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Three Essays on Urban Land Development

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Sungin Ahn

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ABSTRACT OF THE DISSERTATION

Three Essays on Urban Land Development

by

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University of California, Riverside, June 2019
Dr. Richard J. Arnott, Chairperson

This dissertation addresses two major aspects concerning urban land development. The first is the effect of a floor area ratio ("FAR") regulation on land values and urban land development patterns. The model shows that holding all else equal, an increase in the stringency of regulation (a lower maximum-allowed FAR) 1) leads to a decrease in land value, even for those parcels where the regulation may not be binding today; 2) lowers the density of all converted buildings; and 3) hastens the development of all buildings. Our empirical exercise then tests certain aspects of this model in the context of New York city, using data constructed from a publicly available source. Second, this dissertation explores the role of market power in urban land development patterns. Utilizing social surplus analysis, we derive and contrast the optimality conditions for a social planner's and a monopolist's choice of density of housing and the rate at which land is released for development. We then provide a numerical example and a numerical comparative statics analysis. In our numerical examples, the monopolist always develops

at a lower density than is optimal. The rate at which land is released for development, however, will depend on the parameters.

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Introduction

The overarching theme of my research project is the role of land in economics and the analysis of land as a factor of production in the context of public policy and development. While land as a factor of production is unique in that 1) it is supplied perfectly inelastically and 2) apart from agricultural uses it does not depreciate, the economic literature by and large tends to treat land as a type of capital, thus excluding it from most analyses. Nevertheless, it is essential to understand how land fits into the narrative of economists and to develop a microfoundation to analyze its role in the Neoclassical framework. Publications such as Arnott & Stiglitz (1979), which shows that the cost of public goods of an optimally-sized community equals the land rent of that community, or Stiglitz (2015), which shows that land value taxation is not only non-distortionary but also increases productivity point to the importance of fully understanding how we can exploit the properties of land for policy decisions.

While there was a heightened academic interest in land economics in the early 1900s with the publication of Henry George's *Progress and Poverty* and with extensive work by Harry Gunnison Brown, it wasn't until recently that we started seeing significant progress in the subject. Some examples include Richard Arnott's partial equilibrium models of property tax and land development (Arnott (2005); Arnott & Petrova (2006); Arnott & Lewis (1979); Arnott & MacKinnon (1977)), Jan Brueckner's work on the effect of property tax on public policy issues (Brueckner (1986); Brueckner & Kim (2003)), and the vast and rich literature on Georgist economics by Mason Gaffney.

Nevertheless, there are still many unanswered questions to be explored in the field of land economics. What is the optimal development path of land, and how do land use policies affect this path? How does market power impact the rate at which land is developed? My research project attempts to provide some insight into these questions.

The first essay investigates the partial equilibrium effects of a land use regulation which limits the maximum-allowed floor area ratio ("FAR") on a parcel of land. We extend the standard urban land-use model with FAR regulation to a dynamic setting to shed additional light into the development patterns of a city.

The land developer with perfect foresight decides on the timing and the density of multiple conversions of durable housing. We find that increasing the stringency of a FAR regulation 1) decreases land value, even if the regulation may not be binding today; 2) decreases the density of all buildings converted; and 3) speeds up the conversion of all buildings.

This essay contributes to the literature on land-use regulations in several ways. First, we generalize the analysis of a city in a stationary state to that of a growing city. Second, our model provides a guide for a specification of an empirical model to analyze some aspects of FAR regulation not explored in the existing literature. Third, our model provides a supply-side module for a spatial general equilibrium model of a growing city with durable housing.

The second essay is an empirical exercise which tests a few of the predictions of the theoretical model in the first essay. We construct our data using a publicly available source from New York City and find that all else equal, a more stringent FAR regulation is correlated with 1) a lower land value, even for parcels where the regulation is not binding and 2) an earlier time of demolition.

The third essay analyzes how market power affects urban land development patterns in a partial equilibrium model. While some believe that a rapid increase in land prices is due to large landowners holding land off the market, we claim that market power affects the rate at which land is developed - not the total quantity of land developed. We employ social surplus analysis to derive and contrast the equilibrium conditions for the optimal choice of density of housing and rate of land development between a social planner and a monopolist. We then provide a numerical example and conduct a numerical comparative statics analysis. In our numerical exercise, the monopolist develops at a lower density than is optimal. However, the rate at which the monopolist releases land for development will depend on the parameters of the model.

References

[1] Arnott, R.J. (2005). “Neutral Property Taxation.” *Journal of Public Economic Theory*, 7(1):27-50.

[2] Arnott, R.J. & F.D. Lewis (1979). “The Transition of Land to Urban Use.” *Journal of Political Economy*, 87(1):161-169.

[3] Arnott, R.J. & J.G. MacKinnon (1977). “The Effects of the Property Tax: A General Equilibrium Simulation.” *Journal of Urban Economics*, 4(4):389-407.

[4] Arnott, R.J. & P. Petrova (2006). “The Property Tax as a Tax on Value: Deadweight Loss.” *International Tax and Public Finance*, 13(2-3):241-266.

[5] Arnott, R.J. & J. Stiglitz. (1979). “Aggregate Land Rents, Expenditure on Public Goods, and Optimal City Size.” *The Quarterly Journal of Economics*, 93(4):471-500.

[6] Brueckner, J.K. (1986). “A Modern Analysis of the Effects of Site Value Taxation.” *National Tax Journal*, 39:49-58.

[7] Brueckner, J.K. & H. Kim (2003). “Urban Sprawl and the Property Tax.” *International Tax and Public Finance*, 10(1):5-23.

[8] Stiglitz, J.E. (2015). “New Theoretical Perspectives on the Distribution of Income and Wealth Among Individuals: Part IV: Land and Credit.” *NBER Working Paper Series*, 21192.

The Effects of FAR Regulations in a Model of Durable Building with Redevelopment

Abstract

This paper investigates the partial equilibrium effects of a floor area ratio ("FAR") regulation in a growing city with rising rents. We extend the standard urban land-use model with FAR regulation in the existing literature to a dynamic setting to provide additional insights about development patterns in a city with a regulation which restricts the maximum-allowed FAR. We show that a FAR regulation hastens construction and lowers the density of buildings, and we also find the effects of a FAR regulation on land value to be consistent with the existing literature.

1 Introduction

The purpose of this paper is to propose a model which describes the economic effects of introducing a regulation that limits the maximum-allowed floor area ratio ("FAR") on a parcel of land. Our model analyzes the effects of a marginal change on the FAR regulation in a partial equilibrium setting when there is durable housing and redevelopment.

While there are many types of land use regulation, FAR regulation is intended to deal with various forms of land use externalities arising from high population density and tall buildings. For example, FAR regulation combined with other regulations such as total parking space or open space requirements can be used to i) increase sunlight in the streets; ii) reduce traffic congestion; iii) reduce strain on public infrastructure; and iv) reduce wind-tunnel effects, among other things.

First introduced in New York City in 1961, the regulation of FAR was an attempt to improve on the previous policy of restricting the height and shape of buildings. While height restrictions were an attempt to regulate the rapid population growth in New York City, FAR restrictions were a much more effective way of directly limiting population density while decentralizing city planning by giving developers more freedom in deciding on the shape of their buildings.

The literature on the economic impacts of FAR regulation has focused on 1) optimization of FAR regulation policy and 2) measuring the impacts of FAR regulation on land values. Relevant to our paper is a series of papers on measuring the *stringency* of FAR regulation (Brueckner and Sridhar (2012); Brueckner et al (2017); Moon (2018)), defined as the ratio of the theoretical profit-maximizing FAR under no restriction to the maximum-allowed FAR. These papers extend Brueckner's (1987) well-known land use model by including density restriction to examine the cost of FAR regulation to the land developer.

This paper analyzes the effects of FAR regulation on the profit-maximization problem of the land developer ("developer") who also owns the parcel to be developed. The developer decides on the timing and density of a sequence of structures to be built on the parcel under perfect forsight. We make several contributions to the literature on urban land use and density regulation. First, we extend the static, partial equilibrium analysis of the existing literature to a dynamic setting with perfect foresight and multiple conversions of durable housing. Thus, we generalize the analysis of a city in a stationary state to that of a growing city. Second, our paper provides a guide for a specification of an empirical model to analyze some aspects of FAR regulation not investigated in the existing literature. Finally, our model provides a supply-side module for a spatial general equilibrium model of a growing city with durable housing.

As urban development is an innately dynamic problem with time-varying rents, required maintenance, and land conversion decisions to be made, our model provides additional insight for the study of FAR regulations. In a static model, it is assumed either that the FAR regulation is binding for all buildings or that there is no effect of the regulation for buildings for which the regulation is not binding. In our dynamic model, however, a FAR regulation that may not be binding today nevertheless affects the profit-maximizing program prior to its becoming binding. In particular, a marginal relaxation of FAR regulation - given that such a relaxation maintains the current number of building conversions ("conversions") - will result in a sequence of later and higher-density developments for all future buildings. Furthermore, if the FAR regulation is binding for the first and only conversion of land, and if the elasticity of substitution between land and capital is equal to unity as in the existing literature, then there would be no effect of changing the maximum-allowed FAR on the elasticity of land value with respect to the stringency of the regulation.

A companion to this paper provides an empirical test of the model presented here using data on land value and time of construction of parcels that are in between conversions in New York City. We find that the results are consistent with the predictions of the model in that 1) parcels with more stringent FAR regulation tend to be developed earlier and at lower densities and 2) there is a statistically significant effect even for those parcels for which the FAR regulation is not currently binding.

We organize the next sections as follows. In section 2, we discuss the development in the two most relevant literatures - partial equilibrium models of urban development in a growing city with durable housing and multiple conversions and partial equilibrium models of FAR restrictions in a static land use model. In section 3, we provide an illustrative example of our model for a single conversion case to motivate the problem. In section 4, we present the baseline model of urban development in a growing city with durable housing and multiple conversions without FAR restrictions. In section 5, we introduce FAR restrictions into the baseline model and analyze the effects of marginal changes in the maximum-allowed FAR on the optimal development program and on the value of land. In section 6, we provide the concluding remarks.

2 Literature Review

The literature on urban land use model is dense, with countless different specifications. The first analyses of the housing market were Marshallian, and the housing market was viewed as a market for housing services (Arnott (1987)). This view of the housing market was formalized by Muth (1969), and the housing market literature exploded during the seventies with several main lines of development.

Most relevant to our paper is the class of partial equilibrium models of urban development in a growing city with durable housing. One of the first models of this kind was introduced by Richard J. Arnott (Arnott and Lewis (1979); Arnott (1980)). In these models, the developer with perfect foresight decides on the timing and the density of a permanently durable housing to be erected on a single plot of vacant land. By working with perfect foresight Arnott was able to derive important insights not found in models with static or myopic expectations as in the previous literature. For example, in residential location theory the structural density of a building is determined by land rent, but in Arnott's models structural density is determined by land value. Furthermore, the effects of policies will be different based on whether or not the policy was anticipated by developers.

The first paper to introduce multiple conversions with perfect foresight was by Brueckner (1981). Under the assumptions that 1) rents remain constant over time, 2) the quality of the buildings deteriorate over time, and 3) demolition is costless, Brueckner finds that the optimal development program is an infinite sequence of identical buildings. Amin and Capozza (1993) extend the analysis to include growing rents and find that while the qualitative results under multiple conversions is similar to that under a single conversion, the quantitative results differ dramatically.

Also relevant to our study are the stationary, partial equilibrium models by Jan Brueckner (Brueckner and Sridhar (2012); Brueckner et al (2017)) which explore the effects of making incremental changes to the stringency of FAR regulation. These models show that 1) the relaxation of FAR restriction on a property leads to a higher land value and that 2) this increase in land value has a diminishing effect with subsequent relaxation of the FAR restriction.

Our paper is a union of the partial equilibrium models of urban development with rising rents, durable housing, and multiple conversions under perfect foresight and Brueckner's model of anticipated FAR restriction.

2.1 Brueckner's Stationary, Partial Equilibrium Model

It will now be useful to present Brueckner's stationary, partial equilibrium model. Consider a competitive urban land development market in which the developer decides on the optimal density of development. The rent on housing is competitively determined and is dependent on its locational characterstics. The developer's profit-maximization problem can be stated as

$$\max_{S} R(S) = ph(S) - rS \tag{2.1}$$

where R is the per unit rent on land, p is the rent per unit of housing, S is the units of capital invested per unit of land, r is the price per unit of capital, and $h(\cdot)$ is the output of housing per unit of land. The first-order condition for the choice of capital is

$$ph'(S) = r \tag{2.2}$$

with S^* satisfying the equality, so that development takes place when the marginal revenue from the investment of an additional unit of capital is equal to its marginal cost. Now suppose that FAR is limited by regulation, so that the amount of capital that may be invested is limited to $\overline{S} < S^*$. Then, the land rent is given by

$$R\left(\overline{S}\right) = ph\left(\overline{S}\right) - r\overline{S} \tag{2.3}$$

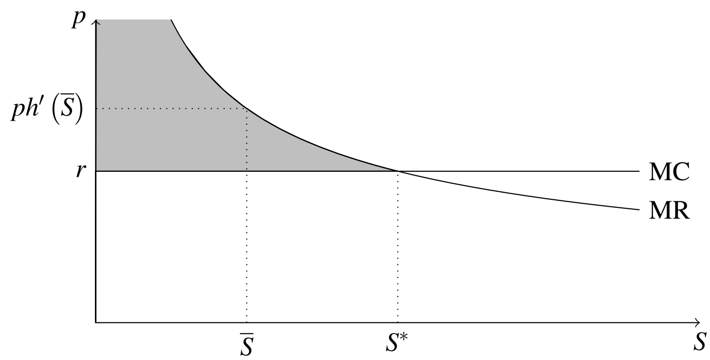


Figure 1.1: Static Model of FAR Restriction

with its derivative with respect to the maximum-allowed FAR being

$$\frac{\partial R}{\partial \overline{S}} = ph'\left(\overline{S}\right) - r > 0 \tag{2.4}$$

which is the additional benefit the developer would gain if he were allowed to invest another unit of capital for his building. In other words, this measures the marginal efficiency loss due to the FAR regulation from the perspective of the developer who does not care about externalities.

Figure 1.1 illustrates the foregoing. Note that the shaded area indicates land rent when the structure is developed optimally.

Now consider the elasticity of land rent with respect to the maximum-allowed capital investment:

$$E_{R:\overline{S}} \equiv \frac{\partial R}{\partial \overline{S}} \frac{\overline{S}}{R} = \frac{\left[ph'\left(\overline{S}\right) - r\right]\overline{S}}{ph\left(\overline{S}\right) - r\overline{S}} \tag{2.5}$$

$$= \frac{\left[h'\left(\overline{S}\right)p - h'\left(S^*\right)p\right]\overline{S}}{ph\left(\overline{S}\right) - \overline{S}h'\left(S^*\right)p} \tag{2.6}$$

$$= \frac{\left[h'\left(\overline{S}\right) - h'\left(S^*\right)\right]\overline{S}}{h\left(\overline{S}\right) - \overline{S}h'\left(S^*\right)} \tag{2.7}$$

where the second equality going from (2.5) to (2.6) comes from substituting in (2.2). Because of the concavity of h, we know that $h'\left(\overline{S}\right)\overline{S} < h\left(\overline{S}\right)$, thus the elasticity is less than unity. Assuming a Cobb-Douglas production function so that $h(S) = S^{\beta}$ where $\beta < 1$ is the output elasticity of land yields

$$E_{R:\overline{S}} = \frac{\left(S^*/\overline{S}\right)^{1-\beta} - 1}{\frac{1}{\beta}\left(S^*/\overline{S}\right)^{1-\beta} - 1} \tag{2.8}$$

so that the elasticity depends on the stringency of the FAR regulation, $\frac{S^*}{\overline{S}}$. Note that all else equal, the regulation becomes more stringent as $\overline{S}$ is lowered.

Taking the derivative of (2.8) with respect to $\frac{S^*}{\overline{S}}$, we can show that

$$\frac{\partial E_{R:\overline{S}}}{\partial \left(S^*/\overline{S}\right)} > 0$$

so that the elasticity is high when the stringency is high. Therefore, the elasticity is diminishing with a marginal relaxation of the FAR regulation. However, as we shall see in the following sections, this result may not always be true in a dynamic setting.

3 Illustrative Example

We begin with an illutrative example to motivate the problem and provide some insight into the features of our model. While the general model assumes multiple conversions, assume for the purpose of illustration that there is only one conversion of land. The plot of land to be developed is initially vacant, and the developer must decide on the optimal timing and density for the transition of land from rural to urban use. Also assume that the urban housing market is competitive with housing rent which grows at a constant rate, g. The developer's profit-maximization problem can be stated as

$$\max_{t,S} V(t,S) = \int_t^\infty p(\tau)\, h(S)\, e^{-\rho\tau} d\tau - rSe^{-\rho t} \tag{3.1}$$

where $V(\cdot)$ is the land value function, t is the time of development, and ρ is the discount rate. The first-order condition with respect to t is

$$p(t)\, h(S) = \rho r S \tag{3.2}$$

with t^* satisfying the equality. Development will take place when the marginal cost of rent foregone is equal to the marginal benefit of delaying construction any further. The first-order condition with respect to S is

$$re^{-\rho t} = h'(S) \int_t^\infty p(\tau)\, e^{-\rho\tau} \tag{3.3}$$

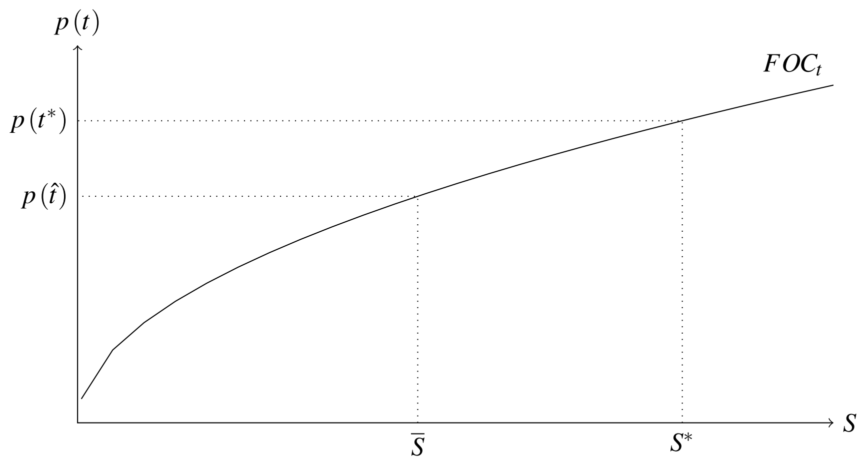


Figure 1.2: Illustrative Example

with S^* satisfying the equality. The building will be constructed to the height where the marginal cost of an additional unit of capital invested is equal to the marginal revenue from the extra housing produced from the additional unit of capital.

