max_{i \in I} \frac{\bar{v_i}^j}{e}$ is an upper bound of the ex-post regret. Although their mechanism has the same regret cap, the designer may favor $(M^*, \mathbf{q}^*, \mathbf{t}^*)$ over their mechanism for reasons outside the model. Specifically, there exists a Pareto ranking between the two mechanisms in the following sense: as long as $\max_{i \in I} \bar{v_i}^j > \bar{v_k}^j$ for some $k \in I$ and some $j \in \mathcal{J}$, it is straightforward to show that in the truth-telling equilibrium, i) the ex-post regret under $(M^*, \mathbf{q}^*, \mathbf{t}^*)$ is weakly lower than that under their mechanism for any $\mathbf{v} \in V$, and ii) the ex-post regret under $(M^*, \mathbf{q}^*, \mathbf{t}^*)$

is strictly lower than that under their mechanism for some $\mathbf{v} \in V$. Intuitively, under their mechanism, the allocation probability is lower for a highest bidder whose upper bound of the values of the good is not the highest, resulting in a higher ex-post regret for such a value profile.

Indeed, under the criterion used in Remark 16, no mechanism from the family of separate second-price auctions[16], denoted by $\mathcal{F}-SSP$, is better than the mechanism $(M^*, \mathbf{q}^*, \mathbf{t}^*)$.

Definition 9. I say a direct mechanism $\mathcal{M} = (V, \mathbf{q}, \mathbf{t})$ is a *dominant-strategy mechanism* if for all $i \in I$, all $\mathbf{v} \in V$, and all $\mathbf{v_i'} \in V_i$,

$$\mathbf{v_i} \cdot \mathbf{q_i}(\mathbf{v}) - t_i(\mathbf{v}) \geq \mathbf{v_i} \cdot \mathbf{q_i}(\mathbf{v_i'}, \mathbf{v_{-i}}) - t_i(\mathbf{v_i'}, \mathbf{v_{-i}}),$$

$$\mathbf{v_i} \cdot \mathbf{q_i}(\mathbf{v}) - t_i(\mathbf{v}) \geq 0.$$

Definition 10. I say a dominant-strategy mechanism $\mathcal{M}_1$ is *undominated* by another dominant-strategy mechanism $\mathcal{M}_2$ if in the truth-telling equilibrium, the ex-post regret under the mechanism $\mathcal{M}_1$ is strictly lower than that under the mechanism $\mathcal{M}_2$ for some $\mathbf{v} \in V$.

Corollary 3. *The mechanism $(M^*, \mathbf{q}^*, \mathbf{t}^*)$ is undominated by any mechanism from $\mathcal{F}-SSP$.*

Proof. Fix any $i \in I$ and any $j \in \mathcal{J}$, consider the value profiles in which $v_i^j \in [0, \bar{v_i}^j]$ and all other values are zeros. Then Proposition 1 in Bergemann and Schlag (2008) implies that the random reserve for the bidder i and the good j in the mechanism $(M^*, \mathbf{q}^*, \mathbf{t}^*)$ is the unique random reserve that minimizes the worst-case ex-post regret in the truth-telling equilibrium for these value profiles. Therefore, if a different (random) reserve were used for the bidder i and the good j, then there would be a value profile with an ex-post regret strictly higher than that under the mechanism $(M^*, \mathbf{q}^*, \mathbf{t}^*)$.

In addition, the specific tie-breaking rule in $(M^*, \mathbf{q}^*, \mathbf{t}^*)$ minimizes the worst-case ex-post regret when there are ties across different tie-breaking rules. To see this, recall that the worst-case

[16] In a separate second-price auction, there may be random reserves, deterministic reserves, or no reserves.

ex-post regret is proportional to the upper bound of values in the one-good one-bidder case and that under the tie-breaking rule in $(M^*, \mathbf{q}^*, \mathbf{t}^*)$, a bidder with the lowest upper bound of values for a good is picked. This finishes the proof. □

Lower Bound on Regret for π^*

Proposition 14. *A lower bound of the expected regret under π^* is $\sum_{j\in\mathcal{J}} \max_{i\in I} \frac{\bar{v}_i{}^j}{e}$.*

Proof. Note that given a joint distribution, minimizing the expected regret across mechanisms and equilibria is equivalent to maximizing the expected revenue across mechanisms and equilibria. Then the revelation principle applies and thus it is without loss to restrict attention to direct mechanisms.

