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A canonical form of the equation of motion of linear dynamical systems

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The equation of motion of a discrete linear system has the form of a second-order ordinary differential equation with three real and square coefficient matrices. It is shown that, for almost all linear systems, such an equation can always be converted by an invertible transformation into a canonical form specified by two diagonal coefficient matrices associated with the generalized acceleration and displacement. This canonical form of the equation of motion is unique up to an equivalence class for non-defective systems. As an important by-product, a damped linear system that possesses three symmetric and positive definite coefficients can always be recast as an undamped and decoupled system.

1. Introduction

The equation of motion of linear systems is one of the most commonly used equations in science and engineering. This equation possesses three real and square coefficient matrices of the same order, and the properties of the coefficient matrices allow the system concerned to be classified. For example, an undamped gyroscopic system possesses a skew-symmetric coefficient of velocity [1]. If a system is elastic and non-circulatory, then the coefficient of displacement is symmetric. And so one may go on. Of particular significance is the class of non-gyroscopic, non-circulatory, passive systems

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characterized by three constant, symmetric and positive definite matrices. For brevity, this class of systems is referred to as passive or as damped linear systems. There should be no denying that the bulk of existing literature on linear vibration and structural dynamics deals implicitly or explicitly with passive systems.

It is well known that the equation of motion of a single-degree-of-freedom passive system can be converted into an undamped system by an invertible transformation. For a multi-degree-of-freedom passive system, this reduction poses a challenge because the equation of motion is usually coupled. The reduction is still permissible under the assumption of classical damping, whereby a passive system can be decoupled by modal analysis into a series of independent single-degree-of-freedom systems. In general, passive systems are non-classically damped, and reduction of the equation of motion of such systems has not been reported in the open literature.

The purpose of this paper is to show that almost all linear systems can be transformed so as to eliminate the coefficient of velocity from their equations of motion. In addition, the remaining two coefficient matrices can be reduced to diagonal forms. This paper builds upon earlier works [2–5] in the decoupling of linear systems. The original impetus was to show that any passive system can be transformed into an undamped one, an important result that has become an offshoot. The organization of the paper is as follows. In §2, the reduction of the equation of motion to a canonical form specified by two diagonal matrices is formulated in mathematical terms and previously known results are reviewed. This is followed in §3 by a concise exposition of an extension of modal analysis to decouple non-defective linear systems in real space. In §4, an explicit transformation to generate the canonical form of the equation of motion of non-defective systems is developed. The reduction of defective linear systems is treated in §5. A summary of findings is provided in §6. Two numerical examples are supplied for illustration.

2. Problem formulation

The equation of motion of an n-degree-of-freedom linear system can be written as

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{f}(t), \tag{2.1}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are arbitrary but real $n \times n$ matrices. The generalized coordinate $\mathbf{q}$ and the excitation $\mathbf{f}(t)$ are n-dimensional column vectors. It is not assumed that $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ possess the familiar properties of symmetry and definiteness, and thus equation (2.1) represents the so-called linear non-conservative systems [6]. However, $\mathbf{M}$ is assumed non-singular. This assumption is not unduly restrictive, as it might be possible to initially decrease the number of degrees of freedom to ensure that $\mathbf{M}$ is non-singular. It will be shown that equation (2.1) can be reduced, by an invertible transformation, to the real decoupled form

$$\ddot{\mathbf{x}} + \mathbf{B}\mathbf{x} = \mathbf{h}(t), \tag{2.2}$$

where $\mathbf{B}$ is a diagonal matrix, and the generalized displacement $\mathbf{x}$ and excitation $\mathbf{h}(t)$ are n-dimensional column vectors. Basically, a transformation will be found to convert $\mathbf{M}$ and $\mathbf{K}$ into diagonal matrices while removing $\mathbf{C}\dot{\mathbf{q}}$ at the same time. The canonical form specified by equation (2.2) is the simplest representation of linear dynamical systems.

When $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are symmetric and positive definite, they are referred to as the mass, damping and stiffness matrices, respectively. In this case, the system is termed a passive or damped linear system. It is well known that passive systems either of a single degree or under classical damping can be reduced to an undamped form. Should equation (2.1) represent a single-degree-of-freedom system, it may be rewritten as

$$m\ddot{q} + c\dot{q} + kq = f(t), \tag{2.3}$$

where m, c and k are just real numbers. Using the invertible transformation [7,8]

$$y = \exp\left(\frac{c}{2m}t\right) q, \tag{2.4}$$

it can be readily verified that equation (2.3) is converted into

$$\ddot{y} + \left(\frac{k}{m} - \frac{c^2}{4m^2}\right) y = \frac{1}{m} \exp\left(\frac{c}{2m}t\right) f(t). \tag{2.5}$$

This undamped form is sometimes referred to as the normal form of a single-degree-of-freedom system. Perhaps it would not be surprising that transformation to an undamped form involves an exponential factor. In free vibration, the response q decays exponentially with any amount of viscous damping. This decay is arrested by the exponential term in equation (2.4), which also exponentially magnifies the excitation of y in equation (2.5).

If a passive system is classically damped, then it can be decoupled by modal analysis into a series of independent single-degree-of-freedom systems. A necessary and sufficient condition [9] under which a passive system is classically damped is

$$\mathbf{C}\mathbf{M}^{-1}\mathbf{K} = \mathbf{K}\mathbf{M}^{-1}\mathbf{C}. \tag{2.6}$$

Proportional damping is just a special case of classical damping. Associated with a passive system is the symmetric eigenvalue problem

$$\mathbf{K}\mathbf{u} = \lambda \mathbf{M}\mathbf{u}. \tag{2.7}$$

Owing to the positive definiteness of $\mathbf{M}$ and $\mathbf{K}$, all eigenvalues λ_j $(j = 1, 2, \ldots, n)$ are real and positive, and the corresponding eigenvectors $\mathbf{u}_j$ are real and orthogonal with respect to either $\mathbf{M}$ or $\mathbf{K}$. Denote the $n \times n$ modal and spectral matrices, respectively, by

$$\mathbf{U} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \end{bmatrix}, \quad \boldsymbol{\Omega} = \mathrm{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n]. \tag{2.8}$$

Upon normalization of the eigenvectors with respect to the mass matrix, the generalized orthogonality of the eigenvectors can be expressed as

$$\mathbf{U}^T\mathbf{M}\mathbf{U} = \mathbf{I}, \qquad \mathbf{U}^T\mathbf{K}\mathbf{U} = \boldsymbol{\Omega}. \tag{2.9}$$