Now suppose that there is an anticipated FAR regulation imposed prior to the developer entering the market so that the amount of housing one can develop is limited to $\overline{h} \equiv h\left(\overline{S}\right)$ for some $\overline{S} < S^*$. In this case, there is only one first-order condition, and (3.2) becomes

$$p(t) = \frac{\rho r \overline{S}}{h\left(\overline{S}\right)} \tag{3.4}$$

with $\hat{t}$ satisfying the optimality condition. The derivative of this expression with respect to the maximum-allowed FAR is

$$\frac{dp(\hat{t})}{d\overline{S}} = \frac{\rho r\left(h\left(\overline{S}\right) - h'\left(\overline{S}\right)\overline{S}\right)}{\left(h\left(\overline{S}\right)\right)^2} > 0$$

where $h\left(\overline{S}\right) - h'\left(\overline{S}\right)\overline{S} > 0$ by the concavity of $h(\cdot)$. Thus, increasing the stringency of the FAR regulation by decreasing $\overline{S}$ will lead to a lower housing rent at the time of development. A lower housing rent in our context corresponds to an earlier point in time, which implies that development will take place at an earlier time.

Figure 1.2 illustrates the foregoing.

Now, given the optimal choice for the timing of construction, the land value is

$$V\left(\hat{t}\left(\overline{S}\right);\overline{S}\right)=\int_{\hat{t}}^{\infty}p\left(\tau\right)h\left(\overline{S}\right)e^{-\rho\tau}d\tau-r\overline{S}e^{-\rho\hat{t}} \tag{3.5}$$

The derivative of land value with respect to the maximum-allowed FAR is

$$\frac{d\hat{V}}{d\overline{S}}=V_{\overline{S}}+V_{\hat{t}}\frac{d\hat{t}}{d\overline{S}} \tag{3.6}$$

where $\hat{V}\equiv V\left(\hat{t}\left(\overline{S}\right);\overline{S}\right)$. But since by the Envelope Theorem $V_{\hat{t}}=0$, we have

$$\begin{aligned}\frac{d\hat{V}}{d\overline{S}}&=V_{\overline{S}}\\&=h'\left(\overline{S}\right)\int_{\hat{t}}^{\infty}p\left(\tau\right)e^{-\rho\tau}d\tau-re^{-\rho\hat{t}}>0\end{aligned} \tag{3.7}$$

so that the FAR regulation decreases the land value. The elasticity of land value with respect to the maximum-allowed FAR is

$$E_{V:\overline{S}}\equiv\frac{V_{\overline{S}}\overline{S}}{V}=\frac{\overline{S}h'\left(\overline{S}\right)\int_{\hat{t}}^{\infty}p\left(\tau\right)e^{-\rho\tau}d\tau-r\overline{S}e^{-\rho\hat{t}}}{h\left(\overline{S}\right)\int_{\hat{t}}^{\infty}p\left(\tau\right)e^{-\rho\tau}d\tau-r\overline{S}e^{-\rho\hat{t}}} \tag{3.8}$$

Once again, as in Brueckner's model, we can see that this is less than one, by the concavity of $h\left(\cdot\right)$.

Next, we investigate the effect of increasing the stringency of the FAR regulation on the elasticity of land value with respect to the maximum-allowed FAR. The effect of changing the maximum-allowed FAR on this elasticity is

$$\frac{dE_{V:\overline{S}}}{d\overline{S}}=\frac{d^2V}{d\overline{S}^2}\left(\frac{\overline{S}}{V}\right)+\frac{dV}{d\overline{S}}\left(\frac{1}{V}\right)\left(1-E_{V:\overline{S}}\right) \tag{3.9}$$

where

$$\begin{aligned}\frac{d^2V}{d\overline{S}^2}=&\frac{\partial}{\partial\overline{S}}\left(\frac{dV}{d\overline{S}}\right)+\frac{\partial}{\partial\hat{t}}\left(\frac{dV}{d\overline{S}}\right)\frac{d\hat{t}}{d\overline{S}}\\=&V_{\overline{S}\overline{S}}+V_{\hat{t}\overline{S}}\frac{d\hat{t}}{d\overline{S}}+V_{\hat{t}}\frac{\partial}{\partial\overline{S}}\left(\frac{d\hat{t}}{d\overline{S}}\right)\end{aligned} \tag{3.10}$$

$$
\begin{aligned}
&+\left[V_{\overline{S}\hat{t}}+V_{\hat{t}\hat{t}}\frac{d\hat{t}}{d\overline{S}}+V_{\hat{t}}\frac{\partial}{\partial\hat{t}}\left(\frac{d\hat{t}}{d\overline{S}}\right)\right]\frac{d\hat{t}}{d\overline{S}} &\quad (3.11)\\
=&V_{\overline{S}\overline{S}}+V_{\hat{t}\overline{S}}\frac{d\hat{t}}{d\overline{S}}\\
&+\left[V_{\overline{S}\hat{t}}+V_{\hat{t}\hat{t}}\frac{d\hat{t}}{d\overline{S}}\right]\frac{d\hat{t}}{d\overline{S}} &\quad (3.12)
\end{aligned}
$$

Under general functional forms and the assumption that rents are growing at a constant rate, g, (3.9) becomes[1]

$$\frac{dE_{V:\overline{S}}}{d\overline{S}}=\frac{\rho h''\overline{S}(1-\sigma)}{hg} \quad (3.13)$$

where $\sigma\equiv-\frac{h'\left(h-h'\overline{S}\right)}{hh''\overline{S}}$ is the elasticity of substitution between land and capital inputs for the production of housing. For $\sigma<1$, the above is negative[2], since $h''<0$.

This indicates that holding the profit-maximizing free market FAR constant, the elasticity would be higher where the land-use regulation is more stringent. In other words, tightening the stringency of the FAR regulation would decrease land values more in areas where the regulation is initially more stringent. If we assume a Cobb-Douglas production function as in the previous literature, $\frac{dE_{V:\overline{S}}}{d\overline{S}}=0$, and the elasticity would be constant for all levels of FAR regulation stringency.

We now turn to the general model to explore how the analysis extends when there are multiple conversions.

[1] See Appendix 1 for proof

[2] The empirical literature suggests that the elasticity of substitution between land and non-land inputs for the production of housing is less than one. For example, see Thorsnes (1997).

4 Baseline Model

Notational Glossary	
$E_{V:\overline{S}}$	Elasticity of land value with respect to the maximum allowed density
g	Growth rate of rent on housing
$h(S)$	Output of housing per unit of land (FAR)
$\overline{h}$	$\left(\equiv h\left(\overline{S}\right)\right)$ The maximum allowed density of housing
J	Index for the final building conversion
$p(t)$	Rent on housing at time t per unit of housing
r	Price of a unit of capital
S_j	Capital investment in housing per unit of land at the jth conversion
S_j^*	Optimal density in the absence of FAR regulation
$\overline{S}$	Maximal allowed capital investment in housing
$\hat{S}_j$	Optimal density under FAR regulation
$\tilde{S}_J$	$\left(\equiv \frac{\overline{S}}{S_J^*}\right)$ The reciprocal of stringency of FAR regulation
t_j	Time at the jth conversion
t_j^*	Optimal timing of development in the absence of FAR regulation
$\hat{t}_j$	Optimal timing of development under FAR regulation
V	Value per unit of land
$\hat{V}$	Optimized land value under FAR regulation
β	Output elasticity of land
ρ	Discount rate

This section derives the baseline model describing the relationship between density regulation and land values. We utilize a partial equilibrium model to isolate and analyze the effect of a density regulation on the optimal development program on a single plot of land. A general equilibrium model may show that a density restriction effectively decreases the total supply of land available for development in the form of airspace, thus leading to an overall increase in land value. However, the goal of our model is to show that the immediate effect of a density restriction on a single plot of land - holding the density of all neighboring plots constant - is to lower the value of that plot of land. This allows us to focus on the cost of the FAR restriction to the developer.

Consider a competitive urban land development market with rising housing rent. The problem facing the developer with perfect foresight is to choose the optimal timing and the density for the initial transition of land from rural to urban use and the stream of subsequent building conversions. The rent on housing is determined competitively and is dependent on its locational characteristics.

The following are the assumptions used throughout this paper:

Assumptions

Assumption 1. *There is no depreciation of housing*

Assumption 2. *Demolition for the purpose of reconstruction is costless.*

Assumption 3. *The housing redevelopment market is competitive.*

Assumption 4. *The developer has perfect foresight with respect to future rent on buildings and the FAR restriction.*

Assumption 5. *The rent function is unbounded, is non-decreasing in time, and has the property that* $p(t) \leq p(0)e^{gt}$ *for some* $g < \rho$.

The housing developer's profit-maximization problem can be stated as

$$\max_{\{t_j,S_j\}} V(t_1,S_1,\cdots) = \sum_{j=1}^{\infty}\left\{\int_{t_j}^{t_{j+1}} p(\tau)h(S_j)e^{-\rho\tau}d\tau - rS_je^{-\rho t_j}\right\} \tag{4.1}$$

where V is the value per unit of land, $p(\cdot)$ is the rent per unit of housing, S_j is the units of capital invested in the jth future conversion, $h(\cdot)$ is the amount of housing per unit of land (FAR), and r is the price per unit of capital. Note that this value is intermittently calculated between conversions, when the previous building is torn down and the next building is erected. Thus, the jth building's contribution to the total land value is calculated when the plot is vacant. Furthermore, this specification of the model implicitly defines the developer's problem as developing a plot of land which is initially vacant at the beginning of time, denoted t_0.[3] As we shall see in the next section, it is prior to this time t_0 when the density restriction is imposed, thus making the restriction an anticipated regulation.

[3]If we wish to model a scenario where there exists a structure at the beginning of time, the profit-maximization problem would become

$$\max_{\{t_j,S_j\}} V(S_0,t_1,S_1,\cdots) = \int_{t_0}^{t_1} p(\tau)h(S_0)e^{-\rho\tau}d\tau + \sum_{j=1}^{\infty}\left\{\int_{t_j}^{t_{j+1}} p(\tau)h(S_j)e^{-\rho\tau}d\tau - rS_je^{-\rho t_j}\right\} \tag{4.2}$$

where $S_0 > 0$ is the density of the initial building. In this paper, we shall assume the former land value function so that the land to be developed is initially vacant.

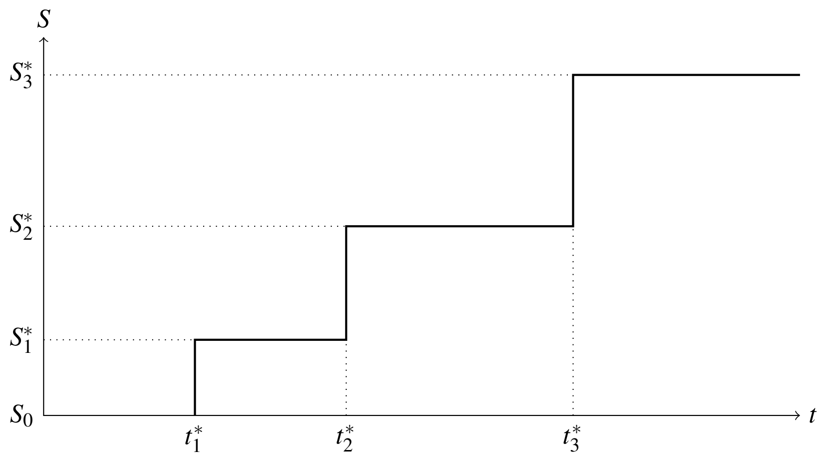


Figure 1.3: Sequential Development

Figure 1.3 illustrates the sequence of building conversions for the optimized development program. The first-order condition for the choice of timing of conversion is

$$(h(S_j) - h(S_{j-1}))\, p(t_j) = \rho r S_j \tag{4.3}$$

where $\left\{t_j^*\right\}_{j=1}^{\infty}$ satisfies the set of optimality conditions and $h(S_0) = 0$. Conversion will take place whenever the rent foregone from waiting an extra period is equal to the benefit of putting off construction for one period. Under our assumptions, there will be an infinite sequence of conversions in the absence of a density restriction.

Proposition 1. *Under Assumptions A1-A5, in the absence of a density restriction there will be an infinite sequence of building conversions.*

Proof. (by contradiction) (from Amin & Capozza (1992)). Consider some nth redevelopment at conversion time t_n and density $h(S_n)$, determined to be the final conversion. But for some S such that $h(S) > h(S_n)$, the variable cost of conversion from density $h(S_n)$ to density $h(S)$ is given by rS. Then, it will be profitable to build to the higher density S if

$$(h(S) - h(S_n)) \int_t^{\infty} p(\tau)\, e^{-\rho\tau} d\tau \geq rS \tag{4.4}$$

Since by Assumption 5 $p(t)$ is a non-decreasing, unbounded function of t, the above equation will eventually be satisfied. Therefore, without any restriction, housing density will increase over time indefinitely. □

The first-order condition for the choice of housing density is

$$re^{-\rho t_j} = h'(S_j) \int_{t_j}^{t_{j+1}} p(\tau) e^{-\rho\tau} d\tau \tag{4.5}$$

with $\left\{S_j^*\right\}_{j=1}^{\infty}$ satisfying the set of optimality conditions. Each structure will be built up to the density where the price a unit of capital is equal to the present value of the marginal revenue from the investment of capital[4].

5 The Effects of an Increase in the Stringency of a FAR Regulation

We now extend the baseline model to treat FAR regulation. Suppose that there is a FAR regulation imposed at some time prior to time t_0 which restricts the density of any future structure to $\overline{h} \equiv h\left(\overline{S}\right)$ for some $\overline{S} > 0$. By Proposition 1, housing density will continue to increase until the FAR restriction is binding for some Jth and final conversion, so that $h\left(\overline{S}\right) < h\left(S_J^*\right)$ (where S_J^* is the unrestricted profit-maximizing density for the Jth conversion). The value function for land now becomes

$$V\left(t_1, S_1, \cdots, t_J, \overline{S}\right) = \sum_{j=1}^{J-1} \left\{ \int_{t_j}^{t_{j+1}} p(\tau) h(S_j) e^{-\rho\tau} d\tau - rS_j e^{-\rho t_j} \right\} + \int_{t_J}^{\infty} p(\tau) h\left(\overline{S}\right) e^{-\rho\tau} d\tau - r\overline{S} e^{-\rho t_J}$$

so that the Jth building is the final structure erected. Thus, there is now a finite number of building conversions if a density regulation is introduced. With this restricted land value function, the set of first-order conditions with respect to the timing of conversion is

$$\begin{cases} \left(h(S_j) - h(S_{j-1})\right) p(t_j) = \rho r S_j & \text{if } j < J \\ \left(h\left(\overline{S}\right) - h(S_{J-1})\right) p(t_J) = \rho r \overline{S} & \text{if } j = J \end{cases} \tag{5.1}$$

and the set of first-order conditions with respect to the density of conversion is

$$\begin{cases} re^{-\rho t_j} = h'(S_j) \int_{t_j}^{t_{j+1}} p(\tau) e^{-\rho\tau} d\tau & \text{if } j < J \\ re^{-\rho t_J} < h'\left(\overline{S}\right) \int_{t_J}^{\infty} p(\tau) e^{-\rho\tau} d\tau & \text{if } j = J \end{cases} \tag{5.2}$$

[4] The second partial derivatives satisfy the conditions for the solutions to be local maxima, as the land value function is a sum of concave functions with derivatives at the critical points being equal to zero.

with $\{\hat{t}_j\}_{j=1}^{J}$ and $\{\hat{S}_j\}_{j=1}^{J-1}$ satisfying the optimality conditions, respectively. Note that the second equation of (5.2) is an inequality, because the final structure is built at a density which is less than optimal. Thus, the marginal revenue from an extra unit investment of capital exceeds the marginal cost of an extra unit of investment of capital.

Given the first-order conditions above, we can write the set of comparative statics equations given the optimized values of timing and density in matrix form as

$$\begin{bmatrix} V_{\hat{t}_1\hat{t}_1} & V_{\hat{t}_1\hat{S}_1} & 0 & \cdots & 0 \\ V_{\hat{S}_1\hat{t}_1} & V_{\hat{S}_1\hat{S}_1} & V_{\hat{S}_1\hat{t}_2} & \ddots & 0 \\ 0 & V_{\hat{t}_2\hat{S}_1} & V_{\hat{t}_2\hat{t}_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & V_{\hat{S}_{J-1}\hat{t}_J} \\ 0 & 0 & \cdots & V_{\hat{t}_J\hat{S}_{J-1}} & V_{\hat{t}_J\hat{t}_J} \end{bmatrix} \begin{bmatrix} \frac{d\hat{t}_1}{d\overline{S}} \\ \frac{d\hat{S}_1}{d\overline{S}} \\ \frac{d\hat{t}_2}{d\overline{S}} \\ \frac{d\hat{S}_2}{d\overline{S}} \\ \vdots \\ \frac{d\hat{t}_J}{d\overline{S}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -V_{\hat{t}_J\overline{S}} \end{bmatrix} \tag{5.3}$$

We can now show that all else equal, reducing the density limit so that the FAR regulation is more stringent hastens development and lowers the density of all sequence of buildings, given that the change in the regulation does not change the number of conversions.

Proposition 2. *Under Assumptions A1-A5, a marginal increase in the stringency of the FAR regulation hastens development for all building conversions.*

Proof. See Appendix 2 □

Proposition 3. *Under Assumptions A1-A5, a marginal increase in the stringency of the FAR regulation lowers the density of development for all building conversions.*

Proof. See Appendix 3 □

Figure 1.4 illustrates the foregoing two Propositions, overlayed on Figure 1.3.

