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THE IRREDUCIBILITY OF THE PRIMAL COHOMOLOGY OF THE THETA DIVISOR OF AN ABELIAN FIVEFOLD

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Abstract. We prove that the primal cohomology of the theta divisor of a very general principally polarized abelian fivefold is an irreducible Hodge structure of level 2.

Contents

Introduction

Let A be a principally polarized abelian variety of dimension $g \geq 4$ with smooth theta divior Θ. By the Lefschetz hyperplane theorem and Poincaré Duality (see, e.g., [IW14]) the cohomology of Θ is determined by that of A except in the middle dimension $g-1$. The primitive cohomology of Θ, in the sense of Lefschetz, is

$$H^{g-1}_{pr}(\Theta) := \operatorname{Ker}\left(H^{g-1}(\Theta,\mathbb{Z}) \xrightarrow{\cup\theta|_\Theta} H^{g+1}(\Theta,\mathbb{Z})\right).$$

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The primal cohomology of Θ is defined as (see [IW14] and [ITW])

$$\mathbb{K} := \mathrm{Ker}(j_* : H^{g-1}(\Theta, \mathbb{Z}) \longrightarrow H^{g+1}(A, \mathbb{Z}))$$

where $j : \Theta \to A$ is the inclusion. This is a Hodge substructure of $H^{g-1}_{pr}(\Theta, \mathbb{Z})$ of rank $g! - \frac{1}{g+1}\binom{2g}{g}$ and level $g-3$ while the primitive cohomology $H^{g-1}_{pr}(\Theta, \mathbb{Z})$ has full level $g-1$.

The primal cohomology is therefore a good test case for the general Hodge conjecture. The general Hodge conjecture predicts that $\mathbb{K}_{\mathbb{Q}} := \mathbb{K} \otimes \mathbb{Q}$ is contained in the image, via Gysin push-forward, of the cohomology of a smooth (possibly reducible) variety of pure dimension $g-3$ (see [IW14]). This conjecture was proved in [IS95] and [ITW] in the cases $g = 4$ and $g = 5$. When $g = 4$, it also follows from the proof of the Hodge conjecture in [IS95] that for (A, Θ) generic, $\mathbb{K}$ is an irreducible Hodge structure (isogenous to the third cohomology of a smooth cubic threefold). When $g = 5$, the cohomology of the variety whose cohomology contains $\mathbb{K}$ is no longer irreducible and the irreducibiity of $\mathbb{K}$ no longer follows from the proof of the Hodge conjecture.

Our main result is the somewhat unexpected (see [KW, 2.9])

Theorem 0.1. For a very general ppav A of dimension 5 with smooth theta divisor Θ. The primal cohomology $\mathbb{K}$ of Θ is an irreducible Hodge structure of level 2.

As explained in [IW14], the above theorem considerably simplifies the proof of the Hodge conjecture in [ITW]: it is no longer necessary to show that the image of the Abel-Jacobi map in [ITW] contains all of $\mathbb{K}$, only that it intersects $\mathbb{K}$ non-trivially.

If A is replaced by a projective space and Θ by a smooth hypersurface, then the primitive and the primal cohomology coincide. The primitive cohomology of a general hypersurface is irreducible (see, e.g., [Lam81, 7.3]).

Our strategy, expalined below, for proving Theorem 0.1 is to use the Mori-Mukai proof [MM83] of the unirationality of $\mathcal{A}_5$.

Let T be an Enriques surface and

$$f : S \longrightarrow T$$

the K3 étale double cover corresponding to the canonical class (which is 2-torsion) $K_T \in \mathrm{Pic}(T)$. Let H be a very ample line bundle on T with $H^2 = 10$. A general element in the linear system $|H| \cong \mathbb{P}^5$ is a smooth curve of genus 6 and such smooth curves are parametrized by the Zariski open subset $|H| \setminus \mathcal{D}$, where $\mathcal{D}$ is the dual variety of the embedding of T in $|H|^*$. For each element

$u \in |H| \setminus \mathcal{D}$, we obtain a nontrivial étale double cover $D_u := f^{-1}(C_u) \to C_u$. Associating to such a cover its Prym variety $P(D_u, C_u)$ defines a morphism from $|H| \setminus \mathcal{D}$ to $\mathcal{A}_5$:

$$\begin{tikzcd} {|H| \setminus \mathcal{D}} \arrow[rr] \arrow[dr, "\mathcal{P}_H"'] & & \mathcal{R}_6 \arrow[dl, "\mathcal{P}"] \\ & \mathcal{A}_5 & \end{tikzcd} .$$

Mori and Mukai [MM83] showed that as we vary (T, H) in moduli, the family of maps $\mathcal{P}_H$ dominates $\mathcal{A}_5$.

The ppav (A, Θ) with singular theta divisor form the Andreotti-Mayer divisor N_0 in $\mathcal{A}_5$ ([Bea77]). The divisor N_0 has two irreducible components θ_{null} and N_0' ([Deb92],[Mum83])) (as divisors, $N_0 = \theta_{null} + 2N_0'$). The theta divisor of a general point $(A, \Theta) \in \theta_{null}$ has a unique node at a two-torsion point while the theta divisor of a general point in N_0' has two distinct nodes x and $-x$.

The primal cohomologies of the theta divisors form a variation of (polarized) Hodge structures over $\mathcal{U} := |H| \setminus (\mathcal{D} \cup \overline{\mathcal{P}_H^{-1}(N_0)})$. Inspired by [Lam81, 7.3], we prove Theorem 0.1 via a detailed study of the monodromy representation

$$\rho : \pi_1(\mathcal{U}) \longrightarrow Aut(\mathbb{K}_{\mathbb{Q}}, \langle , \rangle)$$

where $\langle , \rangle$ is the natural polarization on $\mathbb{K}_{\mathbb{Q}}$ induced by the intersection pairing on $H^4(\Theta, \mathbb{Q})$.

1. Prym varieties associated to a Lefschetz pencil

1.1. **A pencil of double covers.** We denote by

$$\tau : S \longrightarrow S$$

the fixed point free covering involution such that $S/\tau \cong T$. By [Nam85, Prop. 2.3] the invariant subspace of the involution ι^* acting on the Néron Severi group $NS(S)$ is equal to $f^*(NS(T))$. Since the pullback

$$f^* : NS(T) \longrightarrow NS(S)$$

is injective, we deduce that $f^*(NS(T))$ is a rank 10 primitive sublattice in $NS(S)$. It follows that the Picard number of S is greater than or equal to 10. By [Nam85, Prop. 5.6], when T is general in moduli,

$$NS(S) = f^* NS(T). \tag{1.1}$$

Hypothesis: Throughout this paper, we will assume T satisfies (1.1).

Suppose $l \cong \mathbb{P}^1 \subset |H|$ is a Lefschetz pencil, i.e., it is transverse to the dual variety $\mathcal{D}$, hence the singular curves of the pencil consist of finitely many irreducible nodal curves. Denote $\widetilde{T} := Bl_{10}T$ (resp. $\widetilde{S} := Bl_{20}S$) the blow-up of T (resp. S) along the base locus of l (resp. f^*l). We obtain a family of étale double covers parametrized by l:

$$\begin{tikzcd} \widetilde{S} \arrow[rr, "\widetilde{f}"] \arrow[dr, "\pi'"'] & & \widetilde{T} \arrow[dl, "\pi"] \\ & l. & \end{tikzcd}$$

Proposition 1.1. There are 42 singular fibers in the family $\widetilde{T} \xrightarrow{\pi} l$.

