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AN EXISTENCE AND UNIQUENESS THEOREM FOR THE CAUCHY PROBLEM FOR AN INELASTIC MATERIAL WITH MEMORY

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ABSTRACT

The theory of nonlinear semigroups of operators is applied to the problem of existence and uniqueness of solutions of evolutionary equations arising in inelastic materials with memory effects. General nonlinear hereditary laws for the inelastic response of the material are considered. The fading memory property is formulated in terms of an obliviating measure. Suitable restrictions on this measure and on the plastic constitutive mapping are postulated that result in a well-posed initial value problem. For instance, a monotonicity condition is introduced that generalizes the concept of normality of materials with instantaneous response.

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1. Introduction

In this paper, we show that the Cauchy problem for an inelastic material with memory and linear instantaneous elasticity (when restricted to infinitesimal deformations) has a unique solution. The theorem to be proved (Theorem 1 in Section 3) is that when the problem is expressed in the form

$$\dot{\mathbf{x}} + \mathbf{A}\,\mathbf{x} = 0 \quad ; \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{1}$$

where the vector $\mathbf{x}$ consists of the velocity field, the stress field and the field of the past history of stress and $\mathbf{A}$ is a nonlinear operator on a Hilbert space H, then $-\mathbf{A}$ generates a semigroup of contractions. This is equivalent to the existence of a unique solution, and is in fact the way in which the questions of existence and uniqueness are treated within the framework of modern nonlinear semigroup theory [26].

Section 2 is devoted to a survey of the theory of materials with memory, with a view toward finding an appropriate Hilbert space in which to place the unknowns of the problem. This space turns out to be a Lebesgue space with an obliviating measure.

In Section 3 the Cauchy problem is formulated in the aforementioned form and suitable restrictions are imposed on the constitutive mapping. One such restriction is a monotonicity condition which is a generalization of the usual normality rule for inelastic materials.

2. Materials with Memory

Often in physics, the term process is used to signify a collection C of functions of time, $\mathbf{f}(t), \mathbf{g}(t)$, etc., $t \in (-\infty, \infty)$, taking values in a vector space V that may be of finite or infinite dimension. Given any one $\mathbf{f} \in C$, the function $\mathbf{f}^t : [0, \infty) \to V$ defined by

$$\mathbf{f}^t(s) = \mathbf{f}(t - s), \qquad 0 \leqslant s < \infty \tag{2}$$

is called the history of $\mathbf{f}$ up to time t. Different materials and different physical situations are characterized by different constitutive assumptions which place limitations on the class C. Quite frequently in mechanics one encounters a constitutive assumption asserting that the value of a variable $\mathbf{g}(t)$ at time t is given by a mapping* $\mathbf{T}$ operating on the history up to t of another variable $\mathbf{f}$

$$\mathbf{g}(t) = \mathbf{T}\mathbf{f}^t \tag{3}$$

Materials exhibiting such constitutive behavior are commonly referred to as materials with memory.

In a series of papers starting in 1957 GREEN & RIVLIN proposed the use of hereditary constitutive laws for the description of non-linear viscoelastic materials [1,2,3] (originally developed by BOLTZMAN & VOLTERRA for the linear case [21,22]) as an alternative to models using constitutive equations of the rate type [4]. A linearized version of GREEN & RIVLIN's theory, with less severe restrictions on the deformation histories, was established by PIPKIN & RIVLIN [5,6,7].

Frequently, much can be said in a branch of physics from a constitutive law such as (3). In practice, however, the entire history of the body can never be known. The interpretation of the results of an experiment in terms of the theory of materials with memory can be justified only if additional assumptions are made. One such assumption would be that the history of the system prior to the start of the experiment has no appreciable influence on its outcome. This

* In accordance with older usage (e.g., VOLTERRA [22]), such a mapping is sometimes referred to as a functional even though it is not necessarily scalar valued.

physical assumption is realized if one supposes that the memory of the material fades in time. This principle of fading memory can be stated as follows [8]:

Events which occurred in the distant past have less influence in determining the present response than those which occurred in the recent past.