We now turn to the analysis of the FAR regulation's impact on land value. The developer's optimized program yields a land value of

$$\hat{V} = \sum_{k=1}^{J-1} \left\{ \int_{\hat{t}_k}^{\hat{t}_{k+1}} p(\tau)\, h\left(\hat{S}_k\right) e^{-\rho\tau} d\tau - r\hat{S}_k e^{-\rho\hat{t}_k} \right\} + \int_{\hat{t}_J}^{\infty} p(\tau)\, h\left(\overline{S}\right) e^{-\rho\tau} d\tau - r\overline{S} e^{-\rho\hat{t}_J} \tag{5.4}$$

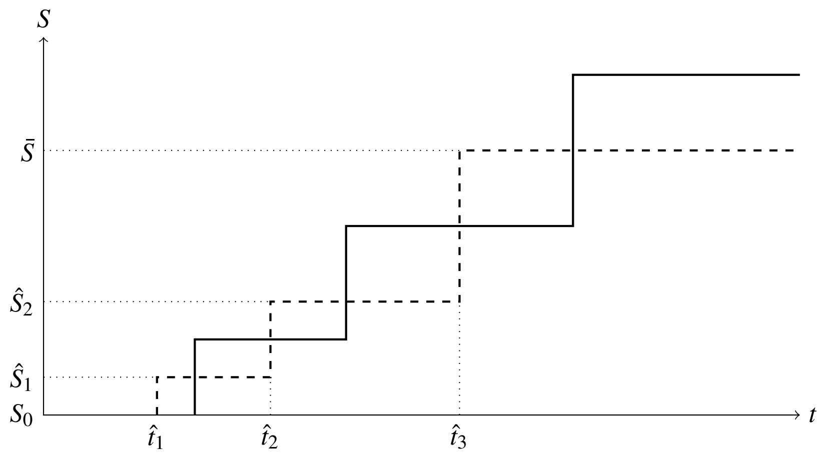


Figure 1.4: Sequential Development with FAR Restriction

where $\hat{V} \equiv V\left(\hat{t}_1\left(\overline{S}\right), \hat{S}_1\left(\overline{S}\right), \cdots, \hat{t}_J\left(\overline{S}\right); \overline{S}\right)$ is the maximized land value under the constraint. The derivative of the above with respect to the maximum-allowed FAR is

$$\frac{d\hat{V}}{d\overline{S}} = V_{\overline{S}} + \sum_{k=1}^{J} V_{\hat{t}_k} \frac{d\hat{t}_k}{d\overline{S}} + \sum_{l=1}^{J-1} V_{\hat{S}_l} \frac{d\hat{S}_l}{d\overline{S}} \tag{5.5}$$

where the subscripts on V denote partial derivatives. Thus, an infinitesimal change in $\overline{h}$ affects the timing and density of development for all building conversions prior to the one where the restriction binds. While it is possible for a change in $\overline{S}$ to lead to a change in the number of conversions (from J cycles to $J-1$ or $J+1$ cycles), we shall only treat those cases where the number of conversions remains the same after a change in the FAR restriction.

Since by the Envelope Theorem $\frac{\partial V}{\partial \hat{t}_k} = 0$ and $\frac{\partial V}{\partial \hat{S}_l} = 0 \; \forall \; k \leq J \;\&\; l \leq J-1$, we have

$$\begin{aligned} \frac{d\hat{V}}{d\overline{S}} &= V_{\overline{S}} \\ &= h'\left(\overline{S}\right) \int_{\hat{t}_J}^{\infty} p(\tau) e^{-\rho\tau} d\tau - re^{-\rho\hat{t}_J} > 0 \end{aligned} \tag{5.6}$$

That is, the changes in the timing and density of building conversions where the restriction do not bind does not have a first-order effect on land value. Nevertheless, the FAR regulation lowers the land value overall. As the FAR regulation is anticipated at the onset, all future development is impacted. Since land value is dependent not only on the current best use of the land, but also on its future development potential, a lowered future development potential in the form of a FAR restriction which will be binding in the future will lower the current value of land.

If there is a marginal change in the FAR regulation, those plots where the FAR regulation was initially more stringent will experience a greater loss in land value. This can be shown by proving that $\frac{d^2\hat{V}}{d\overline{S}^2} < 0$:

Proposition 4. *Under Assumptions A1-A5, a continued increase in the stringency of the FAR regulation will have an increasing effect on the decrease in land value.*

Proof. See Appendix 4 □

In the companion to this paper, we estimate the effect of a percent change in the FAR restriction to the percent change in land value. The following Proposition provides a theoretical basis for the expected estimated value of the empirical exercise:

Proposition 5. *Under Assumptions A1-A5, the elasticity of land value with respect to the maximum-allowed FAR, defined by* $E_{V:\overline{S}} \equiv \frac{V_{\overline{S}}\overline{S}}{V}$ *is less than one.*

Proof. Consider the elasticity of land value with respect to the maximum-allowed capital investment:

$$E_{\hat{V}:\overline{S}} \equiv \frac{V_{\overline{S}}\overline{S}}{\hat{V}} = \frac{\overline{S}h'\left(\overline{S}\right)\int_{\hat{t}_J}^{\infty} p\left(\tau\right)e^{-\rho\tau}d\tau - r\overline{S}e^{-\rho\hat{t}_J}}{\sum_{k=1}^{J-1}\left\{\int_{\hat{t}_k}^{\hat{t}_{k+1}} p\left(\tau\right)h\left(\hat{S}_k\right)e^{-\rho\tau}d\tau - r\hat{S}_k e^{-\rho\hat{t}_k}\right\} + h\left(\overline{S}\right)\int_{\hat{t}_J}^{\infty} p\left(\tau\right)e^{-\rho\tau}d\tau - r\overline{S}e^{-\rho\hat{t}_J}} = \frac{A}{B+C} \tag{5.7}$$

where

$$A \equiv \overline{S}h'\left(\overline{S}\right)\int_{\hat{t}_J}^{\infty} p\left(\tau\right)e^{-\rho\tau}d\tau - r\overline{S}e^{-\rho\hat{t}_J} \tag{5.8}$$

$$B \equiv \sum_{k=1}^{J-1}\left\{\int_{\hat{t}_k}^{\hat{t}_{k+1}} p\left(\tau\right)h\left(\hat{S}_k\right)e^{-\rho\tau}d\tau - r\hat{S}_k e^{-\rho\hat{t}_k}\right\} \tag{5.9}$$

$$C \equiv h\left(\overline{S}\right)\int_{\hat{t}_J}^{\infty} p\left(\tau\right)e^{-\rho\tau}d\tau - r\overline{S}e^{-\rho\hat{t}_J} \tag{5.10}$$

Since h is concave, we know that $h'\left(\overline{S}\right)\overline{S} < h\left(\overline{S}\right)$ and $A < C$. Furthermore, since $B > 0$ as it is the value of land for the first $J-1$ conversions, the elasticity of land value with respect to the capital investment limit is less than one.

□

In Brueckner's model, it was proved that the this elasiticty is greater in areas where the regulation is more stringent. This effect can be analyzed by taking the derivative of the elasticity with respect to the maximum-allowed FAR:

$$\frac{dE_{\hat{V}:\overline{S}}}{d\overline{S}} = \frac{d}{d\overline{S}}\left(\frac{d\hat{V}}{d\overline{S}}\frac{\overline{S}}{\hat{V}}\right) \tag{5.11}$$

$$= \frac{d^2\hat{V}}{d\overline{S}^2}\left(\frac{\overline{S}}{\hat{V}}\right) + \frac{d\hat{V}}{d\overline{S}}\left(\frac{\hat{V} - \overline{S}\frac{d\hat{V}}{d\overline{S}}}{\hat{V}^2}\right) \tag{5.12}$$

$$= \frac{d^2\hat{V}}{d\overline{S}^2}\left(\frac{\overline{S}}{\hat{V}}\right) + \frac{d\hat{V}}{d\overline{S}}\left(\frac{1}{\hat{V}}\right)\left(1 - E_{\hat{V}:\overline{S}}\right) \tag{5.13}$$

If (5.11) were negative, it would indicate that the elasticity of the value of land today with respect to the restricted FAR is higher the more stringent the regulation. While we know that $\frac{d^2\hat{V}}{d\overline{S}^2} < 0$, it is nontrivial to prove that the whole expression above is negative. We leave this proof as a conjecture, based on the results from previous literature and some preliminary work by the author to derive sufficient conditions for which (5.11) is negative.

Conjecture 1. *Under Assumptions A1-A5, the elasticity of land value with respect to the maximum-allowed FAR will be greater in areas with more stringent regulation.*

6 Conclusion

We have presented a model which sheds new light on the effects of FAR regulation in the development of a growing city with rising rents. The model predicts that when the stringency of the FAR regulation increases without changing the total number of conversions, buildings will be erected at an earlier time at lower densities. The FAR regulation will also lower land values at an increasing rate as the regulation becomes more stringent. In the case that there is only one conversion of land from rural to urban use at which the FAR regulation is binding, the percent amount by which land value is decreased from an increase in the stringency of regulation depends on the elasticity of substitution between land and capital for the production of housing. If we assume a Cobb-Douglas production function as in the previous literature, the elasticity of land value with respect to the maximum-allowed FAR will be constant at all levels of regulation stringency.

A limitation to our model is that once multiple conversions are allowed, it is nontrivial to obtain a closed-form solution for the derivative, $\frac{dE_{\hat{V}:\overline{S}}}{d\overline{S}}$, as done in the existing literature. A possible extension to our model which we did not consider is to allow for uncertainty in the change in housing rents. We could modify the model to allow for uncertainty by replacing the developer's perfect foresight with rational expectations and risk neutrality. Despite these limitations, our model provides useful insight into the partial equilibrium effects of FAR regulation and should provide a useful guideline for explaining development patterns in growing cities with FAR regulations.

Appendix 1

Intermediate steps for §3

Here we show the intermediate steps to find the solution (3.13) for the Illustrative Example.

If the maximum-allowed FAR is binding during the initial construction, the land value is a function of $\hat{t}$ so that $V = V\left(\hat{t},\overline{S}\right)$. By the Envelope Theorem, we know $V_{\hat{t}} = 0$. Since this holds for every value of $\overline{S}$, we totally differentiate the above equation and obtain

$$
\begin{aligned}
V_{\hat{t}\hat{t}}d\hat{t} + V_{\hat{t}\overline{S}}d\overline{S} &= 0 \\
\Rightarrow \frac{d\hat{t}}{d\overline{S}} &= -\frac{V_{\hat{t}\overline{S}}}{V_{\hat{t}\hat{t}}} \qquad \text{(A1.1)}
\end{aligned}
$$

We want to find

$$\frac{dE_{\hat{V}:\overline{S}}}{d\overline{S}} = \frac{d^2\hat{V}}{d\overline{S}^2}\left(\frac{\overline{S}}{\hat{V}}\right) + \frac{d\hat{V}}{d\overline{S}}\left(\frac{1}{\hat{V}}\right)\left(1 - E_{\hat{V}:\overline{S}}\right) \qquad \text{(A1.2)}$$

First, we know that

$$
\begin{aligned}
\frac{d^2\hat{V}}{d\overline{S}^2} =& V_{\overline{S}\overline{S}} + V_{\hat{t}\overline{S}}\frac{d\hat{t}}{d\overline{S}} + \left[V_{\overline{S}\hat{t}} + V_{\hat{t}\hat{t}}\frac{d\hat{t}}{d\overline{S}}\right]\frac{d\hat{t}}{d\overline{S}} \qquad \text{(A1.3)} \\
=& V_{\overline{S}\overline{S}} - V_{\hat{t}\overline{S}}\frac{V_{\hat{t}\overline{S}}}{V_{\hat{t}\hat{t}}} - V_{\overline{S}\hat{t}}\frac{V_{\hat{t}_1\overline{S}}}{V_{\hat{t}\hat{t}}} + V_{\hat{t}\hat{t}}\left(\frac{V_{\hat{t}\overline{S}}}{V_{\hat{t}\hat{t}}}\right)^2 \qquad \text{(A1.4)} \\
=& V_{\overline{S}\overline{S}} + V_{\hat{t}\overline{S}}\frac{d\hat{t}_1}{d\overline{S}} < 0 \qquad \text{(A1.5)}
\end{aligned}
$$

Substituting this into $(A1.2)$ yields

$$
\begin{aligned}
\frac{dE_{\hat{V}:\overline{S}}}{d\overline{S}} &= \left(\frac{V_{\overline{S}\overline{S}}V_{\hat{t}\hat{t}} - \left(V_{\hat{t}\overline{S}}\right)^2}{V_{\hat{t}\hat{t}}}\right)\left(\frac{\overline{S}}{V}\right) + V_{\overline{S}}\left(\frac{1}{\hat{V}}\right)\left(1 - \frac{V_{\overline{S}}\overline{S}}{\hat{V}}\right) \qquad \text{(A1.6)} \\
&= \left(\frac{V_{\overline{S}\overline{S}}V_{\hat{t}\hat{t}} - \left(V_{\hat{t}\overline{S}}\right)^2}{V_{\hat{t}\hat{t}}}\right)\left(\frac{\overline{S}}{\hat{V}}\right) + \frac{V_{\overline{S}}}{\hat{V}} - \frac{\left(V_{\overline{S}}\right)^2\overline{S}}{\hat{V}^2} \qquad \text{(A1.7)}
\end{aligned}
$$

Under a common denominator, the above becomes

$$\frac{dE_{\hat{V}:\overline{S}}}{d\overline{S}} = \frac{V_{\overline{S}\overline{S}}V_{\hat{t}\hat{t}}\overline{S}\hat{V} - \left(V_{\hat{t}\overline{S}}\right)^2\overline{S}V}{\hat{V}^2V_{\hat{t}\hat{t}}} + \frac{V_{\overline{S}}V_{\hat{t}\hat{t}}\hat{V} - V_{\hat{t}\hat{t}}\left(V_{\overline{S}}\right)^2\overline{S}}{\hat{V}^2V_{\hat{t}\hat{t}}} \qquad \text{(A1.8)}$$

$$= \frac{\overbrace{\left(V_{\overline{S}\overline{S}}V_{\hat{t}\hat{t}} - \left(V_{\hat{t}\overline{S}}\right)^2\right)\overline{S}\hat{V}}^{(+)}}{\underbrace{\hat{V}^2 V_{\hat{t}\hat{t}}}_{(-)}} + \frac{\overbrace{\left(\hat{V} - V_{\overline{S}}\overline{S}\right)V_{\overline{S}}V_{\hat{t}\hat{t}}}^{(-)}}{\underbrace{\hat{V}^2 V_{\hat{t}\hat{t}}}_{(-)}} \tag{A1.9}$$

Under a general functional form for the production function and the assumption that rents are growing at a constant rate, we have

$$\hat{V} = h\left(\overline{S}\right)\frac{p(0)\,e^{-(\rho-g)\hat{t}}}{\rho - g} - r\overline{S}e^{-\rho\hat{t}} \tag{A1.10}$$

from which we derive

$$V_{\overline{S}} = h'\frac{p(0)\,e^{-(\rho-g)\hat{t}}}{\rho - g} - re^{-\rho\hat{t}} \tag{A1.11}$$

$$V_{\overline{S}\overline{S}} = h''\frac{p(0)\,e^{-(\rho-g)\hat{t}}}{\rho - g} \tag{A1.12}$$

$$V_{\overline{S}\hat{t}} = -h'p(0)\,e^{-(\rho-g)\hat{t}} + \rho re^{-\rho\hat{t}} \tag{A1.13}$$

$$V_{\hat{t}} = -hp(0)\,e^{-(\rho-g)\hat{t}} + \rho r\overline{S}e^{-\rho\hat{t}} \tag{A1.14}$$

$$V_{\hat{t}\hat{t}} = h(\rho - g)\,p(0)\,e^{-(\rho-g)\hat{t}} - \rho^2 r\overline{S}e^{-\rho\hat{t}} \tag{A1.15}$$

Rewriting $A \equiv p(0)\,e^{-(\rho-g)\hat{t}}$ and substituting 3.4 into the above yields

$$\hat{V} = hA\frac{g}{(\rho - g)\,\rho} \tag{A1.16}$$

$$V_{\overline{S}} = A\left(\frac{h'\rho\overline{S} - h(\rho - g)}{(\rho - g)\,\rho\overline{S}}\right) \tag{A1.17}$$

$$V_{\overline{S}\overline{S}} = h''\frac{A}{\rho - g} \tag{A1.18}$$

$$V_{\overline{S}\hat{t}} = A\left(\frac{-h'\overline{S} + h}{\overline{S}}\right) \tag{A1.19}$$

$$V_{\hat{t}\hat{t}} = (-ghA) \tag{A1.20}$$

Substituting the above into $(A1.9)$, we get

$$\frac{dE_{\hat{V}:\overline{S}}}{d\overline{S}} = \frac{h'' \frac{A}{\rho - g}\overline{S}\left(hA\frac{g}{(\rho-g)\rho}\right)(ghA)}{\left(hA\frac{g}{(\rho-g)\rho}\right)^2(ghA)} + \frac{A\left(\frac{h'\rho\overline{S}-h(\rho-g)}{(\rho-g)\rho\overline{S}}\right)\left(hA\frac{g}{(\rho-g)\rho}\right)(ghA)}{\left(hA\frac{g}{(\rho-g)\rho}\right)^2(ghA)} - \frac{\left(A\left(\frac{h'\rho\overline{S}_1-h(\rho-g)}{(\rho-g)\rho\overline{S}}\right)\right)^2\overline{S}(ghA)}{\left(hA\frac{g}{(\rho-g)\rho}\right)^2(ghA)} + \frac{\left(A\left(\frac{-h'\overline{S}_1+h}{\overline{S}}\right)\right)^2\overline{S}\left(ghA\frac{1}{(\rho-g)\rho}\right)}{\left(hA\frac{g}{(\rho-g)\rho}\right)^2(ghA)} \tag{A1.21}$$

which when simplified can be written as

$$\frac{dE_{V:\overline{S}}}{d\overline{S}} = B \times \Big\{h''\overline{S}\rho\overline{S}hg + \left(h'\overline{S}\rho - h(\rho-g)\right)(hg) - \left(h'\overline{S}\rho - h(\rho-g)\right)^2 + \left(-h'\overline{S}+h\right)^2\rho(\rho-g)\Big\} \tag{A1.22}$$

where

$$B = \frac{ghA\left(\frac{1}{(\rho-g)\rho}\right)^2(A)^2\frac{1}{\overline{\overline{S}}}}{\left(hA\frac{g}{(\rho-g)\rho}\right)^2(ghA)} \tag{A1.23}$$

$$= \frac{1}{\overline{S}(hg)^2} \tag{A1.24}$$

But note that

$$\begin{aligned}\frac{dE_{\hat{V}:\overline{S}}}{d\overline{S}} &= \frac{1}{\overline{S}(hg)^2}\left(h''\overline{S}\rho\overline{S}hg + h'\overline{S}g\left(-h'\overline{S}+h\right)\rho\right)\\ &= \frac{1}{(hg)^2}\rho g\left(h''\overline{S}h - h'h'\overline{S} + hh'\right)\\ &= \frac{1}{(hg)^2}\rho g\left(h''\overline{S}h + \frac{h'\left(h-h'\overline{S}\right)}{hh''\overline{S}}\cdot hh''\overline{S}\right)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(hg)^2} \rho g \left(h'' \overline{S} h - \sigma h h'' \overline{S} \right) \\
&= \frac{\rho h'' \overline{S} (1 - \sigma)}{hg}
\end{aligned} \tag{A1.25}
$$

where $\sigma \equiv -\frac{h'\left(h - h'\overline{S}\right)}{hh''\overline{S}}$ is the elasticity of substitution. For $\sigma < 1$, the above is negative, since $h'' < 0$.

Appendix 2

Proof of Proposition 2

We would like to prove that the introduction of a density regulation hastens development for all building cycles. To do this, we Cramer's rule on the matrix of the set of comparative statics equations to show that $\frac{d\hat{t}_1}{d\bar{S}} > 0$, $\frac{d\hat{t}_k}{d\bar{S}} > 0$ for any $k \in (0, J)$, and $\frac{d\hat{t}_J}{d\bar{S}} > 0$.