Using the modal transformation $\mathbf{q} = \mathbf{U}\mathbf{p}$, where $\mathbf{p}$ is an n-dimensional vector of principal coordinates, a passive system represented by equation (2.1) can be converted into

$$\ddot{\mathbf{p}} + \mathbf{D}\dot{\mathbf{p}} + \boldsymbol{\Omega}\mathbf{p} = \mathbf{U}^T\mathbf{f}(t), \tag{2.10}$$

for which the modal damping matrix $\mathbf{D} = \mathbf{U}^T\mathbf{C}\mathbf{U}$ is diagonal under classical damping. To eliminate the damping term in equation (2.10), apply the transformation

$$\mathbf{y} = \exp\left(\frac{1}{2}\mathbf{D}t\right) \mathbf{p}. \tag{2.11}$$

Observe that $\exp(\mathbf{D}t/2)$ is a diagonal matrix. Upon transformation, equation (2.10) is converted into

$$\ddot{\mathbf{y}} + \left(\boldsymbol{\Omega} - \frac{1}{4}\mathbf{D}^2\right) \mathbf{y} = \exp\left(\frac{1}{2}\mathbf{D}t\right) \mathbf{U}^T\mathbf{f}(t). \tag{2.12}$$

The original system becomes undamped and decoupled with respect to the coordinate $\mathbf{y}$, which is connected with $\mathbf{q}$ by

$$\mathbf{q} = \mathbf{U} \exp\left(-\frac{1}{2}\mathbf{D}t\right) \mathbf{y}. \tag{2.13}$$

The key to successful reduction of the equation of motion of classically damped linear systems, as described earlier, is decoupling in real space. In general, there is no reason why equation (2.6) should be satisfied for modal analysis to be applicable. Indeed, experimental modal testing suggests that no physical system is strictly classically damped [10]. The reduction of non-classically damped systems to an undamped form has not been reported in the open literature. In this connection, an attempt was made to reduce the equation of motion of damped gyroscopic systems, for which only the coefficient matrix **C** is non-symmetric, to a form devoid of the velocity term [11]. In addition, the possibility of decoupling equation (2.1) by a time-invariant linear transformation (analogous to modal analysis) was examined [12]. It has been found that a condition equivalent to equation (2.6) is required in both cases.

3. Generalization of modal analysis

Recently, modal analysis has been extended such that almost all linear systems can be decoupled in real space [2–5]. Specifically, a real and invertible transformation has been determined to convert equation (2.1) into

$$\ddot{\mathbf{p}} + \mathbf{D}\dot{\mathbf{p}} + \boldsymbol{\Omega}\mathbf{p} = \mathbf{g}(t), \tag{3.1}$$

for which the $n \times n$ coefficient matrices **D** and $\boldsymbol{\Omega}$ are real and diagonal. Unless equation (2.1) represents a classically damped passive system, **D** and $\boldsymbol{\Omega}$ are not the same as the modal damping and spectral matrices, respectively. There are no scientific restrictions on this extension of modal analysis, which is termed the method of phase synchronization. All parameters required for decoupling are obtained through the solution of the quadratic eigenvalue problem

$$(\mathbf{M}\lambda^2 + \mathbf{C}\lambda + \mathbf{K})\mathbf{v} = \mathbf{0}. \tag{3.2}$$

The system represented by equation (2.1) is said to be non-defective when every repeated eigenvalue of equation (3.2) possesses a full complement of independent eigenvectors. To streamline the presentation, it is assumed that all eigenvalues of equation (3.2) are distinct, which guarantees that the system concerned is non-defective. Relaxation of this assumption to include defective systems, which must possess repeated eigenvalues, will be considered in a subsequent section. Perhaps an alternative viewpoint on repeated eigenvalues should be brought up. If **M**, **C** and **K** are randomly chosen from a uniform distribution, the probability that all eigenvalues of equation (3.2) are distinct is one [2]. In this sense, almost all linear systems are characterized by distinct eigenvalues.

(a) Methodology for decoupling non-defective systems

To provide a concise exposition, an implementation[1] of phase synchronization to decouple non-defective systems with distinct eigenvalues is summarized as a series of tasks. The theory of phase synchronization is expounded in [2–4], and formulas provided in [4] are drawn upon in this presentation.

Task 1. *Solve the quadratic eigenvalue problem (3.2) and index the eigensolutions.*

There are $2n$ eigensolutions, and any complex eigensolutions occur in complex conjugate pairs. Suppose $2c$ eigenvalues are complex and the remaining $2r = 2(n-c)$ are real. The c complex eigenvalues with positive imaginary parts are arranged in order of increasing magnitude of their imaginary parts as the first c eigenvalues such that

$$S_1 = \{\lambda_1, \lambda_2, \ldots, \lambda_c : 0 < \mathrm{Im}[\lambda_1] \leq \mathrm{Im}[\lambda_2] \leq \cdots \leq \mathrm{Im}[\lambda_c]\}. \tag{3.3}$$

Since $\mathrm{Im}[\lambda_j]$ can often be regarded as a frequency of vibration, this is consistent with the convention of arranging frequencies in order of increasing magnitude. Enumerate the remaining c complex eigenvalues, which are the complex conjugates with negative imaginary parts, in such a way that

$$S_3 = \{\lambda_{n+1} = \bar{\lambda}_1,\ \lambda_{n+2} = \bar{\lambda}_2,\ \ldots,\ \lambda_{n+c} = \bar{\lambda}_c\}. \tag{3.4}$$

Thus, $S_1 \cup S_3$ contains the entire set of $2c$ complex eigenvalues. The real eigenvalues are arranged in accordance with a primary–secondary pairing scheme [3]. Among the $2r$ real eigenvalues, the r largest eigenvalues are referred to as primary eigenvalues and the r smallest eigenvalues are

[1] A computer program for decoupling linear systems is available upon request.

termed secondary eigenvalues. Enumerate the r real secondary eigenvalues in order of increasing magnitude such that

$$S_2 = \{\lambda_{c+1}, \lambda_{c+2}, \ldots, \lambda_n : \lambda_{c+1} < \lambda_{c+2} < \cdots < \lambda_n\}. \tag{3.5}$$

Enumerate the remaining r real primary eigenvalues also in order of increasing magnitude so that

$$S_4 = \{\lambda_{n+c+1}, \lambda_{n+c+2}, \ldots, \lambda_{2n} : \lambda_{n+c+1} < \lambda_{n+c+2} < \cdots < \lambda_{2n}\}. \tag{3.6}$$

Thus, $S_2 \cup S_4$ contains the entire set of $2r$ real eigenvalues under the constraint that $\sup S_2 < \inf S_4$. The $2n$ eigenvalues are partitioned into four disjoint subsets. A different indexing scheme for the eigensolutions may be used, subject to the requirement that complex conjugate eigensolutions are always paired.

Task 2. *Normalize the eigenvectors of equation (3.2).*

After the eigensolutions have been indexed, the $2n$ eigenvectors are normalized in accordance with

$$2\lambda_j \mathbf{v}_j^T \mathbf{v}_j + \mathbf{v}_j^T \mathbf{C} \mathbf{v}_j = \lambda_j - \lambda_{n+j} \tag{3.7}$$

and

$$2\lambda_{n+j} \mathbf{v}_{n+j}^T \mathbf{v}_{n+j} + \mathbf{v}_{n+j}^T \mathbf{C} \mathbf{v}_{n+j} = \lambda_{n+j} - \lambda_j \tag{3.8}$$

for $1 \le j \le n$. The above normalization reduces to mass-normalization for undamped or classically damped passive systems [10,13]. This task is optional, and a different scheme for normalizing the eigenvectors may also be used.