Proof. We use the formula

$$\chi_{top}(\widetilde{T}) = \chi_{top}(T) + 10 = \chi_{top}(\mathbb{P}^1)\chi_{top}(C) + N,$$

where C is a smooth fiber in the pencil and N is the number of singular fibers. We obtain $N = 42$. $\square$

Denote C_t the fiber over $t \in l$ of π and D_t the corresponding étale double cover in $\widetilde{S}$ and $\{s_i \in l : i = 1, ..., 42\}$ the 42 points where π is singular.

Proposition 1.2. For any $t \in l$, the étale double cover D_t of C_t is an irreducible curve.

Proof. Suppose D_t is reducible for some t. If C_t is smooth, D_t must be the trivial cover. If C_t has one node, D_t is either the trivial cover or the Wirtinger cover. In either case, the involution ι permutes the two components D_t^1 and D_t^2 of D_t. By (1.1), the class of D_t^i in $NS(S)$ is ι invariant, thus D_t^1 and D_t^2 have the same class in $NS(S)$ and $H = 2D_t^1$. However, since $H^2 = 10$, the class of H in $NS(T)$ is not 2-divisible, a contradiction. $\square$

Corollary 1.3. For a singular fiber $C_{s_i} = C_{pq} := \frac{C}{\{p \sim q\}}$ in the pencil l, the étale double cover $D_{s_i} := D_{pq}$ is obtained by glueing p_i with q_i for $i = 1, 2$ on a nontrivial étale double cover D of C, where $p_i, q_i \in D$ are the inverse images of $p, q \in C$ respectively.

Proof. The étale double cover D_{pq} of C_{pq} is determined by a 2-torsion point in $\mathrm{Pic}^0(C_{pq})$. The statement follows immediately from the irreducibility of D_{s_i} and the exact sequence

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Pic}^0(C_{pq})_2 \xrightarrow{\nu^*} \mathrm{Pic}^0(C)_2 \longrightarrow 0 \ ,$$

where $\nu : C \to C_{pq}$ is the normalization map and the kernel of ν^* is generated by the point of order 2 corresponding to the Wirtinger cover. $\square$

1.2. **The compactified Prym variety.** We describe the compactified Prym variety for the cover $D_{pq} \to C_{pq}$ as in Corollary 1.3. The semiabelian part G_{pq} of the Prym variety is the identity component $\mathrm{Ker}^0(Nm_{pq})$ of $\mathrm{Ker}(Nm_{pq}) \subset \mathrm{Pic}^0(D_{pq})$ in the following commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
1 \arrow[r] & \mathbb{C}^* \arrow[r] \arrow[d] & \mathrm{Ker}(Nm_{pq}) \arrow[r] \arrow[d] & \mathrm{Ker}(Nm) \arrow[r] \arrow[d] & 0 \\
1 \arrow[r] & (\mathbb{C}^*)^2 \arrow[r] \arrow[d] & \mathrm{Pic}^0(D_{pq}) \arrow[r] \arrow[d, "Nm_{pq}"] & \mathrm{Pic}^0(D) \arrow[r] \arrow[d, "Nm"] & 0 \\
1 \arrow[r] & \mathbb{C}^* \arrow[r] \arrow[d] & \mathrm{Pic}^0(C_{pq}) \arrow[r] \arrow[d] & \mathrm{Pic}^0(C) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0. &
\end{tikzcd}$$

It follows immediately that the group scheme G_{pq} is a $\mathbb{C}^*$-extension of the Prym variety $(B, \Xi) := \mathrm{Prym}(D, C)$:

$$1 \longrightarrow \mathbb{C}^* \longrightarrow G_{pq} \longrightarrow B \longrightarrow 0 \ .$$

Let $p : P^\nu \to B$ be the unique $\mathbb{P}^1$ bundle containing G_{pq} and write $P^\nu \setminus G_{pq} = B_0 \amalg B_\infty$, where B_0 and B_∞ are the zero and infinity sections of P^ν.

The compactified 'rank one degeneration' P is constructed as follows (c.f. [Mum83, §1]).

(1) On P^ν, we have $B_0 - B_\infty \backsim_{lin} p^{-1}(\Xi - \Xi_b)$ for a unique $b \in B$. Thus

$$B_0 + p^{-1}\Xi_b \backsim_{lin} B_\infty + p^{-1}\Xi.$$

(2) Let $L^\nu := \mathcal{O}_{P^\nu}(B_0 + p^{-1}\Xi_b)$. Then $L^\nu|_{B_0} \cong \mathcal{O}_B(\Xi)$ and $L^\nu|_{B_\infty} \cong \mathcal{O}_B(\Xi_b)$. Via the Leray spectral sequence for p, we see that $h^0(P^\nu, L^\nu) = 2$ and $B_0 + p^{-1}\Xi_b$, $B_\infty + p^{-1}\Xi$ span $|L^\nu|$.

(3) The compactified Prym variety P is constructed from P^ν by identifying the zero section $B_0 \overset{p}{\cong} B$ with the infinity section $B_\infty \overset{p}{\cong} B$ via translation by $b \in B$. We also denote $\nu : P^\nu \to P$ the normalization morphism.

(4) The line bundle L^ν descends to a line bundle L on P, i.e., $\nu^*L \cong L^\nu$. The linear system $|L|$ is a point.

(5) The theta divisor $\Upsilon \subset P$ is the unique divisor in $|L|$.

Remark 1.4. The $\mathbb{P}^1$ bundle $P^\nu \to B$ contains an open subset $P^\nu \setminus B_\infty$ (resp. $P^\nu \setminus B_0$), which is isomorphic to the total space of $N_{B_0|P^\nu} \cong \mathcal{O}_{B_0}(B_0) \cong \mathcal{O}_B(\Xi - \Xi_b)$ (resp. $\mathcal{O}_B(\Xi_b - \Xi)$). We conclude that $P^\nu \cong \mathbb{P}_B(\mathcal{O}_B(\Xi - \Xi_b) \oplus \mathcal{O}_B(\Xi_b - \Xi))$. In particular $G_{pq} \to B$ and $P^\nu \to B$ are **topologically trivial** $\mathbb{C}^*$ and $\mathbb{P}^1$ bundles, respectively.

Proposition 1.5. For a general rank one degeneration, the normalization Υ^ν of the theta divisor is isomorphic to $Bl_{\Xi\cap\Xi_b}B \subset P^\nu$, the theta divisor $\Upsilon \subset P$ is obtained from Υ^ν by identifying the proper transforms of Ξ and Ξ_b.