The set of comparative statics equations, in a matrix form, is given by

$$\begin{bmatrix} V_{\hat{t}_1\hat{t}_1} & V_{\hat{t}_1\hat{S}_1} & 0 & \cdots & 0 \\ V_{\hat{S}_1\hat{t}_1} & V_{\hat{S}_1\hat{S}_1} & V_{\hat{S}_1\hat{t}_2} & \ddots & 0 \\ 0 & V_{\hat{t}_2\hat{S}_1} & V_{\hat{t}_2\hat{t}_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & V_{\hat{S}_{J-1}\hat{t}_J} \\ 0 & 0 & \cdots & V_{\hat{t}_J\hat{S}_{J-1}} & V_{\hat{t}_J\hat{t}_J} \end{bmatrix} \begin{bmatrix} \frac{d\hat{t}_1}{d\bar{S}} \\ \frac{d\hat{S}_1}{d\bar{S}} \\ \frac{d\hat{t}_2}{d\bar{S}} \\ \frac{d\hat{S}_2}{d\bar{S}} \\ \vdots \\ \frac{d\hat{t}_J}{d\bar{S}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -V_{\hat{t}_J\bar{S}} \end{bmatrix} \tag{A2.1}$$

We show in order that $\frac{d\hat{t}_1}{d\bar{S}} > 0$, $\frac{d\hat{t}_k}{d\bar{S}} > 0$ for any $k \in [1, J]$, and $\frac{d\hat{t}_J}{d\bar{S}} > 0$.

Lemma 1. $\frac{d\hat{t}_1}{d\bar{S}} > 0$.

Proof. Using Cramer's Rule on $(A2.1)$ to find $\frac{d\hat{t}_1}{d\bar{S}}$ yields

$$\frac{d\hat{t}_1}{d\bar{S}} = \frac{\begin{vmatrix} 0 & V_{\hat{t}_1\hat{S}_1} & 0 & \cdots & 0 \\ 0 & V_{\hat{S}_1\hat{S}_1} & V_{\hat{S}_1\hat{t}_2} & \ddots & 0 \\ 0 & V_{\hat{t}_2\hat{S}_1} & V_{\hat{t}_2\hat{t}_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & V_{\hat{S}_{J-1}\hat{t}_J} \\ -V_{\hat{t}_J\bar{S}} & 0 & \cdots & V_{\hat{t}_J\hat{S}_{J-1}} & V_{\hat{t}_J\hat{t}_J} \end{vmatrix}}{\begin{vmatrix} V_{\hat{t}_1\hat{t}_1} & V_{\hat{t}_1\hat{S}_1} & 0 & \cdots & 0 \\ V_{\hat{S}_1\hat{t}_1} & V_{\hat{S}_1\hat{S}_1} & V_{\hat{S}_1\hat{t}_2} & \ddots & 0 \\ 0 & V_{\hat{t}_2\hat{S}_1} & V_{\hat{t}_2\hat{t}_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & V_{\hat{S}_{J-1}\hat{t}_J} \\ 0 & 0 & \cdots & V_{\hat{t}_J\hat{S}_{J-1}} & V_{\hat{t}_J\hat{t}_J} \end{vmatrix}} \tag{A2.2}$$

Applying the Chio pivotal condensation[5] yields

$$\frac{dt_1^*}{d\bar{S}} = \frac{\frac{1}{\left(-V_{\hat{t}_J\bar{S}}\right)^{2J-1-2}} \begin{vmatrix} V_{\hat{t}_J\bar{S}}V_{\hat{t}_1\hat{S}_1} & 0 & \cdots & 0 \\ V_{\hat{t}_J\bar{S}}V_{\hat{S}_1\hat{S}_1} & V_{\hat{t}_J\bar{S}}V_{\hat{S}_1\hat{t}_2} & \ddots & 0 \\ V_{\hat{t}_J\bar{S}}V_{\hat{t}_2\hat{S}_1} & V_{\hat{t}_J\bar{S}}V_{\hat{t}_2\hat{t}_2} & \ddots & \vdots \\ \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & V_{\hat{t}_J\bar{S}}V_{\hat{S}_{J-1}\hat{S}_{J-1}} & V_{\hat{t}_J\bar{S}}V_{\hat{S}_{J-1}\hat{t}_J} \end{vmatrix}}{\det(M)} \tag{A2.3}$$

$$= \frac{\frac{V_{\hat{t}_J\bar{S}}^{2J-2}}{\left(-V_{\hat{t}_J\bar{S}}\right)^{2J-1-2}} \left(V_{\hat{t}_1\hat{S}_1} \cdot V_{\hat{S}_1\hat{t}_2} \cdots V_{\hat{S}_{J-1}\hat{t}_J}\right)}{\det(M)} \tag{A2.4}$$

where the second equality comes from the property of the determinant of a triangular matrix[6] and

$$M \equiv \begin{bmatrix} V_{\hat{t}_1\hat{t}_1} & V_{\hat{t}_1\hat{S}_1} & 0 & \cdots & 0 \\ V_{\hat{S}_1\hat{t}_1} & V_{\hat{S}_1\hat{S}_1} & V_{\hat{S}_1\hat{t}_2} & \ddots & 0 \\ 0 & V_{\hat{t}_2\hat{S}_1} & V_{\hat{t}_2\hat{t}_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & V_{\hat{S}_{J-1}\hat{t}_J} \\ 0 & 0 & \cdots & V_{\hat{t}_J\hat{S}_{J-1}} & V_{\hat{t}_J\hat{t}_J} \end{bmatrix} \tag{A2.5}$$

We now show that $(A2.4)$ is positive. We know that $\det(M) < 0$ since it is a negative-definite Hessian matrix of rank $2J-1$ derived from the developer's profit-maximization problem. The numerator of $(A2.4)$ is negative since the only negative term in the numerator is $\left(-V_{\hat{t}_J\bar{S}}\right)^{2J-1-2}$. Since all other terms are cross partial derivatives of the Hessian matrix, they are positive. Therefore, when the maximum-allowed FAR which will be

[5] Consider a $n \times n$ matrix A. Define a $(n-1) \times (n-1)$ matrix of determinants $B = \left[b_{ij}\right]$ such that $b_{ij} = a_{1,1}a_{i+1,j+1} - a_{1,j+1}a_{i+1,1}$ for $a_{ii} \neq 0$.Then, the Chio pivotal condensation allows us to find $\det(A)$ in terms of $\det(B)$:

$$\det(A) = \frac{\det(B)}{a_{11}^{n-2}}$$

Explicitly,

$$\det(A) = \frac{1}{a_{11}^{n-2}} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{21} & a_{2n} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{31} & a_{3n} \end{vmatrix} \\ \vdots & \vdots & \ddots & \vdots \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{n1} & a_{n2} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{n1} & a_{n3} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{n1} & a_{nn} \end{vmatrix} \end{vmatrix}$$

[6] Let T_n be a triangular matrix of order n. Then, $\det(T_n)$ is equal to the product of all the diagonal elements of T_n.

binding in the future increases while maintaining the original number of building conversions, the construction of the first building is delayed. □

Lemma 2. $\frac{d\hat{t}_k}{d\bar{S}} > 0$ *for any* $k \in [2, J-1]$.

Proof. Using Cramer's Rule on $(A2.1)$ to find $\frac{d\hat{t}_k}{d\bar{S}}$ for any $j \in [2, J-1]$ yields

$$\frac{d\hat{t}_k}{d\bar{S}} = \frac{\begin{vmatrix} V_{\hat{t}_1\hat{t}_1} & V_{\hat{t}_1\hat{S}_1} & \cdots & \overbrace{0}^{(2k-1)\text{th column}} & \cdots & \cdots & 0 \\ V_{\hat{S}_1\hat{t}_1} & V_{\hat{S}_1\hat{S}_1} & \ddots & 0 & \ddots & \ddots & 0 \\ 0 & V_{\hat{t}_2\hat{S}_1} & \ddots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & \ddots & \ddots & V_{\hat{S}_{J-1}\hat{t}_J} \\ 0 & 0 & \cdots & -V_{\hat{t}_J\bar{S}} & \cdots & V_{\hat{t}_J\hat{S}_{J-1}} & V_{\hat{t}_J\hat{t}_J} \end{vmatrix}}{\det(M)} \tag{A2.6}$$

Once again, applying the Chio pivotal condensation yields

$$\frac{d\hat{t}_k}{d\bar{S}} = \frac{\frac{1}{\left(-V_{\hat{t}_J\bar{S}}\right)^{2J-1-2}} \begin{vmatrix} W & X \\ Y & Z \end{vmatrix}}{\det(M)} \tag{A2.7}$$

where W is a $2k \times 2k$ matrix defined by

$$W \equiv \begin{bmatrix} -V_{\hat{t}_J\bar{S}}V_{\hat{t}_1\hat{t}_1} & -V_{\hat{t}_J\bar{S}}V_{\hat{t}_1\hat{S}_1} & \cdots & 0 \\ -V_{\hat{t}_J\bar{S}}V_{\hat{S}_1\hat{t}_1} & -V_{\hat{t}_J\bar{S}}V_{\hat{S}_1\hat{S}_1} & \ddots & 0 \\ 0 & -V_{\hat{t}_J\bar{S}}V_{\hat{t}_2\hat{S}_1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & -V_{\hat{t}_J\bar{S}}V_{\hat{t}_k\hat{S}_k} \\ 0 & \cdots & -V_{\hat{t}_J\bar{S}}V_{\hat{S}_k\hat{t}_k} & -V_{\hat{t}_J\bar{S}}V_{\hat{S}_k\hat{S}_k} \end{bmatrix}$$

X is a $2k \times (2J-2-2k)$ matrix of zeroes, Y is a $(2J-2-2k) \times 2k$ matrix defined by

$$Y \equiv \begin{bmatrix} 0 & 0 & \cdots & -V_{\hat{t}_J\bar{S}}V_{\hat{t}_{k+1}\hat{s}_k} \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

and Z is a $2J-2-2k \times 2J-2-2k$ matrix defined by

$$Z \equiv \begin{bmatrix} V_{\hat{t}_J\bar{S}}V_{\hat{t}_{k+1}\hat{s}_{k+1}} & 0 & \cdots & \cdots & 0 \\ V_{\hat{t}_J\bar{S}}V_{\hat{s}_{k+1}\hat{s}_{k+1}} & V_{\hat{t}_J\bar{S}}V_{\hat{s}_{k+1}\hat{t}_{k+2}} & \ddots & \ddots & 0 \\ 0 & V_{\hat{t}_J\bar{S}}V_{\hat{t}_{k+2}\hat{t}_{k+2}} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & V_{\hat{t}_J\bar{S}}V_{\hat{s}_{J-1}\hat{s}_{J-1}} & V_{\hat{t}_J\bar{S}}V_{\hat{s}_{J-1}\hat{t}_J} \end{bmatrix}$$

Evaluating the determinant of the block triangular matrix[7] and simplifying yields

$$\frac{d\hat{t}_k}{d\overline{S}} = \frac{\frac{1}{\left(-V_{\hat{t}_J\bar{S}}\right)^{2J-1-2}} \det(W) \cdot \det(Z)}{\det(M)} \tag{A2.8}$$

$$= \frac{\frac{1}{\left(-V_{\hat{t}_J\bar{S}}\right)^{2J-1-2}} \left(-V_{\hat{t}_J\bar{S}}\right)^{2k} \det\left(\hat{W}\right) \cdot \left(V_{\hat{t}_J\bar{S}}\right)^{2J-2-2k} \left(V_{\hat{t}_{k+1}\hat{s}_{k+1}} \cdot V_{\hat{s}_{k+1}\hat{t}_{k+2}} \cdots V_{\hat{s}_{J-1}\hat{t}_J}\right)}{\det(M)} \tag{A2.9}$$

where

$$\hat{W} \equiv \begin{bmatrix} V_{\hat{t}_1\hat{t}_1} & V_{\hat{t}_1\hat{s}_1} & \cdots & 0 \\ V_{\hat{s}_1\hat{t}_1} & V_{\hat{s}_1\hat{s}_1} & \ddots & 0 \\ 0 & V_{\hat{t}_2\hat{s}_1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & V_{\hat{t}_k\hat{s}_k} \\ 0 & \cdots & V_{\hat{s}_k\hat{t}_k} & V_{\hat{s}_k\hat{s}_k} \end{bmatrix}$$

[7]The determinant of a block triangular matrix $M \equiv \begin{pmatrix} X & 0 \\ Y & Z \end{pmatrix}$ is the product of the determinants of its diagonal blocks.

We know that $(A2.9)$ is positive, since $\det(M) < 0$, $\left(-V_{\hat{t}_J \bar{S}}\right)^{2J-1-2} < 0$, and $\det\left(\hat{W}\right) > 0$ since $\hat{W}$ is a negative definite matrix of rank $2k$. As in Lemma 1, this shows that increasing the stringency of a FAR restriction which binds in the future but will not change the number of conversions will hasten the development of all buildings prior to the last building cycle. □

Lemma 3. $\frac{d\hat{t}_J}{d\overline{S}} > 0$.

Proof. Using Cramer's Rule on $(A2.1)$ to find $\frac{d\hat{t}_J}{d\overline{S}}$ yields

$$\frac{d\hat{t}_J}{d\overline{S}} = \frac{\begin{vmatrix} V_{\hat{t}_1\hat{t}_1} & V_{\hat{t}_1\hat{S}_1} & 0 & \cdots & 0 \\ V_{\hat{S}_1\hat{t}_1} & V_{\hat{S}_1\hat{S}_1} & V_{\hat{S}_1\hat{t}_2} & \ddots & 0 \\ 0 & V_{\hat{t}_2\hat{S}_1} & V_{\hat{t}_2\hat{t}_2} & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & V_{\hat{t}_J\hat{S}_{J-1}} & -V_{\hat{t}_J\overline{S}} \end{vmatrix}}{\det(M)} \tag{A2.10}$$

Using the recurrence relation for the determinant for a tridiagonal matrix[8] yields

$$\frac{d\hat{t}_J}{d\overline{S}} = \frac{-V_{\hat{t}_J\overline{S}_J}\det(L)}{\det(M)} \tag{A2.11}$$

where

$$L \equiv \begin{bmatrix} V_{\hat{t}_1\hat{t}_1} & V_{\hat{t}_1\hat{S}_1} & 0 & \cdots & 0 \\ V_{\hat{S}_1\hat{t}_1} & V_{\hat{S}_1\hat{S}_1} & V_{\hat{S}_1\hat{t}_2} & \ddots & 0 \\ 0 & V_{\hat{t}_2\hat{S}_1} & V_{\hat{t}_2\hat{t}_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & V_{\hat{t}_{J-1}\hat{S}_{J-1}} \\ 0 & 0 & \cdots & V_{\hat{S}_{J-1}\hat{t}_{J-1}} & V_{\hat{S}_{J-1}\hat{S}_{J-1}} \end{bmatrix} \tag{A2.12}$$

But since matrices L and M are both tridiagonal matrices where $\operatorname{sgn}\det(L) = -\operatorname{sgn}\det(M)$, the above must be postive. That is, when the maximum-allowed FAR increases holding the number of conversions constant, the construction of the final building is delayed. □

[8]Consider a tridiagonal matrix

$$f_n = \begin{bmatrix} a_1 & b_1 & 0 & \cdots & 0 \\ c_1 & a_2 & b_2 & \ddots & 0 \\ 0 & c_2 & a_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & b_{n-1} \\ 0 & 0 & \cdots & c_{n-1} & a_n \end{bmatrix}$$

Then, the determinant of matrix f_n can be computed from a recurrence relation given by

$$f_n = a_n f_{n-1} - c_{n-1} b_{n-1} f_{n-2}$$

Appendix 3

Proof of Proposition 3

We would like to prove that the introduction of a FAR regulation reduces the density of buildings for all conversions.

Proof. Consider the set of comparative statics equations as in Appendix 2:

$$\begin{bmatrix} V_{\hat{t}_1\hat{t}_1} & V_{\hat{t}_1\hat{S}_1} & 0 & \cdots & 0 \\ V_{\hat{S}_1\hat{t}_1} & V_{\hat{S}_1\hat{S}_1} & V_{\hat{S}_1\hat{t}_2} & \ddots & 0 \\ 0 & V_{\hat{t}_2\hat{S}_1} & V_{\hat{t}_2\hat{t}_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & V_{\hat{S}_{J-1}\hat{t}_J} \\ 0 & 0 & \cdots & V_{\hat{t}_J\hat{S}_{J-1}} & V_{\hat{t}_J\hat{t}_J} \end{bmatrix} \begin{bmatrix} \frac{d\hat{t}_1}{d\bar{S}} \\ \frac{d\hat{S}_1}{d\bar{S}} \\ \frac{d\hat{t}_2}{d\bar{S}} \\ \frac{d\hat{S}_2}{d\bar{S}} \\ \vdots \\ \frac{d\hat{t}_J}{d\bar{S}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -V_{\hat{t}_J\bar{S}} \end{bmatrix} \tag{A3.1}$$

By Cramer's Rule, we have for any $k \in [1, J-1]$,

$$\frac{d\hat{S}_k}{d\bar{S}} = \frac{\begin{vmatrix} V_{\hat{t}_1\hat{t}_1} & V_{\hat{t}_1\hat{S}_1} & \cdots & \overbrace{0}^{(2k)\text{th column}} & \cdots & \cdots & 0 \\ V_{\hat{S}_1\hat{t}_1} & V_{\hat{S}_1\hat{S}_1} & \ddots & 0 & \ddots & \ddots & 0 \\ 0 & V_{\hat{t}_2\hat{S}_1} & \ddots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & \ddots & \ddots & V_{\hat{S}_{J-1}\hat{t}_J} \\ 0 & 0 & \cdots & -V_{\hat{t}_J\bar{S}} & \cdots & V_{\hat{t}_J\hat{S}_{J-1}} & V_{\hat{t}_J\hat{t}_J} \end{vmatrix}}{\det(M)} \tag{A3.2}$$

Applying the Chio pivotal condensation yields

$$\frac{d\hat{S}_k}{d\bar{S}} = \frac{\frac{1}{\left(-V_{\hat{t}_J\bar{S}}\right)^{2J-1-2}} \begin{vmatrix} W & X \\ Y & Z \end{vmatrix}}{\det(M)} \tag{A3.3}$$

where W is a $(2k-1)\times(2k-1)$ matrix defined by

$$W=\begin{bmatrix} -V_{\hat{t}_J\bar{S}}V_{\hat{t}_1\hat{t}_1} & -V_{\hat{t}_J\bar{S}}V_{\hat{t}_1\hat{S}_1} & \cdots & 0 \\ -V_{\hat{t}_J\bar{S}}V_{\hat{S}_1\hat{t}_1} & -V_{\hat{t}_J\bar{S}}V_{\hat{S}_1\hat{S}_1} & \ddots & 0 \\ 0 & -V_{\hat{t}_J\bar{S}}V_{\hat{t}_2\hat{S}_1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & -V_{\hat{t}_J\bar{S}}V_{\hat{S}_{k-1}\hat{t}_k} \\ 0 & \cdots & -V_{\hat{t}_J\bar{S}}V_{\hat{t}_k\hat{S}_{k-1}} & -V_{\hat{t}_J\bar{S}}V_{\hat{t}_k\hat{t}_k} \end{bmatrix} \tag{A3.4}$$

