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Ax-Kochen/Ershov style results in model theory of henselian valued fields

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by

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Abstract

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This thesis is concerned with developing a model theoretic understanding of henselian valued fields. The model theory of henselian valued fields has been a major topic of study during the last century, it was initiated by Robinson's model completeness results for algebraically closed valued fields in [Rob56]. Remarkable work has been achieved by Haskell, Hrushovski and Macpherson to understand the model theory of algebraically closed valued fields, more precisely in [HHM05] and [HHM06] they clarify completely the picture for elimination of imaginaries showing that it is sufficient to add the *geometric sorts*. They also develop a notion of stable domination and independence in algebraically closed valued fields, that rather than being understood as a new form of stability should be grasp as a technique to lift ideas from stability to the setting of valued fields. This approach is for example illustrated in [HRK19], where notions of stability (e.g. germs, genericity, domintion, etc) are being used to give a complete description of definable abelian groups in algebraically closed valued fields.

The starting point of this thesis relies on the Ax-Kochen principle, which states that the first order theory of a henselian valued field of equicharacteristic zero or of mixed characteristic, unramified and with perfect residue field is completely determined by the first order theory of its residue field and its value group. A natural principle follows from this theorem: model theoretic questions about the value group itself can be understood by reducing them to questions into the residue field, the value group and their interaction in the field.

A fruitful application of this principle has been achieved to describe the class of definable sets, see for example: [Pas90], [Bas91], [Kuh94]. The next natural step for understanding the model theory of henselian valued fields was obtaining an elimination of imaginaries statement.

The first part of this thesis studies elimination of imaginaries in the setting of henselian valued fields of equicharacteristic zero with residue field algebraically closed. The obtained results are sensitive to the complexity of the value group, which is an ordered abelian group. In the first chapter we study elimination of imaginaries in ordered abelian groups, while in the second chapter we analyze imaginaries in henselian valued fields of equicharacteristic zero.

The second part of this thesis studies domination results in an Ax-Kochen style in the setting of henselian valued fields of equicharacteristic zero. We use a more abstract notion of domination present in [EHM19] that generalizes the definition of stable domination present in [HHM05].

To Mona, Peter and my twin Camilo

Contents

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Chapter 0

0.1 Introduction

Model theory is a branch of mathematical logic that studies *structures* (that is sets equipped with relations, functions and constants) and their *definable sets*, that is the subsets of various cartesian powers that can be defined in terms of these distinguished constants, relations and functions via the logical connectives and quantifiers. For instance, in an algebraically closed field we distinguish the constants 0 and 1 and symbols for the multiplication and the addition. This particular case is well understood: the definable sets are exactly the constructible sets, which are fundamental objects of algebraic geometry.

The domain of interest in model theory is thus extremely broad and encompasses that of algebra and geometry. Its point of view is however very different: model theory is concerned with identifying and studying boundaries between *tame* and *wild* first-order theories (i.e. consistent sets of first-order properties of structures). In other words, model theory studies dividing lines between prototypical tame structures like vector spaces and the field of complex numbers, in which the definable sets are well understood, and wilder structures for which there is no control, such as the ring of integers.

Modern model theory has been heavily influenced by S. Shelah's remarkable work in classification theory [She90]. In the 1970s S. Shelah developed a tremendously profound structure theory for the class of stable theories, in which no first order formula can totally order arbitrarily large sets of tuples. The study of stable theories initiated by S. Shelah, and later refined by many others, brought to the picture tools and ideas that have been the key to solve many problems in other branches of mathematics, such as the Mordell-Lang conjecture for function fields proved by E. Hrushovski.

The *neostability* program seeks to generalize Shelah's work to other dividing lines beyond stability. This program has been tremendously fruitful for several classes of theories, most notably the simple theories , dependent theories and o-minimal theories. These developments have enriched the applications of model theory; the Pila-Wilkie theorem in diophantine geometry is a prime example.

The results presented in this thesis are in model theory and its interactions with algebra, specifically in *henselian* valued fields and related structures. A valued field is a field K equipped with a distinguished subset $\mathcal{O}$, a valuation ring[1]. Examples of valued fields are the p-adic field $\mathbb{Q}_p$ or the Laurent series over the complex numbers $\mathbb{C}((t))$, structures that have played relevant roles in number theory and analysis respectively. Given $\mathcal{O}$ a valuation ring of a field and $\mathcal{M}$ its maximal ideal, we commonly refer to the additive quotient $\mathcal{O}/\mathcal{M}$ as the *residue field*, while the multiplicative quotient $K^{\times}/\mathcal{O}^{\times}$ is an ordered abelian group and it is called the *value group*. A valued field is said to be *henselian* if every non-singular zero of a polynomial in the residue field can be lifted to the field. [2]

One of the most striking results in the model theory of valued fields is the *Ax-Kochen/ Ershov theorem* which roughly states that the first order theory of a henselian valued field of equicharacteristic zero or a henselian valued field of mixed characteristic, unramified [3] and perfect residue field is completely determined by the first order theory of its residue field and its value group. A natural philosophy follows from this theorem: model theoretic questions about the valued field itself can be understood by reducing them to its residue field, its value group and their interaction in the field. This thesis looks towards proving instances of this principle to study model theoretic properties, such as elimination of imaginaries and residue field domination, for a broad class of henselian valued fields in order to develop a unifying and general theory for henselian valued fields.

Towards an Ax-Kochen/Ershov imaginary principle for henselian valued fields

A fruitful application of the Ax-Kochen/Ershov principle describes the class of definable sets of a valued field. For example, in [Pas90] J. Pas proved field quantifier elimination relative to the residue field and the valued group once angular component maps are added to the language. Further studies for the case where no angular map is added were done by S.A. Basarab and F.V. Kuhlmann in [Bas91], [Kuh94] (relative to the RV_n sorts), while the complete picture has been clarified by M. Aschenbrenner, A. Chernikov, A. Gehret and M. Ziegler in [ACGZ20] relative to the power residue sorts and the value group.

There is a more general class of subsets that one could study, called the *interpretable sets*, obtained by taking the quotient of a definable set by a definable equivalence relation. For example, consider a finite dimensional vector space V over K (a field of characteristic zero) where the distinguished structure has a constant for the zero vector, a symbol $+$ for the addition and a map for the scalar multiplication $\cdot : K \times V \to V$. The projective space is

[1] Let K be a field, a subring $A \subseteq K$ is said to be a valuation ring of K if for any element $x \in K\backslash\{0\}$ either $x \in A$ or $x^{-1} \in A$.

[2] More formally, if $p(x) \in \mathcal{O}[x]$ and there is some element $\alpha \in \mathcal{O}$ such that $p(\alpha) \in \mathcal{M}$ while $p'(\alpha) \in \mathcal{O}^{\times}$, then we can find an element $\beta \in \mathcal{O}$ such that $p(\beta) = 0$ and $\beta - \alpha \in \mathcal{M}$.

[3] A valued field is said to be unramified if its value group has a least positive element.

interpretable in V, as it can be obtained by taking the quotient of $V \backslash \{0\}$ by the equivalence relation $E(v, w)$ stating that v and w lie in the same line passing through the origin.

A theory T (that is, a consistent set of axioms) is said to *eliminate imaginaries* if for every model M of T (a model is a structure that satisfies the axioms of T), every definable set $X \subseteq M^n$ and every definable equivalence relation $E(x, y)$ on X, we can find a definable function $f : X \to M^k$ such that for any $x, y \in X$ the relation $E(x, y)$ holds if and only if $f(x) = f(y)$. Hence, elimination of imaginaries is saying that the class of definable sets is closed under taking definable quotients.

The question of whether a first order theory T eliminates imaginaries depends on the language $\mathcal{L}$ (that is, the set of distinguished symbols being used for the relations, functions and constants). Given a structure M we can take its *imaginary extension* M^{eq} where we add a sort S_E for each definable equivalence relation E and a map π_E sending each element to its equivalence class. By construction, the structure M^{eq} eliminates imaginaries, but it does not give a precise description of the interpretable sets of the structure. The real question that one is trying to address while studying elimination of imaginaries is to find a reasonable language, with a clean presentation of the sorts required, which has elimination of imaginaries.

In the case of valued fields, the question of elimination of imaginaries is of course subject to the complexity of its value group and its residue field, as both are interpretable structures in the valued field itself. However, we can look at the value group and the residue field as oracles, for example by providing the entire Shelah's imaginary expansion for each of these sorts. Then, could it be possible to describe the sorts necessary to be added in order to have elimination of imaginaries for the valued field? The problem can be formulated in the following way:

Question 1. *Can one achieve an Ax-Kochen/Ershov style version of elimination of imaginaries for henselian valued fields?*

There has been significant work on elimination of imaginaries of henselian valued fields during the past years. The case for algebraically closed valued fields was finalized by D. Haskell, E. Hrushovski and D. Macpherson in their relevant work [HHM06], where elimination of imaginaries for $ACVF$ (the theory of algebraically closed valued fields) is achieved by adding the geometric sorts S_n (codes for the $\mathcal{O}$-lattices of rank n) and T_n (codes for the residue classes of the elements in S_n). Recent work has been done to achieve elimination of imaginaries in some other examples of henselian valued fields, such as the case of separably closed valued fields of finite imperfection degree in [HKR18], the p-adic case in [HMR18] or enrichments of algebraically closed valued fields in [Rid19].

However, the above results are all obtained for particular instances of henselian valued fields. Obtaining a relative statement for broader classes of henselian valued fields is still an open question. Following the Ax-Kochen/Ershov style principle, it seems natural to attempt first to solve this question by looking at the problem in two orthogonal directions: one by making the residue field as docile as possible and studying which obstruction would the value group bring to the picture; the other, by making the value group very tame and understanding the difficulties that the residue field would contribute to the problem.

The second chapter of this thesis follows the first point of view in the setting of henselian valued fields of equicharacteristic zero, where I assume the residue field to be algebraically closed. The obtained results are sensitive to the complexity of the value group. This path of research requires one to study ordered abelian groups and their complexity independently, since the value group is an ordered abelian group. In 1984 Y. Gurevich and PH. Schmitt initiated the study of the model theoretic complexity of ordered abelian groups in [GS84], where they proved that no ordered abelian group has the independence property. Later, further dividing lines and characterizations have been achieved in [JSW17] for the dp-minimal case, in [HH19], [DG15] and [Far17] for the strongly dependent case and in [ACGZ20] for the distal case. By using the quantifier elimination given by R. Farré in [Far17] for the class of ordered abelian groups with finite spines (see [CH11, Definition 1.5] for a precise description), In the first chapter we prove the following two theorems:

Theorem 2. *Let Γ be an ordered abelian group with finite spines. Then Γ admits weak-elimination of imaginaries once we add the* quotient sorts. *That is, we add a sort for each quotient group Γ/Δ where Δ is a definable convex subgroup of Γ, and sorts for the quotient groups $\Gamma/(\Delta+\ell\Gamma)$ where Δ is a definable convex subgroup and $\ell \in \mathbb{N}_{\geq 2}$.*

The class of ordered abelian groups with finite spines includes the dp-minimal and the strongly dependent ones. An ordered abelian group Γ is dp-minimal if and only if for any prime number p the index $[\Gamma : p\Gamma]$ is finite. From a model theoretic perspective the dp-minimal ordered abelian groups are the least complex among the class of ordered abelian groups. A better statement can be obtained for this case, more precisely:

Theorem 3. *Let Γ be a dp-minimal ordered abelian group, then Γ admits elimination of imaginaries once we add sorts for the quotient groups Γ/Δ where Δ is a convex subgroup, and a set of constants for the elements of the finite groups $\Gamma/\ell\Gamma$ for each $\ell \in \mathbb{N}$.*

Having clarified the picture for the ordered abelian groups, and therefore for the value group, I was able to analyze imaginaries in the class of henselian valued fields of equicharacteristic zero with residue field algebraically closed, for which it was sufficient to add the *stabilizer sorts*. For each $n \in \mathbb{N}^{\geq 1}$ we pick a finite sequence $(I_1, \ldots, I_n)$ where each I_i is an ideal of $\mathcal{O}$. Given $\{e_1, \ldots, e_n\}$ the canonical basis of K^n we define the canonical $\mathcal{O}$-module associated to $(I_1, \ldots, I_n)$ as $C_{(I_1,\ldots,I_n)=}\{e_1x_1 + \cdots + e_nx_n \mid x_i \in I_i\}$. We consider the group of invertible upper triangular matrices $B_n(K)$ and we define the subgroup $\mathrm{Stab}_{(I_1,\ldots,I_n)} = \{M \in B_n(K) \mid MC_{(I_1,\ldots,I_n)} = C_{(I_1,\ldots,I_n)}\}$. The *stabilizer sorts* is the family of quotient groups $B_n(K)/\mathrm{Stab}_{(I_1,\ldots,I_n)}$ for each possible choice of the sequence $(I_1, \ldots, I_n)$. The precise formulation of the theorems obtained in the second chapter are:

Theorem 4. *Let K be a valued field of equicharacteristic zero, residue field algebraically closed and value group with finite spines. Then K admits weak elimination of imaginaries once we add the stabilizer sorts and the quotient sorts for the value group.*

Theorem 5. *Let K be a henselian valued field of equicharacteristic zero, residue field algebraically closed and dp-minimal value group. Then K eliminates imaginaries once we add the stabilizer sorts, the quotient sorts for the value group and constants to distinguish the elements of the finite quotient group $\Gamma/\ell\Gamma$ where $\ell \in \mathbb{N}_{\geq 2}$.*

Residue field domination in henselian valued fields of equicharacteristic zero

Let M be a structure and $A \subseteq M$ be a subset. Given an element $b \in M$ we denote by $\mathrm{tp}(b/A)$ the set of formulas allowing parameters in A that are satisfied by b. The type is essentially giving a description of how the element b relates to the elements in A inside the structure M. In this fashion, we are interested in understanding what is the minimal data that could be required to determine the entire behavior of the element b with respect to the set A. For example, if K is an algebraically closed field and A is a subfield, all the information that we need to know about an algebraic element b over A is its minimal irreducible polynomial over A.

Domination results in valued fields aim to understand the entire type by looking at the behavior of that element in only one part (or sorts) of the structure. In [HHM06] D. Haskell, E. Hrushovski and D. Macpherson developed the notion of *stable domination*, a notion that formalizes how an unstable structure can be governed by its stable part. Their work was fundamental in the study of Berkovich spaces of F. Loeser and E. Hurshovski, and in [HRK19], E. Hrushovski and S. Rideau-Kikuchi studied definable groups in the stably dominated theory of algebraically closed valued fields, by lifting machinery from the stable setting (e.g. genericity, germs, groups with finitely satisfiable generics, etc).

In further work, C. Ealy, D. Haskell and J. Maříková isolated a more abstract notion of domination in [EHM19], without requiring the presence of a stable part of the structure. If T is a complete first order theory and S and Γ are stably embedded sorts, given $C \subseteq A, B$ sets of parameters we say that:

1. the type $\mathrm{tp}(A/C)$ is said to be *dominated by the sort S*, if $S(B)$ being *independent* from $S(A)$ over $S(C)$ implies that $\mathrm{tp}(A/CS(B)) \vdash \mathrm{tp}(A/CB)$.
2. the type $\mathrm{tp}(A/C)$ is said to be dominated by the sort S over Γ if the type $\mathrm{tp}(A/C\Gamma(A))$ is dominated by the sort S.

In particular, in [EHM19], domination results are obtained for the unstable class of real closed valued fields, suggesting that the presence of a stable part in the structure was not fundamental for domination results, and instead of stable domination one should consider domination by the sorts internal to the residue field. There is a tremendously promising line of research regarding domination results: it is a bridge to formalize how strong notions of tameness (stability, simplicity, o-minimality, $NSOP_1$, NTP_2, etc) over the residue field could be used to describe and analyze henselian valued fields.

In the third chapter of this thesis we study domination results in the class of henselian valued fields of equicharacteristic zero in complete generality, in both the Denef-Pas language $\mathcal{L}_{\mathrm{ac}}$ (where an angular component map is added) and the language $\mathcal{L}$ introduced by M. Aschenbrenner, A. Chernikov, A. Gehret and M. Ziegler in [ACGZ20, Subsection 5.4], where the *power residue sorts* (the quotients $k^\times/(k^\times)^n$ where k is the residue field and $n \in \mathbb{N}$) are added.

The results obtained show the reduction in an Ax-Kochen/Ershov style version. More precisely, for T some complete extension of the $\mathcal{L}$-theory of henselian valued fields of equicharacteristic zero we have:

1. *Version* 1*:* Let $C \subseteq L$ be substructures of $\mathfrak{C}$ (the monster model of T) and C a maximal model of T. Then $\mathrm{tp}(L/C)$ is dominated by the value group and the power residue sorts.

2. *Version* 2*:* Let $C \subseteq L$ be substructures of $\mathfrak{C}$ (the monster model of T) and C a maximal model of T. Then $\mathrm{tp}(L/C)$ is dominated over its value group by the sorts internal to the residue field.

In these proofs it becomes clear that the independence notion required is weaker than forking independence, and it is associated to more common notions of independence arising in algebra. The first domination result only requires algebraic independence (in the field theoretic sense) over the residue field, and forking independence for the value group as seen in its reduct as a pure abelian group. For the second version, independence is required only for the quantifier free stable formulas.

Background in model theory

In this section we introduce the required background from model theory.

Codes and elimination of imaginaries

The first appearance of *imaginaries* is due to S. Shelah in [She90]. Through this section we suppose T to be a first order complete $\mathcal{L}$-theory and let $\mathfrak{M}$ be its monster model.

Definition 6. *Let X be a subset of $\mathfrak{M}$ we say that A is a code for X if for all $\sigma \in Aut(\mathfrak{M})$ $\sigma(X) = X$ if and only if σ fixes A setwise.*

We are commonly more interested in finding codes for definable sets, however the definition can be used for more general settings. For example for types where the code is possibly an infinite tuple. We allow A to be an empty set, to be able to code $\mathcal{L}$-definable sets in theories without constants.

Definition 7.

1. *Let X be a definable set and $\phi(\mathbf{x}; \mathbf{y})$ be an $\mathcal{L}$-formula. We say that a tuple a is a canonical parameter for X via ϕ if for all $m \in \mathfrak{M}$ we have that $\phi(\mathfrak{M}; m) = X$ if and only if $m = a$.*

2. *A family of $\mathcal{L}$-definable sets $(X_t)_{t\in D}$ is said to be definable if there is an $\mathcal{L}$-formula $\psi(\mathbf{x}; \mathbf{z})$ and an $\mathcal{L}$-definable set D such that for all $t \in D$ the set X_t is defined by the formula $\psi(\mathbf{x}; t)$. We say that the family admits uniform canonical parameters via $\phi(\mathbf{x}, \mathbf{y})$ if for all $t \in D$, there exists some tuple $a_t \in \mathfrak{M}$ which is a canonical parameter for X_t via ϕ.*

Proposition 8. *Let X be a $\mathcal{L}$ -definable set and $A \subseteq \mathfrak{M}$ be a finite tuple. Then a is a code for X if and only if there exists a finite tuple $a \in A$ which is a canonical parameter for X via some $\mathcal{L}$-formula ϕ and $A \subseteq \mathrm{dcl}(a)$.*

Proof. This is [Rid14, Proposition 0.1.3]. □

Definition 9.

1. *We say that a first order theory T* eliminates imaginaries *if every definable set in every model of T is coded.*

2. *We say that T* uniformly eliminates imaginaries *if every $\mathcal{L}$-definable family of sets admits uniform canonical parameters via some $\mathcal{L}$-formula θ.*

The following is [Rid14, Proposition 0.1.5].

Proposition 10. *Let $\mathcal{L}$ be a multi-sorted language and T be a complete $\mathcal{L}$-first order theory. If there is at least one sort with two constants and every sort contains at least one constant then T eliminates imaginaries if and only if T eliminates imaginaries uniformly.*

Definition 11. *Let T be a complete $\mathcal{L}$-first order theory. Let $M \vDash T$ and D be an $\mathcal{L}$-definable set and $E \subseteq D^2$ be an $\mathcal{L}$-definable equivalence relation on D in M. We say that E is* represented *in M if we can find an $\mathcal{L}$-definable function $f : D \to M^k$ such that $f(x) = f(y)$ if and only if $E(x, y)$.*

A more natural way of thinking about elimination of imaginaries is precisely by making the class of definable sets closed by taking definable quotients, in fact if an equivalence relation E on some set D is represented by a function f then the quotient D/E is in definable bijection with the image of f.

Proposition 12. *Let T be a complete first order theory. The theory T eliminates imaginaries if and only if for all $M \vDash T$ we have that every $\mathcal{L}$-definable equivalence relation in M is represented in M.*

Proof. This is [Rid14, Proposition 0.1.7]. □

Shelah's imaginary extension

The following construction is due to S. Shelah and it is well known in the model theory community as the imaginary expansion. In this construction, we add new sorts in order to make all interpretable sets definable sets, more precisely:

Definition 13. *Let $\mathcal{L}$ be a multi-sorted language and T be a complete first order $\mathcal{L}$ theory. For each $\mathcal{L}$-definable equivalence relation $E \subseteq D^2$ where $D \subseteq S_1 \times \cdots \times S_k$ is an $\emptyset$-definable set where $S_1, \dots, S_k$ are sorts in the language $\mathcal{L}$, we add a new sort S_E and a new function symbol $f_E : D \to S_E$. We let $\mathcal{L}^{eq}$ be the language extending $\mathcal{L}$ obtained by adding all the sorts S_E and the function symbols f_E. Let T^{eq} be the $\mathcal{L}^{eq}$ first order theory:*

$$\begin{aligned} T \cup &\{f_E \text{ is onto} \mid E \text{ is an } \mathcal{L}\text{-definable equivalence relation}\} \\ &\cup \{\forall x, y (f_E(x) = f_E(y) \leftrightarrow E(x,y)) \mid E \text{ is an } \mathcal{L}\text{-definable equivalence relation}\}. \end{aligned}$$

Every model $M \vDash T$ can be naturally extended in to a model M^{eq} of T^{eq} (in a unique way) by interpreting S_E as the quotient D/E and f_E as the canonical projection map.

It is a well known fact that the theory T^{eq} eliminates imaginaries.
Let $\phi(\mathbf{x}; \mathbf{y})$ be a formula and let E_ϕ be the equivalence relation induced by ϕ on $\mathfrak{M}^{|\mathbf{y}|}$ as $E_\phi(y, z)$ if and only if $\forall x \phi(\mathbf{x}, \mathbf{y}) \leftrightarrow \phi(\mathbf{x}, \mathbf{z})$. Given $\mathbf{b} \in \mathfrak{M}^{|\mathbf{y}|}$ let $X = \phi(\mathfrak{M}, b)$. The class b/E_ϕ is called the *code of* X and we denote it by $\ulcorner X \urcorner$.

Some theories are somehow very close to eliminate imaginaries, except for finite finite sets. Those theories are said to have *weak elimination of imaginaries*, more formally:

Definition 14. *Let T be a complete $\mathcal{L}$-first order theory and $\mathfrak{M}$ be its monster model. Let X be a definable set,*

1. *we say that $\mathbf{d} \in \mathfrak{M}$ is a* weak code *for X if there exists a finite number of tuples $(\mathbf{a}_i)_{i \leq k} \in \mathfrak{M}$ such that $\mathbf{a}_0 = \mathbf{d}$ and for all $\sigma \in Aut(\mathfrak{M})$ we have that $\sigma(X) = X$ if and only if $\sigma(\{\mathbf{a}_i \mid i \leq k\}) = \{\mathbf{a}_i \mid i \leq k\}$.*

2. *Let $\phi(\mathbf{x}; \mathbf{y})$ an $\mathcal{L}$-definable formula. We say that a tuple $a \in \mathfrak{M}$ is a weak canonical parameter for X via ϕ if there is a finite number of tuples $(\mathbf{a}_i)_{i \leq k} \subseteq \mathfrak{M}$ such that $\mathbf{a}_0 = \mathbf{a}$ and for all tuples $m \in \mathfrak{M}$ we have $\phi(\mathfrak{M}; m) = X$ if and only if there exists some $i \leq k$ such that $m = \mathbf{a}_i$.*

The following is [Rid14, Proposition 0.1.15].

Proposition 15. *Let T be a complete $\mathcal{L}$-first order theory and $\mathfrak{M}$ its monster model. Let X be a definable set and $\mathbf{a}$ be a finite tuple. The following are equivalent:*

1. *the tuple $\mathbf{a}$ is a weak canonical parameter for X via some $\mathcal{L}$-formula ϕ,*
2. *the tuple $\mathbf{a}$ is a weak code of X,*
3. *$a \in \operatorname{acl}^{eq}(\ulcorner X \urcorner)$ and $\ulcorner X \urcorner \in \operatorname{dcl}^{eq}(\mathbf{a})$.*

Definition 16. *Let T be a complete $\mathcal{L}$-first order theory and $\mathfrak{M}$ be its monster model. We say that T* weakly eliminates imaginaries *if any definable set X has a weak code in $\mathfrak{M}$.*

Definition 17. *Let T be a complete $\mathcal{L}$-first order theory and let $\mathfrak{M}$ be its monster model. We say that it* codes finite sets *if for any finite set S has a code in $\mathfrak{M}$.*

The following is a well known folklore fact.

Fact 18. *Let T be a complete first order $\mathcal{L}$-theory. Suppose that T weakly eliminates imaginaries and codes finite sets then T eliminates imaginaries.*

Relative quantifier elimination

In this subsection we review some model theoretic definitions such as stable embeddeness, orthogonality and relative quantifier elimination. An expert reader can safely skip this subsection.

Definition 19. *Let $\mathcal{L}$ be a multi-sorted languange and $\mathcal{M}$ an $\mathcal{L}$-structure. Consider $\Pi \cup \Sigma$ a partition of the sorts of $\mathcal{L}$. We denote by $\mathcal{L} \restriction_{\Sigma}$ the sublanguage of $\mathcal{L}$ consisting of the sorts for Σ with their relation, function and constant symbols. Then we say that:*

1. *$\mathcal{M}$ Π-eliminates quantifiers if every formula $\phi(x)$ is equivalent to a formula without quantifiers in a sort of Π.*
2. *$\mathcal{M}$ eliminates quantifiers relative to Σ if the theory of $\mathcal{M}^{\Sigma-Mor}$ (obtained by naming all the $\mathcal{L} \restriction_{\Sigma}$-definable sets without parameters with a new predicate)- eliminates quantifiers.*

It is well known that, if $\mathcal{M}$ eliminates quantifiers relatively to Σ, then it eliminates Π-quantifiers.

Definition 20. 1. *A definable subset D of $\mathcal{M}$ is called* stably embedded *if all definable subsets of D^n, $n \in \mathbb{N}$ can be defined with parameters in D.*

2. *Two definable sets D and E of $\mathcal{M}$ are called* orthogonal *if for all formulas*

$$\phi(x_0, \dots, x_n, y_0, \dots, y_r, \bar{a}),$$

where $\bar{a}$ is a tuple of parameters from $\mathcal{M}$, we can find finitely many formulas

$$\theta_i(x_0, \dots, x_n, \bar{a}_i) \text{ and } \eta_i(y_0, \dots, y_R, \bar{a}'_i)$$

with $i < k$ and parameters $\bar{a}_0, \dots, \bar{a}_k, \bar{a}'_0, \dots, \bar{a}'_k$ in $\mathcal{M}$ such that:

$$\phi(D^n, E^r, \bar{a}) = \bigcup_{i<k} \theta_i(D^n, \bar{a}_i) \times \eta_i(E^r, \bar{a}'_i).$$

classification theory

Modern model theory has been significantly influenced by S. Shelah's remarkable work in [She90]. Since then the core goal in model theory has been directed to identify dividing lines among first order theories based on their combinatorical complexity, in an attempt to separate *tame* theories (e.g. vector spaces, algebraically closed fields, etc) from the *wild* theories (e.g. Peano Arithmetic or ZFC set theory), in order to develop a general theory for the *tame* theories and their models. Many of the combinatorical notions that now a days play a relevant role in the classification map, were introduced by S. Shelah and we devote this subsection to summarize these notions.
Trough this section we fix T a complete first order $\mathcal{L}$-theory and we let $\mathfrak{M}$ be its monster model.

stable theories

Definition 21. *Let $\phi(\mathbf{x};\mathbf{y})$ be an $\mathcal{L}$-formula. We say that $\phi(\mathbf{x};\mathbf{y})$ has the* order property *if one can find sequences $\langle a_i \mid i < \omega\rangle$, $\langle b_i \mid i < \omega\rangle$ in $\mathfrak{M}$ such that $\vDash \phi(\mathbf{a}_i, \mathbf{b}_j)$ if and only if $i \leq j$.*

Definition 22. *We say that a formula $\phi(\mathbf{x}, \mathbf{y})$ has the* binary tree property *if there are $|\mathbf{y}|$-tuples $(\mathbf{b}_\sigma \mid \sigma \in 2^{<\omega})$ such that for every $\eta \in 2^\omega$, the partial type $\{\phi(\mathbf{x}, \mathbf{b}_{\eta\restriction i})^{\eta(i)} \mid i < \omega\}$ is consistent.*

By a complete ϕ-type over a set A we mean a maximal consistent set of ϕ-formulas in $\mathcal{L}(A)$. We denote the space of ϕ types over A as $S_\phi(A)$.

Definition 23.

1. *Let $p(\mathbf{x}) \in S_\phi(\mathfrak{M})$, by a ϕ-*definition *of $p(\mathbf{x})$ over a set A we mean a $\mathcal{L}(A)$- formula $d_p\phi(\mathbf{y})$ such that for any parameters $\mathbf{b} \in \mathfrak{M}^{|\mathbf{y}|}$ we have:*

$$\phi(\mathbf{x}; \mathbf{b}) \in p(\mathbf{x}) \quad \text{if and only if } \vDash d_p\phi(\mathbf{b}).$$

2. *Let $p(\mathbf{x})$ be a a global type and A be a set of parameters. We say that $p(\mathbf{x})$ is* definable *over A if for any $\mathcal{L}$-formula $\phi(\mathbf{x}, \mathbf{y})$ there is a ϕ-*definition*. In such case, we refer to the set of ϕ definitions $\{d_p\phi \mid \phi$ is an $\mathcal{L}$-formula $\}$ as the* defining scheme *of p.*

Example 24. *Consider the language $\mathcal{L} = \{E\}$ with a binary relation and let T be the first order theory asserting that E has infinitely many classes all of them infinite. Let $\mathfrak{M}$ be its monster model and fix an element $a_0 \in \mathfrak{M}$, we consider the partial type*

$$\Sigma(x) = \{E(x,a)\} \cup \{x \neq b \mid b \in \mathfrak{M}\}.$$

This theory eliminates quantifiers in the given language thus $\Sigma(x)$ determines a complete type $p(x)$. The defining scheme for this type can be then analyzed as follows:

- *$E(x,b) \in p(x)$ if and only if $\vDash E(b,a)$,*
- *$x = b \in p(x)$ if and only if $b \neq b$.*

The type $p(x)$ is therefore definable over $\{a\}$.

The following statement characterizes stable theories and it is due to S.Shelah [She90].

Theorem 25. *Let $\phi(\mathbf{x};\mathbf{y})$ be a formula. The following are equivalent:*

1. *for any infinite set A, $|S_\phi(A)| \leq |A|$,*
2. *$\phi(\mathbf{x},\mathbf{y})$ does not have the order property,*
3. *$\phi(\mathbf{x},\mathbf{y})$ does not have the binary tree property,*
4. *Any ϕ-type over a set A is definable.*

Proof. This is [She90, Lemma 2.7]. □

Example 26.
1. *The following are examples of stable theories: vector spaces, algebraically closed fields, differentially closed fields of characteristic zero , $\mathbb{Z}$-modules, separably closed fields, etc.*
2. *The following theories are not stable: dense linear orders without endpoints , the first order theory of the random graph, algebraically closed valued fields, Peano arithmetic, etc.*

dependent theories

Definition 27.
1. *A formula $\phi(\mathbf{x};\mathbf{y})$* shatters *a set of size $n \in \mathbb{N}$, say $\{a_1,\dots,a_n\}$ if for any $S \subseteq \{1,\dots,n\}$ we can find a tuple $\mathbf{b}_S$ such that $\vDash \phi(\mathbf{a}_i;\mathbf{b}_S)$ if and only if $i \in S$.*
2. *A formula $\phi(\mathbf{x};\mathbf{y})$ has the* independence property *if for any $n \in \mathbb{N}$ we can find a set of size n shattered by $\phi(\mathbf{x};\mathbf{y})$.*
3. *A formula $\phi(\mathbf{x};\mathbf{y})$ is said to be* dependent *if it does not have the independence property (equivalently called as NIP.)*

4. *A theory is said to be* dependent*(equivalently called NIP) if no $\mathcal{L}$ formula has the independence property.*

If an $\mathcal{L}$-formula $\phi(\mathbf{x};\mathbf{y})$ is dependent, then the maximal integer n for which there is some set A of size n shattered by $\phi(\mathbf{x};\mathbf{y})$ is called the *VC-dimension* of ϕ. If ϕ has the independence property, then its VC-dimension is infinite.

Example 28.

Let T be the first order theory of dense linear orders without endpoints is dependent*. The formula $\phi(x, y) := x \leq y$ is dependent of VC-dimension* 1*.*

Any stable formula $\phi(\mathbf{x}, \mathbf{y})$ is dependent [see [Sim15, Section 2.3.2]].

Let T be the first order theory of the random graph in the language $\mathcal{L} = \{R\}$, then the formula $\phi(x, y) = xRy$ has the independence property, in fact any set of elements is shattered by ϕ.

For a detailed introduction to dependent theories we refer the reader to [Sim15].

dp-rank and dp-minimality

Definition 29. *Let p be a partial type over a set A and let κ be a (finite or infinite) cardinal. We say that $dp - rk(p, A) < \kappa$ if for every family $(I_t \mid t < \kappa)$ of mutually indiscernible sequences over A and $b \vDash p$ there is some $t < k$ such that I_t is indiscernible over Ab.*

Proposition 30. *The following are equivalent:*

1. *the theory T is dependent,*
2. *for every type p and set A, there is some κ such that $dp - rk(p, A) < \kappa$.*

Proof. This is [Sim15, Observation 4.3]. □

Definition 31. *Let T be a dependent theory, we say it is* strongly dpeendent *if for any finite tuple of variables x we have $dp - rk(x = x, \emptyset) < \aleph_0$.*

Example 32. *Let $\mathcal{L} = \{E_i \mid i < \omega\}$ where each of the E_i's are binary predicates, and let T be the $\mathcal{L}$- theory stating that each E_i defines an equivalence relation with infinitely many classes, each of them infinite. We consider the theory $T \subsetneq T_1$ stating that for all x, y if $xE_{i+1}y$ then xE_iy, and each E_i splits into infinitely many E_{i+1}-classes.*
On the other hand, we consider $T \subsetneq T_2$ stating that given $a_0, \ldots, a_{n-1}$ there is some a such that aE_ka_k for every $k < n$.
Then T_1 is strongly dependent, while T_2 is not.

The following is [Sim15, Proposition 4.26].

Proposition 33. *If $dp - rk(x = x, \emptyset) < \aleph_0$ for every variable x with $|x| = 1$, then T is strongly dependent.*

Definition 34. *The theory T is* dp-minimal *if $dp-rk(x=x,\emptyset)=1$, for x a singleton.*

Example 35. *The following theories are dp-minimal: any o-minimal theory, the theory of algebraically closed valued fields, the theory of the p-adics $T=Th(\mathbb{Q}_p)$ and the theory of Presburger arithmetic $T=Th(\mathbb{Z},0,1,+,\leq)$.*

NTP_2 theories

Definition 36. *Let T be a complete first order $\mathcal{L}$-theory. We say that a formula $\phi(\mathbf{x};\mathbf{y})$ has TP_2 (*the tree property of second kind*) if there is an array $(b^t_i : i<\omega, t<\omega)$ of tuples of size $|\mathbf{y}|$ and $k<\omega$ such that:*

1. *for any $\eta:\omega\to\omega$ the conjunction $\bigwedge_{t<\omega}\phi(\mathbf{x};\mathbf{b}^t_{\eta(t)})$ is consistent.*
2. *for any $t<\omega$ the set $\{\phi(\mathbf{x};b^t_i)\mid i<\omega\}$ is k-inconsistent.*

Definition 37. *A $\mathcal{L}$-formula $\phi(\mathbf{x};\mathbf{y})$ is NTP_2 if it does not have the TP_2. We sat that the theroy is NTP_2 if all formulas are NTP_2.*

The following is [Sim15, Proposition 5.31].

Proposition 38. *If T is dependent then it is NTP_2.*

Definition 39. *Let λ be a cardinal. For all $i<\lambda$, $\phi_i(\mathbf{x},\mathbf{y}_i)$ is a $\mathcal{L}$-formula where $\mathbf{x}$ is a common tuple of free variables, $\mathbf{b}_{i,j}$ are elements of $\mathfrak{M}$ of size $|\mathbf{y}_i|$ and k_i is a natural number. Finally, let $p(\mathbf{x})$ be a partial type. We say that $\{\phi_i(\mathbf{x},\mathbf{y}_i),(b_{i,j})_{j\in\omega},k_i\}_{i<\lambda}$ is an* inp-pattern of depth λ in $p(\mathbf{x})$ *if:*

1. *for all $i<\lambda$ the i^{th} row is k_i-inconsistent: any conjunction $\bigwedge_{l=1}^{k_i}\phi_l(\mathbf{x};b_{i,j_l})$ where $j_1<\cdots<j_{k_i}<\omega$ is inconsistent.*
2. *all (vertical) paths are consistent: i.e. for every $f:\lambda\to\omega$, the set $\{\phi_i(\mathbf{x};b_{i,f(i)})\}_{i<\lambda}\cup p(\mathbf{x})$ is consistent.*

Definition 40.
- *Let $p(\mathbf{x})$ be a partial type. The* burden *of $p(\mathbf{x})$ denoted by* $\mathrm{bdn}(p(\mathbf{x}))$ *is the cardinal defined as the supremum of the depths of inp-patterns in $p(\mathbf{x})$.*
- *The cardinal $sup_{S\in\mathcal{S}}\,\mathrm{bdn}(\{x_S=x_S\})$ where x_S is a single variable from the sort S and $\mathcal{S}$ is a set of sorts, is called the* burden *of the theory T and it is denoted as $\mathrm{bdn}(T)$.*

The following is [Che14, Remark 3.3].

Fact 41. *A theory T is NTP_2 if and only if $\mathrm{bdn}(T)<\infty$.*

Theories without the tree property of the second kind, usually called as NTP_2 were studied by A. Chernikov, we refer the reader for further details to [Che14].

Background in Algebra

In this subsection we recall some basic definitions, propositions and examples about valued fields.

Valued fields

Definition 42. *A valued field (K, v) is a field together with a* valuation *map $v : K^\times \to \Gamma$ from K onto an ordered abelian group Γ satisfying the following properties:*

1. *for any $x, y \in K^\times$, $v(xy) = v(x) + v(y)$,*
2. *for any $x, y \in K^\times$, $v(x + y) \geq \min\{v(x), v(y)\}$.*

This map can be extended to K if we add a constant ∞ and we define $v(x) = \infty$ if and only if $x = 0$.

Definition 43. *Let K be a field and $A \subseteq K$ we say that A is* valuation ring *of K if given any element $x \in K^\times$ either $x \in A$ or $x^{-1} \in A$.*

A ring A is a valuation ring if and only if its set of ideals is totally ordered by inclusion. Therefore, any valuation ring has a *unique maximal* ideal [4] that we commonly denote as m_A. Hence $A = A^\times \cup m_A$, where $A^\times$ denotes the set of elements which are invertible in A.

It is well known the one to one correspondence between valuations and valuation rings. If A is a valuation ring of the field K, then we have the disjoint union $K = m_A \cup A^\times \cup (m_A \backslash \{0\})^{-1}$. One can consider the abelian multiplicative quotient group $\Gamma_A := K^\times / A^\times$ written additively. The binary relation $\leq$ on Γ_A is defined as $yA^\times \leq xA^\times$ if and only if $\frac{x}{y} \in A$, where $x, y \in K^\times$. This relation, makes Γ_A into an ordered abelian gorup, adn the natural projection map $v_A := K^\times \to \Gamma_A$ defined by sending an element x to its class $xA^\times$ is a valuation. On the other hand, given a valuation $v : K^\times \to \Gamma$ we can define $\mathcal{O}_v ;= \{x \in K \mid v(x) \geq 0\}$ this is a valuation ring, whose maximal ideal is $m_v = \{x \in K \mid v(x) > 0\}$.

To simplify the notation, given a valued field (K, v) we will indicate as $\mathcal{O}$ its valuation ring and $\mathcal{M}$ its maximal ideal. We commonly refer to the quotient $\mathcal{O}/\mathcal{M}$ as the *residue field*, while we denote as $\Gamma = K^\times / \mathcal{O}^\times$ and refer to it as the *value group*.

Example 44. *The following are common examples of valued fields:*

1. *Let $K = \mathbb{C}(t)$, for each point $a \in \mathbb{C}$ there is a valuation $v_a : K^\times \to \mathbb{Z}$ defined as $v_a(f(t)) = k$ if and only if $f(t) = (t - a)^k \frac{g(t)}{h(t)}$, where $g(t), h(t) \in \mathbb{C}[t]$ and are such that $g(a), h(a) \neq 0$. The following cases can occur:*

[4] Existence is guaranteed by Zorn's Lemma.

- *If* $k > 0$ *then* $f(a) = 0$ *and* k *is the* order of vanishing *of* f *at* a*,*
- *if* $k = 0$*, then* $f(a) \in \mathbb{C}^{\times}$ *and*
- *if* $k < 0$ *then* f *has a* pole *at* a *of order* $-k$*.*

Let $v = v_0$*, the valuation ring associated to this valuation is* $\mathcal{O} := \{\frac{f(t)}{g(t)} \mid f(t), g(t) \in \mathbb{C}[t], g(0) \neq 0\}$*. There is a surjective ring morphism* $\phi : \mathcal{O}_v \to \mathbb{C}$ *defined by sending* $\frac{f(t)}{g(t)}$ *to* $\frac{f(0)}{g(0)}$ *where* $f(t), g(t) \in \mathbb{C}[t]$ *and* $g(0) \neq 0$*. The kernel of this ring morphism is the maximal ideal* $\mathcal{M} = \{\frac{f(t)}{g(t)} \in \mathcal{O} \mid f(0) = 0\}$*, thus it induces an isomorphism between the residue field associated to this valuation and* $\mathbb{C}$*.*

2. *Let* p *be a fixed prime number and let* $K = \mathbb{Q}$ *. We first define the* p*-adic valuation for elements in the ring* $v_p : \mathbb{Z} \to \mathbb{Z}$ *by defining* $v_p(x) = k$ *if and only if* $x = p^k m$ *and* p *does not divide* m*. We extend this map to the fraction field* $\mathbb{Q}$ *in the natural way, that is* $v_p(\frac{x}{y}) = v_p(x) - v_p(y)$*. The valuation ring associated to this valuation is* $\mathcal{O} = \{\frac{a}{b} \in \mathbb{Q} \mid p$ *does not divide* $a\}$*, wile its maximal ideal is* $\mathcal{M} = \{\frac{a}{b} \in \mathcal{O} \mid p$ *divides* $a\}$ *. The residue field in this case is isomorphic to* $\mathbb{F}_p$*.*

Definition 45. *Let* $\mathcal{O}$ *be a valuation ring of* K *and* $\mathcal{O}'$ *be an overring of* $\mathcal{O}$*,and hence a valuation ring of* K*. Then, we say that* $\mathcal{O}$ *is a* coarsening *of* $\mathcal{O}'$ *and* $\mathcal{O}'$ *is a* refinement *of* $\mathcal{O}$*.*

Let $\mathcal{O}$ be a fixed valuation ring of K and $\mathcal{O}'$ be an overring of $\mathcal{O}$. We have $\mathcal{M}' \subseteq \mathcal{M}$, where $\mathcal{M}'$ and $\mathcal{M}$ denote the maximal ideals of $\mathcal{O}'$ and $\mathcal{O}$ respectively. Since $\mathcal{M}'$ is a prime ideal in $\mathcal{O}'$, then it is also a prime ideal of $\mathcal{O}$. Moreover, localizing $\mathcal{O}$ at $\mathcal{M}'$ we can recover $\mathcal{O}'$, in fact $\mathcal{O}' = \mathcal{O}_{\mathcal{M}'}$.

The following is [EP05, Lemma 2.3.1].

Lemma 46. *Let* $\mathcal{O}$ *be a non trivial valuation ring in* K *corresponding to the valuation* $v : K \twoheadrightarrow \Gamma \cup \{\infty\}$*. Then there is a* 1*-to-*1 *correspondence of the convex subgroups* Δ *of* Γ *with the prime ideals* p *of* $\mathcal{O}$*, and hence with the overrings* $\mathcal{O}_p$*. This correspondence is given by:*

$$\Delta \to p_\Delta = \{x \in K \mid v(x) > \delta \textit{ for all } \delta \in \Delta\}$$
$$p \to \Delta_p = \{\gamma \in \Gamma \mid \gamma < v(x) \textit{ and} - \gamma < v(x) \textit{ for all } x \in p\}.$$

Let $\mathcal{O}$ be a valuation ring of K and $v := K \to \Gamma \cup \{\infty\}$ the corresponding valuation. Let p be a prime ideal with corresponding convex subgroup Δ_p in Γ and $\mathcal{O}_p$ the refinement of $\mathcal{O}$. There is a group homomorphism:

$$\phi := \begin{cases} K^\times/\mathcal{O}^\times & \to K^\times/\mathcal{O}_p^\times \\ x\mathcal{O}^\times & \mapsto x\mathcal{O}_p^\times. \end{cases}$$

whose kernel is $\Delta_p \cong \mathcal{O}_p^\times/\mathcal{O}^\times$. The valuation v_p induced by $\mathcal{O}_p$ is therefore obtain from $v := K \to \Gamma \cup \{\infty\}$ simply by taking the quotient of Γ by the convex subgroup Δ.

Hensel's Lemma

Among the class of valued fields we are particularly interested in those that are *henselian.*

Definition 47. *Let (K,v) be a valued field and $\mathcal{O}$ its valuation ring. We say that (K,v) is henselian if the following property is satisfied:* given a polynomial $P(x) \in \mathcal{O}[x]$ if there is some $a \in \mathcal{O}$ such that $v(P(a)) > 0$ while $v(P'(a)) = 0$ then there is some $b \in \mathcal{O}$ such that $a - b \in \mathcal{M}$ and $P(b) = 0$.

The class of henselian valued fields have been of particular interest to model theorist due to the Ax-Kochen Principle. We use [vdDKM+12] as a reference.

Theorem 48 (Ax-Kochen)**.** *Let (K,k,Γ) and (K',k',Γ') be henselian valued fields of equcharacteristic zero, or of mixed characteristic, unramified and with perfect residue field. Then $K \equiv K'$ if and only if $k \equiv k'$ and $\Gamma \equiv \Gamma'$.*

Proof. This is [vdDKM+12, Theorem 7.1]. □

The following Lemma establishes well known equivalences for henselian valued fields. A complete proof can be found in [Jah18, Theorem 3.2].

Lemma 49. *Let (K,v) be a valued field the following are equivalent:*

1. *(K,v) is henselian,*
2. *Every polynomial of the form $X^n + X^{n-1} + a_{n-2}X^{n-2} + \cdots + a_0$ with $a_i \in \mathcal{M}$ for $0 \leq i \leq n-2$ has a zero in K.*
3. *There is a unique extension of v to every finite (algebraic) extension of K.*

Convergence and the topology

Every valued field (K,v) carries naturally a topology, in fact:

Definition 50.
1. *An* open ball *of* radius γ *centered at a is the set $B_\gamma(a) = \{x \in K \mid v(x-a) > \gamma\}$.*
2. *A* close ball *of* radius γ *centered at a is the set $\overline{B_\gamma(a)} = \{x \in K \mid v(x-a) \geq \gamma\}$.*

The topology generated by the open balls is commonly called as the v-topology, and K is a topological field with the v-topology. That is, the field operations $+, -, \cdot : K^2 \to K$ and $^{-1} : K^\times \to K^\times$ are continuous.

Definition 51.
1. *A* well indexed sequence *in K is a sequence $\{a_\alpha\}$ in K whose terms a_α are indexed by an infinite well-ordered set without last element.*

2. *The sequence* $\{a_\alpha\}_{\alpha\in\rho}$ pseudo-converges *to an element* a *if* $\{v(a-a_\alpha)\}_{\alpha\in\rho}$ *is eventually strictly increasing, this is for some* ρ_0 *such that for any* $\eta > \sigma > \rho_0$ *we have* $v(a-a_\eta) > v(a-a_\sigma)$ *. in this case we say that* a *is a* pseudo-limit *of the sequence* $\{a_\alpha\}_{\alpha\in\rho}$*.*

3. *The sequence* $\{a_\alpha\}_{\alpha\in\rho}$ *is* pseudo-cauchy *if there is some element* ρ_0 *such that for any* $\tau > \sigma > \rho > \rho_0$ *we have that* $v(a_\tau - a_\sigma) > v(a_\sigma - a_\rho)$*.*

Immediate extensions and maximality

Let $(K, k, \Gamma) \subseteq (K', k', \Gamma')$ be a valued field extension. We denote as $\mathcal{O}$ the valuation ring of K and $\mathcal{M}$ its prime ideal, while $\mathcal{O}'$ stands for the valuation ring of K' and $\mathcal{M}'$ is its maximal ideal. We identify $k = \mathcal{O}/\mathcal{M}$ as a subfield of $k' = \mathcal{O}'/\mathcal{M}'$ in the usual way, via the map $\phi : \mathcal{O}/\mathcal{M} \to \mathcal{O}'/\mathcal{M}'$ by sending the class $x + \mathcal{M}$ to the class $x + \mathcal{M}'$. Likewise, we can identified the value group $\Gamma = K^\times/\mathcal{O}^\times$ as a subgroup of $\Gamma' = K^\times/(\mathcal{O}')^\times$ via the map $\psi : \Gamma \to \Gamma'$ sending the class $x\mathcal{O}^\times$ to $x(\mathcal{O}')^\times$.
The notion of immediate extensions was originally introduced by Ostrowski, Krull and Kaplanski during the first half of the twentieth century.

Definition 52. *Let* $(K, k, \Gamma) \subseteq (K', k', \Gamma')$ *be a valued field extension. We say that* (K', k', Γ') *is an* immediate extension *of* (K, k, Γ) *if* $k = k'$ *and* $\Gamma = \Gamma'$*.*

Definition 53. *A valued field* (K, k, Γ) *is said to be* maximal *if it has no immediate proper valued field extension.*

The following is a well known fact that characterizes entirely maximal valued fields.

Proposition 54. *A valued field* (K, v) *is maximal if and only if each pseudo-cauchy sequence in* K *has a pseudolimit in* K*.*

Proof. This is [vdDKM+12, Corollary 4.12]. □

Example 55. Hahn fields *given a field* k *and an ordered abelian group* Γ *we can construct a field* $K = k((t^\Gamma))$ *defined to be the set of all formal series* $f(t) = \sum_{\gamma\in\Gamma} a_\gamma t^\gamma$*, with coefficients* $a_\gamma \in k$ *such that the the support of* f*, i.e. supp*$(f) = \{\gamma \in \Gamma \mid a_\gamma \neq 0\}$ *is a well-ordered subset of* Γ*. We can define binary operations on the set* $k((t^\Gamma))$ *as follows:*

- $\sum a_\gamma t^\gamma + \sum b_\gamma t^\gamma = \sum (a_\gamma + b_\gamma) t^\gamma$*, and*
- $\big(\sum a_\gamma t^\gamma\big)\big(b_\gamma t^\gamma\big) = \sum_\gamma \big(\sum_{\alpha+\beta=\gamma}(a_\alpha b_\beta)\big)t^\gamma$*.*

With these operations K *is a field, and* k *is a subfield of* K *via the map* $a \mapsto at^0$*. We define the valuation* $v : K\backslash\{0\} \to \Gamma$ *by* $v\big(\sum a_\gamma t^\gamma\big) := \min\{\gamma \mid a_\gamma \neq 0\}$*. The valuation ring* $\mathcal{O} = \{f(t) \in K \mid supp(f) \subseteq \Gamma^{\geq 0}\}$ *and its maximal ideal* $\mathcal{M} = \{f(t) \in K \mid supp(f) \subseteq \Gamma^{>0}\}$*. For* $f(t) = \sum a_\gamma t^\gamma$ *in* $\mathcal{O}$ *we call* a_0 *the* constant term *of* $f(t)$ *and the map sending* $f(t)$ *to*

its constant term is a surjective ring homomorphism from $\mathcal{O}$ to the field k, whose kernel is $\mathcal{M}$. Hence, the residue field is isomorphic to k.
A particular instance this construction is the Laurent series over the complex numbers, $\mathbb{C}((t))$. Any Hahn field is a maximal field, this is for example guaranteed by [vdDKM+12, Corollary 4.13].

The leading term structure

Through this section we will assume the valued field K to be of equicharacteristic zero. We denote as k its residue field, and Γ its value group. In addition to the maximal ideal $\mathcal{M}$ we can introduce another ideal $\mathcal{M}_\delta$ for each $\delta \in \Gamma^{\geq 0}$ as $\mathcal{M}_\delta = \{x \in \mathcal{O} \mid v(x) > \delta\}$. In particular, $\mathcal{M} = \mathcal{M}_0$. For each $\delta \in \Gamma^{\geq 0}$ the set $(1 + \mathcal{M}_\delta)$ is a subgroup of the multiplicative group $K^\times$, as for any element such that $v(x) > \delta$ we have:

$$v((1+m)^{-1} - 1) = v(1 - (1+m)) - v(1+m) = v(-m) - 0 = v(m) > \delta,$$

so $1 + \mathcal{M}_\delta$ is closed under inverses.

Definition 56. *Let $\delta \geq 0$ in K. The* leading term structure of order δ *is the quotient group $RV_\delta = K^\times/(1 + \mathcal{M}_\delta)$. The quotient map is denoted* $\mathrm{rv}_\delta : K^\times \to RV_\delta$*. As with the value group it is convenient to add an element ∞ in RV_δ as $rv_\delta(0)$. Commonly, for $\delta = 0$ we omit the subscript and simply denote as $RV = RV_0$ and* $\mathrm{rv} = \mathrm{rv}_0$.

The quotient RV_δ naturally carries a multiplicative structure, but it also inherits a partially defined addition from the field via the relation:
$\oplus_\delta(a, b, c) \leftrightarrow \exists x, y, z \in K(a = \mathrm{rv}_\delta(x) \wedge b = \mathrm{rv}_\delta(y) \wedge c = \mathrm{rv}_\delta(z) \wedge x = y = z)$. The sum of a and b is said to be well-defined if there is exactly one c such that $\oplus_\delta(a, b, c)$ holds.
A major reference on the details around the leading term structure in the equicharacteristic zero case has been J. Flenner's PhD thesis [Fle08], we recall some of the most relevant facts present in his work.

Proposition 57. *For all $x, y \in K^\times$ and $\delta \in \Gamma$ the following are equivalent:*

1. $\mathrm{rv}_\delta(x) = \mathrm{rv}_\delta(y)$,
2. $v(x - y) > v(y) + \delta$,
3. $B_{>v(x)+\delta}(x) = B_{>v(y)+\delta}(y)$.

Proof. This is [Fle08, Proposition 1.3.4]. □

Proposition 58. *Suppose $v(x_1 + \cdots + x_n) = \min\{v(x_1), \ldots, v(x_n)\}$. Then $y = \mathrm{rv}_\delta(x_1) + \cdots + \mathrm{rv}_\delta(x_n)$ if and only if $y = \mathrm{rv}_\delta(x_1 + \cdots + x_n)$.*

Proof. This is [Fle08, Proposition 1.3.6]. □

Chapter 1

Elimination of imaginaries in OAG with bounded regular rank

In this paper we study elimination of imaginaries in some classes of pure ordered abelian groups. For the class of ordered abelian groups with bounded regular rank (equivalently with finite spines) we obtain weak elimination of imaginaries once we add sorts for the quotient groups Γ/Δ for each definable convex subgroup Δ, and sorts for the quotient groups $\Gamma/(\Delta + \ell\Gamma)$ where Δ is a definable convex subgroup and $\ell \in \mathbb{N}_{\geq 2}$. We refer to these sorts as the *quotient sorts*. For the dp-minimal case we obtain a complete elimination of imaginaries if we also add constants to distinguish the elements of the finite group $\Gamma/\ell\Gamma$ for each $\ell \in \mathbb{N}_{\geq 2}$.

1.1 Introduction

The model theory of ordered abelian groups has been studied since the sixties, and was initiated by Robinson and Zakon in [RZ60] who studied the completions of *regular* ordered abelian groups (see Definition 70). Later, the study of the elementary properties of ordered abelian groups was continued by Belegradek in [Bel02] for the class of *poly-regular* ordered abelian groups (see Definition 73). Significant achievements on (relative) quantifier elimination, model completion and definability of convex subgroups were achieved by Schmitt in [Sch82] for the general class of ordered abelian groups. More recently, Cluckers and Halupczok obtained a (relative) quantifier elimination for ordered abelian groups in [CH11] in a language that is more aligned with Shelah's imaginary expansion than the one introduced by Schmitt.

The model theoretic classification of certain classes of ordered abelian groups is an area of active research. Results include: the well known result of gurevich that no ordered abelian group has the independence property in [GS84]; the dp-minimal case characterized by Jahnke, Simon and Walsberg in [JSW17]; the strongly dependent case independently obtained by Dolich-Goodrick, Farré and Halevi-Hasson in [DG15][Far17][HH19] (respectively),

and the distal case in [ACGZ20] due to Aschenbrenner, Chernikov, Gehret and Ziegler.
The next natural step regarding the model theory of ordered abelian groups was understanding a reasonable language where one will have elimination of imaginaries. The answer to this problem, interesting in its own sake, has a significant impact in clarifying the problem of elimination of imaginaries for henselian valued fields. At the heart of the model theory of henselian valued fields is the well known *Ax-Kochen/Ershov* theorem, that broadly states that the first order theory of a henselian finitely ramified valued field is completely determined by the first order theory of its residue field and its value group. In a pure henselian valued field, the value group is a pure ordered abelian group and it is interpretable in the structure.
Following the Ax-Kochen principle one can first attempt to solve the problem of elimination of imaginaries for henselian valued fields by following two orthogonal directions:

1. The first one is to make the value group as tame as possible (e.g. to assume that it is definably complete) and to understand the obstacles that the the residue field naturally contributes to the problem. This research path was successfully finalized by Hils and Rideau-Kikuchi in [HRK21a].

2. Alternatively, one can assume the residue field is very tame (e.g. algebraically closed) and study the issues that the complexity of the value group brings to the problem. The work in [Vic21a] clarifies the picture for the equicharacteristic zero case, and this paper is the first milestone towards the solution.

To achieve elimination of imaginaries for pure ordered abelian groups, we use an abstract criterion isolated by Hrushovski in [Hru14] to show the following two results:

Theorem 59. *Let Γ be an ordered abelian group of bounded regular rank (equivalently with finite spines). Then Γ admits weak-elimination of imaginaries once the* quotient sorts *are added.*

Theorem 60. *Let Γ be a dp-minimal ordered abelian group. Then Γ admits elimination of imaginaries once the* quotient sorts *are added, and we add constants to distinguish the cosets of $\ell\Gamma$ in Γ, where $\ell \in \mathbb{N}_{\geq 2}$.*

This chapter is organized as follows:

1. In the first section we present the state of model theory of ordered abelian groups and introduce the class of ordered abelian groups with bounded regular rank.

2. In the second section we characterize the definable end-segments in an ordered abelian group with bounded regular rank and show that they can be coded in the quotient sorts.

3. In the third section we introduce Hrushovski's theorem to achieve a weak elimination of imaginaries result for the class of ordered abelian groups with bounded regular rank.

This criterion requires us to check two conditions: the density of definable types, proved in Proposition 97; and the coding of definable types, proved in Proposition 95.

4. In the last section, we briefly present the main results for pure ordered abelian groups.

The case of direct sums of the integers with the lexicographic order has been done independently by Liccardo in [Lic21], as part of her PhD thesis under D'Aquino. Hils and Mennuni in [HM21] have independently obtained the result for the regular case.