I parameterize the value profile across bidders by $\mathbf{z} = (z_1, z_2, \cdots, z_I) \in [0,1]^I$. Then for any direct mechanism $(\mathbf{q}(\mathbf{z}), \mathbf{t}(\mathbf{z})) = ((\mathbf{q_i}(\mathbf{z}))_{i\in I}, (t_i(\mathbf{z}))_{i\in I})$ where $\mathbf{q_i}(\mathbf{z}) = (q_i^j(\mathbf{z}))_{j\in\mathcal{J}} \in [0,1]^J$ represent the allocation probabilities of the goods to bidder i under the parametrerized value profile $\mathbf{z}$ and $t_i(\mathbf{z}) \in \mathbb{R}$ represents bidder i's payment under $\mathbf{z}$, (BR) together with (PS) imply

$$U_i(z_i) := \sum_{j\in\mathcal{J}(i)} v_i^j(z_i) Q_i^j(z_i) - T_i(z_i) \geq \sum_{j\in\mathcal{J}(i)} v_i^j(z_i) Q_i^j(z_i') - T_i(z_i') \quad \text{for} \quad i \in I, z_i, z_i' \in [0,1], \tag{BIC}$$

$$\sum_{j\in\mathcal{J}(i)} v_i^j(z_i) Q_i^j(z_i) - T_i(z_i) \geq 0 \quad \text{for} \quad i \in I, z_i \in [0,1], \tag{BIR}$$

where $Q_i^j(z_i) = \int_{[0,1]^{I-1}} q_i^j(z_i, \mathbf{z_{-i}}) d\mathbf{z_{-i}}$ and $T_i(z_i) = \int_{[0,1]^{I-1}} t_i(z_i, \mathbf{z_{-i}}) d\mathbf{z_{-i}}$ are the expected allocation of good j to type z_i of bidder i and the expected payment made by type z_i of bidder i respectively, due to the fact that z_i's are independently distributed uniform distributions by the definition of π^*. Note that the allocation of good j for $j \notin \mathcal{J}(i)$ does not appear in either (BIC) or (BIR) because the value for such a good (if any) is zero to bidder i under the joint distribution π^*.

For $z_i' \geq z_i$, (BIC) implies that

$$\sum_{j \in \mathcal{J}(i)} (v_i^j(z_i') - v_i^j(z_i)) Q_i^j(z_i') \geq U_i(z_i') - U_i(z_i) \geq \sum_{j \in \mathcal{J}(i)} (v_i^j(z_i') - v_i^j(z_i)) Q_i^j(z_i). \tag{2}$$

Then $U_i(z_i)$ is Lipschitz, thus absolutely continuous w.r.t. z_i, and so equal to the integral of its derivative. In addition, note that $v_i^j(z_i)$ is differentiable for all z_i but $z_i = 1 - \frac{1}{e}$. Then applying the envelope theorem to (2) at each point of differentiability, I obtain that

$$\frac{\partial U_i(z_i)}{\partial z_i} = \sum_{j \in \mathcal{J}(i)} \frac{\partial v_i^j(z_i)}{\partial z_i} Q_i^j(z_i) = \begin{cases} \sum_{j \in \mathcal{J}(i)} \frac{\bar{v}_i{}^j}{e(1-z_i)^2} Q_i^j(z_i) & \text{if } 0 \leq z_i < 1 - \frac{1}{e}; \\ 0 & \text{if } z_i > 1 - \frac{1}{e}. \end{cases}$$