Proof. Let σ_0, σ_∞ be elements of $H^0(P^\nu, L^\nu)$, such that $div(\sigma_0) = B_0 + p^{-1}\Xi_b$ and $div(\sigma_\infty) = B_\infty + p^{-1}\Xi$. After rescaling, we may assume, under the natural identification $B_0 \overset{p}{\cong} B \overset{p}{\cong} B_\infty$, that $\sigma_0|_{B_\infty}$ and $\sigma_\infty|_{B_0}$ differ by translation by b. Then $\sigma_0 + \sigma_\infty$ descends to a section of L. Since $(\sigma_0+\sigma_\infty)|_{B_0}$ vanishes precisely on Ξ and $(\sigma_0+\sigma_\infty)|_{B_\infty}$ vanishes precisely on Ξ_b, we conclude that for $u \in B\setminus(\Xi\cap\Xi_b)$, $0 \neq (\sigma_0+\sigma_\infty)|_{p^{-1}(u)} \in H^0(\mathcal{O}_{\mathbb{P}^1}(1))$. Thus $\Upsilon^\nu := div(\sigma_0+\sigma_\infty)$ maps one-to-one to B away from $\Xi\cap\Xi_b$. On the other hand, the base locus of the pencil $|L^\nu|$ is clearly $p^{-1}(\Xi\cap\Xi_b)$. Thus $\Upsilon^\nu = m[Bl_{\Xi\cap\Xi_b}B]$, for some integer m, as divisors in P^ν. Since $(\sigma_0+\sigma_\infty)|_{B_0}$ is reduced, $m = 1$. □

2. Numerical calculations

The family of compactified Prym varieties defines a morphism $\rho : l \to \tilde{\mathcal{A}}_5$ where $\tilde{\mathcal{A}}_5$ is the partial compactification of $\mathcal{A}_5$ parametrizing ppav (A, Θ) of dimension 5 and their rank 1 degenerations. This space is a quasi-projective variety and is essentially the blow-up of the open set $\mathcal{A}_5 \amalg \mathcal{A}_4$ in the Baily-Borel compactification $\mathcal{A}_5^*$ along its boundary $\mathcal{A}_4$ ([Igu67]). The coarse moduli space $\tilde{\mathcal{A}}_5$ is the union of $\mathcal{A}_5$ and a divisor Δ parametrizing rank 1 degenerations. Mumford [Mum83] computed the class of the closure of θ_{null} and N_0' in $\tilde{A}_5$ to be

$$[\theta_{null}] = 264\lambda - 32\delta, \tag{2.1}$$

$$[N_0'] = 108\lambda - 14\delta, \tag{2.2}$$

$$[N_0] = [\theta_{null}] + 2[N_0'] = 480\lambda - 60\delta, \tag{2.3}$$

where λ is the first Chern class of the Hodge bundle Λ and δ is the class of Δ.

Lemma 2.1. The degree of $\rho^*\lambda$ is 6.

Proof. The pull-back of the Hodge bundle Λ to l fits in the exact sequence

$$0 \longrightarrow \pi_*\omega_{\widetilde{T}/l} \longrightarrow \pi'_*\omega_{\widetilde{S}/l} \longrightarrow \rho^*\Lambda \longrightarrow 0,$$

where $\omega_{\widetilde{T}/l}$ and $\omega_{\widetilde{S}/l}$ are the relative dualizing sheaves. Thus $c_1(\rho^*\lambda) = c_1(\pi'_*\omega_{\widetilde{S}/l}) - c_1(\pi_*\omega_{\widetilde{T}/l})$. We directly compute that the relative dualizing sheaf $\omega_{\widetilde{T}/l} = K_{\widetilde{T}} \otimes \pi^* K_l^{-1}$ has self intersection number $(\omega_{\widetilde{T}/l})^2 = 30$. Applying Mumford's relation [ACG11, Chapter 13.7] on $\overline{M}_6$, we see that $c_1(\pi_*\omega_{\widetilde{T}/l}) = \frac{30+42}{12} = 6$. Similarly, we compute $c_1(\pi'_*\omega_{\widetilde{S}/l}) = 12$ and therefore $c_1(\rho^*\lambda) = 6$. $\square$

Corollary 2.2. In the pencil l, there are 240 fibers with theta divisor singular at a unique two-torsion point and 60 fibers with theta divisor singular at two points.

Proof. We directly compute $l \cdot [\theta_{null}] = l \cdot (264\lambda - 32\delta) = 240$ and $l \cdot [N'_0] = l \cdot (108\lambda - 14\delta) = 60$. It follows from [SV90, Lemma B] that all these points occur with mutiplicity 1 in the intersection $l \cap N_0$ for a generic choice of l. $\square$

To summarize, we have a family of (compactified) Prym varieties and theta divisors associated to the pencil

$$\Theta \longrightarrow A \longrightarrow l.$$

This family has 240 fibers where theta has a single node, 60 fibers where theta has two nodes, and 42 fibers where theta is as in Proposition 1.5. Furthermore, we have

Proposition 2.3. The total spaces A and Θ are smooth.

Proof. We show that the tangent spaces to A and Θ have dimension 6 and 5 respectively everywhere. Let $p \in A_t$, resp. $p \in \Theta_t$, be a point of the fiber of $A \to l$, resp. $\Theta \to l$, at $t \in l$. If A_t is smooth at p, it follows from [ITW, Proposition 3.1] that, for a generic choice of l, both A and Θ (when $p \in \Theta$) are smooth at p. Assume therefore that A_t is singular at p. In such a case, it follows from the description of Θ_t in Proposition 1.5 that, if $p \in \Theta$, Θ_t is also singular at p. By the description of A_t in Section 1.2, resp. Θ_t in Proposition 1.5, the tangent space to A_t at p, resp. Θ_t at p, has dimension 6, resp. 5. We therefore need to show that the tangent space to the total space A, resp. Θ, is equal to the tangent space of the fiber. The tangent space to the fiber is the kernel of the differential of the map $A \to l$, resp. $\Theta \to l$. Since the map $\Theta \to l$ is the scheme-theoretic restriction of the map $A \to l$, we need to show that the differential of the map $A \to l$ is 0 at p to obtain the smoothness of A at p and also of Θ at p when $p \in \Theta$.

The total space A is the inverse image of the generic line $l \subset |H|$ in the relative Prym variety $P_H \to |H|$ constructed in [AFS15]. By [AFS15, Prop. 3.10, Prop. 4.4, Prop. 5.1], the singular locus of P_H lies above a union of lines or points m_i in $|H|$. We can therefore assume that l does not meet any of the m_i. Furthermore, since all pull-backs are scheme-theoretic and all fibers reduced, the restriction of the differential of $P_H \to |H|$ to A is the differential of the projection $A \to l$. The rank of the differential of $P_H \to |H|$ is not maximal at p, i.e., its image is a proper subspace of the tangent space of $|H|$ at t. Since l is generic, the tangent space of l at t intersects this image in 0. Therefore the differential of $A \to l$ is 0 at p. □

3. General facts about the Clemens-Schmid exact sequence

We briefly review some general facts about the Clemens-Schmid exact sequence. We will apply the general theory in this section to compute the local monodromy representations near the degenerate theta divisors in the pencil.

3.1. **The Clemens-schmid exact sequence.** Let

$$\begin{tikzcd} Y_0 \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d] & Y_t \arrow[l, "i_t"'] \arrow[d] \\ \{0\} \arrow[r] & V & \{t\} \arrow[l] \end{tikzcd}$$

be a one-parameter semistable degeneration (i.e., the total space $\mathcal{Y}$ is smooth and the central fiber Y_0 is reduced with simple normal crossing support) over a small disk V, and $0 \neq t \in \partial V$ a general point. The total space $\mathcal{Y}$ deformation retracts to Y_0. For such a family, the image of the monodromy representation

$$\rho : \pi_1(V \setminus \{0\}, t) \longrightarrow GL(H^\bullet(Y_t))$$

is generated by a unipotent operator $T : H^\bullet(Y_t) \to H^\bullet(Y_t)$, i.e. $(T - Id)^k = 0$ for some integer k [Lan73]. Thus

$$N := \log T := (T - Id) - \frac{1}{2}(T - Id)^2 + \frac{1}{3}(T - Id)^3 + ...$$

is nilpotent.