1.2 Preliminaries

Elimination of imaginaries

Let T be a first order theory and $\mathfrak{M}$ be its monster model. Let $D \subseteq \mathfrak{M}^k$ be some definable set and E some definable equivalence relation over D. The equivalence class $e = a/E$ is said to be an *imaginary element*. Imaginaries in model theory were introduced by Shelah in [She90]. Later in [Mak84], Makkai proposed to construct the many sorted structure $\mathfrak{M}^{eq}$, where we add a sort S_E for each definable equivalence relation E and a map π_E sending each element to its class. Since then, the model theoretic community has presented and studied imaginary elements in this way and refers to the multi-sorted structure $\mathfrak{M}^{eq}$ as the *imaginary expansion of* $\mathfrak{M}$. We call the sorts S_E *imaginary sorts* while we refer to $\mathfrak{M}$ as the *home-sort*. Any formula $\phi(\mathbf{x}, \mathbf{y})$ induces an equivalence relation in $\mathfrak{M}^{|\mathbf{y}|}$ defined as

$$E_\phi(\mathbf{y}_1, \mathbf{y}_2) \text{ if and only if } \forall \mathbf{x}\big(\phi(\mathbf{x}, \mathbf{y}_1) \leftrightarrow \phi(\mathbf{x}, \mathbf{y}_2)\big).$$

Let $\mathbf{b} \in \mathfrak{M}^{|\mathbf{y}|}$ and $X := \phi(\mathbf{x}, \mathbf{b})$. We call the class $\mathbf{b}/E_\phi$ the *code* of X and denote it as $\ulcorner X \urcorner$. We denote by dcl^{eq} and acl^{eq} the definable closure and the algebraic closure in the expansion $\mathfrak{M}^{eq}$.

Definition 61. 1. *We say that T has* elimination of imaginaries *if for any imaginary element e there is a tuple a in the home-sort such that $e \in \mathrm{dcl}^{eq}(a)$ and $a \in \mathrm{dcl}^{eq}(e)$.*

2. *We say that T has* weak elimination of imaginaries *if for any imaginary element e there is a tuple a in the home-sort such that $e \in \mathrm{dcl}^{eq}(a)$ and $a \in \mathrm{acl}^{eq}(e)$.*

3. *We say that T* codes finite sets *if for every model $M \vDash T$ and every finite subset S of M, the code $\ulcorner S \urcorner$ is interdefinable with a tuple of elements in M.*

The following is a folklore fact.

Fact 62. *Let T be a complete multi-sorted theory. If T has weak elimination of imaginaries and codes finite sets then T eliminates imaginaries.*

Ordered abelian groups of bounded regular rank

In this section we summarize several results about the classification of ordered abelian groups and their model theoretic behavior. We start by recalling the following folklore fact.

Fact 63. *Let $(\Gamma, <, +, 0)$ be a non-trivial ordered abelian group. Then the topology induced by the order in Γ is discrete if and only if Γ has a minimal positive element. In this case we say that Γ is* discrete*, otherwise we say that it is* dense*.*

The following notions were isolated in the sixties by Robinson and Zakon in [RZ60] to understand some complete extensions of the theory of ordered abelian groups.

Definition 64. *Let Γ be an ordered abelian group and $n \in \mathbb{N}_{\geq 2}$.*

1. *Let $\gamma \in \Gamma$. We say that γ is n-*divisible *if there is some $\beta \in \Gamma$ such that $\gamma = n\beta$.*
2. *We say that Γ is n-*divisible *if every element $\gamma \in \Gamma$ is n-divisible.*
3. *Γ is said to be n-*regular *if any interval with at least n points contains an n-divisible element.*

Example 65. *We include some examples to illustrate the previous definitions.*

1. *Consider the ordered abelian group $(\mathbb{Z}, +, \leq, 0)$, the elements $2, 4, 6$ are 2-divisible while 1 is not.*
2. *The groups $(\mathbb{Q}, +, \leq, 0)$ and $(\mathbb{Z}, +, \leq, 0)$ are n-regular for each natural number $n \in \mathbb{N}_{\geq 2}$. The group $(\mathbb{Z} \oplus \mathbb{Z}, \leq_{lex}, +, 0)$, where $\leq_{lex}$ is the lexicographic order, is not 2-regular, because the interval $\big((1,-1),(1,4)\big) = \{(1,0),(1,1),(1,2),(1,3)\}$ does not contain a 2-divisible element.*

The following definitions were introduced by Schmitt in [Sch82] and [Sch84].

Definition 66. *We fix an ordered abelian group Γ and $n \in \mathbb{N}_{\geq 2}$. Let $\gamma \in \Gamma$. We define:*

- *$A(\gamma) =$ the largest convex subgroup of Γ not containing γ.*
- *$B(\gamma) =$ the smallest convex subgroup of Γ containing γ.*
- *$C(\gamma) = B(\gamma)/A(\gamma)$.*
- *$A_n(\gamma) =$ the smallest convex subgroup C of Γ such that $B(g)/C$ is n-regular.*
- *$B_n(g) =$ the largest convex subgroup C of Γ such that $C/A_n(\gamma)$ is n-regular.*

In [Sch82, Chapter 2], Schmitt shows that the groups $A_n(\gamma)$ and $B_n(\gamma)$ are definable in the language of ordered abelian groups $\mathcal{L}_{OAG} = \{+, -, \leq, 0\}$ by a first order formula using only the parameter γ.
We recall that the set of convex subgroups of an ordered abelian group is totally ordered by inclusion.

Definition 67. *Let Γ be an ordered abelian group and $n \in \mathbb{N}_{\geq 2}$, we define the n-*regular rank *to be the order type of:*

$$\big(\{A_n(\gamma) \mid \gamma \in \Gamma\backslash\{0\}\}, \subseteq\big).$$

The n-regular rank of an ordered abelian group Γ is a linear order, and when it is finite we can identify it with its cardinal. In [Far17], Farré emphasizes that we can characterize it without mentioning the subgroups $A_n(\gamma)$. The following is [Far17, Remark 2.2].

Definition 68. *Let Γ be an ordered abelian group and $n \in \mathbb{N}_{\geq 2}$, then:*

1. *Γ has n-regular rank equal to 0 if and only if $\Gamma = \{0\}$,*
2. *Γ has n-regular rank equal to 1 if and only if Γ is n-regular and not trivial,*
3. *Γ has n-regular rank equal to m if there are $\Delta_0, \dots, \Delta_m$ convex subgroups of Γ, such that:*
   - *$\{0\} = \Delta_0 < \Delta_1 < \dots < \Delta_m = \Gamma$,*
   - *for each $0 \leq i < m$, the quotient group Δ_{i+1}/Δ_i is n-regular,*
   - *the quotient group Δ_{i+1}/Δ_i is not n-divisible for $0 < i < m$.*

   *In this case we define $RJ_n(\Gamma) = \{\Delta_0, \dots, \Delta_{m-1}\}$. The elements of this set are called the n-*regular jumps*.*

Example 69. *Let $G = \underbrace{\mathbb{Z} \oplus \dots \oplus \mathbb{Z}}_{n-times}$ with the lexicographic order $\leq_{lex}$. The 3-regular rank of G is equal to n. This is witnessed by the sequence:*

$$\{\mathbf{0}\} \leq \underbrace{\{0\} \oplus \dots \oplus \{0\}}_{n-1 \ \ times} \oplus \mathbb{Z} \leq \dots \leq \{0\} \oplus \underbrace{\mathbb{Z} \oplus \dots \oplus \mathbb{Z}}_{n-1-times} \leq \mathbb{Z} \oplus \dots \oplus \mathbb{Z}.$$

Regular groups and poly-regular groups

Definition 70. *An ordered abelian group Γ is said to be* regular *if it is n-regular for all $n \in \mathbb{N}$.*

Example 71. $(\mathbb{Z}, +, \leq, 0)$ *and* $(\mathbb{Q}, +, \leq, 0)$ *are standard examples of regular groups. By [Bel02, Theorem 1.2] any archimedean group is regular.*

Robinson and Zakon in their seminal paper [RZ60] completely characterized the possible completions of the theory of regular groups, obtained by extending the first order theory of ordered abelian groups with axioms asserting that for each $n \in \mathbb{N}$ if an interval contains at least n-elements then it contains an n-divisible element. The following is [RZ60, Theorem 4.7].

Theorem 72. *The possible completions of the theory of regular groups, are:*

1. *the theory of discrete regular groups, and*
2. *the completions of the theory of dense regular groups T_χ where*

$$\chi =: \text{Primes} \to \mathbb{N} \cup \{\infty\},$$

is a function specifying the index $\chi(p) = [\Gamma : p\Gamma]$.

Robinson and Zakon proved as well that each of these completions is the theory of some archimedean group. In particular, any discrete regular group is elementarily equivalent to $(\mathbb{Z}, \leq, +, 0)$. This theory is called the theory of Presburger arithmetic, introduced in 1929 by M. Presburger, who proved that it admits quantifier elimination in the well known *Presburger Language* $\mathcal{L}_{\text{Pres}} = \{0, 1, +, -, <, (\equiv_m)_{m \in \mathbb{N}_{\geq 2}}\}$. Given an ordered abelian group Γ we naturally see it as a $\mathcal{L}_{\text{Pres}}$-structure. The symbols $\{0, +, -, <\}$ take their obvious interpretation. If Γ is discrete, the constant symbol 1 is interpreted as the least positive element of Γ, and by 0 otherwise. For each $m \in \mathbb{N}_{\geq 2}$ the binary relation symbol $\equiv_m$ is interpreted as the equivalence modulo m, i.e. for any $g, h \in \Gamma$ $g \equiv_m h$ if and only if $g - h \in m\Gamma$.
The theory of a dense ordered abelian group admits quantifier elimination in the Presburger language if and only if it is regular. This is a result of Weispfenning in [Wei81, Theorem 2.9].

Definition 73. *Let Γ be an ordered abelian group. We say that it is* poly-regular *if it is elementarily equivalent to a subgroup of the lexicographically ordered group $\mathbb{R}^n$.*

In [Bel02] Belegradek studied poly-regular groups and proved that an ordered abelian group is poly-regular if and only if it has finitely many proper definable convex subgroups, and all the proper definable subgroups are definable over the empty set. In [Wei81, Theorem 2.9] Weispfenning obtained quantifier elimination for the class of poly-regular groups in the language of ordered abelian groups extended with predicates to distinguish the subgroups $\Delta + \ell\Gamma$ where Δ is a convex subgroup and $\ell \in \mathbb{N}_{\geq 2}$.

Ordered abelian groups with bounded regular rank

Definition 74. *Let Γ be an ordered abelian group. We say that it has* bounded regular rank *if it has finite n-regular rank for each $n \in \mathbb{N}_{\geq 2}$. For notation, we will use $RJ(\Gamma) = \bigcup_{n \in \mathbb{N}_{\geq 2}} RJ_n(\Gamma)$.*

The class of ordered abelian groups of bounded regular rank extends the class of poly-regular groups and regular groups. The terminology of *bounded regular rank* becomes clear with the following Proposition (item 3).

Proposition 75. *Let Γ be an ordered abelian group. The following are all equivalent:*

1. *Γ has finite p-regular rank for each prime number p.*
2. *Γ has finite n-regular rank for each $n \geq 2$.*
3. *There is some cardinal κ such that for any $H \equiv \Gamma$, $|RJ(H)| \leq \kappa$.*
4. *For any $H \equiv \Gamma$, any definable convex subgroup of H has a definition without parameters.*
5. *There is some cardinal κ such that for any $H \equiv \Gamma$, H has at most κ definable convex subgroups.*

Moreover, in this case $RJ(\Gamma)$ is the collection of all proper definable convex subgroups of Γ and all are definable without parameters. In particular, there are only countably many definable convex subgroups.

Proof. This is [Far17, Proposition 2.3]. □

Quantifier elimination and the quotient sorts

In [CH11] Cluckers and Halupczok introduced a language $\mathcal{L}_{qe}$ to obtain quantifier elimination for ordered abelian groups relative to the *auxiliary sorts* S_n, T_n and T_n^+, whose precise description can be found in [CH11, Definition 1.5]. This language is similar in spirit to the one introduced by Schmitt in [Sch82], but has lately been preferred by the community as it is more in line with the many-sorted language of Shelah's imaginary expansion $\mathfrak{M}^{eq}$. Schmitt does not distinguish between the sorts S_n, T_n and T_n^+. Instead for each $n \in \mathbb{N}$ he works with a single sort $Sp_n(\Gamma)$ called the *n-spine* of Γ, whose description can be found in [GS84, Section 2]. In [CH11, Section 1.5] it is explained how the auxiliary sorts of Cluckers and Halupczok are related to the n-spines $Sp_n(\Gamma)$ of Schmitt. In [Far17, Section 2], it is shown that an ordered abelian group Γ has bounded regular rank if and only if all the n-spines are finite, and $Sp_n(\Gamma) = RJ_n(\Gamma)$. In this case, we define the regular rank of Γ as the cardinal $|RJ(\Gamma)|$, which is either finite or $\aleph_0$. Instead of saying that Γ is an ordered abelian group with finite spines, we prefer to use the classical terminology of bounded regular rank, as it emphasizes the relevance of the n-regular jumps and the role of the divisibilities to describe the definable convex subgroups.

Definition 76 (The language $\mathcal{L}$)**.** *Let Γ be an ordered abelian group with bounded regular rank. We view Γ as a multi-sorted structure in the language $\mathcal{L}$, where:*

1. *we add a sort for the ordered abelian group Γ, and we equip it with a copy of the language $\mathcal{L}_{\mathrm{Pres}}$ extended with predicates to distinguish each of the convex subgroups $\Delta \in RJ(\Gamma)$. We refer to this sort as the* main sort*.*

2. *For each $\Delta \in RJ(\Gamma)$ we add a sort for each of the ordered abelian groups Γ/Δ, equipped with a copy of the Presburger language* $\mathcal{L}_{\text{Pres}}^{\Delta} = \{0^{\Delta}, 1^{\Delta}, +^{\Delta}, -^{\Delta}, <^{\Delta}, (\equiv_m^{\Delta})_{m\in\mathbb{N}_{\geq 2}}\}$.
We add as well a map $\rho_{\Delta} : \Gamma \to \Gamma/\Delta$, interpreted as the natural quotient map.

In [Far17, Theorem 2.4] Farré obtained a quantifier elimination statement for the class of ordered abelian groups with bounded regular rank in the languange $\mathcal{L}$ extended with a set of constants in the home sort. However, we present a slightly different language where we add the constants for the minimal element in Γ/Δ (if it exists) instead of adding a representative in the home-sort whose projection is the minimal class in Γ/Δ. For this purpose we highlight that the following statement is a direct consequence of [ACGZ20, Proposition 3.14].

Theorem 77. *Let Γ be an ordered abelian group with bounded regular rank. Then Γ admits quantifier elimination in the language $\mathcal{L}$.*

Notation 78. *We will be mainly interested in the description of the definable sets in the* main sort. *For this purpose we will slightly abuse the language, to simplify the notation. For each $k \in \mathbb{Z}$ and $\Delta \in RJ(\Gamma)$ we define $k^{\Delta} = k \cdot 1^{\Delta}$, where 1^{Δ} is the minimal element in Γ/Δ if it exists. We will sometimes indicate k^{Δ} simply as $k + \Delta$. We introduce the following notation:*

1. *We write $\tau(\mathbf{x}) + \Delta < \beta + k + \Delta$ for the formula $\rho_{\Delta}(\tau(\mathbf{x})) <^{\Delta} \rho_{\Delta}(\beta) + k^{\Delta}$.*

2. *We write $\tau(\mathbf{x}) \equiv_{\Delta} \beta + k$ for the formula $\rho_{\Delta}(\tau(\mathbf{x})) = \rho_{\Delta}(\beta) + k^{\Delta}$.*

3. *We write $\tau(\mathbf{x}) \equiv_{\Delta+m\Gamma} \beta + k$ for the formula $\rho_{\Delta}(\tau(\mathbf{x})) \equiv_m^{\Delta} \rho_{\Delta}(\beta) + k^{\Delta}$. The latter is interpreted as $\rho_{\Delta}(\tau(\mathbf{x})) - (\rho_{\Delta}(\beta) + k^{\Delta}) \in m\big(\Gamma/\Delta\big)$.*

Here$\tau(\mathbf{x})$ is a term in the language of ordered abelian groups in m variables, $\mathbf{x} = (x_1, \ldots, x_m)$ and $\beta \in \Gamma$.

Definition 79. *1. A set $S \subset \Gamma$ is said to be an* end-segment *(respectively an initial segment) if for any $x \in S$ and $y \in \Gamma$, $x < y$ (respectively $y < x$) we have that $y \in S$.*

2. *Let $n \in \mathbb{N}_{\geq 2}$, $\Delta \in RJ(\Gamma)$, $\beta \in \Gamma \cup \{-\infty, +\infty\}$ and $\square \in \{\geq, >\}$. $\{\eta \in \Gamma \mid n\eta + \Delta \square \beta + \Delta\}$ is an end-segment of Γ. We call the end-segments of this form* divisibility end-segments. *We define* divisibility initial segments *analogously.*

3. *A* mid-segment *is a non empty set C of the form $C = U \cap L$ where U is a divisibility end-segment and L is a divisibility initial segment.*

4. *A* basic positive congruence formula *is a formula of the form $zx \equiv_{\Delta+l\Gamma} \beta + k$ where $\beta \in \Gamma$, $z, k \in \mathbb{Z}$ and $l \in \mathbb{N}_{\geq 2}$. Likewise, a* basic negative formula *is a formula of the form $zx \not\equiv_{\Delta+l\Gamma} \beta + k$. A* basic congruence formula *is either a basic positive congruence formula or a basic negative formula.*

5. *A* finite congruence restriction *is a finite conjunction of basic congruence formulas.*
6. *A* nice set *is a set of the form $C \cap X$, where C is a mid-segment and X is defined by a finite congruence restriction.*

The following is a direct consequence of quantifier elimination.

Corollary 80. *Let Γ be an ordered abelian group with bounded regular rank. Let $X \subseteq \Gamma$ be a definable set. Then X is a finite union of nice sets.*

We will consider an extension $\mathcal{L}_Q$ of the language $\mathcal{L}$, where for each natural number $n \in \mathbb{N}_{\geq 2}$ and $\Delta \in RJ(\Gamma)$ we add a sort for the quotient group $\Gamma/(\Delta + n\Gamma)$ and a map $\pi_{\Delta}^{n} : \Gamma \to \Gamma/(\Delta + n\Gamma)$.

We will refer to the sorts in the language $\mathcal{L}_Q$ as *quotient sorts.*

The following fact will be very useful to show weak elimination of imaginaries for ordered abelian groups with bounded regular rank.

Fact 81. *Let Γ be an ordered abelian group of finite n-regular rank witnessed by the sequence $\{0\} = \Delta_0 < \Delta_1 < ... < \Delta_l = \Gamma$ and fix some definable convex subgroup H. Then Γ/H is also a group of finite n-regular rank. Moreover, if Γ is an ordered abelian group of bounded regular rank, then $H \in RJ(\Gamma)$ and each coset of Δ_i/H in Γ/H is interdefinable with an element of Γ/Δ_i.*

Proof. Let Γ be an ordered abelian group and H a convex subgroup. Assume that Γ has finite n-regular rank, witnessed by the sequence $\{0\} = \Delta_0 < \Delta_1 < ... < \Delta_l = \Gamma$ and let r be the smallest index such that $\Delta_r \subseteq H \subseteq \Delta_{r+1}$. We aim to show that $\Delta_{r+1}/H < \cdots < \Delta_l/H = \Gamma/H$ witnesses that Γ/H has finite n-regular rank. For each $r \leq i < l$, by the isomorphism theorem $(\Delta_{i+1}/H)/(\Delta_i/H) \cong \Delta_{i+1}/\Delta_i$. As Δ_{i+1}/Δ_i is n-regular and not n-divisible, so is $(\Delta_{i+1}/H)/(\Delta_i/H)$.
The second part of the statement follows immediately by the isomorphism theorem. □ □

A survey of model theoretic results on ordered abelian groups

In 1984 the classification of the model theoretic complexity of ordered abelian groups was initiated by Gurevich and Schmitt in [GS84], who proved that no ordered abelian group has the independence property. During the last years finer classifications have been achieved, and we present the state of the field in this subsection.

Definition 82. *Let Γ be an ordered abelian group and let p be a prime number. We say that p is a* singular prime *if $[\Gamma : p\Gamma] = \infty$. If Γ does not have singular primes we call it* non-singular*.*

The following result corresponds to [JSW17, Proposition 5.1].

Proposition 83. *Let Γ be an ordered abelian group. The following conditions are equivalent:*

1. *Γ is non-singular,*
2. *Γ is dp-minimal.*

The following is [ACGZ20, Theorem 3.13].

Proposition 84. *Let Γ be an ordered abelian group with bounded regular rank (i.e. each $Sp_n(\Gamma)$ is finite). The following statements are equivalent:*

1. *Γ is distal,*
2. *Γ is dp-minimal.*

The following statement was independently achieved in [DG15],[Far17] and [HH19].

Proposition 85. *Let Γ be an ordered abelian group. The following conditions are equivalent:*

1. *Γ is strongly dependent,*
2. *Γ has finite dp-rank,*
3. *Γ has bounded regular rank and finitely many singular primes.*

Moreover, let $\mathcal{P} = \{p \in \mathbb{N} \mid p \text{ is a singular prime}\}$. Then

$$dp-\operatorname{rank}(\Gamma) \leq 1 + \sum_{p \in \mathcal{P}} |RJ_p(G)|.$$

1.3 Definable end-segments

In this subsection we characterize the definable end-segments (or initial segments) in an ordered abelian group with bounded regular rank (equivalently with finite spines). We also show that they can be coded in the quotient sorts.

Definition 86. *Let Γ be an ordered abelian group with bounded regular rank:*

1. *Given $S \subseteq \Gamma$ an end-segment (or an initial segment) we denote by Δ_S the stabilizer of S, i.e. $\Delta_S := \{\beta \in \Gamma \mid \beta + S = S\}$.*
2. *Let $S \subseteq \Gamma$ be an end-segment and $\Delta \in RJ(\Gamma)$. We consider the projection map $\rho_\Delta : \Gamma \to \Gamma/\Delta$, and we will denote $\rho_\Delta(S)$ by S_Δ. One can show that*

$$S_\Delta = \{\gamma \in \Gamma/\Delta \mid \exists y \in S \ \rho_\Delta(y) = \gamma\}$$

is a definable end-segment of Γ/Δ, as it is the projection of an end-segment.

3. *Let $\Delta \in RJ(\Gamma)$ and $S \subseteq \Gamma$ be an end-segment. We say that S is* Δ-decomposable *if it is a union of Δ-cosets.*

4. *Let X and Y be definable sets. We say that Y is* coinitial *(or* cofinal*) in X if for any $y \in X$ there is some element $z \in X \cap Y$ such that $z \leq y$ (respectively $z \geq y$).*

Fact 87. *Let $S \subseteq \Gamma$ be a definable end-segment. Then Δ_S is a definable convex subgroup of Γ, and therefore $\Delta_S \in RJ(\Gamma)$. Furthermore, $\Delta_S = \bigcup_{\Delta \in \mathcal{C}} \Delta$, where*

$$\mathcal{C} = \{\Delta \in RJ(\Gamma) \mid S \text{ is } \Delta\text{-decomposable}\}.$$

Proof. We first show that $\Delta_S \subseteq \bigcup_{\Delta \in \mathcal{C}} \Delta$. Note Δ_S is a definable convex subgroup, so $\Delta_S \in RJ(\Gamma)$. We aim to show that S is Δ_S-decomposable, so it is sufficient to show that for any $\gamma \in S$, $\gamma + \Delta_S \subseteq S$. Fix some $\gamma \in \Gamma$. If $\delta \in \Delta_S$ then $\gamma + \delta \in S$, so $\gamma + \Delta_S \subseteq S$.
We now prove that $\bigcup_{\Delta \in \mathcal{C}} \Delta \subseteq \Delta_S$. Let $\Delta \in \mathcal{C}$ and fix some $\delta \in \Delta$. We want to show that $\delta + S \subseteq S$ and $S \subseteq \delta + S$. Because S is Δ-decomposable, $\gamma + \Delta \subseteq S$ for any $\gamma \in S$. In particular $\gamma + \delta \in S$. As γ is an arbitrary element in S, we conclude that $\delta + S \subseteq S$. It is only left to show that $S \subseteq \delta + S$. Let $\gamma \in S$, then $\gamma - \delta \in S$ because $\gamma + \Delta \subseteq S$. As $\gamma = \delta + (\gamma - \delta) \in \delta + S$, we have $S \subseteq \delta + S$, as required. □ □

Proposition 88. *Let Γ be an ordered abelian group of bounded regular rank. Any definable end-segment is a divisibility end-segment.*

Proof. Let $S \subseteq \Gamma$ be a definable end-segment such that $S \neq \Gamma$. By Fact 87, Δ_S is a definable convex subgroup of Γ and S is Δ_S-decomposable. To simplify the notation we will denote $\hat{\Gamma} = \Gamma/\Delta_S$ and $\hat{S} = S_{\Delta_S} = \rho_{\Delta_S}(S)$. It is sufficient to prove that $\hat{S}$ is a divisibility end-segment in $\hat{\Gamma}$.

Claim 89. *Note that for any $k \in \mathbb{N}$ exactly one of the following occurs:*

- *$\hat{\Gamma}$ is k-regular.*
- *There is a non trivial k-regular convex subgroup Λ_k of $\hat{\Gamma}$ and a coset $\eta + \Lambda_k$ such that $\hat{S} \cap (\eta + \Lambda_k) \neq \emptyset$ and $\hat{S}^c \cap (\eta + \Lambda_k) \neq \emptyset$.*

Proof. Let $\{0\} = \Delta_0 < \Delta_1 < \cdots < \Delta_l = \Gamma$ the sequence of convex subgroups witnessing that Γ has k-finite regular rank equal to l. Let $r \leq l$ be the smallest index such that $\Delta_S \subsetneq \Delta_r$. If $r = l$ then $\hat{\Gamma}$ is k-regular. Otherwise the quotient group $\Lambda_k = \Delta_r/\Delta_S$ satisfies the required conditions. Indeed, as Δ_r/Δ_{r-1} is k-regular so is Δ_r/Δ_S. Additionally, S is not Δ_r-decomposable. If it were, then we would have $\Delta_r \subseteq \Delta_S$ which contradicts $\Delta_S \subsetneq \Delta_r$. Then there is some coset $\eta + \Delta_r$ such that $S \cap (\eta + \Delta_r) \neq \emptyset$ and $S^c \cap (\eta + \Delta_r) \neq \emptyset$ because otherwise S would be Δ_r-decomposable. Thus $\hat{S} \cap (\hat{\eta} + \Lambda_k) \neq \emptyset$ and $\hat{S}^c \cap (\hat{\eta} + \Lambda_k) \neq \emptyset$, where $\hat{\eta} = \eta + \Delta_S$. □ □

We may assume that $\hat{S}$ does not have a minimum because otherwise the statement follows immediately. By Corollary 80 applied to $\hat{\Gamma}$, $\hat{S}$ is a finite union of nice sets $C_i \cap X_i$, where $C_i = U_i \cap L_i$. As $\hat{S}$ is a definable end-segment, it is sufficient to understand the co-initial description of $\hat{S}$. Without loss of generality we may assume that $U_i \subseteq U_1$ for all i. Let $\hat{\Delta} \in RJ(\hat{\Gamma})$. Then $\hat{\Delta} = \Delta/\Delta_S$ for some $\Delta_S \subsetneq \Delta \in RJ(\Gamma)$. Thus there is a coset $\eta + \hat{\Delta}$ such that $\hat{S} \cap (\eta + \hat{\Delta}) \neq \emptyset$ and $\hat{S}^c \cap (\eta + \hat{\Delta}) \neq \emptyset$ because S is not Δ-decomposable.

Hence, each of the congruence formulas involving the groups $\hat{\Delta} + k\hat{\Gamma}$ does not change its truth value over $U_1 \cap (\eta + \hat{\Delta})$. Therefore it does not change its truth value co-initially in U_1. Consider a conjunction of congruence restrictions of the form:

$$C(x) := \big(\bigwedge_{i \leq s} x \equiv_{k_i\hat{\Gamma}} c_i\big) \wedge \big(\bigwedge_{j \leq l} \neg(x \equiv_{r_j\hat{\Gamma}} d_j)\big).$$

Let M be the least common multiple of all the k_i's and r_j's involved in the definition of $C(x)$. By the previous Claim, Γ is M-regular or we can find an M-regular group Λ_M and a coset that intersects $\hat{S}$ and its complement. We first assume the existence of a non-trivial convex subgroup Λ_M and a coset $\eta + \Lambda_M$ such that $\hat{S} \cap (\eta + \Lambda_M) \neq \emptyset$ and $\hat{S}^c \cap (\eta + \Lambda_M) \neq \emptyset$. Let $Y = C(x) \cap (U_1 \cap (\eta + \Lambda_M))$.

Claim 90. *If $Y \neq \emptyset$, then $C(x)$ is co-initial in U_1.*

Proof. Let $x_0 \in Y$ and $U' = \big(U_1 \cap (\eta + \Lambda_M)\big) - x_0$. U' is a definable end-segment of Λ_M without a minimum. Fix an element $\delta \in U'$. As U' does not have a minimum and Λ_M is M-regular, we can find an element $\gamma \in \Lambda_M$ such that $M\gamma \in U'$ and $M\gamma < \delta$. Then $z = M\gamma + x_0 \in Y$ and $z < x_0 + \delta$. Thus $C(x)$ is co-initial in U_1. □ □

Likewise, if $\hat{\Gamma}$ is M-regular we can conclude that $C(x)$ is co-initial in U_1. Consequently, the congruence restrictions are irrelevant in the definition of the end-segment S. It must be the case then that $S = U_1$, as desired. □ □

Though that it may seem like any divisibility cut defined by a formula of the form $nx\Box\beta$ where $n \in \mathbb{N}_{\geq 2}$, $\Box \in \{\geq, >\}$ and $\beta \in \Gamma$ could be coded by β, this statement is false and requires a slightly more delicate treatment. We introduce the following example to motivate the reader to not dismiss the technical work in Proposition 88.

Example 91. *Consider the ordered abelian group $(\mathbb{Z} \oplus \mathbb{Z}, \leq_{lex}, +, 0)$ where $\leq_{lex}$ is the lexicographic order. The definable end-segment $S = \{z \in \mathbb{Z}^2 \mid 2z \geq (1,1)\}$. Note that for any $\beta \in \mathbb{Z}$, S is also defined by the formula $2z \geq (1, \beta)$.*

Lemma 92. *Let Γ be an ordered abelian group of bounded regular rank. Let $\{0\} = \Delta_0 \leq \Delta_1 \leq \cdots \leq \Delta_l = \Gamma$ be the sequence of convex subgroups witnessing that Γ has finite n-regular rank. Then any divisibility end-segment S defined by a formula $nx \Box \beta$ where $n \in \mathbb{N}_{\geq 1}$, $\Box \in \{\geq, >\}$ and $\beta \in \Gamma$ is coded by a tuple of elements in the sorts $\Gamma \cup \{\Gamma/\Delta_i \mid i \leq l\}$.*

Proof. We argue by induction in the n-regular rank of Γ that S can be coded in the sorts $\Gamma \cup \{\Gamma/\Delta_i \mid i \leq l\}$. For the base case, we suppose that Γ is n-regular. We first assume that Γ is dense, and we aim to prove that β and $\ulcorner S \urcorner$ are interdefinable. It is clear that $\ulcorner S \urcorner \in dcl^{eq}(\beta)$. For the converse let σ be any automorphism of the monster model $\mathfrak{M}$ and suppose that $\sigma(\beta) \neq \beta$. Without loss of generality, $\beta < \sigma(\beta)$. By density we can find n-elements in the interval $(\beta, \sigma(\beta))$. By n-regularity and density there is an element δ such that $\beta < n\delta < \sigma(\beta)$. Thus $\sigma(S) \subsetneq S$.
We now assume that Γ is discrete and let 1 be its minimal element. There is a unique natural number $0 \leq i \leq n-1$ such that $\beta+i$ is n-divisible, because $\{\beta, \beta+1, \dots, \beta+(n-1)\}$ is an interval with at least n-elements. Let i_0 be the index such that $\beta+i_0$ is n-divisible. Then $x \in S$ if and only if $nx \geq \beta+i_0$. Thus $\frac{\beta+i_0}{n}$ is the minimal element of S and thereby it is a code for S.

We proceed to show the inductive step, and we consider the sequence $\{0\} = \Delta_0 < \Delta_1 < \dots < \Delta_{l+1} = \Gamma$ witnessing that Γ has n-regular rank equal to $l+1$. Let $\rho_{\Delta_1} : \Gamma \to \Gamma/\Delta_1$ be the canonical projection map, and note that Γ/Δ_1 is an ordered abelian group of n-regular rank l. First we suppose that $\rho_{\Delta_1}(\beta)$ is not n-divisible. Then S is interdefinable with $S_{\Delta_1} = \{\eta \in \Gamma/\Delta_1 \mid n\eta > \rho_{\Delta_1}(\beta)\}$. By the induction hypothesis, such end-segment can be coded in the sorts $\Gamma/\Delta_1 \cup \{(\Gamma/\Delta_1)/(\Delta_i/\Delta_1) \mid 2 \leq i \leq l\}$. As each of the sorts $(\Gamma/\Delta_1)/(\Delta_i/\Delta_1)$ can be canonically identified with Γ/Δ_i, the conclusion of the statement follows.
We consider the case where $\rho_{\Delta_1}(\beta)$ is n-divisible, i.e. there is some $\eta \in \Gamma$ such that $n\rho_{\Delta_1}(\eta) = \rho_{\Delta_1}(\beta)$. Note that $\rho_{\Delta_1}(\eta) = \min(S_{\Delta_1})$. If Δ_1 is discrete, then S has a minimum and this minimal element is a code for S. Thus without loss of generality Δ_1 is dense. We aim to show that β and $\ulcorner S \urcorner$ are interdefinable. In fact, let σ be an automorphism of the monster model $\mathfrak{M}$ fixing $\ulcorner S \urcorner$. We want to show that it fixes also β. We argue by contradiction, and we assume that $\beta < \sigma(\beta)$. As $\rho_{\Delta_1}(\beta) \in dcl^{eq}(\ulcorner S \urcorner)$, we have $\sigma(\beta) - \beta \in \Delta_1$. Fix some element $\eta \in \Gamma$ such that $n\eta + \Delta_1 = \beta + \Delta_1$. We can find elements $\delta_1 < \delta_2 \in \Delta_1$ such that $\beta = n\eta + \delta_1$ and $\sigma(\beta) = n\eta + \delta_2$. By n-regularity and density of Δ_1 we can find an element $\gamma \in \Delta_1$ such that $\delta_1 < n\gamma < \delta_2$, so we have $\beta < n(\gamma + \eta) < \sigma(\beta)$ and hence $S \subsetneq \sigma(S)$, as desired. □ □

Proposition 93. *Let Γ be an ordered abelian group of bounded regular rank, and let $S \subseteq \Gamma$ be a definable end-segment. Then $\ulcorner S \urcorner$ is interdefinable with a tuple of elements in the sorts $\Gamma \cup \{\Gamma/\Delta \mid \Delta \in RJ(\Gamma)\}$. Consequently, any initial segment is also coded in the sorts $\Gamma \cup \{\Gamma/\Delta \mid \Delta \in RJ(\Gamma)\}$.*

Proof. By Proposition 88 it is sufficient to code divisibility end-segments. We may assume that $S = \{\eta \in \Gamma \mid n\eta + \Delta \geq \beta + \Delta\}$. Therefore S is interdefinable with $S_\Delta = \{z \in \Gamma/\Delta \mid nz \geq \rho_\Delta(\beta)\}$; this is a definable end-segment of Γ/Δ. The statement follows immediately from Lemma 92 combined with Fact 81.
The second part of the statement follows by noticing that any initial segment is the complement of an end-segment. □ □

1.4 An abstract criterion to eliminate imaginaries

The following is [Hru14, Lemma 1.17].

Theorem 94. *Let T be a first order theory with home sort K. Let $\mathcal{G}$ be some collection of sorts. If the following conditions all hold, then T has weak elimination of imaginaries in the sorts $\mathcal{G}$.*

1. Density of definable types: *for every non-empty definable set $X \subseteq K$ there is an $acl^{eq}(\ulcorner X \urcorner)$-definable type in X.*
2. Coding definable types: *every definable type in K^n has a code in $\mathcal{G}$ (possibly infinite). That is, if p is any (global) definable type in K^n, then the set $\ulcorner p \urcorner$ of codes of the definitions of p is interdefinable with some (possibly infinite) tuple from $\mathcal{G}$.*

Proof. A very detailed proof can be found in [Joh16, Theorem 6.3]. The first part of the proof shows weak elimination of imaginaries as it is shown that for any imaginary element e we can find a tuple $a \in \mathcal{G}$ such that $e \in dcl^{eq}(a)$ and $a \in acl^{eq}(e)$. □ □

We will use this criterion to prove that any pure ordered abelian group with bounded regular rank admits weak elimination of imaginaries once the quotient sorts are added.

Coding of definable types

In this subsection we show that any definable type $p(\mathbf{x})$ can be coded in the quotient sorts.

Proposition 95. *Let Γ be an ordered abelian group and $p(\mathbf{x}) \in S_n(\Gamma)$ be a definable type. Then $p(\mathbf{x})$ can be coded in the quotient sorts.*

Proof. Let $p(\mathbf{x})$ be a definable type in n variables over Γ. By quantifier elimination (Theorem 77), $p(\mathbf{x})$ is completely determined by formulas of the following forms:

- *First kind:*

$$\phi_1(\mathbf{x}, \beta) := \sum_{i \leq n} z_i x_i + \Delta < \beta + k + \Delta$$

  or

$$\psi_1(\mathbf{x}, \beta) := \sum_{i \leq n} z_i x_i + \Delta > \beta + k + \Delta$$

  where $\beta \in \Gamma$, $\Delta \in RJ(\Gamma)$ and $k, z_i \in \mathbb{Z}$.

- *Second kind:*

$$\phi_2(\mathbf{x}, \beta) := \sum_{i \leq n} z_i x_i \equiv_{\Delta + l\Gamma} \beta + k$$

  where $\beta \in \Gamma$, $\Delta \in RJ(\Gamma)$, $k, z_i \in \mathbb{Z}$ and $l \in \mathbb{N}_{\geq 2}$.

- *Third kind:*

$$\phi_3(\mathbf{x}, \beta) := \sum_{i \leq n} z_i x_i \equiv_{\Delta} \beta + k$$

where $\beta \in \Gamma$, $\Delta \in RJ(\Gamma)$, and $z_i \in \mathbb{Z}$.

The set $\{\beta \in \Gamma \mid \phi_1(\mathbf{x}, \beta) \in p(\mathbf{x})\}$ is an end-segment of Γ, so it can be coded in the quotient sorts by Proposition 88 and 93. Likewise, the set $\{\beta \in \Gamma \mid \psi_1(\mathbf{x}, \beta) \in p(\mathbf{x})\}$ is an initial segment of Γ, and it admits a code in the quotient sorts.
Let $X = \{\beta \in \Gamma \mid \phi_2(\mathbf{x}, \beta) \in p(\mathbf{x})\}$, then X is either empty or we can take $\beta_0 \in X$ and $\ulcorner X \urcorner$ is interdefinable with $\pi^l_{\Delta}(\beta_0) \in \Gamma/(\Delta + l\Gamma)$.
Lastly, $Z = \{\beta \in \Gamma \mid \phi_3(\mathbf{x}, \beta) \in p(\mathbf{x})\}$ is either empty or for any element $\beta_0 \in Z$, we have that $\ulcorner Z \urcorner$ is interdefinable with $\rho_{\Delta}(\beta_0) \in \Gamma/\Delta$. □

□

Density of definable types

In this subsection we prove the first condition required in Hrushovski's criterion: the density of definable types in algebraically closed sets.
The following will be a useful fact to obtain our result.

Fact 96. *Let $X \subseteq \Gamma$ be a definable set without a minimum element. Then there is a $\ulcorner X \urcorner$ definable end-segment S such that X is co-initial in S.*

Proof. Let $I = \{\beta \in \Gamma \mid (-\infty, \beta] \cap X = \emptyset\}$. I is a $\ulcorner X \urcorner$-definable initial segment of Γ. Let $S = \Gamma \backslash I$, it is sufficient to verify that X is co-initial in S. Let $\beta \in S$, then $(-\infty, \beta] \cap X \neq \emptyset$. Because X does not have a minimum, we can find an element $x \in X$ such that $x < \beta$, as required. □ □

Proposition 97. *Let Γ be an ordered abelian group of bounded regular rank and $X \subseteq \Gamma$ a definable set. There is a global type $p(x) \vdash x \in X$ such that $p(x)$ is definable over $\mathrm{acl}^{eq}(\ulcorner X \urcorner)$.*

Proof. Let $X \subseteq \Gamma$ be a 1-definable set. If X has a minimum element a, the statement follows immediately by taking the type of this element. Thus we may assume that X does not have a minimum, by Fact 96 there is a $\ulcorner X \urcorner$-definable end-segment S such that X is co-initial in S. In particular the type:

$$\Sigma^{gen}_S(x) = \{x \in S \cap X\} \cup \{x \notin B \mid B \subsetneq S \text{ and } B \text{ is a definable end-segment}\}$$

is a consistent partial type which is $\ulcorner S \urcorner$-definable.
Let $\pi : \mathbb{N} \to \mathbb{N} \times \mathbb{N}_{\geq 1}$ be some fixed bijection. We now build by induction an increasing sequence of partial consistent types $(\Sigma_i(x) \mid i < \omega)$ in the following way:

- **Stage** 0**:** Set $\Sigma_0(x) = \Sigma^{gen}_S(x)$,

- **Stage** $i+1$**:** Let $\pi(i) = (k, l)$, at this stage we want to decide the congruence modulo the subgroup $\Delta_k + l\Gamma$. To keep the notation simple we assume that $l \geq 2$ and we use the projection map $\pi^l_{\Delta_k} := \Gamma \to \Gamma/(\Delta_k + l\Gamma)$. If $l = 1$ we argue in the same manner to fix the coset of Δ_k and instead we use the projection map $\rho_{\Delta_k} : \Gamma \to \Gamma/\Delta_k$.
  We proceed by cases:
  a) Set $\Sigma_{i+1}(x) = \Sigma_i(x) \cup \{\pi^l_{\Delta_k}(x) \neq \pi^l_{\delta_k}(\beta) \mid \beta \in \Gamma\}$ if it is consistent.
  b) Otherwise, let $A_i = \{\eta_1, \dots, \eta_{r_i}\} \subseteq \Gamma/(\Delta_k + l\Gamma)$ be the finite set of cosets such that $\Sigma_i(x) \cup \{\pi^l_{\Delta_k}(x) = \eta_j\}$ is consistent. Take an element $\hat{\eta} \in A_i$ and set $\Sigma_{i+1}(x) = \Sigma_i(x) \cup \{\pi^l_{\Delta_k}(x) = \hat{\eta}\}$.

Let $\mathfrak{M}$ be the monster model and

$$\mathcal{J} = \{i \in \mathbb{N} \mid \Sigma_i(x) \cup \{\pi^l_{\Delta_k}(x) \neq \pi^l_{\delta_k}(\beta) \mid \beta \in \Gamma\} \text{ is inconsistent}\}$$

Claim 98. *For any $\sigma \in Aut(\mathfrak{M}/\operatorname{acl}^{eq}(\ulcorner X \urcorner))$ the following conditions hold:*

1. *For any $i \in \mathbb{N}$ $\sigma(\Sigma_i(x)) = \Sigma_i(x)$ and*
2. *For any $i \in \mathcal{J}$ $\sigma(A_i) = A_i$.*

In particular, as σ is arbitrary, then $A_i \subseteq \operatorname{acl}^{eq}(\ulcorner X \urcorner)$.

Proof. We argue by induction on i to show that for any $\sigma \in Aut(\mathfrak{M}/\operatorname{acl}^{eq}(\ulcorner X \urcorner))$ we have that $\sigma(\Sigma_i(x)) = \Sigma_i(x)$ and if $i \in \mathcal{J}$ then $\sigma(A_i) = A_i$.
For the base case, fix some $\sigma \in Aut(\mathfrak{M}/\operatorname{acl}^{eq}(\ulcorner X \urcorner))$. Then $\sigma(\Sigma_0(x)) = \Sigma_0(x)$ because $\Sigma^{gen}_S(x)$ is $\ulcorner S \urcorner$-definable and $\ulcorner S \urcorner \in \operatorname{dcl}^{eq}(\ulcorner X \urcorner)$.
Suppose the statement holds for i and fix some $\sigma \in Aut(\mathfrak{M}/\operatorname{acl}^{eq}(\ulcorner X \urcorner))$.
If $\Sigma_{i+1}(x) = \Sigma_i(x) \cup \{\pi^l_{\Delta_k}(x) \neq \pi^l_{\Delta_k}(\beta) \mid \beta \in \Gamma\}$, then

$$\begin{aligned}\sigma(\Sigma_{i+1}(x)) &= \sigma(\Sigma_i(x)) \cup \{\pi^l_{\Delta_k}(x) \neq \pi^l_{\Delta_k}(\sigma(\beta)) \mid \beta \in \Gamma\} \\ &= \Sigma_i(x) \cup \{\pi^l_{\Delta_k}(x) \neq \pi^l_{\Delta_k}(\sigma(\beta)) \mid \beta \in \Gamma\} = \Sigma_{i+1}(x).\end{aligned}$$

Let's assume that $\Sigma_{i+1}(x) = \Sigma_i(x) \cup \{\pi^l_{\Delta_k}(x) = \eta\}$ for some $\eta \in A_i$. We first argue that $\sigma(A_i) = A_i$. By definition of A_i:

$$\mu \in A_i \text{ if and only if } \Sigma_i(x) \cup \{\pi^l_{\Delta_k}(x) = \mu\} \text{ is consistent.}$$

Let $\mu \in A_i$, then $\Sigma_i(x) \cup \{\pi^l_{\Delta_k}(x) = \mu\}$ is consistent. As σ is an automorphism, then

$$\sigma(\Sigma_i(x)) \cup \{\pi^l_{\Delta_k}(x) = \sigma(\mu)\} \text{ is consistent.}$$

By the induction hypothesis, $\sigma(\Sigma_i(x)) = \Sigma_i(x)$. Hence:

$$\Sigma_i(x) \cup \{\pi^l_{\Delta_k}(x) = \sigma(\mu)\} \text{ is consistent.}$$

Consequently, $\sigma(\mu) \in A_i$ and we conclude that $\sigma(A_i) \subseteq A_i$. We argue in a similar manner with σ^{-1} to show that $A_i \subseteq \sigma(A_i)$.
As for any $\sigma \in Aut(\mathfrak{M}/\operatorname{acl}^{eq}(\ulcorner X \urcorner))$, $\sigma(A_i) = A_i$ and A_i is a finite set, then $A_i \subseteq \operatorname{acl}^{eq}(\ulcorner X \urcorner)$. In particular, $\eta \in \operatorname{acl}^{eq}(\ulcorner X \urcorner)$ where $\Sigma_{i+1}(x) = \Sigma_i(x) \cup \{\pi^l_{\Delta_k}(x) = \eta\}$. Then for any $\sigma \in Aut(\mathfrak{M}/\operatorname{acl}^{eq}(\ulcorner X \urcorner))$ we have that $\sigma(\Sigma_{i+1}(x)) = \Sigma_{i+1}(x)$, as required. $\square$

Let $\Sigma_\infty(x) = \bigcup_{i \in \mathbb{N}} \Sigma_i(x)$, this is a partial consistent type and $\Sigma_\infty(x) \vdash x \in X$. By quantifier elimination $\Sigma_\infty(x)$ determines a complete type $p(x)$. Then $p(x) \vdash x \in X$, and $p(x)$ is $acl^{eq}(\ulcorner X \urcorner)$-definable because $p(x)$ is completely determined by the data in $\Sigma_\infty(x)$, which is definable over $\operatorname{acl}^{eq}(\ulcorner X \urcorner)$ by Claim 98 .$\square$ $\square$

1.5 Main Results

Theorem 99. *Let Γ be an ordered abelian group of bounded regular rank (equivalently with finite spines). Then Γ admits weak-elimination of imaginaries in the language $\mathcal{L}_Q$, once the quotient sorts are added.*

Proof. By Theorem 94 it is sufficient to check that we have density of definable types and that we can code definable types in the quotient sorts. The first condition is Proposition 97 and the second one is Proposition 95. $\square$ $\square$

The dp-minimal case

In this section we show that a better statement can be achieved for the *dp*-minimal case.

Definition 100. *Let Γ be an ordered abelian group and H some definable subgroup. A subset $\mathcal{C} \subseteq \Gamma$ is said to be a* complete set of representatives modulo H *if:*

1. *given any $\gamma \in \Gamma$ there is some $\beta \in \mathcal{C}$ such that $\gamma - \beta \in H$.*

2. *for any $\beta \neq \eta \in \mathcal{C}$ we have that $\beta + H \neq \eta + H$.*

Fact 101. *Let Γ be an ordered abelian group, Δ a convex subgroup and $k \in \mathbb{N}$. Let $\mathcal{C}$ be a complete set of representatives of Γ modulo $k\Gamma$, then some subset $\mathcal{C}_0 \subseteq \mathcal{C}$ is a complete set of representatives modulo $\Delta + k\Gamma$.*

Proof. Let $\mathcal{C} \subseteq \Gamma$ be a complete set of representatives of Γ modulo $k\Gamma$ and $\pi_{\Delta}^{k} : \Gamma \to \Gamma/(\Delta + k\Gamma)$ be the projection map. $\pi_{\Delta}^{k}(\mathcal{C}) = \Gamma/(\Delta + k\Gamma)$, because for any $\gamma \in \Gamma$, there is some $\beta \in \mathcal{C}$ such that $\gamma - \beta \in k\Gamma$, in particular $\gamma - \beta \in \Delta + k\Gamma$. For each coset $\eta \in \Gamma/(\Delta + k\Gamma)$ choose an element $c_{\eta} \in \mathcal{C}$ such that $\pi_{\Delta}^{k}(c_{\eta}) = \eta$. The set $\mathcal{C}_0 = \{c_{\eta} \mid \eta \in \Gamma/(\Delta + k\Gamma)\}$ is a complete set of representatives. □

By Proposition 83, an ordered abelian group is *dp*-minimal if and only if it does not have singular primes, i.e. for any p prime number $[\Gamma : p\Gamma] < \infty$. We consider the language $\mathcal{L}_{dp}$ extending $\mathcal{L}_{Q}$, where for each $k \in \mathbb{N}_{\geq 2}$ we add constants for the elements of the finite groups $\Gamma/k\Gamma$.

Corollary 102. *Let Γ be a dp-minimal ordered group. Then Γ admits elimination of imaginaries in the language $\mathcal{L}_{dp}$, where the quotient sorts are added.*

Proof. By Theorem 99 and Fact 62 it is sufficient to show that we can also code finite sets. Let Δ definable convex subgroup and $k \in \mathbb{N}$, the group $\Gamma/(\Delta + k\Gamma)$ is also finite. We first argue that $\Gamma/(\Delta + k\Gamma) \subseteq \mathrm{dcl}(\emptyset)$. Consider the $\emptyset$-definable function

$$\begin{aligned} f :& \Gamma/k\Gamma \to \Gamma/(\Delta + k\Gamma) \\ & \gamma + k\Gamma \to \gamma + (\Delta + k\Gamma). \end{aligned}$$

By Fact 101 f is surjective.
Hence, it is enough to prove that finite sets of tuples in $\mathcal{S} = \{\Gamma/\Delta \mid \Delta \in RJ(\Gamma)\}$ can be coded in the quotient sorts. As each of the sorts Γ/Δ is linearly ordered, there is a definable order induced over the finite products of quotients of Γ/Δ, and thereby any finite set of tuples in $\mathcal{S}$ is already coded in $\mathcal{S}$. □ □

Chapter 2

Elimination of Imaginaries in $\mathbb{C}((\Gamma))$

In this chapter we study elimination of imaginaries in henselian valued fields of equicharacteristic zero and residue field algebraically closed. The results are sensitive to the complexity of the value group. We focus first on the case where the ordered abelian group has finite spines, and then prove a better result for the dp-minimal case. In [Vic21b] it was shown that an ordered abelian with finite spines weakly eliminates imaginaries once one adds sorts for the quotient groups Γ/Δ for each definable convex subgroup Δ, and sorts for the quotient groups $\Gamma/(\Delta + l\Gamma)$ where Δ is a definable convex subgroup and $l \in \mathbb{N}_{\geq 2}$. We refer to these sorts as the *quotient sorts*. In [JSW17] F. Janke, P. Simon and E. Walsberg characterized *dp*-minimal ordered abelian groups as those without singular primes, i.e. for every prime number p $[\Gamma : p\Gamma] < \infty$.

We prove the following two theorems:

Theorem 103. *Let K be a henselian valued field of equicharacteristic zero with residue field algebraically closed and value group of finite spines. Then K admits weak elimination of imaginaries once one adds codes for all the definable $\mathcal{O}$-submodules of K^n for each $n \in \mathbb{N}$, and the quotient sorts for the value group.*

Theorem 104. *Let K be a henselian valued field of equicharacteristic zero, residue field algebraically closed and dp-minimal value group. Then K eliminates imaginaries once one adds codes for all the definable $\mathcal{O}$-submodules of K^n for each $n \in \mathbb{N}$, the quotient sorts for the value group and constants for the finite groups $\Gamma/\ell\Gamma$ where $\ell \in \mathbb{N}$.*

2.1 Introduction

The model theory of henselian valued fields has been a major topic of study during the last century, it was initiated by Robinson's model completeness results for algebraically closed valued fields in [Rob56]. Remarkable work has been achieved by Haskell, Hrushovski and

Macpherson to understand the model theory of algebraically closed valued fields. In a sequence of papers [HHM05] and [HHM06] they developed the notion of stable domination, that rather than being a new form of stability should be understood as a way to apply techniques of stability in the setting of valued fields. Further work of Ealy, Haskell and Maříková in [EHM19] for the setting of real closed convexly valued fields, suggested that the notion of having a stable part of the structure was not fundamental to achieve domination results and indicated that the right notion should be residue field domination or domination by the sorts internal to the residue field. Our main motivation for the present document arises from the natural question of how much further a notion of residue field domination could be extended to broader classes of valued fields to gain a deeper model theoretic insight of henselian valued fields, and the first step is finding a reasonable language where the valued field will eliminate imaginaries.

The starting point in this project relies on the Ax-Kochen theorem, which states that the first order theory of a henselian valued field of equicharacteristic zero or unramified mixed characteristic with perfect residue field is completely determined by the first order theory of its valued group and its residue field. A natural principle follows from this theorem: model theoretic questions about the valued field itself can be understood by reducing them to its residue field, its value group and their interaction in the field.

A fruitful application of this principle has been achieved to describe the class of definable sets. For example, in [Pas90] Pas proved field quantifier elimination relative to the residue field and the value group once angular component maps are added in the equicharacteristic case. Further studies of Basarab and F.V. Kuhlmann show a quantifier elimination relative to the RV sorts [see [Bas91], [Kuh94] respectively].

The question of whether a henselian valued field eliminates imaginaries in a given language is of course subject to the complexity of its value group and its residue field as both are interpretable structures in the valued field itself. The case for algebraically closed valued fields was finalized by Haskell, Hrushovski and Macpherson in their important work [HHM06] , where elimination of imaginaries for $ACVF$ is achieved once the geometric sorts S_n (codes for the $\mathcal{O}$-lattices of rank n) and T_n (codes for the residue classes of the elements in S_n) are added. This proof was later significantly simplified by Will Johnson in [Joh16] by using a criterion isolated by Hurshovski [see [Hru14]].

Recent work has been done to achieve elimination of imaginaries in some other examples of henselian valued fields, as the case of separably closed valued fields in [HKR18], the p-adic case in [HMR18] or enrichments of ACVF in [Rid19].

However, the above results are all obtained for particular instances of henselian valued fields while the more general approach of obtaining a relative statement for broader classes of henselian valued fields is still a very interesting open question.

Following the Ax-Kochen style principle, it seems natural to first attempt to solve this question by looking at the problem in two orthogonal directions: one by making the residue field as docile as possible and studying which troubles would the value group bring into the picture, or by making the value group tame and understanding the difficulties that the residue field would contribute to the problem.
Hils and Rideau [HRK21b] had proved that under the assumption of having a definably complete value group and requiring that the residue field eliminates the $\exists^{\infty}$ quantifier, then any definable set admits a code once the geometric sorts and the linear sorts are added to the language. Any definably complete ordered abelian group is either divisible or a $\mathbb{Z}$-group (i.e. a model of Presburger Arithmetic).

This chapter is addressing the first approach in the setting of henselian valued fields of equicharacteristic zero. We suppose the residue field to be algebraically closed and we obtain results which are sensitive to the complexity of the value group. We first analyze the case where the value group has finite spines. An ordered abelian with finite spines weakly eliminates imaginaries once we add sorts for the quotient groups Γ/Δ for each definable convex subgroup Δ, and sorts for the quotient groups $\Gamma/\Delta + l\Gamma$ where Δ is a definable convex subgroup and $l \in \mathbb{N}_{\geq 2}$. We refer to these sorts as the quotient sorts. The first result that we obtain is:

Theorem 105. *Let K be a valued field of equicharacteristic zero, residue field algebraically closed and value group with finite spines. Then K admits weak elimination of imaginaries once we add codes for all the definable $\mathcal{O}$-submodules of K^n for each $n \in \mathbb{N}$, and the quotient sorts for the value group.*

Later, we prove a better result for the dp-minimal case, this is:

Theorem 106. *Let K be a henselian valued field of equicharacteristic zero, residue field algebraically closed and dp-minimal value group. Then K eliminates imaginaries once we add codes for all the definable $\mathcal{O}$-submodules of K^n for each $n \in \mathbb{N}$, the quotient sorts for the value group and constants for each of the finite groups $\Gamma/\ell\Gamma$ where $\ell \in \mathbb{N}_{\geq 2}$.*

This chapter is organized as follows:

- *Section 2.2:* We introduce the required background, including quantifier elimination statements, the state of the model theory of ordered abelian groups and some results about valued vector spaces.
- *Section 2.3:* We study definable $\mathcal{O}$-modules of K^n.
- *Section 2.4:* We start by presenting Hrushovski's criterion to eliminate imaginaries. We introduce the *stabilizer sorts*, where the $\mathcal{O}$-submodules of K^n can be coded.
- *Section 2.5:* We prove that each of the conditions of Hrushovski's criterion hold. This is the density of definable types in definable sets in 1-variable $X \subseteq K$ and that any

definable type can be coded in the stabilizer sorts and Γ^{eq}. We conclude this section proving the weak elimination of imaginaries of any henselian valued field of equicharacteristic zero, residue field algebraically closed and value group with finite spines down to the stabilizer sorts.

- *Section 2.6:* We show a complete elimination of imaginaries statement when the value group is *dp*-minimal. We prove that any finite set of tuples in the stabilizer sorts can be coded.

2.2 Preliminaries

Quantifier Elimination for valued fields of equicharacteristic zero and residue field algebraically closed

In this section we recall several results relevant for our statement. In particular we state a quantifier elimination relative to the value group in the canonical three sorted language $\mathcal{L}_{val}$ for the class of valued fields of equicharacteristic zero and residue field algebraically closed.

The three-sorted language $\mathcal{L}_{\text{val}}$

We consider valued fields as three sorted structures (K, k, Γ). The first two sorts are equipped with the language of fields $\mathcal{L}_{fields} = \{0, 1, +, \cdot, (\cdot)^{-1}, -\}$, we refer to the first one as the *main field sort* while we call the second one as the *residue field sort*. The third sort is supplied with the language of ordered abelian groups $\mathcal{L}_{OAG} = \{0, <, +, -\}$, and we refer to it as the *value group sort*. We also add constants ∞ to the second sort and the third sort. We introduce a function symbol $v : K \to \Gamma \cup \{\infty\}$, interpreted as the valuation and, we add a map $res : K \to k \cup \{\infty\}$, where $res : \mathcal{O} \to k$ is interpreted as a surjective homomorphism of rings, while for any element $x \in K \backslash \mathcal{O}$ we have $\text{res}(x) = \infty$. We denote this language as $\mathcal{L}_{\text{val}}$.

The Extension Theorem

Let $\mathcal{K} = (K, k, \Gamma)$ be a valued field and $\mathcal{O}$ its valuation ring. A triple $\mathcal{E} = (E, k_E, \Gamma_E)$ is a substructure if E is a subfield of K, k_E is a subfield of k, Γ_E is a subgroup of Γ, $v(E^{\times}) \subseteq \Gamma_E$ and $\text{res}(O_E) \subseteq k_E$ where $O_E = O \cap E$.