X is a $(2k-1)\times(2J-2-2k-1)$ matrix of zeroes, Y is a $(2J-2-2k-1)\times(2k-1)$ matrix defined by

$$Y=\begin{bmatrix} 0 & 0 & \cdots & -V_{\hat{t}_J\bar{S}}V_{\hat{S}_k\hat{t}_k} \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix} \tag{A3.5}$$

and Z is a $(2J-2-2k-1)\times(2J-2-2k-1)$ matrix define by

$$Z=\begin{bmatrix} V_{\hat{t}_J\bar{S}}V_{\hat{S}_k\hat{t}_{k+1}} & 0 & \cdots & \cdots & 0 \\ V_{\hat{t}_J\bar{S}}V_{\hat{t}_{k+1}\hat{t}_{k+1}} & V_{\hat{t}_J\bar{S}}V_{\hat{t}_{k+1}\hat{S}_{k+1}} & \ddots & \ddots & 0 \\ 0 & V_{\hat{t}_J\bar{S}}V_{\hat{S}_{k+1}\hat{S}_{k+1}} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & V_{\hat{t}_J\bar{S}}V_{\hat{S}_{J-1}\hat{S}_{J-1}} & V_{\hat{t}_J\bar{S}}V_{\hat{S}_{J-1}\hat{t}_J} \end{bmatrix} \tag{A3.6}$$

Evaluating the determinant of the block triangular matrix and simplifying yields

$$\frac{d\hat{S}_k}{d\bar{S}}=\frac{\frac{1}{\left(-V_{\hat{t}_J\bar{S}}\right)^{2J-1-2}}\det(W)\cdot\det(Z)}{\det(M)} \tag{A3.7}$$

$$=\frac{\frac{1}{\left(-V_{\hat{t}_J\bar{S}}\right)^{2J-1-2}}\left(-V_{\hat{t}_J\bar{S}}\right)^{2k-1}\det\left(\hat{W}\right)\cdot\left(V_{\hat{t}_J\bar{S}}\right)^{2J-2-2k-1}\left(V_{\hat{S}_k\hat{t}_{k+1}}\cdot V_{\hat{t}_{k+1}\hat{S}_{k+1}}\cdots V_{\hat{S}_{J-1}\hat{t}_J}\right)}{\det(M)} \tag{A3.8}$$

where

$$\hat{W} \equiv \begin{bmatrix} V_{\hat{t}_1\hat{t}_1} & V_{\hat{t}_1\hat{S}_1} & \cdots & 0 \\ V_{\hat{S}_1\hat{t}_1} & V_{\hat{S}_1\hat{S}_1} & \ddots & 0 \\ 0 & V_{\hat{t}_2\hat{S}_1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & V_{\hat{S}_{k-1}\hat{t}_k} \\ 0 & \cdots & V_{\hat{t}_k\hat{S}_{k-1}} & V_{\hat{t}_k\hat{t}_k} \end{bmatrix} \tag{A3.9}$$

We know that $(A3.8)$ is positive, since $\det(M) < 0$, $\left(-V_{\hat{t}_J\bar{S}}\right)^{2J-1-2} < 0$, $\left(-V_{\hat{t}_J\bar{S}}\right)^{2k-1} < 0$, and $\det\left(\hat{W}\right) < 0$ since $\hat{W}$ is a negative definite matrix of rank $2k-1$. This shows that a FAR restriction which binds in the future but does not change the number of conversions will reduce the density of development for all buildings prior to the final conversion. □

Appendix 4

Proof of Proposition 4

Proposition 4 states that the second derivative of the optimized land value function with FAR regulation is negative. To show this, we simplify $\frac{d^2\hat{V}}{d\overline{S}^2}$ into a ratio of matrices and show that the sign of the matrix are opposites of each other.

Lemma 4. $\frac{d^2\hat{V}}{d\overline{S}^2} = \frac{\det(N)}{\det(M)}$ *where matrix N is defined to be*

$$N \equiv \begin{bmatrix} V_{\hat{t}_1\hat{t}_1} & V_{\hat{t}_1\hat{S}_1} & 0 & 0 & \cdots & 0 \\ V_{\hat{S}_1\hat{t}_1} & V_{\hat{S}_1\hat{S}_1} & V_{\hat{S}_1\hat{t}_2} & 0 & \ddots & 0 \\ 0 & V_{\hat{t}_2\hat{S}_1} & V_{\hat{t}_2\hat{t}_2} & V_{\hat{t}_2\hat{S}_2} & \ddots & 0 \\ 0 & 0 & V_{\hat{S}_2\hat{t}_2} & V_{\hat{S}_2\hat{S}_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & V_{\hat{t}_J\overline{S}} \\ 0 & 0 & 0 & \cdots & V_{\overline{S}\hat{t}_J} & V_{\overline{S}\overline{S}} \end{bmatrix}$$

and matrix M is defined as in $(A2.5)$.

Proof. Consider the value function optimized under a FAR restriction:

$$\hat{V} = V\left(\hat{t}_1\left(\overline{S}\right), \hat{S}_1\left(\overline{S}\right), \cdots; \overline{S}\right) \tag{A4.1}$$

Given (5.5), the second derivative of the above can be written as

$$\begin{aligned} \frac{d^2\hat{V}}{d\overline{S}^2} = &\left[\sum_{k=1}^{J} V_{\hat{t}_k\hat{t}_1}\frac{d\hat{t}_k}{d\overline{S}} + \sum_{l=1}^{J-1} V_{\hat{S}_l\hat{t}_1}\frac{d\hat{S}_l}{d\overline{S}} + V_{\overline{S}\hat{t}_1}\right]\frac{d\hat{t}_1}{d\overline{S}} \\ &+ \left[\sum_{k=1}^{J} V_{\hat{t}_k\hat{S}_1}\frac{d\hat{t}_k}{d\overline{S}} + \sum_{l=1}^{J-1} V_{\hat{S}_l\hat{S}_1}\frac{d\hat{S}_l}{d\overline{S}} + V_{\overline{S}\hat{S}_1}\right]\frac{d\hat{S}_1}{d\overline{S}} \\ &+ \cdots + \left[\sum_{k=1}^{J} V_{\hat{t}_k\hat{S}_{J-1}}\frac{d\hat{t}_k}{d\overline{S}} + \sum_{l=1}^{J-1} V_{\hat{S}_l\hat{S}_{J-1}}\frac{d\hat{S}_l}{d\overline{S}} + V_{\overline{S}\hat{S}_{J-1}}\right]\frac{d\hat{S}_{J-1}}{d\overline{S}} \\ &+ \left[\sum_{k=1}^{J} V_{\hat{t}_k\hat{t}_J}\frac{d\hat{t}_k}{d\overline{S}} + \sum_{l=1}^{J-1} V_{\hat{S}_l\hat{t}_J}\frac{d\hat{S}_l}{d\overline{S}} + V_{\overline{S}\hat{t}_J}\right]\frac{d\hat{t}_J}{d\overline{S}} \\ &+ \left[\sum_{k=1}^{J} V_{\hat{t}_k\overline{S}}\frac{d\hat{t}_k}{d\overline{S}} + \sum_{l=1}^{J-1} V_{\hat{S}_l\overline{S}}\frac{d\hat{S}_l}{d\overline{S}} + V_{\overline{S}\overline{S}}\right] \end{aligned} \tag{A4.2}$$

Noting that $V_{\hat{t}_i\hat{t}_j} = V_{\hat{t}_j\hat{t}_i} = 0 \quad \forall i \neq j$ and that $V_{\hat{t}_i\hat{S}_j} = V_{\hat{S}_j\hat{t}_i} = 0 \quad \forall j > i \cup i - j \geq 2$ yields

$$\begin{aligned}
\frac{d^2\hat{V}}{d\overline{S}^2} = & \left[V_{\hat{t}_1\hat{t}_1}\frac{d\hat{t}_1}{d\overline{S}} + V_{\hat{S}_1\hat{t}_1}\frac{d\hat{S}_1}{d\overline{S}} + 0 + \cdots + 0\right]\frac{d\hat{t}_1}{d\overline{S}} \\
& + \left[V_{\hat{t}_1\hat{S}_1}\frac{d\hat{t}_1}{d\overline{S}} + V_{\hat{S}_1\hat{S}_1}\frac{d\hat{S}_1}{d\overline{S}} + V_{\hat{t}_2\hat{S}_1}\frac{d\hat{t}_2}{d\overline{S}} + 0 + \cdots + 0\right]\frac{d\hat{S}_1}{d\overline{S}} \\
& + \cdots + \left[0 + \cdots + 0 + V_{\hat{t}_{J-1}\hat{S}_{J-1}}\frac{d\hat{t}_{J-1}}{d\overline{S}} + V_{\hat{S}_{J-1}\hat{S}_{J-1}}\frac{d\hat{S}_{J-1}}{d\overline{S}} + V_{\hat{t}_J\hat{S}_{J-1}}\frac{d\hat{t}_J}{d\overline{S}} + 0\right]\frac{d\hat{S}_{J-1}}{d\overline{S}} \\
& + \left[0 + \cdots + 0 + V_{\hat{S}_{J-1}\hat{t}_J}\frac{d\hat{S}_{J-1}}{d\overline{S}} + V_{\hat{t}_J\hat{t}_J}\frac{d\hat{t}_J}{d\overline{S}} + V_{\overline{S}\hat{t}_J}\right]\frac{d\hat{t}_J}{d\overline{S}} \\
& + \left[0 + \cdots + 0 + V_{\hat{t}_J\overline{S}}\frac{d\hat{t}_J}{d\overline{S}} + V_{\overline{SS}}\right]
\end{aligned} \tag{A4.3}$$

In the matrix form, the above can be written as

$$\begin{aligned}
\frac{d^2\hat{V}}{d\overline{S}^2} = & \begin{bmatrix} \frac{d\hat{t}_1}{d\overline{S}} & \frac{d\hat{S}_1}{d\overline{S}} & \frac{d\hat{t}_2}{d\overline{S}} & \frac{d\hat{S}_2}{d\overline{S}} & \cdots & 1 \end{bmatrix} \\
& \cdot \begin{bmatrix} V_{\hat{t}_1\hat{t}_1} & V_{\hat{t}_1\hat{S}_1} & 0 & 0 & \cdots & 0 \\ V_{\hat{S}_1\hat{t}_1} & V_{\hat{S}_1\hat{S}_1} & V_{\hat{S}_1\hat{t}_2} & 0 & \ddots & 0 \\ 0 & V_{\hat{t}_2\hat{S}_1} & V_{\hat{t}_2\hat{t}_2} & V_{\hat{t}_2\hat{S}_2} & \ddots & 0 \\ 0 & 0 & V_{\hat{S}_2\hat{t}_2} & V_{\hat{S}_2\hat{S}_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & V_{\hat{t}_J\overline{S}} \\ 0 & 0 & 0 & \cdots & V_{\overline{S}\hat{t}_J} & V_{\overline{SS}} \end{bmatrix} \begin{bmatrix} \frac{d\hat{t}_1}{d\overline{S}} \\ \frac{d\hat{S}_1}{d\overline{S}} \\ \frac{d\hat{t}_2}{d\overline{S}} \\ \frac{d\hat{S}_2}{d\overline{S}} \\ \vdots \\ 1 \end{bmatrix} \\
= & \begin{bmatrix} \frac{dV_{\hat{t}_1}}{d\overline{S}} & \frac{dV_{\hat{S}_1}}{d\overline{S}} & \frac{dV_{\hat{t}_2}}{d\overline{S}} & \frac{dV_{\hat{S}_2}}{d\overline{S}} & \cdots & V_{\overline{SS}} + V_{\hat{t}_J\overline{S}}\frac{d\hat{t}_J}{d\overline{S}} \end{bmatrix} \begin{bmatrix} \frac{d\hat{t}_1}{d\overline{S}} \\ \frac{d\hat{S}_1}{d\overline{S}} \\ \frac{d\hat{t}_2}{d\overline{S}} \\ \frac{d\hat{S}_2}{d\overline{S}} \\ \vdots \\ 1 \end{bmatrix}
\end{aligned}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & V_{\overline{SS}} + V_{\hat{t}_J \overline{S}} \frac{d\hat{t}_J}{d\overline{S}} \end{bmatrix} \begin{bmatrix} \frac{d\hat{t}_1}{d\overline{S}} \\ \frac{d\hat{S}_1}{d\overline{S}} \\ \frac{d\hat{t}_2}{d\overline{S}} \\ \frac{d\hat{S}_2}{d\overline{S}} \\ \vdots \\ 1 \end{bmatrix}$$

$$= \left(V_{\overline{SS}} + V_{\hat{t}_J \overline{S}} \frac{d\hat{t}_J}{d\overline{S}} \right) \tag{A4.4}$$

where the third equality holds, since it is a derivative of a parameter along a maximized value function. We can now write $(A4.4)$ as

$$\begin{aligned} \frac{d^2 \hat{V}}{d\overline{S}^2} &= V_{\overline{SS}} + V_{\hat{t}_J \overline{S}} \frac{d\hat{t}_J}{d\overline{S}} \\ &= \frac{\left(V_{\overline{SS}}\right) \det(M)}{\det(M)} - \frac{\left(V_{\hat{t}_J \overline{S}}\right)^2 \det(L)}{\det(M)} \\ &= \frac{\det(N)}{\det(M)} \end{aligned} \tag{A4.7}$$

where the second equality comes from using the recurrence relation for computing the determinant for a tridiagonal matrix and Cramer's Rule on $\frac{d\hat{t}_J}{d\overline{S}}$[9], and the last equality comes from applying the recurrence relation for the determinant of a tridiagonal matrix again.

Since matrices N and M are both negative definite matrices where $\text{rank}(N) = 2J$ and $\text{rank}(M) = 2J - 1$, we know that $\text{sgn}\det(N) = -\text{sgn}\det(M)$. Thus, $(A4.7)$ is negative. □

[9] Appendix 2

References

[1] Amin, K. & D.R. Capozza. (1993). "Sequential Development." *Journal of Urban Economics*, 34(2):142-158.

[2] Arnott, R.J. (1980). "A Simple Urban Growth Model with Durable Housing." *Regional Science and Urban Economics*, 10:53-76.

[3] Arnott, R.J. (1987). "Economic Theory and Housing." *Handbook of Regional and Urban Economics*, 2:959-988.

[4] Arnott, R.J. & F. Lewis. (1979). "The Transition of Land to Urban Use." *Journal of Political Economy*, 87:161-169.

[5] Brueckner, J.K. (1981). "A Dynamic Model of Housing Production." *Journal of Urban Economics*, 10:1-14.

[6] Brueckner, J.K. (1987). "The Structure of Urban Equilibria: A Unified Treatment of the Muth-Mills Model." *Handbook of Regional and Urban Economics*, 2:821-845.

[7] Brueckner, J.K., S. Fu, Y. Gu, and J. Zhang. (2017). "Measuring the Stringency of Land Use Regulation: The Case of China's Building Height Limits." *The Review of Economics and Statistics*, 99(4):663-677.

[8] Brueckner, J.K. & K.S. Sridhar. (2012). "Measuring Welfare Gains from Relaxation of Land-Use Restrictions: The Case of India's Building-Height Limits." *Regional Science and Urban Economics*, 42(6):1061-1067.

[9] Muth, R.F. (1969). *Cities and Housing*. Chicago: The University of Chicago Press.

[10] Moon, B. (forthcoming). "The Effect of FAR (Floor Area Ratio) Regulations on Land Values: The Case of New York." *Papers in Regional Science*.

[11] Thorsnes, P. (1997). "Consistent Estimates of the Elasticity of Subsitution between Land and Non-Land Inputs in the Production of Housing." *Journal of Urban Economics*, 42(1):98-108.

The Impact of FAR Regulation on Development Patterns and Land Values in New York City

Abstract

This paper explores the relationship between a floor area ratio ("FAR") regulation and land value and the relationship between the regulation and development patterns in New York City. Theory predicts that in a dynamic, partial equilibrium setting, strengthening the stringency of regulation will 1) lead to a decrease in land value, even if the regulation may not be binding today and 2) hasten development. Our data shows that all else equal, a lower maximum-allowed FAR is correlated with lower land values and earlier demolition of buildings.

1 Introduction

This paper provides empirical results consistent with theory developed in Ahn (2019) which extends Brueckner's floor area ratio ("FAR") regulation model to a dynamic setting. Theory predicts that all else equal, 1) strengthening the stringency of regulation (by lowering the maximum-allowed FAR) will lead to a decrease in land value, even if the regulation may not be binding today and 2) strengthening the stringency of regulation will hasten development. In this paper, we investigate these predictions in the context of New York City by estimating the elasticity of land value with respect to the maximum-allowed FAR and by regressing building age at the time of demolition on maximum-allowed FAR. To do this, we measure the effects of deviation in FAR regulation from their spatially-adjusted means on the deviations in land values and the timing at which a building is demolished from their spatially adjusted means through fixed effects models. We find that a regulation that may not be binding today nevertheless is correlated with lower land values and that all else equal, a more stringent regulation is correlated with an earlier demolition.

The set of land use regulations in New York City is notorious for its complexity. The first set of zoning regulations were introduced in 1916 and were designed to restrict the height and set standards for the shape of skyscrapers. Over the years, the 1916 Zoning Resolution was continuously amended in order to adjust for the changing economy, increasing population, and the growth of automobile use. By the 1950s, the 1916 Zoning Resolution came to "[resemble] a torn 'patchwork,' reflecting forty-four tumultuous years of technological, social, and physical change" (Marcus, 1992). As a result, when James Felt was appointed as Chairman of the New York City's Planning Commission in 1956, he put rezoning New York City as his top priority in order to rally public support for the City Planning Commission. That year, Felt commissioned the architectural firm of Voorhees, Walker, Smith, and Smith to conduct a zoning study and propose a new resolution. In 1961, this new Zoning Resolution was approved by the New York City's Board of Estimate to replace the 1916 Zoning Resolution to come into effect. Rather than regulating the height and shape of skyscrapers as its predecessor did, the 1961 Zoning Resolution simply restricted the FAR of a building, which gave developers more freedom in the shape of buildings while giving planners more direct control over population density. Unfortunately, some of the problems of the 1916 Zoning Resolution are manifest also in the 1961 Zoning Resolution: in its current form, the text of New York City's 1961 Zoning Resolution, colloquially referred to as the "zoning text," is 4,338 mind-numbing pages establishing zoning districts, regulations governing land use and development, and all the exceptions to the rules.