It follows from the work of Clemens-Schmid [Cle77], [Sch73] and Steenbrink [Ste76] that one can define mixed Hodge structures on $H^\bullet(Y_t)$, $H^\bullet(\mathcal{Y})$ and $H_\bullet(\mathcal{Y})$ such that we have an exact sequence of mixed Hodge structures (with suitable weight shifts)

$$\longrightarrow H_{2n+2-m}(\mathcal{Y}) \xrightarrow{\alpha} H^m(\mathcal{Y}) \xrightarrow{i_t^*} H^m(Y_t)_{\lim} \xrightarrow{N} H^m(Y_t)_{\lim} \xrightarrow{\beta} H_{2n-m}(\mathcal{Y}) \longrightarrow \tag{3.1}$$

where n is the relative dimension of the fibration, α is the composition

$$H_{2n+2-m}(\mathcal{Y}) \xrightarrow{\text{PD}} H^m(\mathcal{Y}, \partial\mathcal{Y}) \longrightarrow H^m(\mathcal{Y}), \tag{3.2}$$

and β is the composition

$$H^m(Y_t) \xrightarrow{\text{PD}} H_{2n-m}(Y_t) \xrightarrow{i_{t*}} H_{2n-m}(\mathcal{Y}). \tag{3.3}$$

Here 'PD' stands for Poincaré duality. The mixed Hodge structure on $H^\bullet(Y_t)$ is not the usual pure Hodge structure but rather the 'limit mixed Hodge structure' (c.f. Section 3.3). We use the notation $H^\bullet(Y_t)_{\lim}$ to distinguish it from the pure Hodge structure.

3.2. **The weight filtrations on $H^m(\mathcal{Y})$ and $H_m(\mathcal{Y})$.** Denote

$$H^m := H^m(\mathcal{Y}) \cong H^m(Y_0),$$
$$H_m := H_m(\mathcal{Y}) \cong H_m(Y_0).$$

Recall from [Mor84, p. 103] that there is a Mayer-Vietoris type spectral sequence abutting to $H^\bullet(Y_0)$ with E_1 term

$$E_1^{p,q} = H^q(Y_0^{[p]}).$$

Here $Y_0^{[p]}$ is the disjoint union of the codimension p strata of Y_0, i.e.,

$$Y_0^{[p]} := \coprod_{i_0,\ldots,i_p} Z_{i_0} \cap \ldots \cap Z_{i_p}$$

where the Z_{i_j} are distinct irreducible components of Y_0.

The differential d_1

$$\begin{array}{ccc} E_1^{p,q} & \xrightarrow{d_1} & E_1^{p+1,q} \\ \downarrow \cong & & \downarrow \cong \\ H^q(Y_0^{[p]}) & \xrightarrow{d_1} & H^q(Y_0^{[p+1]}) \end{array}$$

is the alternating sum of the restriction maps on all the irreducible components. By [Mor84, p. 103] this sequence degenerates at E_2.

The weight filtration is given by

$$W_k H^m := \oplus_{p+q=m,\ q\le k} E_\infty^{p,q} = \oplus_{p+q=m,\ q\le k} E_2^{p,q}.$$

Therefore the weights on H^m go from 0 to m and

$$Gr_k H^m \cong E_2^{m-k,k} = \frac{\mathrm{Ker}(d_1 : H^k(Y_0^{[m-k]}) \to H^k(Y_0^{[m-k+1]})}{\mathrm{Im}(d_1 : H^k(Y_0^{[m-k-1]}) \to H^k(Y_0^{[m-k]})}.$$

There is also a weight filtration on H_m:

$$W_{-k}H_m := (W_{k-1}H^m)^{\perp}$$

under the perfect pairing between H^m and H_m. With this definition,

$$Gr_{-k}H_m \cong (Gr_k H^m)^{\vee}.$$

3.3. **The limit mixed Hodge structure $H^m(Y_t)_{\lim}$.** The weight filtration associated to the nilpotent operator N has the following form,

$$0 \subset W_0 \subset W_1 \subset \ldots \subset W_{2m} = H^m(Y_t).$$

We refer to [Mor84, pp. 106-109] for the precise definition of the monodromy weight filtration and only summarize the properties we need here.

In the applications in this paper, the nilpotent operator N satisfies

$$N^2 = 0.$$

Thus the monodromy weight filtration satisfies the following

$$\begin{aligned} &W_k = 0 \ \text{ for } \ k \leq m-2, \\ &W_{m-1} = \mathrm{Im}(N), \\ &W_m = \mathrm{Ker}(N), \\ &W_k = H^m(Y_t) \ \text{ for } \ k \geq m+1. \end{aligned}$$

Let $K_t^m := \mathrm{Ker}(N) \subset H^m(Y_t)$ be the monodromy invariant subspace. It inherits an induced weight filtration from $H^m(Y_t)$. The graded pieces of $H^m(Y_t)_{\lim}$ thus satisfy

$$Gr_m H^m(Y_t)_{\lim} \cong Gr_m K_t^m \cong \frac{\mathrm{Ker}(N)}{\mathrm{Im}(N)} \tag{3.4}$$

$$Gr_{m+1}H^m(Y_t)_{\lim} \overset{N}{\cong} Gr_{m-1}H^m(Y_t)_{\lim} \cong Gr_{m-1}K_t^m \cong \mathrm{Im}(N). \tag{3.5}$$

The weight filtrations on H^m and K_t^m are related by the Clemens-Schmid exact sequence. Below are the basic facts we will use (see [Mor84, pp. 107-109])

(1) i_t^* induces an isomorphism

$$Gr_k H^m \xrightarrow{\cong} Gr_k K_t^m \quad \text{for} \quad k \leq m-1. \tag{3.6}$$

(2) There is an exact sequence

$$0 \longrightarrow Gr_{m-2} K_t^{m-2} \longrightarrow Gr_{m-2n-2} H_{2n+2-m} \xrightarrow{\alpha} Gr_m H^m \longrightarrow Gr_m K_t^m \longrightarrow 0 \ . \tag{3.7}$$

The limit Hodge filtration on $H^m(Y_t)_{\lim}$ is given by ([Mor84], [Sch73])

$$F_\infty^p = \lim_{\operatorname{im} z \to \infty} \exp(-zN) F^p(z) \tag{3.8}$$

where $f : U' \to U \setminus \{0\}$, $f(z) = e^{2\pi i z}$ is the universal cover of the punctured disk and F^p is the usual Hodge filtration on $H^m(Y_{f(z)})$ on the fixed underlying space $H^m(Y_t)$.

4. Local monodromy representations near N_0

4.1. **Local monodromy near θ_{null}.** The local monodromy representation on the cohomology of the theta divisor near a general point $(A_0, \Theta_0) \in \theta_{null}$ is given by the classic Picard-Lefschetz formula. Fix a point $p_0 \in l \cap \theta_{null}$ and pick a small disk $U \subset l$ containing p_0. We have a family of theta divisors with smooth total space Θ_U (see Proposition 2.3):

$$\begin{array}{ccc} \Theta_0 & \longrightarrow & \Theta_U \\ \downarrow & & \downarrow \\ p_0 & \longrightarrow & U. \end{array}$$

The local monodromy representation on the cohomology of a general fiber Θ_t for $t \in U \setminus \{p_0\}$

$$\rho : \pi_1(U \setminus \{p_0\}, t) \longrightarrow GL(H^k(\Theta_t))$$

is trivial when $k \neq 4$. When $k = 4$, the Picard-Lefschetz formula (see, for instance, [Voi03, p. 78]) shows that $\rho(\pi_1(U \setminus \{p_0\}, t))$ is generated by

$$\begin{array}{rcl} T_U : H^4(\Theta_t) & \longrightarrow & H^4(\Theta_t) \\ \alpha & \mapsto & \alpha - \langle \alpha, \gamma \rangle \gamma \end{array}$$

where $\langle , \rangle$ is the intersection product on $H^4(\Theta_t)$, and $\gamma \in H^4(\Theta_t)$ is the class of the vanishing 4-sphere with $\langle \gamma, \gamma \rangle = 2$.