Definition 107. *Let $\mathcal{K}_1 = (K_1, k_1, \Gamma_1)$ and $\mathcal{K}_2 = (K_2, k_2, \Gamma_2)$ be valued fields of equicharacteristic zero with residue field algebraically closed.*
Let $\mathcal{E} = (E, k_E, \Gamma_E)$ be a substructure of $\mathcal{K}_1$ a triple $(f, f_r, f_v) = \mathcal{E} \to \mathcal{K}_2$ is said to be an admissible embedding *if it is a $\mathcal{L}_{val}$ isomorphism and $f_v = \Gamma_E \to \Gamma_2$ is a partial elementary map between Γ_1 and Γ_2, i.e for every $\mathcal{L}_{OAG}$ formula $\phi(x_1, \ldots, x_n)$ and tuple $e_1, \ldots, e_n \in \Gamma_E$*

we have that

$$\Gamma_1 \vDash \phi(e_1, \dots, e_n) \text{ if and only if } \Gamma_2 \vDash \phi(f_v(e_1), \dots, f_v(e_n)).$$

Let $\kappa = \max\{|k_E|, |\Gamma_E|\}$*. If* $\mathcal{K}_2$ *is* κ^+*-saturated we say that* (f, f_r, f_v) *is an* admissible map with small domain*.*

Theorem 108. *[The Extension Theorem] The theory of henselian valued fields of equicharacteristic zero with residue field algebraically closed admits quantifier elimination relative to the value group in the language* $\mathcal{L}_{val}$*. That is given* $\mathcal{K}_1 = (K_1, k_1, \Gamma_1)$ *and* $\mathcal{K}_2 = (K_2, k_2, \Gamma_2)$ *henselian valued fields of equicharacteristic zero with residue field algebraically closed, a substructure* $\mathcal{E}$ *of* $\mathcal{K}_1$ *and* $(f, f_r, f_v) : \mathcal{E} \to \mathcal{K}_2$ *and admissible map with small domain, for any* $b \in K_1$ *there is an admissible map* $\hat{f}$ *extending* f *whose domain contains* b*.*

Proof. This is straightforward using the standard techniques to obtain elimination of field quantifiers already present in the area. We refer the reader for example to [vdDKM+12, Theorem 5.21]. The unique step that requires the presence of an angular component map is when for a subfield $E \subseteq K_1$ we want to add an element γ to $v(E^\times)$ and there is some prime number p such that $p\gamma \in v(E^\times)$. For this, take $a \in E$ and $c \in K_1$ be such that $v(a) = p\gamma$ and $v(c) = \gamma$. We first aim to find $b_1 \in K_1$ that is a root of the polynomial $Q(x) \in O_E(x)$, where $Q(x) = x^p - \frac{a}{c^p}$. Let $d = \text{res}(\frac{a}{c^p})$, because k_1 is algebraically closed there is some $z \in k_1$ such that $z^p - d = 0$. Let $\alpha \in K_1$ be such that $\text{res}(\alpha) = z$, then $v(Q(\alpha)) > 0$ while $v(Q'(\alpha)) = 0$, because $p \neq char(k_1)$. Indeed, $Q'(x) = px^{p-1}$ and $z \neq 0$ because $d \neq 0$.
By henselianity we can find $b_1 \in K_1$ that is a root of $Q(x)$. Then $x_1 = (b_1 \cdot c)$ is a pth-root of a.

□

The following is an immediate consequence of relative quantifier elimination.

Corollary 109. *The residue field and the value group are both purely stably embedded and orthogonal to each other.*

Some results on the model theory of ordered abelian groups

In this Subsection we summarize many interesting results about the model theory of ordered abelian groups. We start by recalling the following folklore fact.

Fact 110. *Let* $(\Gamma, \leq, +, 0)$ *be a non-trivial ordered abelian group. Then the topology induced by the order in* Γ *is discrete if and only if* Γ *has a minimum positive element. In this case we say that* Γ *is* discrete*, otherwise we say that it is* dense*.*

The following notions were isolated in the sixties by Robinson and Zakon in [RZ60] to understand some model complete extensions of the theory of ordered abelian groups.

Definition 111. *Let* Γ *be an ordered abelian group and* $n \in \mathbb{N}_{\geq 2}$*.*

1. *Let $\gamma \in \Gamma$. We say that γ is n-*divisible *if there is some $\beta \in \Gamma$ such that $\gamma = n\beta$.*

2. *We say that Γ is n-*divisible *if every element $\gamma \in \Gamma$ is n-divisible.*

3. *Γ is said to be n-*regular *if any interval with at least n points contains an n-divisible element.*

Definition 112. *An ordered abelian group Γ is said to be* regular *if it is n-regular for all $n \in \mathbb{N}$.*

Robinson and Zakon in their seminal paper [RZ60] completely characterized the possible completions of the theory of regular groups, obtained by extending the first order theory of ordered abelian groups with axioms asserting that for each $n \in \mathbb{N}$ if an interval contains at least n-elements then it contains an n-divisible element. The following is [RZ60, Theorem 4.7].

Theorem 113. *The possible completions of the theory of regular groups, are:*

1. *the theory of discrete regular groups, and*

2. *the completions of the theory of dense regular groups T_χ where*

$$\chi =: \text{Primes} \to \mathbb{N} \cup \{\infty\},$$

   is a function specifying the index $\chi(p) = [\Gamma : p\Gamma]$.

Robinson and Zakon proved as well that each of these completions is the theory of some archimedean group. In particular, any discrete regular group is elementarily equivalent to $(\mathbb{Z}, \leq, +, 0)$.

The following definitions were introduced by Schmitt in [Sch82].

Definition 114. *We fix an ordered abelian group Γ and $n \in \mathbb{N}_{\geq 2}$. Let $\gamma \in \Gamma$. We define:*

- *$A(\gamma)$ = the largest convex subgroup of Γ not containing γ.*
- *$B(\gamma)$ = the smallest convex subgroup of Γ containing γ.*
- *$C(\gamma) = B(\gamma)/A(\gamma)$.*
- *$A_n(\gamma)$ = the smallest convex subgroup C of Γ such that $B(g)/C$ is n-regular.*
- *$B_n(g)$ = the largest convex subgroup C of Γ such that $C/A_n(\gamma)$ is n-regular.*

In [Sch82, Chapter 2], Schmitt shows that the groups $A_n(\gamma)$ and $B_n(\gamma)$ are definable in the language of ordered abelian groups $\mathcal{L}_{OAG} = \{+, -, \leq, 0\}$ by a first order formula using only the parameter γ.
We recall that the set of convex subgroups of an ordered abelian group is totally ordered by inclusion.

Definition 115. *Let Γ be an ordered abelian group and $n \in \mathbb{N}_{\geq 2}$, we define the n-*regular rank *to be the order type of:*

$$\big(\{A_n(\gamma) \mid \gamma \in \Gamma \backslash \{0\}\}, \subseteq \big).$$

The n-regular rank of an ordered abelian group Γ is a linear order, and when it is finite we can identify it with its cardinal. In [Far17], Farré emphasizes that we can characterize the n-regular rank without mentioning the subgroups $A_n(\gamma)$. The following is [Far17, Remark 2.2].

Definition 116. *Let Γ be an ordered abelian group and $n \in \mathbb{N}_{\geq 2}$, then:*

1. *Γ has n-regular rank equal to 0 if and only if $\Gamma = \{0\}$,*
2. *Γ has n-regular rank equal to 1 if and only if Γ is n-regular and not trivial,*
3. *Γ has n-regular rank equal to m if there are $\Delta_0, \dots, \Delta_m$ convex subgroups of Γ, such that:*
   - $\{0\} = \Delta_0 < \Delta_1 < \cdots < \Delta_m = \Gamma$,
   - *for each $0 \leq i < m$, the quotient group Δ_{i+1}/Δ_i is n-regular,*
   - *the quotient group Δ_{i+1}/Δ_i is not n-divisible for $0 < i < m$.*

   *In this case we define $RJ_n(\Gamma) = \{\Delta_0, \dots, \Delta_{m-1}\}$. The elements of this set are called the n-*regular jumps.

Definition 117. *Let Γ be an ordered abelian group. We say that it is* poly-regular *if it is elementarily equivalent to a subgroup of the lexicographically ordered group $(\mathbb{R}^n, +, \leq_{lex}, 0)$.*

In [Bel02] Belegradek studied poly-regular groups and proved that an ordered abelian group is poly-regular if and only if it has finitely many proper definable convex subgroups, and all the proper definable subgroups are definable over the empty set. In [Wei81, Theorem 2.9] Weispfenning obtained quantifier elimination for the class of poly-regular groups in the language of ordered abelian groups extended with predicates to distinguish the subgroups $\Delta + \ell\Gamma$ where Δ is a convex subgroup and $\ell \in \mathbb{N}_{\geq 2}$.

Definition 118. *Let Γ be an ordered abelian group. We say that it has* bounded regular rank *if it has finite n-regular rank for each $n \in \mathbb{N}_{\geq 2}$. For notation, we will use $RJ(\Gamma) = \bigcup\limits_{n \in \mathbb{N}_{\geq 2}} RJ_n(\Gamma)$.*

The class of ordered abelian groups of bounded regular rank extends the class of poly-regular groups and regular groups. The terminology of *bounded regular rank* becomes clear with the following Proposition (item 3).

Proposition 119. *Let Γ be an ordered abelian group. The following are all equivalent:*

1. *Γ has finite p-regular rank for each prime number p.*
2. *Γ has finite n-regular rank for each $n \geq 2$.*
3. *There is some cardinal κ such that for any $H \equiv \Gamma$, $|RJ(H)| \leq \kappa$.*
4. *For any $H \equiv \Gamma$, any definable convex subgroup of H has a definition without parameters.*
5. *There is some cardinal κ such that for any $H \equiv \Gamma$, H has at most κ definable convex subgroups.*

Moreover, in this case $RJ(\Gamma)$ is the collection of all proper definable convex subgroups of Γ and all are definable without parameters. In particular, there are only countably many definable convex subgroups.

Proof. This is [Far17, Proposition 2.3]. □

The first results about the model completions of ordered abelian groups appear in [RZ60] (1960), where the notion of n-regularity was isolated.

Definition 120.
1. *Let Γ be an ordered abelian group and $\gamma \in \Gamma$, we say that γ is n-divisible if there is some $\beta \in \Gamma$ such that $\gamma = n\beta$.*
2. *Let $n \in \mathbb{N}_{\geq 2}$. An ordered abelian group Γ is said to be n-*regular *if any interval with at least n-points contains an n-divisible element.*
3. *Let Γ be an ordered abelian group, we say that it is* regular *if it is n-regular for all $n \in \mathbb{N}_{\geq 2}$.*

Quantifier elimination and the quotient sorts

In [CH11] Cluckers and Halupczok introduced a language $\mathcal{L}_{qe}$ to obtain quantifier elimination for ordered abelian groups relative to the *auxiliary sorts* S_n, T_n and T_n^+, whose precise description can be found in [CH11, Definition 1.5]. This language is similar in spirit to the one introduced by Schmitt in [Sch82], but has lately been preferred by the community as it is more in line with the many-sorted language of Shelah's imaginary expansion $\mathfrak{M}^{eq}$. Schmitt does not distinguish between the sorts S_n, T_n and T_n^+. Instead for each $n \in \mathbb{N}$ he works with a single sort $Sp_n(\Gamma)$ called the *n-spine* of Γ, whose description can be found in [GS84, Section 2]. In [CH11, Section 1.5] it is explained how the auxiliary sorts of Cluckers and Halupczok are related to the n-spines $Sp_n(\Gamma)$ of Schmitt. In [Far17, Section 2], it is shown that an ordered abelian group Γ has bounded regular rank if and only if all the n-spines are finite, and $Sp_n(\Gamma) = RJ_n(\Gamma)$. In this case, we define the regular rank of Γ as the cardinal $|RJ(\Gamma)|$, which is either finite or $\aleph_0$. Instead of saying that Γ is an ordered abelian group with finite spines, we prefer to use the classical terminology of bounded regular rank, as it emphasizes the relevance of the n-regular jumps and the role of the divisibilities to describe the definable convex subgroups.

We define the *Presburger Language* $\mathcal{L}_{\text{Pres}} = \{0, 1, +, -, <, (P_m)_{m\in\mathbb{N}_{\geq 2}}\}$. Given an ordered abelian group Γ we naturally see it as a $\mathcal{L}_{\text{Pres}}$-structure. The symbols $\{0, +, -, <\}$ take their obvious interpretation. If Γ is discrete, the constant symbol 1 is interpreted as the least positive element of Γ, and by 0 otherwise. For each $m \in \mathbb{N}_{\geq 2}$ the symbol P_m is a unary predicate interpreted as $m\Gamma$.

Definition 121. *[The language $\mathcal{L}_b$] Let Γ be an ordered abelian group with bounded regular rank, we view Γ as a multi-sorted structure where:*

1. *We add a sort for the ordered abelian group Γ, and we equip it with a copy of the language $\mathcal{L}_{\text{Pres}}$ extended with predicates to distinguish each of the convex subgroups $\Delta \in RJ(\Gamma)$. We refer to this sort as the* main sort*.*

2. *We add a sort for each of the ordered abelian groups Γ/Δ, equipped with a copy of the language $\mathcal{L}^{\Delta}_{\text{Pres}} = \{0^{\Delta}, 1^{\Delta}, +^{\Delta}, -^{\Delta}, <^{\Delta}, (P^{\Delta}_m)_{m\in\mathbb{N}_{\geq 2}}\}$. We add as well a map $\rho_\Delta : \Gamma \to \Gamma/\Delta$, interpreted as the natural projection map.*

Remark 122. *To keep the notation as simple and clear as possible, for each $\Delta \in RJ(\Gamma)$ and $n \in \mathbb{N}_{\geq 2}$ and $\beta \in \Gamma/\Delta$ we will write $\beta \in n(\Gamma/\Delta)$ instead of $P^{\Delta}_n(\beta)$.*

The following statement is a direct consequence of [ACGZ20, Proposition 3.14].

Theorem 123. *Let Γ be an ordered abelian group with bounded regular rank. Then Γ admits quantifier elimination in the language $\mathcal{L}_b$.*

We will consider an extension of this language that we will denote as $\mathcal{L}_{bq}$, where for each natural number $n \geq 2$ and $\Delta \in RJ(\Gamma)$ we add a sort for the quotient group $\Gamma/(\Delta + n\Gamma)$ and a map $\pi^n_\Delta : \Gamma \to \Gamma/(\Delta + n\Gamma)$. We will refer to the sorts in the language $\mathcal{L}_{bq}$ as *quotient sorts*. The following is [Vic21b, Theorem 5.1].

Theorem 124. *Let Γ be an ordered abelian group with bounded regular rank. Then Γ admits weak elimination of imaginaries in the language $\mathcal{L}_{bq}$, i.e. once one adds all the quotient sorts.*

Definable end-segments in ordered abelian groups with bounded regular rank

Definition 125.

1. *A non-empty set $S \subset \Gamma$ is said to be an end-segment if for any $x \in S$ and $y \in \Gamma$, $x < y$ we have that $y \in S$.*

2. *Let $n \in \mathbb{N}$, $\Delta \in RJ(\Gamma)$, $\beta \in \Gamma \cup \{-\infty\}$ and $\square \in \{\geq, >\}$. The set:*

$$S^\Delta_n(\beta) := \{\eta \in \Gamma \mid n\eta + \Delta \square \beta + \Delta\}$$

   is an end-segment *of Γ. We call any of the end-segments of this form as* divisibility end-segments.

3. *Let $S \subseteq \Gamma$ be a definable end-segment and $\Delta \in RJ(\Gamma)$. We consider the projection map $\rho_\Delta : \Gamma \to \Gamma/\Delta$, and we write S_Δ to denote $\rho_\Delta(S)$. This is a definable end-segment of Γ/Δ.*

4. *Let $\Delta \in RJ(\Gamma)$ and $S \subseteq \Gamma$ an end-segment. We say that S is* Δ-decomposable *if it is a union of Δ-cosets.*

5. *We denote as Δ_S the* stabilizer of S, *i.e.* $\Delta_S := \{\eta \in \Gamma \mid \eta + S = S\}$.

Definition 126. *Let Γ be an ordered abelian group. Let $S, S' \subseteq \Gamma$ be definable end-segments. We say that S is a* translate *of S' if there some $\beta \in \Gamma$ such that $S = \beta + S'$. Given a family $\mathcal{S}$ of definable end-segments we say that $\mathcal{S}$ is* complete *if every definable end-segment is a translate of some $S' \in \mathcal{S}$.*

Fact 127. *Let Γ be an ordered abelian group with bounded regular rank. Let $\beta, \gamma \in \Gamma$, $\Delta \in RJ(\Gamma)$ and $n \in \mathbb{N}_{\geq 2}$. If $\beta - \gamma \in \Delta + n\Gamma$ then $S^\Delta_n(\gamma)$ is a translate of $S^\Delta_n(\beta)$.*

The following is [Vic21b, Proposition 3.3].

Proposition 128. *Let Γ be an ordered abelian group of bounded regular rank. Any definable end-segment is a divisibility end-segment.*

Remark 129. *Let Γ be an ordered abelian group and Δ be a convex subgroup. Any complete set of representatives in Γ modulo $k\Gamma$ for $k \in \mathbb{N}$ is also a complete set of representative of Γ modulo $\Delta + k\Gamma$. Moreover, there is and $\emptyset$-definable surjective function $f : \Gamma/k\Gamma \to \Gamma/(\Delta + k\Gamma)$.*

Proof. For the first part of the statement take $\gamma, \beta \in \Gamma$, if $\gamma - \beta \in k\Gamma$ then $\gamma - \beta \in \Delta + k\Gamma$. For the second part, consider the $\emptyset$-definable function:

$$\begin{aligned} f : \Gamma/k\Gamma &\to \Gamma/(\Delta + k\Gamma) \\ \gamma + k\Gamma &\to \gamma + (\Delta + k\Gamma). \end{aligned}$$

This function is surjective by the first part of the statement. □

Corollary 130. *Let Γ be an ordered abelian group with bounded regular rank. For each $n \in \mathbb{N}_{\geq 2}$ let $\mathcal{C}_n$ be a complete set of representatives of the cosets $n\Gamma$ in Γ. Define $\mathcal{S}_n^{\Delta} := \{S_n^{\Delta}(\beta) \mid \beta \in \mathcal{C}_n\}$. Then $\mathcal{S} = \bigcup_{\Delta \in RJ(\Gamma), n \in \mathbb{N}_{\geq 2}} \mathcal{S}_n^{\Delta}$ is a complete family.*

Proof. It is an immediate consequence of Proposition 128, Fact 127 and Remark 129. □

The following is [Vic21b, Fact 4.1].

Fact 131. *Let $S \subseteq \Gamma$ be a definable end-segment. Then Δ_S is a definable convex subgroup of Γ, therefore $\Delta_S \in RJ(\Gamma)$. Furthermore, $\Delta_S = \bigcup_{\Delta \in \mathcal{C}} \Delta$, where*

$$\mathcal{C} = \{\Delta \in RJ(\Gamma) \mid S \text{ is } \Delta\text{-decomposable}\}.$$

Definition 132. *Let $S \subseteq \Gamma$ be a definable end-segment. Let*

$$\Sigma_S^{gen}(x) := \{x \in S\} \cup \{x \notin B \mid B \subsetneq S \text{ and } B \text{ is a definable end-segment } \}.$$

We refer to this partial type as the generic type in S. *This partial type is $\ulcorner S \urcorner$-definable.*

The dp-minimal case

In 1984 the classification of the model theoretic complexity of ordered abelian groups was initiated by Gurevich and Schmitt, who proved that no ordered abelian group has the independence property. During the last years finer classifications have been achieved, in particular dp-minimal ordered abelian groups have been characterized in [JSW17].

Definition 133. *Let Γ be an ordered abelian group and let p be a prime number. We say that p is a* singular prime *if $[\Gamma : p\Gamma] = \infty$.*

The following result corresponds to [JSW17, Proposition 5.1].

Proposition 134. *Let Γ be an ordered abelian group, the following conditions are equivalent:*

1. *Γ does not have singular primes,*
2. *Γ is dp-minimal.*

Definition 135. *[The language $\mathcal{L}_{dp}$] Let Γ be a dp-minimal ordered abelian group. We consider the language extension $\mathcal{L}_{dp}$ of $\mathcal{L}_{bq}$ [see Definition 121] where for each $n \in \mathbb{N}_{\geq 2}$ we add a set of constant for the elements of the finite group $\Gamma/n\Gamma$.*

The following is [Vic21b, Corollary 5.2].

Corollary 136. *Let Γ be a dp-minimal ordered abelian group. Then Γ admits elimination of imaginaries in the language $\mathcal{L}_{dp}$.*

The following will be a very useful fact.

Fact 137. *Let Γ be a dp-minimal ordered abelian group and let $S \subseteq \Gamma$ be a definable end-segment. Then any complete type $q(x)$ extending $\Sigma_S^{gen}(x)$ is $\ulcorner S \urcorner$-definable.*

Proof. Let $\Sigma_S^{gen}(x)$ be the generic type of S and $q(x)$ be any complete extension. $\Sigma_S^{gen}(x)$ is $\ulcorner S \urcorner$-definable, and by Theorem 123 $q(x)$ is completely determined by the quantifier free formulas. It is sufficient to verify that for each $\Delta \in RJ(\Gamma)$, $k \in \mathbb{Z}$ and $n \in \mathbb{N}$ the set:

$$Z = \{\beta \in \Gamma \mid \big(\rho_\Delta(x) - \rho_\Delta(\beta) + k^\Delta \in n(\Gamma/\Delta)\big) \in q(x)\}$$

is $\ulcorner S \urcorner$-definable. First, we note that there is a canonical one to one correspondence

$$g := (\Gamma/\Delta)/n\big(\Gamma/\Delta\big) \to \Gamma/(\Delta + n\Gamma).$$

Let $c = g(k^\Delta + n(\Gamma/\Delta)) \in \mathrm{dcl}^{eq}(\emptyset)$. Take $\mu \in \Gamma/n\Gamma$ be such that $\pi_k(x) = \mu \in q(x)$. Let f be the $\emptyset$-definable function given by Remark 129. Then $\beta \in Z$ if and only if $\vDash \pi_\Delta^n(\beta) = f(\mu) + c$, and $f(\mu) + c \in \mathrm{dcl}^{eq}(\emptyset)$. □

We conclude this subsection with the following Remark, that simplifies the presentation of a complete family in the dp-minimal case.

Remark 138. *Let Γ be a dp-minimal ordered abelian group. For each $n \in \mathbb{N}_{\geq 2}$ let Ω_n be a finite set of constants in Γ to distinguish representatives for each of the cosets of $n\Gamma$ in Γ. Let $\mathcal{S}_n^\Delta := \{S_n^\Delta(d) \mid d \in \Omega^n\}$. The set $\mathcal{S}_{dp} = \bigcup_{\Delta \in RJ(\Gamma), n \in \mathbb{N}_{\geq 2}} \mathcal{S}_n^\Delta$ is a complete family whose elements are all definable over $\emptyset$.*

Henselian valued fields of equicharacteristic zero with residue field algebraically closed and value group with bounded regular rank

The main goal of this section is to describe the 1-definable subsets $X \subseteq K$, where K is a valued field with residue field algebraically closed and with value group of bounded regular rank.

The language $\mathcal{L}$

Let (K, v) be a valued field of equicharactieristic zero, whose residue field is algebraically closed and whose value group is of bounded regular rank. We will view this valued field as an $\mathcal{L}$-structure, where $\mathcal{L}$ is the language extending $\mathcal{L}_{\text{val}}$ in which the value group sort is equipped with the language $\mathcal{L}_b$ described in Definition 121. Let T be the complete $\mathcal{L}$-first ordered theory of (K, v). (In particular, we are fixing a complete theory for the value group)

Corollary 139. *The first order theory T admits quantifier elimination in the language $\mathcal{L}$.*

Proof. This is a direct consequence of Theorem 108 and Theorem 123. □

Description of definable sets in 1-variable

In this Subsection we give a description of the definable subsets in 1-variable $X \subseteq K$, where $K \models T$. We denote as $\mathcal{O}$ its valuation ring.

Definition 140. *Let $(K, \mathcal{O})$ be a henselian valued field of equicharacteristic zero and let Γ be its value group. Let Δ be a convex subgroup of Γ then the map:*

$$v_\Delta : \begin{cases} K & \to \Gamma/\Delta \\ x & \mapsto v(x) + \Delta, \end{cases}$$

is a henselian valuation on K and it is commonly called as the coarsened valuation induced by Δ. *Note that $v_\Delta = \rho_\Delta \circ v$.*

The following is a folklore fact.

Fact 141. *There is a one-to-one correspondence between the $\mathcal{O}$-submodules of K and the end-segments of Γ. Given $M \subseteq K$ an $\mathcal{O}$-submodule, we have that $S_M := \{v(x) \mid x \in M\}$ is an end-segment of Γ. We refer to S_M as the end-segment induced by M. And given an end-segment $S \subseteq \Gamma$, the set $M_S := \{x \in K \mid v(x) \in S\}$ is an $\mathcal{O}$-submodule of K.*

Definition 142.
1. *Let M and N be $\mathcal{O}$-submodules of K, we say that M is a* scaling *of N if there is some $b \in K$ such that $M = bN$.*
2. *A family $\mathcal{F}$ of definable $\mathcal{O}$-submodules of K is said to be* complete *if any definable submodule $M \subseteq K$ is a scaling of some $\mathcal{O}$-submodule $N \in \mathcal{F}$.*

Fact 143. *Let $\mathcal{F} = \{M_S \mid S \in \mathcal{S}\}$, where $\mathcal{S}$ is the complete family of definable end-segments described in Corollary 130. Then $\mathcal{F}$ is a complete family of $\mathcal{O}$-submodules of K.*

Definition 144.
1. *Let $w : K \to \Gamma_w$ be a valuation, $\gamma \in \Gamma_w$ and $a \in K$. The* closed ball of radius γ centered at a according to the valuation w *is the set of the form $\bar{B}_\gamma(a) = \{x \in K \mid \gamma \leq w(x-a)\}$, and the* open ball of radius γ centered at a according to the valuation w *is the set of the form $B_\gamma(a) = \{x \in K \mid \gamma < w(x-a)\}$.*

2. *A* swiss cheese according to the valuation w *is a set of the form* $A\backslash(B_1 \cup \cdots \cup B_n)$ *where for each* $i \leq n$, $B_i \subsetneq A$ *and the* B_i *and* A *are balls according to the original valuation* $w : K \to \Gamma_w$.

3. *A* 1*-torsor of* K *is a set of the form* $a + bI$ *where* $a, b \in K$ *and* $I \in \mathcal{F}$.

4. *A* generalized swiss cheese *is either a singleton element in the field* $\{a\}$ *or a set of the form* $A\backslash(B_1 \cup \cdots \cup B_n)$ *where* A *is a* 1*-torsor for each* $i \leq n$, $B_i \subsetneq A$ *and the* B_i *is either a* 1*-torsor or a singleton element* $\{b_i\}$ *of the field.*

5. *A* basic positive congruence formula in the valued field *is a formula of the form* $zv_\Delta(x - \alpha) - \rho_\Delta(\beta) + k_\Delta \in n(\Gamma/\Delta)$, *where* $k, z \in \mathbb{Z}$, $\alpha \in K$, $\beta \in \Gamma$, $n \in \mathbb{N}_{\geq 2}$ *and* $k_\Delta = k \cdot 1^\Delta$, *where* 1^Δ *is the minimum positive element of* Γ/Δ *if it exists.*

6. *A* basic negative congruence formula in the valued field *is a formula of the form* $zv_\Delta(x - \alpha) - \rho_\Delta(\beta) + k_\Delta \notin n(\Gamma/\Delta)$, *where* $k, z \in \mathbb{Z}$, $\alpha \in K$, $\beta \in \Gamma$, $n \in \mathbb{N}_{\geq 2}$ *and* $k_\Delta = k \cdot 1^\Delta$, *where* 1^Δ *is the minimum positive element of* Γ/Δ *if it exists.*

7. *A* basic congruence formula in the valued field *is either a* basic positive congruence formula in the valued fied *or a* basic negative congruence formula in the valued field.

8. *A* finite congruence restriction in the valued field *is a finite conjunction of basic congruence formulas in the valued field.*

9. *A* nice set *is a set of the form* $S \cap C$ *where* S *is a generalized swiss cheese and* C *is the set defined by a finite congruence restriction in the valued field.*

To describe completely the definable subsets of K we will need the following lemmas, which permit us to reduce the valuation of a polynomial into the valuation of linear factor of the form $v(x - a)$. We recall a definition and some results present in [Fle08] that will be useful for this purpose.

Definition 145. *Let* (K, w) *be a henselian valued field,* $\alpha \in K$ *and* S *a swiss cheese. Let* $p(x) \in K[x]$, *we define:*

$$m(p, \alpha, S) := \max\{i \leq d \mid \exists x \in S \ \forall j \leq d \ \big(w(a_i(x - \alpha)^i) \leq w\big(a_j(x - \alpha)^j\big)\big\},$$

where the a_i *are the coefficients of the expansion of* p *around* α, *i.e.* $p(x) = \sum_{i \leq d} a_i(x - \alpha)^i$.
Thus $m(p, \alpha, S)$ *is the highest order term in* p *centered at* α *which can have minimal valuation (among the other terms of* p*) in* S.

The following is [Fle08, Proposition 3.4].

Proposition 146. *Let $\mathcal{K}$ be a valued field of characteristic zero. Let $p(x) \in K[x]$ and S be a* swiss cheese *in K. Then there are (disjoint) sub-swiss cheeses $T_1, \dots, T_n \subseteq S$ and $\alpha_1, \dots, \alpha_n \in K$ such that $S = \bigcup_{1 \leq i \leq n} T_i$, where for all $x \in T_i$ $w(p(x)) = w\big(a_{im_i}(x-\alpha_i)^{m_i}\big)$, where $p(x) = \sum_{n=0}^{d} a_{in}(x-\alpha_i)^n$ and $m_i = m(p, \alpha_i, T_i)$. Furthermore, $\alpha_1, \dots, \alpha_k$ can be taken algebraic over the subfield of K generated by the coefficients of $p(x)$.*

Though the preceding proposition is stated for a single polynomial, the same result will hold for any finite number of polynomials Σ. To obtain the desired decomposition, simply apply the proposition to each $p(x) \in \Sigma$, then intersect the resulting partitions to get one that works for all $p(x) \in \Sigma$, using the fact that intersection of two swiss cheeses is again a swiss cheese.

Fact 147. *Let (K, w) be a henselian valued field of equicharacteristic zero, and $Q_1(x), Q_2(x) \in K[x]$ be two polynomials in a single variable. Let $R = \{x \in K \mid Q_2(x) = 0\}$. There is a finite union of swiss cheeses $K = \bigcup_{i \leq k} T_i$, coefficients $\epsilon_i \in K$, elements $\gamma_i \in \Gamma$ and integers $z_i \in \mathbb{Z}$ such that for any $x \in T_i \backslash R$:*

$$w(Q_1(x)) - w(Q_2(x)) = \gamma_i + z_i w(x - \epsilon_i).$$

Proof. The statement is a straightforward computation after applying Proposition 146, and it is left to the reader. □

Proposition 148. *Let $K \vDash T$, for each $\Delta \in RJ(\Gamma)$ let $v_\Delta : K \to \Gamma/\Delta$ be the coarsened valuation induced by Δ. Let $Q_1(x), Q_2(x) \in K[x]$ and $R = \{x \in K \mid Q_1(x) = 0$ or $Q_2(x) = 0\}$. Let $X \subseteq K \backslash R$ be the set defined by a formula of the form:*

$$\gamma \leq^{\Delta} v_\Delta(Q_1(x)) - v_\Delta(Q_2(x)) \text{ or } v_\Delta\left(\frac{Q_1(x)}{Q_2(x)}\right) - \gamma \in n\left(\Gamma/\Delta\right);$$

where $\gamma \in \Gamma/\Delta$ and $n \in \mathbb{N}$. Then X is a finite union of nice sets.

Proof. First we observe that a swiss cheese with respect to the coarsened valuation v_Δ is a generalized swiss cheese with respect to v. The statement follows by a straightforward computation after applying Fact 147, and it is left to the reader. □

We conclude this section by characterizing the definable sets in 1-variable.

Theorem 149. *Let $K \vDash T$ and $X \subseteq K$ be a definable set. Then X is a finite union of nice sets.*

Proof. By Corollary 139 , X is a boolean combination of sets defined by formulas of the form $\gamma \leq^{\Delta} v_{\Delta}(Q_1(x)) - v_{\Delta}(Q_2(x))$ or $v_{\Delta}\left(\dfrac{Q_1(x)}{Q_2(x)}\right) - \gamma \in n\,(\Gamma/\Delta)$, where $\Delta \in RJ(\Gamma)$, $\gamma \in \Gamma/\Delta$ and $n \in \mathbb{N}_{\geq 2}$. By Proposition 148 each of these formulas defines a finite union of nice sets. Because the intersection of two generalized swiss cheeses is again a generalized swiss cheese and the complement of a generalized swiss cheese is a finite union of generalized swiss cheeses the statement follows. $\square$

$\mathcal{O}$-modules and homomorphisms in maximal valued fields

In this section we recall some results about modules over maximally complete valued fields. We follow ideas of Kaplansky in [Kap52] to characterize the $\mathcal{O}$-submodules of finite dimensional K-vector spaces.

Definition 150. 1. *Let K be a valued field and $\mathcal{O}$ its valuation ring. We say that K is* maximal*, if whenever $\alpha_r \in K$ and (integral or fractional) ideals I_r are such that the congruences $x - \alpha_r \in I_r$ are pairwise consistent, then there exists in K a simultaneous solution of all the congruences.*

2. *Let K be a valued field and $M \subseteq K^n$ be an $\mathcal{O}$-module. We say that M is* maximal *if whenever ideals $I_r \subseteq \mathcal{O}$ and elements $s_r \in M$ are such that $x - s_r \in I_r M$ is pairwise consistent in M, then there exists in M a simultaneous solution of all the congruences.*

3. *Let $N \subseteq K^n$ be an $\mathcal{O}$-submodule. Let $x \in N$ we say that x is* α-divisible in *N if there is some $n \in N$ such that $x = \alpha n$.*

We start by recalling a very useful fact.

Fact 151. *Let K be a henselian valued field of equicharacteristic zero, then there is an elementary extension $K \prec K'$ that is maximal.*

Proof. Let K be a henselian valued field of equicharacteristic zero, let T be its $\mathcal{L}_{\text{val}}$-complete first order theory and $\mathfrak{C}$ the monster model of T. By [vdDKM$^+$12, Lemma 4.30] there is some maximal immediate extension of $K \subseteq F \subseteq \mathfrak{C}$. By [vdDKM$^+$12, Theorem 7.12] $K \prec F$. $\square$

The following is [Kap52, Lemma 5].

Lemma 152. *Let K be a maximal valued field, then any (integral or fractional) ideal I of $\mathcal{O}$ is maximal as an $\mathcal{O}$-submodule of K. Moreover, any finite direct sum of maximal $\mathcal{O}$-modules is also maximal.*

Fact 153. *Let $N \subseteq K$ be a non-trivial $\mathcal{O}$-submodule. Let $n \in N \backslash \{0\}$ then $N = nI$ where I is a copy of K, $\mathcal{O}$ or an (integral or fractional) ideal of $\mathcal{O}$.*

Definition 154. *Let K be a field and $n \in \mathbb{N}_{\geq 1}$, we say that a set $\{a_1, \dots, a_n\}$ is an* upper triangular basis *of the vector space K^n if it is a K-linearly independent set and the matrix $[a_1, \dots, a_n]$ is upper triangular.*

Theorem 155. *Let K be a maximal valued field and $n \in \mathbb{N}_{\geq 1}$. Let $N \subseteq K^n$ be an $\mathcal{O}$-submodule then N is maximal, and N is definably isomorphic to a direct sum of copies of K, $\mathcal{O}$ and (integral or fractional) ideals of $\mathcal{O}$. Moreover, if $N \cong \bigoplus_{i \leq n} I_i$ where each I_i is either a copy of K, $\mathcal{O}$ and (integral or fractional) ideals of $\mathcal{O}$ one can find an upper triangular basis $\{a_1, \dots, a_n\}$ of K^n such that $N = \{a_1 x_1 + \dots + a_n x_n \mid x_i \in I_i\}$. In this case we say that $[a_1, \dots, a_n]$ is a* representation matrix for the module *N.*

Proof. We proceed by induction on n, the base case is given by Fact 153 and Lemma 152. For the inductive step, let $\pi : K^{n+1} \to K$ be the projection into the last coordinate and let $M = \pi(N)$. We consider the exact sequence of $\mathcal{O}$-modules $0 \to N \cap \big(K^n \times \{0\}\big) \to N \to M \to 0$. By induction, $N \cap (K^n \times \{0\})$ is maximal and of the required form. And there is an upper triangular basis $\{a_1, \dots, a_n\}$ of $K^n \times \{0\}$ such that $[a_1, \dots, a_n]$ is a representation matrix for $N \cap (K^n \times \{0\})$. If $M = \{0\}$ we are all set, so we may take $m \in M$ such that $m \neq 0$.

Claim 156. *There is some element $x \in N$ such that $\pi(x) = m$ and for any $\alpha \in \mathcal{O}$, if m is α-divisible in M then x is α-divisible in N.*

Proof. Let $J = \{\alpha \in \mathcal{O} \mid m \text{ is } \alpha\text{-divisible in } M\}$. For each $\alpha \in J$, let $m_\alpha \in M$ be such that $m = \alpha m_\alpha$ and take $n_\alpha \in \pi^{-1}(m_\alpha) \cap N$. Fix an element $y \in N$ satisfying $\pi(y) = m$ and let $s_\alpha = y - \alpha n_\alpha \in N \cap (K^n \times \{0\})$.
Consider $\mathcal{S} = \{x - s_\alpha \in \alpha N \cap (K^n \times \{0\}) \mid \alpha \in J\}$ this is system of congruences in $N \cap (K^n \times \{0\})$. We will argue that it is pairwise consistent. Let $\alpha, \beta \in \mathcal{O}$, then either $\frac{\alpha}{\beta} \in \mathcal{O}$ or $\frac{\beta}{\alpha} \in \mathcal{O}$ (or both). Without loss of generality assume that $\frac{\alpha}{\beta} \in \mathcal{O}$, then:

$$s_\alpha - s_\beta = (y - \alpha n_\alpha) - (y - \beta n_\beta) = \beta n_\beta - \alpha n_\alpha = \beta \underbrace{\left(n_\beta - \frac{\alpha}{\beta} n_\alpha\right)}_{\in N \cap (K^n \times \{0\})}$$

Thus s_α is a solution to the system $\{x - s_\alpha \in \alpha N \cap (K^n \times \{0\})\} \cup \{x - s_\beta \in \beta N \cap (K^n \times \{0\})\}$. By maximality of $N \cap (K^n \times \{0\})$ we can find an element $z \in N \cap (K^n \times \{0\})$ such that z is a simultaneous solution to the whole system of congruences in $\mathcal{S}$. Let $x = y - z \in N$, then x satisfies the requirements. Indeed, for each $\alpha \in J$, we had chosen $z - s_\alpha \in \alpha N \cap (K^n \times \{0\})$, so $z = s_\alpha + \alpha w$ for some $w \in N \cap (K^n \times \{0\})$. Thus, $x = y - z = y - s_\alpha - \alpha w = y - (y - \alpha n_\alpha) - \alpha w = \alpha(n_\alpha - w) \in \alpha N$, as desired. □

Let $s : M \to N$ be the map sending an element αm to αx, where $\alpha \in K$. As N is a torsion free module, s is well defined. One can easily verify that s is a homomorphism such that $\pi \circ s = id_M$. Thus, N is the direct sum of $N \cap (K^n \times \{0\})$ and $s(M)$, so it is maximal by Lemma 152. Moreover, $[a_1, \dots, a_n, x]$ is a representation matrix for N, as required. □

Proposition 157. *Let K be a maximal valued field. Let $M, N \subseteq K$ be $\mathcal{O}$-submodules. For any $\mathcal{O}$-homomorphism $h : M \to K/N$ there is some $a \in K$ such that for any $x \in M$, $h(x) = ax + N$.*

Proof. By Fact 153 $M = bI$ where I is a copy of K, $\mathcal{O}$ or an (integral or fractional ideal) of $\mathcal{O}$. It is sufficient to prove the statement for $b = 1$. Let $S_I = \{v(y) \mid y \in I\}$ be the end-segment induced by I. Let $\{\gamma_\alpha \mid \alpha \in \kappa\}$ be a co-initial decreasing sequence in S_I. Choose an element $x_\alpha \in K$ such that $v(x_\alpha) = \gamma_\alpha$, then for each $\alpha < \beta < \kappa$, $x_\beta\mathcal{O} \subseteq x_\alpha\mathcal{O}$ and $I = \bigcup\limits_{\alpha\in\kappa} x_\alpha\mathcal{O}$.

Claim 158. *For each $\alpha \in \kappa$ there is an element $a_\alpha \in K$ such that for all $x \in x_\alpha\mathcal{O}$ we have $h(x) = a_\alpha x + N$.*

For each α choose an element y_α such that $h(x_\alpha) = y_\alpha + N$ and let $a_\alpha = x_\alpha^{-1}y_\alpha$. Fix an element $x \in x_\alpha\mathcal{O}$, then:

$$h(x) = h(x_\alpha \underbrace{(x_\alpha^{-1}x)}_{\in\mathcal{O}}) = \left(x_\alpha^{-1}x\right) \cdot h(x_\alpha) = (x_\alpha^{-1}x) \cdot (a_\alpha x_\alpha + N) = a_\alpha x + N.$$

Claim 159. *Given $\beta < \alpha < \kappa$, then $a_\beta - a_\alpha \in x_\beta^{-1}N$.*

Note that $x_\beta \in x_\beta\mathcal{O} \subseteq x_\alpha\mathcal{O}$, by Claim 158 we have $h(x_\beta) = a_\alpha x_\beta + N = a_\beta x_\beta + N$, then $(a_\alpha - a_\beta)x_\beta \in N$. Hence, $(a_\alpha - a_\beta) \in x_\beta^{-1}N$.

Claim 160. *Without loss of generality we may assume that for any $\alpha < \kappa$ there is some $\alpha < \alpha' < \kappa$ such that for any $\alpha' < \alpha'' < \kappa$ $\;a_\alpha - a_{\alpha''} \notin x_{\alpha''}^{-1}N$.*

Suppose the statement is false. Then there is some α such that for any $\alpha < \alpha'$ we can find $\alpha' < \alpha''$ such that $a_\alpha - a_{\alpha''} \in x_{\alpha''}^{-1}N$. Define:

$$h^* : \begin{cases} I & \to K/N \\ x & \to a_\alpha x + N. \end{cases}$$

We will show that for any $x \in I$, $h(x) = h^*(x)$. Fix an element $x \in I$, since $< \gamma_\alpha \mid \alpha \in \kappa >$ is coinitial and decreasing in S_I we can find an element $\alpha' > \alpha$ such that $v(x) > \gamma_{\alpha'}$, so $x \in x_{\alpha'}\mathcal{O} \subseteq x_{\alpha''}\mathcal{O}$. Then

$$(a_\alpha - a_{\alpha'})x = \underbrace{(a_\alpha - a_{\alpha''})x}_{\in x_{\alpha''}^{-1}xN\subseteq N} + \underbrace{(a_{\alpha''} - a_{\alpha'})x}_{\in x_{\alpha'}^{-1}xN\subseteq N}.$$

we conclude that $(a_\alpha - a_{\alpha'})x \in N$. By Claim 158 we have $h(x) = a_{\alpha'}x + N$, thus $h^*(x) = h(x)$ and h^* witnesses the conclusion of the statement.

Claim 161. *There is a subsequence $< b_\alpha \mid \alpha \in \operatorname{cof}(\kappa) >$ of $< a_\alpha \mid \alpha \in \kappa >$ that is pseudo-convergent.*

Proof. Let $g : \mathrm{cof}(\kappa) \to \kappa$ be a cofinal function in κ i.e. for any $\delta \in \kappa$ there is some $\alpha \in \mathrm{cof}(\kappa)$ such that $g(\alpha) > \delta$. We construct the desired sequence by transfinite recursion in $\mathrm{cof}(\kappa)$, building a strictly increasing function $f : \mathrm{cof}(\kappa) \to \kappa$ satisfying the following conditions:

1. for each $\alpha < \mathrm{cof}(\kappa)$ we have $b_\alpha = a_{f(\alpha)}$ and $f(\alpha) > g(\alpha)$,
2. for any $\alpha < \mathrm{cof}(\kappa)$ the sequence $(b_\eta \mid \eta < \alpha)$ is pseudo-convergent. This is, given $\eta_1 < \eta_2 < \eta_3 < \alpha$
$$v(b_{\eta_3} - b_{\eta_2}) > v(b_{\eta_2} - b_{\eta_1}),$$
3. for each $\alpha < \mathrm{cof}(\kappa)$ we have that: for any $\eta < \alpha$, and $f(\alpha) < \eta' < \kappa$
$$v(a_{\eta'} - b_\alpha) > v(b_\eta - b_\alpha) \text{ and } a_{\eta'} - b_\alpha \notin x_{\eta'}^{-1}N.$$

For the base case, set $b_0 = a_0$ and $f(0) = g(0) + 1$. Suppose that for $\mu < \mathrm{cof}(\kappa)$, $f \restriction_\mu$ has been defined and $< b_\eta \mid \eta < \mu >$ has been constructed. Let $\mu^* = \sup\{f(\eta) \mid \eta < \mu\}$, by Claim 160 (applied to $\alpha = \max\{\mu^*, g(\mu)\}$) there is some $\max\{\mu^*, g(\mu)\} < v < \kappa$ satisfying the following property:

$$\textit{for any } v < \eta' < \kappa,\ a_\alpha - a_{\eta'} \notin x_{\eta'}^{-1}N.$$

Set $f(\mu) = v$ and $b_\mu = a_v$. We continue verifying that the three conditions are satisfied. The first condition $b_\mu = a_{f(\mu)}$ and $f(\mu) > g(\mu)$ follows immediately by construction.

We continue checking that $(b_\eta \mid \eta \leq \mu)$ is a pseudo-convergent sequence. Fix $\eta_1 < \eta_2 < \mu$ we must show that $v(b_\mu - b_{\eta_2}) > v(b_{\eta_2} - b_{\eta_1})$. By construction $b_\mu = a_{f(\mu)} = a_v$ and $f(\mu) = v > \mu^* \geq f(\eta_2)$. Since the third condition holds for η_2 we must have $v(a_v - b_{\eta_2}) > v(b_{\eta_2} - b_{\eta_1})$, as required.

Lastly, we verify that the third condition holds for μ. Let $\eta < \mu$ and $v = f(\mu) < \eta'$, we aim to show $v(a_{\eta'} - b_\mu) > v(b_\mu - b_\eta)$. Suppose by contradiction that this inequality does not hold, then $\frac{b_\mu - b_\eta}{a_{\eta'} - b_\mu} \in \mathcal{O}$. Because the third condition holds for η and by construction $v = f(\mu) > f(\eta)$, we have that $b_\mu - b_\eta = a_v - a_{f(\eta)} \notin x_v^{-1}N$.
By Claim 159 $a_{\eta'} - b_\mu = a_{\eta'} - a_v \in x_v^{-1}N$, then:

$$b_\mu - b_\eta = \underbrace{\frac{b_\mu - b_\eta}{a_{\eta'} - b_\mu}}_{\in \mathcal{O}}(a_{\eta'} - b_\mu) \in x_v^{-1}N,$$

which leads us to a contradiction. It is only left to show that for any $f(\mu) < \eta' < \kappa$ we have that $a_{\eta'} - b_\mu \notin x_{\eta'}^{-1}N$. By construction, we have chose $b_\mu = a_v$ where $\alpha = \max\{g(\mu), \mu^*\} < v < \kappa$ and for any $v < \eta^{'} < \kappa$ we have:

$$a_\alpha - a_{\eta'} \notin x_{\eta'}^{-1}N.$$

As $\alpha < f(\mu)$, by Claim 159 $a_\alpha - a_{f(\mu)} \in x_\mu^{-1}N \subseteq x_{\eta'}^{-1}N$.

Fix $\eta' > v = f(\mu)$, then $a_{\eta'} - b_\mu = a_{\eta'} - a_{f(\mu)} \notin x_{\eta'}^{-1}N$. Otherwise,

$$a_{\eta'} - a_\alpha = \underbrace{(a_{\eta'} - a_{f(\mu)})}_{\in x_{\eta'}^{-1}N} + \underbrace{(a_{f(\mu)} - a_\alpha)}_{\in x_{\eta'}^{-1}N} \in x_{\eta'}^{-1}N \text{ because } x_\eta^{-1}\text{N is an } \mathcal{O}\text{-submodule of } K,$$

which leads us to a contradiction.

□

Since K is maximal there is some $a \in K$ that is a pseudolimit of $< b_\alpha \mid \alpha \in \mathrm{cof}(\kappa) >$. We aim to prove that $h(x) = ax + N$ for $x \in I$. Fix an element $x \in I$. The function f is cofinal in κ because of the first condition combined with the fact that g is cofinal in κ. We can find some $\alpha \in \mathrm{cof}(\kappa)$ such that $x \in x_{f(\alpha)}\mathcal{O} \subseteq I$. By Claim 158 $h(x) = a_{f(\alpha)}x + N$, hence it is sufficient to prove that $(a - a_{f(\alpha)})x \in N$. As $x \in x_{f(\alpha)}\mathcal{O}$ it is enough to show that $(a - a_{f(\alpha)}) = (a - b_\alpha) \in x_{f(\alpha)}^{-1}N$. Let $\alpha < \beta < \kappa$, by Claim 159 $(b_\beta - b_\alpha) = (a_{f(\beta)} - a_{f(\alpha)}) \in x_{f(\alpha)^{-1}}N$. Also, $v(a - a_{f(\alpha)}) = v(a_{f(\beta)} - a_{f(\alpha)})$ thus $(a - a_{f(\alpha)}) = u(a_{f(\beta)} - a_{f(\alpha)})$ for some $u \in \mathcal{O}^\times$, thus $(a - a_{f(\alpha)}) \in x_{f(\alpha)}^{-1}N$, as desired. □

Valued vector spaces

We introduce valued vector spaces and some facts that will be required through this chapter. An avid and curious reader can consult [Kuh, Section 2.3] for a more exhaustive presentation. Through this section we fix (K, Γ, v) a valued field and V a K-vector space.

Definition 162. *A tuple* $(V, \Gamma(V), val, +)$ *is a* valued vector space *structure if:*

1. $\Gamma(V)$ *is a linear order,*
2. *there is an action* $+ : \Gamma \times \Gamma(V) \to \Gamma(V)$ *which is order preserving in each coordinate,*
3. $val : V \to \Gamma(V)$ *is a map such that for all* $v, w \in V$ *and* $\alpha \in K$ *we have:*
   - $val(v + w) \geq \min\{val(w), val(v)\}$,
   - $val(\alpha v) = v(\alpha) + val(v)$.

The following Fact is [Joh16, Remark 1.2].

Fact 163. *Let* V *be a finite dimensional valued vector space over* K*, then the action of* $\Gamma(K)$ *over* $\Gamma(V)$ *has finitely many orbits. In fact,* $|\Gamma(V)/\Gamma(K)| \leq dim_K(V)$.

Definition 164. *Let* $(V, \Gamma(V), val, +)$ *be a valued vector space:*

1. *Let* $a \in V$ *and* $\gamma \in \Gamma(V)$*. A* ball *in* V *is a set of the form:*

$$\overline{Ball_\alpha(a)} = \{x \in V \mid val(x - a) \geq \gamma\} \text{ or } Ball_\alpha(a) = \{x \in V \mid val(x - a) > \gamma\}.$$

2. *We say that* $(V, \Gamma(V), val, +)$ *is* maximal *if every nested family of balls in* V *has non-empty intersection.*

Definition 165. *Let* $(V, \Gamma(V), val, +)$ *be a valued vector space and let* W *be a subspace of* V*. Then* $(W, \Gamma(W), val, +)$ *is also a valued vector space, where* $\Gamma(W) = \{val(w) \mid w \in W\}$*. We say that:*

1. W is maximal in V *if every family of nested balls*

$$\{Ball_\alpha(x_\alpha) \mid \alpha \in S\}, \text{ where } S \subseteq \Gamma(W) \text{ and for each } \alpha \in S \ x_\alpha \in W.$$

that has non-empty intersection in V *has non-empty intersection in* W*.*

2. $W \leq V$ *has the* optimal approximation property *if for any* $v \in V \backslash W$ *the set*

$$\{val(v - w) \mid w \in W\}$$

attains a maximum.

The following is a folklore fact.

Fact 166. *Let* $(V, \Gamma(V), val, +)$ *be a valued vector space, and* W *a subspace of* V *the following statements are equivalent:*

1. W *is maximal in* V*,*

2. W *has the optimal approximation property in* V*.*

Additionally, if W *is maximal then it is maximal in* V*.*

We conclude this subsection with the definition of separated basis.

Definition 167. *Let* $(V, \Gamma(V), val, +)$ *be a valued vector space. Assume that* V *is a* K*-vector space of dimension* n*. A basis* $\{v_1, \ldots, v_n\} \subseteq V$ *is a* separated basis *if for any* $\alpha_1, \ldots, \alpha_n \in K$ *we have that:*

$$val(\sum_{i \leq n} \alpha_i v_i) = \min\{val(\alpha_i v_i) \mid i \leq n\}.$$

2.3 Definable modules

In this section we study definable $\mathcal{O}$-submodules in henselian valued fields of equicharacteristic zero.

Corollary 168. *Let* (F, v) *be a henselian valued field of equicharacteristic zero and* N *be a definable* $\mathcal{O}$*-submodule of* F^n*. Then* N *is definably isomorphic to a direct sum of copies of* F*,* $\mathcal{O}$*, or (integral or fractional) ideals of* $\mathcal{O}$*. Moreover, if* $N \cong \oplus_{i \leq n} I_i$ *there is some upper triangular basis* $\{a_1, \ldots, a_n\}$ *of* F^n *such that* $[a_1, \ldots, a_n]$ *is a representation matrix of* N*.*

Proof. By Fact 151 we can find F' an elementary extension of F that is maximal, so we can apply Theorem 155. As the statement that we are trying to show is first order expressible, it must hold as well in F. □

Corollary 169. *Let (F, v) be a henselian valued field of equicharacteristic zero and let $N, M \subseteq F$ be a definable $\mathcal{O}$-submodules. Then for any definable $\mathcal{O}$-homomorphism $h : M \to K/N$. Then there is some $b \in F$ satisfying that for any $y \in M$, $h(y) = by + N$.*

Proof. By Fact 151 we can find an elementary extension $F \prec F'$ that is maximal. The statement follows by applying Proposition 157, because it is first order expressible. □

Definable modules in valued fields of equicharacteristic zero with residue field algebraically closed and value group with bounded regular rank

Let (K, v) be a henselian valued field of equicharacteristic zero with residue field algebraically closed and value group with bounded regular rank. Let $\mathcal{O}$ be its valuation ring and T be the complete $\mathcal{L}$-first order theory of (K, v). In this section we study the definable $\mathcal{O}$-modules and torsors. Let $\mathcal{I}'$ be the complete family of $\mathcal{O}$-submodules of K described in Fact 143. From now on we fix a complete family $\mathcal{I} = \mathcal{I}' \backslash \{0, K\}$.

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Remark 170. *If $K \vDash T$, then $N \cong \oplus_{i \leq n} I_i$, where each $I_i \in \mathcal{F} \cup \{0, K\}$. This follows because $\mathcal{F}$ is a complete family of $\mathcal{O}$-modules.*

Definition 171. *Let $K \vDash T$. A* definable torsor *U is a coset in K^n of a definable $\mathcal{O}$-submodule of K^n, if $n = 1$ we say that U is a 1-torsor. Let U be a definable 1-torsor, we say that U is:*

1. closed *if it is a translate of a submodule of K of the form $a\mathcal{O}$.*
2. *it is* open *if it is either K or a translate of a submodule of the form aI for some $a \in K$, where $I \in \mathcal{F} \backslash \mathcal{O}$.*

Definition 172. *Let $(I_1, \ldots, I_n) \in \mathcal{F}^n$ be a fixed tuple.*

1. *An $\mathcal{O}$-module $M \subseteq K^n$ is of* type $(I_1, \ldots, I_n)$ *if $M \cong \bigoplus_{i \leq n} I_i$.*
2. *An $\mathcal{O}$-module $M \subseteq K^n$ of type $(\mathcal{O}, \ldots, \mathcal{O})$ is said to be an $\mathcal{O}$-lattice of rank n.*
3. *A torsor Z is* of type $(I_1, \ldots, I_n)$, *if $Z = \bar{d} + M$ where $M \subseteq K^n$ is an $\mathcal{O}$-submodule of K^n of type $(I_1, \ldots, I_n)$.*

Proposition 173. *Let Z be a torsor of type $(I_1, \ldots, I_n)$. Then there is some $\mathcal{O}$-module $L \subseteq K^{n+1}$ of type $(I_1, \ldots, I_n, \mathcal{O})$ such that $\ulcorner Z \urcorner$ and $\ulcorner L \urcorner$ are interdefinable.*

Proof. Let $N \subseteq K^n$ be the $\mathcal{O}$-submodule and take $\bar{d} \in K^n$ be such that $Z = \bar{d} + N$. Let $N_2 = N \times \{0\}$ which is an $\mathcal{O}$ submodule of K^{n+1} and let $\bar{b} = \begin{bmatrix} \bar{d} \\ 1 \end{bmatrix}$. Define the $\mathcal{O}$-module of K^{n+1}:

$$L_{\bar{d}} := N_2 + \bar{b}\mathcal{O} = \{ \begin{bmatrix} n + \bar{d}r \\ r \end{bmatrix} \mid r \in \mathcal{O}, n \in N\}.$$

By a standard computation, one can verify that the definition of $L_{\bar{d}}$ is independent of the choice of $\bar{d}$, i.e. if $\bar{d} - \bar{d}' \in N$ then $L_{\bar{d}} = L_{\bar{d}'}$. So we can denote $L = L_{\bar{d}}$, and we aim to show that L and Z are interdefinable. It is clear that $\ulcorner L \urcorner \in dcl^{eq}(\ulcorner Z \urcorner)$, while $\ulcorner Z \urcorner \in dcl^{eq}(\ulcorner L \urcorner)$ because $Z = \pi_{2 \leq n+1}\big(L \cap (K^n \times \{1\})\big)$ where $\pi_{2\leq n+1} : K^{n+1} \to K^n$ is the projection into the last n-coordinates. □

Definable 1-$\mathcal{O}$-modules

In this subsection we study the quotient modules of 1-dimensional modules.

Notation 174. *Let $M \subseteq K$ be a definable $\mathcal{O}$-module. We denote by $S_M := \{v(x) \mid x \in M\}$ the* end-segment induced by M. *We recall as well that we write $\mathcal{F}$ to denote the complete family of $\mathcal{O}$-submodules of K previously fixed.*

Definition 175. *A* definable 1-$\mathcal{O}$-module *is an $\mathcal{O}$-module which is definably isomorphic to a quotient of a definable $\mathcal{O}$-submodule of K by another, i.e. something of the form aI/bJ where $a, b \in K$ and $I, J \in \mathcal{F} \cup \{0, K\}$.*

The following operation between $\mathcal{O}$-modules will be particularly useful in our setting.

Definition 176. *Let N, M be $\mathcal{O}$-submodules of K, we define the* colon module

$$Col(N : M) = \{x \in K \mid xM \subseteq N\}.$$

It is a well known fact from Commutative Algebra that $Col(N : M)$ is also an $\mathcal{O}$-module.

