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Analytic and Combinatorial Features of Stable Polynomials

by

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Committee in charge:

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Abstract

Analytic and Combinatorial Features of Stable Polynomials

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Doctor of Philosophy in Mathematics

University of California, Berkeley

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We investigate two main overarching topics in the theory of stable polynomials.

1. **Differential and Difference Operators**. We first study root properties of differential and difference operators. This includes the location of roots, the spacing between roots, and the effect of linear operators on the roots. Our results in this direction are mainly for the purpose of theory, with motivation being derived from previous applications which rely on such results.

2. **Applications to Graphs**. We next study some examples of the connection between graphs and polynomials. This includes a study of polynomial capacity and its relation to bipartite matchings, as well as a study of the roots of the independence polynomial and implications of various properties. Our results in this direction are applications of the theory of stable polynomials, and demonstrates their use in combinatorics.

In the vein of (1), we extend the root bound of [47] to demonstrate the submodular nature of the roots of real-rooted polynomials. This natural extension leads to further questions on generalizations to multivariate polynomials, which are important steps on the path to understanding the barrier arguments of Marcus-Spielman-Srivastava. We further settle a conjecture of Brändén-Krasikov-Shapiro on a finite difference convolution which preserves the spacing of roots of real-rooted polynomials. This convolution generalizes the one studied by Marcus-Spielman-Srivastava, and the connections between these results yield interesting open problems which have not yet been fully explored.

Towards (2), we combine the Borcea-Brändén characterization of stability preservers with Gurvits' capacity results to produce a theory of capacity preservers. We use this to give a new proof of Csikvári's lower bound on the matchings of a biregular bipartite graph. In another direction, we give a new proof of the Chudnovsky-Seymour result on the real-rootedness of the independence polynomial of a claw-free graph. We also prove root bounds similar to that of Heilmann-Lieb for a class of independence polynomials, using an interesting multivariate extension of Godsil's divisibility relations for the matching polynomial.

Contents

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Next, I want to thank Nikhil Srivastava, who is essentially another advisor to me. Nikhil and I have rather different views on most every subject we share interest in, and this has been fantastic as a graduate student trying to learn. There is no time that I would speak to him without coming away with a new perspective, or at very least an important and well-placed asking of "why?". Even comments and intuitions on the basics of polynomial theory were always enlightening. The purpose and conceptual features of polynomials given in the introduction of this thesis, and even the format of many of the sections, are highly influenced by Nikhil.

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Chapter 1

Introduction

Polynomials are central objects of study in mathematics and other nearby subjects. They are perhaps the simplest nice functions we can think of, and yet their properties give rise to wide range of rich applications in almost every field that touches mathematics. What allows for this is a large number of interpretations, each pointing to intuitions that have developed over the history of the subject. We will mainly focus on the following three, under the influence of a generally analytic mindset:

- *Analytic*: Can properties like convexity and positivity be used to obtain desirable inequalities?

- *Combinatorial*: Can we count things in a graph (e.g.) by storing information in the coefficients of a polynomial?

- *Algebraic*: Can we translate information about the zeros of polynomials into analytic and combinatorial properties?

In what follows, the interplay of the *analytic* and the *combinatorial* will often be mediated by the *algebraic*. In other words, in applications of our results we want to determine analytic and combinatorial information, and we often use properties about the zeros of polynomials to get it. This form is a deep thread in the analytic theory of polynomials, and the intuition for the usefulness of this form comes from two classical observations: Newton's identities and Newton's inequalities.

1.1 Two Classical Observations

First, suppose we have a univariate monic polynomial written in the following form:

$$p(x) = \prod_{i=1}^{d}(x - r_i)$$

If we wish to write this polynomial in terms of its coefficients, the process is easy via Newton's identities. The coefficients are in fact given by the elementary symmetric polynomials:

$$p(x) = \sum_{k=0}^{d} (-1)^k e_k(r_1, ..., r_d) x^{d-k} \qquad \text{where} \qquad e_k(r_1, ..., r_d) := \sum_{\substack{S \subset [d] \\ |S|=k}} \prod_{i \in S} r_i$$

The other direction—determining the roots from the coefficients—is a substantially harder problem. In fact, there is an argument to be had that a big portion of the analytic theory of polynomials is attempting to solve this reverse problem. The moral of the story: it is easier to determine coefficient (combinatorial) information from root (algebraic) information than the other way around.