One checks immediately that

$$T_U^2 = Id. \tag{4.1}$$

4.2. **Local monodromy near N_0'.** Next we fix a point $p_0 \in l \cap N_0'$ and a small disk $U \subset l$ containing p_0. The central fiber Θ_0 of the family Θ_U has two ordinary double points x and $-x$.

If we make a degree two base change $V \to U$ ramified at p_0:

$$\begin{tikzcd} \Theta_V \arrow[r] \arrow[d] & \Theta_U \arrow[d] \\ V \arrow[r] & U, \end{tikzcd}$$

then blow up the two singular points of Θ_V, we obtain a family

$$\begin{tikzcd} \widetilde{\Theta}_0 \arrow[r] \arrow[d] & \widetilde{\Theta}_V \arrow[d] \\ p_0 \arrow[r] & V, \end{tikzcd}$$

where the central fiber $\widetilde{\Theta}_0 = \Theta_0' \cup Q_1 \cup Q_2$ is reduced with simple normal crossing support. Here Θ_0' is the blow-up of Θ_0 at the two singular points and $Q_1 \cong Q_2$ are smooth quadric 4-folds. The double loci $\Theta_0' \cap Q_1$ and $\Theta_0' \cap Q_2$ are smooth quadric 3-folds.

Since $V \to U$ is a degree 2 ramified cover, the local monodromy operator T_V for the family $\widetilde{\Theta}_V \to V$ is equal to $T_U^2 \in GL(H^4(\Theta_t))$.

Proposition 4.1. Notation as above, $T_V = T_U^2 = Id \in GL(H^4(\Theta_t))$.

Proof. Since the central fiber $\widetilde{\Theta}_0 = \Theta_0' \cup Q_1 \cup Q_2$ only has a double locus, we have

$$Gr_k H^4(\Theta_t) = 0$$

for $k \neq 3, 4, 5$. Since $H^3(\Theta_0' \cap Q_1) \oplus H^3(\Theta_0' \cap Q_2)) = 0$, we conclude

$$Gr_5 H^4(\Theta_t) \cong Gr_3 H^4(\Theta_t) \cong Gr_3 H^4(\widetilde{\Theta}_0) = \mathrm{Im}(N_V) = 0,$$

where $N_V := \log T_V = 0$. Therefore $T_V = Id$. □

5. Local monodromy near the boundary Δ

Near the boundary Δ, the family of Prym varieties $A_U \to U$ parametrized by a small disk $U \subset l$ has smooth general fiber (A_t, Θ_t) and central fiber (P, Υ) as in Proposition 1.5. We use the Clemens-Schmid exact sequence to compute the monodromy action.

5.1. **The semi-stable reduction.** Making a ramified base change $V \to U$ of order 2 of the family

$$\begin{tikzcd} A_V \arrow[r] \arrow[d] & A_U \arrow[d] \\ V \arrow[r] & U, \end{tikzcd}$$

and then blowing up the singular locus $P \setminus G_{pq}$ of A_V, we obtain a family $\widetilde{A}_V \to V$.

Proposition 5.1. The central fiber $\widetilde{A}_0$ of the family $(\widetilde{A}_V, \widetilde{\Theta}_V) \to V$ is the union of two copies P_1^ν and P_2^ν of P^ν, with $B_0 \subset P_1^\nu$ identified with $B_\infty \subset P_2^\nu$ via the identity map and $B_\infty \subset P_1^\nu$ identified with $B_0 \subset P_2^\nu$ via translation by b. The intersection $P_1^\nu \cap P_2^\nu = B_{0\infty} \amalg B_{\infty 0}$ is the disjoint union of two copies of B.

Proof. Clearly the main component $P_1^\nu \cong P^\nu$. We will show the exceptional divisor P_2^ν is also isomorphic to P^ν. In the semistable family $\widetilde{A}_V \to V$, we have

$$N^\vee_{B_{0\infty}/P_1^\nu} \cong N_{B_{0\infty}/P_2^\nu}.$$

Therefore P_2^ν contains the total space of $\mathcal{O}_B(\Xi_b - \Xi) \cong \mathcal{O}_{B_0}(-B_0) \cong N_{B_{0\infty}|P_2^\nu} = P_2^\nu \setminus B_{\infty 0}$ as a Zariski open subset. Apply the same argument to $B_{\infty 0}$, we see that P_2^ν also contains the total space of $\mathcal{O}_B(\Xi - \Xi_b) \cong N_{B_{\infty 0}|P_2^\nu} = P_2^\nu \setminus B_{0\infty}$ as an open subset. We conclude that $P_2^\nu \cong \mathbb{P}_B(\mathcal{O}_B(\Xi - \Xi_b) \oplus \mathcal{O}_B(\Xi_b - \Xi)) \cong P^\nu$. The statement about the gluing follows from the fact that after contracting P_2^ν, the infinity and zero sections of P_1^ν are identified via translation by b. □

Corollary 5.2. The central fiber $\widetilde{\Theta}_0$ of the family $(\widetilde{A}_V, \widetilde{\Theta}_V) \to V$ is the union $\Upsilon^\nu \cup Q_\Xi$, where $\Upsilon^\nu = Bl_{\Xi \cap \Xi_b} B$ and the conic bundle Q_Ξ is the restriction of $P_2^\nu \to B$ to Ξ. The intersection $\Upsilon^\nu \cap Q_\Xi = \Xi_{0\infty} \amalg \Xi_{\infty 0}$ is the disjoint union of two copies of Ξ.

Proof. Immediate. □

5.2. **The weight filtration on $H^m(\widetilde{A}_0)$.** By Section 3.2 and Proposition 5.1, the weight filtration on $H^m(\widetilde{A}_0)$ only has the following possibly nontrivial graded pieces

$$Gr_m H^m(\widetilde{A}_0) = \mathrm{Ker}(d_1 : H^m(P_1^\nu) \oplus H^m(P_2^\nu) \longrightarrow H^m(B_{0\infty}) \oplus H^m(B_{\infty 0}))$$

and

$$Gr_{m-1} H^m(\widetilde{A}_0) = \mathrm{Coker}(d_1 : H^{m-1}(P_1^\nu) \oplus H^{m-1}(P_2^\nu) \longrightarrow H^{m-1}(B_{0\infty}) \oplus H^{m-1}(B_{\infty 0}))$$

Proposition 5.3. We have

$$Gr_m H^m(\widetilde{A}_0) \cong H^{m-2}(B) \oplus H^m(P^\nu),$$

and

$$Gr_{m-1} H^m(\widetilde{A}_0) \cong H^{m-1}(B).$$

Proof. By Remark 1.4, $P^\nu \to B$ is a topologically trivial $\mathbb{P}^1$ bundle. The statements then follow easily from Proposition 5.1 and the Künneth formula. $\square$

Corollary 5.4. The monodromy weight filtration on $H^m(A_t)_{\lim}$ satisfies

$$Gr_{m+1} H^m(A_t)_{\lim} \cong Gr_{m-1} H^m(A_t)_{\lim} \cong H^{m-1}(B).$$

Furthermore, $\dim_{\mathbb{C}} Gr_m H^m(A_t)_{\lim} = \binom{10}{m} - 2\binom{8}{m-1}$.