Task 3. *Construct the decoupled equation (3.1) using the eigenvalues and eigenvectors of equation (3.2).*

Using the indexed eigensolutions, assemble the following $n \times n$ matrices:

$$\boldsymbol{\Lambda}_1 = \text{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n], \quad \boldsymbol{\Lambda}_2 = \text{diag}[\lambda_{n+1}, \lambda_{n+2}, \ldots, \lambda_{2n}] \tag{3.9}$$

and

$$\mathbf{V}_1 = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}, \quad \mathbf{V}_2 = \begin{bmatrix} \mathbf{v}_{n+1} & \mathbf{v}_{n+2} & \cdots & \mathbf{v}_{2n} \end{bmatrix}. \tag{3.10}$$

The real and diagonal coefficients of equation (3.1) are given by

$$\mathbf{D} = -(\boldsymbol{\Lambda}_1 + \boldsymbol{\Lambda}_2), \quad \boldsymbol{\Omega} = \boldsymbol{\Lambda}_1 \boldsymbol{\Lambda}_2. \tag{3.11}$$

The excitation $\mathbf{g}(t)$ of equation (3.1) is given in terms of $\mathbf{f}(t)$ by

$$\mathbf{g}(t) = \left(\mathbf{D} + \mathbf{I}\frac{\mathrm{d}}{\mathrm{d}t}\right)\mathbf{G}_1 \mathbf{f}(t) + \mathbf{G}_2 \mathbf{f}(t), \tag{3.12}$$

where $\mathbf{G}_1$ and $\mathbf{G}_2$ are real $n \times n$ matrices computed in accordance with

$$\mathbf{G}_1 = [(\mathbf{V}_1\boldsymbol{\Lambda}_1 - \mathbf{V}_2\boldsymbol{\Lambda}_2\mathbf{V}_2^{-1}\mathbf{V}_1)^{-1} + (\mathbf{V}_2\boldsymbol{\Lambda}_2 - \mathbf{V}_1\boldsymbol{\Lambda}_1\mathbf{V}_1^{-1}\mathbf{V}_2)^{-1}]\mathbf{M}^{-1} \tag{3.13}$$

and

$$\mathbf{G}_2 = [\boldsymbol{\Lambda}_1(\mathbf{V}_1\boldsymbol{\Lambda}_1 - \mathbf{V}_2\boldsymbol{\Lambda}_2\mathbf{V}_2^{-1}\mathbf{V}_1)^{-1} + \boldsymbol{\Lambda}_2(\mathbf{V}_2\boldsymbol{\Lambda}_2 - \mathbf{V}_1\boldsymbol{\Lambda}_1\mathbf{V}_1^{-1}\mathbf{V}_2)^{-1}]\mathbf{M}^{-1}. \tag{3.14}$$

Task 4. *Construct the real decoupling transformations in the configuration and state spaces.*

Assemble the following real $n \times n$ matrices:

$$\mathbf{T}_1 = (\mathbf{V}_1\boldsymbol{\Lambda}_2 - \mathbf{V}_2\boldsymbol{\Lambda}_1)(\boldsymbol{\Lambda}_2 - \boldsymbol{\Lambda}_1)^{-1}, \quad \mathbf{T}_2 = (\mathbf{V}_2 - \mathbf{V}_1)(\boldsymbol{\Lambda}_2 - \boldsymbol{\Lambda}_1)^{-1}. \tag{3.15}$$

The configuration-space decoupling transformation can be expressed as

$$\mathbf{q} = \left(\mathbf{T}_1 + \mathbf{T}_2\frac{\mathrm{d}}{\mathrm{d}t}\right)\mathbf{p} - \mathbf{T}_2\mathbf{G}_1\mathbf{f}(t). \tag{3.16}$$

When cast in the state space, the decoupling transformation takes the form

$$\begin{bmatrix} \mathbf{p}(t) \\ \dot{\mathbf{p}}(t) \end{bmatrix} = \mathbf{S}\begin{bmatrix} \mathbf{q}(t) \\ \dot{\mathbf{q}}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{G}_1\mathbf{f}(t) \end{bmatrix}, \tag{3.17}$$

where the $2n \times 2n$ real and invertible matrix $\mathbf{S}$ is given by

$$\mathbf{S} = \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \boldsymbol{\Lambda}_1 & \boldsymbol{\Lambda}_2 \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 & \mathbf{V}_2 \\ \mathbf{V}_1\boldsymbol{\Lambda}_1 & \mathbf{V}_2\boldsymbol{\Lambda}_2 \end{bmatrix}^{-1}. \tag{3.18}$$

The upper half of equation (3.17) yields a configuration-space mapping from $\mathbf{q}$ to $\mathbf{p}$ that is an inverse of equation (3.16). When $t = 0$, equation (3.17) generates the initial values $\mathbf{p}(0)$ and $\dot{\mathbf{p}}(0)$ of equation (3.1). The decoupling transformations in both the configuration and state spaces are nonlinear for non-homogeneous systems and linear for homogeneous systems.

(b) Relationship with modal analysis

The decoupling procedure expounded earlier is a direct extension of modal analysis. If equation (2.1) represents an undamped passive system with a mass-normalized modal matrix $\mathbf{U}$, then the eigenvectors of equation (3.2) are such that $\mathbf{V}_1 = \mathbf{V}_2 = \mathbf{U}$ up to arbitrary signs in the columns of $\mathbf{U}$. Consequently,

$$\mathbf{T}_1 = \mathbf{U}, \quad \mathbf{T}_2 = \mathbf{0}. \tag{3.19}$$

In this case, the configuration-space decoupling transformation represented by equation (3.16) reduces to the modal transformation $\mathbf{q} = \mathbf{U}\mathbf{p}$. With different indexing schemes, phase synchronization generates all possible decoupled forms into which a linear system can be transformed in real space [3,4].

4. Generation of the canonical form

An explicit transformation is developed in this section to convert equation (2.1) into the canonical form specified by equation (2.2). When the eigenvalues of equation (3.2) are distinct, equation (2.1) can be decoupled into equation (3.1) by either the configuration-space transformation (3.16) or the state-space transformation (3.17). To eliminate the velocity term in equation (3.1), apply the transformation

$$\mathbf{x} = \exp\left(\frac{1}{2}\mathbf{D}t\right)\mathbf{p}. \tag{4.1}$$

Upon transformation, equation (3.1) is converted into equation (2.2), for which

$$\mathbf{B} = \boldsymbol{\Omega} - \frac{1}{4}\mathbf{D}^2 \tag{4.2}$$

is a real diagonal matrix, and

$$\mathbf{h}(t) = \exp\left(\frac{1}{2}\mathbf{D}t\right)\mathbf{g}(t) = \exp\left(\frac{1}{2}\mathbf{D}t\right)\left\{\left(\mathbf{D} + \mathbf{I}\frac{\mathrm{d}}{\mathrm{d}t}\right)\mathbf{G}_1\mathbf{f}(t) + \mathbf{G}_2\mathbf{f}(t)\right\}. \tag{4.3}$$