Next, suppose we have a univariate real-rooted polynomial with non-negative coefficients:

$$p(x) = \sum_{k=0}^{d} c_k x^k = c_d \prod_{i=1}^{d} (x + r_i) \qquad \text{where} \qquad r_i \geq 0 \text{ and } c_k \geq 0$$

Newton's inequalities then tell us that the coefficients of p are ultra log-concave, and similarly p itself is log-concave (as a function) on the set of positive reals. That is:

$$\left(\frac{c_k}{\binom{d}{k}} \right)^2 \geq \left(\frac{c_{k-1}}{\binom{d}{k-1}} \right) \left(\frac{c_{k+1}}{\binom{d}{k+1}} \right) \qquad \text{and} \qquad \frac{d^2}{dx^2} \log(p)(x) \leq 0 \text{ for } x > 0$$

However, these properties do not characterize real-rooted polynomials. And so the same sort of thing happens: it is easier to determine analytic information from algebraic information than the other way around. (Of course, the recently developed theories of strongly log-concave/completely log-concave/Lorentzian polynomials present a counterargument to this point.)

1.2 The Role of Stability

The point is then to hold on to zero location information for as long as possible, before transferring to the combinatorial or analytic information that we actually want. To do this, we need some robust inductive structure around the notion of real-rootedness. That's where stable polynomials come in to play.

Stability is a multivariate generalization of real-rootedness, and one of the most interesting conceptual ideas in the theory is the fact that more variables actually reduce complexity and make computation and interpretation easier. Specifically, multivariate polynomials allow association of variables to elements of a given combinatorial object (e.g., to edges of a graph). This yields a rich theory of stable polynomials, with strong connections to combinatorics. Classic instances of stable polynomias include the multivariate matching polynomial

and the spanning tree polynomial for any given graph. The role that polynomials often play in these applications is that of conceptual unification: various natural operations that one may apply to a given type of object can often be represented as natural operations applied to associated polynomials. For the matching polynomial deletion and contraction correspond to certain derivatives, and for the spanning tree polynomial this idea extends to the minors of a matroid in general.

So certain linear operators correspond to combinatorial operations, but what about analytic information? Is there some generalization of Newton's inequalities to multivariate stable polynomials? In fact, for real stable polynomials we have the strongly Rayleigh inequalities (Theorem 2.2.7, see also [9]) which more or less play this role. These inequalities will be particularly important in Chapter 5 when we study the notion of polynomial capacity (Definition 5.0.1).

That is: what makes stable polynomials so important is that they are at the intersection of the three intuitions discussed in the introduction. First, they are defined via a condition on their zeros. Second, the class of such polynomials is preserved by various operators with combinatorial interpretations (see Proposition 2.2.4 below). And finally, they are equivalently defined via certain analytic log-concavity statements (see Theorem 2.2.7). This prompts a general framework for solving problems with stable polynomials.

combinatorial objects → simple polys → stability preservers → combinatorial/analytic info

In this thesis we will investigate various instances of the different pieces of this framework and discuss their applications. The thesis itself is broken up into two main parts. The first studies zero location properties of some specific preservers: differential and finite difference operators. The second studies how to use zero location properties of polynomials to prove combinatorial and analytic results on graphs. We now discuss these two parts in more detail.

1.3 Differential and Finite Difference Operators

In the first part of this thesis (Chapters 3 and 4), the central object of study is the additive convolution (also called the Walsh convolution [64] and the finite free convolution [47]), along with its generalizations. (There is also a Grace-Szegö multiplicative convolution [62] which we will discuss in those chapters as well, but to a lesser extent.) Given two univariate polynomials f and g of degree at most d, we will denote this bilinear function as follows:

$$f \boxplus^d g := \frac{1}{d!} \sum_{k=0}^{d} \partial_x^k f \cdot (\partial_x^{d-k} g)(0)$$

This notation is suggestive, as these convolutions can be thought of producing polynomials whose roots are contained in the Minkowski sum of complex discs containing the roots of the input polynomials. When the inputs have real roots, this fact also holds in terms of real intervals containing the roots. This convolution also has a strong relation to differential

operators: every constant coefficient differential operator on polynomials of bounded degree can be written as $f \mapsto f \boxplus^d g$ for some g.