Lemma 177. *Let $K \vDash T$. Let A be a 1-definable $\mathcal{O}$-module. Suppose that $A = A_1/A_2$, where $A_2 \leq A_1$ are $\mathcal{O}$-submodules of K. Then the $\mathcal{O}$-module $Hom_{\mathcal{O}}(A, A)$ is definably isomorphic to the 1-definable $\mathcal{O}$-module*

$$\big(Col(A_1 : A_1) \cap Col(A_2 : A_2)\big)/Col(A_2, A_1).$$

Proof. By Fact 151 without loss of generality we may assume K to be maximal, because the statement is first order expressible. Let

$$\mathcal{B} = \{f : A_1 \to A_1/A_2 \mid f \text{ is a homomorphism and } A_2 \subseteq ker(f)\}.$$

$\mathcal{B}$ is canonically in one-to-one correspondence with $Hom_{\mathcal{O}}(A, A)$. By Corollary 169, for every homomorphism $f \in \mathcal{B}$ there is some $b_f \in K$ satisfying that for any $x \in A_1$, $f(x) = b_f x + A_2$ and we say that b_f *is a linear representation of* f.

Claim 178. *Let $f \in \mathcal{B}$ if b_f is a linear representation of f, then $b_f \in Col(A_1 : A_1) \cap Col(A_2 : A_2)$.*

Proof. First we verify that $b_f \in Col(A_1 : A_1)$. Let $x \in A_1$, by hypothesis $f(x) = b_f x + A_2 \in A_1/A_2$. Then there is some $y \in A_1$ such that $b_f x + A_2 = y + A_2$ and therefore $b_f x - y \in A_2 \subseteq A_1$. Consequently, $b_f x \in y + A_1 = A_1$, and as x is an arbitrary element we conclude that $b_f \in Col(A_1 : A_1)$. We check now that $b_f \in Col(A_2 : A_2)$, and we fix an element $x \in A_2$. By hypothesis, $b_f x + A_2 = A_2$ so $b_f x \in A_2$, and as $x \in A_2$ is an arbitrary element we conclude that $b_f \in Col(A_2 : A_2)$. □

Claim 179. *Let $f \in \mathcal{B}$ if b_f, b'_f are linear representations of f, then $b_f - b'_f \in Col(A_2 : A_1)$*

Proof. Let $x \in A_1$, by hypothesis $f(x) = b_f x + A_2 = b'_f x + A_2$, so $(b_f - b'_f)x \in A_2$. Because x is arbitrary in A_1 we have that $(b_f - b'_f) \in Col(A_2 : A_1)$. □

We consider the map $\phi : \mathcal{B} \to \big(Col(A_1 : A_1) \cap Col(A_2 : A_2)\big)/Col(A_2 : A_1)$ that sends an $\mathcal{O}$-homomorphism f to the coset $b_f + Col(A_1 : A_2)$. By Claim 179 such map is well defined. By a standard computation ϕ is an injective $\mathcal{O}$-homomorphism. To show that ϕ is surjective, let

$$b \in Col(A_1 : A_1) \cap Col(A_2 : A_2),$$

and consider $f_b : A_1 \to A_1/A_2$, the map that sends the element x to $bx + A_2$. Because $b \in Col(A_2 : A_2)$, for any $x \in A_2$ we have that $bx \in A_2$ thus $A_2 \subseteq ker(f_b)$. Consequently, $f_b \in \mathcal{B}$ and $\phi(f_b) = b + Col(A_1 : A_2)$. □

Lemma 180. *Let $n \in \mathbb{N}_{\geq 2}$ and $M \subseteq K^n$ be an $\mathcal{O}$-module.*

1. *Let $\pi^{n-1} : K^n \to K^{n-1}$ be the projection into the first $(n-1)$-coordinates and $B_{n-1} = \pi^{n-1}(M)$. Take $A_1 \subseteq K$ be the $\mathcal{O}$-module such that $ker(\pi^{n-1}) = M \cap (\{0\}^{n-1} \times K) = (\{0\}^{n-1} \times A_1)$.*

2. *Let $\pi_n : K^n \to K$ be the projection into the last coordinate and $B_1 = \pi_n(M)$. Let $A_{n-1} \subseteq K^{n-1}$ be the $\mathcal{O}$-module such that $ker(\pi_n) = M \cap (K^{n-1} \times \{0\}) = (A_{n-1} \times \{0\})$.*

Then $A_{n-1} \leq B_{n-1}$ and both lie in K^{n-1}, and $A_1 \leq B_1$ and both lie in K. The map $\phi : B_{n-1} \to B_1/A_1$ given by $b \mapsto a + A_1$ where $(b, a) \in M$, is a well defined homomorphism of $\mathcal{O}$-modules whose kernel is A_{n-1}. In particular, $B_{n-1}/A_{n-1} \cong B_1/A_1$. Furthermore if M is definable, ϕ is also definable.

Proof. Let $\bar{m} \in A_{n-1}$, then $(\bar{m}, 0) \in M$ thus $\pi^{n-1}(\bar{m}, 0) = \bar{m} \in B_{n-1}$. We conclude that A_{n-1} is a submodule of B_{n-1}. Likewise $A_1 \leq B_1$. For the second part of the statement, it is a straightforward computation to verify that the map $\phi : B_{n-1} \to B_1/A_1$(defined as in the statement) , is a well defined surjective homomorphism of $\mathcal{O}$-modules whose kernel is A_{n-1}. Lastly, the definability of ϕ follows immediately by the definability of M. □

2.4 The Stabilizer sorts

An abstract criterion to eliminate imaginaries

We start by recalling Hurshovski's criterion, The following is [Hru14, Lemma 1.17].

Theorem 181. *Let T be a first order theory with home sort K (meaning that $\mathfrak{M}^{eq} = dcl^{eq}(K)$). Let $\mathcal{G}$ be some collection of sorts. If the following conditions all hold, then T has weak elimination of imaginaries in the sorts $\mathcal{G}$.*

1. Density of definable types: *for every non-empty definable set $X \subseteq K$ there is an $acl^{eq}(\ulcorner X \urcorner)$-definable type in X.*
2. Coding definable types: *every definable type in K^n has a code in $\mathcal{G}$ (possibly infinite). This is, if p is any (global) definable type in K^n, then the set $\ulcorner p \urcorner$ of codes of the definitions of p is interdefinable with some (possibly infinite) tuple from $\mathcal{G}$.*

Proof. A very detailed proof can be found in [Joh16, Theorem 6.3]. The first part of the proof shows weak elimination of imaginaries as it is shown that for any imaginary element e we can find a tuple $a \in \mathcal{G}$ such that $e \in dcl^{eq}(a)$ and $a \in acl^{eq}(e)$. □ □

We start by describing the sorts that are required to be added to apply this criterion and show that any valued field of equicharacteristic zero, with residue field algebraically closed and value group with bounded regular rank admits weak elimination of imaginaries.

Definition 182. *For each $n \in \mathbb{N}$, let $\{e_1, \dots, e_n\}$ be the canonical basis of K^n and $(I_1, \dots, I_n) \in \mathcal{F}^n$.*

1. *Let $C_{(I_1,\dots,I_n)} = \{\sum_{1 \leq i \leq n} x_i e_i \mid x_i \in I_i\}$, we refer to this module as* the canonical $\mathcal{O}$-submodule of K^n of type $(I_1, \dots, I_n)$.
2. *We denote as $B_n(K)$ the multiplicative group of $n \times n$-upper triangular and invertible matrices.*
3. *We define the subgroup $Stab_{(I_1,\dots,I_n)} = \{A \in B_n(K) \mid AC_{(I_1,\dots,I_n)} = C_{(I_1,\dots,I_n)}\}$.*
4. *Let $\Lambda_{(I_1,\dots,I_n)} := \{M \mid M \subseteq K^n$ is an $\mathcal{O}$-module of type $(I_1, \dots, I_n)\}$.*
5. *Let $\mathcal{U}_n \subseteq (K^n)^n$ be the set of n-tuples $(\bar{b}_1, \dots, \bar{b}_n)$, such that $B = [\bar{b}_1, \dots, \bar{b}_n]$ is an invertible upper triangular matrix. We define the equivalence relation $E_{(I_1,\dots,I_n)}$ on $\mathcal{U}_n$ as:*
$$E_{(I_1,\dots,I_n)}\big(\bar{a}_1, \dots, \bar{a}_n; \bar{b}_1, \dots, \bar{b}_n\big) \text{ holds if and only if}$$
$(\bar{a}_1, \dots, \bar{a}_n)$ and $(\bar{b}_1, \dots, \bar{b}_n)$ generate the same $\mathcal{O}$-module of type $(I_1, \dots, I_n)$, i.e.
$$\{\sum_{1 \leq i \leq n} x_i \bar{a}_i \mid x_i \in I_i\} = \{\sum_{1 \leq i \leq n} x_i \bar{b}_i \mid x_i \in I_i\}.$$

6. *We denote as $\bar{\rho}_{(I_1,\dots,I_n)}$ the canonical projection map:*

$$\bar{\rho}_{(I_1,\dots,I_n)} : \begin{cases} \mathcal{U}_n & \to \mathcal{U}_n/E_{(I_1,\dots,I_n)} \\ (\bar{a}_1,\dots,\bar{a}_n) & \mapsto [(\bar{a}_1,\dots,\bar{a}_n)]_{E_{(I_1,\dots,I_n)}} \end{cases}$$

Remark 183. *1. The set $\{\ulcorner M \urcorner \mid M \in \Lambda_{(I_1,\dots,I_n)}\}$ can be canonically identified with $B_n(K)/Stab_{(I_1,\dots,I_n)}$. Indeed, by Corollary 168 given any $\mathcal{O}$-module M of type $(I_1,\dots,I_n)$ we can find an upper triangular basis $\{\bar{a}_1,\dots,\bar{a}_n\}$ of K^n such that $[a_1,\dots,a_n]$ is a matrix representation of M. The code $\ulcorner M \urcorner$ is interdefinable with the coset*

$$[a_1,\dots,a_n]Stab_{(I_1,\dots,I_n)}.$$

2. *Fix some $n \in \mathbb{N}_{\geq 2}$ and let $(I_1,\dots,I_n)$ be a fixed tuple. The sort $B_n(K)/Stab_{(I_1,\dots,I_n)}$ is in definable bijection with the equivalence classes of $\mathcal{U}_n/E_{(I_1,\dots,I_n)}$. In fact we can consider the $\emptyset$-definable map:*

$$f : \begin{cases} \mathcal{U}_n/E_{(I_1,\dots,I_n)} & \to B_n(K)/Stab_{(I_1,\dots,I_n)} \\ [(\bar{a}_1,\dots,\bar{a}_n)]_{E_{(I_1,\dots,I_n)}} & \mapsto [\bar{a}_1,\dots,\bar{a}_n]Stab_{(I_1,\dots,I_n)}. \end{cases}$$

We denote as $\rho_{(I_1,\dots,I_n)} = \mathcal{U}_n \to B_n(K)/Stab_{(I_1,\dots,I_n)}$ the composition maps $\rho_{(I_1,\dots,I_n)} = f \circ \bar{\rho}_{(I_1,\dots,I_n)}$.

Definition 184. *[The stabilizer sorts] We consider the language $\mathcal{L}_{\mathcal{G}}$ extending the three sorted language $\mathcal{L}$ (defined in Subsection 2.2), where:*

1. *We equipped the value group with the multi-sorted language $\mathcal{L}_{bq}$ introduced in Subsection 2.2.*

2. *For each $n \in \mathbb{N}$ we consider the parametrized family of sorts $B_n(K)/Stab_{(I_1,\dots,I_n)}$ and maps*

$$\rho_{(I_1,\dots,I_n)} : \mathcal{U}_n \to B_n(K)/Stab_{(I_1,\dots,I_n)}$$

where $(I_1,\dots,I_n) \in \mathcal{F}^n$.

We refer to the sorts in the language $\mathcal{L}_{\mathcal{G}}$ as the stabilizer sorts*. We denote as $\mathcal{G}$ their union, i.e.*

$$K \cup k \cup \Gamma \cup \{\Gamma/\Delta \mid \Delta \in RJ(\Gamma)\} \cup \{\Gamma/\Delta + n\Gamma \mid \Delta \in RJ(\Gamma), n \in \mathbb{N}_{\geq 2}\} \\ \cup \{B_n(K)/Stab_{(I_1,\dots,I_n)} \mid n \in \mathbb{N}, (I_1,\dots,I_n) \in \mathcal{F}^n\}.$$

Remark 185. *The geometric sorts for the case of ACVF are a particular instance of the stabilizer sorts. Let S_n denotes the set of $\mathcal{O}$-lattices of K^n of rank n, these are simply the $\mathcal{O}$-modules of type $\underbrace{(\mathcal{O},\dots,\mathcal{O})}_{n-times}$. For each $\Lambda \in S_n$, let $res(\Lambda) = \Lambda \otimes_{\mathcal{O}} k = \Lambda/\mathcal{M}\Lambda$, which is a*

k -vector space of dimension n.
Let $T_n = \bigcup_{\Lambda \in S_n} res(\Lambda) = \{(\Lambda, x) \mid \Lambda \in S_n, x \in res(\Lambda)\}$. Each of these torsors is considered in the stabilizer sorts as the code of an $\mathcal{O}$ module of type $(\mathcal{M}, \dots, \mathcal{M}, \mathcal{O})$, because any torsor of the form $a+\mathcal{M}\Lambda$ for some $\Lambda \in S_n$ can be identified with an $\mathcal{O}$-module of type $(\underbrace{\mathcal{M}, \dots, \mathcal{M}}_{n-times}, \mathcal{O})$ (see Proposition 173).

An explicit description of the stabilizer sorts

In this subsection we state an explicit description of the subgroups $Stab_{(I_1,\dots,I_n)}$.

Notation 186. *For each $\Delta \in RJ(\Gamma)$ we denote as $\mathcal{O}_\Delta$ the valuation ring of K of the coarsened valuation $v_\Delta : K^\times \to \Gamma/\Delta$ induced by Δ.*

Fact 187. *Let $I \in \mathcal{F}$ and let $S_I = \{v(x) \mid x \in I\}$. Then*

$$Stab(I) = \mathcal{O}_{\Delta_{S_I}}^\times = \{x \in K \mid v(x) \in \Delta_{S_I}\}.$$

Proof. This is an immediate consequence of Fact 131. □

Proposition 188. *Let $n \in \mathbb{N}$, and $(I_1, \dots, I_n) \in \mathcal{F}^n$. Then*

$$\begin{aligned} Stab_{(I_1,\dots,I_n)} = \{((a_{i,j})_{1 \le i,j \le n} \in B_n(K) \mid a_{ii} \in \mathcal{O}_{\Delta_{S_{I_i}}}^\times \\ \text{and } a_{ij} \in Col(I_i, I_j) \text{ for each } 1 \le i < j \le n \ \} \end{aligned}$$

Proof. This is a straightforward computation and it is left to the reader. □

2.5 Weak Elimination of imaginaries for henselian valued field with value group with bounded regular rank

Let (K, v) be a henselian valued field of equicharacteristic zero, with residue field algebraically closed and value group with bounded regular rank. Let T be its complete $\mathcal{L}_\mathcal{G}$-first order theory and $\mathfrak{M}$ its monster model.
In this section we show that both conditions required by Hrushovski's criterion to obtain weak elimination of imaginaries down to the stabilizer sorts hold.

Density of definable types

In this section we prove density of definable types for 1-definable sets $X \subseteq \mathfrak{M}$. There are two ways to tackle this problem. One can either use the quantifer elimination (see Corollary 139) and obtain a *canonical* decomposition of X into nice sets $T_i \in \operatorname{acl}^{eq}(\ulcorner X \urcorner)$ and then build a global type $p(x) \in x \in T_i$ which is $\operatorname{acl}^{eq}(\ulcorner T_i \urcorner)$. This approach was successfully achieved by Holly in [Hol95] for the case of $ACVF$ and real closed valued fields, and her work essentially gives a way to code one-definable sets in the main field down to the geometric sorts. It is worth pointing out, that finding *a canonical decomposition* is often a detailed technical work. Instead of following this strategy, we follow a different approach that exploits the power of generic types, which are definable partial types.

Definition 189. *Let $U \subseteq \mathfrak{M}$ be a definable 1-torsor, let*

$$\Sigma_U^{gen}(x) = \{x \in U\} \cup \{x \notin B \mid B \subsetneq U \mid B \textit{ is a proper sub-torsor of } U\}.$$

This is a $\ulcorner U \urcorner$-definable partial type.

Proposition 190. *Let $U \subseteq \mathfrak{M}$ be a definable closed 1-torsor. Then there is a* unique *complete global type $p(x)$ extending $\Sigma_U^{gen}(x)$ i.e. $\Sigma_U^{gen}(x) \subseteq p(x)$. Moreover, $p(x)$ is $\ulcorner U \urcorner$-definable.*

Proof. Let $a \in U$ and $Y_a = \{v(x-a) \mid x \in U\} \subseteq \Gamma$. As U is a closed $\mathcal{O}$-module then Y_a has a minimum element γ. For any other element $b \in U$, we have that $\gamma = \min(Y_a) = \min(Y_b)$, thus $\gamma \in \operatorname{dcl}^{eq}(\ulcorner U \urcorner)$.
By quantifier elimination (see Corollary 139) it is sufficient to show that $\Sigma_U^{gen}(x)$ determines also the congruence and coset formulas. Let c be a realization of $\Sigma_U^{gen}(x)$ and then for any $a \in U(\mathfrak{M})$ we have that $v(c-a) = \gamma$. Let $p(x) = \operatorname{tp}(c/\mathfrak{M})$, $\Delta \in RJ(\Gamma)$, $\ell \in \mathbb{N}$ and $\beta \in \Gamma$. If $a \in U(\mathfrak{M})$ then:

$$\vDash v_\Delta(c-a) - \rho_\Delta(\beta) + k^\Delta \in \Delta \text{if and only if}$$

$$\vDash \phi_\Delta^k(\beta) := \gamma - \rho_\Delta(\beta) + k^\Delta, \text{ and}$$

$$\vDash v_\Delta(c-a) - \rho_\Delta(\beta) + k^\Delta \in \ell\big(\Gamma/\Delta\big) \text{ if and only if}$$

$$\vDash \psi_\Delta^k(\beta) := \gamma - \rho_\Delta(\beta) + k^\Delta \in \ell(\Gamma/\Delta).$$

We observe that $\psi_\Delta^k(\beta)$ and $\phi_\Delta^k(\beta)$ are $\mathcal{L}(\operatorname{dcl}^{eq}(\ulcorner U \urcorner))$-formulas, and their definition is completely independent from the choice of c.
If $a \notin U(\mathfrak{M})$, then for any $b \in U(\mathfrak{M})$ we have that $v(c-a) = v(b-a)$. Therefore

$$\vDash v_\Delta(c-a) - \rho_\Delta(\beta) + k^\Delta \in \Delta \text{ if and only if}$$

$$\vDash \epsilon_\Delta^k(a,\beta) := \exists b \in U\big(v_\Delta(b-a) - \rho_\Delta(\beta) + k^\Delta\big), \text{ and}$$

$$\vDash v_\Delta(c-a) - \rho_\Delta(\beta) + k^\Delta \in \ell\big(\Gamma/\Delta\big) \text{ if and only if}$$

$$\vDash \eta_{\Delta}^{k}(a,\beta) := \exists b \in U\big(v_{\Delta}(b-a) - \rho_{\Delta}(\beta) + k^{\Delta} \in \ell(\Gamma/\Delta)\big).$$

Both formulas $\epsilon_{\Delta}^{k}(a,\beta)$ and $\eta_{\Delta}^{k}(a,\beta)$ are $\mathcal{L}(\ulcorner U \urcorner)$-definable and completely independent from the choice of c.
We conclude that $p(x)$ is a $\ulcorner U \urcorner$- definable type. Furthermore, for any possible realization $c \vDash \Sigma_{U}^{gen}(x)$ we obtain the same scheme of definition. Hence, there is a *unique* extension $p(x)$ of $\Sigma_{U}^{gen}(x)$. □

Theorem 191. *For every non-empty definable set $X \subseteq \mathfrak{M}$, there is a $acl^{eq}(\ulcorner X \urcorner)$-definable global type $p(x) \vdash x \in X$.*

Proof. Let $X \subseteq \mathfrak{M}$ be a 1-definable set.

Claim 192. *There is a 1-torsor U such that $\ulcorner U \urcorner \in \mathrm{acl}^{eq}(\ulcorner X \urcorner)$ and the partial type:*

$$\Sigma_{U}^{gen}(x) \cup \{x \in X\} \text{ is consistent.}$$

Proof. Let $\mathcal{F}$ be the family of closed balls B such that $B \cap X \neq \emptyset$. We say that $B_1 \sim B_2$ if and only if $B_1 \cap X = B_2 \cap X$. This is a $\ulcorner X \urcorner$-definable equivalence relation over $\mathcal{F}$.
Let $\pi : \mathcal{F} \to \mathcal{F}/\sim$, the natural $\ulcorner X \urcorner$-definable map sending a closed ball to its class $[B]_{\sim}$.
For each class $\mu \in \mathcal{F}/\sim$ the set $U_{\mu} = \bigcap_{B \in \mathcal{F}, \pi(B)=\mu} B$ is a μ-definable 1-torsor. Moreover, for any $B \in \mathcal{F}$, $B \cap X = U_{\mu} \cap X$ if and only if $\pi(B) = \mu$. In particular, if B is a proper closed subball of U_{μ}, then $\pi(B) \neq \mu$.
Then set $\mathcal{F}/\sim$ admits a partial $\ulcorner X \urcorner$-definable order defined as:

$$\mu_1 \lhd \mu_2 \text{ if and only } B_1 \cap X \subsetneq B_2 \cap X \text{ where } \pi(B_1) = \mu_1 \text{ and } \pi(B_2) = \mu_2.$$

The set $\big(\mathcal{F}/\sim, \lhd\big)$ is a tree with a maximal element $\mu_0 \in \mathrm{acl}^{eq}(\ulcorner X \urcorner)$. This class is obtained by taking the projection of a ball B_0 such that $B_0 \cap X = X$. By quantifier elimination (see Corollary 139) X is a finite union of nice sets, thus such ball B_0 exists.

For each $\mu \in \mathcal{F}/\sim$, we write $P(\mu)$ to denote the set of immediate predecessors of μ (if they exists). This is

$$P(\mu) := \{\beta \in \mathcal{F}/\sim \ \mid \beta \lhd \mu \text{ and} \neg \exists z (\beta \lhd z \lhd \mu)\}.$$

If $\Sigma_{U_{\mu}}^{gen}(x) \cup \{x \in X\}$ is inconsistent then $P(\mu)$ is finite and has size at least 2. Indeed, $\Sigma_{U_{\mu}}^{gen}(x) \cup \{x \in X\}$ is consistent if and only if

$$\{x \in U_{\mu}\} \cup \{x \notin B \mid B \subseteq U \text{ is a closed ball}\} \cup \{x \in X\} \text{ is consistent.}$$

Hence, if $\Sigma_{U_{\mu}}^{gen}(x) \cup \{x \in X\}$ is inconsistent, by compactness we can find finitely many disjoint closed balls $B_1, \dots, B_k$ such that $B_i \cap X \neq \emptyset$ and

$$U_{\mu} \cap X \subseteq \bigcup_{i \leq k} B_i$$

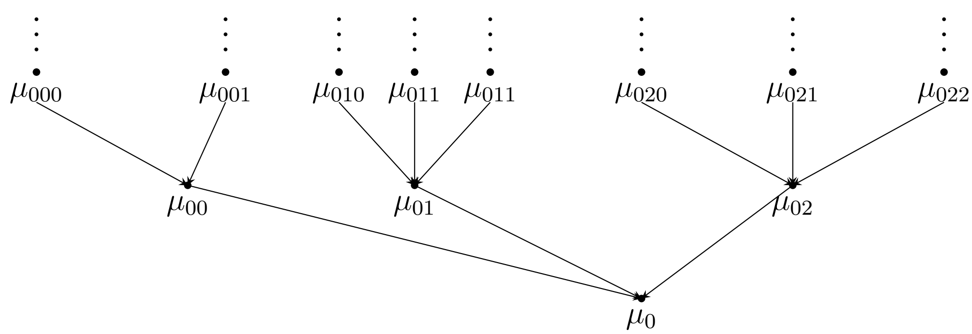
Let $\beta_i = \pi(B_i) \triangleleft \mu$. Then $P(\mu) = \{\beta_i \mid i \leq k\} \subseteq \mathrm{acl}^{eq}(\ulcorner X \urcorner, \mu)$.

We now start looking for the 1-torsor $U \in \mathrm{acl}^{eq}(\ulcorner X \urcorner)$ such that $\Sigma_U^{gen}(x) \cup \{x \in X\}$ is consistent.
Let $\mu_0 \in \mathrm{acl}^{eq}(\ulcorner X \urcorner)$ be the maximal element of $(\mathcal{F}/\sim, \triangleleft)$, if $\Sigma_{U_{\mu_0}}^{gen}(x) \cup \{x \in X\}$ is consistent, the torsor U_{μ_0} satisfies the required conditions. We may assume that $\Sigma_{U_{\mu_0}}^{gen}(x) \cup \{x \in X\}$ is inconsistent, thus it has finitely many predecessors $P(\mu_0) \subseteq \mathrm{acl}^{eq}(\ulcorner X \urcorner)$. For each $\beta \in P(\mu_0)$ exactly one of the following cases hold:

1. $\Sigma_{U_\beta}^{gen}(x) \cup \{x \in X\}$ is consistent, then the torsor U_β satisfies the required conditions of the claim; or

2. $\Sigma_{U_\beta}^{gen}(x) \cup \{x \in X\}$ is inconsistent, and β has finitely many predecessors $P(\beta) \subseteq \mathrm{acl}^{eq}(\ulcorner X \urcorner, \beta) \subseteq \mathrm{acl}^{eq}(\ulcorner X \urcorner)$.

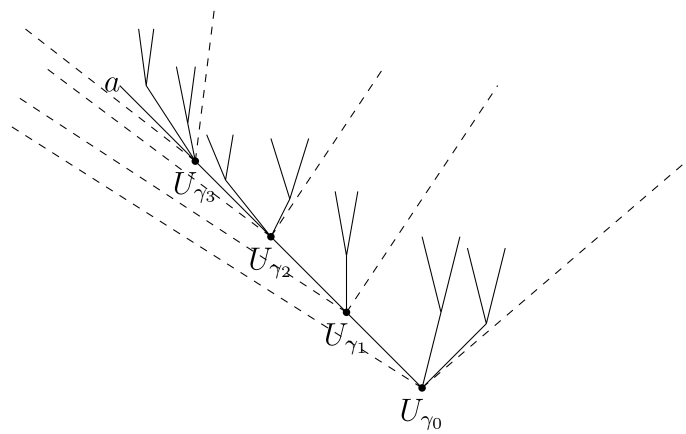
By iterating this process for each of the predecessors, we build a discrete tree $\mathcal{T} \subseteq \mathcal{F}/\sim$ of finite ramification.



Hence, it is sufficient to argue that every path in this tree is finite. Suppose by contradiction that a path is infinite, then we can find an infinite decreasing sequence $< \gamma_i \mid i \in \mathbb{N} >$ of elements in $\mathcal{F}/\sim$ such that $U_{\gamma_0} = U_{\mu_0}$, and:

1. for each $i \in \mathbb{N}$, $P(\gamma_i)$ is finite and of size at least 2. Given $\eta_1 \neq \eta_2 \in P(\gamma_i)$ we have that $U_{\eta_1} \cap U_{\eta_2} = \emptyset$. And U_{η_1} is a proper subtorsor of U_{γ_i}.

2. For each $\mu \in \mathcal{F}/\sim$, $U_\mu \subseteq U_{\gamma_i}$ for some $i \in \mathbb{N}$, or there is some $i \in \mathbb{N}$ such that $U_\mu \subseteq U_\eta$ for some $\eta \in P(\gamma_i) \backslash \{\gamma_{i+1}\}$.



By compactness we can find an element $a \in \mathfrak{M}$ such that $a \in \bigcap_{i \in \mathbb{N}} U_{\gamma_i}$. We note that $\{U_\mu \mid \mu \in \mathcal{F}/\sim\}$ is a uniform definable family of 1-torsors. Then we can define the set

$$D = \{x \in K \mid \exists \mu \in \mathcal{F}/\sim \ x \in U_\mu \text{ and } a \notin U_\mu\},$$

but this set is not a finite union of nice sets (by the conditions in 2.5), which leads us to a contradiction. □

Proof. If U is a closed 1-torsor, we let c be a realization of $\Sigma_U^{gen}(x) \cup \{x \in X\}$. By Proposition 190 the type $p(x) = \operatorname{tp}(c/\mathfrak{M}) \vdash x \in X$ is $\ulcorner U \urcorner$-definable. The statement follows as $\ulcorner U \urcorner \in \operatorname{acl}^{eq}(\ulcorner X \urcorner)$.
We may assume that U is an open torsor. We observe that for any realization $c \vDash \Sigma_U^{gen}(x)$ given $a \neq a' \in U((M))$ we have that $v(c-a) = v(c-a')$. Let $\pi := \mathbb{N} \to \mathbb{N} \times \mathbb{N}_{\geq 1}$ be a fixed bijection. We build an increasing sequence of partial consistent types $\big(\Sigma_k(x) \mid k \in \mathbb{N})$ by induction:

- **Stage** 0**:** Let $\Sigma_0(x) := \Sigma_U^{gen}(x) \cup \{x \in X\}$.
- **Stage** $k+1$**:** Let $\pi(k) = (n, \ell)$. At this stage we decide the congruence modulo $\Delta_n + \ell\Gamma$. To simplify the notation we will assume that $\ell \geq 2$, otherwise the argument will follow in a similar manner (instead of working with $\ell(\Gamma/\Delta_n)$ we argue with Γ/Δ_n). Let

$$\Lambda_k(x) := \Sigma_k(x) \cup \{v_{\Delta_n}(x-a) - \rho_{\Delta_n}(\beta) \notin \ell(\Gamma/\Delta_n) \mid a \in U(\mathfrak{M}), \beta \in \Gamma\}.$$

If the partial type $\Lambda_k(x)$ is consistent, then we set $\Sigma_{k+1}(x) = \Lambda(x)$. Otherwise, let

$$A_i = \{\mu \in \Gamma/(\Delta_n + \ell\Gamma) \mid \Sigma_k(x) \cup \{\pi^\ell_{\Delta_n}(v(x-a)) = \mu \mid a \in U(\mathfrak{M})\} \mid \text{is consistent}\}.$$

A_i is a finite set. We set

$$\Sigma_{k+1}(x) := \Sigma_k(x) \cup \{\pi^\ell_{\Delta_n}(v(x-a)) = \mu \mid a \in U(\mathfrak{M})\}.$$

Let $\mathcal{J} = \{k \in \mathbb{N}_{\geq 1} \mid \Lambda_k(x) \text{ is inconsistent }\}$.

Claim 193. *For all $k \in \mathbb{N}$ we have that for any automorphism $\sigma \in Aut(\mathfrak{M}/\operatorname{acl}^{eq}(\ulcorner X \urcorner)$, $\sigma(\Sigma_k(x)) = \Sigma_k(x)$ and if $k \in \mathcal{J}$ then $\sigma(A_k) = A_k$. In particular, $A_k \subseteq \operatorname{acl}^{eq}(\ulcorner X \urcorner)$ for all $k \in \mathcal{J}$.*

Proof. We proceed by induction, for the base case $k = 0$ the statement follows because $\ulcorner U \urcorner \in \operatorname{acl}^{eq}(\ulcorner X \urcorner)$. We assume that for any $\sigma \in Aut(\mathfrak{M}/\ulcorner X \urcorner)$ we have that $\sigma(\Sigma_k(x)) = \Sigma_k(x)$. We fix $\tau \in Aut(\mathfrak{M}/\ulcorner X \urcorner)$ and we aim to show that $\tau(\Sigma_{k+1}(x)) = \Sigma(\Sigma_{k+1}(x))$. If $\Lambda_k(x)$, then:

$$\begin{aligned}
\tau\big(\Sigma_{k+1}(x)\big) &= \tau\big(\Sigma_k(x) \cup \{v_{\Delta_n}(x-a) - \rho_{\Delta_n}(\beta) \notin \ell(\Gamma/\Delta_n) \mid a \in U(\mathfrak{M}), \beta \in \Gamma\}\big)\\
&= \Sigma_k(x) \cup \{v_{\Delta_n}(x-\tau(a)) - \rho_{\Delta_n}(\tau(\beta)) \notin \ell(\Gamma/\Delta_n) \mid a \in U(\mathfrak{M}), \beta \in \Gamma\}\\
&= \Sigma_{k+1}(x).
\end{aligned}$$

If $\Lambda_k(x)$ is not consistent then $k \in \mathcal{J}$. And we first argue that $\tau(A_k) = A_k$. By definition of A_k, given $\mu \in A_k$ then

$$\Sigma_k(x) \cup \{\pi^{\ell}_{\Delta_n}(v(x-a)) = \mu \mid a \in U(\mathfrak{M})\} \text{ is consistent,}$$

because τ is an isomorphism,

$$\begin{aligned}
&\tau\big(\Sigma_k(x) \cup \{\pi^{\ell}_{\Delta_n}(v(x-a)) = \mu \mid a \in U(\mathfrak{M})\}\big)\\
&\qquad = \Sigma_k(x) \cup \{\pi^{\ell}_{\Delta_n}(v(x-\tau(a))) = \tau(\mu) \mid a \in U(\mathfrak{M})\}\\
&\qquad = \Sigma_k(x) \cup \{\pi^{\ell}_{\Delta_n}(v(x-a)) = \tau(\mu) \mid a \in U(\mathfrak{M})\} \text{ is consistent,}
\end{aligned}$$

hence $\tau(\mu) \in A_i$. We conclude that $\tau(A_i) = A_i$, and because τ is an arbitrary element in $Aut(\mathfrak{M}/\operatorname{acl}^{eq}(\ulcorner X \urcorner))$ we conclude that $A_i \subseteq \operatorname{acl}^{eq}\big(\operatorname{acl}^{eq}(\ulcorner X \urcorner)\big) = \operatorname{acl}^{eq}(\ulcorner X \urcorner)$. In particular, for any $\mu \in A_i$, $\tau(\mu) = \mu$. Consequently,

$$\begin{aligned}
\tau\big(\Sigma_{k+1}(x)\big) &= \Sigma_k(x) \cup \{\pi^{\ell}_{\Delta_n}(v(x-\tau(a))) = \mu \mid a \in U(\mathfrak{M})\}\\
&= \Sigma_k(x) \cup \{\pi^{\ell}_{\Delta_n}(v(x-a)) = \mu \mid a \in U(\mathfrak{M})\} = \Sigma_{k+1}(x), \text{ as required.}
\end{aligned}$$

□

Let $\Sigma_\infty(x) := \bigcup_{k \in \mathbb{N}} \Sigma_k(x)$, by construction this is a consistent partial type $\operatorname{acl}^{eq}(\ulcorner X \urcorner)$-definable and $\Sigma_\infty(x) \vdash x \in X$. By quantifier elimination, $\Sigma_\infty(x)$ determines a complete global type $p(x) \vdash x \in X$. This type $p(x)$ is $\operatorname{acl}^{eq}(\ulcorner X \urcorner)$-definable as $\Sigma_\infty(x)$ is. □

□

Coding definable types

In this subsection we prove that any definable type can be coded in the stabilizer sorts $\mathcal{G}$. Let $\mathbf{x} = (x_1, \ldots, x_k)$ be a tuple of variables in the main field sort. By quantifier elimination any definable type $p(\mathbf{x})$ over a model K is completely determined by the boolean combinations formulas of the form:

1. $Q_1(\mathbf{x}) = 0$,
2. $v_\Delta(Q_1(\mathbf{x})) \leq v_\Delta(Q_2(\mathbf{x}))$,
3. $v_\Delta\left(\frac{Q_1(\mathbf{x})}{Q_2(\mathbf{x})}\right) - k_\Delta \in n(\Gamma/\Delta)$,
4. $v_\Delta\left(\frac{Q_1(\mathbf{x})}{Q_2(\mathbf{x})}\right) = k_\Delta$.

where $Q_1(\mathbf{x}), Q_2(\mathbf{x}), \in K[X_1, \ldots, X_k]$, $n \in \mathbb{N}_{\geq 2}$, $\Delta \in RJ(\Gamma)$, $k \in \mathbb{Z}$ and $k_\Delta = k \cdot 1_\Delta$ where 1_Δ is the minimal element of Γ/Δ if it exists. We will approximate such a type by considering for each $l \in \mathbb{N}$ the definable vector space D_l/I_l, where D_l is the set of polynomials of degree at most l and I_l is the subspace of D_l of polynomials $Q(\mathbf{x})$ such that $Q(\mathbf{x}) = 0$ is a formula in $p(\mathbf{x})$. The formulas of the second kind, essentially give D_l/I_l a valued vector space structure with all the coarsened valuations, while the formulas of the third and forth kind simply impose some binary relations in the linear order $\Gamma(D_l/I_l)$. This philosophy reduces the problem of coding definable types into finding a way to code the possible valuations that could be induced over some power of K while taking care as well for the congruences.

The following is [Joh16, Lemma 3.3].

Fact 194. *Let K be any field. Let V be a subspace of K^n then V can be coded by a tuple of K, and V and K^n/V have a $\ulcorner V \urcorner$-definable basis.*

We start by coding the $\mathcal{O}$-submodules of K^n.

Lemma 195. *Let $K \vDash T$ and $M \subseteq K^n$ be a definable $\mathcal{O}$-submodule. Then the code $\ulcorner M \urcorner$ can be coded in the stabilizer sorts.*

Proof. Let V^+ the span of M and V^- the maximal K-subspace of K^n contained in M. By Fact 194 the subspaces V^+ and V^- can be coded by a tuple c in K, and the quotient vector space V^+/V^- admits a c-definable basis. Hence V^+/V^- can be identified over c with some power K^m. And $\ulcorner M \urcorner$ is interdefinable over c with the code of the image M/V^- in K^m. But this image is an $\mathcal{O}$-submodule of K^m of type $(I_1, \ldots, I_m) \in \mathcal{F}^m$ so it admits a code in $B_m(K)/Stab_{(I_1,\ldots,I_n)}$. So M admits a code in the stabilizer sorts, as required. □

Definition 196. *[Valued relation] Let $K \vDash T$, and Γ be its value group. Let V be some finite dimensional K-vector space and $R \subseteq V \times V$ be a definable subset that defines a total pre-order. We say that R is a* valued relation *if there is an interpretable valued vector space structure $(V, \Gamma(V), val, +)$ in K such that $(v, w) \in R$ if and only if $val(v) \leq val(w)$.*

In fact, given a relation $R \subseteq V \times V$ that defines a total pre-order satisfying that:

- for all $v, w \in V \quad (v, v+w) \in R$ or $(w, v+w) \in R$,
- for all $v \in V \quad (v, v) \in R$,
- for all $v, w \in V$ and $\alpha \in K$, if $(v, w) \in R$ then $(\alpha v, \alpha w) \in R$.

We can define an equivalence relation E_R over V as $E_R(v, w) \leftrightarrow (v, w) \in R \wedge (w, v) \in R$. The set $\Gamma(V) = V/E_R$ is therefore interpretable in K and we call it as the *linear order induced by R*. Let $val : V \to \Gamma(V)$ be the canonical projection map that sends each vector to its class. We can naturally define an action of Γ over $\Gamma(V)$ as:

$$+ : \begin{cases} \Gamma \times \Gamma(V) & \to \Gamma(V) \\ (\alpha, [v]_{E_R}) & \mapsto [av]_{E_R}, \text{ where } a \in K \text{ satisfying } v(a) = \alpha. \end{cases}$$

This is a well defined map by the third condition imposed over R. The structure $(V, \Gamma(V), val, +)$ is an interpretable valued vector space structure over V and we refer to it as the *valued vector space structure induced by R.*

Lemma 197. *Let K be a model of T and let $R \subset K^n \times K^n$ be a binary relation inducing a valued vector space structure $(K^n, \Gamma(K^n), val, +)$ over K^n. Then we can find a basis $\{v_1, \ldots, v_n\}$ of K^n such that:*

1. *It is a separated basis for val, this is given any set of coefficients $\lambda_1, \ldots, \lambda_n \in K$,*

$$val\big(\sum_{i \leq n} \lambda_i v_i\big) = \min\{v(\lambda_i) + val(v_i) \mid i \leq n\}.$$

2. *For each $i \leq n$, $\gamma_i = val(v_i) \in dcl^{eq}(\ulcorner R \urcorner)$.*

Proof. Because the statement we are proving is first order expressible, by Fact 151 we may assume that K is maximal. We proceed by induction on n. For the base case, note that $K = span_K\{1\}$ then $\gamma = val(1) \in dcl^{eq}(\ulcorner R \urcorner)$. We assume the statement for n and we want to prove it for $n+1$. Let $W = K^n \times \{0\}$, $val_W = v \restriction_W$, $\Gamma(W) = \{val(w) \mid w \in W\}$, and $R_W = R \cap (W \times W)$. Then $(W, \Gamma(W), val_W, +)$ is a valued vector space structure over W and $\ulcorner R_W \urcorner \in \mathrm{dcl}^{eq}(\ulcorner R \urcorner)$. The subspace W admits an $\emptyset$-definable basis, so it can be canonically identified with K^n. By the induction hypothesis we can find $\{w_1, \ldots, w_n\}$ a separated basis of W such that $val_W(w_i) \in \mathrm{dcl}^{eq}(\ulcorner R_W \urcorner) \subseteq \mathrm{dcl}^{eq}(\ulcorner R \urcorner)$. As W is finite dimensional it is maximal by Lemma 152. By Fact 166 W has the optimal approximation property in K^{n+1}. We can therefore define the valuation over the quotient space K^{n+1}/W as follows:

$$val_{K^{n+1}/W} : \begin{cases} \big(K^{n+1}/W\big) & \to \Gamma(K^n) \\ v + W & \mapsto \max\{val(v + w_0) \mid w_0 \in W\}. \end{cases}$$

Define $R_{K^{n+1}/W}=\{(w_1+W,w_2+W)\mid val_{K^{n+1}/W}(w_1+W)\leq val_{K^{n+1}/W}(w_2+W)\}$, which is a valued relation over the quotient space K^{n+1}/W. As $K^{n+1}/W=K^{n+1}/(K^n\times\{0\})$ is definably isomorphic over the $\emptyset$-set to K, we can find a non zero coset $v+W$ such that $val_{K^{n+1}/W}(v+W)\in dcl^{eq}(\ulcorner R_{K^{n+1}/W}\urcorner)\subseteq dcl^{eq}(\ulcorner R\urcorner)$. Let $w^*\in W$ be a vector where the maximum of $\{val_{K^{n+1}/W}(v+w)\mid w\in W\}$ is attained, i.e. $val_{K^{n+1}/W}(v+W)=val(v+w^*)$. It is sufficient to show that $\{w_1,\dots,w_n,v+w^*\}$ is a separated basis for K^{n+1}.
Let $\alpha\in K$, we show that for any $w\in W$ $val((v+w^*)+\alpha w)=\min\{val(v+w^*),val(\alpha w)\}$. If $val(v+w^*)\neq val(\alpha w)$ then $val((v+w^*)+\alpha w)=\min\{val(v+w^*),val(\alpha w)\}$. So let's assume that $\gamma=val(v+w^*)=val(\alpha w)$, by the ultrametric inequality $val((v+w^*)+\alpha w)\geq\gamma$. By the maximal choice of w^*, we have that $val((v+w^*)+\alpha w)\leq val(v+w^*)=\gamma$. So $val((v+w^*)+\alpha w)=\min\{val(v+w^*),val(\alpha w)\}$ as required.

□

Theorem 198. *Let K be a model of T and Γ its value group. Let R be a definable valued relation over K^n and $(K^n,\Gamma(K^n),val,+)$ be the valued vector space structure induced by R. Then $\ulcorner R\urcorner$ is interdefinable with a tuple of elements in the stabilizer sorts and there is an $\ulcorner R\urcorner$- definable bijection $\Gamma(K^n)$ and finitely many disjoint copies of Γ (all contained in Γ^s, where s is the number of Γ-orbits over $\Gamma(K^n)$).*

Proof. As the statement that we are trying to prove is first order expressible, without loss of generality we may assume that K is maximal. Let R be a valued relation over K^n and let $(K^n,\Gamma(K^n),val,+)$ be the valued vector space structure induced by R. By Lemma 197, we can find a separated basis $\{v_1,\dots,v_n\}$ of K^n, such that for each $i\leq n$, $val(v_i)\in dcl^{eq}(\ulcorner R\urcorner)$. Let $\{\gamma_1,\dots,\gamma_s\}\subseteq\{val(v_i)\mid i\leq n\}$ be a complete set of representatives of the orbits of Γ over the linear order $\Gamma(K^n)$, this is:

$$\Gamma(K^n)=\dot{\bigcup_{i\leq s}}\Gamma+\gamma_i.$$

For each $i\leq s$, we define $B_i:=\{x\in K^n\mid val(x)\geq\gamma_i\}$. Each B_i is an $\mathcal{O}$-submodule of K^n, so by Lemma 195 $\ulcorner B_i\urcorner$ is interdefinable with a tuple in the stabilizer sorts. The valued vector space structure over K^n is completely determined by the closed balls containing 0, and each of these ones is of the form αB_i for some $\alpha\in K$ and $i\leq s$. Thus the code $\ulcorner R\urcorner$ is interdefinable with the tuple $(\ulcorner B_1\urcorner,\dots,\ulcorner B_s\urcorner)$. We conclude that $\ulcorner R\urcorner$ can be coded in the stabilizer sorts.
For the second part of the statement, consider the map:

$$\begin{cases} f:\dot{\bigcup}_{i\leq s}\Gamma+\gamma_i & \to\Gamma^s\\ \alpha+\gamma_i & \mapsto(0,\dots,0,\underbrace{\alpha}_{i-\text{th coordinate}},0,\dots,0).\end{cases}$$

As $\{\gamma_1,\dots,\gamma_s\}\subseteq dcl^{eq}(\ulcorner R\urcorner)$ this is a $\ulcorner R\urcorner$-definable bijection between $\Gamma(K^n)$ to finitely many disjoint copies of Γ, contained in Γ^s. □

Theorem 199. *Let* $p(\mathbf{x})$ *be a definable global type in* $\mathfrak{M}^n$*. Then* $p(\mathbf{x})$ *can be coded in* $\mathcal{G}\cup\Gamma^{eq}$*.*

Proof. Let $p(\mathbf{x})$ be a definable global type, and let K be a small model where $p(\mathbf{x})$ is defined. Let $q(\mathbf{x}) = p(\mathbf{x}) \upharpoonright_K$ it is sufficient to code $q(\mathbf{x})$.

For each $\ell \in \mathbb{N}$ let D_ℓ be the space of polynomials in $K[X_1, \dots, X_n]$ of degree less or equal than ℓ. This is a finite dimensional K-vector space with an $\emptyset$-definable basis. Let $I_\ell := \{Q(\bar{x}) \in D_\ell \mid Q(\bar{x}) = 0 \in q(\bar{x})\}$, this is a subspace of D_ℓ. Let $R_\ell := \{(Q_1(\mathbf{x}), Q_2(\mathbf{x})) \in D_\ell \times D_\ell \mid v(Q_1(\mathbf{x})) \leq v(Q_2(\mathbf{x})) \in q(\mathbf{x})\}$, this relation induces a valued vector space structure on the quotient space $V_\ell = D_\ell / I_\ell$. Let $(V_\ell, \Gamma(V_\ell), val_\ell, +_\ell)$ be the valued vector space structure induced by R_ℓ over V_ℓ.
For each $\Delta \in RJ(\Gamma)$ and $k \in \mathbb{Z}$, a formula of the form $v_\Delta(Q_1(\mathbf{x})) = v_\Delta(Q_2(\mathbf{x})) + k_\Delta$ determines a definable relation $\phi^k_\Delta \subseteq \Gamma(V_\ell)^2$, defined as:

$$(val_\ell(Q_1(\mathbf{x})), val_\ell(Q_2(\mathbf{x})) \in \phi^k_\Delta \text{ if and only if } v_\Delta(Q_1(\mathbf{x})) = v_\Delta(Q_2(\mathbf{x})) + k_\Delta \in q(\mathbf{x}).$$

Similarly, for each $\Delta \in RJ(\Gamma)$, $k \in \mathbb{Z}$ and $n \in \mathbb{N}_{\geq 2}$ we consider the definable binary relation $\psi^{n,k}_\Delta \subseteq \Gamma(V_\ell)^2$ determined as:

$$(val_\ell(Q_1(\mathbf{x})), val_\ell(Q_2(\mathbf{x}))) \in \psi^{k,n}_\Delta \text{ if and only if } v_\Delta(Q_1(\mathbf{x})) - v_\Delta(Q_2(\mathbf{x})) + k_\Delta \in n(\Gamma/\Delta) \in q(\mathbf{x}).$$

Likewise, for each $\Delta \in RJ(\Gamma)$ and $k \in \mathbb{Z}$ we consider the definable binary relations $\theta^k_\Delta \subseteq \Gamma(V_\ell)^2$ defined as:

$$(val_\ell(Q_1(\mathbf{x})), val_\ell(Q_2(\mathbf{x}))) \in \theta^k_\Delta \text{ if and only if } v_\Delta(Q_1(\mathbf{x})) < v_\Delta(Q_2(\mathbf{x})) + k_\Delta \in q(\mathbf{x}).$$

Let

$$\begin{aligned}\mathcal{S}_\ell = \{&\phi^k_\Delta \mid \Delta \in RJ(\Gamma), k \in \mathbb{Z}\} \cup \{\psi^{k,n}_\Delta \mid \Delta \in RJ(\Gamma), k \in \mathbb{Z}, n \in \mathbb{N}_{\geq 2}\} \\ &\cup \{\theta^k_\Delta \mid \Delta \in RJ(\Gamma), k \in \mathbb{Z}\}\end{aligned}$$

We denote as $\mathcal{V}_\ell = (V_\ell, \Gamma(V_\ell), val_\ell, +_\ell, \mathcal{S}_\ell)$ the valued vector space over V_ℓ with the enriched structure over the linear order $\Gamma(V_\ell)$. By quantifier elimination (see Corollary 139), the type $q(\mathbf{x})$ is completely determined by boolean combinations of formulas of the form:

- $Q_1(x) = 0$,
- $v_\Delta(Q_1(\mathbf{x})) < v_\Delta(Q_2(\mathbf{x}))$,
- $v_\Delta\left(\frac{Q_1(\mathbf{x})}{Q_2(\mathbf{x})}\right) - k_\Delta \in n(\Gamma/\Delta)$,
- $v_\Delta\left(\frac{Q_1(\mathbf{x})}{Q_2(\mathbf{x})}\right) = k_\Delta$.

where $Q_1(\mathbf{x}), Q_2(\mathbf{x}), \in K[X_1, \dots, X_k]$, $n \in \mathbb{N}_{\geq 2}$, $\Delta \in RJ(\Gamma)$, $k \in \mathbb{Z}$ and $k_\Delta = k \cdot 1_\Delta$ where 1_Δ is the minimum positive element of Γ/Δ if it exists. Hence the type $p(\mathbf{x})$ is entirely determined (and determines completely) by the sequence of valued vector spaces with enriched structure over the linear order $(\mathcal{V}_\ell \mid \ell \in \mathbb{N})$.
By Fact 194 for each $\ell \in \mathbb{N}$ we can find codes $\ulcorner I_\ell \urcorner$ in the home sort for the $I_\ell' s$. After naming these codes, each quotient space $V_\ell = D_\ell / I_\ell$ has a definable basis, so it can be definably identified with some power of K. Therefore, without loss of generality we may assume that the underlying set of the valued vector space with enriched structure V_ℓ is some power of K.
By Theorem 198, the relation R_ℓ admits a code $\ulcorner R_\ell \urcorner$ in the stabilizer sorts. Moreover, there is a $\ulcorner R_\ell \urcorner$ definable bijection $f : \Gamma(V_\ell) \to \Gamma^s$, where $s \in \mathbb{N}_{\geq 2}$ is the number of Γ-orbits over $\Gamma(V_\ell)$.
In particular, for each $\Delta \in RJ(\Gamma)$, $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ the definable relations ϕ^k_Δ, $\psi^{k,n}_\Delta$ and θ^k_Δ are interdefinable over $\ulcorner R \urcorner$ with $f(\phi^k_\Delta)$, $f(\psi^{k,n}_\Delta)$ and $f(\theta^k_\Delta)$, all subsets of Γ^{2s}. Consequently, the type $q(\mathbf{x})$ can be coded in the sorts $\Gamma \cup \Gamma^{eq}$, as every definable subset D in some power of Γ admits a code in Γ^{eq}. □

Theorem 200. *Let K be a valued field of equicharacteristic zero, residue field algebraically closed and value group with bounded regular rank. Then K admits weak elimination of imaginaries in the language $\mathcal{L}_\mathcal{G}$, where the stabilizer sorts are added.*

Proof. By Theorem 181, K admits weak elimination of imaginaries down to the sorts $\mathcal{G} \cup \Gamma^{eq}$, where $\mathcal{G}$ are the stabilizer sorts. In fact, Hrushovski's criterion requires us to verify the following two conditions:

1. the density of definable types, this is Theorem 191, and
2. the coding of definable types, this is Theorem 199.

By Corollary 109 the value group Γ is stably embedded. By Theorem 124, the ordered abelian group with bounded regular rank Γ admits weak elimination of imaginaries once one adds the quotient sorts,

$$\{\Gamma/\Delta \mid \Delta \in RJ(\Gamma)\} \cup \{\Gamma/\Delta + n\Gamma \mid \Delta \in RJ(\Gamma), n \in \mathbb{N}_{\geq 2}\}.$$

We conclude that K admits weak elimination of imaginaries down to the stabilizer sorts $\mathcal{G}$. □

2.6 Elimination of imaginaries for henselian valued field with dp-minimal value group

Let (K, v) be a henselian valued field of equicharacteristic zero, residue field algebraically closed and dp-minimal value group. We see K as a multisorted structure in the language $\hat{\mathcal{L}}$ extending the language $\mathcal{L}_\mathcal{G}$ (described in Definition 184), where the value group is equipped with the language $\mathcal{L}_{dp}$ described in Subsection 2.2. Let $\mathcal{I}'$ be the complete family of $\mathcal{O}$-submodules of K described in Fact 143. From now on we fix a complete family $\mathcal{F} = \mathcal{I}' \backslash \{0, K\}$.

We refer to these sorts as the *stabilizer sorts* and we denote their union

$$\mathcal{G} = K \cup k \cup \Gamma \cup \{\Gamma/\Delta \mid \Delta \in RJ(\Gamma)\} \cup \{B_n(K)/\operatorname{Stab}(I_1, \dots, I_n) \mid (I_1, \dots, I_n) \in \mathcal{I}^n\}$$

Remark 201. *If we work with the complete family $\mathcal{I}$ of end-segments given by Remark 138, each of $\mathcal{O}$-modules in $\mathcal{I}$ is definable over the empty set. In this setting we are adding a finite set of constants Ω_n in Γ choosing representatives of $n\Gamma$ in Γ for each $n \in \mathbb{N}$. The results we obtain in this section will hold in the same manner if we work with this language instead.*

Our main goal is the following Theorem.

Theorem 202. *Let K be a henselian valued field of equicharacteristic zero, residue field algebraically closed and dp-minimal value group. Then K eliminates imaginaries in the language $\hat{\mathcal{L}}$, where the stabilizer sorts are added.*

Definition 203. *We say that a multi-sorted first order theory T* codes finite sets *if for every model $M \vDash T$, and every finite subset $S \subseteq M$, the code $\ulcorner S \urcorner$ is interdefinable with a tuple of elements in M.*

The following is a folklore fact (see for example [Poi83]).

Fact 204. *Let T be a complete multi-sorted theory. If T has weak elimination of imaginaries and codes finite sets then T eliminates imaginaries.*

In view of Theorem 200 and Fact 204 it is only left to show that any finite set can be coded in $\mathcal{G}$.

Definition 205.
1. *An equivalence relation E on a set X is said to be* proper *if it has at least two different equivalence classes. It is said to be* trivial *if for any $x, y \in X$ we have $E(x, y)$ if and only if $x = y$.*
2. *A finite set F is* primitive over A *if there is no proper non-trivial $(\ulcorner F \urcorner \cup A)$-definable equivalence relation on F. If F is primitive over $\emptyset$ we just say that it is* primitive.

To code finite sets we need numerous smaller results. This section is organized as follows:

1. In the first subsection we analyze the stable and stably embedded multi-sorted structure $VS_{k,C}$, consisting of the k-vector spaces $\operatorname{red}(s)$, where s is some $\mathcal{O}$-lattice definable over C, an arbitrary imaginary set of parameters. This structure has elimination of imaginaries by results of Hurshovski in [Hru12].
2. In the second subsection we introduce the notion of germ of a definable function f over a definable type p. We prove that germs can be coded in the stabilizer sorts.

3. In the third subsection later we show that the code of any $\mathcal{O}$-submodule $M \subseteq K^n$ is interdefinable with the code of its projection to the last coordinate and the germ of the function describing each of the fibers. We show that the same statement holds for torsors.

4. In the forth subsection we prove several results on coding finite sets in the one-dimensional case, e.g. if F is a primitive finite set of 1-torsors then it can be coded in $\mathcal{G}$.

5. in the fifth subsection we carry a simultaneous induction to prove that any finite set $F \subseteq \mathcal{G}^r$ can be coded in the stabilizer sorts, and any definable function $f : F \to \mathcal{G}$ admits a code in the stabilizer sorts.

6. In the sixth we state the result on full elimination of imaginaries down to the stabilizer sorts.

The multi-sorted structure of k-vector spaces

By Corollary 109 the residue field k is stably embedded and it is a strongly minimal structure, because it is an algebraically closed field. This enables us to construct, over any imaginary base set of parameters C, a part of the structure that naturally inherits stability-theoretic properties from the residue field. Given a $\mathcal{O}$-lattice $s \subseteq K^n$ we have $\mathrm{red}(s) = s/\mathcal{M}s$ is a k-vector space.

Definition 206. *For any imaginary set of parameters C, we let $VS_{k,C}$ be the many-sorted structure whose sorts are the k vector spaces* $\mathrm{red}(s)$ *where $s \subseteq K^n$ is an $\mathcal{O}$-lattice of rank n definable over C. Each sort* $\mathrm{red}(s)$ *is equipped with its k-vector space structure. In addition, $VS_{k,C}$ has any C-definable relation on products of the sorts.*

Definition 207. *A definable set D is said to be* internal to the residue field *if there is a finite set of parameters $F \subseteq \mathcal{G}$ such that $D \subseteq \mathrm{dcl}^{eq}(kF)$.*

Each of the structures $\mathrm{red}(s)$ is internal to the residue field, and the parameters needed to witness the internality lie in $\mathrm{red}(s)$, so in particular each of the k-vector spaces $\mathrm{red}(s)$ is stably embedded. The entire multi-sorted structure $VS_{k,C}$ is also stably embedded and stable, and in this subsection we will prove that it eliminates imaginaries.

Notation 208. *We recall that given an $\mathcal{O}$-submodule M of K, we write S_M to denote the end-segment induced by M, i.e.$\{v(x) \mid x \in M\}$.*

We recall some definitions from [Hru12] to show that $VS_{k,C}$ eliminates imaginaries.

Definition 209. *Let t be a theory of fields (possibly with additional structure). A t-*linear structure *$\mathcal{A}$ is a structure with a sort k for a model of t, and addional sorts $(V_i \mid i \in I)$ denoting finite-dimensional vector spaces. Each V_i has (at least) a k-vector space structure, and $dim V_i < \infty$. We assume that:*

1. *k is stably embedded,*

2. *the induced structure on k is precisely given by t,*

3. *The V_i are closed under tensor products and duals.*

Moreover, we say it is flagged *if for any finite dimensional vector space V there is a flitration $V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = V$ by subspaces, with $dimV_i = i$ and V_i is one of the distinguished sorts.*

The following is [Hru12, Lemma 5.2].

Lemma 210. *If k is an algebraically closed field and $\mathcal{A}$ is a flagged k-linear structure, then $\mathcal{A}$ admits elimination of imaginaries.*

Notation 211. *Let A be an $\mathcal{O}$-module. Let $\mathcal{M}A = \{xa \mid x \in \mathcal{M},\ a \in A\}$ we denote as* $\mathrm{red}(A)$ *the quotient $\mathcal{O}$-module $A/\mathcal{M}A$.*

We observe that $\mathrm{red}(A) = A/\mathcal{M}A$ is canonically isomorphic to $A \otimes_{\mathcal{O}} k$.

Fact 212. *Let $A \subseteq K^n$ and $B \subseteq K^m$ be $\mathcal{O}$-lattices. Then* $\mathrm{red}(A)\otimes_k\mathrm{red}(B)$ *can be canonically identified with* $\mathrm{red}(A \otimes_{\mathcal{O}} B)$.