Thus,

$$U_i(z_i) = \begin{cases} U_i(0) + \int_0^{z_i} [\sum_{j \in \mathcal{J}(i)} \frac{\bar{v}_i{}^j}{e(1-\tilde{z}_i)^2} Q_i^j(\tilde{z}_i)] d\tilde{z}_i & \text{if } 0 \leq z_i < 1 - \frac{1}{e}; \\ U_i(0) + \int_0^{1-\frac{1}{e}} [\sum_{j \in \mathcal{J}(i)} \frac{\bar{v}_i{}^j}{e(1-\tilde{z}_i)^2} Q_i^j(\tilde{z}_i)] d\tilde{z}_i & \text{if } z_i \geq 1 - \frac{1}{e}. \end{cases}$$

Therefore, the expected revenue from bidder i

$$
\begin{aligned}
\int_0^1 T_i(z_i)dz_i &= \int_0^1 [\sum_{j\in\mathcal{J}(i)} v_i^j(z_i)Q_i^j(z_i) - U_i(z_i)]dz_i \\
&= \int_0^{1-\frac{1}{e}} \{\sum_{j\in\mathcal{J}(i)} v_i^j(z_i)Q_i^j(z_i) - U_i(0) - \int_0^{z_i}[\sum_{j\in\mathcal{J}(i)} \frac{\bar{v}_i{}^j}{e(1-\tilde{z}_i)^2}Q_i^j(\tilde{z}_i)]d\tilde{z}_i\}dz_i + \\
&\int_{1-\frac{1}{e}}^1 \{\sum_{j\in\mathcal{J}(i)} v_i^j(z_i)Q_i^j(z_i) - U_i(0) - \int_0^{1-\frac{1}{e}}[\sum_{j\in\mathcal{J}(i)} \frac{\bar{v}_i{}^j}{e(1-\tilde{z}_i)^2}Q_i^j(\tilde{z}_i)]d\tilde{z}_i\}dz_i \\
&\le \int_0^{1-\frac{1}{e}} \{\sum_{j\in\mathcal{J}(i)} v_i^j(z_i)Q_i^j(z_i) - \int_0^{z_i}[\sum_{j\in\mathcal{J}(i)} \frac{\bar{v}_1{}^j}{e(1-\tilde{z}_i)^2}Q_i^j(\tilde{z}_i)]d\tilde{z}_i\}dz_i + \\
&\int_{1-\frac{1}{e}}^1 \{\sum_{j\in\mathcal{J}(i)} v_i^j(z_i)Q_i^j(z_i) - \int_0^{1-\frac{1}{e}}[\sum_{j\in\mathcal{J}(i)} \frac{\bar{v}_i{}^j}{e(1-\tilde{z}_i)^2}Q_i^j(\tilde{z}_i)]d\tilde{z}_i\}dz_i \\
&= \sum_{j\in\mathcal{J}(i)} \{\int_0^{1-\frac{1}{e}} [(v_i^j(z_i) - (1-\frac{1}{e}-z_i)\frac{\bar{v}_i{}^j}{e(1-z_i)^2})Q_i^j(z_i)]dz_i + \\
&\int_{1-\frac{1}{e}}^1 [v_i^j(z_i)Q_i^j(z_i) - \int_0^{1-\frac{1}{e}}[\frac{\bar{v}_i{}^j}{e(1-\tilde{z}_i)^2}Q_i^j(\tilde{z}_i)]d\tilde{z}_i]dz_i\} \\
&= \sum_{j\in\mathcal{J}(i)} \{\int_0^{1-\frac{1}{e}} [(v_i^j(z_i) - (1-z_i)\frac{\bar{v}_i{}^j}{e(1-z_i)^2})Q_i^j(z_i)]dz_i + \int_{1-\frac{1}{e}}^1 [(v_i^j(z_i)Q_i^j(z_i)]dz_i\} \\
&= \sum_{j\in\mathcal{J}(i)} \int_{1-\frac{1}{e}}^1 [\bar{v}_i{}^j Q_i^j(z_i)]dz_i \le \sum_{j\in\mathcal{J}(i)} \frac{\bar{v}_i{}^j}{e},
\end{aligned}
$$