There is a long history of studying differential operators that preserve the set of univariate polynomials with only real roots. A classic result in this direction is: given a real-rooted polynomial $p(t)$, it is easy to see that $p(\partial_t)$ preserves the set of real-rooted polynomials. When one bounds the degree of the input polynomial though, the set of all differential operators preserving real-rootedness is actually larger than just those of the above form. The set of such differential operators in this case has connections to the classical Walsh [64] additive convolution. Namely, if $\boxplus^d$ denotes the Walsh additive convolution for polynomials of degree at most n (also recently known as the finite free convolution; e.g. see [48]) then $p(\partial_t)$ preserves the set of real-rooted polynomials of degree at most d if and only if there is some real-rooted q such that $p(\partial_t)r(t) = (r \boxplus^d q)(t)$ for all r.

Recently, there has been interest in understanding how certain differential operators preserving real-rootedness affect the roots of the input polynomial. Much of this interest derives from the notion of interlacing families, heavily studied by Marcus, Spielman, and Srivastava in their collection of papers ([52],[49],[51]) containing their celebrated resolution of Kadison-Singer. Most uses of interlacing families share the same loose goal: to study spectral properties of random combinatorial objects. To do this, one equates random combinatorial operations on the objects to differential operators on associated characteristic polynomials. Then, understanding the spectrum of the random objects is reduced to understanding how the roots of certain polynomials are affected by differential operators preserving real-rootedness.

The most robust way to study the effects of a differential operator on roots comes from framework of Marcus, Spielman, and Srivastava. They associate an R-transform to polynomials, inspired from free probability theory, which gives tight bounds on the movement of the largest root via the additive convolution mentioned above. This framework was used in particular in [52] to prove the existence of Ramanujan bipartite graphs. The strength of their framework is that it gives tight largest root bounds for a general class of differential operator preserving real-rootedness, replacing many of the ad hoc methods used before to study specific desired operators.

Generalizing the MSS Root Bound

Some combinatorial objects require the use of multivariate methods to analyze. Here the associated polynomials are real stable, and for these methods there is no general framework in place to study the analytic effects of the linear operators on the roots. We consider the following to be one of the large unanswered questions around the recent resurgence of interest in finite free convolutions: How does the multivariate additive convolution affect root information?

In an attempt to better understand the multivariate case, we expand upon the previous results of Marcus-Spielman-Srivastava and provide more general results about how *all* roots are of a given polynomial are affected by finite free convolutions. To do this, we first expand

their bound on the movement of the largest root to all differential operators preserving real-rootedness. Further, we utilize the theory of hyperbolic polynomials to give more interesting root bounds on interior roots (other roots besides the largest).

With these results in hand, we state a number of conjectures (and some counterexamples) in the direction of stronger univariate results on interior roots and of analogous multivariate results. Proving similar multivariate results seems to be a hard problem in general. But the hope is that by better fleshing out the details of the additive convolution in the univariate case, one can better abstract to the multivariate case to handle problems such as Kadison-Singer, the Paving conjecture, and Heilman-Lieb root bounds.

Finite Difference Convolution

In [10], the authors show an interesting property of the additive convolution which is outside of the scope of the discussion thus far: it can only increase root mesh, which is defined as the minimum absolute difference between any pair of roots of a given polynomial. That is, the mesh of the output polynomial is at least as large as the mesh of either of the input polynomials.

There are natural generalized convolution operators which also preserve root mesh, as well as a multiplicative variant called logarithmic root mesh (minimum ratio between roots). Lamprecht studied this q-multiplicative convolution (where the q- should give the connotation of q-binomial coefficients), showing that it preserves logarithmic root mesh of at least q [39]. That said, we will study a finite difference version of the additive convolution called the b-additive convolution (or finite difference convolution), and in particular we will demonstrate that it preserves root mesh of at least b. This resolves a conjecture of Brändén, Krasikov, and Shapiro given in [10], where they investigated root mesh preservation properties of finite difference operators on polynomials of all degrees.

The finite difference convolution is defined as follows.

$$p \boxplus_b^n r := \frac{1}{n!} \sum_{k=0}^{n} \Delta_b^k p \cdot (\Delta_b^{n-k} r)(0)$$

Here, Δ_b is a finite b-difference operator, defined as:

$$\Delta_b : p \mapsto \frac{p(x) - p(x - b)}{b}$$

By limiting $b \to 0$, we recover the usual additive convolution. That is this convolution is some discrete generalization of the additive convolution, which gives some motivation for why it is interesting to study.

First, by generalizing in general one inevitably loses some structure of the original object. This can help to focus one's attention on the "right" properties in order to solve some problem or prove some conjecture. For example some of the conjectures for the classical additive convolution, mentioned above, might become easier if converted to conjectures for the b-additive convolution and then observed in this new light.