Proof. This is a straightforward computation and it is left to the reader. $\square$

Remark 213. *Given $A \subseteq K^n$ and $B \subseteq K^m$ $\mathcal{O}$-lattices, there is some $\mathcal{O}$-lattice $C \subseteq K^{mn}$ such that $A \otimes_{\mathcal{O}} B$ is canonically identified with C. This isomorphism induces as well a one to one correspondence between* $\mathrm{red}(A \otimes_{\mathcal{O}} B)$ *and* $\mathrm{red}(C)$.

Proof. Given K^n and K^m two vector spaces, the tensor product $K^n \otimes K^m$ is a K vector space whose basis is $\{e_i \otimes e_j \mid i \leq n, j \leq m\}$ and it is canonically identified with K^{nm}, via a linear map ϕ that extends the bijection between the basis sending $e_i \otimes e_j$ to e_{ij}. Given $A \subseteq K^n$ and $B \subseteq K^m$ $\mathcal{O}$-lattices, then $A \otimes_{\mathcal{O}} B$ is an $\mathcal{O}$-lattice of $K^n \otimes K^m$ and we denote as $C = \phi(A \otimes_{\mathcal{O}} B)$. This map induces as well an identification between $\mathrm{red}(A \otimes_{\mathcal{O}} B)$ and $\mathrm{red}(C)$ such that the following map commutes:

$$\begin{tikzcd} A \otimes_{\mathcal{O}} B \arrow[r, "\phi"] \arrow[d, "\mathrm{red}"] & C \arrow[d, "\mathrm{red}"] \\ \mathrm{red}(A \otimes_{\mathcal{O}} B) \arrow[r, "i"] & \mathrm{red}(C) \end{tikzcd}$$

$\square$

Fact 214. *Let $A \subseteq K^n$ and $B \subseteq K^m$ be $\mathcal{O}$-lattices. Then there is an isomorphism*

$$\phi : \mathrm{red}(Hom_{\mathcal{O}}(A, B)) \to Hom_k(\mathrm{red}(A), \mathrm{red}(B)),$$

where for any $f \in Hom_{\mathcal{O}}(A, B)$ and $a \in A$:

$$\phi(f+\mathcal{M}Hom_{\mathcal{O}}(A,B)):\begin{cases}\mathrm{red}(A) & \to \mathrm{red}(B)\\ a+\mathcal{M}A & \mapsto f(a)+\mathcal{M}B.\end{cases}$$

Proof. This is a straightforward computation and it is left to the reader. □

Remark 215. *Given an $\mathcal{O}$-lattice $A \subseteq K^n$, then $Hom_{\mathcal{O}}(A,\mathcal{O})$ can be canonically identified with some $\mathcal{O}$-module C of K^n. So there is a correspondence between* $\mathrm{red}(Hom_{\mathcal{O}}(A,\mathcal{O}))$ *and* $\mathrm{red}(C)$.

Proof. Let A be an $\mathcal{O}$-lattice of K^n. By linear algebra K^n can be identified with its dual space $(K^n)^*$. Let

$$A^* = \{T \in (K^n)^* \mid \text{for all } \ a \in A,\ T(a) \in \mathcal{O}\}.$$

A^* is canonically identified with $Hom_{\mathcal{O}}(A,\mathcal{O})$ via the map that sends a transformation T to $T\restriction_A$. Also A^* is isomorphic to some $\mathcal{O}$-lattice C of K^n, as there is a canonical isomorphism between K^n and its dual space. So we have a definable $\mathcal{O}$-isomorphism ϕ between $Hom_{\mathcal{O}}(A,\mathcal{O})$ and C, and this correspondence induces an identification $\hat{\phi}$ between $\mathrm{red}(Hom_{\mathcal{O}}(A,O))$ and $\mathrm{red}(C)$ making the following diagram commute:

$$\begin{tikzcd} Hom_{\mathcal{O}}(A,\mathcal{O}) \arrow[r, "\phi"] \arrow[d, "\mathrm{red}"] & C \arrow[d, "\mathrm{red}"] \\ \mathrm{red}(Hom_{\mathcal{O}}(A,\mathcal{O})) \arrow[r, "\bar{\phi}"] & \mathrm{red}(C) \end{tikzcd}$$

□

Remark 216. *Let $A \subseteq K^n$ be an $\mathcal{O}$-lattice. There is a sequence of $\mathcal{O}$-lattices $< A_i \mid i \leq n >$ such that $< \mathrm{red}(A_i) \mid i \leq n >$ is a flag of* $\mathrm{red}(A)$ *and for each $i \leq n$, $\ulcorner A_i \urcorner \in \mathrm{dcl}^{eq}(\ulcorner A \urcorner)$.*

Proof. We proceed by induction on n, the base case is trivial. Let $A \subseteq K^{n+1}$, and $\pi_{n+1} : K^{n+1} \to K$ be the projection into the last coordinate. Let $B \subseteq K^n$ be the $\mathcal{O}$-lattice such that

$$\ker(\pi_{n+1}) = B \times \{0\} = A \cap (K^n \times \{0\}).$$

We observe that $\ulcorner B \urcorner, \ulcorner \pi_{n+1}(A) \urcorner \in \mathrm{dcl}^{eq}(\ulcorner A \urcorner)$. By Corollary 168 B is a direct summand of A, so we have the exact splitting sequence

$$0 \to B \to A \to \pi_{n+1}(A) \to 0.$$

Consequently,

$$0 \to \mathcal{M}B \to \mathcal{M}A \to \mathcal{M}\pi_{n+1}(A) \to 0 \text{ and}$$
$$0 \to \mathrm{red}(B) \to \mathrm{red}(A) \to \mathrm{red}(\pi_{n+1}(A)) \to 0$$

are exact sequences that split. By the induction hypothesis, there is a sequence $\{0\} \leq A_1 \leq \cdots \leq A_n = B$ such that $< \text{red}(A_i) \mid i \leq n >$ is a flag of $\text{red}(B)$, $dim(\text{red}(A_i)) = i$ and $\ulcorner A_i \urcorner \in \text{dcl}^{eq}(\ulcorner B \urcorner) \subseteq \text{dcl}^{eq}(\ulcorner A \urcorner)$. Let $A_{n+1} = A$, the sequence $< A_i \mid i \leq n+1 >$ satisfies the required conditions. $\square$

Theorem 217. *Let $C \subseteq K^{eq}$, then $VS_{k,C}$ has elimination of imaginaries.*

Proof. The sorts $\text{red}(s)$ where s is a $\mathcal{O}$-lattice of K^n and $\text{dcl}^{eq}(C)$-definable form the multi-sorted structure $\text{VS}_{k,C}$. Each $\text{red}(s)$ carries a k-vector space structure. $VS_{k,C}$ is closed under tensor operation by Remark 213 and Fact 212. It is closed under duals by Remark 215 and Fact 214. By Remark 216 each sort $\text{red}(s)$ where s is an $\mathcal{O}$-lattice admits a complete filtration by C-definable vector spaces. Therefore, $\text{VS}_{k,C}$ is a flagged k-linear structure, so the statement is immediate consequence of 210. $\square$

Germs of functions

In this subsection we show how to code the germ of a definable function f over a definable type $p(\mathbf{x})$ in the stabilizer sorts.

Definition 218. *Let T be a complete first order theory and $M \vDash T$. Let $B \subseteq M$ and p be a B-definable type whose solution set is P. Let f be an M-definable function whose domain contains P. Suppose that $f = f_c$ is defined by the formula $\phi(x, y, c)$ (so $f_c(x) = y$). We say that f_c and $f_{c'}$* have the same germ on P *if the formula $f_c(x) = f_{c'}(x)$ lies in p. By the definability of p the equivalence relation $E_\phi(c, c')$ that states f_c and $f_{c'}$ have the same germ on P is definable over B.The* germ of f_c on P *is defined to be the class of c under the equivalence relation $E_\phi(y, z)$, which is an element in M^{eq}. We write $germ(f, p)$ to denote the code for this equivalence class.*

Definition 219. *Let p be a global type definable over B and let C a set of parameters. We say that a realization a of p is sufficiently generic over BC if $a \vDash p \restriction_{BC}$.*

We start proving some results that will be required to show how to code the germs of a definable function f over a definable type p in the stabilizer sorts.
Let $U \subseteq K$ be a 1-torsor, we recall Definition 189, where we defined the $\ulcorner U \urcorner$-definable partial type.

$$\Sigma_U^{gen}(x) = \{x \in U\} \cup \{x \notin B \mid B \subsetneq U \text{ is a proper subtorsor of } U\}.$$

We refer to this type as *the generic type of U*.
When considering complete extensions of $\Sigma_U^{gen}(x)$ one finds an important distinction between the closed and the open case. In Proposition 190 we proved that whenever U is a closed 1-torsor then $\Sigma_U^{gen}(x)$ admits a unique complete extension. The open case inherits a higher level of complexity.

Proposition 220. *Let U be an open 1-torsor, then any completion of the generic type of U is $\ulcorner U \urcorner$-definable.*

Proof. Let $c \vDash \Sigma_U^{gen}(x)$, it is sufficient to prove that $p(x) = \text{tp}(c/\mathfrak{M})$ is $\ulcorner U \urcorner$-definable. Let $\Delta \in RJ(\Gamma)$, $\ell \in \mathbb{N}$, $\beta \in \Gamma$, $k \in \mathbb{Z}$. First, we observe that for any $a, a' \in U(\mathfrak{M})$ we have that $v(c-a) = v(c-a')$, because c realizes the generic type of U. In particular,

$$v_\Delta(x-a)-\rho_\Delta(\beta)+k^\Delta \in \ell(\Gamma/\Delta) \in p(x) \text{ if and only if} v_\Delta(x-a')-\rho_\Delta(\beta)+k^\Delta \in \ell(\Gamma/\Delta) \in p(x).$$

Pick some element $\delta \in \Gamma$ such that $\rho_\Delta(\delta) = k^\Delta$, and let $\mu = \pi_\Delta^\ell(v(c-a)+\delta) \in \text{dcl}^{eq}(\emptyset)$. Then, for any $a \in U(\mathfrak{M})$ we have:

$$v_\Delta(x-a) - \rho_\Delta(\beta) + k^\Delta \in \ell(\Gamma/\Delta) \in p(x) \text{ if and only if } \pi_\Delta^\ell(\beta) = \mu.$$

If $a \notin U(\mathfrak{M})$, then $v(c-a) = v(b-a)$ for any $b \in U(\mathfrak{M})$. Hence:

$$v_\Delta(x-a) - \rho_\Delta(\beta) + k^\Delta \in \ell(\Gamma/\Delta) \in p(x) \text{ if and only if}$$
$$\underbrace{\exists b \in U\big(v_\Delta(b-a) - \rho_\Delta(\beta) + k^\Delta \in \ell(\Gamma/\Delta)\big)}_{\phi(a,\beta)}.$$

Let

$$\psi(a,\beta) := \big(a \in U \wedge \pi_\Delta^\ell(\beta) = \mu\big) \vee \big(a \notin U \wedge \phi(a,\beta)\big).$$

Hence,

$$v_\Delta(x-a) - \rho_\Delta(\beta) + k^\Delta \in \ell(\Gamma/\Delta) \in p(x) \text{ if and only if } \psi(a,\beta).$$

Note that $\ulcorner \psi(x,z) \urcorner \in \text{dcl}^{eq}(\ulcorner U \urcorner)$.
We continue showing a $\text{dcl}^{eq}(\ulcorner U \urcorner)$-definable scheme for the coset formulas. Let $a \in U(\mathfrak{M})$ and consider the definable end-segment $S_a = \{v(x-a) \mid x \in U\}$. For any $a \neq a'$, $S_a = S_{a'}$, thus we write S to denote this set. Note that $\ulcorner S \urcorner \in \text{dcl}^{eq}(\ulcorner U \urcorner)$ and let $\Delta \in RJ(\Gamma)$. We recall that we denote by S_Δ the set $\rho_\Delta(S)$, which is a definable end-segment in Γ/Δ. If S_Δ has a minimum element $\gamma \in \text{dcl}^{eq}(S_\Delta) \subseteq \text{dcl}^{eq}(\ulcorner S \urcorner)$ then for any $a \in U(\mathfrak{M})$ we have:

$$v_\Delta(x-a) = \rho_\Delta(\beta) + k^\Delta \in p(x) \text{ if and only if } \rho_\Delta(\beta) + k^\Delta = \gamma.$$

If S_Δ does not have a minimum, then for any $a \in U(\mathfrak{M})$,

$$v_\Delta(x-a) = \rho_\Delta(\beta) + k^\Delta \in p(x) \text{ if and only if } \beta \neq \beta.$$

Finally, for $a \notin U(\mathfrak{M})$ we have that $v(c-a) = v(b-a)$ for any $b \in U(\mathfrak{M})$, therefore for $\Delta \in RJ(\Gamma)$ and $k \in \mathbb{Z}$ we have that:

$$v_\Delta(x-a) = \rho_\Delta(\beta) + k^\Delta \in p(x) \text{ if and only if } \exists b \in U\big(v_\Delta(b-a) = \rho_\Delta(\beta) + k^\Delta\big).$$

Consequently, for each quantifier free formula $\phi(x,y)$, we have shown the existence of a formula $d\phi(y)$ such that $\ulcorner d\phi(y) \urcorner \in \text{dcl}^{eq}(\ulcorner U \urcorner)$ and $\phi(x,b) \in p(x)$ if and only if $\vDash d\phi(b)$. By quantifier elimination (see Corollary 139), the type $p(x)$ is completely determined by the quantifer free formulas, we conclude that $p(x)$ is $\ulcorner U \urcorner$-definable. $\square$

Corollary 221. *Let U be a definable 1-torsor, then each completion $p(x)$ of $\Sigma_U^{gen}(x)$ is $\ulcorner U \urcorner$-definable.*

Proof. This follows immediately by combining Proposition 190 and Proposition 220. □

Proposition 222. *Let $M \subseteq \mathfrak{M}$ be a definable $\mathcal{O}$-module and let $p(x)$ be a global type containing the generic type of M. Then $p(x)$ is stabilized additively by $M(\mathfrak{M})$ i.e. if c is a realization of $p(x)$ and $a \in M(\mathfrak{M})$ then $a + c$ is a realization of $p(x)$.*

Proof. Let c be a realization of the type $p(x)$, $a \in M(\mathfrak{M})$ and $d = c + a$. As $\Sigma_M^{gen}(x) \subseteq p(x)$, $c \in A$ and $c \notin U$ for any proper subtorsor $U \subseteq A$. First we argue that $d \vDash \Sigma_M^{gen}(x)$. Because A is a $\mathcal{O}$-submodule of K (in particular closed under addition) we have $d \in A$. And if there is a subtorsor $U \subsetneq A$ such that $d \in U$, then $c \in -a + U \subsetneq A$ contradicting that $c \vDash \Sigma_A^{gen}(x)$. For any $\Delta \in RJ(\Gamma)$, element $z \in M(\mathfrak{M})$ and realization $b \vDash \Sigma_A^{gen}(x)$ we have $v_\Delta(z - b) = v_\Delta(b)$. Thus for any $n \in \mathbb{N}$ and $\beta \in \Gamma/\Delta$:

$$v_\Delta(b - z) - \beta \in n(\Gamma/\Delta) \text{ if and only if } v_\Delta(b) - \beta \in n(\Gamma/\Delta).$$

We conclude that d and c must satisfy the same congruence and coset formulas, because c is a realization of the generic type of M, $a \in M(\mathfrak{M})$ and $v_\Delta(d) = v_\Delta(c + a) = v_\Delta(c - (-a)) = v_\Delta(c)$. □

Corollary 223. *Let $M \subseteq \mathfrak{M}$ be a definable $\mathcal{O}$-module. Let $p(x)$ be a global type containing the generic type of M. Let $a \in M(\mathfrak{M})$, then a is the difference of two realizations of $p(x)$ i.e. we can find $c, d \vDash p(x)$ such that $a = c - d$.*

Proof. Let c be a realization of $p(x)$ and fix $a \in M(\mathfrak{M})$. By Proposition 222, $d = c - a$ is also a realization of $p(x)$. The statement now follows because $a = c - d$. □

Proposition 224. *Let $(I_1, \ldots, I_n) \in \mathcal{I}^n$, for every $\mathcal{O}$-module M of type $(I_1, \ldots, I_n)$ we can find a type $p_M(\bar{x}_1, \ldots, \bar{x}_n) \in S_{n\times n}(K)$ such that:*

1. *$p_M(\mathbf{x})$ is definable over $\ulcorner M \urcorner$,*
2. *A realization of $p_M(\mathbf{x})$ is a matrix representation of M. This is if $(\bar{d}_1, \ldots, \bar{d}_n) \vDash p_M(\mathbf{x})$ then $[\bar{d}_1, \ldots, \bar{d}_n]$ is a representation matrix for M.*

Proof. Let $\mathfrak{M}$ be the monster model.

Step 1: We define a partial type $\Sigma_{(I_1,\ldots,I_n)}$ satisfying condition i) and ii) for the canonical module $C_{(I_1,\ldots,I_n)}$. Such a type is left-invariant under the action of $\mathrm{Stab}(I_1, \ldots, I_n)$.

We consider the set $\mathcal{J} = \{(i, j) \mid 1 \leq i, j \leq n\}$, and we equip it with a linear order defined as:

$$(i, j) < (i', j') \text{ if and only if } j < j' \text{ or } j = j' \wedge i' > i.$$

And we consider an enumeration of $\mathcal{J} = \{v_1, \ldots, v_{n^2}\}$ such that $v_1 < v_2 < \cdots < v_{n^2}$. By Proposition 188

$$\begin{aligned} \mathrm{Stab}_{(I_1,\ldots,I_n)} = \{((a_{i,j})_{1\le i,j\le n} \in B_n(K) \mid a_{ii} \in \mathcal{O}^{\times}_{\Delta_{S_{I_i}}} \\ \wedge\, a_{ij} \in Col(I_i, I_j) \text{ for each } 1 \le i < j \le n \}. \end{aligned}$$

Hence, for each $1 \le m \le n^2$ let:

$$p_{v_m}(x) = \begin{cases} tp(0) & \text{if } v_m = (i,j) \text{ where } 1 \le j < i \le n, \\ \Sigma^{gen}_{\mathcal{O}_{\Delta_{S_{I_i}}}}(x) & \text{if } v_m = (i,i) \text{ for some } 1 \le i \le n, \\ \Sigma^{gen}_{Col(I_i,I_j)}(x) & \text{if } v_m = (i,j) \text{ where } 1 \le i < j \le n. \end{cases}$$

Consider the partial definable type $\Sigma_{C_{(I_1,\ldots,I_n)}} = p_{v_{n^2}} \otimes \cdots \otimes p_{v_1}$. Given a realization of this type $(b_{v_{n^2}}, \ldots, b_{v_1}) \models \Sigma_{C_{(I_1,\ldots,I_n)}}$ let

$$B = \begin{bmatrix} b_{v_n} & b_{v_{2n}} & \ldots & b_{v_{n^2}} \\ \vdots & \vdots & \vdots & \vdots \\ b_{v_1} & b_{v_{n+1}} & \ldots & b_{v_{n(n-1)+1}} \end{bmatrix}$$

By construction B is an upper triangular matrix such that $(B)_{i,j} \in Col(I_i, I_j)$ for $1 \le i < j \le n$ and $(B)_{ii} \in \mathcal{O}^{\times}_{\Delta_{S_{I_i}}}$, thus its column vectors constitute a basis for the canonical module. To check left invariance, it is sufficient to take $A \in \mathrm{Stab}_{(I_1,\ldots,I_n)}(\mathfrak{M})$ and argue that for each $1 \le m \le n^2$ the element $(AB)_{v_m}$ is a realization of generic type p_{v_m} sufficiently generic over $\mathfrak{M} \cup \{(AB)_{v_k} \mid k < m\}$. Suppose that $v_m = (i,j)$, then $(AB)_{v_m} = (AB)_{ij} = \displaystyle\sum_{k=i}^{j} a_{ik} b_{kj} = a_{ii} b_{ij} + \cdots + a_{ij} b_{jj}$.

In the fixed enumeration we guarantee that b_{ij} is chosen sufficiently generic over $\mathfrak{M} \cup \{b_{kj} \mid i < k \le j\}$. For each $i < k \le j$, $a_{ik} b_{kj} \in Col(I_i, I_j)$, thus we have that $v(a_{ii} b_{ij}) = v((AB)_{ij})$. Consequently, $(AB)_{ij}$ is a realization of p_{v_m} generic over $\mathfrak{M}$ together with all the elements b_{kl} where (k,l) appears earlier in the enumeration than $(i.j)$.

Step 2: For any $\mathcal{O}$-module $M \subseteq K^n$ of type $(I_1, \ldots, I_n)$ there is an $\ulcorner M \urcorner$-definable type p_M, such that any realization of p_M is a representation matrix for M.

Let $T = \mathfrak{M}^n \to \mathfrak{M}^n$ be a linear transformation whose representation matrix is upper triangular and T sends the canonical module $C_{(I_1,\ldots,I_n)}$ to M. And let $\Sigma_M = T(\Sigma_{C_{(I_1,\ldots,I_n)}})$, its definition is independent from the choice of T, because given two linear transformations with upper triangular representation matrices $[T]$ and $[T']$ which send the canonical $\mathcal{O}$-module of type $(I_1, \ldots, I_n)$ to M, we have that $[T']^{-1}[T] \in \mathrm{Stab}_{(I_1,\ldots,I_n)}$ and the type $\Sigma_{C_{(I_1,\ldots,I_n)}}$ is left invariant under the action of such group. Thus, Σ_M is $\ulcorner M \urcorner$-definable and given $B \models \Sigma_M$, the type $\mathrm{tp}(B/\mathfrak{M})$ is still $\ulcorner M \urcorner$-definable by Corollary 221. □

Theorem 225. *Let X be a definable subset of K^n and let $p(\mathbf{x}) \vdash \mathbf{x} \in X$ be a global type definable over $\ulcorner X \urcorner$. Let $f = X \to \mathcal{G}$ be a definable function. Then the p-germ of f is coded in $\mathcal{G}$ over $\ulcorner X \urcorner$.*

Proof. We first assume that $f : X \to B_n(K)/\operatorname{Stab}_{(I_1,\dots,I_n)}$. Let

$$B = \operatorname{dcl}_{\mathcal{G}}(\operatorname{germ}(f,p), \ulcorner X \urcorner) = \operatorname{dcl}^{eq}(\operatorname{germ}(f,p), \ulcorner X \urcorner) \cap \mathcal{G}.$$

Suppose that f is c-definable, and let $q = \operatorname{tp}(c/B)$ and Q be its set of realizations. Fix some $c' \in Q$. We denote by f' the function obtained by replacing the parameter c by c' in the formula defining f. Let M be a small model containing Bcc'.

Step 1: For any realization $\mathbf{a} \vDash p(\mathbf{x}) \upharpoonright_M$ we have $f(\mathbf{a}) = f'(\mathbf{a})$.
Let $\mathbf{a}$ be a realization of $p(\mathbf{x}) \upharpoonright_M$. Let $u_{f(\mathbf{a})}(\mathbf{y})$ be the definable type over $\ulcorner f(\mathbf{a}) \urcorner$ given by Proposition 224. Given any realization $\mathbf{d} = (\bar{d}_1, \dots, \bar{d}_n) \vDash u_{f(\mathbf{a})}(\mathbf{y})$, $[\bar{d}_1, \dots, \bar{d}_n]$ is a representation matrix for the module $f(\mathbf{a})$. In particular, $f(\mathbf{a}) = \rho_{(I_1,\dots,I_k)}(\bar{d}_1, \dots, \bar{d}_k)) \in \operatorname{dcl}^{eq}(\mathbf{d})$. Let $\mathbf{d}$ be a realization of $u_{f(\mathbf{a})}(\mathbf{y}) \upharpoonright_M$ and let $r(\mathbf{x}, \mathbf{y}) = \operatorname{tp}(\mathbf{a}, \mathbf{d}/M)$, then the type $r(\mathbf{x}, \mathbf{y})$ is B-definable, and therefore B-invariant.

Because $\operatorname{tp}(c/B) = \operatorname{tp}(c'/B)$ we can find an automorphism $\sigma \in Aut(\mathfrak{M}/B)$ sending c to c'. Then $u_{f'(\mathbf{a})} = \sigma(u_{f(\mathbf{a})})$, which is a definable type over $\ulcorner f'(\mathbf{a}) \urcorner$. Let $\mathbf{d}'$ be a realization of $u_{f'(\mathbf{a})} \upharpoonright_M$. Let $r'(\mathbf{x}, \mathbf{y}) = \operatorname{tp}(\mathbf{a}, \mathbf{d}'/M)$, then $\sigma(r(\mathbf{x}, \mathbf{y})) = r'(\mathbf{x}, \mathbf{y}) = r(\mathbf{x}, \mathbf{y})$ by the B-invariance of $r(\mathbf{x}, \mathbf{y})$, so $\operatorname{tp}(\mathbf{a}, \mathbf{d}/M) = \operatorname{tp}(\mathbf{a}, \mathbf{d}'/M)$. Since $f(\mathbf{a}) \in \operatorname{dcl}^{eq}(\mathbf{d})$ and $f'(\mathbf{a}) \in \operatorname{dcl}^{eq}(\mathbf{d}')$, we must have that $\operatorname{tp}(\mathbf{a}, f(\mathbf{a})/M) = \operatorname{tp}(\mathbf{a}, f'(\mathbf{a})/M)$ and since f and f' are both definable over M this implies that $f(\mathbf{a}) = f'(\mathbf{a})$.

Step 2: The $\operatorname{germ}(f,p)$ *is coded in the stabilizer sorts $\mathcal{G}$ over $\ulcorner X \urcorner$.*
Firstly, note that for any $\mathbf{a} \vDash p(\mathbf{x}) \upharpoonright_{Bcc'}$ it is the case that $f(\mathbf{a}) = f'(\mathbf{a})$. In fact, by Step 1 $f(x) = f'(x) \in \operatorname{tp}(a/M)$ and $f(x) = f'(x)$ is a formula in $\operatorname{tp}(a/Bcc')$. Then f and f' both have the same p-germ. Since $p(\mathbf{x})$ is definable over $B = \operatorname{dcl}_{\mathcal{G}}(B)$ the equivalence relation E stating that f and f' have both the same p-germ is B-definable. Since for any realization $\mathbf{a} \vDash p(\mathbf{x}) \upharpoonright_{Bcc'}$ it is the case that $f(\mathbf{a}) = f'(\mathbf{a})$, the class $E(x,c)$ is B-invariant, therefore $\operatorname{germ}(f,p)$ is definable over $B = \operatorname{dcl}_{\mathcal{G}}(B)$.

We continue arguing that the statement for $f = X \to B_n(K)/\operatorname{Stab}_{(I_1,\dots,I_n)}$ is sufficient to conclude the entire result. For each $\Delta \in RJ(\Gamma)$ there is a canonical isomorphism $\Gamma/\Delta \cong K^{\times}/\mathcal{O}_{\Delta}^{\times}$, where $\mathcal{O}_{\Delta}$ is the valuation ring of the coarsened valuation v_{Δ} induced by Δ. The functions whose image lie in Γ/Δ are being considered in the previous case, because $\operatorname{Stab}(\mathcal{O}_{\Delta}) = \mathcal{O}_{\Delta}^{\times}$. By Proposition 173 any definable function $f = X \to k = \mathcal{O}/\mathcal{M}$ can be seen as a function whose image lies in $B_2(K)/\operatorname{Stab}_{(\mathcal{M},\mathcal{O})}$.
It is only left to consider the case where the target set is K. The proof follows in a very similar manner as the case for $f : X \to B_n(K)/\operatorname{Stab}_{(I_1,\dots,I_n)}$. Let $\mathbf{a} \vDash p(x)$. We let $a \vDash p(x)$ and let $r(x,y) := \operatorname{tp}(\mathbf{a}, f(\mathbf{a})/M)$, this is a B-definable type by Theorem

199, in particular B-invariant. Likewise, $r'(x,y) := \text{tp}(\mathbf{a}, f'(\mathbf{a})/M)$ is B-invariant, thus $\text{tp}(\mathbf{a}, f(\mathbf{a})/M) = \text{tp}(\mathbf{a}, f'(\mathbf{a})/M)$. Since f and f' are both definable over M, this implies that $f(\mathbf{a}) = f'(\mathbf{a})$. The rest of the proof follows exactly as in the second step. □

Some useful lemmas

In this subsection we prove several lemmas that will be required to code finite sets.

Notation 226. *Let $U \subseteq K$ be a 1- torsor, $U = a + bI$ where $I \in \mathcal{I}$. Let $A = bI$, we consider the definable equivalence relation over U given by: $E(b,b')$ if and only if $b - b' \in \mathcal{M}A$. We write $red(U)$ to denote the definable quotient U/E.*
We write $p_U(x)$ to denote some type centered in U extending the generic type of U $\Sigma_U^{gen}(x)$ which is $\ulcorner U \urcorner$-definable. If U is closed such type is unique (see Proposition 190) and for the open case there are several choices for this type, but all of them are $\ulcorner U \urcorner$ -definable by Proposition 220.

Lemma 227. *Let $F = \{B_1, \ldots, B_n\}$ be a primitive finite set of 1- torsors. Let $W = \{\{x_1, \ldots, x_n\} \mid x_i \in B_i\}$, and $W^* = \{\ulcorner\{x_1, \ldots, x_n\}\urcorner \mid \{x_1, \ldots, x_n\} \in W\}$ Then there is a $\ulcorner W^* \urcorner$- definable type q concentrated on W^*. Furthermore, given b^* a realization of q sufficiently generic over a set of parameters C, if we take B the finite set coded by b^*, then if $b \in B$ is the element that belongs to B_i then b_i is a sufficiently generic realization of some type $p_{B_i}(x)$, which is $\ulcorner B_i \urcorner$-definable and extends the generic type of B_i. Lastly, the types $p_{B_i}(x)$ are compatible under the action of $Aut(\mathfrak{M}/\ulcorner F \urcorner)$ meaning that if $\sigma \in Aut(\mathfrak{M}/\ulcorner F \urcorner)$ and $\sigma(B_i) = B_j$ then $\sigma(p_{B_i}(x)) = p_{B_j}(x)$.*

Proof. We focus first on the construction of the type q, and later we show that it satisfies the required conditions. Suppose that each 1-torsor $B_i = c_i + b_i I_i$ for some $I_i \in \mathcal{I}$. By transitivity all the balls are of the same type $I \in \mathcal{I}$ and for all $1 \leq i, j \leq n$ we have that $v(b_i) = v(b_j)$. Hence, we may assume that each B_i is of the form $c_i + bI$ for some fixed $c_i, b \in K$ and $I \in \mathcal{I}$. We argue by cases:

1. *Case 1: All the 1-torsors B_i are closed.*
   For each $i \leq n$, let $p_{B_i}(x)$ be the unique $\ulcorner B_i \urcorner$-definable type given by Proposition 190. Define $r(x_1, \ldots, x_n) = p_{B_1}(x_1) \otimes \cdots \otimes p_{B_n}(x_n)$, this is $\ulcorner W^* \urcorner$-definable type. Let $(a_1, \ldots, a_n) \vDash r(x_1, \ldots, x_n)$ and let $q = \text{tp}(\ulcorner\{a_1, \ldots, a_n\}\urcorner/\mathfrak{M})$. This type is well defined independently of the choice of the order, because each type $p_{B_i}(x)$ is generically stable, thus it commutes with any definable type by [Sim15][Proposition 2.33]. The type q is $\ulcorner W^* \urcorner$-definable and centered at W^*.

2. *Case 2: All the 1-torsors B_i are open, i.e. $I \in \mathcal{I} \backslash \{\mathcal{O}\}$.*
   Let $S_{bI} = v(b) + S_I = \{v(b) + v(x) \mid x \in I\}$, this is a definable end-segment of Γ with no minimal element. Let $r(y)$ be the $\ulcorner S_{bI} \urcorner$ definable type given by Fact 137, extending the partial generic type $\Sigma_{S_{bI}}^{gen}(y)$. Fix elements $a = \{a_1, \ldots, a_n\} \in W(\mathfrak{M})$ and $\delta \vDash r(y)$, we define $C(a, \delta) = \{C_1(a), \ldots, C_n(a)\}$, where each $C_i(a)$ is the closed ball around a_i

of radius δ. For each $i \leq n$ we take $p_{C_i(a)}(x)$ the unique extension of the generic type of $C_i(a)$ given by Proposition 190, this type is $\ulcorner C_i(a) \urcorner$-definable. Let q^a_δ be the symmetrized generic type of $C_1(a) \times \cdots \times C_n(a)$, i.e. we take $\operatorname{tp}(\ulcorner \{b_1, \ldots, b_n\} \urcorner / \mathfrak{M}\delta)$ where $(b_1, \ldots, b_n)$ is a realization of the generically stable type $p_{C_1(a)} \otimes \cdots \otimes p_{C_n(a)}$. Let q^a be the definable global type satisfying that $d \vDash q^a$ if and only if there is some $\delta \vDash r(y)$ and $d \vDash q^a_\delta$.

Claim 228. *The type q^a does not depend on the choice of a.*

Proof. Let $a' = \{a'_1, \ldots, a'_n\} \in W(\mathfrak{M})$ and $\delta \vDash r(y)$. For each $i \leq n$, $a_i, a'_i \in B_i$ meaning that $a_i - a'_i \in bI$ i.e. $v(a_i - a'_i) \in S_{bI} = v(b) + S_I$ and note that $v(a_i - a'_i) \in \Gamma(\mathfrak{M})$. By construction, $\delta \in S_{bI}$ and $\delta < v(a_i - a'_i)$, thus the closed ball of radius δ concentrated on a_i is the same closed ball of radius δ concentrated on a'_i. As the set of closed balls $C(a, \delta) = C(a', \delta)$ we must have that $q^a_\delta = q^{a'}_\delta$, and since this holds for any $\delta \vDash r(y)$ we conclude that q^a does not depend on the choice of a and we simply denote it as q. □

This type q is $\ulcorner W^* \urcorner$-definable and it is centered in W^*. This finalizes the construction of the type q that we are looking for.

We continue checking that the type q that we have constructed satisfies the other properties that we want. In both cases, by construction, if b^* is a sufficiently generic realization of q over C and B is the finite set coded by b^* if we take b_i the unique element of B that lies on B_i then b_i realizes the generic type $\Sigma^{gen}_{B_i}(x)$. By Corollary 221, the type $\operatorname{tp}(b_i/\mathfrak{M})$ is $\ulcorner B_i \urcorner$-definable. If the torsors are closed, then the types $p_{B_i}(x)$ are all compatible under the action of $Aut(\mathfrak{M}/\ulcorner F \urcorner)$ as there is a unique complete extension of the generic type of B_i, this is guaranteed by Proposition 190. We now work the details for the open case, let's fix $\sigma \in Aut(\mathfrak{M}/\ulcorner F \urcorner)$ and assume that $\sigma(B_i) = B_j$. The type $r(y)$ is $\ulcorner F \urcorner$-definable, thus $\sigma(r(y)) = r(y)$. By construction, for all $k \leq n$ the type $p_{B_k}(x)$ that we are fixing is the unique extension of generic type of some closed ball $C_\delta(a_k)$ where $a_k \in B_i$ and $\delta \vDash r(y)$. And for any $a, a' \in B_i$ and $\delta, \delta' \vDash r(y)$, $C_\delta(a) = C_{\delta'}(a')$. If $\sigma(B_i) = B_j$, then $\sigma(b_i) \in B_j$ and

$$\sigma(C_\delta(b_i)) = C_{\sigma(\delta)}(\sigma(b_i)) = C_\delta(b_j).$$

By Proposition 190 there is a unique complete extension of the generic type of the closed ball $C_\delta(a_k)$ for each $k \leq n$, thus $\sigma(p_{B_i}(x)) = p_{B_j}(x)$ as desired. □

Notation 229. *Let $M \subseteq K^n$ be a non-trivial definable $\mathcal{O}$-module and let $Z = \bar{d} + M$ be a torsor. Let $\pi_n = K^n \to K$ be the projection into the last coordinate. Consider the function that describes the fiber in Z of each element at the projection, this is $h_Z(x) = \{y \in K^{n-1} \mid (y, x) \in Z\}$.*

Fact 230. *Let M be a $\mathcal{O}$-submodule of K^n. Then for any $x, z \in \pi_n(M)$ we have that*

$$h_M(x) + h_M(y) = h_M(x + y).$$

Furthermore, if $Z = \bar{b} + M \in K^n/M$ *is a torsor, then for any* $d_1, d_2 \in \pi_n(Z)$ *we have that* $d_1 - d_2 \in \pi_n(N)$ *and:*

$$h_N(d_1 - d_2) = h_Z(d_1) - h_Z(d_2).$$

Proof. This is a straightforward computation and it is left to the reader. □

Lemma 231. *Let* $n \geq 2$ *be a natural number and* $M \subseteq K^n$ *be a definable* $\mathcal{O}$*-submodule. Then* $\ulcorner M \urcorner$ *is interdefinable with* $(\ulcorner \pi_n(M) \urcorner, \mathrm{germ}(h_M, p_{\pi_n(M)}))$*, where* $p_{\pi_n}(M)$ *is any complete extension of the generic type of* $\pi_n(M)$*.*

Proof. Let M_1 and M_2 be $\mathcal{O}$-modules of the same type. Suppose that $A = \pi_n(M_1) = \pi_n(M_2)$, and $\mathrm{germ}(h_{M_1}, p_A) = \mathrm{germ}(h_{M_2}, p_A)$. We must show that M_1 and M_2 are the same $\mathcal{O}$-module.

Let c be a realization of the type $p_A(x)$ sufficiently generic over $\ulcorner M_1 \urcorner \ulcorner M_2 \urcorner$ and $d = c - y$. By Proposition 222 d is a realization of $p_A(x)$ sufficiently generic over $\ulcorner M_1 \urcorner \ulcorner M_2 \urcorner$, and $y = c - d$. As $\mathrm{germ}(h_{M_1}, p_A) = \mathrm{germ}(h_{M_2}, p_A)$, we have that $h_{M_1}(c) = h_{M_2}(c)$ and $h_{M_1}(d) = h_{M_2}(d)$. By Fact 230, $h_{M_1}(y) = h_{M_1}(c) - h_{M_1}(d) = h_{M_2}(c) - h_{M_2}(d) = h_{M_2}(y)$. Consequently, $M_1 = M_2$ as desired. □

Corollary 232. *Let* $n \geq 2$ *be a natural number and* $N \subseteq K^n$ *be a definable* $\mathcal{O}$*-submodule. Let* $Z = \bar{b} + N$ *be a torsor, then* $\ulcorner Z \urcorner$ *is interdefinable with* $(\ulcorner \pi_n(Z) \urcorner, \mathrm{germ}(h_Z, p_{\pi_n(Z)}))$*, where* $p_{\pi_n}(Z)$ *is any global type containing the generic type of* $\pi_n(Z)$*.*

Proof. We first show that $\ulcorner Z \urcorner$ is interdefinable with $\big(\ulcorner \pi_n(Z) \urcorner, \ulcorner N \urcorner, \mathrm{germ}(h_Z, p_{\pi_n(Z)})\big)$. Let $Z_1 = \bar{b}_1 + N$ and $Z_2 = \bar{b}_2 + N$ torsors, and suppose that $A = \pi_n(Z_1) = \pi_n(Z_2)$. Let c be a realization of the type $p_A(x)$ sufficiently generic over $\ulcorner Z_1 \urcorner \ulcorner Z_2 \urcorner$, then $h_{Z_1}(c) = h_{Z_2}(c)$. If $Z_1 \neq Z_2$, then they must be disjoint because they are different cosets of N. But if $h_{Z_1}(c) = h_{Z_2}(c)$ then $Z_1 \cap Z_2 \neq \emptyset$, so $Z_1 = Z_2$.
We continue showing that N is definable over $(\ulcorner \pi_n(Z) \urcorner, \mathrm{germ}(h_Z, p_{\pi_n(Z)}))$. We will find a global type $p_{\pi_n(N)}(x)$ extending the generic type of $\pi_n(N)$ such that

$$(\ulcorner \pi_n(N) \urcorner, \mathrm{germ}(h_N, p_{\pi_n(N)})) \in \mathrm{dcl}^{eq}(\ulcorner \pi_n(Z) \urcorner, \mathrm{germ}(h_Z, p_{\pi_n(Z)})).$$

By Lemma 231, this guarantees that $\ulcorner N \urcorner \in \mathrm{dcl}^{eq}(\ulcorner \pi_n(Z) \urcorner, \mathrm{germ}(h_Z, p_{\pi_n(Z)}))$.
First, let $y' \in \pi_n(Z)$, then $\pi_n(N) = \{y - y' \mid y \in \pi_n(Z)\}$. As this definition is independent from the choice of y', we have $\ulcorner \pi_n(N) \urcorner \in \mathrm{dcl}^{eq}(\ulcorner \pi_n(Z) \urcorner)$.

Claim 233. *Let* $q(x_2, x_1) = p_{\pi_n(Z)}(x_2) \otimes p_{\pi_n(Z)}(x_1)$ *and* $(d_2, d_1) \vDash q(x_2, x_1)$*. Then* $\Sigma^{gen}_{\pi_n(N)}(y) \subseteq \mathrm{tp}(d_2 - d_1/\mathfrak{M})$*.*

Proof. We proceed by contradiction and we assume the existence of some proper $\mathfrak{M}$-definable subtorsor $B \subseteq \pi_n(N)$ such that $d_2 - d_1 \in B$. Then $d_2 \in d_1 + B \subsetneq \pi_n(Z)$, and $d_1 + B$ is a proper $\mathfrak{M}d_1$-definable torsor of $\pi_n(Z)$, but this contradicts that d_2 is a sufficiently generic realization of $\Sigma^{gen}_{\pi_n(Z)}(x)$ over $\mathfrak{M}d_1$. □

Let $p_{\pi_n(N)}(x) = \text{tp}(d_2 - d_1/\mathfrak{M})$, we observe that such type is independent of the choices of d_1 and d_2 as the congruence and coset formulas are completely determined in the type $q(x_2, x_1)$.
It is only left to show that $\text{germ}(h_N, p_{\pi_n(N)}) \in \text{dcl}^{eq}(\ulcorner \pi_n(Z) \urcorner, \text{germ}(h_Z, p_{\pi_n(Z)}))$.
Let $\sigma \in Aut(\mathfrak{M}/(\ulcorner \pi_n(Z) \urcorner, \text{germ}(h_Z, p_{\pi_n(Z)})))$, we will show that $h_N(x) = h_{\sigma(N)}(x) \in p_{\pi_n(N)}(x)$. Because $p_{\pi_n(N)}(x)$ is $\ulcorner \pi_n(N) \urcorner$-definable then $\sigma(p_{\pi_n(N)}(x)) = p_{\pi_n(N)}(x)$. As $\sigma(\text{germ}(h_Z, p_{\pi_n(Z)})) = \text{germ}(h_Z, p_{\pi_n(Z)})$, then $h_Z(x) = h_{\sigma(Z)}(x) \in p_{\pi_n(Z)}$. Let $C = \{\ulcorner Z \urcorner, \ulcorner \sigma(Z) \urcorner, \ulcorner N \urcorner, \ulcorner \sigma(N) \urcorner\}$. In particular, if $d_1 \vDash p_{\pi_n(Z)} \upharpoonright_C$ and $d_2 \vDash p_{\pi_n(Z)} \upharpoonright_{Cd_1}$ then $h_Z(d_1) = h_{\sigma(Z)}(d_1)$ and $h_Z(d_2) = h_{\sigma(Z)}(d_2)$. By Fact 230,

$$h_N(d_2 - d_1) = h_Z(d_2) - h_Z(d_1) = h_{\sigma(Z)}(d_2) - h_{\sigma(Z)}(d_1) = h_{\sigma(N)}(d_2 - d_1).$$

Consequently $\sigma(\text{germ}(h_N, p_{\pi_n(N)})) = \text{germ}(h_N, p_{\pi_n(N)})$. Because σ is arbitrary, we conclude that

$$\text{germ}(h_N, p_{\pi_n(N)}) \in \text{dcl}^{eq}(\ulcorner \pi_n(Z) \urcorner, \text{germ}(h_Z, p_{\pi_n(Z)})), \text{ as required.}$$

□

Some coding lemmas

Lemma 234. *Let A be a definable $\mathcal{O}$-lattice in K^n and $U \in K^n/A$ be a torsor. Let B be the $\mathcal{O}$-lattice in K^{n+1} that is interdefinable with U (given by Proposition 173).Then there is a $\ulcorner U \urcorner$- definable injection :*

$$f = \begin{cases} \text{red}(U) & \rightarrow \text{red}(B) \\ b + \mathcal{M}A & \mapsto (b, 1) + \mathcal{M}B. \end{cases}$$

Proof. We recall how the construction of B was achieved. Given any $\bar{d} \in U$, we can represent $B = A_2 + \begin{bmatrix} \bar{d} \\ 1 \end{bmatrix} \mathcal{O}$, where $A_2 = \{0\} \times A$. This definition is independent from the choice of $\bar{d}$. We consider the $\ulcorner U \urcorner$ definable injection $\phi = U \rightarrow B$ that sends each element $\bar{b}$ to $\begin{bmatrix} \bar{b} \\ 1 \end{bmatrix}$. The interpretable sets $\text{red}(U)$ and $\text{red}(B) = B/\mathcal{M}B$ are both $\ulcorner U \urcorner$-definable. It follows by a standard computation that for any $b, b' \in U$, $b - b' \in \mathcal{M}A$ if and only if $\begin{bmatrix} b \\ 1 \end{bmatrix} - \begin{bmatrix} b' \\ 1 \end{bmatrix} \in \mathcal{M}B$. This shows that the map f is a $\ulcorner U \urcorner$-definable injection. □

Lemma 235. *Let F be a primitive finite set of* 1*-torsors, then F can be coded in $\mathcal{G}$.*

Proof. If $|F| = 0$ or $|F| = 1$ the statement follows clearly. So we may assume that $|F| > 1$. By primitivity all the torsors in F are translates of the same $\mathcal{O}$-submodule of K. Indeed, there are some $b \in K$ and $I \in \mathcal{I}$ that for any $t \in F$ there is some $a_t \in K$ satisfying $t = a_t + bI$.

Moreover, there is some $\delta \notin v(b) + S_I$ such that for any two different torsors $t, t' \in F$ if $x \in t$ and $y \in t'$ then $v(x-y) = \delta$. Let $T = \bigcup_{t \in F} t$. We define

$$J_F = \{Q(x) \in K[x] \mid Q(x) \text{ has degree at most } |F| \text{ and for all } x \in T,\\ v(Q(x)) \in v(b) + (|F|-1)\delta + S_I\}.$$

Step 1: $\ulcorner J_F \urcorner$ is interdefinable with $\ulcorner F \urcorner$.
Observe that J_F is definable over $\ulcorner F \urcorner$, because $v(b), \ulcorner T \urcorner, \delta$ lie in $\mathrm{dcl}^{eq}(\ulcorner F \urcorner)$. Hence, it is sufficient to prove that we can recover F from J_F. For this we will show that given a monic polynomial $Q(x) \in K[x]$ with exactly $|F|$-different roots in K each of multiplicity one, we have that $Q(x) \in J_F$ if and only if $Q(x)$ satisfies all the following condition:

Condition: Let $\{\beta_1, \dots, \beta_{|F|}\} \subseteq K$ be the set of all the roots of $Q(x)$ (note that all of them are different). For each $1 \leq i \leq |F|$ there is some $t \in F$ such that $\beta_i \in t$. And all the roots of $Q(x)$ lie in different torsors, i.e. if $i \neq j$, take $t, t' \in F$ such that $\beta_i \in t$ and $\beta_j \in t'$ then $t \neq t'$.

We first show that a monic polynomial $Q(x)$ with exactly $|F|$-different roots in K each of multiplicity one satisfying the condition above belongs to J_F. Let $R = \{\beta_1, \dots, \beta_{|F|}\} \subseteq K$ the set of all the (different) roots of $Q(x)$. Let $x \in T$, then there is some $t \in F$ such that $x \in t$. Let β_i be the root of $Q(x)$ that belongs to t, then $x, \beta_i \in t$, so $v(x - \beta_i) \in v(b) + S_I$. For any other index $j \neq i$, let $t' \in F$ be such that $\beta_j \in t'$, because $t \neq t'$, $v(x - \beta_j) = \delta$. Summarizing we have:

$$v\big(Q(x)\big) = v\Big(\prod_{k \leq |F|} (x - \beta_k)\Big) = \underbrace{v(x-\beta_i)}_{\in v(b)+S_I} + \underbrace{\sum_{j \neq i} v(x - \beta_j)}_{=(|F|-1)\delta} \in v(b) + (|F|-1)\delta + S_I.$$

Consequently, $Q(x) \in J_F$. For the converse, let $Q(x) \in J_F$ be a monic polynomial with exactly $|F|$-different roots $R = \{\beta_1, \dots, \beta_{|F|}\} \subseteq K$. We show that $Q(x)$ satisfies the condition, i.e. each root belongs to some torsor $t \in F$ and any two different roots belong to different torsors of F.

Claim 236. *Given any torsor $t \in F$, there is a unique root $\beta \in R$ such that for all elements $x \in t$, $v(x - \beta) > \delta$.*

Proof. Let $t \in F$ be a fixed torsor. We first show the existence of some root $\beta \in R$ such that for any $x \in t$, we have $v(x - \beta) > \delta$. We argue by contradiction, so let $t \in F$ and assume that there is no root $\beta \in B$ such that $v(x - \beta) > \delta$ for all $x \in t$. Then for each element $x \in t$ we have:

$$v(Q(x)) = v\Big(\prod_{i \leq |F|} (x - \beta_i)\Big) = \sum_{i \leq |F|} v(x - \beta_i) \leq |F|\delta.$$

In this case $Q(x) \notin J_F$, because $|F|\delta \notin v(b)+(|F|-1)\delta+S_I$ as $\delta \notin v(b)+S_I$. This concludes the proof for existence.
For uniqueness, let $\{t_1, \dots, t_{|F|}\}$ be some fixed enumeration of F. Let $\beta_i \in R$ be such that for all $x \in t_i$ we have $v(x-\beta_i) > \delta$. We first argue that for any $i \neq j$, we must have that $\beta_i \neq \beta_j$. Suppose by contradiction that $\beta_i = \beta_j = \beta$, and let $x \in t_i$ and $y \in t_j$, then:

$$\delta = v(x-y) = v((x-\beta)+(\beta-y)) \geq \min\{v(x-\beta), v(y-\beta)\} > \delta.$$

The uniqueness now follows because $|F| = |R|$. $\square$

By Claim 236, we can fix an enumeration $\{t_i \mid i \leq |F|\}$ of F such that for any $x \in t_i$, $v(x-\beta_i) > \delta$. We note that if $j \neq i$, then for any $x \in t_i$ we have that $v(x-\beta_j) = \delta$. In fact, fix some $y \in t_j$, as $v(y-\beta_j) > \delta$ we have:

$$v(x-\beta_j) = v((x-y)+(y-\beta_j)) = \min\{\underbrace{v(x-y)}_{=\delta}, \underbrace{v(y-\beta_j)}_{>\delta}\} = \delta.$$

Claim 237. *For each $i \leq |F|$ we have that $\beta_i \in t_i$.*

Proof. We fix some $i \leq |F|$. Thus, for any $x \in t_i$:

$$v(Q(x)) = v\Big(\prod_{k \leq |F|} v(x-\beta_k)\Big) = v(x-\beta_i) + \sum_{j \neq i} v(x-\beta_j) = v(x-\beta_i) + (|F|-1)\delta.$$

Because $Q(x) \in J_F$, we must have that $v(x-\beta_i) \in v(b)+S_I$. Thus $\beta_i \in t_i$. Moreover, by construction, if $i \neq j$ then $t_i \neq t_j$. $\square$

Step 2: F admits a code in the geometric sorts.
By the first step F is interdefinable with J_F. The latter one is an $\mathcal{O}$-module, so by Lemma 195 it admits a code in the stabilizer sorts $\mathcal{G}$. $\square$

Lemma 238. *Let F be a primitive finite set of 1-torsors such that $|F| > 1$. There is a $\ulcorner F \urcorner$-definable $\mathcal{O}$-lattice $s \subseteq K^2$ and an $\ulcorner F \urcorner$-definable injective map $g = F \to \mathrm{VS}_{k,\ulcorner s \urcorner}$.*

Proof. Let F be a primitive finite set of 1-torsors. By primitivity, there is some $d \in K$ and $I \in \mathcal{I}$ such that for any $t \in F$ there is some $a_t \in K$ satisfying $t = a_t + dI$. Moreover, there is some $\delta \in \Gamma \backslash (v(d)+S_I)$ such that for any pair of different torsors $t, t' \in F$, and $x \in t$, $y \in t'$ we have $v(x-y) = \delta$. Let $T = \bigcup_{t \in F} t$, and take elements $c \in T$ and $b \in K$ such that $v(b) = \delta$. Let $U = c + b\mathcal{O}$. Then U is the smallest closed 1-torsor that contains all the elements of F. Note that U is definable over $\ulcorner F \urcorner$. Let h be the map sending each element of F to the unique class that contains it in $\mathrm{red}(U)$. By construction, such a map must be injective and it is $\ulcorner F \urcorner$-definable. Let s be the $\mathcal{O}$-lattice in K^2, whose code is interdefinable with $\ulcorner U \urcorner$ (given by Proposition 173). By Lemma 234 there is a $\ulcorner s \urcorner$-definable injection $f : \mathrm{red}(U) \to \mathrm{red}(s)$. Let $g = f \circ h$, the composition map $g = F \to \mathrm{VS}_{k,\ulcorner s \urcorner}$ satisfies the required conditions. $\square$

Lemma 239. *Let F be a finite set of 1-torsors and let $f : F \to \mathcal{G}$ be a definable function. Suppose that F is primitive over $\ulcorner f \urcorner$, then:*

1. *for any set of parameters C if $f(F) \subseteq VS_{k,C}$ then f is coded in $\mathcal{G}$ over C,*
2. *if $f(F) \subseteq K$ then f is coded in $\mathcal{G}$,*
3. *if $f(F)$ is a finite set of 1-torsors of the same type $I \in \mathcal{I}$. Then f is coded in $\mathcal{G}$.*

Proof. In all the three cases, we may assume that $|F| > 1$, otherwise the statement clearly follows. Also, by primitivity of F over $\ulcorner f \urcorner$, f is either constant or injective. If it is constant and equal to c, the tuple $(\ulcorner F \urcorner, c)$ is a code for f. By Lemma 235 $\ulcorner F \urcorner$ admits a code in $\mathcal{G}$, so $(\ulcorner F \urcorner, c)$ is interdefinable with a tuple in the stabilizer sorts. In the following arguments we assume that f is an injective function and that $|F| \geq 2$.

1. By Lemma 235 $\ulcorner F \urcorner \in \mathcal{G}$. Let s be the $\mathcal{O}$-lattice of K^2 and $g : F \to VS_{k,C\ulcorner s \urcorner}$ the injective map given by Lemma 238. Both s and g are $\ulcorner F \urcorner$-definable. Let $F^* = g(F) \subseteq VS_{k,C\ulcorner s \urcorner}$, the map $f \circ g^{-1} : F^* \to VS_{k,C\ulcorner s \urcorner}$ can be coded in $\mathcal{G}$ by Theorem 217. Hence, the tuple $(\ulcorner f \circ g^{-1} \urcorner, \ulcorner F \urcorner)$ is a code of f over C, because g is a $\ulcorner F \urcorner$-definable bijection, and $(\ulcorner f \circ g^{-1} \urcorner, \ulcorner F \urcorner)$ is interdefinable with a tuple of elements in $\mathcal{G}$.

2. Let $D = f(F) \subseteq K$, this is a finite set in the main field so it can be coded by a tuple d of elements in K. Because F is primitive over $\ulcorner f \urcorner$, then D is a primitive set. Thus, there is some $\delta \in \Gamma$ such that for any pair of different elements $x, y \in D$ $v(x - y) = \delta$. Let $b \in K$ be such that $v(b) = \delta$, take $x \in D$ and let $U = x + b\mathcal{O}$, this is the smallest closed 1-torsor containing D. The elements of D all lie in different classes of $\mathrm{red}(U)$ and let $g : D \to \mathrm{red}(U)$ be the definable map sending each element $x \in D$ to the unique element in $\mathrm{red}(U)$ that contains x. Both U and g are $\ulcorner D \urcorner$-definable, and therefore d-definable. By Proposition 173, there is an $\mathcal{O}$-lattice $s \subseteq K^2$, whose code is interdefinable with $\ulcorner U \urcorner$. Let $h : \mathrm{red}(U) \to \mathrm{red}(s)$ the $\ulcorner U \urcorner$- definable injective map given by Lemma 234. Both U and h are d-definable. By (1) of this statement the function $h \circ g \circ f : F \to VS_{k,\ulcorner s \urcorner}$ can be coded in $\mathcal{G}$. Since f and $h \circ g \circ f$ are interdefinable over d, the statement follows.

3. Let $D = f(F)$ then D must be a primitive set of 1-torsors because F is primitive over $\ulcorner f \urcorner$ (in particular, there are $I \in \mathcal{I}$ and $b \in K$ and elements $a_t \in K$ such that for each $t \in f(F)$ we have $t = a_t + bI$). By Lemma 235, we may assume $\ulcorner D \urcorner$ is a tuple in the stabilizer sorts. Let $s \subseteq K^2$ and $g : D \to \mathrm{red}(s) \subseteq VS_{k,\ulcorner s \urcorner}$ the injective map given by Lemma 238. Both s and g are $\ulcorner D \urcorner$-definable. By part (1) of this statement the composition $g \circ f$ can be coded in $\mathcal{G}$, and as g is a $\ulcorner D \urcorner$-definable bijection the tuple $(\ulcorner g \circ f \urcorner, \ulcorner D \urcorner)$ is interdefinable with $\ulcorner f \urcorner$.

$\square$

Coding of finite sets of tuples in the stabilizer sorts

We start by recalling some terminology from previous sections for sake of clarity.

Notation 240. *Let $M \subseteq K^n$ be an $\mathcal{O}$-module, and $(I_1, \dots, I_n) \in \mathcal{I}$ be such that $M \cong \oplus_{i \leq n} I_i$. For any torsor $Z = \bar{d} + M \in K^n/M$ we say that Z is of type $(I_1, \dots, I_n)$ and it has* complexity *n. We denote by $\pi_n : K^n \to K$ the projection to the last coordinate and for a torsor $Z = \bar{d} + M \in K^n/M$ we write as $A_Z = \pi_n(Z)$. We recall as well the notation introduced in 229 for the function that describes the fiber in Z of each element at the projection, this is $h_Z(x) = \{y \in K^{n-1} \mid (y, x) \in Z\}$.*

Definition 241. *Let F be a finite set of torsors, the complexity of F is the maximum complexity of the torsors $t \in F$.*

The following is a very useful fact that we will use repeatedly.

Fact 242. *Let F be a finite set of torsors, then there is a finite set $F' \subseteq \mathcal{G}$ such that $\ulcorner F \urcorner$ and $\ulcorner F' \urcorner$ are both interdefinable. In particular, any definable function $f : F \to P$, where P is a finite set of torsors or $P \subseteq \mathcal{G}$, is interdefinable with a fuction $g : F' \to \mathcal{G}$, where $F' \subseteq \mathcal{G}$.*

Proof. The statement follows immediately by Proposition 173. □

The main goal of this section is the following theorem.

Theorem 243. *For every $m \in \mathbb{N}_{\geq 1}$ the following hold:*

- *I_m: For every $r > 0$ and finite set $F \subseteq \mathcal{G}^r$ of size m then $\ulcorner F \urcorner$ is interdefinable with a tuple of elements in $\mathcal{G}$.*
- *II_m: For every $F \subseteq \mathcal{G}$ of size m and $f : F \to \mathcal{G}$ a definable function, then $\ulcorner f \urcorner$ is interdefinable with a tuple in $\mathcal{G}$.*

We will prove this statement by induction on m, we note that for $m = 1$ the statements I_m and II_m follow trivially. We now assume that I_k and II_k hold for each $k \leq m$ and we want to show I_{m+1} and II_{m+1}. In order to keep the steps of the proof easier to follow we break the proof into some smaller steps. We write each step as a proposition to make the document more readable.