There is no unique way to render this intuitive principle of fading memory in mathematical terms. Loosely speaking, fading memory is achieved when the constitutive functional **T** is continuous in the space of histories X with respect to some "obliviating topology," that poses a less restrictive characterization of "closeness" for histories whose support lies in the distant past than for histories with support lying in the recent past. Thus, the fading memory property is dependent upon the topology chosen for the space of histories X.

A number of different obliviating topologies have been proposed in the past as appropriate for the characterization of the fading memory property. A precise meaning for this concept was first achieved by COLEMAN & NOLL [9,10,11,12] in a series of papers in the years 1960-63 and by WANG [13]. They assumed the space V to be a normed space with norm $|\cdot|$, and identified the space of histories X with the Lebesgue space $L^p([0,\infty), V,\mu)$, with the usual norm

$$||\mathbf{f}||_\mu = \left[\int_0^\infty |\mathbf{f}(s)|^p \, d\mu(s)\right]^{1/p} \tag{4}$$

Here, μ denotes a non-trivial, positive Borel measure over $[0,\infty)$, such that for every Borel subset E of $[0,\infty)$ and $a \in R^+$

$$\mu(E + a) \leqslant \mu(E) \tag{5}$$

Such measures are termed obliviating measures [13].

The mathematical formalisms that have been proposed for the study of the fading memory property are not restricted, by any means, to norm topologies. For instance, WANG [14] in his second theory based his developments on the compact convergence topology, while PERZYNA [15] proposed the use of general metric topologies. For the problem at hand, however, the L^p setting proves the most convenient, and is considered in the sequel.

Given a history $\mathbf{f}: [0,\infty) \to V$, the following related histories are often of interest: The "static continuation" of $\mathbf{f}$ by the amount $a > 0$ is defined to be the function $\mathbf{R}(a)\mathbf{f}: [0,\infty) \to V$ defined by

$$[\mathbf{R}(a)\mathbf{f}](s) = \begin{cases} \mathbf{f}(0) & 0 \leqslant s \leqslant a \\ \mathbf{f}(s-a) & a < s < \infty \end{cases} \tag{6}$$

while the a-section of $\mathbf{f}$ is the function $\mathbf{L}(a)\mathbf{f}: [0,\infty) \to V$ given by

$$[\mathbf{L}(a)\mathbf{f}](s) = \mathbf{f}(s+a), \quad 0 \leqslant s \leqslant \infty \tag{7}$$

A constant history, $\mathbf{f}(s) = const.$, is commonly referred to as an equilibrium history. Equilibrium histories $\mathbf{f}$ have the property that $\mathbf{R}(a)\mathbf{f} = \mathbf{L}(a)\mathbf{f} = \mathbf{f}$ for every $a > 0$, i.e., they are the fixed points of the operators $\mathbf{R}(a)$ and $\mathbf{L}(a)$.

On physical grounds, the normed space of histories X with norm $||\cdot||$ is subject to three elementary requirements:

A) All equilibrium histories must be included in X.

B) For every history $\mathbf{f} \in X$, each one of its static continuations $\mathbf{R}(a)\mathbf{f}$, $a > 0$ must also be in X. In other words, the domain of the semigroup $\mathbf{R}(a)$ is assumed to be all of X. Moreover, if the distance $||\mathbf{f}_1 - \mathbf{f}_2||$ between two histories is zero, then the distance between their static continuations $\mathbf{R}(a)\mathbf{f}_1$ and $\mathbf{R}(a)\mathbf{f}_2$ by any given amount $a > 0$ must also be zero.

C) For every history $\mathbf{f} \in X$, each one of its a-sections $\mathbf{L}(a)\mathbf{f}$, $a > 0$ must also belong to X. In other words, the domain of the semigroup $\mathbf{L}(a)$ is assumed to be all of X.

In what follows, the space of histories X will be identified with the space $L^p([0,\infty), V, \mu)$, for some obliviating measure μ, and with the norm (4). This being the case, postulate (A) is equivalent to the requirement that $\mu([0,\infty)) < \infty$.