The typical rationale of a land use regulation such as the 1916 and 1961 Zoning Resolutions of New York City is to deal with various forms of land use externalities. For example, some justification used to advance building height and density restrictions include i) to increase sunlight in the streets; ii) to reduce wind-tunnel effects; iii) to reduce traffic congestion; and iv) to reduce the cost of providing certain public services. Optimal regulation requires that these externalities are internalized, such that the marginal social benefit of strengthening the regulation equals the corresponding marginal social cost. But because many of the benefits of regulation are difficult to quantify, most of the empirical literature on land use regulations in economics has focused on its costs. In particular, the majority of research focuses on measuring the effects of regulation on the overall levels of housing prices and quantities (see Gyourko and Molloy (2015) for a survey on this topic).

In a more recent literature, Brueckner introduced the idea of measuring the *stringency* of land use regulation (Brueckner and Sridhar (2012); Brueckner et al (2017); Brueckner and Singh (2018); Moon (2018)). Measuring the stringency of FAR regulation, as opposed to the cost and magnitude of regulation, allows us to gauge the extent to which the land use regulation causes development patterns and land values to diverge from the theoretical free-market levels in the absence of regulation even if the theoretically optimal levels are unknown, as long as the parameters and the functional forms of the theoretical model are known (see Brueckner and Singh (2018)). In Brueckner's papers, this is accomplished by estimating the elasticity of land value with respect to the maximum-allowed FAR. In particular, he finds that for parcels where the FAR regulation is binding, this elasticity is positive and less than 1 - a marginal increase in the stringency of FAR regulation decreases land value.

In this paper, we test some of the new predictions of a dynamic version of Brueckner's FAR regulation model from its companion paper, Ahn (2019). Our theoretical model analyzes the partial equilibrium effects of a FAR regulation in a growing city with durable structures and rising rents on the profit-maximization problem of a land developer with perfect foresight. The land developer who also owns the parcel to be developed decides on the timing and density of a sequence of structures to be built. While Brueckner's papers show that a binding FAR regulation lowers land values, our model predicts that a FAR regulation that may not be binding today nevertheless affects a parcel's land value by lowering the future development potential on that parcel. Our model also predicts that in a dynamic setting, increasing the stringency of the regulation hastens building conversions. The first-order condition with respect to timing of conversion dictates that a conversion will occur when the marginal rent foregone from postponing development is equal to the corresponding marginal benefit. This occurs at an earlier time if the maximum-allowed FAR is lowered by a marginal amount, given that the number of conversions remains the same. A possible explanation for this result is that since our model utilizes a linear cost function and a concave production function, the marginal benefit of postponing development decreases more than the marginal cost of postponing development when the maximum-allowed FAR is lowered. As a result, it becomes more profitable to build at an earlier date.

Our results contribute to the literature on the economic effects of land use regulations in a few ways. Our first result shows that a more stringent FAR regulation lowers land value even for parcels where the regulation is not binding at the time the data was extracted. This reaffirms the importance of understanding the short-term impact of land use regulations even if the policies may seem to be irrelevant today. Second, we find that all else equal, buildings with a lower maximum-allowed FAR (equivalent to higher stringency) are demolished at an earlier time. Thus, land use regulations both affect land values and alter development patterns[1].

We organize the next sections as follows. In section 2, we present a series of testable predictions of the model, which provides a guide for a specification of the empirical model that we will utilize in this paper. In

[1]Our paper provides evidence of the private cost of FAR regulations, but since the regulations were imposed to deal with perceived negative density externalities, this by itself does not indicate whether or not regulations are socially desirable.

section 3, we discuss how we construct our data. In section 4, we present our empirical strategy and the results. We then provide the concluding remarks in section 5.

2 Testable Predictions of the Model

This section outlines the core results of the theoretical model. For a more detailed exposition on the model, see the companion paper (Ahn (2019)).

Consider a competitive urban land development market in a growing city with durable buildings. Suppose that there is also a FAR regulation imposed at some time before the developer takes possession of the property at $t = 0$, so that the regulation is anticipated in the developer's decisions. In the context of New York City, this would correspond to a developer who purchases a property some time after the 1961 Zoning Resolution has come into effect. Further assume that the developer takes the discount rate to be constant and that he has perfect foresight. The developer is then faced with the problem of choosing the optimal timing and the density for the series of building conversions, until the final one when the FAR limit is binding. Once the FAR limit is binding, the developer has no incentive to redevelop further, as by assumption structures are durable and there is zero depreciation or maintenance costs[2].

Now consider the profit-maximization problem for the developer at $t = 0$ who owns a parcel of land that is subjct to a FAR restriction:

$$\max_{\{t_j,S_j\}} V\left(S_0,t_1,S_1,\cdots;\overline{S}\right) = \int_0^{t_1} p(\tau)\,h(S_0)\,e^{-\rho\tau}d\tau + \sum_{j=1}^{J-1}\left\{\int_{t_j}^{t_{j+1}} p(\tau)\,h(S_j)\,e^{-\rho\tau}d\tau - rS_je^{-\rho t_j}\right\} + \int_{t_J}^{\infty} p(\tau)\,h\left(\overline{S}\right)e^{-\rho\tau}d\tau - r\overline{S}e^{-\rho t_J} \tag{2.1}$$

where V is the value per unit of land, $p(\cdot)$ is the rent per unit floor area, t_j is the year in which the jth building is constructed, S_j is the units of capital invested in the jth future conversion, S_0 is the density of the building already existing on site prior to the developer taking possession of the parcel, $h(\cdot)$ is the sqft of building per unit of land (FAR), r is the price per unit of capital, ρ is the discount rate, $\overline{S}$ is the maximum amount of capital that the developer is able to invest given the FAR regulation, and J is the final building converted when the FAR regulation is binding. The first addend of (2.1) is the income received from the building existing on

[2]With depreciation and/or maintenance costs, the developer may want to eventually redevelop some time in the future. See Arnott et al (1983) and Brueckner (1981) for examples.

the parcel when the developer initially gains possession of the parcel. The second and third addends are the income the developer receives from subsequent conversions of the parcel of land, until the final conversion is made at year t_J when the regulation is binding[3].

We are interested in the partial equilibrium effects that FAR regulation has on land values and development patterns. That is, how are the deviation in land values and timing of development from their spatially adjusted means related to differences in the stringency of FAR regulation? This model provides two propositions to address these issues.

2.1 The Effect of FAR Regulation on Land Values

Proposition 1. *A marginal increase in the stringency of the FAR regulation lowers land value, even if the regulation may not be binding today.*

The reasoning for this Proposition is intuitive. Even if the FAR regulation is not binding today, the anticipation of the regulation binding in the future lowers the development potential of the parcel. A lowered development potential is manifested in the form of a lower profitability of the parcel and therefore its value.

To test this Proposition, we estimate the elasticity of land value with respect to the maximum-allowed FAR. The existing literature shows that, for buildings where the maximum-allowed FAR is already binding, the elasticity of land value with respect to the maximum-allowed FAR is positive (Brueckner et al, 2017). The implication of this result is that a marginal increase in the stringency of regulation will lead to a decrease in land value. Ahn (2019) proves that this result remains true even if the FAR regulation is not binding today. This paper will show that the results from the existing literature remain robust even for parcels where the FAR restriction was not binding at the time the data was collected.

2.2 The Effect of FAR Regulation on the Timing of Development

Proposition 2. *A marginal increase in the stringency of the FAR regulation hastens development for all building conversions.*

[3]For the purpose of our empirical exercise, we construct data on the value of land between conversions since all the plots we consider have previously been developed.

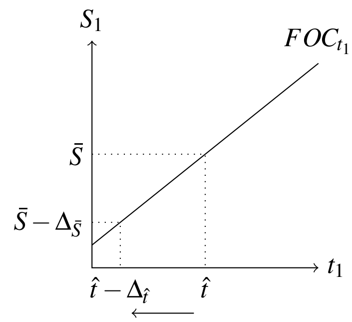


Figure 2.1: Optimal Timing of Conversion on t-S Coordinate Plane

Consider the developer's profit-maximization problem. The set of first-order conditions for this problem with respect to the timing of conversion is

$$\begin{cases} \left(h\left(S_j\right)-h\left(S_{j-1}\right)\right)p\left(t_j\right)=\rho r S_j & \text{if } j<J \\ \left(h\left(\overline{S}\right)-h\left(S_{J-1}\right)\right)p\left(t_J\right)=\rho r\overline{S} & \text{if } j=J \end{cases} \tag{2.2}$$

with $\left\{\hat{t}_j\right\}_{j=1}^{J}$ satisfying the equalities. The set of first-order conditions with respect to the density of conversion is

$$\begin{cases} re^{-\rho t_j}=h'\left(S_j\right)\int_{t_j}^{t_{j+1}}p\left(\tau\right)e^{-\rho\tau}d\tau & \text{if } j<J \\ re^{-\rho t_J}<h'\left(\overline{S}\right)\int_{t_J}^{\infty}p\left(\tau\right)e^{-\rho\tau}d\tau & \text{if } j=J \end{cases} \tag{2.3}$$

with $\left\{\hat{S}_j\right\}_{j=1}^{J-1}$ satisfying the optimality conditions. Given these first-order conditions, a marginal change in $\overline{S}$ creates a chain effect and changes the development patterns of all other conversions. Namely, a marginal change in $\overline{S}$ changes the optimality condition for the timing of the final conversion, which in turn changes the density of the second-to-last conversion, and so on, and our problem can be solved by using backward induction. Ahn (2019) proves that all else equal a marginal decrease in $\overline{S}$ - an increase in the stringency of regulation - will lead to all conversions taking place at earlier times and at lower densities. That is, the paper shows that $\frac{dt_j}{d\overline{S}}>0\;\forall j\in[1,J]$. Thus, if two parcels of land are similar in characteristics but have different FAR restrictions, the jth conversion for the parcel with the lower $\overline{S}$ will occur earlier.

We illustrate the foregoing with Figure 2.1, which plots the set of first-order conditions on a $t-S$ coordinate plane, assuming that the FAR regulation is binding during the first and initial conversion. Suppose that the initial maximum-allowed FAR is given by $\overline{S}$. Then, there is only one first-order condition, FOC_{t_1}, and the

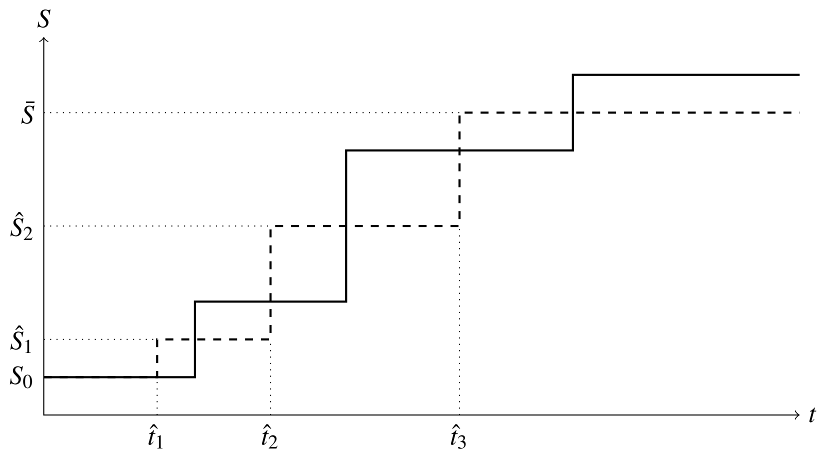


Figure 2.2: Sequential Development with FAR Restriction

optimal timing for conversion is given by $\hat{t}$. Now, if the FAR regulation becomes more stringent so that the maximum-allowed FAR is now $\overline{S} - \Delta_{\overline{S}}$, then the new profit-maximizing timing is found by evaluating FOC_t at $\overline{S} - \Delta_{\overline{S}}$. Since FOC_t has a positive slope, a lower density corresponds to an earlier conversion time.

Figure 2.2 is a visual representation of the change in development patterns because of the FAR regulation. The solid line represents the development pattern in the absense of a FAR restriction, and the dashed line represents the development pattern when a FAR restriction is introduced. Each conversion takes place at an earlier time and at lower densities, until restriction is binding for the final conversion (the third conversion in this example).

3 Data

We construct two sets of data in order to address two aspects of our model: 1) FAR regulation depresses land value for parcels where the regulation is not binding and 2) FAR regulation hastens development.

The first set of data is from Moon (2018). Since we use the same data as in Moon (2018), the restricted sample used in this paper and the sample used in Moon's paper represent the same population. In particular, all of our data comes from New York City's Open Data, which is a joint project between the Mayor's Office of Data Analytics (MODA) and the Department of Information Technology and Telecommunications (DoITT) to provide the public with city related data for free. The core of our data comes from Primary Land Use Tax Lot Output ("PLUTO"), which provides information on every plot of land in the five boroughs of New York. Some of the relevant information that this dataset provides includes lot area, maximum allowable FAR, built FAR, the borough in which the property is located, the year that a property was built, and BBL (borough, tax block, & lot) code. Our extract of the data comes from September 2016 with an initial sample size of 834,182 parcels.

To construct the first set of data, we extract rolling sales records from 2003-2016, provided by the Department of Finance. To clean the data, we remove irregular sales records, some of which may correspond to transactions that were not at arm's length and others to errors, as some properties were transferred at prices as low as $0 or $1. We then match the rolling sales data and the PLUTO data with information on demolition and new building permits from the Department of Buildings. Namely, we reconcile all the extracts by BBL code for each property, then we isolate all properties that were issued both a demolition permit and a new building permit within a two year frame after a sales takes place. This gives us a very accurate measure of vacant land values; a building that is demolished to be converted immediately has a value close to $0, and the value of the parcel that was sold right before demolition wholly consists of vacant land value. Once we construct this data, we are left with a sample size of 2,720. The significant loss in sample size from the original extract is due to the requirement for inclusion in the restricted sample of parcels that the building on the site be demolished and that a redevelopment permit be issued within two years of the demolition. Many buildings in the data were issued a demolition permit without a corresponding new building permit, indicating that some properties may have been purchased to be used as a parking lot or held as a vacant lot for development in the future.

The second set of data is constructed by calculating the age of each building when it is demolished. In particular, we subtract the year in which a building was erected from the year when a demolition permit was issued at the end of the building's life. This information is used to test the prediction that all else equal, a building with a lower FAR limit will be demolished at an earlier age for a conversion. The idea behind this approach is that if two buildings were constructed at a similar date, comparing the age of the buildings at demolition would allow us to determine the effect of FAR regulation. After constructing this data, we are left with a sample size of 7,398.

Table 2.1 shows the summary statistics for the two sets of constructed data. The notable aspect about our data is that the mean Built FAR is lower than the mean maximum-allowed FAR, which indicates that the restriction does not bind for some of the buildings.

Table 2.1: Summary Statistics

Vacant Land Value Data		
	Mean	S.D
Lot Area (sqft)	35387	328609
Max FAR	2.38	2.13
Built FAR	1.73	2.60
Sale Price ($)	1405280	5260532
Price Per square foot ($)	310	1053
N	2720	
Building Age at Demolition Data		
	Mean	S.D.
Year Built	1931	24.7
Age (years)	76.4	25.6
Max FAR	2.6	2.0
Built FAR	1.2	1.4
N	7318	

4 Empirical Strategy & Results

We address two aspects of the model in our study: 1) the effect of FAR regulation differentials on land value for those properties for which the FAR limit is not binding and 2) the effect of FAR regulation differentials on the timing at which a conversion takes place. To do this, we control for spatial heterogeneity across parcels through a fixed effects model and estimate the effect of deviations in maximum-allowed FAR from the spatially-adjusted means on deviations on land values from their spatially-adjusted means.

4.1 Estimating the Effect of FAR Regulation Differentials on Land Value

Our model predicts that a marginal increase in the stringency of FAR regulation lowers land value. That is, all else equal a parcel with a higher maximum-allowed FAR will have a higher land value. This is shown in the model by the fact that the elasticity of land value with respect to the maximum-allowed FAR is positive. To identify the effect of FAR regulation differentials on land value, we regress the log of land value on the log of the maximum-allowed FAR. This gives us the estimate for the elasticity of land value with respect to the maximum-allowed FAR. We include a sales year fixed effect and a zip code fixed effect to control for variation in sales prices across different years and for neighborhood characteristics. The equation for this regression is given by

$$\log\left(V_{ict}\right) = \alpha_t + \beta_c + \theta \log\left(FAR_{ict}\right) + \varepsilon_{ict} \tag{4.1}$$

Table 2.2: Effect of FAR Regulation on Land Value

VARIABLES	(1) LN(Land Price/lot area)	(2) LN(Land Price/lot area) MaxFAR>BuiltFAR
Ln(FAR)	0.432***	0.421***
	(0.0713)	(0.0881)
Constant	6.528***	7.077***
	(0.216)	(0.257)
Zip Code FE	Yes	Yes
Sales Year FE	Yes	Yes
Observations	2,720	2,244
R-squared	0.585	0.484

Robust standard errors in parentheses
*** p<0.01, ** p<0.05, * p<0.1

where V is the land value, FAR is the maximum-allowed FAR, t is the vector of dummies for the year in which building i was sold, and c is the vector of dummies for zip code. Note that this equation does not control for characteristics of the buildings that were sold.

This is because in our data, buildings that were sold were going to demolished anyway and their characteristics do not affect land value.

Table 2.2 gives the estimation results of the correlation between FAR regulation on land value. Column (1) shows the estimation results for the aggregate data, while column (2) shows the estimation results for samples for which the FAR limit is not binding. The results are consistent even for parcels where the FAR is not binding - we find the elasticity of land value with respect to maximum-allowed FAR to be positive, significant, and less than 1 for parcels that have non-binding FAR limits. For these parcels, we find that a 1% increase in the maximum-allowed FAR is associated with a 0.42% increase in land value.

4.2 Estimating the Effect of FAR Regulation Differential on Timing of Conversion

To estimate the effect of FAR regulation differentials on the timing of conversion, we regress the age at which a building is demolished on maximum-allowed FAR. Theory predicts that if two buildings are similar in characteristics, the building with a lower maximum-allowed FAR will be converted at an earlier time. If two buildings were constructed at a similar time, have similar neighborhood characteristics, and differ only in the maximum-allowed FAR, the timing of conversion can be estimated by comparing the age at which each of the buildings are demolished.