Consequently, when recast in the generalized coordinate $\mathbf{x}$, equation (2.1) takes on a decoupled form devoid of the velocity term. To determine a configuration-space transformation between $\mathbf{q}$ and $\mathbf{x}$, combine equations (3.16) and (4.1) to yield

$$\mathbf{q} = \left(\mathbf{T}_1 + \mathbf{T}_2\frac{\mathrm{d}}{\mathrm{d}t}\right)\exp\left(-\frac{1}{2}\mathbf{D}t\right)\mathbf{x} - \mathbf{T}_2\mathbf{G}_1\mathbf{f}(t). \tag{4.4}$$

Alternatively, a state-space transformation can be determined by combining equations (3.17) and (4.1) to obtain

$$\begin{bmatrix} \mathbf{x}(t) \\ \dot{\mathbf{x}}(t) \end{bmatrix} = \begin{bmatrix} \exp(\mathbf{D}t/2) & \mathbf{0} \\ (\mathbf{D}/2)\exp(\mathbf{D}t/2) & \exp(\mathbf{D}t/2) \end{bmatrix} \left\{ \mathbf{S}\begin{bmatrix} \mathbf{q}(t) \\ \dot{\mathbf{q}}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{G}_1\mathbf{f}(t) \end{bmatrix} \right\}. \tag{4.5}$$

When $t = 0$, the above state-space transformation generates the initial values $\mathbf{x}(0)$ and $\dot{\mathbf{x}}(0)$ of the canonical form (2.2). The transformations given by equations (4.4) and (4.5) are both real, nonlinear and invertible. In the reduction of equation (2.1), the canonical form specified by equation (2.2) is the simplest representation that one may achieve.

The generation of the canonical form defined by equation (2.2) is certainly applicable to passive systems, which are characterized by three symmetric and positive definite coefficient matrices. Consequently, a solution to the well-trodden problem of reducing a damped linear system to an undamped form has been provided herein.

(a) Uniqueness of the canonical form

How many different canonical forms, of the type defined by equation (2.2) into which equation (2.1) can be reduced, are there? It is obvious that the canonical form (2.2) is unique if phase synchronization generates a unique decoupled system represented by equation (3.1). However, phase synchronization can be implemented with different indexing and normalization schemes. For a given indexing scheme, the coefficient matrices $\mathbf{D}$ and $\boldsymbol{\Omega}$ of equation (3.1) are independent of the normalization of eigenvectors because they are constructed from the eigenvalues. As a result, the homogenous part of equation (2.2) remains unchanged by normalization. By contrast, the excitation $\mathbf{h}(t)$ of equation (2.2) is dependent on the eigenvectors of equation (3.2) and, therefore, on the normalization used. However, normalization has no physical significance and is just a matter of convenience. For a given indexing scheme, the canonical form (2.2) is unique up to the normalization of eigenvectors.

There remains the question of equivalence due to different indexing schemes. Two decoupled systems are regarded as the same if their component equations coincide; the order in which the component equations appear is immaterial. Hence, indexing schemes that re-order the component equations of equation (2.2) are considered equivalent. Any indexing scheme must pair the complex conjugate eigensolutions. For a given normalization scheme, there is only one decoupled system associated with equation (3.1) if all eigenvalues are complex, and, therefore, only one canonical form defined by equation (2.2). If there are $2r$ distinct real eigenvalues of equation (3.2), then there are

$$N = \frac{\binom{2r}{2}\binom{2r-2}{2}\binom{2r-4}{2}\cdots\binom{2}{2}}{r!} = \frac{(2r)!}{2^r r!} \tag{4.6}$$

different ways to pair the real eigensolutions [3]. Indeed, using a fixed normalization but different indexing schemes, there are N different decoupled systems associated with equation (3.1), and hence N different canonical forms defined by equation (2.2). These N canonical forms usually have different homogeneous parts. For a non-defective system with repeated eigenvalues, the number of different canonical forms is less than N. It can be stated that various indexing and normalization schemes generate an equivalence class of canonical forms of the type defined by equation (2.2). However, there are not more than N members of this equivalence class that are essentially different with different homogeneous parts.

(b) An illustrative example

Consider a two-degree-of-freedom system governed by

$$\ddot{\mathbf{q}} + \begin{bmatrix} 0.1 & 0.2 \\ 0.1 & 0.3 \end{bmatrix}\dot{\mathbf{q}} + \begin{bmatrix} 0.7 & 0.3 \\ 0.5 & 0.4 \end{bmatrix}\mathbf{q} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}\sin 2t, \tag{4.7}$$

with initial values $\mathbf{q}(0) = \mathbf{0}$ and $\dot{\mathbf{q}}(0) = \mathbf{0}$. This is a realization of equation (2.1) with non-symmetric coefficient matrices. Solution of the quadratic eigenvalue problem (3.2) yields

$$\boldsymbol{\Lambda}_1 = \begin{bmatrix} -0.0402 + 0.3683i & 0 \\ 0 & -0.1598 + 0.9599i \end{bmatrix}, \quad \boldsymbol{\Lambda}_2 = \bar{\boldsymbol{\Lambda}}_1 \tag{4.8}$$

and

$$\mathbf{V}_1 = \begin{bmatrix} 0.4756 + 0.1059i & 0.7404 - 0.0497i \\ -0.9092 + 0.0139i & 0.6698 + 0.0632i \end{bmatrix}, \quad \mathbf{V}_2 = \bar{\mathbf{V}}_1. \tag{4.9}$$

The eigenvectors are normalized in accordance with equations (3.7) and (3.8). Since all eigenvalues are complex and distinct, there is only one canonical form of the type defined by equation (2.2), unique up to the normalization of eigenvectors. The real and diagonal coefficients of the decoupled equation (3.1) are given by

$$\mathbf{D} = \begin{bmatrix} 0.0804 & 0 \\ 0 & 0.3196 \end{bmatrix}, \quad \boldsymbol{\Omega} = \begin{bmatrix} 0.1373 & 0 \\ 0 & 0.9470 \end{bmatrix}. \tag{4.10}$$

From equations (3.13) and (3.14),

$$\mathbf{G}_1 = \begin{bmatrix} -0.0374 & 0.1698 \\ -0.1770 & 0.2236 \end{bmatrix}, \quad \mathbf{G}_2 = \begin{bmatrix} 0.6808 & -0.7721 \\ 0.9234 & 0.4728 \end{bmatrix}. \tag{4.11}$$

It can be verified that the canonical form (2.2) is specified by

$$\ddot{\mathbf{x}} + \begin{bmatrix} 0.1357 & 0 \\ 0 & 0.9215 \end{bmatrix} \mathbf{x} = \mathbf{h}(t), \tag{4.12}$$

for which

$$\mathbf{h}(t) = \begin{bmatrix} (-0.4144\cos 2t + 1.4362\sin 2t)\mathrm{e}^{0.0402t} \\ (-0.8012\cos 2t + 0.3226\sin 2t)\mathrm{e}^{0.1598t} \end{bmatrix}. \tag{4.13}$$