where the first inequality holds because (BIR) implies that $U_i(0) \ge 0$, the third equality is obtained via integration by parts, the last equality holds because $v_i^j(z_i) - (1-z_i)\frac{\bar{v}_i{}^j}{e(1-z_i)^2} = 0$ for $0 \le z_i < 1-\frac{1}{e}$ and $v_i^j(z_i) = \bar{v}_i{}^j$ for $z_i > 1-\frac{1}{e}$, and the last inequality holds because $Q_i^j(z_i) \le 1$.

Then, the expected revenue from all the bidders

$$
\begin{aligned}
\sum_{i=1}^I \int_0^1 T_i(z_i)dz_i &\le \sum_{i\in I}\sum_{j\in\mathcal{J}(i)} \frac{\bar{v}_i{}^j}{e} \\
&= \sum_{j\in\mathcal{J}} \max_{i\in I} \frac{\bar{v}_i{}^j}{e},
\end{aligned}
$$

where the equality holds by the definition of $\mathcal{J}(i)$.

Now, the expected regret

$$\sum_{i\in I}\int_0^1 \sum_{j\in\mathcal{J}(i)} v_i^j(z_i)dz_i - \sum_{i=1}^{I}\int_0^1 T_i(z_i)dz_i = \sum_{j\in\mathcal{J}} \max_{i\in I}\frac{2\bar{v}_i{}^j}{e} - \sum_{i=1}^{I}\int_0^1 T_i(z_i)dz_i$$
$$\geq \sum_{j\in\mathcal{J}} \max_{i\in I}\frac{\bar{v}_i{}^j}{e},$$

where the term $\sum_{i\in I}\int_0^1\sum_{j\in\mathcal{J}(i)} v_i^j(z_i)dz_i$ is the full surplus given the joint distribution π^* and the equality holds by direct calculation and by the definition of $\mathcal{J}(i)$. Thus, $\sum_{j\in\mathcal{J}}\max_{i\in I}\frac{\bar{v}_i{}^j}{e}$ is a lower bound of the expected regret under π^*. □

Remark 17. In the Step 1 of the definition of π^*, it is important that for each good, only one bidder is selected when there are ties. Otherwise, there would be competition for some good, resulting in a lower expected regret.

Remark 18. One may be tempted to consider the following joint distribution over value profiles as a candidate for a worst-case distribution. There is only one bidder, bidder i, whose values for the goods are non-zero; in addition, the bidder i's values for the goods follow the comonotonic equal-revenue distribution. The bidder i is selected such that $i \in \arg\max_{i\in I}\sum_{j\in\mathcal{J}}\frac{\bar{v}_i{}^j}{e}$. Then by an argument similar to the proof of Proposition 14 , a lower bound of the expected regret under this joint distribution is $\max_{i\in I}\sum_{j\in\mathcal{J}}\frac{\bar{v}_i{}^j}{e}$. However, this lower bound is lower than $\sum_{j\in\mathcal{J}}\max_{i\in I}\frac{\bar{v}_i{}^j}{e}$ in general. Thus, this joint distribution is not "bad" enough for the designer and is not a worst-case distribution in general. Intuitively, this joint distribution may ignore a bidder whose upper bound of the values of a given good is the highest among the bidders, resulting in an expected regret not high enough. This motivates the Step 1 of the definition of π^*.