Proposition 244. *Let F be finite set of torsors of size at most $m+1$, then $\ulcorner F \urcorner$ is interdefinable with a tuple of elements in $\mathcal{G}$. Furthermore, any definable function $f : S \to F$, where S is a finite set of at most $m+1$-torsors and F is a finite set of torsors can be coded in $\mathcal{G}$.*

Proof. We will start by proving the following statements by a simultaneous induction on n:

- A_n: Any set F of torsors of size at most $m+1$ of complexity at most n can be coded in $\mathcal{G}$.

- B_n : Every definable function $f : S \to F$, where S is a finite set of at most $m+1$-torsors and F is a finite set of torsors of complexity at most n can be coded in $\mathcal{G}$.

We observe first that we may assume in A_n that F is a primitive set of size $m+1$. If $|F| \leq m$ the statement follows immediately by Fact 242 combined with I_k for each $k \leq m$. So we may assume that F has $m+1$ elements. If F is not primitive, then we can find a non trivial equivalence E relation definable over $\ulcorner F \urcorner$, and let $C_1, \dots, C_l$ be the equivalence classes. For each $i \leq l$ $|C_i| \leq m$, by Fact 242 and because I_k holds for each $k \leq m$ $\ulcorner C_i \urcorner$ is interdefinable with a tuple c_i of elements in $\mathcal{G}$. Because $l \leq m$ and I_l holds, we can find a code c in the stabilizer sorts of the set $\{c_1, \dots, c_l\}$. The code $\ulcorner F \urcorner$ is interdefinable with $c \in \mathcal{G}$.
Likewise, for B_n we may assume that S is primitive over $\ulcorner f \urcorner$. Otherwise, there is a $\big(\ulcorner f \urcorner \cup \ulcorner S \urcorner\big)$-definable equivalence relation E on S and let $C_1, \dots, C_l$ be the equavalence classes of this relation. For each $i \leq l$, $|C_i| \leq m$ and let $f_i = f \restriction_{C_i}$. By Fact 242, for each $i \leq l$ $\ulcorner f_i \urcorner$ is interdefinable with a map $g_i : S_i \to \mathcal{G}$ where $S_i \subseteq \mathcal{G}$ and $|S_i| \leq m$. Because II_k holds for each $k \leq m$, f_i admits a code c_i in $\mathcal{G}$. Because I_l holds, we can find a code c for the finite set $\{c_1, \dots, c_l\}$. The codes $\ulcorner f \urcorner$ and c are interdefinable.
We continue arguing for the base case $n = 1$. The statement A_1 holds by Lemma 235, while B_1 is given by (3) of Lemma 239. We now assume that A_n and B_n hold and we prove A_{n+1} and B_{n+1}.
First we prove that A_{n+1} holds. Let F be a primitive finite set of torsors of size $m+1$. By primitivity all the torsors in F are of the same type. For each $Z \in F$ we write A_Z to denote the projection of Z into the last coordinate. By primitivity of F the projections to the last coordinate are either all equal or all different. We argue by cases:

1. *Case 1: All the projections are equal, i.e. $A = A_Z$ for all $Z \in F$.*

   Proof. For each $x \in A$, the set of fibers $\{h_Z(x) \mid Z \in F\}$ is a finite set of torsors of size at most $m+1$ of complexity at most n. By the induction hypothesis A_n it admits a code in the stabilizer sorts. By compactness we can uniformize such codes, and we can define the function $g : A \to \mathcal{G}$ by sending the element x to the code $\ulcorner \{\ulcorner h_Z(x) \urcorner \mid Z \in F\} \urcorner$. This is a $\ulcorner F \urcorner$-definable function. Let $p_A(x)$ be a global type extending the generic type of A, it is $\ulcorner A \urcorner$-definable by Corollary 221. By Theorem 225 the germ of g over p_A can be coded in $\mathcal{G}$ over $\ulcorner A \urcorner$. By Corollary 232 for any $Z \in F$ the code $\ulcorner Z \urcorner$ is interdefinable with the tuple $(\ulcorner A \urcorner, \mathrm{germ}(h_Z, p_A))$, then $\ulcorner F \urcorner$ is interdefinable with $(\ulcorner A \urcorner, \ulcorner \{\mathrm{germ}(h_Z, p_A) \mid Z \in F\} \urcorner)$.

   Claim 245. $\mathrm{germ}(g, p_A)$ *is interdefinable with the code* $\ulcorner \{\mathrm{germ}(h_Z, p_A) \mid Z \in F\} \urcorner$ *over* $\ulcorner A \urcorner$.

   Proof. We first prove that $\mathrm{germ}(g, p_A) \in \mathrm{dcl}^{eq}(\ulcorner A \urcorner, \ulcorner \{\mathrm{germ}(h_Z, p_A) \mid Z \in F\} \urcorner)$.
   Let $\sigma \in Aut(\mathfrak{M}/\ulcorner \{\mathrm{germ}(h_Z, p_A) \mid Z \in F\} \urcorner, \ulcorner A \urcorner)$, we want to show that $\sigma(\mathrm{germ}(g, p_A)) = \mathrm{germ}(\sigma(g), p_A) = \mathrm{germ}(g, p_A)$. Let B the set of all the parameters required to define

all the objects that have been mentioned so far. It is therefore sufficient to argue that for any realization c of $p_A(x)$ sufficiently generic over B we have $\sigma(g)(c) = g(c)$, where $\sigma(g) : A \to \mathcal{G}$ is the function given by sending the element x to the code $\ulcorner\{\ulcorner h_{\sigma(Z)}(x)\urcorner \mid Z \in F\}\urcorner$. Note that

$$\sigma(\{\mathrm{germ}(h_Z, p_A) \mid Z \in F\}) = \{\mathrm{germ}(h_{\sigma(Z)}, p_A) \mid Z \in F\} = \{\mathrm{germ}(h_Z, p_A) \mid Z \in F\},$$

because $\sigma(\ulcorner\{\mathrm{germ}(h_Z, p_A) \mid Z \in F\}\urcorner) = \ulcorner\{\mathrm{germ}(h_Z, p_A) \mid Z \in F\}\urcorner$. As a result, for any realization c of $p_A(x)$ sufficiently generic over B we must have that $\{\ulcorner h_Z(c)\urcorner \mid Z \in F\} = \{\ulcorner h_{\sigma(Z)}(c)\urcorner \mid Z \in F\}$ so $g(c) = \sigma(g)(c)$, as desired.
For the converse, let $\sigma \in Aut(\mathfrak{M}/\ulcorner A\urcorner, \mathrm{germ}(g, p_A))$ we want to show that

$$\sigma(\ulcorner\{\mathrm{germ}(h_Z, p_A) \mid Z \in F\}\urcorner) = \ulcorner\{\mathrm{germ}(h_Z, p_A) \mid Z \in F\}\urcorner.$$

Let c be a realization of $p_A(x)$ sufficiently generic over B by hypothesis $g(c) = \sigma(g)(c)$. Then:

$$g(c) = \ulcorner\{\ulcorner h_Z(c)\urcorner \mid Z \in F\}\urcorner = \ulcorner\{\ulcorner h_{\sigma(Z)}(c)\urcorner \mid Z \in F\}\urcorner = \sigma(g)(c).$$

Therefore, for each $Z \in F$ there is some $Z' \in F$ such that $h_Z(c) = h_{\sigma(Z')}(c)$ and this implies that $\mathrm{germ}(h_Z, p_A) = \mathrm{germ}(h_{\sigma(Z')}, p_A)$. Thus

$$\sigma\big(\{\mathrm{germ}(h_Z, p_A) \mid Z \in F\}\big) = \{\mathrm{germ}(h_{\sigma(Z)}, p_A) \mid Z \in F\} = \{\mathrm{germ}(h_Z, p_A) \mid Z \in F\}.$$

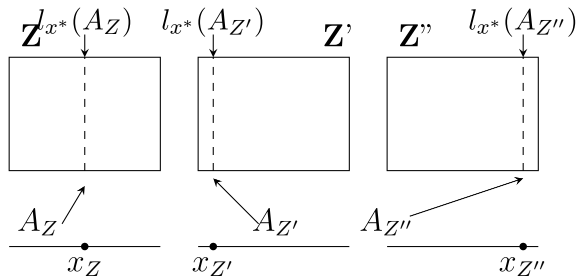
We conclude that $\sigma(\ulcorner\{\mathrm{germ}(h_Z, p_A) \mid Z \in F\}\urcorner) = \ulcorner\{\mathrm{germ}(h_Z, p_A) \mid Z \in F\}\urcorner$, as desired. □

Consequently, F is coded by the tuple $(\ulcorner A\urcorner, \mathrm{germ}(g, p_A))$ which is a sequence of elements in $\mathcal{G}$. □

2. *Case 2: All the projections are different i.e. $A_Z \neq A_{Z'}$ for all $Z \neq Z' \in F$.*

   Proof. To simplify the notation fix some enumeration of the projections $\{A_Z \mid Z \in F\}$ say $\{A_1, \dots, A_n\}$. Let $W = \{\{x_1, \dots, x_n\} \mid x_i \in A_i\}$, such set is independent from the choice of the enumeration. Each set $\{x_1, \dots, x_n\} \in W$ admits a code in the home sort K, because fields uniformly code finite sets. We denote by $W^* = \{\ulcorner\{x_1, \dots, x_n\}\urcorner \mid \{x_1, \dots, x_n\} \in W\}$, i.e. the set of all these codes.
   For each $x^* \in W^*$, we define the function $f_{x^*} : S \to K$ that sends $A_Z \mapsto x_Z$, where x_Z is the unique element in the set coded by x^* that belongs to A_Z. Let $l_{x^*} : S \to \mathcal{G}$ the function given by sending $A_Z \mapsto \ulcorner h_Z(f_{x^*}(A_Z))\urcorner$.
   This map sends the projection A_Z to the code of the fiber in the module Z at the point x_Z, which is the unique point in the set coded by x^* that belongs to A_Z [See Figure 2].

Figure 2



Claim 246. *For each $x^* \in W^*$ the functions f_{x^*} and l_{x^*} can be coded in $\mathcal{G}$.*

Proof. We argue first for the function f_{x^*}. If S is primitive over $\ulcorner f_{x^*} \urcorner$ the statement follows by Lemma (2) 239. If S is not primitive over $\ulcorner f_{x^*} \urcorner$ then there is an equivalence relation E definable over $\left(S \cup \ulcorner f_{x^*} \urcorner\right)$ and let $C_1, \dots, C_l$ be the equivalence classes of E. For each $i \leq l$, $|C_i| \leq m$ and let $f^i_{x^*} = f_{x^*} \restriction_{C_i} : C_i \to K$. For each $i \leq l$ $\ulcorner f_i \urcorner$ is interdefinable with a tuple c_i of elements in $\mathcal{G}$, this follows by combining Fact 242 and II_k for each $k \leq m$. Because I_l holds, the set $\{c_1, \dots, c_l\}$ admits a code c in the stabilizer sorts. Then $\ulcorner f_{x^*} \urcorner$ and c are interdefinable.
For the function l_{x^*}, the statement follows immediately by the induction hypothesis B_n. $\square$

By compactness we can uniformize all such codes, so we can define the function $g : W^* \to \mathcal{G}$ by sending $x^* \mapsto (\ulcorner f_{x^*} \urcorner, \ulcorner l_{x^*} \urcorner)$.

By Lemma 227 there is some $\ulcorner W^* \urcorner$-definable type $q(x^*) \vdash x^* \in W^*$. The second part of Lemma 227 also guarantees that given d^* a generic realization of q over a set of parameters B, if we take Y the set coded by d^* and b is the element in Y that belongs to A_Z then b is a sufficiently generic realization over B of some type $p_{A_Z}(x)$ which is $\ulcorner A_Z \urcorner$-definable and extends the generic type of A_Z.We recall as well that the types $p_{A_Z}(x)$ given by Lemma 227 are all compatible under the action of $Aut(\mathfrak{M}/\ulcorner F \urcorner)$, this is for any $\sigma \in Aut(\mathfrak{M}/\ulcorner F \urcorner)$ if $\sigma(Z) = Z'$ then $\sigma(p_{A_Z}(x)) = p_{A_{\sigma(Z)}}(x)$. By Theorem 225 the germ of g over q can be coded in the stabilizer sorts $\mathcal{G}$ over $\ulcorner W^* \urcorner \in \mathrm{dcl}^{eq}(\ulcorner S \urcorner)$. By Lemma 238 we may assume $\ulcorner S \urcorner \in \mathcal{G}$.

Claim 247. *The tuple* $(\mathrm{germ}(g, q), \ulcorner S \urcorner) \in \mathcal{G}$ *is interdefinable with* $\ulcorner F \urcorner$.

Proof. It is clear that $(\mathrm{germ}(g, q), \ulcorner S \urcorner) \in \mathrm{dcl}^{eq}(\ulcorner F \urcorner)$. For the converse, let $\sigma \in Aut(\mathfrak{M}/\,\mathrm{germ}(g, q), \ulcorner S \urcorner)$ we want to show that $\sigma(F) = F$. By Corollary 232 the code of each torsor $Z \in F$ is interdefinable with the pair $(A_Z, \mathrm{germ}(h_Z, p_{A_Z}))$. Hence it is sufficient to argue that:

$$\sigma(\{(A_Z, \mathrm{germ}(h_Z, p_{A_Z})) \mid Z \in F\}) = \{(A_Z, \mathrm{germ}(h_Z, p_{A_Z})) \mid Z \in F\}.$$

We have that $\sigma(\ulcorner W^* \urcorner) = \ulcorner W^* \urcorner$ because $\sigma(S) = S$. Therefore $\sigma(\text{germ}(g,q)) = \text{germ}(\sigma(g),q) = \text{germ}(g,q)$. Let B be the set of parameters required to define all the objects that have been mentioned so far. For any realization d^* of the type q sufficiently generic over B we have $g(d^*) = \sigma(g)(d^*)$, where $\sigma(g)$ is the function sending an element x^* in W^* to the tuple $(\sigma(f)_{x^*}, \sigma(l)_{x^*})$. As a result, $(\ulcorner f_{d^*} \urcorner, \ulcorner l_{d^*} \urcorner) = (\ulcorner \sigma(f)_{d^*} \urcorner, \ulcorner \sigma(l)_{d^*} \urcorner)$. Let $D = \{d_t \mid t \in S\}$ be the set of elements coded by d^*. The action of σ is just permuting the elements of the graph f_{d^*}, because $\ulcorner f_{d^*} \urcorner = \ulcorner \sigma(f)_{d^*} \urcorner$.
The function $f_{d^*} : S \to K$ sends a 1-torsor t to the unique element $d_t \in D$ such that $d_t \in t$, then σ is sending the pair (t, d_t) to $(\sigma(t), d_{\sigma(t)})$, where $d_{\sigma(t)}$ is a realization $p_{\sigma(t)}(x)$ sufficiently generic over B. By assumption, we also have that $\ulcorner l_{d^*} \urcorner = \ulcorner \sigma(l)_{d^*} \urcorner$, thus the action of σ is a bijection among the elements of the graph of l_{d^*}. Consequently, for any $t \in S$ there is some unique $t' \in S$ such that $\sigma((t, h_Z(d_t))) = (\sigma(t), h_{\sigma(Z)}(d_{\sigma(t)})) = (t', h_{\sigma(Z)}(d_{t'}))$. Thus $\sigma(Z)$ is a torsor whose projection is $t \in S$, and $d_t \in t$ is a realization of the type $p_t(x)$ sufficiently generic over B. As a result, $\sigma\big(t, \text{germ}(h_Z, p_t)\big) = \big(t', \text{germ}(h_{\sigma(Z)}, p_{t'})\big)$. We conclude that:

$$\begin{aligned}&\sigma\big(\{(A_Z, \text{germ}(h_Z, p_{A_Z})) \mid Z \in F\}\big) = \sigma\big(\big\{(t, \text{germ}(h_Z, p_t)) \mid t \in S\big\}\big)\\ &\quad = \{(t', \text{germ}(h_{\sigma(Z)}, p_{t'})) \mid t' \in S\} = \{(A_Z, \text{germ}(h_Z, p_{A_Z})) \mid Z \in F\}, \text{ as desired.}\end{aligned}$$

□

This finalizes the proof for *Case* 2. □

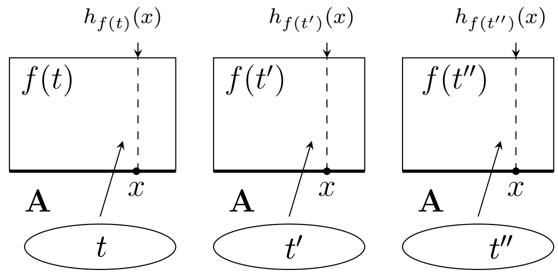
Consequently A_{n+1} holds. We prove B_{n+1}, i.e. every definable function $f : S \to F$ where S is a finite set of at most $m+1$ torsors and F is a finite set of torsors of complexity at most n can be coded in $\mathcal{G}$. We recall that without loss of generality we may assume that S is primitive over $\ulcorner f \urcorner$, so F is also a primitive set. By primitivity f is either constant or injective, if it is constant equal to c then $\ulcorner f \urcorner$ is interdefinable with $(\ulcorner S \urcorner, c)$. By Proposition 173 and Lemma 235 this tuple is interdefinable with a tuple in $\mathcal{G}$. Thus we may assume that f is an injective function. By primitivity of F, all the torsors in F are of the same type and the projections to the last coordinate are either all equal or all different. We proceed again by cases.

1. *Case* 1*: The projections are all equal, i.e. there is a torsor A such that $A = A_Z$ for all $Z \in F$.*

   Proof. We fix $p_A(x)$ be some global type extending the generic type of A, it is $\ulcorner A \urcorner$-definable by Corollary 221. Let $f : S \to F$ be a definable injective map. For each $x \in A$ we define the function $g_x : S \to \mathcal{G}$ by sending $t \mapsto \ulcorner h_{f(t)}(x) \urcorner$.

   This is the function that sends each torsor $t \in S$ to the fiber at x of the torsor $f(t) \in F$. [See Figure 1].

Figure 1



By the induction hypothesis B_n for each $x \in A$, the function g_x can be coded in $\mathcal{G}$ because its range is of lower complexity. By compactness we can uniformize such codes, so we can define the function $r : A \to \mathcal{G}$ by sending $x \mapsto \ulcorner g_x \urcorner$.
By Theorem 225 the germ of r over $p_A(x)$ can be coded in $\mathcal{G}$ over $\ulcorner A \urcorner$. By Lemma 235 the set S admits a code $\ulcorner S \urcorner$ in the stabilizer sorts.

Claim 248. *The code $\ulcorner f \urcorner$ is interdefinable with $(\ulcorner A \urcorner, \ulcorner S \urcorner, \mathrm{germ}(r, p_A))$, and the later is a sequence of elements in $\mathcal{G}$.*

Proof. It is clear that $(\ulcorner A \urcorner, \ulcorner S \urcorner, \mathrm{germ}(r, p_A)) \in \mathrm{dcl}^{eq}(\ulcorner f \urcorner)$.We want to show that $\ulcorner f \urcorner \in \mathrm{dcl}^{eq}(\ulcorner A \urcorner, \ulcorner S \urcorner, \mathrm{germ}(r, p_A))$. Let $\sigma \in Aut(\mathfrak{M}/\ulcorner A \urcorner, \ulcorner S \urcorner, \mathrm{germ}(r, p_A))$. By Corollary 232 for each torsor $Z \in F = \{f(t) \mid t \in S\}$, the code $\ulcorner Z \urcorner$ is being identified with the tuple $(\ulcorner A \urcorner, \mathrm{germ}(h_Z, p_A))$. Thus, the function f is interdefinable over $\ulcorner A \urcorner$ with the function: $f' : S \to \mathcal{G}$ that sends $t \mapsto \mathrm{germ}(h_{f(t)}, p_A)$. So, it is sufficient to argue that $\sigma(\ulcorner f' \urcorner) = \ulcorner f' \urcorner$. Let B be the set of parameters required to define all the objects that have been mentioned so far. For any realization c of $p_A(x)$ sufficiently generic over B we must have that $r(c) = \sigma(r)(c)$. Because $\mathrm{germ}(r, p_A) = \sigma(\mathrm{germ}(r, p_A)) = \mathrm{germ}(\sigma(r), p_A)$. By definition, $r(c) = \ulcorner g_c \urcorner$ and $\sigma(r)(c) = \ulcorner \sigma(g)_c \urcorner$, where $\sigma(g)_c : S \to \mathcal{G}$ is the function that sends $t \mapsto \ulcorner h_{\sigma(f)(t)}(c) \urcorner$.
For any torsor $t \in S$ there must be a unique element $t' \in S$ such that $\sigma(t') = t$ and $h_{f(t)}(c) = h_{\sigma(f)(\sigma(t'))}(c)$, as $g_c = \sigma(g)_c$. The later implies that $\mathrm{germ}(h_{f(t)}, p_A) = \mathrm{germ}(h_{\sigma(f)(\sigma(t'))}, p_A)$. We conclude that $\sigma(t', \mathrm{germ}(h_{f(t')}, p_A)) = (t, \mathrm{germ}(h_{f(t)}, p_A))$ meaning that σ is acting as a bijection among the elements in the graph of f'. Therefore, $\sigma(\ulcorner f' \urcorner) = \ulcorner f' \urcorner$, as desired. □

This completes the proof for the first case. □

2. *Case 2: All the projections are different i.e. $A_Z \neq A_{Z'}$ for all $Z \neq Z' \in F$.*

   Proof. Let $f : S \to F$ be a definable injective function where S is a finite set of 1-torsors primitive over $\ulcorner f \urcorner$. We consider the definable function that sends each torsor $t \in S$ to the code of the projection into the last coordinate of the torsor $f(t) \in F$, more explicitly:

$$\pi_{n+1} \circ f : \begin{cases} S & \to \mathcal{G} \\ t & \mapsto \ulcorner \pi_{n+1}(f(t)) \urcorner. \end{cases}$$

By Lemma (3) 239, $\pi \circ f$ can be coded in $\mathcal{G}$, and by A_{n+1} the finite set F is coded by a tuple in $\mathcal{G}$. It is sufficient to show the following claim:

Claim 249. *The code $\ulcorner f \urcorner$ is interdefinable with the tuple $(\ulcorner \pi \circ f \urcorner, \ulcorner F \urcorner)$, which is a tuple in the stabilizer sorts.*

Proof. Clearly $(\ulcorner \pi \circ f \urcorner, \ulcorner F \urcorner) \in \operatorname{dcl}^{eq}(\ulcorner f \urcorner)$. Note that $\ulcorner S \urcorner \in \operatorname{dcl}^{eq}(\ulcorner \pi \circ f \urcorner)$ because S is the domain of the given function, we can describe the function $f : S \to F$ by sending $t \mapsto \ulcorner Z_t \urcorner$, where Z_t is the unique torsor in F such that $\ulcorner \pi_{n+1}(Z_t) \urcorner = (\pi \circ f)(t)$, we conclude that $\ulcorner f \urcorner \in \operatorname{dcl}^{eq}(\ulcorner \pi \circ f \urcorner, \ulcorner F \urcorner)$. As a consequence, f is coded in $\mathcal{G}$ by the tuple $(\ulcorner \pi \circ f \urcorner, \ulcorner F \urcorner)$. □

This finalizes the proof for the second case. □

Consequently, A_n and B_n hold for all $n \in \mathbb{N}$. The statement follows. □

We continue arguing that I_{m+1} holds for $r = 1$.

Proposition 250. *Let $F \subseteq \mathcal{G}$ be a finite set of size $m + 1$ then F admits a code in $\mathcal{G}$.*

Proof. If F is not primitive we show that $\ulcorner F \urcorner$ can be coded in $\mathcal{G}$, by using Fact 242 and the induction hypothesis I_k for $k \leq m$. We may assume that F is a primitive set, so all the elements of F lie in the same sort. If F is either contained in the main field or the residue field, then F is coded by a tuple of elements in the same field, because fields code uniformly finite sets. If $F \subseteq \Gamma/\Delta$ for some $\Delta \in RJ(\Gamma)$ the statement follows as there is a definable order over the elements of F. If $F \subseteq B_n(K)/\operatorname{Stab}_{(I_1,\dots,I_n)}$ for some $n \geq 2$, by Proposition 244 F admits a code in $\mathcal{G}$. (Indeed, $\mathcal{O}$-modules are in particular torsors). □

We continue showing that II_{m+1} holds, we first prove the following statement.

Proposition 251. *Let F be a finite set of torsors of size $m+1$ and $f : F \to P$ be a definable bijection, where P is a finite set of torsors. Suppose that F is primitive over $\ulcorner f \urcorner$, then $\ulcorner f \urcorner$ is interdefinable with a tuple of elements in $\mathcal{G}$.*

Proof. We proceed by induction on the complexity of the torsors in F. The base case follows directly by Proposition 244. We assume the statement for any set of torsors F with complexity n and we prove it for complexity $n + 1$. By primitivity all the projections into the last coordinate are either equal or all distinct. For each torsor $Z \in F$ we denote as A_Z the projection of Z into the last coordinate. We argue by cases:

1. *Case* 1*: All the projections are equal and let* $A = A_Z$ *for all* $Z \in F$*.*
   For each $x \in A$, let $\mathcal{I}_x = \{\ulcorner h_Z(x) \urcorner \mid Z \in F\}$ which describes the set of fibers at x. We define $B = \{x \in A \mid |\mathcal{I}_x| = |F|\}$ which is a $\ulcorner F \urcorner$-definable set. For each $y \in B$ we consider the map $g_y : \mathcal{I}_y \to P$ defined by sending $h_Z(y) \mapsto f(Z)$, which is the function that sends each fiber to the image of the torsor under f. By the induction hypothesis we can find a code $\ulcorner g_y \urcorner$ in $\mathcal{G}$, and by compactness we can uniformize such codes. Therefore we can define the function: $r : B \to \mathcal{G}$ by sending $y \mapsto \ulcorner g_y \urcorner$.
   Let $p_A(x)$ be a global complete type containing the generic type of A, it is $\ulcorner A \urcorner$- definable by Corollary 221. By Corollary 232, $p_A(x) \vdash x \in B$. In fact, if we fix a realization of the generic type c of $p_A(x)$ sufficiently generic over $\{\ulcorner Z \urcorner \mid Z \in F\}$, and $Z \neq Z' \in F$ then the fibers $h_Z(c)$ and $h_{Z'}(c)$ must be different. By Theorem 225 the germ of r over $p_A(x)$ can be coded in $\mathcal{G}$ over $\ulcorner A \urcorner$.

Claim 252. *The code* $\ulcorner f \urcorner$ *is interdefinable with* $(\mathrm{germ}(r, p_A), \ulcorner F \urcorner)$ *which is a tuple in the stabilizer sorts* $\mathcal{G}$*.*

Proof. Clearly $(\mathrm{germ}(r, p_A), \ulcorner F \urcorner) \in \mathrm{dcl}^{eq}(\ulcorner f \urcorner)$. We will argue that for any automorphism
$\sigma \in Aut(\mathfrak{M}/\ulcorner F \urcorner, \mathrm{germ}(r, p_A))$ we have $\sigma(\ulcorner f \urcorner) = \ulcorner f \urcorner$. As each torsor $Z \in F$ is being identified with the tuple $(\ulcorner A \urcorner, \mathrm{germ}(h_Z, p_A))$, and $\ulcorner A \urcorner \in \mathrm{dcl}^{eq}(\ulcorner F \urcorner)$ then it is sufficient to argue that:

$$\sigma\big(\{(\mathrm{germ}(h_Z, p_A), f(Z)) \mid Z \in F\}\big) = \{(\mathrm{germ}(h_Z, p_A), f(Z)) \mid Z \in F\}.$$

For any $Z \in F$ there is a unique torsor $Z' \in F$ such that $\sigma(Z') = Z$, because $\sigma(\ulcorner F \urcorner) = \ulcorner F \urcorner$. Let D be the set of parameters required to define all the objects that have been mentioned so far. For any realization c of the type $p_A(x)$ sufficiently generic over D we have $r(c) = \sigma(r)(c)$, because $\sigma(\mathrm{germ}(r, p_A)) = \mathrm{germ}(\sigma(r), p_A)$. Consequently $r(c) = \ulcorner g_c \urcorner = \ulcorner \sigma(g)_c \urcorner = \sigma(r)(c)$. In particular, $h_{\sigma(Z')}(c) = h_Z(c)$ which implies that $\mathrm{germ}(h_{\sigma(Z')}, p_A) = \mathrm{germ}(h_Z, p_A)$. In addition, $\sigma(f)(\sigma(Z')) = \sigma(g)(h_{\sigma(Z')}(c)) = g_c(h_Z(c)) = f(Z)$. Therefore,

$$\sigma\big(\{(\mathrm{germ}(h_Z, p_A) f(Z)) \mid Z \in F\}\big) = \{(\mathrm{germ}(h_Z, p_A), f(Z)) \mid Z \in F\}, \text{as desired.}$$

□

2. *Case* 2*: All the projections are different. i.e. for all* $Z \neq Z' \in F$ *we have* $A_Z \neq A_{Z'}$*.*
   By Proposition 244 we can find a code in the stabilizer sorts for F, and $\ulcorner F \urcorner \in \mathrm{dcl}^{eq}(\ulcorner f \urcorner)$ as it is the domain of this function. Let $S = \{A_Z \mid Z \in F\}$ and define the function $g : S \to F$ by sending $A_Z \mapsto Z$, where Z is the unique torsor in F satisfying that $\pi_{n+1}(Z) = A_Z$. Clearly g is a $\ulcorner F \urcorner$-definable bijection. We consider the map $f \circ g : S \to P$ that sends $A_Z \mapsto f(Z)$. By Proposition 244, the function $f \circ g$ admits a code in the stabilizer sorts.

Claim 253. *The code $\ulcorner f \urcorner$ is interdefinable with the tuple $(\ulcorner f \circ g \urcorner, \ulcorner F \urcorner)$ which is a tuple in the stabilizer sorts.*

Proof. It is clear that $(\ulcorner f \circ g \urcorner, \ulcorner F \urcorner) \in \mathrm{dcl}^{eq}(\ulcorner f \urcorner)$. For the converse note that S is definable over $\ulcorner f \circ g \urcorner$ as it is its domain. As F is given, we can define the function $\pi : F \to S$ that sends $Z \mapsto A_Z$. This is the map that sends each torsor to its projection into the last coordinate. We observe that $f = (f \circ g) \circ \pi$, in fact $f(Z) = (f \circ g)(A_Z)$. So $\ulcorner f \urcorner \in \mathrm{dcl}^{eq}(\ulcorner f \circ g \urcorner, \ulcorner F \urcorner)$. □

□

Proposition 254. *For every $F \subseteq \mathcal{G}$ finite set of size $m+1$ and definable function $f : F \to \mathcal{G}$, the code $\ulcorner f \urcorner$ is interdefinable with a tuple of elements in $\mathcal{G}$.*

Proof. Without loss of generality we may assume that F is primitive over $\ulcorner f \urcorner$. Otherwise, there is a $(\ulcorner F \urcorner \cup \ulcorner f \urcorner)$-definable equivalence relation on F and we let $C_1, \dots, C_l$ be the equivalence classes. For each $i \leq l$ we have $|C_i| \leq m$ and let $f_i = f \restriction_{C_i}$. By the induction hypothesis, for each $k \leq m$ II_k the code $\ulcorner f_i \urcorner$ is interdefinable with a tuple $c_i \in \mathcal{G}$. Because $l \leq m$ and I_l holds the set $\{c_1, \dots, c_l\}$ admits a code $c \in \mathcal{G}$. Then $\ulcorner f \urcorner$ and c are interdefinable. Hence, we may assume that F is primitive over $\ulcorner f \urcorner$. By primitivity f is either constant or injective. If f is constant equal to some c then $\ulcorner f \urcorner$ is interdefinable with the tuple $(\ulcorner F \urcorner, c)$, which lies in the stabilizer sorts by Proposition 250. Summarizing, we may assume that f is an injective function and F is primitive over $\ulcorner f \urcorner$. By primitivity all the torsors of F lie in the same sort.
If F is contained in the residue field, then F is interdefinable with the code of a finite set of 1-torsors of type $\mathcal{M}$ and the statement follows by Proposition 251. If $F \subseteq B_n(K)/\operatorname{Stab}_{(I_1,\dots,I_n)}$ for some $n \geq 2$, the statement follows by Proposition 251, because $\mathcal{O}$-modules are torsors. If $F \subseteq \Gamma/\Delta$ for some $\Delta \in RJ(\Gamma)$, then we can list the elements of F in increasing order $\gamma_1 < \dots < \gamma_{m+1}$, and the tuple $(\gamma_i, f(\gamma_i))_{1 \leq i \leq m+1}$ lies in the stabilizer sorts and is interdefinable with the code of f.
It is therefore left to consider the case where $F \subseteq K$. We may assume that $\ulcorner F \urcorner$ is a tuple of elements in the main field, as fields code finite sets. Let U be the smallest closed torsor that contains all the elements of F, this is a $\ulcorner F \urcorner$-definable set. Let g the function that sends each element $x \in F$ to the unique class of $\mathrm{red}(U)$ that contains such element. Let s be the $\mathcal{O}$- lattice whose code is interdefinable with $\ulcorner U \urcorner$, and let $h = \mathrm{red}(U) \to \mathrm{red}(s)$ be the map given by Lemma 234. Let $D = h \circ g(F)$, which is an $\ulcorner F \urcorner$-definable finite subset of $\mathrm{red}(s)$. By Proposition 251, the composition $f \circ g^{-1} \circ h^{-1} = D \to \mathcal{G}$ can be coded in the stabilizer sorts $\mathcal{G}$. As $h \circ g = F \to D$ is a $\ulcorner F \urcorner$-definable bijection, then f is interdefinable with the tuple $(\ulcorner F \urcorner, \ulcorner f \circ g^{-1} \circ h^{-1} \urcorner)$ which is a sequence of elements in $\mathcal{G}$. □

Finally, we conclude proving that I_{m+1} holds for $r > 0$.

Proposition 255. *For any $r > 0$ let $F \subseteq \mathcal{G}^r$ be a finite set of size $m+1$. Then F can be coded in $\mathcal{G}$.*

Proof. Let $r > 0$ and F be a finite set of $\mathcal{G}^r$ of size $m+1$. Suppose that F is not primitive, that means that we can find a non trivial equivalence E relation definable over $\ulcorner F \urcorner$, and let $C_1, \dots, C_l$ be such classes. For each $i \leq l$, $|C_i| \leq m$, because I_k holds for each $k \leq m$ we can find a code $c_i \in \mathcal{G}$. As $l \leq m$ by I_l holds, we can find a code c in the stabilizer sorts of the set $\{c_1, \dots, c_l\}$, because $l < m+1$. The code $\ulcorner F \urcorner$ is interdefinable with c.
We assume that F is a primitive set. Let $\pi_i = \mathcal{G}^r \rightarrow \mathcal{G}$ be the projection into the $i-th$ coordinate. By primitivity of F each projection π_i es either constant or injective. As $|F| > 1$ there must be an index $1 \leq i_0 \leq r$ such that π_{i_0} is injective and $F_0 = \pi_{i_0}(F)$ is a primitive finite subset of $\mathcal{G}$. By Proposition 250 we can find a code $\ulcorner F_0 \urcorner$ in $\mathcal{G}$. For each other index $i \neq i_0$, by Proposition 254 we have that $\pi_i \circ \pi_{i_0}^{-1} = F_0 \rightarrow \mathcal{G}$ can be coded in the stabilizer sorts. Then $\ulcorner F \urcorner$ is interdefinable with the tuple $(\ulcorner F_0 \urcorner, (\ulcorner \pi_i \circ \pi_{i_0}^{-1} \urcorner)_{i \neq i_0})$ which is a tuple in the stabilizer sorts, as required. □

This completes the induction on the cardinality of the set F. Because I_m holds for each $m \in \mathbb{N}$ we can conclude with the following statement.

Theorem 256. *Let $r > 0$ and $F \subseteq \mathcal{G}^r$, then $\ulcorner F \urcorner$ is interdefinable with a tuple of elements in $\mathcal{G}$.*

Putting everything together

We conclude this section with our main theorem.

Theorem 257. *Let K be a henselian valued field of equicharacteristic zero, residue field algebraically closed and dp-minimal value group. Then K eliminates imaginaries in the language $\hat{\mathcal{L}}$, where the stabilizer sorts are added.*

Proof. By Theorem 200, K has weak elimination of imaginaries down to the stabilizer sorts. By Fact 204 it is sufficient to show that finite sets can be coded, this is guaranteed by Theorem 256. □

Chapter 3

Residue field domination in henselian valued fields of equicharacteristic zero

The model theory of henselian valued fields has been a major topic of study during the last century. It was initiated by Robinson's model completeness results of algebraically closed valued fields in [Rob56]. Remarkable work has been achieved by Haskell, Hrushovski and Macpherson to understand the model theory of algebraically closed valued fields. In a sequence of papers [HHM05] and [HHM06], they developed the notion of stable domination, that essentially established how an algebraically closed valued field is governed by its stable part.
In further work Ealy, Haskell and Maříková present in [EHM19] , introduced an abstract form the notion of domination present in [HHM05], Let T be a complete first order theory and let S and Γ be stably embedded sorts, and $C \subseteq A, B$ be sets of parameters in the monster model $\mathfrak{C}$.

Definition 258. 1. *the type* $\mathrm{tp}(A/C)$ *is said to be* dominated by the sort S, *if whenever* $S(B)$ *is* independent *from* $S(A)$ *over* $S(C)$ *then that* $\mathrm{tp}(A/CS(B)) \vdash \mathrm{tp}(A/CB)$.

2. *the type* $\mathrm{tp}(A/C)$ *is said to be dominated by the sort* S *over* Γ *if the type* $\mathrm{tp}(A/C\Gamma(A))$ *is dominated by the sort* S.

In [EHM19] domination results for the setting of real closed convexly valued fields are proved, which suggests that the presence of a stable part of the structure is not fundamental to achieve domination results and indicates that the right notion should be residue field domination or domination by the internal sorts to the residue field in broader classes of henselian valued fields.
Our main motivation arises from the natural question of how much further a notion of residue field domination could be extended to broader classes of valued fields to gain a deeper model theoretic insight of henselian valued field.
Stable domination has played a fundamental role in understanding the model theory of algebraically closed valued fields, and more precisely it has served as a bridge to lift ideas from

stability theory to the setting of valued fields. For example, Hrushovski and Rideau-Kikuchi in [HRK19], have shown that for any definable abelian group A in a model of $ACVF$ we can find a definable group $\Lambda \subseteq \Gamma^n$, where Γ is the value group, and a definable homomorphism $\lambda : A \to \Lambda$, such that $H := ker(\lambda)$ is limit stably dominated [see [HRK19, Definition 5.6]].
In this chapter we study domination results for henselian valued fields of equicharacteristic zero. The general strategy to show domination of a type $\mathrm{tp}(A/C)$ by a sort S is taking a partial elementary map σ witnessing $\mathrm{tp}(A/CS(B)) = \mathrm{tp}(A'/CS(B))$ and finding an automorphism $\hat{\sigma}$ of the monster model that extends σ and fixes CB. For each of the statements, we specify precisely which notion of *independence* is required to extend the isomorphism, in fact not the entire power of forking independence is needed.
It is still an open question to find a reasonable language in which a henselian valued field of equicharacteristic zero eliminates imaginaries. Some positive results have been obtained in certain classes of henselian valued fields of equicharacteristic zero, see for example [HRK21a] and [Vic21a].
Therefore, we start by studying domination results for henselian valued fields of equicharacteristic zero for elements in the home-sort. A valued field $(K, \mathcal{O})$ where $\mathcal{O}$ is its valuation ring and $\mathcal{M}$ the maximal ideal, can be considered in several different languages. For instance, a valued field can be seen as a three sorted structure in the language $\mathcal{L}_{\text{val}}$ [see Definition 267], where the first two sorts are equipped with the language of fields while the third one is provided with the language of ordered abelian groups $\mathcal{L}_{OAG} = \{0, <, +, -\}$ extended by a constant ∞. We interpret the first sort as the main field, the second one as the residue field and the third one as the value group Γ and ∞, where $\gamma < \infty$ for all $\gamma \in \Gamma$ and $\gamma + \infty = \infty + \gamma = \infty$ for all $\gamma \in \Gamma \cup \{\infty\}$.
A natural extension of $\mathcal{L}_{\text{val}}$ is the language where an angular component map is added, we denote this extension by $\mathcal{L}_{\text{ac}}$ [see Definition 272].
In [ACGZ20], Aschenbrenner, Chernikov, Gehret and Ziegler introduced a multi-sorted language $\mathcal{L}$ extending $\mathcal{L}_{\text{val}}$ in which one obtains elimination of field quantifiers for any henselian valued field of equicharacteristic zero. In this extension, we expand the structure $(\mathbf{k}, \Gamma \cup \{\infty\})$ by adding a new sort $\mathbf{k}/(\mathbf{k}^{\times})^n$ for every $n \geq 2$ together with the natural surjections $\pi_n : \mathbf{k} \to \mathbf{k}/(\mathbf{k}^{\times})^n$. A precise description of the language $\mathcal{L}$ is given in Definition 280. The multi-sorted structure $(\mathbf{k}/(\mathbf{k}^{\times})^n \mid n \in \mathbb{N})$ will play a fundamental role in the domination results, so we refer to them as the *power residue sorts*. We identify $\mathbf{k}/\mathbf{k}^0$ with the residue field $\mathbf{k}$.

This chapter is organized as follows:

1. Section 3.1: we briefly summarize the relative quantifier elimination already known for henselian valued fields in equicharacteristic zero.

2. Section 3.2: we prove that over a maximal model, a valued field is dominated by the value group and the power residue sorts in the language $\mathcal{L}$ introduced by Aschenbrenner, Chernikov, Gehret and Ziegler. We show as well that over a maximal field, a valued field is dominated by the residue field and the value group sort in the $\mathcal{L}_{\text{ac}}$-language.

3. Section 3.3: we prove that over a maximal model forking is determined by Shelah's imaginary expansion of the power residue sorts (residue sort) and Shelah's imaginary expansion of the value group in the language $\mathcal{L}$ (or $\mathcal{L}_{\text{ac}}$). We assume the theory of the residue field to be NTP_2.

4. Section 3.4: we show that over a maximal model a valued field is dominated by the sorts internal to the residue field over the value group in the language $\mathcal{L}$.

3.1 Preliminaries

Finer valuations, places and separated bases

To gain our final statement of domination of the internal sorts to the residue field, we will need to construct a refined valuation induced by the composition of some places. In this subsection, we recall some basics facts about refinements and places. We refer the reader interested in further details to [EP05, Section 2.3].

Definition 259. *Let $\mathcal{O}$ be a valuation ring of K and $\mathcal{O}'$ be an overring of $\mathcal{O}$, and hence a valuation ring of K. Then, we say that $\mathcal{O}$ is a* coarsening *of $\mathcal{O}'$ and $\mathcal{O}'$ is a* refinement *of $\mathcal{O}$.*

Let $\mathcal{O}$ be a fixed valuation ring of K and $\mathcal{O}'$ be an overring of $\mathcal{O}$. We have $\mathcal{M}' \subseteq \mathcal{M}$, where $\mathcal{M}'$ and $\mathcal{M}$ denote the maximal ideals of $\mathcal{O}'$ and $\mathcal{O}$ respectively. Since $\mathcal{M}'$ is a prime ideal in $\mathcal{O}'$, then it is also a prime ideal of $\mathcal{O}$. Moreover, localizing $\mathcal{O}$ at $\mathcal{M}'$ we can recover $\mathcal{O}'$, in fact $\mathcal{O}' = \mathcal{O}_{\mathcal{M}'}$.

The following is [EP05, Lemma 2.3.1].

Lemma 260. *Let $\mathcal{O}$ be a non trivial valuation ring in K corresponding to the valuation $v : K \twoheadrightarrow \Gamma \cup \{\infty\}$. Then there is a 1-to-1 correspondence of the convex subgroups Δ of Γ with the prime ideals p of $\mathcal{O}$, and hence with the overrings $\mathcal{O}_p$. This correspondence is given by:*

$$\begin{aligned} \Delta \to p_{\Delta} &= \{x \in K \mid v(x) > \delta \text{ for all } \delta \in \Delta\} \\ p \to \Delta_p &= \{\gamma \in \Gamma \mid \gamma < v(x) \text{ and } -\gamma < v(x) \text{ for all } x \in p\}. \end{aligned}$$

Let $\mathcal{O}$ be a valuation ring of K and $v : K \to \Gamma \cup \{\infty\}$ the corresponding valuation. Let p be a prime ideal with corresponding convex subgroup Δ_p in Γ and $\mathcal{O}_p$ the refinement of $\mathcal{O}$. There is a group homomorphism:

$$\phi : \begin{cases} K^{\times}/\mathcal{O}^{\times} & \to K^{\times}/\mathcal{O}_p^{\times} \\ x\mathcal{O}^{\times} & \mapsto x\mathcal{O}_p^{\times}. \end{cases}$$

whose kernel is $\Delta_p \cong \mathcal{O}_p^\times/\mathcal{O}^\times$. The valuation v_p induced by $\mathcal{O}_p$ is therefore obtain from $v := K \to \Gamma \cup \{\infty\}$ simply by taking the quotient of Γ by the convex subgroup Δ_p.

Definition 261. *Let M and C be valued fields. Let $m_1, \ldots, m_k$ be elements of M, we write* $\mathrm{Vect}_C(m_1, \ldots, m_k)$ *for the C-vector space generated by $\{m_1, \ldots, m_k\}$. We say that $\{m_1, \ldots, m_k\}$ is* separated over C *if for all $c_1, \ldots, c_k \in C$, we have:*

$$v\big(\sum_{i=1}^{k} c_i m_i\big) = \min\{v(c_i m_i) \mid 1 \leq i \leq k\}.$$

In particular, it is a basis for $\mathrm{Vect}_C(m_1, \ldots, m_k)$. *In general, we say that M has the* separated basis property *over C if every finite dimensional C-subspace* $\mathrm{Vect}_C(m_1, \ldots, m_k)$ *where $\{m_1, \ldots, m_k\} \subseteq M$ has a separated basis.*
If $C \subseteq M$, a separated basis is said to be good *if in addition for all $1 \leq i, j \leq k$, either $v(m_i) = v(m_j)$ or $v(m_i) - v(m_j) \notin \Gamma_C$, and we say that M has the* good separated basis property *over C if every finite dimensional C-subspace* $\mathrm{Vect}_C(m_1, \ldots, m_k)$ *where $\{m_1, \ldots, m_k\} \subseteq M$ has a good separated basis.*

The following is a folklore fact, details can be found for example in [HHM05, Lemma 12.2].

Fact 262. *Suppose C is maximally complete. Then every valued field extension M of C has the good separated basis property.*

Definition 263. *Let K and L be fields. A map $\phi : K \to L \cup \{+\infty\}$ is a* place over K *if for any $x, y \in K$:*

- $\phi(x + y) = \phi(x) + \phi(y)$,
- $\phi(x \cdot y) = \phi(x) \cdot \phi(y)$,
- $\phi(1) = 1$.

Here for all $a \in L$, the following operations are defined $a + \infty = \infty + a = \infty$, and $a \cdot \infty = \infty \cdot a = \infty \cdot \infty = \infty$. While the operations $\infty + \infty, 0 \cdot \infty, \infty \cdot 0$ are not.

The following proposition states the correspondence between places and valuations over a field. This a folklore fact, for example see [EP05, Exercise 2.5.4].

Proposition 264. *Let K and L be fields and $\phi : K \to L \cup \{\infty\}$ be a place over K. Then $\mathcal{O} = \phi^{-1}(L)$ is a valuation ring of K whose maximal ideal is $\mathcal{M} = \phi^{-1}(\{0\})$ and its residue field is $\phi(K)\backslash\{\infty\}$. Moreover, given a valuation ring $\mathcal{O}$ of K whose maximal ideal is $\mathcal{M}$ the map:*

$$\phi : K \to \mathcal{O}/\mathcal{M} \cup \{+\infty\}$$
$$\begin{cases} x \to x + \mathcal{M} \text{ if } x \in \mathcal{O}, \\ x \to \infty \text{ if } x \in K\backslash\mathcal{O}. \end{cases}$$

is a place of K.

Notation 265. *Let Γ be an ordered abelian group, and let $\gamma, \delta \in \Gamma$. We write $\gamma \ll \delta$ to indicate that $n\gamma < \delta$ for all $n \in \mathbb{N}$.*

We conclude this subsection stating a lemma that we will need to obtain a domination result by the internal sorts of the residue field over the value group.

Lemma 266. *Let $v : L \to \Gamma$ be a valuation on a field L. Let $p : L \to \mathrm{res}(L) \cup \{\infty\}$ be the place corresponding to the valuation v and F be a subfield of $\mathrm{res}(L)$ and $p' : \mathrm{res}(L) \to F$ be a place which is the identity on F. Let p^* the composition of places $p' \circ p : L \to F$, and $v^* : L \to \Gamma^*$ the induced valuation. Suppose that $a \in L$ with $p(a) \subseteq \mathrm{res}(L) \backslash \{0\}$ and $p^*(a) = 0$. Then:*

1. *if $\Delta = \{v^*(x), -v^*(x) \mid x \in L,\ p(x) \notin \{\infty, 0\}\ , p^*(x) = 0\} \cup \{0_{\Gamma^*}\}$. Then Δ is a convex subgroup of Γ^* and there is an isomorphism of ordered abelian groups $g : \Gamma^*/\Delta \to \Gamma$ such that $g \circ v^* = v$,*
2. *if $b \in L$ with $v(b) > 0$, then $0 < v^*(a) \ll v^*(b)$,*
3. *let $M \subseteq L$ be a subfield. If $(r_1, \ldots, r_n)$ is a separated basis of the M-vector subspace $\mathrm{Vect}_M(r_1, \ldots, r_n)$ according to the valuation v^* then it is also a separated basis according to the valuation v. Furthermore, if $v^*\big(\sum_{i=1}^{n} r_i m_i\big) = \min\{v^*(r_i m_i) \mid i \leq n\} = v^*(r_j m_j)$, then $v\big(\sum_{i=1}^{n} r_i m_i\big) = \min\{v(r_i m_i) \mid i \leq n\} = v(r_j m_j)$.*

Proof. The first and the second statement are proved in [HHM05, Lemma 12.16]. The third one is a standard computation that we leave to the reader. □

Valued fields and relative quantifier elimination

In this section, we summarize many results on valued fields that will be used through the paper. There are many languages in which one can view a valued field K to obtain field quantifier elimination statements, we introduce them in detail and state their corresponding relative quantifier elimination. In this paper we are only concerned about henselian valued fields of equicharacteristic zero.

Definition 267. *[The $\mathcal{L}_{\mathrm{val}}$-language] Let $(K, \mathcal{O})$ be a henselian valued field of equicharacteristic zero, where $\mathcal{O}$ is the valuation ring and $\mathcal{M}$ is the maximal ideal of $\mathcal{O}$. Any henselian valued field can be seen as a three sorted structure $(K, \mathbf{k}, \Gamma)$ where the first two sorts are equipped with the language of fields while the third one is provided with the language of ordered abelian groups $\mathcal{L}_{OAG} = \{0, <, +, -\}$ extended by a constant ∞. We interpret the first sort as the main field, the second one as the residue field and the third one as the value* *group Γ with a constant ∞, where $\gamma < \infty$ for all $\gamma \in \Gamma$ and $\gamma + \infty = \infty + \gamma = \infty$ for all $\gamma \in \Gamma \cup \{\infty\}$. We add a function symbol $v : K \to \Gamma \cup \{\infty\}$ by extending the valuation to a monoid morphism sending $v(0) = \infty$. We add as well a function symbol* $\mathrm{res} : K \to \mathbf{k}$, *which sends an element $a \in \mathcal{O}$ to its residue class* $\mathrm{res}(a) = a + \mathcal{M}$, *while* $\mathrm{res}(a) = 0$ *for any element $a \in K \backslash \mathcal{O}$. We refer to this language as the $\mathcal{L}_{\mathrm{val}}$-language.*

Notation 268. *Let L be a henselian valued field we will denote as k_L its residue field and Γ_L its value group.*

Certain classes of henselian valued fields of equicharacteristic zero eliminate field quantifiers in the $\mathcal{L}_{\mathrm{val}}$-language. For example, the following is a well known fact.

Theorem 269. *Let K be a henselian valued field of equicharacteristic zero with residue field algebraically closed, then K eliminates quantifiers relative to the value group in the language $\mathcal{L}_{\mathrm{val}}$.*

An immediate consequence of this theorem is the following statement.

Corollary 270. *Let $(K, \mathbf{k}, \Gamma \cup \{\infty\}, \mathrm{res}, v)$ be a henselian valued field of equicharacteristic zero with residue field algebraically closed, then $\mathbf{k}$ and Γ are purely stably embedded and orthogonal to each other.*

Definition 271. *Let $(K, \mathbf{k}, \Gamma)$ be a valued field an* angular component map *is a map* $\mathrm{ac} : K \to \mathbf{k}$ *that satisfies the following conditions:*

- $\mathrm{ac}(0) = 0$,
- *for all $x \in \mathcal{O}^{\times}$* $\mathrm{ac}(x) = x + \mathcal{M} = \mathrm{res}(x)$,
- *for all $x, y \in K$* $\mathrm{ac}(xy) = \mathrm{ac}(x)\,\mathrm{ac}(y)$.

Definition 272. *[The $\mathcal{L}_{\mathrm{ac}}$-language] We denote by $\mathcal{L}_{\mathrm{ac}}$ the expansion of $\mathcal{L}_{\mathrm{val}}$ where an angular component map is added to the language.*

In [Pas90, Theorem 4.1] Pas proved that any henselian valued field of equicharacteristic zero eliminates field quantifiers in the $\mathcal{L}_{\mathrm{ac}}$-language, we include the statement for sake of completeness.
Let $\mathcal{K} = (K, \mathbf{k}, \Gamma, \mathrm{res}, v, \mathrm{ac})$ be a valued field of equicharacteristic zero. A good substructure of $\mathcal{K}$ is a triple $\mathcal{E} = (E, \mathbf{k}_{\mathcal{E}}, \Gamma_{\mathcal{E}})$ such that:

- E is a subfield of K,
- $\mathbf{k}_{\mathcal{E}}$ is a subfield of $\mathbf{k}$ with $\mathrm{ac}(E) \subseteq \mathbf{k}_{\mathcal{E}}$ (In particular, $\mathrm{res}(\mathcal{O}_E) \subseteq \mathbf{k}_{\mathcal{E}}$),
- $\Gamma_{\mathcal{E}}$ is an ordered abelian subgroup of Γ with $v(E^{\times}) \subseteq \Gamma_{\mathcal{E}}$.

Definition 273. *Let $\mathcal{K}$ and $\mathcal{K}'$ be henselian valued fields of equicharacteristic zero seen as $\mathcal{L}_{\mathrm{ac}}$-structures and let $\mathcal{E} = (E, \mathbf{k}_{\mathcal{E}}, \Gamma_{\mathcal{E}}), \mathcal{E}' = (E', \mathbf{k}_{\mathcal{E}'}, \Gamma_{\mathcal{E}'})$ be good substructures of $\mathcal{K}$ and $\mathcal{K}'$ respectively. A triple $\mathbf{f} = (f, f_r, f_v)$ is said to be a good map, if $f : E \to E'$ and $f_r : \mathbf{k}_{\mathcal{E}} \to \mathbf{k}_{\mathcal{E}'}$ are field isomorphisms and $f_v : \Gamma_{\mathcal{E}} \to \Gamma_{\mathcal{E}'}$ is a $\mathcal{L}_{OAG}$- ordered group isomorphism such that:*

- *$f_r(\mathrm{ac}(a)) = \mathrm{ac}'(f(a))$ for all $a \in E$ and f_r is a partial elementary map between the fields $\mathbf{k}$ and $\mathbf{k}'$,*
- *$f_v(v(a)) = v^{'}(f(a))$ for all $a \in E^{\times}$, and f_v is a partial elementary map between the ordered abelian groups Γ and Γ'.*

Theorem 274. *[Pas] Let $\mathcal{K}$ and $\mathcal{K}'$ be two henselian valued fields of equicharacteristic zero in the $\mathcal{L}_{\mathrm{ac}}$-language. Let $\mathbf{f} : \mathcal{E} \to \mathcal{E}'$ be a good map, then $\mathbf{f}$ is elementary.*

The following statement is an immediate consequence of Theorem 274.

Corollary 275. *Let $\mathcal{K} = (K, \mathbf{k}, \Gamma, \mathrm{res}, v, \mathrm{ac})$ be a henselian valued field of equicharacteristic zero. Then the residue field and the value group are purely stably embedded and are orthogonal to each other.*

Given $(K, \mathbf{k}, \Gamma)$ a valued field we denote as $RV^{\times}$ the multiplicative quotient group $K^{\times}/(1 + \mathcal{M})$ and $\mathrm{rv} : K^{\times} \to RV^{\times}$ the canonical projection map. By adding a constant 0_{RV} we can naturally extend this map sending the element 0 to 0_{RV}, so we denote $RV = RV^{\times} \cup \{0_{RV}\}$. For any $a \in \mathcal{O}\backslash\mathcal{M}$ the class $a(1 + \mathcal{M})$ only depends on the coset $a + \mathcal{M}$, so we obtain a group embedding $i : \mathbf{k}^{\times} \to RV^{\times}$ by sending the element $a + \mathcal{M} \in \mathbf{k}^{\times}$ to $a(1 + \mathcal{M}) \in RV^{\times}$. We can also consider the group morphism $v_{\mathrm{rv}} : RV^{\times} \to \Gamma$ induced by the valuation map $v : K^{\times} \to \Gamma$, defined as $v_{\mathrm{rv}}(a(1+\mathcal{M})) = v(a)$. In fact, given two elements in the main field sort $a, b \in K$ if $a(1+\mathcal{M}) = b(1+\mathcal{M})$ then $v(a) = v(b)$. Therefore, we have a pure exact sequence:

$$1 \to \mathbf{k}^{\times} \to RV^{\times} \to \Gamma \to 0,$$

which can be naturally extended to a short exact sequence of monoids by adding some constants, i.e.

$$1 \to \mathbf{k} \to RV \to \Gamma \cup \{\infty\} \to 0.$$

Besides the induced multiplication, RV also inherits a partially defined addition from K, via the ternary relation:

$$\oplus(a, b, c) \leftrightarrow \exists x, y, z \in K(a = \mathrm{rv}(x) \wedge b = \mathrm{rv}(y) \wedge c = \mathrm{rv}(z) \wedge x + y = z).$$

We consider the three sorted structure $(\mathbf{k}, RV, \Gamma \cup \{\infty\})$ with the language $\mathcal{L}_{rv} = \mathcal{L}_r \cup \mathcal{L}_g \cup \{\cdot_{rv}, i, v_{rv}\}$, where $\mathcal{L}_r$ is a copy of the language of fields for the first sort, $\mathcal{L}_g$ is the language of ordered abelian groups extended by a constant ∞ i.e. $\{0_g, +_g, -_g, <_g, \infty\}$, i is a function symbol interpreted as the monoid morphism $i : \mathbf{k} \to RV$ and v_{rv} is a function symbol interpreted as the monoid morphism $v_{\mathrm{rv}} : RV \to \Gamma$.

Building on work of Basarab in [Bas91], Kuhlmann proved in [Kuh94] that any henselian valued field of equicharacteristic zero eliminates field quantifiers relative to the structure $(\mathbf{k}, RV, \Gamma)$. We use Flenner as a reference, the following is [Fle08, Proposition 3.3.1].

Proposition 276. *[Kulhmann] Let K be a henselian valued field of equicharacteristic zero, the theory of $(K, \mathbf{k}, RV, \Gamma \cup \{\infty\}, \mathrm{res}, v, v_{\mathrm{rv}}, i)$ eliminates field quantifiers.*

The following is a direct consequence of the relative quantifier elimination to RV.

Corollary 277. *Let K be a henselian valued field of equicharacteristic zero, the structure $(\mathbf{k}, RV, \Gamma \cup \{\infty\})$ is purely stably embedded.*

Kulhmann's statement reduces the elimination of field quantifiers to the structure $(\mathbf{k}, RV, \Gamma\cup \{\infty\})$. For certain classes of henselian valued fields of equicharacteristic zero the structure $(\mathbf{k}, RV, \Gamma \cup \{\infty\})$ eliminates RV quantifiers in the language $\mathcal{L}_{\mathrm{rv}}$.