Table 2.3: Effect of FAR Regulation on the Timing of Conversion

VARIABLES	(1) Age	(2) Age	(3) Age	(4) Age
FAR	0.708***	0.255***	0.702***	0.247***
	(0.0492)	(0.0648)	(0.0493)	(0.0649)
Distance (Distance from ESB)			-0.00893	-0.0179***
			(0.00570)	0.00637)
Constant	186.7***	180.2***	186.7***	180.3***
	(0.219)	(0.505)	(0.221)	(0.506)
Built Year FE	Yes	Yes	Yes	Yes
Built FAR FE	Yes	No	Yes	No
By-Borough Built FAR FE	No	Yes	No	Yes
Observations	7,318	7,318	7,318	7,318
R-squared	0.915	0.914	0.915	0.914

Notes: In columns 2 and 4, by-borough built FAR fixed effects were included.
Robust standard errors in parentheses
*** p<0.01, ** p<0.05, * p<0.1

We include a built year fixed effect based on the decade[4] that a building was erected for two reasons. First, as discussed above, comparing the age of demolition of buildings that were constructed at similar times, allows us to determine the differences in the timing of conversion due to differences in FAR regulation. Second, we assume that the timing at which a building is demolished will heavily depend on the year at which the building was built. Buildings that were constructed at a similar time will have similar characteristics and be composed of similar materials and with similar technology.

We also include a built FAR fixed effect, with the built FAR of each building rounded to the nearest multiple of 0.5. This allows us to further control for property characteristics, under the assumption that two buildings that were constructed at a similar time to a similar density must also have similar characteristics. Thus, our exercise involves comparing the timing of demolition of two buildings that were built at the same time and at the same density, but differ only in their maximum-allowed FAR.

In the simplest case, this estimation equation is given by

$$AGE_{isl} = \alpha_l + \lambda_s + \theta FAR_{isl} + \varepsilon_{isl} \tag{4.2}$$

where l is the vector of dummies for built year and s is the vector of dummies built FAR.

To further account for differences across parcels, we include by-borough fixed effects for the built FAR of

[4]Using bins of 3 years and 5 years also yields a similar result.

buildings prior to conversion[5]. In particular, we interact borough dummies with the built FAR of buildings, rounded to the nearest multiple of 0.5. The resulting estimation equation is

$$AGE_{ibsl} = \alpha_l + \lambda_{bs} + \gamma * X_{ibsl} + \theta FAR_{ibsl} + \varepsilon_{ibsl} \tag{4.3}$$

where b is the vector of dummies for borough and X_{ibsl} includes the distance of building i from the ESB.

Table 2.3 gives the estimation results for the correlation between maximum-allowed FAR and a building's age at demolition. As the theory predicts, the coefficient on *FAR* is positive, indicating that buildings with a relatively more stringent regulation are demolished at an earlier time. In particular, if we do not include by-borough built FAR fixed effects, we find that a increasing the maximum-allowed FAR by 1 corresponds to a building being demolished 0.71 of a year later. If we include by-borough built FAR fixed effects, we find that a 1 unit increase in the maximum-allowed FAR corresponds to a building being demolished 0.25 of a year later.

A limitation of our study is that the data may be censored on the right. That is, if the theory is correct, some properties with relatively loose FAR regulation may be demolished at a later date, which we would not be able to observe in the data. Censoring of data is most likely to lead to a downward bias of our estimates. If we include buildings that are built in the relevant time frame but are demolished at a later date, it is most likely to drive up the estimates we have in Table 2.3.

5 Conclusion

We have presented an empirical model which tests a few aspects of Ahn's (2019) extension of Brueckner's (Brueckner and Sridhar (2012); Brueckner et al (2017); Brueckner and Singh (2018); Moon (2018)) model of FAR regulation to a dynamic setting. In particular, we find that all else equal, a more stringent FAR regulation is correlated with lower land values even for buildings where the regulation is not binding and earlier conversion of buildings. A direction for future research involves testing another aspect of Ahn's (2019) model, which states that increasing the stringency of regulation will lead to lower density of buildings at each conversion.

[5] Using by-zip code built FAR fixed effects yields similar results.

References

[1] Ahn, S. (2019). "The Effects of FAR Regulations in a Model of Durable Buildings with Redevelopment." PhD diss., University of California, Riverside.

[2] Brueckner, J.K., S. Fu, Y. Gu, and J. Zhang. (2017). "Measuring the Stringency of Land Use Regulation: The Case of China's Building Height Limits." *The Review of Economics and Statistics*, 99(4):663-677.

[3] Brueckner, J.K. & R. Singh (2018). "Stringency of Land-Use Regulation: Building Heights in US Cities." *CESifo Working Paper Series No. 6978*.

[4] Brueckner, J.K. & K.S. Sridhar. (2012). "Measuring Welfare Gains from Relaxation of Land-Use Restrictions: The Case of India's Building-Height Limits." *Regional Science and Urban Economics*, 42(6):1061-1067.

[5] Gyourko, J. & R. Molloy. (2015). "Regulation and Housing Supply." *Handbook of Regional and Urban Economics*, 5:1289-1337.

[6] Marcus, N. (1992). "New York City Zoning – 1961-1991: Turning Back the Clock – But With an Up-To-The-Minute Social Agenda." *Fordham Urban Law Journal*, 19(3):707-726.

[7] Moon, B. (2018). "The Effect of FAR (Floor Area Ratio) Regulations on Land Values: The Case of New York." *Papers in Regional Science*, (forthcoming)

Market Power and Urban Land Development

Abstract

This paper investigates the relationship between market power and urban housing development patterns in a two-period, partial equilibrium model of a durable housing rental market. Since we employ a social surplus analysis, the planner's equilibrium coincides with the market outcome under perfect competition. Thus, we contrast the monopoly equilibrium and the perfectly competitive equilibrium. We find the optimality conditions for the social planner and the monopolist, then provide numerical examples and a numerical comparative statics analysis. In all our numerical examples, we find that the monopolist builds at a lower density than the social planner in each period, but the rate at which land is released for development will depend on the model parameters.

1 Introduction

This paper explores how market power affects the density of housing and the time path of development of an urban land developer. In particular, we contrast the decisions of a monopolist with those of a social planner. Since we employ social surplus analysis, the planner's decisions coincide with the market outcome under perfect competition. Thus, we contrast the monopoly equilibrium and the perfectly competitive equilibrium.

We utilize a two-period, partial equilibrium model of a durable housing rental market in which the developer must determine both the density of housing and the rate of land development. We find that in choosing the optimal density, the social planner equates demand to marginal cost, while the monopolist equates discounted marginal revenue to marginal cost. Furthermore, in choosing the rate at which to release land for development, the social planner requires that net marginal social surplus (demand minus the cost of developing land) grows at the rate of interest, while the monopolist requires that his net discounted marginal revenue (discounted marginal revenue minus the cost of developing land) grows at the rate of interest. We also provide a numerical example, along with a numerical comparative statics exercise. In all our numerical results, in each period the monopolist develops at a lower density than does the social planner, but the rate at which land is released for development will depend on the parameters.

While there are a myriad of reasons for rapid increases in land prices, large developers and landowners have often been made the scapegoats for unaffordable housing. The usual argument is that those with market power in the urban land development market intentionally hold land off the market to restrict supply and artificially increase land values. Some of the existing literature seems to be in accord with this sentiment. For example, Markusen and Scheffman (1978) and Vousden (1981) show in monocentric city models that there will be “leapfrog” development when there are large landowners. That is, there may be rings of unoccupied land strictly within the boundary of the city. This is because withholding land in the interior of the city raises the rent gradient more than withholding land at the edge of the city.

Monocentric city models such as these treat land as a standard monopoly good that is supplied elastically. While such class of models originally developed by Alonso (1964), Mills (1972), and Muth (1969) are useful in explaning the spatial distribution of population in a city through spatial differences in housing density, they do not adequately address the question regarding vacant land development patterns. Rather, models which view land as being supplied inelastically and which differentiate vacant land from the improvement upon it are better tools to answer how different economic factors affect the way in which land is developed.

The most relevant paper which shall guide the economics of our model is Stiglitz’s (1976) work on the rate of extration of exhaustible resources. In Stiglitz’s model, the stock of the exhaustible resource is assumed to be fixed, and the supplier must decide on the quantity of oil to be extracted over multiple periods. His central case with zero extraction costs and constant elasticity demand schedules shows that the monopolist’s and the first-best rates of extraction are identical. While in the competitive equilibrium the seller must be indifferent between selling his good this period and the next period, thus setting $p_1 = \frac{p_2}{1+r}$, the monopolist equates the marginal revenues in both periods so that $MR_1 = \frac{MR_2}{1+r}$. With constant demand elasticity over time, price is equally proportional to marginal revenue in both periods, and the two equilibria will be the same. To see this, suppose that $p_t = kq_t^{\beta}$ for $k > 0$, $-1 < r < 0$, $q_1 + q_2 \leq \overline{q}$, and $t = 1,2$. Then, $MR_t = k(1+r)q_t^{\beta}$, and the optimality condition for both the competitive seller and the monopolist are identical.

Figure 3.1 illustrates the foregoing. θ_1 is the amount of resources extracted in the first period, while θ_2 is the amount of resources extracted in the second period. If the elasticity of demand is constant over time, then the ratio of price to marginal revenue is equal in each period, and the equilibrium for the competitive firm and the monopolist are identical.

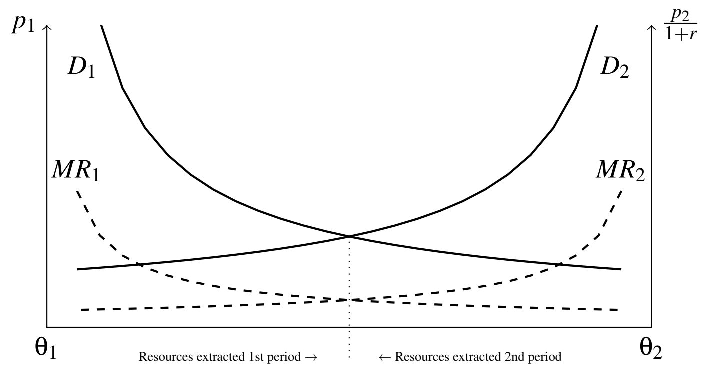


Figure 3.1: Stiglitz (1976) Base Model

Stiglitz (1976) further shows that under certain conditions such as increasing elasticity of demand or positive extraction costs, the rate of extraction between the first-best equilibrium and the monopolist equilibrium may differ. Mills' (1980) model may be interpreted as a special case of Stiglitz's model with increasing elasticity of demand. He shows in a dynamic, partial equilibrium model with a fixed stock of homogeneous land and increasing elasticity of demand that the pace at which land is developed decreases with an increase in market power among landowners. However, Mills' model cannot explain the role of market power in prices and housing density, as it does not differentiate vacant land from the density of housing.

In this paper, we present a dynamic, partial equilibrium model with a fixed stock of homogeneous land to analyze the role of market power in urban land development patterns. Our paper contributes to the literature on urban land development by extending Stiglitz's (1976) model in a few but meaningful ways. First, our exhaustible resource, land, is used as an input into the production of a durable good. Second, our exhaustible resource is used as one of two inputs to production, so that the demand for land is a derived demand. These extensions provide additional insights in the role of market power in urban land development. In particular, we find that while the monopolist may indeed produce a lower quantity of housing than is optimal, this result is driven by his lower choice of housing density in each period, rather than from withholding land from the market. Furthermore, we find that under certain specifications of the model, the social planner may actually withhold more land than does the monopolist.

We organize the remainder of the paper as follows. In section 2, we present the general model of the decisions of the social planner and the monopolist. In section 3, we provide a numerical example and a numerical comparative statics analysis. In section 4, we provide the concluding remarks.

2 Model

Notational Glossary	
$p_t(\cdot)$	Inverse demand for housing in period t
r	Discount rate
$C(\cdot)$	Cost function for the total production of housing
$c(\cdot)$	Cost function for the construction of housing per unit area of land
s_t	Floor area of housing per unit area of land developed in period t
θ_t	Proportion of land released for development in period t $(0 \leq \theta_t \leq 1)$
h_t	$(= s_t\theta_t)$ Total floor area of housing produced in period t

2.1 Model Set-Up

Consider a housing rental market with durable housing and no depreciation. The developer's problem is to decide on both the density of housing (s_t) and the rate at which to develop the large tract of land he owns (θ_t) over two discrete periods. We assume the following in our model:

Assumption 1. *Housing is rented out in each period and not sold.*

Assumption 2. *Housing is durable and there is no depreciation.*

Assumption 3. *The total amount of land to be developed is fixed and normalized to 1.*

Assumption 4. *All land to be developed is homogeneous.*

Assumption 5. *The cost of production for housing is given by* $C(h_t) = C(\theta_t s_t) \equiv \theta_t c(s_t)$ *with* $c' > 0$ *and* $c'' > 0$ *such that the marginal cost is increasing in density and constant in land.*

We assume that housing is rented out in each period rather than being sold, since if a durable goods monopolist were to sell his goods over a period of time, he would have to end up selling all of his goods at marginal cost (Coase, 1972). We first derive the conditions for a first-best optimum. Then, we analyze the urban land development decisions of a monopolist.

2.2 Social Planner

Denote the inverse demand functions by $p_1(h_1)$ and $p_2(h_1+h_2)$ for period 1 and period 2, respectively, where h_t is the total floor area of housing produced in period t. Because housing is assumed to be durable, any housing produced in the first period can also be rented out in the second period. Let SS_t be the social surplus for period t. Then, we have

$$SS_1 = \int_0^{h_1} p_1(u)\,du - C(h_1) \tag{2.1}$$

$$SS_2 = \int_0^{h_1+h_2} p_2(u)\,du - C(h_2) \tag{2.2}$$

By Assumption 5, we know that it is cost-minimizing for all of the available land to be developed. Thus, we set $\theta_2 = 1 - \theta_1$, and we have

$$h_1 = \theta_1 s_1 \tag{2.3}$$

$$h_2 = (1-\theta_1)\,s_2 \tag{2.4}$$

$$C(h_1) = \theta_1 c(s_1) \tag{2.5}$$

$$C(h_2) = (1-\theta_1)\,c(s_2) \tag{2.6}$$

The social planner would like to maximize discounted social surplus with respect to θ_1, s_1, and s_2. With (2.3)-(2.6) substituted in, we get

$$\begin{aligned} DSS =& SS_1 + \frac{SS_2}{1+r} \\ =& \int_0^{\theta_1 s_1} p_1(u)\,du - \theta_1 c(s_1) + \frac{\int_0^{\theta_1 s_1 + (1-\theta_1)s_2} p_2(u)\,du - (1-\theta_1)\,c(s_2)}{1+r} \end{aligned} \tag{2.7}$$

where r is the discount rate.

The first-order condition with respect to s_1 is

$$p_1(\theta_1 s_1) + \frac{p_2(\theta_1 s_1 + (1-\theta_1)\,s_2)}{1+r} = c'(s_1) \tag{2.8}$$

so that if development occurs in the first period, then the density of housing hould be chosen such that the discounted marginal benefit from an additional floor area per unit area of land is equal to the marginal cost.

Similarly, the first-order condition with respect to s_2 is

$$p_2\left(\theta_1 s_1 + (1-\theta_1)\, s_2\right) = c'\left(s_2\right) \tag{2.9}$$

so that if development occurs in the first period, then structural density should be chosen such that the marginal benefit from an additional floor area per unit area of land is equal to the marginal cost. The two first-order conditions with respect to the first and second period housing density can be solved simultaneously to obtain the density which maximizes social surplus in each period in terms of θ_1 and the parameters:

$$s_1^* \equiv s_1^*\left(\theta_1\right) \tag{2.10}$$

$$s_2^* \equiv s_2^*\left(\theta_1\right) \tag{2.11}$$

The first-order condition with respect to θ_1 is

$$p_1\left(\theta_1 s_1\right) s_1 - c\left(s_1\right) + \frac{p_2\left(\theta_1 s_1 + (1-\theta_1)\, s_2\right)\left(s_1 - s_2\right) - c\left(s_2\right)}{1+r} = 0 \tag{2.12}$$

which can be rewritten as

$$\left[p_1\left(\theta_1 s_1\right) + \frac{p_2\left(\theta_1 s_1 + (1-\theta_1)\, s_2\right)}{1+r}\right] s_1 - c\left(s_1\right) = \frac{p_2\left(\theta_1 s_1 + (1-\theta_1)\, s_2\right) s_2 - c\left(s_2\right)}{1+r} \tag{2.13}$$

The above equation equates the discounted marginal surplus from the development of an extra unit area of land in period 1 to the discounted marginal surplus from the development of an extra unit area of land in period 2. Now consider this first-order condition with respect to land evaluated at the profit maximizing densities:

$$\begin{aligned}\left[p_1\left(\theta_1 s_1^*\left(\theta_1\right)\right) + \frac{p_2\left(\theta_1 s_1^*\left(\theta_1\right) + (1-\theta_1)\, s_2^*\left(\theta_1\right)\right)}{1+r}\right] s_1^*\left(\theta_1\right) - c\left(s_1^*\left(\theta_1\right)\right) \\ = \frac{p_2\left(\theta_1 s_1^*\left(\theta_1\right) + (1-\theta_1)\, s_2^*\left(\theta_1\right)\right) s_2^*\left(\theta_1\right)}{1+r} - \frac{c\left(s_2^*\left(\theta_1\right)\right)}{1+r}\end{aligned} \tag{2.14}$$

If the left-hand side of (2.14) is greater than the right-hand side when evaluated at $\theta_1 = 1$, then all land is developed in the first period. Likewise, if the right-hand side is greater than the left-hand side when evaluated at $\theta_1 = 0$, then all land is developed in the second period. Otherwise, the above equation can be used to solve for θ_1^*, the optimal rate at which to develop land to maximize social surplus.