Remark 19. What if the designer knows nothing about the joint distribution over bidders' value profiles? That is, bidders' values can be unbounded. I argue that the regret cap for any mechanism that secures bidders' participation will be infinity. To see this, consider a joint distribution that puts all probability masses on a single value profile in which bidder i has a large positive value of θ for good j, bidder i's values for the other goods are zeros and the other bidders' values for all the goods are zeros. Recall that given a joint distribution over value profiles, the revelation principle applies and it is without loss to restrict attention to direct mechanisms. In addition, the expected revenue is generated from selling good j to bidder i only, as the other values are zeros and the mechanism secures bidders' participation. Consider a revenue-maximizing (and therefore regret-minimizing) direct mechanism, let $Q_i^j(x)$ denote the expected allocation probability of good j to bidder i given a bidder i's report of x about his value for good j. Note that $Q_i^j(x)$ is non-decreasing in x by the incentive compatible constraint. Then the expected revenue is $T_i^j(\theta) = \theta Q_i^j(\theta) - \int_0^\theta Q(x)dx$. Define $\lim_{v_i^j \to \infty} Q_i^j(v_i^j) := \kappa$. By definition, for any $\varepsilon > 0$, there exists a $t \geq 0$ such that $Q_i^j(v_i^j) \geq \kappa - \varepsilon$ for any $v_i^j \geq t$. Then $\int_0^\theta Q_i^j(x)dx = \int_0^t Q_i^j(x)dx + \int_t^\theta Q_i^j(x)dx \geq (\theta - t)(\kappa - \varepsilon)$, so $T_i^j(\theta) \leq \theta\kappa - (\theta - t)(\kappa - \varepsilon) = \varepsilon\theta + t(\kappa - \varepsilon)$, and the expected regret is $\theta - T_i^j(\theta) \geq (1-\varepsilon)\theta - t(\kappa - \varepsilon)$. As ε can be chosen to be arbitrarily small, the expected regret goes to infinity as θ goes to infinity[17].

Theorem 13 follows immediately from Proposition 13 and 14.

3.5 Special Cases

In this section, I present the results for two special cases in which $I = 1$ and $J = 1$ respectively, which correspond to multi-dimensional screening (Section 3.5.1) and single-good auction (Section 3.5.2).

3.5.1 Multi-Dimensional Screening: $I = 1$

Let the mechanism $(M_1^*, \mathbf{q}_1^*, t_1^*)$ (resp, the joint distribution π_1^*) be the specialization of the mechanism $(M^*, \mathbf{q}^*, \mathbf{t}^*)$ (resp, the joint distribution π^*) to the case in which $I = 1$. I omit their descriptions for brevity. Note that the mechanism $(M_1^*, \mathbf{q}_1^*, t_1^*)$ is a *separate randomized*

[17]This proof is similar to the proof of Proposition 1 in Carrasco et al. (2017).

posted-price mechanism: each good is sold separately with a random posted price. In the joint distribution $\pi_{\mathbf{1}}^*$, the marginal distribution of each good is an equal-revenue distribution and the values across goods are comonotonic.

Corollary 4 (Multi-Dimensional Screening)**.** *If $I = 1$, then the mechanism $(M_1^*, \mathbf{q}_{\mathbf{1}}^*, t_1^*)$ is a minimax regret mechanism with the regret cap of $\sum_{j\in\mathcal{J}} \frac{\bar{v_1}^j}{e}$. The joint distribution $\pi_{\mathbf{1}}^*$ is a worst-case distribution.*

Proof. The proof is a straightforward adaptation of the proof of Theorem 13 to the case in which $I = 1$. □

Remark 20. There are very limited results in multi-dimensional screening for other correlation structures. McAfee et al. (1989) show that with independent continuous distributions, separate selling is essentially never optimal. Therefore an independent joint distribution, where the marginal distributions remain the same but the values across the goods are independent, is not a worst-case distribution.

Remark 21. There is another minimax regret mechanism for the multi-dimensional screening: a randomized grand bundling. It can be described as follows. The designer sells the bundle of all the goods only. Let b be the bid for the bundle of all the goods. If $b > \sum_{j\in\mathcal{J}} \frac{\bar{v_1}^j}{e}$, then allocate the bundle with a probability of $1 + \ln \frac{b}{\sum_{j\in\mathcal{J}} \bar{v_1}^j}$ and charge a price of $b - \sum_{j\in\mathcal{J}} \frac{\bar{v_1}^j}{e}$; otherwise, no goods are allocated and the buyer (the bidder 1) pays nothing. It is straightforward to show that the regret cap of this mechanism is $\sum_{j\in\mathcal{J}} \frac{\bar{v_1}^j}{e}$.