Proof. By (3.5) and (3.6), $Gr_{m+1} H^m(A_t)_{\lim} \cong Gr_{m-1} H^m(A_t)_{\lim} \cong Gr_{m-1} H^m(\widetilde{A}_0)$ which is isomorphic to $H^{m-1}(B)$ by Proposition 5.3. The second part follows from Sequence (3.7). $\square$

5.3. **The weight filtration on $H^m(\widetilde{\Theta}_0)$.** By Section 3.2 and Proposition 5.2, the weight filtration on $H^m(\widetilde{\Theta}_0)$ only has the following possibly nontrivial graded pieces

$$Gr_m H^m(\widetilde{\Theta}_0) = \mathrm{Ker}(d_1 : H^m(\Upsilon^\nu) \oplus H^m(Q_\Xi) \longrightarrow H^m(\Xi_{0\infty}) \oplus H^m(\Xi_{\infty 0}))$$

and

$$Gr_{m-1} H^m(\widetilde{\Theta}_0) = \mathrm{Coker}(d_1 : H^{m-1}(\Upsilon^\nu) \oplus H^{m-1}(Q_\Xi) \longrightarrow H^{m-1}(\Xi_{0\infty}) \oplus H^{m-1}(\Xi_{\infty 0}))$$

Proposition 5.5. For $m \leq 4$,

$$Gr_m H^m(\widetilde{\Theta}_0) \cong H^m(B) \oplus H^{m-2}(\Xi \cap \Xi_b) \oplus H^{m-2}(\Xi),$$

and for all m,

$$Gr_{m-1} H^m(\widetilde{\Theta}_0) \cong H^{m-1}(\Xi).$$

Proof. By Corollary 5.2, $H^m(\Upsilon^\nu) \cong H^m(B) \oplus H^{m-2}(\Xi \cap \Xi_b)$ and the restriction map $H^m(\Upsilon^\nu) \to H^m(\Xi_{0\infty})$ can be identified with the map

$$H^m(B) \oplus H^{m-2}(\Xi \cap \Xi_b) \xrightarrow{(j^*, i_*)} H^m(\Xi).$$

Thus the image of

$$H^m(\Upsilon^\nu) \longrightarrow H^m(\Xi_{0\infty}) \oplus H^m(\Xi_{\infty 0})$$

is contained in the image of

$$H^m(Q_\Xi) \cong H^m(\Xi) \oplus H^{m-2}(\Xi) \longrightarrow H^m(\Xi_{0\infty}) \oplus H^m(\Xi_{\infty 0}),$$

which is equal to the diagonal of $H^m(\Xi_{0\infty}) \oplus H^m(\Xi_{\infty 0})$. Thus

$$Gr_{m-1}H^m(\widetilde{\Theta}_0) \cong H^{m-1}(\Xi).$$

Next we compute $Gr_m H^m(\widetilde{\Theta}_0) \subset H^m(\Upsilon^\nu) \oplus H^m(Q_\Xi)$. By the previous discussion, for any $x \in H^m(\Upsilon^\nu)$, we can find $y \in H^m(Q_\Xi)$ such that $(x, y) \in Gr_m H^m(\widetilde{\Theta}_0)$. Thus we have an exact sequence

$$0 \longrightarrow H^{m-2}(\Xi) \longrightarrow Gr_m H^m(\widetilde{\Theta}_0) \longrightarrow H^m(\Upsilon^\nu) \longrightarrow 0$$

Therefore, we have a noncanonical isomorphism

$$Gr_m H^m(\widetilde{\Theta}_0) \cong H^{m-2}(\Xi) \oplus H^m(\Upsilon^\nu) \cong H^m(B) \oplus H^{m-2}(\Xi \cap \Xi_b) \oplus H^{m-2}(\Xi)$$

□

Corollary 5.6. The monodromy weight filtration on $H^m(\Theta_t)_{\lim}$ satisfies

$$Gr_{m+1}H^m(\Theta_t)_{\lim} \cong Gr_{m-1}H^m(\Theta_t)_{\lim} \cong H^{m-1}(\Xi).$$

Furthermore, $\dim_{\mathbb{C}} Gr_m H^m(\Theta_t)_{\lim} = h^m(\Theta_t) - 2h^{m-1}(\Xi)$.

Proof. Analogous to the proof of Corollary 5.4. □

5.4. **The vanishing cocycles near the boundary.** Let $Z \xrightarrow{\iota} |H| \cong \mathbb{P}^5$ be the 2-to-1 cover ramified exactly along $\Gamma := \mathcal{D} + \overline{\mathcal{P}_H^{-1}(N_0)}$ and set $X := \iota^{-1}l$, $\mathcal{V} := Z \setminus \Gamma$. Note that Z exists since Γ has even degree by Proposition 1.1 and Corollary 2.2. The curve X is a 2-to-1 cover of l ramified along $X \cap \Gamma$. After base change to X and blowing up the singular locus of each singular theta divisor, we obtain a family $(\widetilde{A}, \widetilde{\Theta})$ with general fiber (A_t, Θ_t).

$$\begin{tikzcd}
\Theta_t \arrow[r, "i_t"] \arrow[d, "j_t"'] & \widetilde{\Theta} \arrow[d, "j"] \\
A_t \arrow[r, "h_t"] \arrow[d] & \widetilde{A} \arrow[d, "p"] \\
\{t\} \arrow[r] & X.
\end{tikzcd}$$

The total spaces of $\widetilde{A}$ and $\widetilde{\Theta}$ are smooth and the local pictures are described in Sections 4.1, 4.2 and 5.1.

For each s_i, $i = 1, ..., 42$, corresponding to the degeneration in Section 1 (also see Section 5.1), choose a small disk $V_i \ni s_i$ and pick a general point $t_i \in V_i$. Let $\gamma_i \subset X$ be a general path connecting t with t_i. The family $\widetilde{\Theta}|_{\cup\gamma_i}$ deformation retracts to Θ_t. Thus we have induced **diffeomorphisms**

$$\psi_i : \Theta_t \longrightarrow \Theta_{t_i}.$$

Over each V_i we have the Clemens-Schmid exact sequences (3.1) for the degenerations of the abelian varieties and their theta divisors

$$\begin{tikzcd} \arrow[r] & H^m(\widetilde{\Theta}_{V_i}) \arrow[r, "i^*_{t_i}"] \arrow[d, "j_*"] & H^m(\Theta_{t_i})_{\lim} \arrow[r, "N_i"] \arrow[d, "j_{t_i*}"] & H^m(\Theta_{t_i})_{\lim} \arrow[r, "\beta_i"] \arrow[d, "j_{t_i*}"] & H_{10-m}(\widetilde{\Theta}_{V_i}) \arrow[r] \arrow[d, "j_*"] & {} \\ \arrow[r] & H^{m+2}(\widetilde{A}_{V_i}) \arrow[r] & H^{m+2}(A_{t_i})_{\lim} \arrow[r] & H^{m+2}(A_{t_i})_{\lim} \arrow[r] & H_{10-m}(\widetilde{A}_{V_i}) \arrow[r] & {} . \end{tikzcd} \tag{5.1}$$

Here $j_* : H^m(\widetilde{\Theta}_{V_i}) \to H^{m+2}(\widetilde{A}_{V_i})$ is defined to be the transpose of $j^* : H_c^{10-m}(\widetilde{A}_{V_i}) \to H_c^{10-m}(\widetilde{\Theta}_{V_i})$ under Poincaré duality and is a morphism of mixed Hodge structures [ITW, ??].