2.3 Monopoly

Now consider the monopoly developer's discounted profit function:

$$\begin{aligned}\pi^m =& p_1(\theta_1 s_1)\theta_1 s_1 + \frac{p_2(\theta_1 s_1 + (1-\theta_1)s_2)(\theta_1 s_1 + (1-\theta_1)s_2)}{1+r} \\ & - \theta_1 c(s_1) - \frac{(1-\theta_1)c(s_2)}{1+r}\end{aligned} \tag{2.15}$$

The first-order condition with respect to s_1 is

$$\begin{aligned}\frac{\partial p_1(\theta_1 s_1)}{\partial h_1}\theta_1 s_1 + p_1(\theta_1 s_1) + \frac{\frac{\partial p_2(\theta_1 s_1+(1-\theta_1)s_2)}{\partial h_1}(\theta_1 s_1 + (1-\theta_1)s_2) + p_2(\theta_1 s_1 + (1-\theta_1)s_2)}{1+r} \\ = c'(s_1)\end{aligned} \tag{2.16}$$

where $h_1 = \theta_1 s_1$. Recognizing that the right-hand side of (2.16) is the monopolist's marginal revenue with respect to density in the first period, this can be rewritten as

$$MR_{1,s}(s_1, s_2, \theta_1) = c'(s_1) \tag{2.17}$$

where $MR_{t,s}(s_1, s_2, \theta_1)$ is the marginal revenue with respect to density in period t. If development occurs in the first period, then the density of housing is chosen such that the discounted marginal revenue from an additional floor area of housing per unit area of land is equal to its marginal cost. The first-order condition with respect to s_2 is

$$\frac{\partial p_2(\theta_1 s_1 + (1-\theta_1)s_2)}{\partial h_2}(\theta_1 s_1 + (1-\theta_1)s_2) + p_2(\theta_1 s_1 + (1-\theta_1)s_2) = c'(s_2) \tag{2.18}$$

where $h_2 = (1-\theta_1)s_2$. This can be rewritten as

$$MR_{2,s}(s_1, s_2, \theta_1) = c'(s_2) \tag{2.19}$$

Once again, if development occurs in the second period, then the density of housing is chosen such that the marginal revenue from an additional floor area of housing per unit area of land is equal to its marginal cost. As in the previous section, the two first-order conditions with respect to density in each period can be solved simultaneously to yield the profit-maximizing density with respect to θ_1:

$$s_1^m = s_1^m(\theta_1) \tag{2.20}$$

$$s_2^m = s_2^m(\theta_1) \tag{2.21}$$

The first-order condition with respect to θ_1 is

$$\frac{\partial p_1(h_1)}{\partial h_1} s_1 \theta_1 s_1 + p_1(h_1) s_1 + \frac{\frac{\partial p_2(h_1+h_2)}{\partial h_1} s_1 (\theta_1 s_1 + (1-\theta_1) s_2) + p_2(h_1+h_2) s_1}{1+r} - c(s_1)$$
$$= \frac{\frac{\partial p_2(h_1+h_2)}{\partial h_2} s_2 (\theta_1 s_1 + (1-\theta_1) s_2) + p_2(h_1+h_2) s_2}{1+r} - \frac{c(s_2)}{1+r} \tag{2.22}$$

or

$$MR_{1,\theta_1} - c(s_1) = \frac{MR_{2,\theta_1} - c(s_2)}{1+r} \tag{2.23}$$

where MR_{t,θ_1} is the marginal revenue with respect to land in period t. Land is developed in such a away that the discounted marginal profit from developing an additional unit area of land in the first period is equal to the discounted marginal profit from developing an additional unit area of land in the second period. Note that the profit-maximizing condition for the monopolist is that net discounted marginal revenues rise at the rate of interest. Now suppose that the first-order condition with respect to θ_1 is evaluated at the profit-maximizing densities:

$$MR_{1,\theta_1} - c(s_1^m(\theta_1)) = \frac{MR_{2,\theta_1} - c(s_2^m(\theta_1))}{1+r} \tag{2.24}$$

As in the case of the social planner, if the left-hand side of (2.22) is greater than the right-hand side when evaluated at $\theta_1 = 1$, then all development occurs in the first period, and vice-versa.

3 Numerical Example & Comparative Statics Results

In this section, we provide numerical comparative statics results using parameter values which yield interior solutions. We first solve a numerical example in a baseline model, then present the effects of changes in parameters. Assume that period-specific inverse demand curves have the form $a_t - bh_t$, that the housing marginal cost curve is $c + ds_t$, and that there is no fixed cost so that $c(s_t) = cs_t + \frac{ds_t^2}{2}$ for $t = 1,2$. For the baseline model, assume that $b = c = d = r = 1$, $a_1 = 5$, and $a_2 = 15$. Thus, the increase in demand from the first to the second period is in the form of a parallel shift outward of the demand curve.

3.1 Social Planner

First consider the social planner who wishes to maximize the discounted social surplus, given by

$$DSS = \int_0^{\theta_1 s_1} (5-u)\,du - \theta_1\left(s_1 + \frac{s_1^2}{2}\right) + \frac{\int_0^{\theta_1 s_1 + (1-\theta_1)s_2} (15-u)\,du - (1-\theta_1)\left(s_2 + \frac{s_2^2}{2}\right)}{2} \tag{3.1}$$

Maximizing with respect to the variables of interest yields

$$s_1: \quad 5 - \theta_1 s_1 + \frac{15 - \theta_1 s_1 - (1-\theta_1)s_2}{2} = 1 + s_1 \tag{3.2}$$

$$s_2: \quad 15 - \theta_1 s_1 - (1-\theta_1)s_2 = 1 + s_2 \tag{3.3}$$

$$\theta_1: \quad \left(5 - \theta_1 s_1 + \frac{15 - \theta_1 s_1 - (1-\theta_1)s_2}{2}\right)s_1 - \left(s_1 + \frac{s_1^2}{2}\right) = \frac{(15 - \theta_1 s_1 - (1-\theta_1)s_2)s_2}{2} - \frac{\left(s_2 + \frac{s_2^2}{2}\right)}{2} \tag{3.4}$$

Solving the three equations simultaneously yields

$$s_1^* = 5.38 \tag{3.5}$$

$$s_2^* = 7.61 \tag{3.6}$$

$$\theta_1^* = 0.54 \tag{3.7}$$

3.2 Monopolist

Now consider the monopolist, who would like to maximize his discounted profits given by

$$\pi^m = (5 - \theta_1 s_1)\theta_1 s_1 + \frac{(15 - \theta_1 s_1 - (1-\theta_1)s_2)(\theta_1 s_1 + (1-\theta_1)s_2)}{2} - \theta_1\left(s_1 + \frac{s_1^2}{2}\right) - \frac{(1-\theta_1)\left(s_2 + \frac{s_2^2}{2}\right)}{2} \tag{3.8}$$

The three first-order conditions are given by

$$s_1: \qquad (5-2\theta_1 s_1) + \frac{(15-2\theta_1 s_1 - 2(1-\theta_1)s_2)}{2} = 1 + s_1 \tag{3.9}$$

$$s_2: \qquad (15-2\theta_1 s_1 - 2(1-\theta_1)s_2) = 1 + s_2 \tag{3.10}$$

$$\theta_1: \qquad \left(5-2\theta_1 s_1 + \frac{15-2\theta_1 s_1 - 2(1-\theta_1)s_2}{2}\right)s_1 - \left(s_1 + \frac{s_1^2}{2}\right) = \frac{(15-2\theta_1 s_1 - 2(1-\theta_1)s_2)s_2}{2} - \frac{\left(s_2 + \frac{s_2^2}{2}\right)}{2} \tag{3.11}$$

Solving the three equations simultaneously yields

$$s_1^m = 3.64 \tag{3.12}$$

$$s_2^m = 5.14 \tag{3.13}$$

$$\theta_1^m = 0.47 \tag{3.14}$$

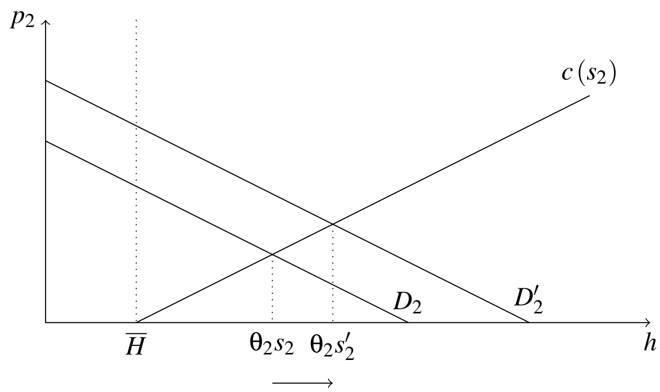
Thus, with the chosen parameters and functional forms, the monopolist will produce a lower density of housing in each period and will withhold more land in the first period compared to the social planner. Furthermore, the overall quantity of housing built by the monopolist will be lower than that of the social planner.

3.3 Comparative Statics

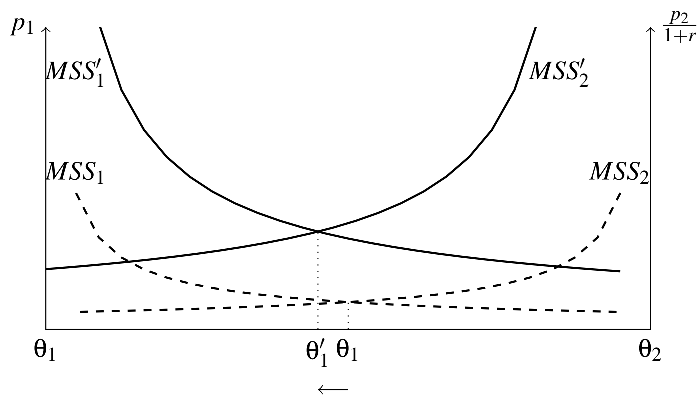
3.3.1 Changes in a_2

Suppose that there is an increase in a_2 so that second period demand shifts outward in a parallel fashion and there is an increase in second period demand. We would like to show how this change affects various aspects of the model.

Consider the social planner's problem in the second period. Holding all else fixed, an increase in a_2 leads to an increase in s_2, as it will become possible to increase social surplus by doing so. To see this, consider Figure 3.2A. Suppose that $\overline{H}$ is the total amount of housing constructed in the first period. Then, holding all else fixed, an increase in demand from D_2 to D_2' will increase the amount of housing produced in the second period from $\theta_2 s_2$ to $\theta_2 s_2'$, with a corresponding increase in the social surplus. Similarly, an increase in the second-period demand leads to an increase in the first-period density. Since housing is durable and any housing



(a) Second Period Density



(b) Rate of Land Development

Figure 3.2: Effect of a change in a_2

constructed in the first period can also be rented out in the second period, the social planner is able to increase social surplus by increasing the density of housing in the first period.

Figure 3.2B illustrates the effect of increasing a_2 on the rate of land development. Suppose that, as is in the case of our numerical example, the equilibrium density in the second period is greater than that of the first period. A marginal increase in a_2 increases both the first period and the second period marginal social surplus with respect to θ_1 by $\frac{s_1}{2}$ and $\frac{s_2}{2}$, respectively. Since by assumption $s_2 > s_1$, the marginal social surplus in the second period increases by more than that in the first period. Thus, the new point of interception between the new marginal social surplus curves occur at a lower θ_1. Thus, an increase in a_2 leads to more land being withheld in the first period.

Figure 3.3 compares the effect of changing a_2 on the equilibrium density and rate of land development for the social planner and the monopolist. As discussed above, an increase in demand leads to an increase in housing density in both periods and a decrease in the amount of land released for development in the first

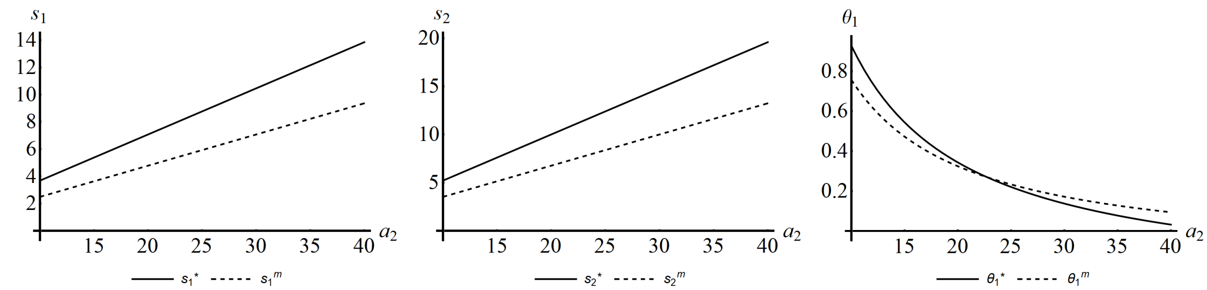


Figure 3.3: Comparative statics changing parameter a_2

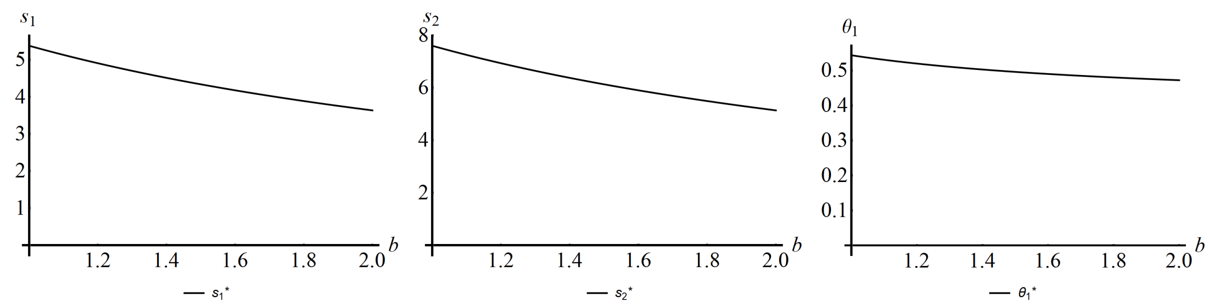


Figure 3.4: Comparative statics changing parameter b

period. Furthermore, as in our numerical example, the monopolist's density in both periods is consistently lower than that of the social planner for all values of a_2. Surprisingly, however, a sufficient increase in a_2 will lead the monopolist to develop more land than the social planner in the first period.

3.3.2 Changes in b

While b is the slope of the second period inverse demand curve, it can also be interpreted as "market power" given the assumption of linear inverse demand curves. That is, since the slope of the marginal revenue curve for a monopolist is twice the slope of the inverse demand curve, an increase in b from 1 to 2 corresponds to the marginal revenue curves of firms with increasing market power.

Figure 3.4 illustrates the foregoing. As market power is increased, housing density in both periods and the rate at which land is developed are decreased. It is important to note, however, that the results in the previous section point to the fact that if the second period demand is sufficiently high, an increase in market power will lead to an increase - not a decrease - in the amount of land that is developed in the first period.

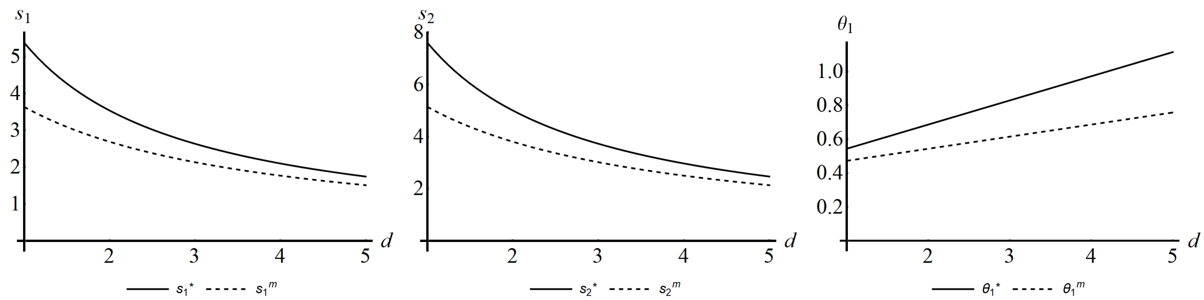


Figure 3.5: Comparative statics changing parameter d

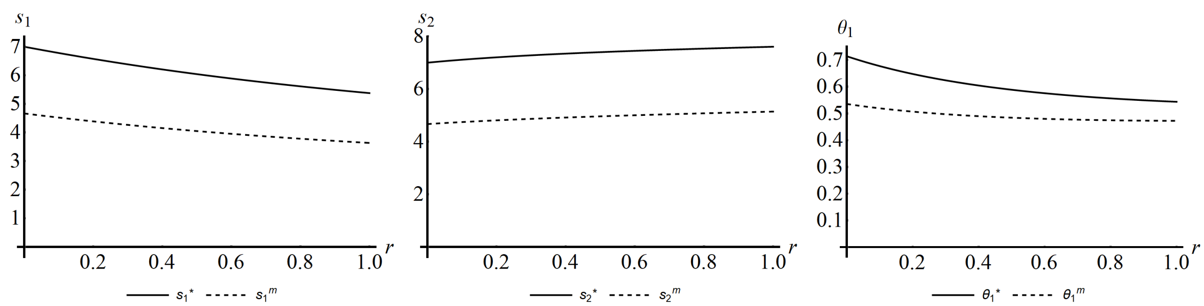


Figure 3.6: Comparative statics changing parameter r

3.3.3 Changes in d

Figure 3.5 illustrates the effect of increasing the marginal cost of development. The numerical results indicate that an increase in marginal cost leads to a decrease in housing density in both periods, but the rate of this decrease is more rapid the more competitive the market. This points to the fact that increased market power allows the producer to place more of the burden of the increased cost on the consumers.

3.3.4 Changes in r

Figure 3.6 shows the effect of increasing the discount rate on development patterns. The immediate effect of an increase in the interest rate is to shift both the second and the first period marginal social surplus curves downward. In our numerical example, the first period marginal social surplus curve shifts more than the second period marginal social surplus curve, and as a result, the amount of land developed in the first period decreases with an increase in interest rate. Furthermore, an increase in the interest rate decreases the second period marginal revenue of housing constructed in the first period, thus decreasing the density of housing in the first period. This reduction in the production of housing in the first period increases marginal revenue of housing produced in the second period, leading to an increase in the density of housing in the second period.

4 Conclusion

We have presented a model which analyzes the role of market power in urban land development patterns. Our model treats land as being inelastically supplied and as an input for the production of housing. The model predicts that in determining the density of housing, the social planner will equate discounted marginal benefit from an additional floor area of housing to its marginal cost, while the monopolist will equate discounted marginal revenue from an additional floor area of housing to its marginal cost. Furthermore, in determining the amount of land to be relased for development, the social planner's optimality condition requires that the net marginal social surplus of developing land increases at the rate of interest, while the monopolist's optimality condition requires that the net discounted marginal revenue increases at the rate of interest. Our numerical example shows that the monopolist will build at a lower density in both periods and release land for development at a slower rate than is optimal.

While a popular belief is that the rapid rise in housing is due to large landowners holding land off the market, our analysis shows that it is the rate at which land is developed - not the total quantity - that is affected by market power. Stiglitz (1976) shows that in the problem of the extraction of exhaustible resources, the change in the elasticity of demand is what affects the rate at which extraction occurs. We conjecture that this will be true also in our model. A direction for future research includes providing numerical examples involving demand schedules involving different elasticities and an analytical comparative statics exercise.

References

[1] Alonso, W. (1964). *Location and Land Use*. Cambridge: Harvard University Press.

[2] Markussen, J.R. & D.T. Scheffman. (1978). "Ownership Concentration and Market Power in Urban Land Markets." *The Review of Economic Studies*, 45(3):519-526.

[3] Mills, D.E. (1980). "Market Power and Land Development Timing." *Land Economics*, 56(1):10-20.

[4] Mills, E.S. (1972). *Studies in the Structure of the Urban Economy*. Baltimore: Johns Hopkins Press.

[5] Muth, R.F. (1969). *Cities and Housing*. Chicago: University of Chicago Press.

[6] Stiglitz, J.E. (1976). "Monopoly and the Rate of Extraction of Exhaustible Resources." *The American Economic Review*, 66(4):655-661.

[7] Vousden, N. (1981). "Market Power in a Non-Malleable City." *The Review of Economic Studies*, 48(1):3-19.