3.5.2 Single-Good Auction: $J = 1$

Let the mechanism $(M^{1*}, \mathbf{q}^{\mathbf{1}*}, \mathbf{t}^{\mathbf{1}*})$ (resp, the joint distribution $\pi^{\mathbf{1}*}$) be the specialization of the mechanism $(M^*, \mathbf{q}^*, \mathbf{t}^*)$ (resp, the joint distribution π^*) to the case in which $J = 1$. I omit their descriptions for brevity. Note that the mechanism $(M^{1*}, \mathbf{q}^{\mathbf{1}*}, \mathbf{t}^{\mathbf{1}*})$ is a *second-price auction with random reserves*: the single good is auctioned via a second-price auction with bidder-specific random reserves. In the joint distribution $\pi^{\mathbf{1}*}$, only the bidder with the highest upper bound of the values for the good has a positive value (breaking ties arbitrarily) and the marginal distribution of this bidder's value is an equal-revenue distribution.

Corollary 5 (Single-Good Auction)**.** *If $J = 1$, then the mechanism $(M^{1*}, \mathbf{q}^{\mathbf{1}*}, \mathbf{t}^{\mathbf{1}*})$ is a minimax regret mechanism with the regret cap of $\max_{i\in I} \frac{\bar{v}_i^1}{e}$. The joint distribution $\pi^{\mathbf{1}*}$ is a worst-case distribution.*

Proof. The proof is a straightforward adaptation of the proof of Theorem 13 to the case in which $J = 1$. □

3.6 Discussion

3.6.1 Solution Concept

In this paper, I consider the class of all mechanisms that secure bidders' participation and the worst Bayes Nash Equilibrium for the designer. The solution concept follows from a recent literature on informationally robust mechanism design, e.g., Du (2018) and Brooks and Du (2021). Several remarks can be made in sequence. First, if we assume that the class of mechanisms is the set of dominant-strategy mechanisms and that the truth-telling equilibrium is played, then the same result will hold by a simple extension of the current proofs. This is because under the constructed worst-case distribution, the expected regret under the best dominant-strategy mechanism is the same as that under the best Bayesian incentive-compatible mechanism. Second, for the main result, it is not crucial that adversarial nature has to pick the worst equilibrium. That is, we can allow adversarial nature to pick the best equilibrium for the designer, and the same result will still hold. So the main result may be a priori surprising result: the class of the mechanisms is much wider than the set of dominant-strategy mechanisms, yet, a dominant-strategy mechanism emerges as a minimax regret mechanism.

3.6.2 Comparative Statics

It is instructive to discuss some comparative statics assuming that there is no trivial good or bidder, i.e., $\bar{v}_i{}^j > 0$ for any $i \in I$ and $j \in \mathcal{J}$. First, the minimax regret is strictly increasing in J. To understand this, note that the comonotonic structure in the worst-case distribution reduces multi-dimensional screening to single-dimensional screening, then when adding a new good, the minimax regret will increase by the amount of the minimax regret when there is only this new good. Second, the minimax regret is weakly increasing in I. To understand this, note that the zero values in the worst-case distribution eliminate the competition[18] for a given good, then as the full surplus weakly increases with I, the minimax regret also weakly increases with I (strictly increases with I when the new bidder's upper bound of the values of some good is higher than that of any previous bidder). Third, for the symmetric case where the upper bounds of the values for a given good are the same across bidders, or $\bar{v}_i{}^j = \bar{v}_k{}^j$ for any $i \in I$, any $k \in I$ and any $j \in \mathcal{J}$, the *average* minimax regret (the minimax regret divided by I) is strictly decreasing in I. To understand this, note that when adding a symmetric bidder, the full surplus does not change given the worst-case distribution, and, again, there is still no competition for any good. Then, the minimax regret remains the same and thus the average minimax regret is strictly decreasing in I.