Denote $\mathbb{V}_i^m := \psi_i^* \operatorname{Ker} \beta_i = \psi_i^* \operatorname{Im}(N_i) = \psi_i^* Gr_{m-1} H^m(\Theta_{t_i})_{\lim} \subset H^m(\Theta_t)_{\lim}$.

Proposition 5.7. The space $\mathbb{V}_i$ is the space of 'local vanishing m-cocycles', i.e., cohomology classes whose Poincaré dual vanishes in Θ_{V_i}.

Proof. This follows immediately from the definition of β_i in (3.3). □

By Corollary 5.6, we have

$$\operatorname{Im}(N_i) = Gr_{m-1} H^m(\Theta_{t_i})_{\lim} \overset{i^*_{t_i}}{\cong} Gr_{m-1} H^m(\widetilde{\Theta}_{V_i}) \cong H^{m-1}(\Xi).$$

When $m = 4$, we can further rewrite the above isomorphisms as

$$Gr_3 H^4(\Theta_{t_i})_{\lim} \cong H^3(\Xi) \cong H^3(B) \oplus \mathbb{H}'_i \cong j^*_{t_i} Gr_3 H^4(A_{t_i})_{\lim} \oplus \mathbb{H}'_i, \tag{5.2}$$

where $\mathbb{H}'_i \subset H^3(\Xi)$ is the primal cohomology of Ξ in B, which is 10-dimensional. Let $\mathbb{H}_i \subset \mathbb{V}_i^4 \subset H^4(\Theta_t)$ be the image of $\mathbb{H}'_i$ under the composition

$$H^3(B) \oplus \mathbb{H}'_i \cong Gr_3 H^4(\Theta_{t_i})_{\lim} \subset H^4(\Theta_{t_i})_{\lim} \xrightarrow{\psi_i^*} H^4(\Theta_t).$$

6. Global monodromy

Let $H^m(\Theta_t)_{var} := \operatorname{Ker}(i_{t*} : H^m(\Theta_t) \to H^{m+2}(\widetilde{\Theta}))$ and $H^m(A_t)_{var} := \operatorname{Ker}(h_{t*} : H^m(A_t) \to H^{m+2}(\widetilde{A}))$ be the variable cohomology of Θ_t in $\widetilde{\Theta}$ and A_t in $\widetilde{A}$, respectively.

6.1. **The primal cohomology and the variable cohomology.** The next four propositions describe the variable middle cohomology $H^4(\Theta_t)_{var}$ and its relation with the primal cohomology $\mathbb{K}_t$.

Proposition 6.1. The variable cohomology $H^m(\Theta_t)_{var}$ is equal to $\sum_{i=1}^{42} \mathbb{V}_i^m$.

Proof. By Equation (4.1) and Proposition 4.1, when the theta divisor has one or two nodes, the local monodromy representation is trivial after we make a base change of order 2. Thus from the Clemens-Schmid sequence, there is no 'local vanishing cocycles' near these singular theta divisors. Therefore the space of vanishing cocycles is generated by the 'local vanishing cocyles' near Θ_{s_i}, $i = 1, ..., 42$. □

Proposition 6.2. The pull-back maps $i_t^* : H^4(\widetilde{\Theta}) \to H^4(\Theta_t)$ and $(j \circ i_t)^* : H^4(\widetilde{A}) \to H^4(\Theta_t)$ have the same image. As a consequence, $H^4(\Theta_t)_{var} = (\mathrm{Ker}(j \circ i_t)_* : H^4(\Theta_t) \to H^8(\widetilde{A}))$.

Proof. Choose another general point $u \neq t$ in X. Write $W := X \setminus \{u\}$, and $(\widetilde{A}_W, \widetilde{\Theta}_W) := (p^{-1}(W), (p \circ j)^{-1}(W))$.

Consider the Gysin sequence

$$\begin{tikzcd} \arrow[r] & H^4(\widetilde{A}) \arrow[r] \arrow[d, "j^*"] & H^4(\widetilde{A}_W) \arrow[r, "Res"] \arrow[d, "j_W^*"] & H^3(A_u) \arrow[r, "h_{u*}"] \arrow[d, "j_u^*", "\cong"'] & H^5(\widetilde{A}) \arrow[r] \arrow[d, "j^*"] & {} \\ \arrow[r] & H^4(\widetilde{\Theta}) \arrow[r] & H^4(\widetilde{\Theta}_W) \arrow[r, "Res"] & H^3(\Theta_u) \arrow[r, "i_{u*}"] & H^5(\widetilde{\Theta}) \arrow[r] & {} \end{tikzcd}$$

where Res denotes Griffiths' residue map. We claim that $j_W^* : H^k(\widetilde{A}_W) \to H^k(\widetilde{\Theta}_W)$ is an isomorphism for $k \leq 4$ and injective for $k = 5$ (this is the Lefschetz hyperplane theorem in a slightly modified setting). To this end, apply the long exact sequence of singular cohomology of the pair $(\widetilde{A}_W, \widetilde{\Theta}_W)$. The relative cohomology $H^k(\widetilde{A}_W, \widetilde{\Theta}_W)$ is isomorphic to $H_{12-k}(\widetilde{A}_W \setminus \widetilde{\Theta}_W)$ [Voi03, p. 33]. Note that $\widetilde{\Theta}$ is p-ample, and therefore $\widetilde{\Theta} + kA_u$ is ample in $\widetilde{A}$ for some $k > 0$. We conclude that the open set $\widetilde{A}_W \setminus \widetilde{\Theta}_W = \widetilde{A} \setminus (\widetilde{\Theta} \cup A_u)$ is affine, thus has the homotopy type of a CW-complex of real dimension 6. Therefore $H_{12-k}(\widetilde{A}_W \setminus \widetilde{\Theta}_W) = 0$ for $k \leq 6$, which implies the claim.

By Proposition 6.1 and Corollaries 5.4 and 5.6, $H^3(A_u)_{var} := \mathrm{Ker}(h_{u*}) \cong H^3(\Theta_u)_{var}$, thus by the Gysin sequence and the fact that j_W^* is an isomorphism when $k = 4$, the restriction map $H^4(\widetilde{\Theta}) \to H^4(\widetilde{\Theta}_W)$ has the same image as the composition $H^4(\widetilde{A}) \to H^4(\widetilde{A}_W) \xrightarrow{j_W^*} H^4(\widetilde{\Theta}_W)$. Taking the restriction map from $H^4(\widetilde{\Theta}_W)$ to $H^4(\Theta_t)$, the first statement follows immediately.

The second statement follows from the fact that Gysin push-forward is the transpose of the pull-back map. □

Proposition 6.3. The primal cohomology $\mathbb{K}_t := \mathrm{Ker}(j_{t*} : H^4(\Theta_t) \to H^6(A_t))$ is contained in the variable cohomology $H^4(\Theta_t)_{var}$.

Proof. By Proposition 6.2, we have $H^4(\Theta_t)_{var} = (\mathrm{Ker}(j \circ i_t)_* : H^4(\Theta_t) \to H^8(\widetilde{A}))$, which implies $\mathbb{K}_t \subset H^4(\Theta_t)_{var}$. □

Proposition 6.4. The primal cohomology $\mathbb{K}_t$ is equal to $\sum_{i=1}^{42} \mathbb{H}_i$.