It was shown by COLEMAN & MIZEL [16] that postulates (A) and (B) force a fundamental distinction between the past, $(s > 0)$, and the present, $(s = 0)$: They imply that μ must have an atom at 0, and be absolutely continuous over $(0,\infty)$ with respect to the Lebesgue measure. In other words, μ admits the representation

$$\mu = \mu_r + \mu_o \delta \tag{8}$$

where μ_r is Lebesgue-absolutely continuous, δ denotes the Dirac-delta measure with support $\{0\}$ and $\mu_o \geqslant 0$. Moreover, this decomposition is unique, by the Lebesgue Decomposition Theorem. Also, it follows from the Radon-Nikodym Theorem that there exists a Lebesgue-measurable function $k : (0,\infty) \rightarrow R^+$ such that

$$\int_E d\mu_r = \int_E k(s)\, ds \tag{9}$$

for all Borel sets $E \epsilon B$. The influence function k is then termed the Radon-Nikodym derivative of μ_r with respect to the Lebesgue measure. Postulates (A) and (B) also require that $k(s)$ must decay to 0 at least as $o(1/s)$ as $s \rightarrow 0$ [16]. This in turn implies that μ is an obliviating measure that has the fading memory property in the sense of WANG. Moreover, postulate (C) requires that $k(s)$ cannot decay to zero too fast as $s \rightarrow \infty$, either. In fact, it follows [16] that, in order for a-sections to be always defined, the rate of decay of $k(s)$ has to be at most exponential. Finally, under assumptions (A), (B) and (C), it can be shown [16] that the families operators $\mathbf{R}(a)$ and $\mathbf{L}(a)$, $a > 0$, form in fact two strongly continuous semigroups of bounded operators in X.

As a consequence of the representation (8) of the influence measure μ, the norm on X now takes the form

$$||\mathbf{f}|| = \left[(\mu_o |\mathbf{f}(0)|)^p + ||\mathbf{f}_r||_r^p\right]^{1/p} \tag{10}$$

where $\mathbf{f}_r$ signifies the restriction of $\mathbf{f}$ to $(0,\infty)$, or "past history," and

$$||\mathbf{f}_r||_r = \left[\int_0^\infty |\mathbf{f}_r(s)|^p \, d\mu_r(s)\right]^{1/p} = \left[\int_0^\infty |\mathbf{f}_r(s)|^p \, k(s)\, ds\right]^{1/p} \tag{11}$$

Thus, the space of histories X can be expressed, algebraically and topologically, as the direct sum $X = V + X_r$ of V and the space X_r of past histories. For a material with the fading memory property, if μ_o is chosen to be 0, the continuity of $\mathbf{T}$ requires that it be defined solely over X_r, and the material does not exhibit instantaneous response. On the other hand, if μ_r is chosen to be the trivial measure, the continuity of $\mathbf{T}$ requires that it be defined over V only, and the material does not exhibit memory effects.

3. An Existence and Uniqueness Theorem

The nonlinear theory of semigroups has experienced considerable progress over the past recent years, and has been successfully applied to a variety of problems in physics and engineering [20,26,27]. At present, it constitutes an extensive subject in itself as well as in its applications. To the physicist and the engineer, the main interest in semigroup theory lies in its application to the study of linear and nonlinear initial value problems. This section presents a detailed example in which nonlinear semigroup theory is applied to the problem of existence and uniqueness for evolutionary equations arising in inelastic materials with memory effects.

The results obtained here are a generalization of a linear version obtained by NAVARRO [17]. NAVARRO's Theorem is concerned with linear thermoelastic materials with memory. Suitable restrictions on the (linear) constitutive laws of the material are imposed that insure compliance with the conditions of the Hille-Yosida Theorem and thus result in existence and uniqueness of the solution.

In the present approach, general nonlinear hereditary laws are considered. Following COLEMAN & NOLL, the fading memory property is formulated in terms of an obliviating measure, in the manner outlined in the preceding section. Suitable restrictions on this measure and on the constitutive mapping are discussed that result in a well-posed initial value problem. For instance, a monotonicity condition is proposed for the hereditary constitutive mapping that generalizes the concept of normality of materials with instantaneous inelastic response.