3.6.3 Digital Goods

Consider a related problem in which the designer auctions digital goods[19] to I bidders, e.g., e-books, mobile apps, online courses, etc. Each bidder demands at most one unit of the good. Bidder i has a private value $v_i \in [0, \bar{v}_i]$. The designer aims to minimize the worst-case expect regret. The formal objective function can be similarly defined. Indeed, this problem may be interpreted as a special case of the model: there are I different goods, but each bidder values

[18]The competition would increase with I for general joint distributions. For example, consider a joint distribution in which bidders' values of a given good follow *i.i.d.* uniform distributions. It is straightforward to show that the expected regret under an optimal mechanism would eventually go to 0 as I goes to infinity given this joint distribution.

[19]A digital goods auction is an auction in which the designer has an unlimited supply of the same good.

only one of the goods and the good each bidder values is different. Under this interpretation, adversarial nature's ability is "constrained" in that the set of possible joint distributions is smaller than the previous one. Note however that the worst-case distribution in Theorem 13 is not excluded. Then a direct implication of Theorem 13 is that a *separate randomized posted-price mechanism* as follows is a minimax regret mechanism for this problem:

$$q_i(v_1, v_2, \cdots, v_I) = \begin{cases} 1 + \ln \frac{v_i}{\bar{v}_i} & \text{if } \frac{\bar{v}_i}{e} \leq v_i \leq \bar{v}_i; \\ 0 & \text{if } 0 \leq v_i < \frac{\bar{v}_i}{e}. \end{cases}$$

And the payment rule is characterized by the envelope theorem. Note that the allocation to bidder i depends on bidder i's value only. In addition, an *independent equal-revenue distribution* as follows is a worst-case distribution: the marginal distribution of v_i follows an equal-revenue distribution whose cumulative distribution function is

$$\pi_i(v_i) = \begin{cases} 1 - \frac{\bar{v}_i}{e v_i} & \text{if } \frac{\bar{v}_i}{e} \leq v_i < \bar{v}_i; \\ 1 & \text{if } v_i = \bar{v}_i. \end{cases}$$

And the values across bidders are independent.

3.7 Concluding Remarks

In this paper, I characterize a simple minimax regret mechanism for auctioning multiple goods given general upper bounds of values. It is worth noting that the proposed mechanism is strategy-proof. Hence, it (essentially[20]) remains a minimax regret mechanism even without the assumption of a common prior among bidders. Critically, I drop the extreme assumption made by the traditional mechanism design literature that the designer knows the joint distribution over value profiles, but impose an equally extreme assumption that the designer has no distributional

[20] A strategy-proof mechanism can be slightly perturbed so that truth-telling is the unique equilibrium.

information except for the upper bounds of values, on which the result heavily relies. I believe the truth lies in intermediate cases, which are interesting to further explore. I further conjecture that separation remains a property in many other informational environments.