Proposition 278. *Let K be a henselian valued field of equicharacteristic zero with residue field algebraically closed, the structure $(\mathbf{k}, RV, \Gamma \cup \{\infty\})$ eliminates quantifiers relative to the value group in the language $\mathcal{L}_{\mathrm{rv}}$.*

Proof. This follows by a standard back and forth argument using that $k^{\times}$ is divisible. □

The elimination of RV quantifiers in the more general setting was later obtained by Aschenbrenner, Chernikov, Gehret and Ziegler in [ACGZ20]. They extend the language adding a new sort for each $n \in \mathbb{N}$ denoted as $\mathbf{k}^{\times}/(\mathbf{k}^{\times})^n$ which is an abelian group and we extend it adding an element ∞ such that for each $a \in \mathbf{k}^{\times}/(\mathbf{k}^{\times})^n$, $a \cdot \infty = \infty$. For each $n \in \mathbb{N}$ we denote this sort as $\mathcal{A}_n$, and we refer to the multi-sorted structure $\mathcal{A} = (\mathcal{A}_n \mid n \in \mathbb{N})$ as the *power residue sorts*. We add surjective maps $\pi_n : \mathbf{k}^{\times} \to \mathbf{k}^{\times}/(\mathbf{k}^{\times})^n$, which can be naturally extended to a monoid morphism $\pi_n : \mathbf{k} \to \mathcal{A}_n$. We add maps $\rho_n : RV \to \mathcal{A}_n$, interpreted as $\rho_n(0) = \infty$, over $v_{rv}^{-1}(n\Gamma)$, we define ρ_n as the composition of the group morphisms:

$$v_{rv}^{-1}(n\Gamma) \subseteq RV \to RV^n \cdot i(\mathbf{k}^{\times}) \to (RV^n \cdot i(\mathbf{k}^{\times}))/RV^n \cong \mathbf{k}^{\times}/(RV^n \cap \mathbf{k}^{\times}) \cong \mathbf{k}^{\times}/(\mathbf{k}^{\times})^n,$$

and the map is equal to zero outside of $v_{rv}^{-1}(n\Gamma)$. Let $\mathcal{L}_{rvqe} = \mathcal{L}_{rv} \cup \{\rho_n, \pi_n \mid n \in \mathbb{N}\}$.
The following is [ACGZ20, Remark 4.4].

Corollary 279. *The structure $(\mathcal{A}, RV, \Gamma\cup\{\infty\}, \{\pi_n, \rho_n \mid n \in \mathbb{N}\})$ eliminates RV quantifiers. In particular, $\mathcal{A}$ and $\Gamma \cup \{\infty\}$ are purely stably embedded and are orthogonal to each other.*

Combining Proposition 276 and Corollary 279 Aschenbrenner, Chernikov, Gehret and Ziegler obtained as well a field quantifier elimination for henselian valued fields of equicharacteristic zero relative to the power residue sorts and the value group in the following language:

Definition 280. *[The Language $\mathcal{L}$] Consider the expansion of $\mathcal{L}_{\mathrm{val}}$ obtained by adding the power residue sorts $\mathcal{A} = (\mathcal{A}_n \mid n \in \mathbb{N})$. We also add the surjective maps $\pi_n : \mathbf{k} \to \mathcal{A}_n$, and we interpret them as the described. For each $n \in \mathbb{N}$ we add a map $\mathrm{res}^n : K \to \mathcal{A}_n$ interpreted in the following way: if $v(a) \notin n\Gamma$ set $\mathrm{res}^n(a) = 0$. Otherwise, let b be any element of K with $nv(b) = v(a)$ and set $\mathrm{res}^n(a) = \pi_n\big(\mathrm{res}\left(\frac{a}{b^n}\right)\big)$. We denote this expansion of $\mathcal{L}_{\mathrm{val}}$ by $\mathcal{L}$.*

Note that for each $a \in K$, $\mathrm{res}^n(a) = \rho_n(rv(a))$. The following is a direct consequence of [ACGZ20, Theorem 5.15].

Theorem 281. *A henselian valued field of equicharacteristic zero eliminates field quantifiers in the language $\mathcal{L}$.*

The following statement is an immediate consequence of [ACGZ20, Theorem 5.15].

Corollary 282. *Let K be a henselian valued field of equicharacteristic zero seen as a $\mathcal{L}$-structure. The power residue sorts and the value group are purely stably embedded and are orthogonal to each other.*

Definition 283 (The $\mathcal{L}_{RV}$-language)**.** *We write $\mathcal{L}_{RV}$ to denote the extension of the language $\mathcal{L}$ where we add as well a sort for the monoid RV where we equip the exact sequence $(\mathbf{k}, RV, \Gamma \cup \{\infty\})$ with the language $\mathcal{L}_{rvqe}$.*

Some remarks in ordered abelian groups

In 1984 Gurevich and Schmitt [GS84] showed that every ordered abelian group is NIP. In [Sch84], Schmitt investigated deeply the model completeness of theories of ordered abelian groups and obtained a (*relative*) quantifier elimination to the spines, whose description can be found in [GS84, Section 2]. Later, Cluckers and Halupczok in [CH11, Definition 1.5] introduced a language $\mathcal{L}_{CH}$-extending $\mathcal{L}_{OAG} = \{+, -, 0, <\}$ and obtained a (*relative*) quantifier elimination to the *auxiliary sorts*, whose definition can be found in [CH11, Section 1.2]. The language $\mathcal{L}_{CH}$ has been more often used by the model theory community as it is more in line with Shelah's imaginary expansion. The following is [CH11, Corollary 1.10].

Corollary 284. *Let G be an ordered abelian group, for any function $f : G^n \to G$ which is $\mathcal{L}_{OAG}$-definable with parameters from a set B, there exists a partition of G^n into finitely many B-definable sets and for each such set A, f is linear. This is, there are finitely many elements $r_1, \ldots, r_n, s \in \mathbb{Z}$ with $s \neq 0$ and $b \in \mathrm{dcl}(B)$ such that for any $\dashv \in A$ we have*

$$f(a_1, \ldots, a_n) = \frac{1}{s}\Big(\sum_{i \leq n} r_i a_i + b\Big)$$

.

Let G be an ordered abelian group, we extend the language $\mathcal{L}_{CH}$ by adding a set of constants $\mathcal{C}$ to name each element of $\mathrm{dcl}(\emptyset)$, and we denote this extension as $\mathcal{L}_{CH}^*$. An immediate consequence is the following fact.

Fact 285. *Let G be an ordered abelian group and $B \subseteq G$. Then $\mathrm{dcl}(B) = \big(\mathbb{Q} \otimes B\big) \cap G$.*

This fact will play a fundamental role to weaken the necessary hypothesis to obtain domination results for henselian valued fields of equicharacteristic zero. We will denote as $\mathcal{L}_{\mathrm{val}}^*$ and $\mathcal{L}_{\mathrm{ac}}^*$ the extension of the language obtain by adding a set of constants to the main field $\Sigma = \{t_d \mid d \in \mathrm{dcl}(\emptyset) \cap \Gamma\}$ such that $v(t_d) = d$ for each element $d \in \mathrm{dcl}(\emptyset) \cap \Gamma$.
The following is [JSW17, Proposition 5.1].

Proposition 286. *An ordered abelian group is dp-minimal if and only if for every prime number p, $[\Gamma : p\Gamma]$ is finite.*

Independence notions

Trough this paper we will use several notions of independence. We begin by recalling a few basic properties of forking.

Definition 287. *A formula $\phi(x, b)$* divides *over C if there is a sequence $(b_i)_{i<\omega}$ in $\mathrm{tp}(b/C)$ with $b = b_0$ such that $\{\phi(x, b_i) \mid i < \omega\}$ is m-inconsistent. We say that $\phi(x, b)$* forks *if $\phi(x, b) \vdash \bigvee_{i \leq k} \psi_i(x, b_i)$ where each formula $\psi_i(x, b_i)$ divides over C. We say that $\mathrm{tp}(a/Cb)$ forks (respectively divides) over C if some formula in the type forks (or divides) over C. We write $a \mathop{\smile\!\!\!\!\!\!|}_C b$ if the type $\mathrm{tp}(a/Cb)$ does not fork over C, and write $a \mathop{\smile\!\!\!\!\!\!|}_C^d b$ to indicate that the type $\mathrm{tp}(a/Cb)$ divides over C.*

In many theories the relation of forking independence have been completely characterized. For example, in the theory of algebraically closed fields, forking independence coincides with algebraic independence. Let C, E and F be fields, and suppose that $C \subseteq E \cap F$ we will write $E \mathop{\smile\!\!\!\!\!\!|}_C^{alg} F$ to indicate that E and F are algebraically independent over C.

Forking independence in abelian groups

In [Pre03], the model theory of modules is extensively studied. We will be interested in applying some of the results in [Pre03] to the reduct of the value group to the language of groups $\mathcal{L}_{AG} = \{+, -, 0\}$. It is well known that modules are stable, and every abelian group is a $\mathbb{Z}$-module. We recall some of the necessary notions to characterize forking independence in abelian groups. Throughout this section we consider the $\mathcal{L}_{AG}$ first order theory of some torsion free group and we denote as $\mathfrak{G}$ its monster model.
We recall some of the well known facts about stable theories.

Fact 288. *Let T be a complete first order theory and assume that T is stable and $M \vDash T$. Let $p \in S_n(M)$ then p is stationary. Furthermore, for any set of parameters $M \subseteq A$ and $q \in S_n(A)$ such that $p \subseteq q$ the following conditions are equivalent:*

1. *q is a non-forking extension of p,*
2. *q is a heir extension of p (i.e. every formula represented in q is also represented in p),*
3. *q is a co-heir extension of p (i.e. for every formula $\phi(\mathbf{x}, \mathbf{a}) \in q$ is finitely satisfiable in M, this is there is some $\mathbf{m} \subseteq M$ such that $\vDash \phi(\mathbf{m}, \mathbf{a})$.)*

Definition 289 (*p.p. formula*)**.** *A p.p. formula $\phi(\mathbf{v})$ is an $\mathcal{L}_{AG}$ formula of the form*

$$\exists w_1, \dots, w_l \big(\bigwedge_{j=1}^{k} \sum_{i=1}^{n} r_{j_i} v_i + \sum_{i=1}^{l} s_{j_i} w_i = 0 \big),$$

where s_{j_i}, $r_{j_i} \in \mathbb{Z}$, and $\mathbf{v} = (v_1, \dots, v_n)$ is a tuple of variables.

Given a *p.p.* formula if we replace the last $(n-i)$- variables by a tuple of parameters $\bar{a} = (a_i, \dots, a_n)$, the formula $\phi(v_1, \dots, v_{i-1}, \bar{a})$ defines a coset of $\phi(v_1, \dots, v_{i-1}, \bar{0})$, which defines a subgroup of $\mathfrak{G}^i$.

Definition 290 (*p.p.*-type)**.** *Let $\mathbf{c}$ be a tuple and A some set of a parameters, the p.p. type of $\mathbf{c}$ over A is the set of p.p. $\mathcal{L}_{AG}(A)$-formulas that $\mathbf{c}$ satisfies. This is:*

$$\text{p. p. -type}(\mathbf{c}/A) = \{\phi(\mathbf{v}, \mathbf{a}) \mid \ \phi(\mathbf{v}, \mathbf{a}) \textit{ is a } \mathcal{L}_{AG}(A) \textit{ p.p. formula and} \vDash \phi(\mathbf{c}, \mathbf{a})\}.$$

If p is a *p.p.*-type over A and $\phi(\mathbf{v}, \mathbf{y})$ is an $\mathcal{L}_{AG}$-formula, then we say that it is *represented* in p if there is some tuple $\mathbf{a} \subseteq A$ such that $\phi(\mathbf{v}, \mathbf{a}) \in p$. We consider the type definable group

$$G(p) = \{\phi(\mathbf{v}, \bar{0}) \mid \phi(\mathbf{v}, \mathbf{y}) \text{ is represented in } p\}.$$

It is well known that in stable theories to characterize the non-forking extensions of p the group $G(p)$ would not be the right invariant to consider, but instead one might be more interested in its connected component $G^0(p) = \bigcap_{H \in F} H$, where

$$\mathcal{F} = \{H \mid H \text{ is a subgroup of some } G \in G(p) \text{ and } [G : H] \text{ is finite}\}.$$

The following is [Pre03, Theorem 5.3].

Theorem 291. *Let p be a type and suppose that q is any extension of p. Then q is a non-forking extension of p if and only if $G^0(p) = G^0(q)$. In particular, for any type p if $G(p) = G^0(p)$ then p is stationary.*

This statement allows us to characterize forking independence for arbitrary set of parameters.

Corollary 292. *Let $A, B, C \subseteq \mathfrak{G}$, then $B \mathop{\smile\!\!\!\!\!\!\mid}_A C$ if and only if for every p.p-formula $\psi(\mathbf{x}, \mathbf{y}, \mathbf{z})$ and tuples $\mathbf{b} \subseteq B, \mathbf{c} \subseteq C$ and $\mathbf{a} \subseteq A$ such that $\vDash \psi(\mathbf{b}, \mathbf{c}, \mathbf{a})$, there is some p.p. formula $\phi(\mathbf{x}, \mathbf{w})$ and a tuple $\mathbf{a}' \subseteq A$ such that $\vDash \phi(\mathbf{b}, \mathbf{a}')$ and $[\phi(\mathbf{x}, \bar{0}) : \phi(\mathbf{x}, \bar{0}) \wedge \psi(\mathbf{x}, \bar{0}, \bar{0})]$ is finite..*

We conclude this subsection by introducing the notion of independence that we will be using for the value group. Let $(\Gamma, +, -, \leq, 0)$ be a non-trivial ordered abelian group. Let T be its complete $\mathcal{L}_{OAG}$-theory and $\mathfrak{G}$ be its monster model. We let $\mathfrak{G} \restriction_{\mathcal{L}_{AG}}$ be its reduct to the language of abelian group, this is purely a torsion free abelian group.

Definition 293. *Let $A, B, C \subseteq \mathfrak{G}$, then $A \mathop{\smile\!\!\!\!\!\!\mid}^s_C B$ if and only if tp $\restriction_{\mathcal{L}_{AG}} (A/BC)$ does not fork over C if and only if $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ in the stable structure $\mathfrak{G} \restriction_{\mathcal{L}_{AG}}$.*

We will use the following fact repeatedly.

Fact 294. *Let A and B be subgroup of $\mathfrak{G}$ and let $C \subseteq A \cap B$ be a common subgroup. If $A \mathop{\smile\!\!\!\!\!\!\mid}^s_C B$, then $A \cap B \subseteq \operatorname{dcl}(C)$.*

Proof. This follows by forking independence for stable formulas, including $x = y$. If $a \in A \cap B \setminus \operatorname{acl}(C)$, then the formula $x = a \in \operatorname{tp}(A/B)$ and divides over C, because we can find an infinite non constant indiscernible sequence $(a_i)_{i=0}^{\infty}$ in $\operatorname{tp}(a/C)$ and $\{x = a_i\}_{i=0}^{\infty}$ is 2-inconsistent. We conclude that a must be algebraic over C, and as there is a definable order in the home sort $a \in \operatorname{dcl}(C)$. □

3.2 Domination by the power residue sorts and the value group

In this section we study domination of the type of a valued fields by the power residue sorts and the value group in each of the languages. We would like to highlight that the required ingredients to carry out the argument are the existence of separated basis and a relative quantifier elimination statement. We obtain the following results:

1. In the $\mathcal{L}$-language the type of a valued field over a maximal model is dominated by the power residue sorts and the value group.

2. In the $\mathcal{L}_{\text{ac}}$-language the type of a valued field over a maximal field is dominated by the residue sort and the value group.

3. For the theory of henselian valued fields of equicharacteristic zero with residue field algebraically closed, the type of a valued field over a maximal field is dominated by the residue sort and the value group in the $\mathcal{L}^*_{\text{val}}$-language.

The following is [EHS22, Lemma 2.5].

Proposition 295. *Let L and M be valued fields with $C \subseteq L \cap M$ be a common valued subfield. Assume that:*

1. $\Gamma_L \cap \Gamma_M = \Gamma_C$,

2. k_M and k_L are linearly disjoint over k_C,

3. M (or L) have the good separated basis property over C.

Then M (or L) has the separated basis property over L (or M respectively). Therefore, L and M are linearly disjoint over C, $\Gamma_{C(L,M)}$ is the group generated by Γ_L and Γ_M over Γ_C and $k_{C(L,M)}$ is the field generated by k_M and k_L over k_C.

The following is a direct consequence of Proposition 295, but we bring details to the picture to clarify some of our arguments.

Fact 296. *Let L and M be valued fields with $C \subseteq L \cap M$ a common valued subfield. Assume that:*

1. $\Gamma_L \cap \Gamma_M = \Gamma_C$,

2. k_M and k_L are linearly disjoint over k_C,

3. M (or L) has the good separated basis property over C.

Let $a \in \mathcal{O}^{\times}_{C(L,M)}$ then there are elements

$$l^1_1, \dots, l^1_k, l^2_1, \dots, l^2_s, l \in \mathcal{O}_L \text{ and } m^1_1, \dots, m^1_k, m^2_1, \dots, m^2_s, m \in \mathcal{O}_M,$$

such that:

$$\operatorname{res}(a) = \big(1 + \sum_{i \leq k} \operatorname{res}(l^1_i) \operatorname{res}(m^1_i)\big)\big(1 + \sum_{i \leq s} \operatorname{res}(l^2_i) \operatorname{res}(m^2_i)\big)^{-1} \operatorname{res}(l) \operatorname{res}(m).$$

Proof. Let $a = \frac{y_1}{y_2}$ where $y_1, y_2 \in C[L, M]$. By Proposition 295 L and M are linearly disjoint over C and M (or L) has the separated basis property over L (or M). Suppose that $y_i = \sum_{j \leq n_i} \hat{l}^i_j \hat{m}^i_j$, by hypothesis $v(y_1) = v(y_2) = \gamma$. As M (or L) has the separated basis property over L (or M) there is some index j^i_0 such that $\gamma = v(\hat{l}^i_{j^i_0} \hat{m}^i_{j^i_0})$ and let $I_i = \{j \leq n_i \mid v(\hat{l}^i_j \hat{m}^i_j) = \gamma\}$. Hence

$$y_i = \hat{l}^i_{j^i_0} \hat{m}^i_{j^i_0} \big(1 + \sum_{j \leq n_i, j \neq j^i_0} \big(\frac{\hat{l}^i_{j^i} \hat{m}^i_{j^i}}{\hat{l}^i_{j^i_0} \hat{m}^i_{j^i_0}}\big)\big).$$

Claim 297. *Given elements $l \in L$ and $m \in M$ such that $v(lm) = 0$ there is some element $c \in C$ satisfying $v(lc) = 0$ and $v(mc^{-1}) = 0$.*

Proof. Let $l \in L$ and $m \in M$ be such that $v(lm) = 0$ then $v(l) = -v(m) \in \Gamma_L \cap \Gamma_M = \Gamma_C$ so there is some $c \in C$ such that $v(lc) = 0$ and $v(mc^{-1}) = 0$. $\square$

In particular, for any $j \in I_i$, since $v(\hat{l}^i_j \hat{m}^i_j) = v(\hat{l}^i_{j^i_0} \hat{m}^i_{j^i_0})$ we have that $v\big(\frac{\hat{l}^i_j}{\hat{l}^i_{j^i_0}} \frac{\hat{m}^i_{j^i_0}}{\hat{m}^i_j}\big) = 0$, so we can find elements $c^i_j \in C$ such that $v\big(\frac{\hat{l}^i_j}{\hat{l}^i_{j^i_0}} c^i_j\big) = 0$ and $v\big((c^i_j)^{-1} \frac{\hat{m}^i_{j^i_0}}{\hat{m}^i_j}\big) = 0$. Let $l^i_j = \frac{\hat{l}^i_j}{\hat{l}^i_{j^i_0}} c^i_j$ and $m^i_j = (c^i_j)^{-1} \frac{\hat{m}^i_{j^i_0}}{\hat{m}^i_j}$. Moreover, $\gamma = v(l^1_{j^1_0} m^1_{j^1_0}) = v(l^2_{j^2_0} m^2_{j^2_0})$, thus we can find an element $c \in C$ such that $v\big(\frac{l^1_{j^1_0}}{l^2_{j^2_0}} c\big) = 0$ and $v\big(\frac{\hat{m}^1_{j^1_0}}{\hat{m}^2_{j^2_0}} c^{-1}\big) = 0$, we set $l = \frac{\hat{l}^1_{j^1_0}}{\hat{l}^1_{j^2_0}} c$ and $m = \frac{\hat{m}^1_{j^1_0}}{\hat{m}^2_{j^2_0}} c^{-1}$.
Then:

$$\begin{aligned}
\operatorname{res}(a) = \operatorname{res}(\frac{y_1}{y_2}) &= \operatorname{res}\big(1 + \sum_{j \le n_1, j \neq j^1_0} \frac{\hat{l}^1_j \hat{m}^1_j}{\hat{l}^1_{j^1_0} \hat{m}^1_{j^1_0}}\big)\big(\operatorname{res}\big(1 + \sum_{j \le n_2, j \neq j^2_0} \frac{\hat{l}^2_j \hat{m}^2_j}{\hat{l}^2_{j^2_0} \hat{m}^2_{j^2_0}}\big)\big)^{-1} \operatorname{res}\big(\frac{\hat{l}^1_{j^1_0} \hat{m}^1_{j^1_0}}{\hat{l}_{j^2_0} \hat{m}^2_{j^2_0}}\big) \\
&= \operatorname{res}\big(1 + \sum_{j \in I_1, j \neq j^1_0} \big(\frac{\hat{l}^1_j \hat{m}^1_j}{\hat{l}^1_{j^1_0} \hat{m}^1_{j^1_0}}\big)\big)\big(\operatorname{res}\big(1 + \sum_{j \in I_2, j \neq j^2_0} \big(\frac{\hat{l}^2_j \hat{m}^2_j}{\hat{l}^2_{j^2_0} \hat{m}^2_{j^2_0}}\big)\big)\big)^{-1} \operatorname{res}(l) \operatorname{res}(m) \\
&= \big(1 + \sum_{j \in I_1, j \neq j^1_0} \operatorname{res}\big(\frac{\hat{l}^1_{j^1} \hat{m}^1_{j^1}}{\hat{l}^1_{j^1_0} \hat{m}^1_{j^1_0}}\big)\big)\big(1 + \sum_{j \in I_2, j \neq j^2_0} \operatorname{res}\big(\frac{\hat{l}^2_{j^2} \hat{m}^2_{j^2}}{\hat{l}^2_{j^2_0} \hat{m}^2_{j^2_0}}\big)\big)^{-1} \operatorname{res}(l) \operatorname{res}(m) \\
&= \big(1 + \sum_{j \in I_1, j \neq j^1_0} \operatorname{res}(l^1_j) \operatorname{res}(m^1_j)\big)\big(1 + \sum_{j \in I_2, j \neq j^2_0} \operatorname{res}(l^2_j) \operatorname{res}(m^2_j)\big)^{-1} \operatorname{res}(l) \operatorname{res}(m) \text{ as required.}
\end{aligned}$$

□

We start by recalling some propositions about extending a given isomorphism of valued fields in the reduct $\mathcal{L}_{\text{val}}$. The following is [EHS22, Proposition 2.6].

Proposition 298. *Let L and M be valued fields with $C \subseteq L \cap M$ a common valued subfield. Assume that $\Gamma_L \cap \Gamma_M = \Gamma_C$, k_L and k_M are linearly disjoint over k_C and that L or M has the good separated basis property. Let $\sigma : L \to L'$ be a $\mathcal{L}_{\text{val}}$ valued field isomorphism which is the identity on $C\Gamma_L k_L$. Then σ can be extended to a $\mathcal{L}_{\text{val}}$- valued field isomorphism f from $C(L, M)$ to $C(L', M)$ which is the identity on M and $f \restriction_L = \sigma$.*

We continue arguing that without loss of generality we may assume the $\mathcal{L}_{\text{val}}$-isomorphism to fix the residue field and the value group of M instead.

Proposition 299. *Let L and M be valued fields with $C \subseteq L \cap M$ a common valued subfield. Assume that:*

1. $\Gamma_L \cap \Gamma_M = \Gamma_C$,
2. *k_M and k_L are linearly disjoint over k_C,*
3. *M or L have the good separated basis property over C.*

And let $\sigma : C(L, M) \to C(L', M')$ be an $\mathcal{L}_{\text{val}}$-isomorphism fixing $Ck_M\Gamma_M$, such that $\sigma(L) = L^{'}$ and $\sigma(M) = M'$. Then there is an $\mathcal{L}_{\text{val}}$-isomorphism $\tau : C(L, M) \to C(L', M)$ such that τ is the identity on M and $\tau \restriction_L = \sigma \restriction_L$.

Proof. Let $\sigma : C(L, M) \to C(L', M')$ be the given $\mathcal{L}_{\text{val}}$ isomorphism fixing $Ck_M\Gamma_M$. We want to find an $\mathcal{L}_{\text{val}}$-isomorphism $\tau : C(L, M) \to C(L', M)$ which is the identity on M and such that $\tau \restriction_L = \sigma \restriction_L$. We consider the restriction map $\sigma^{-1} \restriction_{M'}: M' \to M$, an $\mathcal{L}_{\text{val}}$-isomorphism fixing $Ck_M\Gamma_M$. By Proposition 298 there is an $\mathcal{L}_{\text{val}}$-isomorphism $\phi : C(M', L') \to C(M, L')$ that extends $\sigma^{-1} \restriction_{M'}: M' \to M$ and is the identity on L'. Let $\tau : C(L, M) \to C(L', M)$ be the $\mathcal{L}_{\text{val}}$-isomorphism given by the composition $\tau = \phi \circ \sigma$; this map satisfies the required conditions. $\square$

We conclude this section by restating our result in terms of domination for the class of henselian valued fields of equicharacteristic zero with residue field algebraically closed. We first recall a general fact about regular extensions:

Fact 300. *Let F be a field, E a regular field extension of F and R be any other field extension of F. If $E \mathop{\smile\!\!\!\!\!\!|}_F^{alg} R$ then E and R are linearly disjoint over F.*

Proof. See [Alg02, Theorem 4.12 Chapter VIII]. $\square$

Corollary 301. *Let T be some $\mathcal{L}^*_{\text{val}}$ complete extension of the first order theory of henselian valued fields of equicharacteristic zero with residue field algebraically closed. Let $\mathfrak{M}$ be its monster model and C a maximal field. Let $C \subseteq L$ be a valued field extension such that Γ_L/Γ_C is a torsion free extension and k_L is a regular extension of k_C. Then $\text{tp}(L/C)$ is dominated by the residue field and the value group, this is for any field extension $C \subseteq M$ such that $k_M \mathop{\smile\!\!\!\!\!\!|}_{k_C}^{alg} k_L$ and $\Gamma_M \mathop{\smile\!\!\!\!\!\!|}_{\Gamma_C}^{s} \Gamma_L$ we have $\text{tp}(L/Ck_M\Gamma_M) \vdash \text{tp}(L/M)$.*

Proof. Let $C \subseteq M$ be a field extension such that $k_M \mathop{\smile\!\!\!\!\!\!|}_{k_C}^{alg} k_L$ and $\Gamma_M \mathop{\smile\!\!\!\!\!\!|}_{\Gamma_C}^{s} \Gamma_L$. By Fact 300 k_M and k_L are linearly disjoint over k_C. As $\Gamma_M \mathop{\smile\!\!\!\!\!\!|}_{\Gamma_C}^{s} \Gamma_L$, by Fact 294 $\Gamma_M \cap \Gamma_L \subseteq \text{dcl}(\Gamma_C)$. Combining Fact 285 together with the hypothesis of Γ_L/Γ_C is torsion free we obtain that $\Gamma_M \cap \Gamma_L \subseteq \Gamma_C$. Let $L' \vDash \text{tp}(L/Ck_M\Gamma_M)$ and let $\sigma : L \to L'$ be a partial elementary map fixing $Ck_M\Gamma_M$, as the hypothesis of Proposition 298 are satisfied, we can find an automorphism τ of $\mathfrak{M}$ fixing M and extending σ. The map τ must be elementary by Theorem 269, because its restriction to the value group is a partial elementary map of $\Gamma_{\mathfrak{M}}$. We conclude that $\text{tp}(L/Ck_M\Gamma_M) \vdash \text{tp}(L/M)$, as required. $\square$

In the following remark we indicate how forking independence relates to the notions of independence required in Corollary 301.

Remark 302. *Let T be some $\mathcal{L}^*_{\text{val}}$ complete extension of the first order theory of henselian valued fields of equicharacteristic zero with residue field algebraically closed. Let $\mathfrak{M}$ be its monster model, and L, M substructures. Let $C \subseteq L \cap M$ be a common subfield, then if $k_M\Gamma_M \mathop{\smile\!\!\!\!\!\!|}_C k_L\Gamma_L$ we have $k_M \mathop{\smile\!\!\!\!\!\!|}_{k_C}^{alg} k_L$ and $\Gamma_M \mathop{\smile\!\!\!\!\!\!|}_{\Gamma_C}^{s} \Gamma_L$.*

Proof. By Corollary 270, the residue field and the value group are stably embedded and orthogonal to each other, hence $k_M \mathop{\smile\!\!\!\!\!\!|}_{k_C} k_L$ and $\Gamma_M \mathop{\smile\!\!\!\!\!\!|}_{\Gamma_C} \Gamma_L$. Forking independence in the residue field implies in particular algebraic independence, so $k_M \mathop{\smile\!\!\!\!\!\!|}^{alg}_{k_C} k_L$. Forking independence in the value group guarantees forking independence in the reduct to $\mathcal{L}_{AG}$ so $\Gamma_M \mathop{\smile\!\!\!\!\!\!|}^{s}_{\Gamma_C} \Gamma_L$. □

Domination by the residue sort and the value group in the language $\mathcal{L}$.

In this subsection, we let T be some complete extension of the $\mathcal{L}$-theory of henselian valued fields of equicharacteristic zero and we let $\mathfrak{C}$ be its monster model. Given L an $\mathcal{L}$ substructure of $\mathfrak{C}$ and $n \in \mathbb{N}$ we set $(\mathcal{A}_n)_L = \{\mathrm{res}^n(l) \mid l \in L\}$, and $\mathcal{A}_L = ((\mathcal{A}_n)_L \mid n \in \mathbb{N})$. The main goal of this subsection is proving that the type of a valued field over a maximal model C is dominated by the power residue sorts and the value group.

Theorem 303. *Let L and M be substructures of $\mathfrak{C}$, and let C be a maximal model of T which is also a common substructure of L and M. If*

1. $\Gamma_L \mathop{\smile\!\!\!\!\!\!|}^{s}_{\Gamma_C} \Gamma_M$,
2. *k_M and k_L are linearly disjoint over k_C.*

Then $\mathrm{tp}(L/C\mathcal{A}_M\Gamma_M) \vdash \mathrm{tp}(L/M)$.

Proof. Let $L' \models \mathrm{tp}(L/C\mathcal{A}_M\Gamma_M)$ and let σ be a partial elementary map sending L to L' fixing $C\mathcal{A}_M\Gamma_M$. By Fact 294, $\Gamma_L \cap \Gamma_M \subseteq \mathrm{dcl}(\Gamma_C) = \Gamma_C$ because C is definably closed. By Proposition 299, there is an $\mathcal{L}_{\mathrm{val}}$ valued field isomorphism $\tau : C(L, M) \to C(L', M)$ which is the identity on M and $\tau\restriction_L = \sigma$. By Proposition 295, L and M are linearly disjoint over C, L has the good basis property over M, the value group $\Gamma_{C(L,M)}$ is the group generated by Γ_M and Γ_L over Γ_C and the residue field $k_{C(L,M)}$ is the field generated by k_L and k_M over k_C. In particular, any element $x \in C[L, M]$ can be represented as $x = \sum_{i\leq n} l_i m_i$ and

$$v(x) = v(\sum_{i\leq n} l_i m_i) = \min\{v(l_i) + v(m_i) \mid i \leq n\}.$$

As τ is an $\mathcal{L}_{val}$-isomorphism, we have:

$$\tau(v(x)) = \tau\big(\min\{v(l_i) + v(m_i) \mid i \leq n\}\big) = \min\{\sigma(v(l_i)) + v(m_i) \mid i \leq n\} = v(\tau(x)),$$

and because $\sigma : L \to L'$ is a partial elementary map fixing Γ_M, the restriction map $\tau : \Gamma_{C(L,M)} \to \Gamma_{C(L',M)}$ is a partial elementary map. We want to extend the $\mathcal{L}_{\mathrm{val}}$-isomorphism to a $\mathcal{L}$-isomorphism, we start by proving the following claim.

Claim 304. *Fix $n \in \mathbb{N}$ and $x \in C(L, M)$ be such that $v(x) \in n\Gamma$. Then there are $a \in \mathcal{O}^{\times}_{C(L,M)}$, $l \in L$ and $m \in M$ such that $\mathrm{res}^n(x) = \pi_n(\mathrm{res}(a))\,\mathrm{res}^n(l)\,\mathrm{res}^n(m)$.*

Proof. Let $x \in C(L, M)$, as L has the separated basis property over M there are $l' \in L$ and $m' \in M$ such that $v(x) = v(l') + v(m')$. Let $\phi(x, y) = \exists\gamma(x + y = n\gamma)$, because $\vDash \phi(v(l'), v(m'))$ the $\mathcal{L}_{AG}$-formula $\phi(x, y)$ is represented in the type $\mathrm{tp}(v(l')/\Gamma_M)$. Because $\Gamma_L \mathop{\smile\!\!\!\!\!\!\mid}^{s}_{\Gamma_C} \Gamma_M$, by Fact 288 $\mathrm{tp}_{\mathcal{L}_{AG}}(v(l')/\Gamma_M)$ is a heir extension of $\mathrm{tp}_{\mathcal{L}_{AG}}(v(l')/\Gamma_C)$ so we can find some element $c \in C$ such that $\vDash \phi(v(l'), v(c))$. Take $l = l'c \in L$ and $m = m'c^{-1}$, then $v(x) = v(l) + v(m)$ where $v(l), v(m) \in n\Gamma$. Let $a = \frac{x}{lm}$, so $x = a(lm)$ and

$$\mathrm{res}^n(x) = \mathrm{res}^n(a)\,\mathrm{res}^n(lm) = \pi_n(\mathrm{res}(a))\,\mathrm{res}^n(l)\,\mathrm{res}^n(m), \text{ as desired.}$$

□

Let $x, y \in C(L, M)$, $n \in \mathbb{N}$ be such that $\mathrm{res}^n(x) = \mathrm{res}^n(y)$, $a, b \in \mathcal{O}^{\times}_{C(L,M)}$ and $l_1, l_2 \in L$ and $m_1, m_2 \in M$ satisfying $\mathrm{res}^n(x) = \pi_n(\mathrm{res}(a))\,\mathrm{res}^n(l_1)\,\mathrm{res}^n(m_1)$ and

$$\mathrm{res}^n(y) = \pi_n(\mathrm{res}(b))\,\mathrm{res}^n(l_2)\,\mathrm{res}^n(m_2).$$

Thus:

$$\mathrm{res}^n(x) = \mathrm{res}^n(y) \text{ if and only if } \pi_n(\mathrm{res}(a))\,\mathrm{res}^n(l_1)\,\mathrm{res}^n(m_1) = \pi_n(\mathrm{res}(b))\,\mathrm{res}^n(l_2)\,\mathrm{res}^n(m_2).$$

By Fact 296 the equality $\pi_n(\mathrm{res}(a))\,\mathrm{res}^n(l_1)\,\mathrm{res}^n(m_1) = \pi_n(\mathrm{res}(b))\,\mathrm{res}^n(l_2)\,\mathrm{res}^n(m_2)$ can be expressed by a formula in $\mathrm{tp}(L/C\mathcal{A}_M\Gamma_M)$, as σ is an elementary map the same formula holds for the elements in $\sigma(L)$. As a result,

$$\pi_n(\mathrm{res}(\tau(a)))\,\mathrm{res}^n(\tau(l_1))\,\mathrm{res}^n(m_1) = \pi_n(\mathrm{res}(\tau(b)))\,\mathrm{res}^n(\tau(l_2))\,\mathrm{res}^n(m_2)$$

so $\mathrm{res}^n(\tau(x)) = \mathrm{res}^n(\tau(y))$.
Hence we can naturally extend the $\mathcal{L}_{\mathrm{val}}$-isomorphism τ to an $\mathcal{L}$-isomorphism, by taking maps $\tau_n : (\mathcal{A}_n)_{C(L,M)} \to (\mathcal{A}_n)_{C(L',M)}$ sending the residue class $\mathrm{res}^n(x)$ to $\mathrm{res}^n(\tau(x))$. Then $\mathbf{t} = \tau \cup \{\tau_n \mid n \in \mathbb{N}\}$ is a $\mathcal{L}$-isomorphism from $C(L, M)$ into $C(L', M)$ satisfying the following properties:

1. $\mathbf{t} \restriction_M = \mathrm{id}_M$ and $\tau \restriction_L = \sigma$,
2. $\mathbf{t} : \Gamma_{C(L,M)} \to \Gamma_{C(L',M)}$ is a partial elementary map in $\Gamma_{\mathfrak{C}}$ because $\Gamma_{C(L,M)}$ is the value group generated by Γ_M and Γ_L over Γ_C and σ fixes Γ_M,
3. $\mathbf{t} : \mathcal{A}_{(C(L,M))} \to \mathcal{A}_{(C(L',M))}$ is a partial elementary map in $\mathcal{A}_{\mathfrak{C}}$.This follows by the fact that σ is a partial elementary map fixing $\mathcal{A}_M$ combined with Claim 304 and Fact 296.

By Theorem 281 τ is a partial elementary map and therefore can be extended to an automorphism of $\mathfrak{C}$. As a result $\mathrm{tp}(L/M) = \mathrm{tp}(L'/M)$ as required. □

We conclude this section by restating our result in terms of domination.

Corollary 305. *Let $C \subseteq L$ be substructures of $\mathfrak{C}$ with C a maximal model of T. Then* $\mathrm{tp}(L/C)$ *is dominated by the value group and the power residue sorts, this is for any field extension $C \subseteq M$ such that $k_M \mathop{\smile\!\!\!\!\!\!\mid}^{alg}_{k_C} k_L$ and $\Gamma_M \mathop{\smile\!\!\!\!\!\!\mid}^{s}_{\Gamma_C} \Gamma_L$ we have* $\mathrm{tp}(L/C\mathcal{A}_M\Gamma_M) \vdash \mathrm{tp}(L/M)$

Proof. We want to show that $\mathrm{tp}(L/C\mathcal{A}_M\Gamma_M) \vdash \mathrm{tp}(L/M)$. Because C is a model and the residue field is of characteristic zero, k_L is a regular extension of k_C. By hypothesis $k_M \mathop{\smile\!\!\!\!\!\!\mid}^{alg}_{k_C} k_L$, by Fact 300 k_L and k_M must be linearly disjoint over k_C. By Fact 262, L has the good separated basis property over C. Hence, the hypothesis of Theorem 303 are satisfied, so $\mathrm{tp}(L/C\mathcal{A}_M\Gamma_M) \vdash \mathrm{tp}(L/M)$ as required. □

The following remark emphasizes how forking independence relates to the required independence conditions in Corollary 305.

Remark 306. *Let C and L be as in Corollary 305. Let $C \subseteq M$ be a field extension, such that $\mathcal{A}_M\Gamma_M \mathop{\smile\!\!\!\!\!\!\mid}_{\mathcal{A}_C\Gamma_C} \mathcal{A}_L\Gamma_L$ then $k_M \mathop{\smile\!\!\!\!\!\!\mid}^{alg}_{k_C} k_L$ and $\Gamma_M \mathop{\smile\!\!\!\!\!\!\mid}^{s}_{\Gamma_C} \Gamma_L$.*

Proof. Let $M \supseteq C$ be another structure such that $\mathcal{A}_M\Gamma_M \mathop{\smile\!\!\!\!\!\!\mid}_{\mathcal{A}_C\Gamma_C} \mathcal{A}_L\Gamma_L$, because the sorts $\mathcal{A}$ and Γ are orthogonal and purely stably embedded this is equivalent to $\mathcal{A}_M \mathop{\smile\!\!\!\!\!\!\mid}_{\mathcal{A}_C} \mathcal{A}_L$ and $\Gamma_L \mathop{\smile\!\!\!\!\!\!\mid}_{\Gamma_C} \Gamma_M$. In particular in the reduct to $\mathcal{L}_{AG}$ it must be the case that $\Gamma_M \mathop{\smile\!\!\!\!\!\!\mid}^{s}_{\Gamma_C} \Gamma_L$. Because $\mathcal{A}_L \mathop{\smile\!\!\!\!\!\!\mid}_{\mathcal{A}_C} \mathcal{A}_M$, in particular k_L and k_M are algebraically independent over k_C. □

Domination by the residue field and the value group in the $\mathcal{L}_{\mathrm{ac}}$-language

In this subsection we prove a domination result for henselian valued fields of equicharacteristic zero in the language $\mathcal{L}_{\mathrm{ac}}$, using Theorem 274.
Adding an angular component simplifies significantly the henselian valued field, in fact it corresponds to having the exact sequence

$$1 \to k^{\times} \to RV \to \Gamma \to 0,$$

to split. However, it should be noted that adding an angular component increases the set of definable sets, so it is interesting to understand as well domination results in this framework by its own sake.
Through this section T is some complete extension of the $\mathcal{L}_{\mathrm{ac}}$-theory of henselian valued fields of equicharacteristic zero and $\mathfrak{C}$ is the monster model. Given M a substructure of $\mathfrak{C}$, we will write $k(M)$ to denote $\mathrm{dcl}(M) \cap k_{\mathfrak{C}}$ and we observe that $\mathrm{ac}(M) \subseteq k(M)$.

Theorem 307. *Let L and M be good substructures of $\mathfrak{C}$, and let C be a maximal model of T which is also a common substructure of L and M. If the following conditions hold*

1. *k_M and k_L are linearly disjoint over k_C,*
2. *$\Gamma_M \cap \Gamma_L = \Gamma_C$,*

3. M or L have the good separated basis property over C,

then $\mathrm{tp}(L/Ck(M)\Gamma_M) \vdash \mathrm{tp}(L/M)$.

Proof. As in Theorem 303 we start taking $L' \vDash \mathrm{tp}(L/Ck(M)\Gamma_M)$, and σ a partial elementary map sending L to L' fixing $Ck(M)\Gamma_M$. By Proposition 299, there is a $\mathcal{L}_{\mathrm{val}}$ valued field isomorphism $\tau : C(L, M) \to C(L', M)$ which is the identity on M and $\tau \restriction_L = \sigma$.
By Proposition 295, L and M are linearly disjoint over C, M (or L) has the separated basis property over L (or M), the value group $\Gamma_{C(L,M)}$ is the group generated by Γ_M and Γ_L over Γ_C and the residue field $k_{C(L,M)}$ is the field generated by k_L and k_M over k_C. In particular, any element $x \in C[L, M]$ can be represented as $x = \sum_{i \leq n} l_i m_i$ and

$$v(x) = v(\sum_{i \leq n} l_i m_i) = \min\{v(l_i) + v(m_i) \mid i \leq n\}.$$

As τ is an $\mathcal{L}_{\mathrm{val}}$-isomorphism, we have:

$$\tau(v(x)) = \tau\big(\min\{v(l_i) + v(m_i) \mid i \leq n\}\big) = \min\{\sigma(v(l_i)) + v(m_i) \mid i \leq n\} = v(\tau(x)).$$

We want to extend the $\mathcal{L}_{\mathrm{val}}$-isomorphism to a $\mathcal{L}_{\mathrm{ac}}$-isomorphism, so it is sufficient to verify that τ respects also the angular component map.

Claim 308. *Let $x \in C[L, M]$ then there are $a \in \mathcal{O}^{\times}_{C(L,M)}$, $l \in L$ and $m \in M$ such that $x = alm$. In particular, $\tau(\mathrm{ac}(x)) = \mathrm{ac}(\tau(x))$ and $\mathrm{ac}(x) = \mathrm{res}(a)\,\mathrm{ac}(l)\,\mathrm{ac}(m)$.*

Proof. Let $x \in C[L, M]$ and suppose that $x = \sum_{i \leq n} l_i m_i$. Because M (or L) has the separated basis property over L (or M) there is some $i_0 \leq n$ such that $v(x) = v(l_{i_0} m_{i_0})$. Let $a = \frac{x}{l_{i_0} m_0} \in \mathcal{O}^{\times}_{C(L,M)}$, then

$$\mathrm{ac}(x) = \mathrm{ac}(l_{i_0} m_{i_0} a) = \mathrm{ac}(l_{i_0})\,\mathrm{ac}(m_{i_0})\,\mathrm{ac}(a) = \mathrm{ac}(l_{i_0})\,\mathrm{ac}(m_{i_0})\,\mathrm{res}(a).$$

Note that $\tau(x) = \tau(a)\sigma(l_{i_0})m_{i_0}$. Thus:

$$\begin{aligned} \tau(\mathrm{ac}(x)) &= \tau(\mathrm{ac}(l_{i_0})\,\mathrm{ac}(m_{i_0})\,\mathrm{res}(a)) = \tau(\mathrm{ac}(l_{i_0}))\tau(\mathrm{ac}(m_{i_0}))\tau(\mathrm{res}(a)) \\ &= \mathrm{ac}(\sigma(l_{i_0}))\,\mathrm{ac}(m_{i_0})\,\mathrm{res}(\tau(a)) = \mathrm{ac}(\sigma(l_{i_0})m_{i_0}\tau(a))) = \mathrm{ac}(\tau(x)), \text{ as required.} \end{aligned}$$

□

We conclude that τ is an $\mathcal{L}_{\mathrm{ac}}$-isomorphism, and because $\Gamma_{C(L,M)}$ is the group generated by Γ_L and Γ_M over Γ_C and σ fixes Γ_M, then $\tau \restriction \Gamma_{C(L,M)} \to \Gamma_{C(L',M)}$ in an elementary map in Γ. Combining Claim 308, the fact that the residue field $k_{C(L,M)}$ is the field generated by k_L and k_M over k_C and $\mathrm{ac}(M) \subseteq k(M)$ is fixed by σ, we can conclude that $\tau \restriction \mathrm{ac}(C(L, M))$ is an elementary map in $\mathbf{k}$. By Theorem 274, such map must be elementary. □

We restate our result in terms of domination, and we highlight that we required weaker hypothesis compare to Corollary 305.

Corollary 309. *Let T be some complete extension of the $\mathcal{L}_{\mathrm{ac}}^{*}$ first order theory of henselian valued fields of equicharacteristic zero and let $\mathfrak{C}$ be its monster model. Let $C \subseteq L$ be substructures of $\mathfrak{C}$, with C maximal, k_L a regular extension of k_C and Γ_L/Γ_C torsion free. Then* $\mathrm{tp}(L/C)$ *is dominated by the value group and the residue field, this is for any field extension $C \subseteq M$ if $k_M \mathop{\smile\!\!\!\!\!\!\mid}_{k_C}^{alg} k_L$ and $\Gamma_M \mathop{\smile\!\!\!\!\!\!\mid}_{\Gamma_C}^{s} \Gamma_L$ then* $\mathrm{tp}(L/Ck_M\Gamma_M) \vdash \mathrm{tp}(L/M)$.

Proof. The argument follows in a very similar manner as Corollary 301. □

Remark 310. 1. *As in Remark 302, using the purely stable embeddeness and orthogonality between the residue field and the value group one can obtain that forking independence implies the independence conditions required in 309, this is if $k_M\Gamma_M \mathop{\smile\!\!\!\!\!\!\mid}_{C} k_L\Gamma_L$ implies that $k_M \mathop{\smile\!\!\!\!\!\!\mid}_{k_C}^{alg} k_L$ and $\Gamma_M \mathop{\smile\!\!\!\!\!\!\mid}_{\Gamma_C}^{s} \Gamma_L$.*

2. *A similar version of Corollary 309 can be obtained for the language $\mathcal{L}_{\mathrm{ac}}$ without adding the constants and requiring C to be a model of T. The proof is similar to Corollary 305.*

3.3 Forking over maximal models in NTP_2 henselian valued fields

In this section we apply the domination results obtained in Section 3.2 to show that forking independence over maximal models is controlled by Shelah's imaginary expansion of the value group and Shelah's imaginary expansion of the residue field in the class of henselian valued fields of equicharacteristic zero which are NTP_2.
We start by introducing some notation:

Notation 311. 1. *Through this section we will work with a slight refinement of the languages introduced in Subsection 3.1. We denote as $\mathcal{L}'$, $\mathcal{L}'_{\mathrm{ac}}$ and $\mathcal{L}'_{\mathrm{val}}$ the extension of $\mathcal{L}$, $\mathcal{L}_{\mathrm{ac}}$ and $\mathcal{L}_{\mathrm{val}}$ (respectively); where the residue field is equipped with the multi-sorted Shelah's imaginary expansion k^{eq} as well as the value group is endorsed with the language of Γ^{eq}. Likewise*

2. *Let T be a complete first order theory in the language $\mathcal{L}'$. Given $\mathcal{S}$ a family of $\mathcal{L}'$ sorts and A a set of parameters, we write $\mathcal{S}(A)$ to denote* $\mathrm{dcl}(A) \cap \mathcal{S}$.

It is well known that in general forking and dividing are different notions, however, they do coincide in a very large class of theories (sometimes over arbitrary sets of parameters or only over models). In [CK12] Chernikov and Kaplan shown that if a theory is NTP_2 then forking and dividing over models are the same. The following is a folklore fact, and it is the left-transitivity of dividing in any theory.

Fact 312. *Let T be a complete first order theory and $\mathfrak{M}$ its monster model. Let $C \subseteq \mathfrak{M}$ be a set of parameters, $a, b, d \in \mathfrak{M}$, if $d \mathop{\smile\!\!\!\!\!\!\;\mid}^d_C b$ and $a \mathop{\smile\!\!\!\!\!\!\;\mid}^d_{Cd} b$ then $ad \mathop{\smile\!\!\!\!\!\!\;\mid}^d_C b$.*

In [Che14, Theorem 7.6] Chernikov proved that a henselian valued field of equicharacteristic zero in the $\mathcal{L}_{\mathrm{ac}}$ language is NTP_2 if and only if its residue field is NTP_2. Later in [Tou18, Theorem 3.11] P. Touchard proved that if $\mathcal{K} = (K, \mathbf{k}, \Gamma, \mathrm{ac}, res, v)$ is a henselian valued field of equicharacteristic zero then $\mathrm{bdn}(\mathcal{K}_{\mathrm{ac}}) = \mathrm{bdn}(\mathbf{k}) + \mathrm{bdn}(\Gamma)$, where $\mathrm{bdn}(X)$ is the burden of the definable set X as defined in [Tou18, Definition 1.12]. He also showed that if a valued field of equicharacteristic zero is considered in the language $\mathcal{L}$ then $\mathrm{bdn}(\mathcal{K}) = \max_{n \geq 0}(\mathrm{bdn}(k^\times/(k^\times)^n) + \mathrm{bdn}(n\Gamma))$, therefore a henselian valued field of equicharacteristic zero is NTP_2 if and only if its residue field is NTP_2.

Lemma 313. *Let C be some set of parameters and a, b tuples in the main field sort.*

1. *In the $\mathcal{L}'$ language, $\mathcal{A}(Ca)\Gamma(Ca) \mathop{\smile\!\!\!\!\!\!\;\mid}_C b$ if and only if $\mathcal{A}(Ca)\Gamma(Ca) \mathop{\smile\!\!\!\!\!\!\;\mid}_C \mathcal{A}^{eq}(Cb)\Gamma^{eq}(Cb)$.*
2. *In the $\mathcal{L}'_{\mathrm{ac}}$ language, $k(Ca)\Gamma(Ca) \mathop{\smile\!\!\!\!\!\!\;\mid}_C b$ if and only if $k(Ca)\Gamma(Ca) \mathop{\smile\!\!\!\!\!\!\;\mid}_C k^{eq}(Cb)\Gamma^{eq}(Cb)$.*
3. *For the theory of henselian valued fields of equicharacteristic zero with residue field algebraically closed, in the $\mathcal{L}'_{\mathrm{val}}$-language $k(Ca)\Gamma(Ca) \mathop{\smile\!\!\!\!\!\!\;\mid}_C b$ if and only if*
$$k(Ca)\Gamma(Ca) \mathop{\smile\!\!\!\!\!\!\;\mid}_C k(Cb)\Gamma^{eq}(Cb).$$

Proof. We start by proving the first statement. The left to right direction is clear, because if $\mathcal{A}(Ca)\Gamma(Ca) \mathop{\smile\!\!\!\!\!\!\;\mid}_C b$ then $\mathcal{A}(Ca)\Gamma(Ca) \mathop{\smile\!\!\!\!\!\!\;\mid}_C \mathrm{acl}(Cb)$, and $\mathcal{A}^{eq}(Cb)\Gamma^{eq}(Cb) \subseteq \mathrm{acl}(Cb)$. We proceed to prove the converse.
Suppose that $\mathcal{A}(Ca)\Gamma(Ca) \mathop{\smile\!\!\!\!\!\!\;\mid}_C \mathcal{A}^{eq}(Cb)\Gamma^{eq}(Cb)$, because $\mathcal{A}$ and Γ are orthogonal to each other, this is equivalent to $\mathcal{A}(Ca) \mathop{\smile\!\!\!\!\!\!\;\mid}_C \mathcal{A}^{eq}(Cb)$ and $\Gamma(Ca) \mathop{\smile\!\!\!\!\!\!\;\mid}_C \Gamma^{eq}(Cb)$. By Corollary 282 Γ and $\mathcal{A}$ are orthogonal to each other and are purely stably embedded, thus $\mathcal{A}(Ca)\Gamma(Ca) \mathop{\smile\!\!\!\!\!\!\;\mid}_C b$ if and only if $\mathcal{A}(Ca) \mathop{\smile\!\!\!\!\!\!\;\mid}_C b$ and $\Gamma(Ca) \mathop{\smile\!\!\!\!\!\!\;\mid}_C b$. Hence, it is sufficient to prove that $\mathcal{A}(Ca) \mathop{\smile\!\!\!\!\!\!\;\mid}_C b$ and $\Gamma(Ca) \mathop{\smile\!\!\!\!\!\!\;\mid}_C b$.

Claim 314. *$\mathcal{A}(Ca) \mathop{\smile\!\!\!\!\!\!\;\mid}_C b$ and $\Gamma(Ca) \mathop{\smile\!\!\!\!\!\!\;\mid}_C b$.*

We proceed by contradiction, and we assume that $\mathrm{tp}(\mathcal{A}(Ca)/Cb)$ forks over C to show that $\mathrm{tp}(\mathcal{A}(Ca)/C\mathcal{A}^{eq}(Cb))$ forks over C. We can find a formula $\phi(\bar{x}, b) \in \mathrm{tp}(\mathcal{A}(Ca)/Cb)$, and finite set of formulas $\{\psi_i(\bar{x}, d_i) \mid i \leq n\}$ such that $\phi(\bar{x}, b) \vdash \bigvee_{i \leq l} \psi_i(\bar{x}, d_i)$, where each formula $\psi_i(\bar{x}, d_i)$ divides over C.
As $\mathcal{A}$ is purely stably embedded, the subset of $\mathcal{A}^n$ defined by $\phi(\bar{x}, b)$ is also defined by a formula $\eta(\bar{x}, e)$ where e is a tuple of elements in $\mathcal{A}^{eq}(Cb)$. By a similar argument, the set defined by each formula $\psi_i(\bar{x}, d_i)$ is also defined by a formula $\epsilon_i(\bar{x}, f_i)$ where f_i is a tuple of

elements in $\mathcal{A}^{eq}(Cd_i)$. Because $\phi(\bar{x}, b)$, $\eta(\bar{x}, e)$ define the same set, as $\psi_i(\bar{x}, d_i)$ and $\epsilon_i(\bar{x}, f_i)$ do, then it is also the case that $\eta(\bar{x}, e) \vdash \bigvee_{i \leq l} \epsilon_i(\bar{x}, f_i)$. Since $\eta(\bar{x}, e) \in \mathrm{tp}(\mathcal{A}(Ca)/\mathcal{A}^{eq}(Cb))$ it is sufficient to argue that $\epsilon_i(\bar{x}, f_i)$ also divides over C.
Each formula $\psi_i(\bar{x}, d_i)$ divides over C, so we can find an infinite sequence $\langle b_j \mid j < \omega \rangle$ in the type $\mathrm{tp}(d_i/C)$ such that $b_0 = d_i$ and $\{\psi_i(\bar{x}, b_j) \mid j < \omega\}$ is m_i-inconsistent. Let σ_j be an automorphism of the monster model sending b_0 to b_j and fixing C. Let $g_j = \sigma_j(f_i)$, then $g_0 = f_i$, $\langle g_j \mid j < \omega \rangle$ is in the type $\mathrm{tp}(f_i/C)$. As a result, $\{\epsilon_i(\bar{x}, g_j) \mid j < \omega\}$ is also m_i-inconsistent, because $\psi_i(\bar{x}, b_j)$ and $\epsilon_i(\bar{x}, g_j)$ define the same subset of $\mathcal{A}^n$. Consequently, each $\epsilon_i(\bar{x}, f_i)$ divides over C, so $\mathrm{tp}(\mathcal{A}(Ca)/C\mathcal{A}^{eq}(Cb))$ forks over C. We conclude that if $\mathcal{A}(Ca) \mathop{\smile\!\!\!\!\!\!|}_C \mathcal{A}^{eq}(Cb)$ then $\mathcal{A}(Ca) \mathop{\smile\!\!\!\!\!\!|}_C b$. Likewise, one can show that if $\Gamma(Ca) \mathop{\smile\!\!\!\!\!\!|}_C \Gamma^{eq}(Cb)$ then $\Gamma(Ca) \mathop{\smile\!\!\!\!\!\!|}_C b$. This concludes the proof of the right to left direction.
Likewise, we can conclude similarly the second and the third statement. In fact, the proof only requires that the residue field and the value group are orthogonal to each other and are purely stably embedded. This is guaranteed by Corollary 275 and 270 respectively. $\square$

Theorem 315. *Let C be some maximal model and assume that the residue field is NTP_2.*

1. *In the $\mathcal{L}'$-language, $a \mathop{\smile\!\!\!\!\!\!|}_C b$ if and only if $\mathcal{A}(Ca)\Gamma(Ca) \mathop{\smile\!\!\!\!\!\!|}_C \mathcal{A}^{eq}(Cb)\Gamma^{eq}(Cb)$.*
2. *In the $\mathcal{L}'_{\mathrm{ac}}$-language, $a \mathop{\smile\!\!\!\!\!\!|}_C b$ if and only if $k(Ca)\Gamma(Ca) \mathop{\smile\!\!\!\!\!\!|}_C k^{eq}(Cb)\Gamma^{eq}(Cb)$.*

Proof. We start proving the first statement, the left to right direction is clear. We assume that
$\mathcal{A}(Ca)\Gamma(Ca) \mathop{\smile\!\!\!\!\!\!|}_C \mathcal{A}^{eq}(Cb)\Gamma^{eq}(Cb)$. By Corollary 282 $\mathcal{A}$ and Γ are purely stably embedded and orthogonal to each other, so this is equivalent to $\mathcal{A}(Ca) \mathop{\smile\!\!\!\!\!\!|}_C \mathcal{A}^{eq}(Cb)$ and $\Gamma(Ca) \mathop{\smile\!\!\!\!\!\!|}_C \Gamma^{eq}(Cb)$. Because $\mathcal{A}(Ca) \mathop{\smile\!\!\!\!\!\!|}_C \mathcal{A}^{eq}(Cb)$, then $k(Ca) \mathop{\smile\!\!\!\!\!\!|}^{alg}_{k_C} k(Cb)$. As C is a model and the residue field is of characteristic zero, $k(Ca)$ is a regular extension of k_C so we can apply Fact 300 to conclude that $k(Ca)$ and $k(Cb)$ are linearly disjoint over k_C. Because $\Gamma(Ca) \mathop{\smile\!\!\!\!\!\!|}_C (Cb)$, then $\Gamma(Ca) \mathop{\smile\!\!\!\!\!\!|}^{s}_{\Gamma_C} \Gamma(Cb)$, as Γ is a stably embedded sort and we are considering the reduct to the language of abelian groups.

Claim 316. $a \mathop{\smile\!\!\!\!\!\!|}^{d}_{C\mathcal{A}(Ca)\Gamma(Ca)} b$.