We proceed to formulate the problem. It is supposed that the body occupies a bounded region Ω in R^N with smooth boundary $\partial\Omega$, and that the reference configuration is stress-free. The symbols $\mathbf{v}$, $\boldsymbol{\sigma}$ and $\dot{\boldsymbol{\epsilon}}^p$ will be used to signify the velocity, stress and inelastic strain rate fields over Ω, respectively. Infinitesimal deformations are assumed throughout. Then, the equations of motion for the body can be expressed

$$\begin{aligned} \frac{d\mathbf{v}}{dt} &= \frac{1}{\rho}\nabla\cdot\boldsymbol{\sigma} \\ \frac{d\boldsymbol{\sigma}}{dt} &= D\cdot(\nabla\mathbf{v} - \dot{\boldsymbol{\epsilon}}^p) \end{aligned} \tag{12}$$

where $\rho > 0$ represents the mass density and D the elastic compliances of the material, which are assumed to be symmetric and positive definite. For simplicity, the boundary conditions are assumed to be homogeneous and of the Dirichlet type.

It is also assumed that the inelasticity of the material exhibits memory effects so that the inelastic strain rates $\dot{\epsilon}^p(\omega,t)$ at every point $\omega \in \Omega$ and time t are related to the stress history at the same point, say $\sigma^t(\omega,s)$, $0 \leqslant s < \infty$, through a nonlinear mapping

$$D \cdot \dot{\epsilon}^p(\omega,t) = T\sigma^t(\omega) \tag{13}$$

The unknowns of the problem are, therefore, the velocity, stress and stress history fields, $\mathbf{v}(t)$, $\boldsymbol{\sigma}(t)$ and $\boldsymbol{\sigma}^t$, respectively, for all times $t > 0$. The evolution of the system is assumed to take place in the Hilbert space $H = H_u \times H_\tau \times H_\Lambda$, where H_u and H_τ are the velocity and stress field spaces which are taken to coincide with the Lebesgue space $L^2(\Omega)$ with the inner products

$$<\mathbf{v}_1,\mathbf{v}_2>_{H_u} = \int_\Omega \rho\, \mathbf{v}_1 \cdot \mathbf{v}_2 \, d\omega \tag{14}$$

and

$$<\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2>_{H_\tau} = \int_\Omega \boldsymbol{\sigma}_1 \cdot \mathbf{C} \cdot \boldsymbol{\sigma}_2 \, d\omega \tag{15}$$

where $\mathbf{C} = \mathbf{D}^{-1}$. On the other hand, H_Λ denotes the space of past history stress fields, which is taken to coincide with the Lebesgue space $L^2([0,\infty), H_\tau, \mu)$, with the inner product

$$<\boldsymbol{\Lambda}_1, \boldsymbol{\Lambda}_2>_{H_\Lambda} = \int_0^\infty <\boldsymbol{\Lambda}_1(s), \boldsymbol{\Lambda}_2(s)>_{H_\tau} d\mu_r(s) \tag{16}$$

where μ_r is an obliviating measure, with the properties discussed in the preceding section. This in turn endows H with the following inner product

$$<\begin{Bmatrix} \mathbf{v}_1 \\ \boldsymbol{\sigma}_1 \\ \boldsymbol{\Lambda}_1 \end{Bmatrix}, \begin{Bmatrix} \mathbf{v}_2 \\ \boldsymbol{\sigma}_2 \\ \boldsymbol{\Lambda}_2 \end{Bmatrix}> = <\mathbf{v}_1, \mathbf{v}_2>_{H_u} + <\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2>_{H_\tau} + <\boldsymbol{\Lambda}_1, \boldsymbol{\Lambda}_2>_{H_\Lambda} \tag{17}$$

It is noted that in the present formulation the space H_τ plays the role of the space V in the introduction. Furthermore, in the norm associated with (17), the term $||\boldsymbol{\sigma}||_{H_\tau}$ concerns the instantaneous values of the stress field and is analogous to the first term in the right hand side of (10), while the term $||\boldsymbol{\Lambda}||_{H_\Lambda}$ is associated with the past history of the stress field and therefore corresponds to the norm (11).