3.8 Appendix

An *additional information structure* consists of a measuable set of additional information S_i for each bidder i, with $S = \times_{i=1}^{I} S_i$, and a joint distribution $\delta \in \Delta(V \times S)$. An additional information structure is denoted by $\mathcal{T} = (S, \delta)$. I say $\mathcal{T}$ is $\pi-consistent$ if the marginal of δ on V is π, i.e., for every measurable $\tilde{V} \subseteq V$, $\delta(\tilde{V} \times S) = \pi(\tilde{V})$. The set of all $\pi-consistent$ additional information structures is denoted by $\mathbf{T}(\pi)$. As before, each bidder i knows his private value vector $\mathbf{v_i} \in V_i$. And π is their common prior. But, before playing a mechanism, each bidder i may observe a signal $\mathbf{s_i} \in S_i$ from an additional information structure $\mathcal{T} \in \mathbf{T}(\pi)$. And $\mathcal{T}$ is their common knowledge. The definition of and the requirement for a mechanism are the same as before. Given a mechanism $\mathcal{M}$ and a common prior π and an additional information structure $\mathcal{T} \in \mathbf{T}(\pi)$, I have a game of incomplete information. With slight abuse of notations, a *Bayes Nash Equilibrium* (BNE) of the game is a strategy profile $\sigma = (\sigma_i)$, $\sigma_i : V_i \times S_i \to \Delta(M_i)$, such that σ_i is best response to σ_{-i}: Let $U_i(\mathbf{v_i}, \mathbf{s_i}, \mathcal{M}, \pi, \mathcal{T}, \sigma) = \int_{\mathbf{v_{-i}}, \mathbf{s_{-i}}} U_i(\mathbf{v_i}, (\sigma_i(\mathbf{v_i}, \mathbf{s_i}), \sigma_{-i}(\mathbf{v_{-i}}, \mathbf{s_{-i}}))) d\delta(\mathbf{v_{-i}}, \mathbf{s_{-i}} | \mathbf{v_i}, \mathbf{s_i})$ where $U_i(\mathbf{v_i}, (\sigma_i(\mathbf{v_i}, \mathbf{s_i}), \sigma_{-i}(\mathbf{v_{-i}}, \mathbf{s_{-i}})))$ is the multilinear extension of U_i in Equation (1), then for any $i, \mathbf{v_i}, \mathbf{s_i}, \sigma'_i$,

$$U_i(\mathbf{v_i}, \mathbf{s_i}, \mathcal{M}, \pi, \mathcal{T}, \sigma) \geq U_i(\mathbf{v_i}, \mathbf{s_i}, \mathcal{M}, \pi, \mathcal{T}, (\sigma'_i, \sigma_{-i})). \tag{BR'}$$

The set of all Bayes Nash Equilibria for a given mechanism $\mathcal{M}$ and a given common prior π and a given additional information structure $\mathcal{T} \in \mathbf{T}(\pi)$ is denoted by $\Sigma(\mathcal{M}, \pi, \mathcal{T})$.

Given a common prior π and an additional information structure $\mathcal{T} \in \mathbf{T}(\pi)$, the expected regret is $ER'(\mathcal{M}, \pi, \mathcal{T}, \sigma) = \int_{\mathbf{v},\mathbf{s}} \{\sum_{j=1}^{J} \max_{i \in I} v_i^j - \sum_{i=1}^{I} t_i(\sigma(\mathbf{v}, \mathbf{s}))\} d\delta(\mathbf{v}, \mathbf{s})$. The designer evaluates a mechanism by its worst-case expected regret across all possible common priors and consistent additional information structures and equilibria. Formally, the designer evaluates a mechanism $\mathcal{M}$ by $GER'(\mathcal{M}) = \sup_{\pi \in \Delta(V)} \sup_{\mathcal{T} \in \mathbf{T}(\pi)} \sup_{\Sigma(\mathcal{M},\pi,\mathcal{T})} ER'(\mathcal{M},\pi,\mathcal{T},\sigma)$. The designer's goal is to find a mechanism with the minimal worst-case expected regret. Formally, the designer aims to find a mechanism, referred to as a *min-3max regret mechanism*, that solves the following problem:

$$\inf_{\mathcal{M}} GER'(\mathcal{M}). \tag{MRM'}$$

Theorem 13'. *The mechanism* $(M^*, \mathbf{q}^*, \mathbf{t}^*)$ *is a min-3max regret mechanism.*

Proof. For adversarial nature's strategy, let the common prior be π^* and the set of additional information S be a singleton. The proof of Theorem 13 then applies. □

Intuitively, adversarial nature cannot generate strictly more expected regret even though it can use additional information structures because the mechanism $(M^*, \mathbf{q}^*, \mathbf{t}^*)$ is strategy-proof.

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