Using equation (4.4), the configuration-space transformation between $\mathbf{q}$ and $\mathbf{x}$ can be expressed as

$$\mathbf{q} = \left(\mathbf{E}(t) + \mathbf{F}(t)\frac{\mathrm{d}}{\mathrm{d}t}\right)\mathbf{x} + \begin{bmatrix} 0.0388 \\ 0.0342 \end{bmatrix} \sin 2t, \tag{4.14}$$

where

$$\mathbf{E}(t) = \begin{bmatrix} 0.4756e^{-0.0402t} & 0.7404e^{-0.1598t} \\ -0.9092e^{-0.0402t} & 0.6698e^{-0.1598t} \end{bmatrix} \tag{4.15}$$

and

$$\mathbf{F}(t) = \begin{bmatrix} 0.2874e^{-0.0402t} & -0.0518e^{-0.1598t} \\ 0.0377e^{-0.0402t} & 0.0658e^{-0.1598t} \end{bmatrix}. \tag{4.16}$$

The state-space transformation that reduces equation (4.7) to equation (4.12) is given by equation (4.5), for which

$$\begin{bmatrix} \exp\left(\dfrac{\mathbf{D}t}{2}\right) & \mathbf{0} \\ \left(\dfrac{\mathbf{D}}{2}\right)\exp\left(\dfrac{\mathbf{D}t}{2}\right) & \exp\left(\dfrac{\mathbf{D}t}{2}\right) \end{bmatrix} = \begin{bmatrix} e^{0.0402t} & 0 & 0 & 0 \\ 0 & e^{0.1598t} & 0 & 0 \\ 0.0402e^{0.0402t} & 0 & e^{0.0402t} & 0 \\ 0 & 0.1598e^{0.1598t} & 0 & e^{0.1598t} \end{bmatrix}, \tag{4.17}$$

$$\mathbf{S} = \begin{bmatrix} 0.6941 & -0.7286 & -0.0374 & 0.1698 \\ 0.9281 & 0.5045 & -0.1770 & 0.2236 \\ -0.0587 & -0.0567 & 0.6808 & -0.7721 \\ 0.0121 & -0.0363 & 0.9234 & 0.4728 \end{bmatrix} \tag{4.18}$$

and

$$\begin{bmatrix} \mathbf{0} \\ \mathbf{G}_1\mathbf{f}(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -0.2072\sin 2t \\ -0.4006\sin 2t \end{bmatrix}. \tag{4.19}$$

The initial values of equation (4.12) are $\mathbf{x}(0) = \mathbf{0}$ and $\dot{\mathbf{x}}(0) = \mathbf{0}$. To examine the effect of normalization, let the eigenvectors $\mathbf{v}_j$ $(j = 1, 2)$ be normalized in such a way that the state eigenvectors $\left[\mathbf{v}_j \quad \lambda_j\mathbf{v}_j\right]^{\mathrm{T}}$ have unit Euclidean norm. In this case,

$$\mathbf{V}_1 = \begin{bmatrix} 0.4308 + 0.1028i & 0.5309 \\ -0.8265 & 0.4751 + 0.0773i \end{bmatrix}, \quad \mathbf{V}_2 = \bar{\mathbf{V}}_1. \tag{4.20}$$

The homogeneous part of equation (4.12) remains unchanged because it is constructed from the eigenvalues. However, the excitation $\mathbf{h}(t)$ in equation (4.12) becomes

$$\mathbf{h}(t) = \begin{bmatrix} (-0.5877\cos 2t + 1.5759\sin 2t)\mathrm{e}^{0.0402t} \\ (-1.1927\cos 2t + 0.4078\sin 2t)\mathrm{e}^{0.1598t} \end{bmatrix}. \tag{4.21}$$

The transformation given either by equation (4.4) or equation (4.5) also changes with normalization in such a way that equation (4.12) with $\mathbf{h}(t)$ specified by equation (4.21) is generated. As explained earlier, canonical forms generated by different normalization schemes are regarded as equivalent.

5. Defective linear systems

In this section, formulas presented previously will be generalized to reduce defective linear systems to the canonical form specified by equation (2.2). In addition, any type of linear system not previously considered can be treated by this generalization. When an eigenvalue λ_k of equation (3.2) occurs m_k times and a full complement of m_k independent eigenvectors cannot be found, equation (2.1) is defective; the $\rho_k < m_k$ eigenvectors $\mathbf{v}_j^k$ $(j = 1, 2, \ldots, \rho_k)$ must be supplemented by $m_k - \rho_k$ generalized eigenvectors $\mathbf{v}_{\rho_k+\ell}^k$ $(\ell = 1, 2, \ldots, m_k - \rho_k)$. These generalized eigenvectors are defined by the sequence [14]

$$\left.\begin{aligned} \mathbf{Q}(\lambda_k)\mathbf{v}_{\rho_k+1}^k + \mathbf{Q}'(\lambda_k)\mathbf{v}_{\rho_k}^k &= \mathbf{0}, \\ \mathbf{Q}(\lambda_k)\mathbf{v}_{\rho_k+2}^k + \mathbf{Q}'(\lambda_k)\mathbf{v}_{\rho_k+1}^k + \frac{1}{2}\mathbf{Q}''(\lambda_k)\mathbf{v}_{\rho_k}^k &= \mathbf{0}, \\ &\vdots \\ \mathbf{Q}(\lambda_k)\mathbf{v}_{m_k}^k + \mathbf{Q}'(\lambda_k)\mathbf{v}_{m_k-1}^k + \frac{1}{2}\mathbf{Q}''(\lambda_k)\mathbf{v}_{m_k-2}^k &= \mathbf{0}, \end{aligned}\right\} \tag{5.1}$$

where

$$\mathbf{Q}(\lambda_k) = \mathbf{M}\lambda_k^2 + \mathbf{C}\lambda_k + \mathbf{K}, \quad \mathbf{Q}'(\lambda_k) = 2\mathbf{M}\lambda_k + \mathbf{C}, \quad \mathbf{Q}''(\lambda_k) = 2\mathbf{M}. \tag{5.2}$$

Once a complete set of vectors is obtained for every defective eigenvalue, it is then possible to convert equation (2.1) into the decoupled system represented by equation (3.1). Afterwards, equation (4.1) can be applied to convert equation (3.1) into the canonical form (2.2). While defective systems do not typically arise in practical applications, they have received attention from a number of authors [15,16]. As demonstrated in [5], the decoupling of defective systems is a delicate procedure that can easily vary on a case-by-case basis, but regardless it is always possible to recast equation (2.1) in the canonical form (2.2).