The pointwise application of the mapping T in eq. (13) (modulo null sets) defines a mapping from H_τ into itself that we shall denote by $\mathbf{T}$. With this notation, the equations of motion (12) can be rephrased

$$\begin{aligned} \frac{d}{dt}\mathbf{v}(t) &= \frac{1}{\rho}\nabla\cdot\boldsymbol{\sigma}(t) \\ \frac{d}{dt}\boldsymbol{\sigma}(t) &= D\cdot\nabla\mathbf{v}(t) - \mathbf{T}\boldsymbol{\sigma}^{t} \end{aligned} \tag{18}$$

For these equations to be complete, they have to be supplemented with an evolutionary equation for $\boldsymbol{\sigma}^t$. Clearly, as time goes on, the history $\boldsymbol{\sigma}^t$ shifts to the right, and the current values of the stress field $\boldsymbol{\sigma}$ are fed in from the left into $\boldsymbol{\sigma}^t$. The infinitesimal generator for this type of evolution is readily found to be [18]

$$\left(\frac{d}{dt}\boldsymbol{\sigma}^t\right)(\omega, s) = -\frac{\partial}{\partial s}\boldsymbol{\sigma}^t(\omega, s) \tag{19}$$

along with the compatibility condition

$$\boldsymbol{\sigma}^t(0) = \boldsymbol{\sigma}(t) \tag{20}$$

The partial derivative in the right hand side of (19) is taken in the distributional sense.

Eqs. (18), (19) and (20), together with initial conditions

$$\begin{aligned} \mathbf{v}(0) &= \mathbf{v}_0 \\ \boldsymbol{\sigma}(0) &= \boldsymbol{\sigma}_0 \\ \boldsymbol{\sigma}^{t=0} &= \boldsymbol{\sigma}^0 \end{aligned} \tag{21}$$

define and initial value problem in H. This initial value problem can be rephrased in a more compact fashion by introducing the notation

$$\mathbf{x}(t) = \begin{Bmatrix} \mathbf{v}(t) \\ \boldsymbol{\sigma}(t) \\ \boldsymbol{\sigma}^t \end{Bmatrix} ; \quad -\mathbf{A} = \begin{bmatrix} 0 & \frac{1}{\rho}\nabla\cdot & 0 \\ D\cdot\nabla & 0 & -\mathbf{T} \\ 0 & 0 & -\frac{\partial}{\partial s} \end{bmatrix} ; \quad \mathbf{x}_0 = \begin{Bmatrix} \mathbf{v}_0 \\ \boldsymbol{\sigma}_0 \\ \boldsymbol{\sigma}^0 \end{Bmatrix} \tag{22}$$

With this notation, eqs. (18), (19) and (21) can be expressed as

$$\dot{\mathbf{x}}(t) + \mathbf{A}\,\mathbf{x}(t) = 0 \quad ; \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{23}$$

The question now arises of whether $-\mathbf{A}$ generates a semigroup of contractions $\mathbf{S}(t)$ in H. To tackle this problem, we postulate the following additional restrictions on the constitutive mapping **T**.

A) **T** is defined in all of H_Λ and is Lipschitz continuous.

B) For every $\sigma_1, \sigma_2 \in H_\tau$ and $\Lambda_1, \Lambda_2 \in H_\Lambda$,

$$<\sigma_1 - \sigma_2, \mathbf{T}\Lambda_1 - \mathbf{T}\Lambda_2>_{H_\tau} = <\sigma_1 - \sigma_2, D\cdot(\dot{\epsilon}_1^p - \dot{\epsilon}_2^p)>_{H_\tau} = <\sigma_1 - \sigma_2, \dot{\epsilon}_1^p - \dot{\epsilon}_2^p>_{L^2} \geqslant 0$$

C) The influence function $k(s)$ for μ_r is assumed to be absolutely continuous and monotonically decreasing.