Proof. The morphism $j_* : H^4(\widetilde{\Theta}_{V_i}) \to H^6(\widetilde{A}_{V_i})$ in (5.1) is a morphism of mixed Hodge structures. The induced morphism

$$\begin{tikzcd} Gr_3 H^4(\widetilde{\Theta}_{V_i}) \arrow[r] \arrow[d, "\cong"] & Gr_5 H^6(\widetilde{A}_{V_i}) \arrow[d, "\cong"] \\ H^3(\Xi) \arrow[r] & H^5(B) \end{tikzcd}$$

is Gysin pushforward. By construction, $\mathbb{H}' \subset Gr_3 H^4(\widetilde{\Theta}_{V_i}) \subset H^4(\widetilde{\Theta}_{V_i})$ is contained in $\mathrm{Ker}(j_*)$. Thus by sequence (5.1), $i_{t_i}^* \mathbb{H}' \subset \mathrm{Ker}(j_{t_i *} : H^4(\Theta_{t_i}) \to H^6(A_{t_i}))$, or equivalently, $\mathbb{H}_i \subset \mathbb{K}_t$. It remains to show $\mathbb{K}_t \subset \sum_{i=1}^{42} \mathbb{H}_i$. To this end, pick any $\alpha \in \mathbb{K}_t$, by Proposition 6.1 and Equation (5.2), we can write $\alpha = \sum_{i=1}^{42}(x_i + y_i)$, where $x_i \in j_t^* H^4(A_t)$ and $y_i \in \mathbb{H}_i \subset \mathbb{K}_t$. From the direct sum decomposition

$$H^4(\Theta_t) = j_t^* H^4(A_t) \oplus \mathbb{K}_t,$$

we conclude $\sum_{i=1}^{42} x_i = 0$ and $\alpha \in \sum_{i=1}^{42} \mathbb{H}_i$. □

6.2. **The proof of the main theorem.** From now on we will abuse notation by considering N_i in (5.1) as an endomorphism on $H^4(\Theta_t)$ via ψ_i^* and then restricting it to $\mathbb{K}_t$. With the new notation, $N_i : \mathbb{K}_t \to \mathbb{K}_t$ satisfies

$$N_i^2 = 0, \tag{6.1}$$

$$N_i(\mathbb{K}_t) = \mathbb{H}_i. \tag{6.2}$$

Since the monodromy operator preserves the intersection product $\langle , \rangle$ on $\mathbb{K}_t$, N_i also satisfies the equality

$$\langle N_i(x), y \rangle + \langle x, N_i(y) \rangle = 0 \tag{6.3}$$

for any $x, y \in \mathbb{K}_t$.

Each N_i induces a 'limit mixed Hodge structure' $\mathbb{K}^i_{\lim}$ on $\mathbb{K}_t$ as in Section 3.3.

Lemma 6.5. We have $\cap_{i=1}^{42}\operatorname{Ker}(N_i) = 0$.

Proof. Equation (6.3) implies that $\langle N_i(x), y\rangle = 0$ for any $x \in \mathbb{K}_t$ and $y \in \operatorname{Ker}(N_i)$. Thus $\operatorname{Ker}(N_i) \perp \mathbb{H}_i$. Any element in $\cap_{i=1}^{42}\operatorname{Ker}(N_i)$ is therefore perpendicular to all $\mathbb{H}_i$, $i = 1, ..., 42$. The statement now follows immediately from Proposition 6.4 and the fact that the intersection product is nondegenerate. $\square$

Lemma 6.6. With the notation of Section 5.4, all $\mathbb{H}_i, i = 1, ..., 42$ are conjugate under the monodromy representation

$$\rho : \pi_1(\mathcal{V}, t) \longrightarrow \operatorname{Aut}(\mathbb{K}_t, \langle,\rangle).$$

Proof. For any $i \neq j$, choose a path δ' in l connecting t_i and t_j. By perturbing δ', we can assume δ' does not intersect the inverse image of N_0. We can lift δ' to a path $\delta \subset X \cap \mathcal{V}$ as a smooth section over δ' in the tubular neighborhood of the smooth locus $\mathcal{D}^0$ of $\mathcal{D}$ in $\mathcal{V}$. A $\mathcal{C}^\infty$-trivialization of the total space of the theta divisors over δ induces a map on cohomology, which sends $\mathbb{H}'_i \subset H^4(\Theta_{t_i})$ to $\mathbb{H}'_j \subset H^4(\Theta_{t_j})$. This precisely means that under the monodromy action, $\rho(\gamma_i \cdot \delta \cdot \gamma_j^{-1})$ sends $\mathbb{H}_i$ to $\mathbb{H}_j$. $\square$

Proof. of **Theorem 0.1**. It suffices to show that for very general $t \in X \cap \mathcal{V}$, $\mathbb{K}_t$ is an irreducible Hodge structure. Suppose $0 \subsetneqq \mathbb{F}_t \subset \mathbb{K}_t$ is a rational Hodge substructure, then $\mathbb{F}_t$ is an invariant subspace under the action of the Mumford-Tate group $MT(\mathbb{K}_t)$. For very general t, $MT(\mathbb{K}_t)$ contains the identity component $I_\mathcal{V}$ of the **algebraic monodromy group** $G_\mathcal{V}$, i.e., the Zariski closure in $GL(\mathbb{K}_t)$ of the monodromy group $\rho(\pi_1(\mathcal{V}))$, (c.f. [Sch11, Prop. 6]), thus by further passing to a finite étale cover $\mathcal{V}'$ of $\mathcal{V}$, we can assume $\mathbb{F}_t$ is invariant under $\rho(\pi_1(\mathcal{V}'))$. Therefore, we obtain a local subsystem $\mathbb{F}_{\mathcal{V}'} \subset \mathbb{K}_{\mathcal{V}'}$ over $\mathcal{V}'$.

Note that

$$I_{\mathcal{V}'} = I_\mathcal{V},$$

since $I_{\mathcal{V}'} \subset I_\mathcal{V}$ is of finite index and $I_\mathcal{V}$ is connected. Moreover, $T_i = \exp(N_i) \in I_\mathcal{V} = I_{\mathcal{V}'}$. (Because T_i is in the image of the exponential map $\exp : gl(\mathbb{K}_t) \to GL(\mathbb{K}_t)$.) We conclude that $\mathbb{F}_t$ is invariant under T_i and therefore N_i. Each N_i then induces a 'limit mixed Hodge structure' $\mathbb{F}^i_{\lim}$ on $\mathbb{F}_t$.

By Lemma 6.5, for any $0 \neq x \in \mathbb{F}_t$, $x \notin \operatorname{Ker}(N_i)$ for some i, thus $0 \neq N_i(x) \in \mathbb{F}_t \cap \mathbb{H}_i = \mathbb{F}_t \cap W_3\mathbb{K}^i_{\lim} = W_3\mathbb{F}^i_{\lim}$. Since $\mathbb{H}_i = W_3\mathbb{K}^i_{\lim}$ is an irreducible pure Hodge structure (follows from the main result of [IS95]), we conclude $\mathbb{H}_i \subset \mathbb{F}_t$. By Lemma 6.6, the $\mathbb{H}_i$ are conjugate under the monodrmy group $\pi_1(\mathcal{V})$, thus $\mathbb{H}_i \subset \mathbb{F}_t$ for all i and, by Proposition 6.4, $\mathbb{F}_t = \mathbb{K}_t$. $\square$

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