Proof. Let $p(x, C\mathcal{A}(Ca)\Gamma(Ca)) = \mathrm{tp}(a/C\mathcal{A}(Ca)\Gamma(Ca), b)$. It is sufficient to argue that no formula $\phi(\bar{x}, b) \in p(x, C\mathcal{A}(Ca)\Gamma(Ca), b)$ divides over $C\mathcal{A}(Ca)\Gamma(Ca)$. Let $\langle b_i \mid i \in \omega \rangle$ a $C\mathcal{A}(Ca)\Gamma(Ca)$-indiscernible sequence in the type $\mathrm{tp}(b/C\mathcal{A}(Ca)\Gamma(Ca))$. Let σ_i be an automorphism of $\mathfrak{C}$ fixing $C\mathcal{A}(Ca)\Gamma(Ca)$ sending b to b_i. By Theorem 303 we can find an automorphism τ_i of $\mathfrak{C}$ which is the identity on $\mathrm{dcl}(Ca)$ and whose restriction to $\mathrm{dcl}(Cb)$ coincides with σ_i. In particular,

$$\models \phi(a, b) \leftrightarrow \models \phi(\tau_i(a), \tau_i(b)) \leftrightarrow \models \phi(a, b_i) \text{ for any } i < \omega.$$

so $\{\phi(x, b_i) \mid i < \omega\}$ is consistent, and we conclude that $\phi(x, b)$ does not divide over $C\mathcal{A}(Ca)\Gamma(Ca)$ as required. □

Combining Claim 316 with Lemma 313 we have that $a \mathop{\smile\!\!\!\!\!\!\mid}^{d}_{C\mathcal{A}(Ca)\Gamma(Ca)} b$ and $\mathcal{A}(Ca)\Gamma(Ca) \mathop{\smile\!\!\!\!\!\!\mid}^{d}_{C} b$ so we can apply Fact 312 to conclude that $a\mathcal{A}(Ca)\Gamma(Ca) \mathop{\smile\!\!\!\!\!\!\mid}^{d}_{C} b$. As forking is equal to dividing over models in NTP_2 theories we have $a\mathcal{A}(Ca)\Gamma(Ca) \mathop{\smile\!\!\!\!\!\!\mid}_{C} b$. Because $\mathcal{A}(Ca)\Gamma(Ca) \subseteq \operatorname{acl}(Ca)$, this is equivalent to $a \mathop{\smile\!\!\!\!\!\!\mid}_{C} b$.
Likewise, we can conclude the second statement for the $\mathcal{L}'_{\text{ac}}$-language, using Theorem 307 and Corollary 275 instead. We observe that there is no need to work with the extension $\mathcal{L}^*_{\text{ac}}$, as the independence assumption over the value group implies that $\Gamma(Ca) \cap \Gamma(Cb) \subseteq \Gamma(C) = \Gamma_C$, because C is definably closed. □

Proposition 317. *Let T be some complete extension of the $\mathcal{L}^*_{dp}$- first order theory of henselian valued fields of equicharacteristic zero with residue field algebraically closed and whose value group is dp-minimal. Let $\mathfrak{C}$ be its monster model and $C \subseteq \mathfrak{C}$ be a maximal field. Let $a, b \in \mathfrak{C}$ and suppose that $k(Ca)$ is a regular extension of k_C and $\Gamma(Ca)/\Gamma_C$ is torsion free. We have that $a \mathop{\smile\!\!\!\!\!\!\mid}_{C} b$ if and only if $k(Ca)\Gamma(Cb) \mathop{\smile\!\!\!\!\!\!\mid}_{C} k(Cb)\hat{\Gamma}(Cb)$. Where $\hat{\Gamma} = \Gamma \cup \{\Gamma/\Delta \mid \Delta$ is a convex subgroup$\}$.*

Proof. The proof follows by a very similar argument as in Theorem 315, applying Proposition 299 instead. In fact, $\Gamma_M \cap \Gamma_L \subseteq \operatorname{dcl}(\Gamma_C) = (\mathbb{Q} \otimes \Gamma_C) \cap \Gamma_{\mathfrak{C}}$ and as $\Gamma(Ca)/\Gamma_C$ is torsion free we have that $\Gamma_M \cap \Gamma_L = \Gamma_C$. Also, the independence in the residue field together with the assumption of $k(Ca)$ being a regular extension of k_C guarantees that $k(Ca)$ and $k(Cb)$ are linearly disjoint over k_C. We can apply the equivalence between forking and dividing over sets of parameters in the main sort by a result of Cotter and Starchenko [CS12, the remarks preceding Proposition 2.6 together with Corollary 5.6].

□

3.4 Domination by the internal sorts to the residue field in the language $\mathcal{L}_{RV}$

In this section we investigate domination of a field by the sorts internal to the residue field over the value group in the $\mathcal{L}_{RV}$- languange. We start by fixing some notation.

Notation 318. *Given a field F we denote as F^{alg} its field algebraic closure.*

Let K be an $\mathcal{L}_{RV}$-structure, we will write k_K to denote the residue field of K, Γ_K to denote its value group and $RV_K = \{rv(k) \mid k \in K\}$.
Any henselian valued field K can be naturally embedded into a model of $ACVF$, in fact we can simply take the algebraic closure of K with the unique extension of v to K^{alg}. We denote by $\mathcal{O}$ the valuation ring of K and $\mathcal{M}$ its prime ideal, while $\mathcal{O}^{alg}$ is the valuation ring of K^{alg} and $\mathcal{M}^{alg}$ indicates its maximal ideal. Hence, the sort RV_K can be naturally

embedded into $RV_{K^{alg}}$, by sending the class $x(1+\mathcal{M})$ to $x(1+\mathcal{M}^{alg})$. Likewise, there is a natural embedding from the residue field of K into the residue field of K^{alg}, where for $x \in \mathcal{O}$ we send the class $x+\mathcal{M}$ to $x+\mathcal{M}^{alg}$.

In [HHM06, Section 3.1] Haskell, Hrushovski and Macpherson introduced the well known *geometric language* $\mathcal{L}_{\mathcal{G}}$, in which $ACVF$ eliminates imaginaries (see [HHM06, Theorem 1.0.1]). Through this section we will equip any model of $ACVF$ with the language $\mathcal{L}_{ACVF}$ extending the language of $\mathcal{L}_{\mathcal{G}}$ and a RV-sort.

Notation 319. *Let T be the $\mathcal{L}_{RV}$-theory of henselian valued fields of equicharacteristic zero. We will denote by $\mathfrak{C}$ its monster model, which can be embedded into the monster model $\mathfrak{U}$ of $ACVF$. Through this section we will work in both theories, so we emphasize the notation that we will be using to distinguish both theories. We will simply denote as* dcl, acl, *or* $\mathrm{tp}(A/C)$ *the definable closure, algebraic closure or the type in the language $\mathcal{L}_{RV}$. While we emphasize that* $\mathrm{dcl}_{ACVF}, \mathrm{acl}_{ACVF}$ *or* tp_{ACVF} *indicate the definable closure, algebraic closure or the type in the geometric language. We recall our notation, given S a stably embedded sort in the $\mathcal{L}_{RV}$- theory and $A \subseteq \mathfrak{C}$ a set of parameters we denote as $S(A) = S \cap \mathrm{dcl}(A)$, while if S is a stably embedded sort in $ACVF$ we indicate by $S_{ACVF}(A) = S \cap \mathrm{dcl}_{ACVF}(A)$.*

Definition 320. 1. *A definable set E is said to be* internal *to a definable set D if there is some finite set of parameters F such that $E \subseteq \mathrm{dcl}^{eq}(F \cup D)$.*

2. *A family of definable sets $\{E_i\}_{i\in I}$ is said to be internal to a definable set D if for each $i \in I$ we have that E_i is internal to D.*

Definition 321. *Let $S \subseteq \Gamma_{\mathfrak{C}}$ and M be a substructure of $\mathfrak{C}$ such that $S \subseteq \Gamma_M$. We write*

$$\mathrm{kInt}_S^M = \{k_M\} \cup \{RV_M \cap v_{RV}^{-1}(\gamma) \mid \gamma \in S\}.$$

For each $\gamma \in S$, $RV_{\mathfrak{C}} \cap v_{RV}^{-1}(\gamma)$ is internal to the residue field and the parameters required to witness the internality lie in $RV_{\mathfrak{C}} \cap v_{RV}^{-1}(\gamma)$. Indeed, given $b(1+\mathcal{M}) \in RV_{\mathfrak{C}} \cap v_{RV}^{-1}(\gamma)$ the map $f : \mathcal{O}^{\times}/(1+\mathcal{M}) \to RV \cap v_{RV}^{-1}(\gamma)$ sending the element $x(1+\mathcal{M})$ to $\big(b(1+\mathcal{M})\big)\cdot\big(x(1+\mathcal{M})\big) = bx(1+\mathcal{M})$ is a bijection. In particular, for each $\gamma \in S$, $RV_{\mathfrak{C}} \cap v_{RV}^{-1}(\gamma)$ is stably embedded and so it is $\mathrm{kInt}_S^{\mathfrak{C}}$.
Similarly, we can consider the structures $RV_{\mathfrak{U}} \cap v_{RV}^{-1}(\gamma)$, $\mathrm{kInt}_S^{\mathfrak{U}}$ and obtain the same results in this setting.

In the case of $ACVF$, let $C \subseteq \mathfrak{U}$ a set of parameters and L a substructure of $\mathfrak{U}$, then $\mathrm{acl}_{ACVF}(\mathrm{kInt}_{\Gamma_L}^M)$ is precisely the part of $\mathfrak{U}^{eq}$ which is internal to the residue field and contained in sets definable over C and $\Gamma_{ACVF}(L)$ (see [HHM05, Lemma 12.9]). In $ACVF$ the residue field is an algebraically closed field, so it has a strongly minimal theory and forking independence coincides with algebraic independence in the field theoretic sense. In [HHM06, Lemma 2.6.2], Haskell, Hrushovski and Macpherson characterize the definable sets that are internal to the residue field precisely as those that are stable and stably embedded, or more precisely as those that have finite Morley rank with the induced structure. In particular, in $ACVF$ the multi-sorted structure $\mathrm{acl}_{ACVF}(\mathrm{kInt}^M_{\Gamma_L})$ is naturally equipped with a well-behaved notion of independence, which is simply forking independence in stable theories.

We will not investigate this in the more general setting of henselian valued fields of equicharacteristic zero. Instead we use the fact that any henselian valued field of equicharacteristic zero can be naturally embedded into a model of $ACVF$ and we use the well-behaved notion of independence induced there, which in our setting corresponds to independence for the quantifier free and stable formulas.

Definition 322. *Let L and M be substructures of $\mathfrak{C}$ such that* $\mathrm{dcl}(L) = L$ *and* $\Gamma_L \subseteq \Gamma_M$. *We consider these structures embedded in the monster model $\mathfrak{U}$ of $ACVF$. Suppose that $\Gamma_L \subseteq \Gamma_M$ and let $C \subseteq L \cap M$ be a common valued field. We say that* $\mathrm{kInt}^L_{\Gamma_L} \mathop{\smile\!\!\!\!\!\mid}^{qfs}_{C\Gamma_L} \mathrm{kInt}^M_{\Gamma_L}$ *in $\mathfrak{C}$ if and only if* $\mathrm{acl}_{ACVF}(\mathrm{kInt}^L_{\Gamma_L}) \mathop{\smile\!\!\!\!\!\mid}_{C\Gamma_L} \mathrm{acl}_{ACVF}(\mathrm{kInt}^M_{\Gamma_L})$ *in $\mathfrak{U}$.*

Our next goal is showing that types over maximally complete bases are dominated by the sorts internal to the residue field, to achieve this final milestone we will need Lemma 327, which generalizes [HHM05, Lemma 12.9 and 12.10], both obtained for algebraically closed substructures.
The following is a well-known fact for valued field extensions, we use [vdDKM+12] as a reference.

Fact 323. *Let $C \subseteq L$ be a valued field extension, where $\mathcal{O}_C$ is the valuation ring of C and $\mathcal{O}_L$ is the valuation ring of L. Let $(b_i \mid i \in \mathcal{I})$ be a sequence of elements of $\mathcal{O}_L^\times$ such that* $\mathrm{res}(b_i)$ *in $\mathbf{k}_L$ is algebraically independent over $\mathbf{k}_C$. And let $(a_j \mid j \in J)$ be a family of elements of $L^\times$ such that the family $\big(v(a_j) \mid j \in J\big)$ in $\mathbb{Q} \otimes \Gamma_L$ is $\mathbb{Q}$-linearly independent over $\mathbb{Q} \otimes \Gamma_C$. Assume that $I \cap J = \emptyset$ and define $d_k \in L$ for $k \in I \cup J$ by $d_i = b_i$ for $i \in I$ and $d_j = a_j$ for $j \in J$. Then:*

1. *$\big(d_k \mid k \in I \cup J\big)$ in L is algebraically independent over C, and*
2. *if $C \subseteq L$ is an extension of finite transcendence degree, then*

$$\mathrm{trdeg}(\mathbf{k}_L/\mathbf{k}_C) + \dim_{\mathbb{Q}}(\mathbb{Q} \otimes \Gamma_L/\mathbb{Q} \otimes \Gamma_C) \leq \mathrm{trdeg}(L/C).$$

The second statement is known as the Zariski-Abhyankar Inequality *and it is a direct consequence of the first one.*

Proof. This is [vdDKM+12, Lemma 3.24 and Corollary 3.25]. □

Fact 324. *Let L be a substructure of $\mathfrak{U}$, then*

$$\Gamma_{ACVF}(L) = \mathbb{Q} \otimes \Gamma_L \text{ and}$$
$$k_{ACVF}(L) \subseteq \mathrm{acl}_{ACVF}(L) \cap \mathbf{k} = k_L^{alg}.$$

Proof. Let L be a substructure of $\mathfrak{U}$ then $L \subseteq L^{alg} \subseteq \mathfrak{U}$ and $L^{alg} \models ACVF$. In particular, $\Gamma_{ACVF}(L) \subseteq \Gamma_{ACVF}(L^{alg})$. It is therefore sufficient to argue that $\Gamma_{ACVF}(L^{alg}) = \mathbb{Q} \otimes \Gamma_L$. Let $\gamma \in \Gamma_{ACVF}(L^{alg})$ and $\phi(x, l)$ be some formula witnessing that γ is in the definable closure of L^{alg}, so $l \in L^{alg}$. As Γ is purely stably embedded there is some $\mathcal{L}_{OAG}$-formula $\psi(x, a)$ such that

$$\mathfrak{U} \models \forall x\big((\phi(x, l) \leftrightarrow \psi(x, a)\big).$$

Because L^{alg} is a model

$$L^{alg} \models \exists a \in \Gamma^n \forall x\big((\phi(x, l) \leftrightarrow \psi(x, a)\big).$$

Thus, we can find some element $a \in \Gamma^n_{L^{alg}}$ such that $L^{alg} \models \psi(\gamma, a)$ and $|\psi(L^{alg}, a)| = 1$. By quantifier elimination of $ODAG$, $\psi(x, a)$ must be equivalent to a formula $x = \tau(a)$, where τ is a term in the language $\mathcal{L}_{OAG}$. Thus $\gamma \in \Gamma_{L^{alg}} = \mathbb{Q} \otimes \Gamma_L$.
We continue arguing for the residue field. It is clear that $k_{ACVF}(L) \subseteq \operatorname{acl}_{ACVF}(L) \cap \mathbf{k}$. Thus it is sufficient to argue that $\operatorname{acl}_{ACVF}(L) \cap \mathbf{k} = k_L^{alg}$. Let $b \in \operatorname{acl}_{ACVF}(L) \subseteq \operatorname{acl}_{ACVF}(L^{alg})$ and $\phi(x, l)$ the formula witnessing that b is algebraic over L. Because L^{alg} is a model and $\mathbf{k}$ is purely stably embedded, there is some formula $\psi(x, z)$ in the language of fields such that:

$$L^{alg} \models \exists d \in k^r\big(\forall x\big(\phi(x, l) \leftrightarrow \psi(x, d)\big)\big).$$

Consequently, we can find some tuple $d \in k_{L^{alg}}$ such that $L^{alg} \models \psi(b, d)$ and $\psi(L^{alg}, d)$ is finite. By quantifier elimination in ACF, $b \in k_{L^{alg}} = k_L^{alg}$. $\square$

Notation 325. *Let k_C be a subfield of the residue field and $\mathbf{a} = \langle a_i \mid i \leq n\rangle$ be a tuple of elements in the residue field, we denote as $k_C\langle a_i \mid i \leq n\rangle$ the field generated by k_C and the tuple $\mathbf{a}$, i.e. $k_C\langle a_i \mid i \leq n\rangle$.*

In the case of $ACVF$ the following statement is an immediate consequence of the Zariski-Abhyankar inequality.

Corollary 326. *Let $C \subseteq L$ be a valued field extension. Let $\mathcal{O}_C$ and $\mathcal{O}_L$ be the valuation rings of C and L respectively. Let $\langle b_i \mid i \leq r\rangle$ be a sequence of elements of $\mathcal{O}_L^\times$ such that* $\operatorname{res}(b_i)$ *in $\mathbf{k}_L$ is algebraically independent over $\mathbf{k}_C$. And let $\langle a_j \mid j \leq s\rangle$ be a sequence of elements of $L^\times$ such that $\langle v(a_j) \mid j \leq s\rangle$ in $\mathbb{Q} \otimes \Gamma_L$ is $\mathbb{Q}$-linearly independent over $\mathbb{Q} \otimes \Gamma_C$. Let E be the field generated by C and $\langle b_i \mid i \leq r\rangle$ and $\langle a_j \mid j \leq s\rangle$, then:*

- $\Gamma_{ACVF}(E) \subseteq (\mathbb{Q} \otimes \Gamma_C) \oplus \bigoplus_{j \leq s}(\mathbb{Q} \otimes v(a_j))$, *and*
- $k_{ACVF}(E) \subseteq \operatorname{acl}_{ACVF}(E) \cap \mathbf{k} \subseteq (k_C\langle \operatorname{res}(b_i) \mid i \leq r\rangle)^{alg}$.

In particular, if for each $j \leq s$ we let $d_j = \operatorname{rv}(a_j)$ then:

- $\Gamma_{ACVF}(Cd_1, \ldots, d_s, \operatorname{res}(b_1), \ldots, \operatorname{res}(b_r)) \subseteq (\mathbb{Q} \otimes \Gamma_C) \oplus \bigoplus_{j \leq s}(\mathbb{Q} \otimes v(a_j))$ *and,*

- $$k_{ACVF}(Cd_1, \dots, d_s, \mathrm{res}(b_1), \dots, \mathrm{res}(b_r)) \subseteq$$
  $$\mathrm{acl}_{ACVF}(Cd_1, \dots, d_s, \mathrm{res}(b_1), \dots, \mathrm{res}(b_r)) \cap \mathbf{k} \subseteq (k_C\langle \mathrm{res}(b_i) \mid i \leq r\rangle)^{alg}.$$

Proof. Let $C \subseteq L$ and $\langle a_j \mid j \leq s\rangle$, $\langle b_i \mid i \leq r\rangle$ be tuples as in the statement. Let E_0 be the field generated by C and the tuple $\langle b_i \mid i \leq r\rangle$. By Fact 323 and Fact 324:

$$r = \mathrm{trdeg}(E_0/C) \geq \mathrm{trdeg}(k_{ACVF}(E_0)/k_C) + \dim_{\mathbb{Q}}(\Gamma_{ACVF}(E_0)/\Gamma_{ACVF}(C)),$$

because $\mathrm{trdeg}(k_{ACVF}(E_0)/k_C) \geq r$, then $\mathrm{trdeg}(k_{ACVF}(E_0)/k_C) = r$ and

$$\dim_{\mathbb{Q}}(\Gamma_{ACVF}(E_0)/\Gamma_{ACVF}(C)) = 0$$

. In particular,

$$\Gamma_{ACVF}(E_0) \subseteq \Gamma_{ACVF}(C) = \mathbb{Q} \otimes \Gamma_C$$

, and

$$k_{ACVF}(E_0) \subseteq \mathrm{acl}_{ACVF}(E_0) \cap \mathbf{k} = k_{E_0}^{alg} = (k_C\langle \mathrm{res}(b_i) \mid i \leq r\rangle)^{alg}.$$

Let E be the field generated by E_0 and $\langle a_i \mid i \leq s\rangle$. Again by Fact 323,

$$s = \mathrm{trdeg}(E/E_0) \geq \mathrm{trdeg}(k_{ACVF}(E)/k_{ACVF}(E_0)) + \dim_{\mathbb{Q}}(\Gamma_{ACVF}(E)/\Gamma_{ACVF}(E_0)).$$

Because $\langle v(a_i) \mid i \leq s\rangle \subseteq \Gamma_{ACVF}(E)$ and $\Gamma_{ACVF}(E_0) \subseteq \mathbb{Q} \otimes \Gamma_C$, then

$$s = \dim_{\mathbb{Q}}(\Gamma_{ACVF}(E)/\Gamma_{ACVF}(E_0))$$

. Thus, $\mathrm{trdeg}(k_{ACVF}(E)/k_{ACVF}(E_0)) = 0$. Summarizing all the above, we conclude that:

- $k_{ACVF}(E) \subseteq \mathrm{acl}_{ACVF}(E) \cap \mathbf{k} = k_E^{alg} = (k_C\langle \mathrm{res}(b_i) \mid i \leq r\rangle)^{alg}$, and
- $$\Gamma_{ACVF}(E) \subseteq (\mathbb{Q} \otimes \Gamma_C) \oplus \bigoplus_{j \leq s} (\mathbb{Q} \otimes v(a_j))$$
  , as required.

The second part of the statement follows immediately by the fact that $\mathrm{res}(b_j) \in \mathrm{dcl}_{ACVF}(b_j)$ and $d_i = \mathrm{rv}(a_i) \in \mathrm{dcl}_{ACVF}(a_i)$. □

Lemma 327. *Let L, M be substructures of $\mathfrak{U}$, the monster model of $ACVF$. Let C be a common substructure of L and M and suppose that $\Gamma_L \subseteq \Gamma_M$. Let*

- *$A = \{a_i \mid i \in R\} \subseteq L$ be such that $\{v(a_i) \mid i \in R\}$ is a maximally $\mathbb{Q}$-linearly independent set of $\Gamma_L \subseteq \Gamma(L) = \mathbb{Q} \otimes \Gamma(L)$ over Γ_C.*
- *$E = \{e_i \mid i \in R\} \subseteq M$ satisfying $v(e_i) = v(a_i)$,*

- $B = \{b_j \mid j \in S\} \subseteq \mathcal{O}_L^{\times}$ *such that* $\{\mathrm{res}(b_j) \mid j \in S\}$ *is a transcendence base of* $k_L \subseteq \mathrm{acl}(L) \cap \mathbf{k} = k_L^{alg}$ *over* k_C.

The following statements are equivalent:

1. *The set*
$$\{\mathrm{res}\,(\frac{a_i}{e_i}), \mathrm{res}(b_j) \mid j \in S, i \in R\} \textit{ is algebraically independent over } k_M.$$
2. *The structures* $\mathrm{acl}_{ACVF}(\mathrm{kInt}^L_{\Gamma_L})$, $\mathrm{acl}_{ACVF}(\mathrm{kInt}^M_{\Gamma_L})$ *are independent over* $C\Gamma_L$.

Proof. Let (a_i), (e_i) and $\mathrm{res}(b_j)$ satisfying the required hypothesis. Let d_i be the code of the open ball $B_{v(a_i)}(a_i) = \{x \in \mathfrak{U} \mid v(x - a_i) > v(a_i)\}$, note that this code is inter-definable with the class $\mathrm{rv}(a_i) \in \mathrm{dcl}_{ACVF}(a_i)$. For notational convenience we will assume that R and S are finite and equal to $\{1, \ldots, r\}$ and $\{1, \ldots, s\}$ respectively, as the more general argument follows in a similar manner by applying the argument to any finite sequence.

Claim 328. *The set* $D = \{d_1, \ldots, d_r, \mathrm{res}(b_1), \ldots, \mathrm{res}(b_s)\} \subseteq \mathrm{acl}_{ACVF}(\mathrm{kInt}^L_{\Gamma_L})$ *is algebraically independent over* $C\Gamma_L$. *(In the model theoretic sense)*

Proof. We proceed by contradiction and we argue by cases. Suppose the existence of some index $j_0 \leq s$ such that $\mathrm{res}(b_{j_0}) \in \mathrm{acl}_{ACVF}(D_0 C\Gamma_L)$, where
$$D_0 = D \backslash \{\mathrm{res}(b_{j_0})\}$$
. Let $B_0 = B \backslash \{b_{j_0}\}$ and note that
$$\mathrm{res}(b_{j_0}) \in \mathrm{acl}_{ACVF}(D_0 C\Gamma_L) \subseteq \mathrm{acl}_{ACVF}(C(A, B_0))$$
, where $C(A, B_0)$ denotes the field generated over C by A and B_0. By Corollary 326,
$$\mathrm{res}(b_{j_0}) \in \mathrm{acl}_{ACVF}(C(A, B_0)) \cap \mathbf{k} \subseteq (k_C\langle \mathrm{res}(b_j) \mid j \leq s, j \neq j_0\rangle)^{alg}.$$
This contradicts the choice of the elements $(b_i \mid i \leq s)$. We now assume that for some index $i_0 \leq s$ such that $d_{i_0} \in \mathrm{acl}_{ACVF}(C D_0 \Gamma_L)$ where $D_0 = D \backslash \{d_{i_0}\}$. Let
$$\begin{aligned} E_0 &= \mathrm{acl}(C(A_0, B)) \text{ where} \\ A_0 &= A \backslash \{a_{i_0}\} \end{aligned}$$
, and $C(A_0, B)$ denotes the field generated by A_0 and B over C. Let $G = RV_{E_0}$, by Corollary 326 and Fact 324:

- $k_{E_0} = \mathrm{acl}_{ACVF}(E_0) \cap \mathbf{k} = (k_C < \mathrm{res}(b_j) \mid j \leq s >)^{alg}$, and

- $\Gamma_{E_0} = \Gamma_{ACVF}(E_0) = (\mathbb{Q} \otimes \Gamma_C) \oplus \bigoplus_{j \neq j_0} (\mathbb{Q} \otimes v(a_j)) = v_{RV}(G)$.

Moreover by construction

$$\Gamma_L \subseteq (\mathbb{Q} \otimes \Gamma_C) \oplus \bigoplus_{j \leq r} (\mathbb{Q} \otimes v(a_j))$$

. Let $\gamma = v(a_{i_0})$ and let $\phi(x, \gamma)$ be the $\mathcal{L}(CD_0)$-formula witnessing that $d_{i_0} \in \mathrm{acl}_{ACVF}(CD_0\Gamma_L)$. Because the residue field is infinite, we can find an element $d \in RV_{\mathfrak{U}} \cap v^{-1}(\gamma)$, such that $\mathfrak{U} \vDash \neg\phi(d, \gamma)$. To simplify the notation, we write $\hat{\Gamma}$ to denote $(\mathbb{Q} \otimes \Gamma_C) \oplus \bigoplus_{j \leq r} (\mathbb{Q} \otimes v(a_j))$.

By Corollary 326,

$$k_{ACVF}(CD_0 \cup \{d\}) \subseteq (k_C\langle \mathrm{res}(b_j) \mid j \leq s \rangle)^{alg} = k_{E_0} \textit{ and}$$
$$k_{ACVF}(CD) \subseteq (k_C\langle \mathrm{res}(b_j) \mid j \leq s \rangle)^{alg} = k_{E_0}.$$

Let $G_1 = G \cdot d_{i_0}^{\mathbb{Z}}$ and $G_2 = G \cdot d^{\mathbb{Z}}$ and consider the partial isomorphism:

$$\begin{aligned} f :& G_1 \to G_2 \\ & g d_{i_0}^n \to g d^n, \text{ where } n \in \mathbb{Z}. \end{aligned}$$

Let $f_v = id_{\hat{\Gamma}}$, and $f_r = id_{k_{E_0}}$, then the triple $(f_r, f, f_v) : (k_{E_0}, G_1, \hat{\Gamma}) \to (k_{E_0}, G_2, \hat{\Gamma})$ is a partial isomorphism in the $\mathcal{L}_{rv}$ language [See Definition 3.1]. By Proposition 278 the partial isomorphism (f_r, f, f_v) must be an elementary map. In particular $\mathrm{tp}(d/CD_0\Gamma_L) = \mathrm{tp}(d_{i_0}/CD_0\Gamma_L)$, but this leads us to a contradiction because

$$\mathfrak{U} \vDash \neg\phi(d, \gamma) \wedge \phi(d_{i_0}, \gamma)$$

. □

We can now prove the equivalence between (1) and (2). By Claim 328 $D \subseteq \mathrm{acl}_{ACVF}(\mathrm{kInt}^L_{\Gamma_L})$ is algebraically independent (in the model theoretic sense) over $C\Gamma_L$. Each fiber $RV_{\mathfrak{U}} \cap v^{-1}(\gamma)$ is stably embedded and internal to the residue field (which eliminates imaginaries), so it must be a strongly minimal set. Therefore, algebraic independence in the model theoretic sense over $C\Gamma_L$ coincides with forking independence in the stable sense, in particular $MR(D/C\Gamma_L) = s + r$. Thus,

$$\mathrm{acl}_{ACVF}(\mathrm{kInt}^L_{\Gamma_L}) \underset{C\Gamma_L}{\downarrow} \mathrm{acl}_{ACVF}(\mathrm{kInt}^M_{\Gamma_L})$$

if and only if

$$MR(D/\,\mathrm{acl}_{ACVF}(\mathrm{kInt}^M_{\Gamma_L})) = r + s.$$

As $\mathrm{acl}_{ACVF}(\mathrm{kInt}^{\mathfrak{U}}_{\Gamma_L})$ is stably embedded,

$$MR(D/\,\mathrm{acl}_{ACVF}(\mathrm{kInt}^M_{\Gamma_L})) = r + s$$

if and only if $MR(D/M) = r + s$. Because each element d_i is interdefinable over M with $\operatorname{res}\left(\frac{a_i}{e_i}\right)$, then $MR(D/M) = r + s$ if and only if

$$MR\big(\{\operatorname{res}\big(\frac{a_1}{e_1}\big), \dots, \operatorname{res}\big(\frac{a_r}{e_r}\big), \operatorname{res}(b_1), \dots, \operatorname{res}(b_s)\}/M\big) = r + s.$$

As the residue field is purely stably embedded,

$$MR(\{\operatorname{res}\big(\frac{a_1}{e_1}\big), \dots, \operatorname{res}\big(\frac{a_r}{e_r}\big), \operatorname{res}(b_1), \dots, \operatorname{res}(b_s)\}/M) = r + s$$

if and only if

$$MR\big(\{\operatorname{res}\big(\frac{a_1}{e_1}\big), \dots, \operatorname{res}\big(\frac{a_r}{e_r}\big), \operatorname{res}(b_1), \dots, \operatorname{res}(b_s)\}/k_M\big) = r + s,$$

thus $\{\operatorname{res}\left(\frac{a_1}{e_1}\right), \dots, \operatorname{res}\left(\frac{a_r}{e_r}\right), \operatorname{res}(b_1), \dots, \operatorname{res}(b_s)\}$ is algebraically independent over k_M. We conclude therefore the equivalence between (1) and (2) as required. $\square$

We emphasize that in the following statement we work for T a complete extension of the $\mathcal{L}_{RV}$-theory of henselian valued fields of equicharacteristic zero, and we let $\mathfrak{C}$ be its monster model. The following theorem generalizes ideas present in [HHM05, Proposition 12.15], we include all details for sake of completeness.

Notation 329. *Given a set of parameters $B \subseteq \mathbf{k}_{\mathfrak{C}}$ we will denote as $\operatorname{cl}(B)$ the field theoretic algebraic closure of B inside of $\mathbf{k}_{\mathfrak{C}}$.*
We recall that given a substructure M of $\mathfrak{C}$ for each $n \in \mathbb{N}$ we denote as

$$(\mathcal{A}_n)_M = \{\operatorname{res}^n(m) \mid m \in M\} \quad \textit{and} \quad \mathcal{A}_M = ((\mathcal{A}_n)_M \mid n \in \mathbb{N}).$$

Theorem 330. *Let L and M be substructures of $\mathfrak{C}$ and let $C \subseteq L \cap M$ be a common substructure which is a maximal model of T. We suppose:*

1. $\Gamma_L \subseteq \Gamma_M$ *and* $\Gamma(L) = \Gamma_L$,
2. $\operatorname{kInt}^L_{\Gamma_L} \mathop{\smile\!\!\!\!\!\!\mid}^{qfs}_{C\Gamma_L} \operatorname{kInt}^M_{\Gamma_L}$,
3. *L has finite transcendence degree over C.*

Then $\operatorname{tp}(L/C\Gamma_L\mathcal{A}_M \operatorname{kInt}^M_{\Gamma_L}) \vdash \operatorname{tp}(L/M)$.

Proof. Let $L' \models tp(L/C\Gamma_L\mathcal{A}_M \operatorname{kInt}^M_{\Gamma_L})$ and let σ be an automorphism of $\mathfrak{C}$ fixing $C\Gamma_L\mathcal{A}_M \operatorname{kInt}^M_{\Gamma_L}$ taking L to L'.

Step 1: Without loss of generality we may assume that σ fixes Γ_M.

Proof. Let $\beta \in \Gamma_M$ such that $\sigma(\beta) = \beta'$. Because Γ is stably embedded it is sufficient to prove that β and β' realize the same type over $\Gamma(L' \,\mathrm{kInt}^M_{\Gamma_L})$. So, we can take an automorphism of the structure τ fixing $L' \,\mathrm{kInt}^M_{\Gamma_L}$ sending β' to β and we may replace σ by $\tau \circ \sigma$. To show that $\mathrm{tp}(\beta/\Gamma(L' \,\mathrm{kInt}^M_{\Gamma_L})) = \mathrm{tp}(\beta'/\Gamma(L' \,\mathrm{kInt}^M_{\Gamma_L}))$ we will argue that $\Gamma(L' \,\mathrm{kInt}^M_{\Gamma_L}) = \Gamma_L$. Let f be a L'-definable function from $\mathrm{kInt}^M_{\Gamma_L}$ to Γ. We aim to prove that for each $x \in \mathrm{kInt}^M_{\Gamma_L}$ we have that $f(x) \in \Gamma(L')$. For each $\gamma \in \Gamma_L$, the function f takes finitely many values on $RV_M \cap v_{RV}^{-1}(\gamma) \subseteq RV_{\mathfrak{C}} \cap v_{RV}^{-1}(\gamma)$, because the power residue sorts and the value group are orthogonal to each other. Hence, for each $\gamma \in \Gamma_L$ the set $f\big(RV_M \cap v^{-1}(\gamma)\big)$ is finite, thus algebraic over L'. Then $f\big(RV_M \cap v^{-1}(\gamma)\big) \subseteq \mathrm{acl}(L') \cap \Gamma = \Gamma(L') = \Gamma_{L'} = \Gamma_L$, as required. □

As in Proposition 12.15 in [HHM05] we start by perturbing the valuation on $C(L, M)$.

Step 2 : There is some valuation $\hat{v}$ on $C(L, M)$ finer that v satisfying the following properties

- $\Gamma_{(L,\hat{v})} \cap \Gamma_{(M,\hat{v})} = \Gamma_{(C,\hat{v})}$,
- $k_{(L,\hat{v})}$ and $k_{(M,\hat{v})}$ are linearly disjoint over $k_{(C,\hat{v})}$,
- for any element $x \in M$, we have that $v(x) = \hat{v}(x)$.

Proof. We choose elements $\{a_i \mid i \in r\}, \{b_j \mid j \in s\} \subseteq L$ and $\{e_i \mid i \in r\} \subseteq M$ satisfying the hypotheses of Lemma 327. By hypothesis,

$$\mathrm{kInt}^L_{\Gamma_L} \underset{C\Gamma_L}{\overset{qfs}{\downarrow}} \mathrm{kInt}^M_{\Gamma_L}$$

thus $\mathrm{acl}_{ACVF}(\mathrm{kInt}^L_{\Gamma_L}) \downarrow_{C\Gamma_L} \mathrm{acl}_{ACVF}(\mathrm{kInt}^M_{\Gamma_L})$, so we can apply Lemma 327 and we obtain that:

$$\{\mathrm{res}\,\big(\frac{a_1}{e_1}\big), \ldots, \mathrm{res}\,\big(\frac{a_r}{e_r}\big), \mathrm{res}(b_1), \ldots, \mathrm{res}(b_s)\}$$

are algebraically independent (in the field theoretic sense) over k_M. For each $i \leq r$ we define:

$$R^i = \mathrm{cl}(k_M, \mathrm{res}\,\big(\frac{a_1}{e_1}\big), \ldots, \mathrm{res}\,\big(\frac{a_i}{e_i}\big), \mathrm{res}(b_1), \ldots, \mathrm{res}(b_s)).$$

In particular:

$$R^{(0)} = \mathrm{cl}(k_M, \mathrm{res}(b_1), \ldots, \mathrm{res}(b_s)) = \mathrm{cl}(k_M, k_L),$$
$$R^{(r)} = \mathrm{cl}\,\big(k_M, k_L, \mathrm{res}\,\big(\frac{a_1}{e_1}\big), \ldots, \mathrm{res}\,\big(\frac{a_r}{e_r}\big)\big).$$

Let $p_i : R^{i+1} \to R^i \cup \{\infty\}$ be a place which fixes R^i and $p_i\big(\mathrm{res}\,\big(\frac{a_i}{e_i}\big)\big) = 0$, such map can be found by the algebraic independence of $\{\mathrm{res}\,\big(\frac{a_i}{e_i}\big), \ldots, \mathrm{res}\,\big(\frac{a_r}{e_r}\big), \mathrm{res}(b_1), \ldots, \mathrm{res}(b_s)\}$ over k_M.

Let $p_v : C(L,M) \to k_{C(L,M)} \cup \{\infty\}$ be the place corresponding to the valuation ring over $C(L,M)$ given by Proposition 264, and fix a place $p^* : k_{C(L,M)} \to R^r \cup \{\infty\}$ fixing R^r.
We take the place $\hat{p} : C(L,M) \to \mathrm{cl}(k_L, k_M) \cup \{\infty\}$ given by taking the composition

$$\hat{p} = p_0 \circ p_1 \circ \cdots \circ p_{r-1} \circ p^* \circ p_v.$$

Let $\hat{v}$ be the valuation over $C(L,M)$ induced by $\hat{p}$, which is a refinement of the original valuation v. Because each of the places is the identity map on k_M, we may identify the valued field (M,v) with $(M,\hat{v})$ and the value groups $\Gamma_{(M,v)}$ and $\Gamma_{(M,\hat{v})}$. So we now have two valuations v and $\hat{v}$ induced over $C(L,M)$, and the construction ensures that the following conditions are satisfied:

- if $x \in M$, then $v(x) = \hat{v}(x)$ and if $x, y \in L$ and $v(x) \leq v(y)$ then $\hat{v}(x) \leq \hat{v}(y)$,
- and by Lemma 266 (2) for any $x \in C(L,M)$ with $v(x) > 0$ we have:
$$0 < \hat{v}\big(\frac{a_1}{e_1}\big) << \cdots << \hat{v}\big(\frac{a_r}{e_r}\big) << \hat{v}(x).$$

Likewise, we can identify (L,v) and $(L,\hat{v})$ and their value groups, as all the places are the identity map as well over k_L. However, it is impossible to identify (M,v) and $(M,\hat{v})$, and (L,v) with $(L,\hat{v})$ simultaneously. We should not identify (L,v) and $(L,\hat{v})$. For each $i \leq r$, let $\eta_i = \hat{v}\big(\frac{a_i}{e_i}\big)$ and $\hat{v}(e_i) = v(e_i) = \epsilon_i$.
Let

$$\Delta = \{\hat{v}(x), -\hat{v}(x) \mid x \in C(L,M), p_v(x) \notin \{0,\infty\} \text{ and } \hat{p}(x) = 0\} \cup \{0_{\Gamma_{(C(L,M),\hat{v})}}\}.$$

By Lemma 266 (1) Δ is a convex subgroup of $\Gamma_{(C(L,M),\hat{v})}$, that contains the subgroup generated by $\{\eta_i \mid i \leq r\}$. (Because for each $i \leq r$, $p_v\big(\frac{a_i}{e_i}\big) \notin \{0,\infty\}$ while $\hat{p}\big(\frac{a_i}{e_i}\big) = 0$).
We continue to show that the refined valuation satisfies the independence conditions over the value group and its residue field.

Claim 331. $\Gamma_{(M,\hat{v})} \cap \Gamma_{(L,\hat{v})} = \Gamma_{(C,\hat{v})}$.

Proof. Take elements $m \in M$ and $l \in L$ such that $\hat{v}(m) = \hat{v}(l)$. By hypothesis $\{v(a_i) \mid i \leq r\} \subseteq \Gamma_{(L,v)}$ and it is a $\mathbb{Q}$-linearly independent set over $\Gamma_{(C,v)}$, hence $v(l)$ must belong to the pure hull of the subgroup generated by $\{v(a_i) \mid i \leq r\}$ and $\Gamma_{(C,v)}$, thus we can find we can find $p_i \in \mathbb{Z}$,$\gamma \in \Gamma_C$ and $k \in \mathbb{N}^{\geq 1}$ such that:

$$kv(l) = \sum_{i=1}^{r} p_i v(a_i) + \gamma = \sum_{i=1}^{r} p_i v(e_i) + \gamma = \sum_{i=1}^{r} p_i \epsilon_i + \gamma. \tag{3.1}$$

Because $\Gamma_{(L,v)}$ and $\Gamma_{(L,\hat{v})}$ are isomorphic,

$$k\hat{v}(l) = \sum_{i=1}^{r} p_i \hat{v}(a_i) + \gamma = \sum_{i=1}^{r} p_i \eta_i + \sum_{i=1}^{r} p_i \epsilon_i + \gamma. \tag{3.2}$$

Because $\{v(a_i) \mid i \leq r\}$ is a $\mathbb{Q}$-linearly independent set over $\Gamma_{(C,v)} = \Gamma_{(C,\hat{v})}$ and

$$\{v(a_i) \mid i \leq r\} = \{v(e_i) \mid i \leq r\} \subseteq \Gamma_{(M,v)} = \Gamma_{(M,\hat{v})},$$

then $\{\epsilon_1, \dots, \epsilon_r\}$ is also a $\mathbb{Q}$-linearly independent set over $\Gamma_{(C,v)} \subset \Gamma_{(M,v)} = \Gamma_{(M,\hat{v})}$. Thus we can extend it to a maximal set of elements in Γ_M which are $\mathbb{Q}$-linearly independent set over Γ_C, say $\{\epsilon_1, \dots, \epsilon_r\} \cup \{\mu_\alpha \mid \alpha \in \lambda\}$. Hence, we can find indices $\{\mu_{\alpha_1}, \dots, \mu_{\alpha_t}\}$ such that

$$s\hat{v}(m) = \sum_{i=1}^{r} r_i\epsilon_i + \sum_{i=1}^{t} q_i\mu_{\alpha_i} + \gamma', \text{ where } r_i, q_i \in \mathbb{Z},\ \gamma' \in \Gamma_C \text{ and } s \in \mathbb{N}^{\geq 1}.$$

Since $\hat{v}(l) = \hat{v}(m)$, we must have that $s(k\hat{v}(l)) = k(s\hat{v}(m))$ thus:

$$\underbrace{\sum_{i=1}^{r} sp_i\eta_i}_{=\delta\in\Delta} + \underbrace{\sum_{i=1}^{r}(sp_i - kr_i)\epsilon_i - \sum_{i=1}^{t} kq_i\mu_{\alpha_i} + (s\gamma - k\gamma')}_{=\beta\in\Gamma_M} = 0, \tag{3.3}$$

Because the elements $\{\eta_1, \dots, \eta_r\}$ are infinitesimal with respect to $\Gamma^{>0}_{(M,v)}$, the equation 3.3 is satisfied if and only if

$$\delta = \sum_{i=1}^{r} sp_i\eta_i = 0 \text{ and } \beta = \sum_{i=1}^{r}(sp_i - kr_i)\epsilon_i - \sum_{i=1}^{t} kq_i\mu_{\alpha_i} + (s\gamma - k\gamma') = 0.$$

As $0 < \eta_1 << \eta_2 << \dots << \eta_r$, then $\sum_{i=1}^{r} sp_i\eta_i = 0$ if and only if $p_i = 0$ for all $i \leq r$. Then,

$$-\sum_{i=1}^{r} kr_i\epsilon_i - \sum_{i=1}^{t} kq_i\mu_{\alpha_i} + (s\gamma - k\gamma') = 0$$

but the $\mathbb{Q}$-linear independence of $\{\epsilon_1, \dots, \epsilon_r\} \cup \{\mu_{\alpha_1}, \dots, \mu_{\alpha_t}\}$ over Γ_C implies that $r_i = 0$ for all $i \leq r$ and $q_i = 0$ for all $i \leq t$. Summarizing we have that $k\hat{v}(l) = \gamma$ and $s\hat{v}(m) = \gamma'$, thus γ is k-divisible and γ' must be s-divisible. Hence:

$$\hat{v}(l) = \frac{\gamma}{k} = \frac{\gamma'}{s} = \hat{v}(m).$$

As C is a model of T, Γ_C must be definably closed, thus $\frac{\gamma}{k} = \frac{\gamma'}{s} \in \Gamma_C$, as required. □

Claim 332. *$k_{(M,\hat{v})}$ and $k_{(L,\hat{v})}$ are linearly disjoint over $k_{(C,\hat{v})}$.*

Proof. By the second hypothesis, so k_L and k_M are algebraically independent over k_C. Because C is a model, $k_C = \mathrm{cl}(k_C)$ and since the residue field is of characteristic zero, it must be the case that $k_C \leq k_L$ is a regular extension. By Fact 300, k_L and k_M are linearly disjoint over k_C. Each of the places p^j and p^* are the identity map over k_M and k_L, so is their composition

$$p_0 \circ p_1 \circ \cdots \circ p_{r-1} \circ p^* : k_{(C(L,M),v)} \to \mathrm{cl}(k_M, k_L).$$

Because $k_{(M,v)}$ and $k_{(L,v)}$ are linearly disjoint over $k_{(C,v)}$ then $k_{(M,\hat{v})}$ and $k_{(L,\hat{v})}$ must be linearly disjoint over $k_{(C,\hat{v})} = k_{(C,v)}$. □

□

Step 3: We find the $\mathcal{L}_{val}$-isomorphism $\hat{\sigma}$ extending σ which is the identity on M.

Proof. By Step 2 there is some valuation $\hat{v}$ over $C(L,M)$ finer than v satisfying the following conditions:

- $\Gamma_{(M,\hat{v})} \cap \Gamma_{(L,\hat{v})} = \Gamma_{(C,\hat{v})}$,
- $k_{(M,\hat{v})}$ and $k_{(L,\hat{v})}$ are linearly disjoint over $k_{(C,\hat{v})}$,
- for any element $x \in M$, we have that $v(x) = \hat{v}(x)$.

Because $C \subseteq M$, then the valuations v and $\hat{v}$ coincide over C. In particular $(C,\hat{v})$ is maximal, so by Fact 262 $(M,\hat{v})$ has the good separated basis property over C. By Proposition 295 M has the separated basis property over $(L,\hat{v})$, M and L are linearly disjoint over C, the value group of $\Gamma_{(C(L,M),\hat{v})}$ is the group generated by $\Gamma_{(L,\hat{v})}$ and $\Gamma_{(M,\hat{v})}$ over $\Gamma_{(C,\hat{v})}$ and the residue field $k_{(C(L,M),\hat{v})}$ is the field generated by $k_{(L,\hat{v})}$ and $k_{(M,\hat{v})}$ over $k_{(C,\hat{v})}$.
We consider the field $C(\sigma(L),\sigma(M))$ with the valuation $\hat{v}$ such that $\sigma : (C(L,M),\hat{v}) \to (C(\sigma(L),\sigma(M)),\hat{v})$ is an $\mathcal{L}_{\mathrm{val}}$-isomorphism, which fixes $k_M \subset Int^M_{\Gamma_L}\Gamma_L$ and Γ_M by Step 1. By Proposition 299, we can find an $\mathcal{L}_{\mathrm{val}}$-isomorphism $\tau : (C(L,M),\hat{v}) \to (C(\sigma(L),M),\hat{v})$ which is the identity on M and $\tau\restriction_{(L,\hat{v})}: (L,\hat{v}) \to (\sigma(L),\hat{v})$.

We want to show that $\tau : C(L,M) \to C(\sigma(L),M)$ induces as well an $\mathcal{L}_{\mathrm{val}}$-isomorphism with the original valuation v. Let $x \in C[L,M]$, without loss of generality we may assume that

$$x = \sum_{i=1}^{n} l_i m_i$$

where $(m_1,\dots,m_n)$ is a separated basis of $Vect_L(m_1,\dots,m_n)$ according to the valuation $\hat{v}$, thus

$$\hat{v}(x) = \hat{v}\big(\sum_{i=1}^{n} l_i m_i\big) = \min\{\hat{v}(l_i m_i) \mid i \le n\} = \hat{v}(m_j l_i) \text{ for some } j \le n.$$

As $\tau : (C(L, M), \hat{v}) \to (C(\sigma(L), M), \hat{v})$ is an $\mathcal{L}_{\text{val}}$-isomorphism, $(m_1, \ldots, m_n)$ is a separated basis of $\text{Vect}_{\sigma(L)}(m_1, \ldots, m_n)$ and

$$\hat{v}(\tau(x)) = \hat{v}\big(\sum_{i=1}^{n} \sigma(l_i)m_i\big) = \min\{\hat{v}(\sigma(l_i)m_i) \mid i \leq n\} = \hat{v}(\sigma(l_j)m_j) = \tau(\hat{v}(l_j m_j)).$$

By Lemma 266(3) $(m_1, \ldots, m_n)$ is also a separated basis of

$$\text{Vect}_L(m_1, \ldots, m_n) \text{ and } \text{Vect}_{\sigma(L)}(m_1, \ldots, m_n)$$

with respect to the valuation v. Consequently,

$$\begin{aligned} v(x) &= v(m_j l_j) = v(m_j) + v(l_j) = v(m_j) + \sigma(v(l_j)) \\ &= v(m_j) + v(\sigma(l_j)) = v(m_j \sigma(l_j)) = v(\tau(x)). \end{aligned}$$

As $x \in C[L, M]$ is an arbitrary element, we conclude that the value group of $C(L, M)$ and $C(\sigma(L), M)$ according to the valuation v is Γ_M and τ acts as the identity on Γ_M. $\square$

Hence τ is also a $\mathcal{L}_{\text{val}}$-isomorphism between the valued field structures $(C(L, M), v)$ and $(C(\sigma(L), M), v)$ which acts as the identity on M and coincides with σ on L.

Step 4: We extend the $\mathcal{L}_{\text{val}}$-isomorphism to a $\mathcal{L}$-isomorphism.

We want to extend the $\mathcal{L}_{\text{val}}$-isomorphism to a $\mathcal{L}$-isomorphism, thus we first want to extend the isomorphism by adding maps $\tau_n : (\mathcal{A}_n)_{C(L,M)} \to (\mathcal{A}_n)_{C(\sigma(L),M)}$.
Let $a \in C(L, M)$ we say that $\text{rv}(a)$ is *representable with parameters over* $\text{kInt}^M_{\Gamma_L}$ if there are $l_1, \ldots, l_s \in L$ and $m_1, \ldots, m_s \in M$ such that

$$\text{rv}(a) = \big(\sum_{i \leq s} \text{rv}(l_i)\,\text{rv}(m_i)\big)$$

where for each $i \leq s$, $\text{rv}(m_i) \in \text{kInt}^M_{\Gamma_L}$.

Claim 333. *For each $x \in C[L, M]$ there are $\hat{m} \in M$ and $a \in C(L, M)$, such that $v(a) = 0$, $\text{rv}(x) = \text{rv}(a)\,\text{rv}(\hat{m})$ and $\text{rv}(a)$ is representable with parameters over $\text{kInt}^M_{\Gamma_L}$. Furthermore, $x = a\hat{m}$ and $a = \sum_{i \leq n} l_i m_i$ for some $l_i \in L$ and $m_i \in M$, thus*

$$\tau(x) = \tau(a)\tau(\hat{m}) = \big(\sum_{i \leq n} \sigma(l_i)m_i\big)\hat{m}$$

.

Proof. Fix an element $x \in C[L, M]$, and suppose $x = \sum_{i \le n} l_i m_i$. Because M has the separated basis property over L according to the valuation v,

$$v(x) = v(\sum_{i \le n} l_i m_i) = \min\{v(l_i m_i) \mid i \le n\} = v(l_{i_0} m_{i_0}).$$

As $\Gamma_L \subseteq \Gamma_M$ there is some $\hat{m} \in M$ such that $v(l_{i_0} m_{i_0}) = v(\hat{m})$. Let $I = \{i \le n \mid v(l_i m_i) = v(\hat{m})\}$ and

$$a = \frac{x}{\hat{m}} = \sum_{i \le n} l_i \frac{m_i}{\hat{m}}$$

.

For each $i \in I$, $v(\frac{m_i}{\hat{m}}) = -v(l_i) = -\lambda_i \in \Gamma_L$, thus $\operatorname{rv}\left(\frac{m_i}{\hat{m}}\right) \in \mathrm{kInt}^M_{\Gamma_L}$. Then,

$$\operatorname{rv}(a) = \sum_{i \in I} \operatorname{rv}(l_i) \operatorname{rv}\left(\frac{m_i}{\hat{m}}\right)$$

where each

$$\operatorname{rv}\left(\frac{m_i}{\hat{m}}\right) \in \mathrm{kInt}^M_{\Gamma_L}.$$

Summarizing, we have that $x = a\hat{m}$, so $\operatorname{rv}(x) = \operatorname{rv}(a)\operatorname{rv}(\hat{m})$ where $\operatorname{rv}(a)$ is representable with parameters over $\mathrm{kInt}^M_{\Gamma_L}$. For the second part of the statement, we simply notice that $\tau(a) = \dfrac{\tau(x)}{\tau(\hat{m})} = \sum_{i \le n} \sigma(l_i) \dfrac{m_i}{\hat{m}}$, as required. □

Thus, given $x_1, x_2, y_1, y_2 \in C[L, M]$ we can find elements

$$m_1, n_1, m_2, n_2 \in M, a_1, b_1, a_2, b_2 \in \mathcal{O}^{\times}_{C(L,M)}$$

such that $x_1 = a_1 m_1,\ x_2 = a_2 m_2,\ y_1 = b_1 n_1$ and $y_2 = b_2 n_2$.
We argue that for each $n \in \mathbb{N}$, if $v\big(\frac{x_1}{x_2}\big), v\big(\frac{x_2}{y_2}\big) \in n\Gamma$ and $\rho_n\big(\operatorname{rv}\big(\frac{x_1}{x_2}\big)\big) = \rho_n\big(\operatorname{rv}\big(\frac{y_1}{y_2}\big)\big)$ then

$$\rho_n\Big(\operatorname{rv}\Big(\frac{\tau(x_1)}{\tau(x_2)}\Big)\Big) = \rho_n\Big(\operatorname{rv}\Big(\frac{\tau(y_1)}{\tau(y_2)}\Big)\Big).$$

Note that:

$$\begin{gathered}
\rho_n\Big(\operatorname{rv}\Big(\frac{x_1}{x_2}\Big)\Big) = \rho_n\Big(\operatorname{rv}\Big(\frac{y_1}{y_2}\Big)\Big) \text{ if and only if} \\
\rho_n\Big(\operatorname{rv}\Big(\frac{a_1}{b_1}\frac{m_1}{n_1}\Big)\Big) = \rho_n\Big(\operatorname{rv}\Big(\frac{a_2}{b_2}\frac{m_2}{n_2}\Big)\Big), \\
\text{if and only if } \rho_n(\operatorname{rv}(a_1))\rho_n(\operatorname{rv}(b_1)^{-1})\rho_n\Big(\operatorname{rv}\Big(\frac{m_1}{n_1}\Big)\Big) = \\
\rho_n(\operatorname{rv}(a_2))\rho_n(\operatorname{rv}(b_2)^{-1})\rho_n\Big(\operatorname{rv}\Big(\frac{m_2}{n_2}\Big)\Big),
\end{gathered}$$

where $\mathrm{rv}(a_1), \mathrm{rv}(b_1), \mathrm{rv}(a_2)$ and $\mathrm{rv}(b_2)$ are representable with parameters in $\mathrm{kInt}^M_{\Gamma_L}$ and

$$\rho_n\big(\mathrm{rv}\,(\frac{m_1}{n_1})\big), \rho_n\big(\mathrm{rv}\,(\frac{m_2}{n_2})\big) \in \mathcal{A}_M.$$

In particular, the equality $\rho_n\big(\mathrm{rv}\,(\frac{x_1}{x_2})\big) = \rho_n\big(\mathrm{rv}\,(\frac{y_1}{y_2})\big)$ can be represented by a formula satisfied by L using parameters in $C\,\mathrm{kInt}^M_{\Gamma_L}\,\mathcal{A}_M$. As $\sigma : L \to L'$ is an elementary map fixing $C\,\mathrm{kInt}^M_{\Gamma_L}\,\mathcal{A}_M$, the same formula must hold for $\sigma(L)$, thus

$$\rho_n\big(\mathrm{rv}\,(\frac{\tau(x_1)}{\tau(x_2)})\big) = \rho_n\big(\mathrm{rv}\,(\frac{\tau(y_1)}{\tau(y_2)})\big).$$

Hence, for each $n \in \mathbb{N}$, we can naturally define the map $\tau_n : (\mathcal{A}_n)_{C(L,M)} \to (\mathcal{A}_n)_{C(\sigma(L),M)}$, where for $x, y \in C[L, M]$ we define

$$\tau_n\Big(\rho_n\big(\mathrm{rv}\,(\frac{x}{y})\big)\Big) = \rho_n\big(\mathrm{rv}\,(\frac{\tau(x)}{\tau(y)})\big).$$

Take $\mathbf{t} = \tau \cup \{\tau_n \mid n \in \mathbb{N}\}$, then $\mathbf{t} : C(L, M) \to C(\sigma(L, M))$ is a $\mathcal{L}_{RV}$-isomorphism which satisfies the following conditions:

1. $\mathbf{t} \restriction_{\mathcal{A}_{C(L,M)}} : \mathcal{A}_{C(L,M)} \to \mathcal{A}_{C(\sigma(L,M))}$ is partial elementary map of $\mathcal{A}_{\mathfrak{C}}$. This follows by Claim 333 combined with the fact that $\sigma : L \to L'$ is a partial elementary map fixing $C\,\mathrm{kInt}^M_{\Gamma_L}\,\mathcal{A}_M$.
2. $\mathbf{t} \restriction_{\Gamma_{C(L,M)}} : \Gamma_{C(L,M)} \to \Gamma_{C(\sigma(L,M))}$ is partial elementary map of $\Gamma_{\mathfrak{C}}$. In fact,
$$\Gamma_{C(L,M)} = \Gamma_M = \Gamma_{C(\sigma(L),M)}$$
and $\mathbf{t}$ acts as the identity on the value group.

By quantifier elimination relative to the power residue sorts and the value group, the partial isomorphism τ must be an elementary map. It coincides with σ over L and is the identity on M, so we conclude that $\mathrm{tp}(L/M) = \mathrm{tp}(L'/M)$, as desired. □

We restate the result in terms of domination.

Corollary 334. *Let L be an elementary substructure of $\mathfrak{C}$ and let $C \subseteq L$ be a maximal model of T. Then the type $\mathrm{tp}(L/C)$ is dominated over its value group by the sorts internal to the residue field, that is for any field extension $C\Gamma_L \subseteq M$ such that $kInt^M_{\Gamma_L} \mathop{\smile\!\!\!\!\!\!\downarrow}^{qfs}_{\Gamma_L C} kInt^L_{\Gamma_L}$ we have $\mathrm{tp}(L/C\Gamma_L kInt^M_{\Gamma_L}) \vdash \mathrm{tp}(L/C\Gamma_L M)$.*

Proof. Let $C\Gamma_L \subseteq M$ such that $kInt^M_{\Gamma_L} \mathop{\smile\!\!\!\!\!\!\downarrow}^{qfs}_{C\Gamma_L} kInt^L_{\Gamma_L}$. We aim to prove that

$$\mathrm{tp}(L/C\Gamma_L kInt^M_{\Gamma_L}) \vdash \mathrm{tp}(L/C\Gamma_L M),$$

this is given an elementary map $\sigma : L \to L'$ fixing $C\Gamma_L kInt^M_{\Gamma_L}$ we can find an automorphism τ extending σ which is the identity on M. This is precisely the conclusion of Theorem 330. □

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