Example 1. As an example of a material complying with postulates (A), (B) and (C), consider one with instantaneous response only, of the linear viscoplastic type. In this case, a closed convex set **C**, or functional elastic domain is assumed to exist in H_τ, such that the inelastic strain rate field is given by an associated flow rule

$$D\cdot\dot{\epsilon}^p = \frac{1}{\lambda}(\mathbf{I} - \mathbf{P_C})\,\sigma \tag{24}$$

where λ is a viscosity parameter and $\mathbf{P_C}$ denotes the (closest point) projection onto **C**. Clearly, the mapping **T** is defined in all of H_Λ and is Lipschitz continuous [23], so that postulate (A) is satisfied. On the other hand, the monotonicity condition (B) now takes the form

$$<\sigma_1 - \sigma_2, D\cdot(\dot{\epsilon}_1^p - \dot{\epsilon}_2^p)>_{H_\tau} = \frac{1}{\lambda}<\sigma_1 - \sigma_2, (\mathbf{I} - \mathbf{P_C})\,\sigma_1 - (\mathbf{I} - \mathbf{P_C})\,\sigma_2>_{H_\tau} =$$

$$\frac{1}{\lambda}<\mathbf{P_C}\sigma_1 + (\mathbf{I} - \mathbf{P_C})\,\sigma_1 - \mathbf{P_C}\sigma_2 - (\mathbf{I} - \mathbf{P_C})\,\sigma_2, (\mathbf{I} - \mathbf{P_C})\,\sigma_1 - (\mathbf{I} - \mathbf{P_C})\,\sigma_2>_{H_\tau} =$$

$$\frac{1}{\lambda}<\mathbf{P_C}\sigma_1 - \mathbf{P_C}\sigma_2, (\mathbf{I} - \mathbf{P_C})\,\sigma_1 - (\mathbf{I} - \mathbf{P_C})\sigma_2>_{H_\tau} + \frac{1}{\lambda}\|(\mathbf{I} - \mathbf{P_C})\sigma_1 - (\mathbf{I} - \mathbf{P_C})\,\sigma_2\|_{H_\tau}^2 \geqslant 0$$

which is always greater or equal to zero [23] and postulate (B) is also satisfied. It is thus seen that the normality rule of materials with an instantaneous viscoplastic response results in monotonicity in the sense of postulate (B). On these grounds, postulate (B) can be regarded as a generalization of the concept of normality.

Remark. It is noted that the domain **D**(**A**) of the operator **A** is $H^1 = H_u^1 \times H_\tau^1 \times H_\Lambda^1$, where H_u^1 and H_τ^1 and H_Λ^1 signify the Sobolev spaces associated with H_u, H_τ and H_Λ, respectively, H_u^1 being further restricted to those velocity fields satisfying the homogeneous boundary conditions in the usual trace sense. In particular, $H_\Lambda^1 = \{\Lambda \in H_\Lambda \;\; s.t. \;\; \frac{\partial \Lambda}{\partial s} \in H_\Lambda\}$, where the partial derivative is taken in the distributional sense. We recall that, by the Sobolev embedding Lemma [19], H_Λ^1 is a subset of the set of continuous functions $C([0,\infty), H_\tau)$, and, hence, point values of the Λ's in $\mathbf{D}(\mathbf{A})$, as in eq. (20), are well-defined.

Theorem 1. The operator $-\mathbf{A}$ generates a semigroup of contractions in H.

Proof. It is shown in this proof that the operator $\mathbf{A}$ is maximal monotone. Then, it follows from a theorem by KOMURA [24] that $-\mathbf{A}$ generates a semigroup of contractions in $\overline{\mathbf{D}(\mathbf{A})} = H$. We start by splitting the operator $\mathbf{A}$ into two parts

$$-\mathbf{A} = \begin{pmatrix} 0 & \frac{1}{\rho}\nabla\cdot & 0 \\ D\cdot\nabla & 0 & 0 \\ 0 & 0 & -\frac{\partial}{\partial s} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathbf{T} \\ 0 & 0 & 0 \end{pmatrix} \equiv -(\tilde{\mathbf{W}} + \mathbf{T}) \tag{25}$$

Here, the operator $\tilde{\mathbf{W}}$ is linear and densely defined, with $\mathbf{D}(\tilde{\mathbf{W}}) = \mathbf{D}(\mathbf{A}) = H^1$. Next we show that it is also monotone. To this end, consider $\mathbf{x}_1, \mathbf{x}_2 \in \mathbf{D}(\tilde{\mathbf{W}})$. Then

$$< \tilde{\mathbf{W}}\,\mathbf{x}_1 - \tilde{\mathbf{W}}\,\mathbf{x}_2, \mathbf{x}_1 - \mathbf{x}_2 > =$$
$$-<\nabla\cdot(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2), \mathbf{v}_1 - \mathbf{v}_2>_{H_u} - <D\cdot\nabla(\mathbf{v}_1 - \mathbf{v}_2), \boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2>_{H_\tau} + <\frac{\partial}{\partial s}(\Lambda_1 - \Lambda_2), \Lambda_1 - \Lambda_2>_{H_\Lambda}$$