(a) Decoupling of defective systems

In general, for homogeneous systems with $\mathbf{f}(t) = \mathbf{0}$, equations (2.1) and (3.1) are connected in the state space by a real, invertible and time-varying transformation given by [5]

$$\begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix} = \begin{bmatrix} \mathbf{V}_q \\ \mathbf{V}_q\mathbf{J}_q \end{bmatrix} e^{\mathbf{J}_q t} e^{-\mathbf{J}_p t} \begin{bmatrix} \mathbf{V}_p \\ \mathbf{V}_p\mathbf{J}_p \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ \dot{\mathbf{p}} \end{bmatrix} = \mathbf{T}(t) \begin{bmatrix} \mathbf{p} \\ \dot{\mathbf{p}} \end{bmatrix} = \begin{bmatrix} \mathbf{T}_1(t) & \mathbf{T}_2(t) \\ \mathbf{T}_3(t) & \mathbf{T}_4(t) \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ \dot{\mathbf{p}} \end{bmatrix}. \tag{5.3}$$

In the above expression, $\mathbf{J}_q$ and $\mathbf{J}_p$ are $2n \times 2n$ Jordan matrices of the indexed eigenvalues on the diagonal, where $\mathbf{J}_p$ is usually a modified form of $\mathbf{J}_q$ whose structure imposes the eigenvalue pairing scheme required for decoupling. The $n \times 2n$ matrix $\mathbf{V}_q$ contains the eigenvectors and generalized eigenvectors associated with the indexed eigenvalues in $\mathbf{J}_q$, while the structure of the $n \times 2n$ matrix $\mathbf{V}_p$ enforces the pairing scheme imposed by $\mathbf{J}_p$. The coefficient matrices of equations (2.1) and (3.1) are related by the $2n \times 2n$ real and invertible transformation matrix $\mathbf{T}(t)$ according to

$$\begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\boldsymbol{\Omega} & -\mathbf{D} \end{bmatrix} = \mathbf{T}^{-1}(t) \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix} \mathbf{T}(t) - \mathbf{T}^{-1}(t)\dot{\mathbf{T}}(t). \tag{5.4}$$

To decouple equation (2.1) when the excitation is included, consider the state-space transformation

$$\begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix} = \mathbf{T}(t) \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \end{bmatrix}. \tag{5.5}$$

After casting equation (2.1) in the state space as

$$\begin{bmatrix} \dot{\mathbf{q}} \\ \ddot{\mathbf{q}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{f}(t) \end{bmatrix}, \tag{5.6}$$

substitute equation (5.5) into equation (5.6), pre-multiply the result by $\mathbf{T}^{-1}(t)$, and use relationship (5.4) to obtain

$$\begin{bmatrix} \dot{\mathbf{p}}_1 \\ \dot{\mathbf{p}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\boldsymbol{\Omega} & -\mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{g}_1(t) \\ \mathbf{g}_2(t) \end{bmatrix}, \tag{5.7}$$

where

$$\begin{bmatrix} \mathbf{g}_1(t) \\ \mathbf{g}_2(t) \end{bmatrix} = \mathbf{T}^{-1}(t) \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{f}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{G}_1(t)\mathbf{f}(t) \\ \mathbf{G}_2(t)\mathbf{f}(t) \end{bmatrix}. \tag{5.8}$$

Extracting the upper and lower halves of equation (5.7), eliminating the coordinate $\mathbf{p}_2$, and comparing the result to equation (3.1) reveals that

$$\mathbf{p}_1 = \mathbf{p}, \quad \mathbf{p}_2 = \dot{\mathbf{p}} - \mathbf{G}_1(t)\mathbf{f}(t) \tag{5.9}$$

and the excitation

$$\mathbf{g}(t) = \left(\mathbf{D} + \mathbf{I}\frac{\mathrm{d}}{\mathrm{d}t}\right) \mathbf{G}_1(t)\mathbf{f}(t) + \mathbf{G}_2(t)\mathbf{f}(t). \tag{5.10}$$

Consequently, from equation (5.5), the decoupling transformation in the state space is

$$\begin{bmatrix} \mathbf{p}(t) \\ \dot{\mathbf{p}}(t) \end{bmatrix} = \mathbf{T}^{-1}(t) \begin{bmatrix} \mathbf{q}(t) \\ \dot{\mathbf{q}}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{G}_1(t)\mathbf{f}(t) \end{bmatrix}. \tag{5.11}$$

The corresponding configuration-space decoupling transformation is given by

$$\mathbf{q} = \left(\mathbf{T}_1(t) + \mathbf{T}_2(t)\frac{\mathrm{d}}{\mathrm{d}t}\right)\mathbf{p} - \mathbf{T}_2(t)\mathbf{G}_1(t)\mathbf{f}(t). \tag{5.12}$$

When $t = 0$, equation (5.11) generates the initial values $\mathbf{p}(0)$ and $\dot{\mathbf{p}}(0)$ of equation (3.1).

Decoupling a defective system represented by equation (2.1) is less systematic than in the non-defective case, as the process for constructing the coefficient matrices $\mathbf{D}$ and $\boldsymbol{\Omega}$ of equation (3.1) varies with the number of real eigenvalues and with the geometric multiplicities of the defective eigenvalues. Moreover, it is generally not possible to simplify the time-varying transformation matrix $\mathbf{T}(t)$ in equation (5.3) to a more explicit and descriptive form, as exemplified by equation (3.18) when the system is non-defective. However, a special case in which simplification occurs is when all eigenvalues are complex. Suppose $2N < 2n$ of these eigenvalues are distinct and, for simplicity, each defective eigenvalue has unit geometric multiplicity (i.e. each has one associated eigenvector). The latter assumption is simply a matter of convenience and can be relaxed with care [5]. Let m_k ($k = 1, 2, \ldots, N$) denote the algebraic multiplicity (the number of occurrences) of each unique eigenvalue λ_k with positive imaginary part. Associated with λ_k is an $m_k \times m_k$ Jordan block

$$\mathbf{J}_k = \begin{bmatrix} \lambda_k & 1 & 0 & \cdots & 0 \\ 0 & \lambda_k & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_k & 1 \\ 0 & \cdots & 0 & 0 & \lambda_k \end{bmatrix} = \lambda_k \mathbf{I}_{m_k} + \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} = \boldsymbol{\Lambda}_k + \mathbf{N}_k. \tag{5.13}$$

Under the assumption of unit geometric multiplicity, λ_k has a single eigenvector $\mathbf{v}_1^k$ and $m_k - 1$ generalized eigenvectors $\mathbf{v}_j^k$ $(j = 2, 3, \ldots, m_k)$ that are computed according to equations (5.1) and (5.2). Arrange these vectors in an $n \times m_k$ matrix

$$\mathbf{V}_k = \begin{bmatrix} \mathbf{v}_1^k & \mathbf{v}_2^k & \cdots & \mathbf{v}_{m_k}^k \end{bmatrix}. \tag{5.14}$$

Compile the N Jordan blocks $\mathbf{J}_k$ and the N matrices $\mathbf{V}_k$ of eigenvectors and generalized eigenvectors to form the $n \times n$ matrices

$$\mathbf{J} = \mathrm{diag}[\mathbf{J}_1, \mathbf{J}_2, \ldots, \mathbf{J}_N], \quad \mathbf{V} = \begin{bmatrix} \mathbf{V}_1 & \mathbf{V}_2 & \cdots & \mathbf{V}_N \end{bmatrix}. \tag{5.15}$$