Integrating by parts and making use of (16)

$$< \tilde{\mathbf{W}}\,\mathbf{x}_1 - \tilde{\mathbf{W}}\,\mathbf{x}_2, \mathbf{x}_1 - \mathbf{x}_2 > =$$
$$<\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2, \nabla(\mathbf{v}_1 - \mathbf{v}_2)>_{L^2} - <\nabla(\mathbf{v}_1 - \mathbf{v}_2), \boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2>_{L^2} + <\frac{\partial}{\partial s}(\Lambda_1 - \Lambda_2), \Lambda_1 - \Lambda_2>_{H_\Lambda} =$$
$$\int_0^\infty <\frac{\partial}{\partial s}(\Lambda_1(s) - \Lambda_2(s)), \Lambda_1(s) - \Lambda_2(s)>_{H_\tau} k(s)\, ds = \frac{1}{2}\int_0^\infty \frac{\partial}{\partial s}||\Lambda_1(s) - \Lambda_2(s)||_{H_\tau}^2\, k(s)\, ds$$

Integrating by parts with respect to s

$$< \tilde{\mathbf{W}}\,\mathbf{x}_1 - \tilde{\mathbf{W}}\,\mathbf{x}_2, \mathbf{x}_1 - \mathbf{x}_2 > = \frac{1}{2}k(0)\,||\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2||_{H_\tau}^2 - \frac{1}{2}\int_0^\infty ||\Lambda_1(s) - \Lambda_2(s)||_{H_\tau}^2\, k'(s)\, ds \geqslant 0$$

which is always greater than 0 by virtue of postulate (C). Moreover, $\tilde{\mathbf{W}}$ clearly defines the semigroup

$$\mathbf{S}_{\tilde{\mathbf{W}}}(t)\,\mathbf{x}_0 = \begin{cases} \begin{Bmatrix} \mathbf{v}(t) \\ \boldsymbol{\sigma}(t) \end{Bmatrix} = \mathbf{S}_{\mathbf{W}}(t) \begin{Bmatrix} \mathbf{v}_0 \\ \boldsymbol{\sigma}_0 \end{Bmatrix} \\ \boldsymbol{\sigma}^t(s) = \begin{cases} \boldsymbol{\sigma}^0(s-t), & s > t \\ \boldsymbol{\sigma}(s), & 0 < s < t \end{cases} \end{cases} \tag{26}$$

where $\mathbf{S_W}(t)$ denotes the unitary group generated by the linear elasticity wave operator and $\sigma^t(s)$ merely records the stress trajectory defined by $\mathbf{S_W}(t)$. Clearly, the stress histories $\sigma^t(s)$ defined in (26) belong to H^1, for every $t > 0$, due to the fact that the trajectory $\sigma(t)$ determined by $\mathbf{S_W}(t)$ is differentiable almost everywhere [20]. Moreover, the semigroup (26) is a C_o semigroup of bounded operators, due to the C_o character and boundedness of $\mathbf{S_W}(t)$ and of the right shift in H_Λ, [16,18]. In other words, the operator $\tilde{\mathbf{W}}$ generates a linear C_o semigroup of bounded operators, and, therefore, it is maximal monotone by the Hille-Yosida Theorem.

On the other hand, $\mathbf{T}$ is monotone, Lipschitz continuous and its domain, $\mathbf{D(T)} = H$, contains the domain of $\tilde{\mathbf{W}}$, H^1, by postulates (A) and (B). Hence, by a result by CRANDALL & PAZY [25], the operator $\tilde{\mathbf{W}} + \mathbf{T}$ is maximal monotone, and $-\mathbf{A} = -(\tilde{\mathbf{W}} + \mathbf{T})$ generates a semigroup of contractions in H, by KOMURA's Theorem [24]. ////

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