Likewise, construct the following $n \times n$ matrices from the N diagonal matrices $\boldsymbol{\Lambda}_k$ and the N nilpotent matrices $\mathbf{N}_k$ defined in equation (5.13):

$$\boldsymbol{\Lambda} = \mathrm{diag}[\boldsymbol{\Lambda}_1, \boldsymbol{\Lambda}_2, \ldots, \boldsymbol{\Lambda}_N], \quad \mathbf{N} = \mathrm{diag}[\mathbf{N}_1, \mathbf{N}_2, \ldots, \mathbf{N}_N]. \tag{5.16}$$

Note that the matrices $\boldsymbol{\Lambda}$ and $\mathbf{N}$ commute in multiplication. For this special case of a defective system, the decoupling transformation is such that [5]

$$\mathbf{J}_q = \mathrm{diag}[\mathbf{J}, \bar{\mathbf{J}}], \quad \mathbf{V}_q = \begin{bmatrix} \mathbf{V} & \bar{\mathbf{V}} \end{bmatrix}, \quad \mathbf{J}_p = \mathrm{diag}[\boldsymbol{\Lambda}, \bar{\boldsymbol{\Lambda}}], \quad \mathbf{V}_p = \begin{bmatrix} \mathbf{I} & \mathbf{I} \end{bmatrix}, \tag{5.17}$$

where the coefficient matrices of the decoupled equation (3.1) are given by

$$\mathbf{D} = -(\boldsymbol{\Lambda} + \bar{\boldsymbol{\Lambda}}), \quad \boldsymbol{\Omega} = \boldsymbol{\Lambda}\bar{\boldsymbol{\Lambda}}. \tag{5.18}$$

In other words, equation (3.1) comprises N collections of m_k identical, independent single-degree-of-freedom systems with generally different excitations and initial values. Note that the Jordan matrices of equation (5.17) imply that the corresponding decoupling transformation preserves the eigenvalues of equation (2.1) but not the geometric multiplicities. Based on equation (5.17), the state transformation matrix $\mathbf{T}(t)$ defined in equation (5.3) becomes

$$\mathbf{T}(t) = \begin{bmatrix} \mathbf{T}_1(t) & \mathbf{T}_2(t) \\ \mathbf{T}_3(t) & \mathbf{T}_4(t) \end{bmatrix} = \begin{bmatrix} \mathbf{V} & \bar{\mathbf{V}} \\ \mathbf{VJ} & \overline{\mathbf{VJ}} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \boldsymbol{\Lambda} & \bar{\boldsymbol{\Lambda}} \end{bmatrix}^{-1} \begin{bmatrix} e^{\mathbf{N}t} & \mathbf{0} \\ \mathbf{0} & e^{\mathbf{N}t} \end{bmatrix}, \tag{5.19}$$

where the $n \times n$ sub-matrices

$$\mathbf{T}_1(t) = (\mathbf{V}\bar{\boldsymbol{\Lambda}} - \bar{\mathbf{V}}\boldsymbol{\Lambda})(\bar{\boldsymbol{\Lambda}} - \boldsymbol{\Lambda})^{-1} e^{\mathbf{N}t}, \quad \mathbf{T}_2(t) = (\bar{\mathbf{V}} - \mathbf{V})(\bar{\boldsymbol{\Lambda}} - \boldsymbol{\Lambda})^{-1} e^{\mathbf{N}t}. \tag{5.20}$$

As a result, equation (5.8) yields

$$\mathbf{G}_1(t) = e^{-\mathbf{N}t}[(\mathbf{VJ} - (\overline{\mathbf{VJ}})\bar{\mathbf{V}}^{-1}\mathbf{V})^{-1} + (\overline{\mathbf{VJ}} - (\mathbf{VJ})\mathbf{V}^{-1}\bar{\mathbf{V}})^{-1}]\mathbf{M}^{-1} \tag{5.21}$$

and

$$\mathbf{G}_2(t) = e^{-\mathbf{N}t}[\boldsymbol{\Lambda}(\mathbf{VJ} - (\overline{\mathbf{VJ}})\bar{\mathbf{V}}^{-1}\mathbf{V})^{-1} + \bar{\boldsymbol{\Lambda}}(\overline{\mathbf{VJ}} - (\mathbf{VJ})\mathbf{V}^{-1}\bar{\mathbf{V}})^{-1}]\mathbf{M}^{-1}. \tag{5.22}$$

It is generally not possible to express the transformation matrix $\mathbf{T}(t)$ in an explicit form such as equation (5.19) when some of the defective eigenvalues are real. Additional details of the decoupling of equation (2.1) when it possesses defective real eigenvalues are provided in [5].

(b) Transformation to the canonical form

After a defective system has been converted into a decoupled system represented by equation (3.1), the canonical form (2.2) is obtained through application of transformation (4.1). In this case, the diagonal coefficient matrix $\mathbf{B}$ is still given by equation (4.2), and the excitation has the form

$$\mathbf{h}(t) = \exp\left(\frac{1}{2}\mathbf{D}t\right)\mathbf{g}(t) = \exp\left(\frac{1}{2}\mathbf{D}t\right)\left\{\left(\mathbf{D} + \mathbf{I}\frac{\mathrm{d}}{\mathrm{d}t}\right)\mathbf{G}_1(t)\mathbf{f}(t) + \mathbf{G}_2(t)\mathbf{f}(t)\right\}. \tag{5.23}$$

Combining equations (4.1) and (5.12) yields the configuration-space transformation relating $\mathbf{q}$ and $\mathbf{x}$:

$$\mathbf{q} = \left(\mathbf{T}_1(t) + \mathbf{T}_2(t)\frac{\mathrm{d}}{\mathrm{d}t}\right)\exp\left(-\frac{1}{2}\mathbf{D}t\right)\mathbf{x} - \mathbf{T}_2(t)\mathbf{G}_1(t)\mathbf{f}(t). \tag{5.24}$$

When equations (4.1) and (5.11) are combined, the transformation connecting equations (2.1) and (2.2) in the state space is obtained:

$$\begin{bmatrix}\mathbf{x}(t)\\ \dot{\mathbf{x}}(t)\end{bmatrix} = \begin{bmatrix}\exp(\mathbf{D}t/2) & \mathbf{0}\\ (\mathbf{D}/2)\exp(\mathbf{D}t/2) & \exp(\mathbf{D}t/2)\end{bmatrix}\left\{\mathbf{T}^{-1}(t)\begin{bmatrix}\mathbf{q}(t)\\ \dot{\mathbf{q}}(t)\end{bmatrix} + \begin{bmatrix}\mathbf{0}\\ \mathbf{G}_1(t)\mathbf{f}(t)\end{bmatrix}\right\}. \tag{5.25}$$

Note that equations (5.23)–(5.25) hold for any defective system. If a defective system has all complex conjugate eigenvalues, then the matrices $\mathbf{D}$ and $\boldsymbol{\Omega}$ that characterize the canonical form (2.2) are as specified in equation (5.18), and the matrices $\mathbf{G}_1(t)$, $\mathbf{G}_2(t)$, $\mathbf{T}_1(t)$, $\mathbf{T}_2(t)$ and $\mathbf{T}(t)$ in equations (5.23)–(5.25) are given by equations (5.19)–(5.22). Should this system be non-defective, then the matrices $\mathbf{N} = \mathbf{0}$ and $\mathbf{J} = \boldsymbol{\Lambda}$. Taking $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}_1$, $\bar{\boldsymbol{\Lambda}} = \boldsymbol{\Lambda}_2$, $\mathbf{V} = \mathbf{V}_1$ and $\bar{\mathbf{V}} = \mathbf{V}_2$, it is easy to verify that all formulae for transforming equation (2.1) into the canonical form (2.2) reduce to their non-defective counterparts.

(c) Numerical example of a defective system

A two-degree-of-freedom system is governed by

$$\ddot{\mathbf{q}} + \begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}\dot{\mathbf{q}} + \begin{bmatrix}5 & -1\\ -1 & 10\end{bmatrix}\mathbf{q} = \begin{bmatrix}1\\ -2\end{bmatrix}\cos t, \tag{5.26}$$

with initial values $\mathbf{q}(0) = \mathbf{0}$ and $\dot{\mathbf{q}}(0) = \mathbf{0}$. Solution of the quadratic eigenvalue problem (3.2) reveals that the system is defective with a repeated complex eigenvalue such that

$$\mathbf{J} = \begin{bmatrix}-1 + i\sqrt{6} & 1\\ 0 & -1 + i\sqrt{6}\end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix}-i\sqrt{6}/2 & 5/2\\ 1 & 0\end{bmatrix} \tag{5.27}$$

and

$$\boldsymbol{\Lambda} = \left(-1 + i\sqrt{6}\right)\mathbf{I}, \quad \mathbf{N} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}. \tag{5.28}$$

The real and diagonal coefficients of the decoupled equation (3.1) are given by

$$\mathbf{D} = 2\mathbf{I}, \quad \boldsymbol{\Omega} = 7\mathbf{I}. \tag{5.29}$$

From equations (5.21) and (5.22),

$$\mathbf{G}_1(t) = \begin{bmatrix}0 & -t/6\\ 0 & 1/6\end{bmatrix}, \quad \mathbf{G}_2(t) = \begin{bmatrix}-t/2 & t/6 + 5/6\\ 1/2 & -1/6\end{bmatrix}. \tag{5.30}$$

The canonical form (2.2) for equation (5.26) is then specified by

$$\ddot{\mathbf{x}} + \begin{bmatrix}6 & 0\\ 0 & 6\end{bmatrix}\mathbf{x} = \mathbf{h}(t), \tag{5.31}$$

where the excitation

$$\mathbf{h}(t) = \frac{e^t}{6}\begin{bmatrix} -8\cos t - t(\cos t + 2\sin t) \\ \cos t + 2\sin t \end{bmatrix}. \tag{5.32}$$

The coordinates **q** and **x** are related in the configuration space by transformation (5.24):

$$\mathbf{q} = \left(\begin{bmatrix} 0 & 5e^{-t}/2 \\ e^{-t} & te^{-t} \end{bmatrix} + \begin{bmatrix} -e^{-t}/2 & -te^{-t}/2 \\ 0 & 0 \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \right) \mathbf{x}. \tag{5.33}$$

Reduction of equation (5.26) to equation (5.31) is accomplished in the state space by equation (5.25), for which

$$\begin{bmatrix} \exp(\mathbf{D}t/2) & \mathbf{0} \\ (\mathbf{D}/2)\exp(\mathbf{D}t/2) & \exp(\mathbf{D}t/2) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} e^t, \tag{5.34}$$

$$\mathbf{T}(t) = \begin{bmatrix} -1/2 & 5/2 - t/2 & -1/2 & -t/2 \\ 1 & t & 0 & 0 \\ 7/2 & 7t/2 - 1/2 & 1/2 & t/2 + 2 \\ 0 & 1 & 1 & t \end{bmatrix} \tag{5.35}$$

and

$$\begin{bmatrix} \mathbf{0} \\ \mathbf{G}_1(t)\mathbf{f}(t) \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 0 \\ 0 \\ t\cos t \\ -\cos t \end{bmatrix}. \tag{5.36}$$

The initial values of equation (5.31) computed from equation (5.25) are $\mathbf{x}(0) = \mathbf{0}$ and $\dot{\mathbf{x}}(0) = [0, \ -1/3]^T$. As in the non-defective case, the canonical form generated is dependent on the normalization of eigenvectors. For example, if instead

$$\mathbf{V} = \begin{bmatrix} -0.1250 - 0.2041i & 0.5 - 0.2552i \\ 0.1667 - 0.1021i & 0.0680i \end{bmatrix}, \tag{5.37}$$

then the excitation $\mathbf{h}(t)$ in equation (5.31) becomes

$$\mathbf{h}(t) = e^t \begin{bmatrix} -8.7438\cos t + 2.5124\sin t - t(3.4545\cos t + 0.9091\sin t) \\ 3.4545\cos t + 0.9091\sin t \end{bmatrix}. \tag{5.38}$$

The homogeneous part of equation (5.31) is unaffected by normalization because it is constructed only from the eigenvalues. The transformation given either by equation (5.24) or equation (5.25) changes with normalization in such a way that equation (5.31) with $\mathbf{h}(t)$ specified by equation (5.38) is generated.

6. Conclusion

It has been shown that almost all linear systems governed by equation (2.1) can be reduced to a canonical form specified by equation (2.2), a decoupled equation devoid of the velocity term and with the identity matrix as the coefficient of acceleration. While an exhaustive derivation has been provided only for non-defective systems with distinct eigenvalues, the reduction is applicable to both non-defective and defective linear systems possessing either symmetric or non-symmetric coefficient matrices. Major findings are summarized in the following statements.

1. All parameters required to construct the invertible transformation to convert equation (2.1) into equation (2.2) are obtained through the solution of the quadratic eigenvalue problem (3.2). For systems with distinct eigenvalues, the transformation is given either by equation (4.4) or equation (4.5), both of which are nonlinear.

2. For non-defective systems, different indexing and normalization schemes generate an equivalence class of canonical forms of the type defined by equation (2.2). If there are $2r$ real eigenvalues of equation (3.2), then not more than N members of this equivalence class have different homogeneous parts, where N is given by equation (4.6). If all eigenvalues of equation (3.2) are complex, the canonical form (2.2) is unique up to the normalization of eigenvectors.
3. As an important by-product, a solution to the well-trodden problem of reducing a damped passive system to an undamped form has been provided.

Almost all linear systems are non-defective with distinct eigenvalues, and an emphasis has been placed on such systems. In the reduction of the equation of motion, the canonical form specified by equation (2.2) is the simplest representation of linear systems. Two examples have been supplied for illustration.

Data accessibility. This work does not involve any experimental data.
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