Second, root mesh properties and finite differences have more direct connections to certain combinatorial objects. For instance, the identity element of the finite difference convolution is the rising factorial polynomial, and the coefficients of this polynomial are the Stirling numbers of the first kind. This suggests connection between mesh and counting permutations and partitions.

That said, we do not explore these potential connections here. We only mention them to motivate the study of these generalized convolutions. Instead we will discuss two very different methods for resolving the conjecture of Brändén, Krasikov, and Shapiro. One is analytic in nature, giving a way to transfer results from multiplicative convolutions to additive ones; the second simply generalizes Lamprecht's proof in the multiplicative case. We direct the reader to Chapter 4 for further discussion.

1.4 Applications to Graph Theory

In the second part of this thesis (Chapters 5 and 6), we discuss ways in which stability and related properties of polynomials can be used to prove graph-theoretic results. In Chapter 5, we combine Gurvits' notion of polynomial capacity with the Borcea-Brändén characterization to achieve a theory of capacity preserving operators. Through one of Gurvits' (many) ideas, we use this theory to give a new proof of Csikvári's strengthening of Friedland's lower matching conjecture. In Chapter 6, we then use a weaker multivariate notion of stability to give a new simpler proof of Chudnovsky and Seymour's famous result on the real-rootedness of the independence polynomial of a claw-free graph. We then extend a matching polynomial divisibility result of Godsil to multivariate matching polynomials and to a certain class of claw-free independence polynomials.

Polynomial Capacity and Bipartite Matchings

The particular line into which Chapter 5 falls begins with the work of Gurvits, who in a series of papers (e.g., see [35]) gave a vast generalization of the Van der Waerden lower bound for permanents of doubly stochastic matrices and the Schrijver lower bound on the number of perfect matchings of regular graphs. In particular, he showed that a related inequality holds for real stable polynomials in general, and then derives each of the referenced results as corollaries. His inequality describes how much the derivative can affect a particular analytic quantity called the capacity of a polynomial. In fact, one should interpret his result is as a statement about the capacity preservation properties of the derivative. We define α-capacity here, but leave discussion of his result to Chapter5

$$\mathrm{Cap}_\alpha(p) := \inf_{x>0} \frac{p(x)}{x_1^{\alpha_1} \cdots x_n^{\alpha_n}}$$

For those familiar with the real stability literature, the concept of preservation properties of a linear operator (specifically that of the derivative here) is not new. Perhaps the most

essential result in the theory is the Borcea-Brändén characterization [7], which essentially says that the symbol of a linear operator T holds the real stability preservation information of T. In Chapter 5, we make use of this concept by showing that the symbol also holds the capacity preservation information of T. That is, we combine the ideas of Gurvits and of Borcea and Brändén to create a theory of capacity preserving operators. Our main result in this direction says that the amount a stability preserver can affect the capacity of the input polynomial is related to the capacity of its symbol. We delay the formal statement of this result until Chapter 5.

Using this, we are able to reprove a few results. The first of these are essentially results mentioned above that Gurvits was able to obtain using his theorem: the Van der Waerden lower bound (see [21] and [20] for the original resolution of this conjecture) and Schrijver's inequality [57]. We also reprove Gurvits' theorem using the capacity preservation theory, which amounts to a very basic computation for the partial derivative.

The main application is a new proof of Csikvári's bound on the number of k-matchings of a biregular bipartite graph [17]. (This possibility was suggested to us by Gurvits.) This result generalizes Schrijver's inequality and is actually stronger than Friedland's lower matching conjecture (see [25]). The computations involved in this new proof never exceed the level of basic calculus. This was one of the most remarkable features of Gurvits' original result, and this theme continues to play out here. We state Csikvári's result now.

Theorem 5.2.6 (Csikvári). *Let G be an (a,b)-biregular bipartite graph with (m,n)-bipartitioned vertices (so that $am = bn$ is the number of edges of G). Then the number of size-k matchings of G is bounded as follows:*

$$\mu_k(G) \geq \binom{n}{k}(ab)^k \frac{m^m(ma-k)^{ma-k}}{(ma)^{ma}(m-k)^{m-k}}$$

Beyond these specific applications, one of the main purposes of this chapter is to unify the various results that fit into the lineage of the concept of capacity. Some of these are inequalities for specific combinatorial quantities ([35], [33], [34]), some are approximation algorithms for those quantities ([2], [60]), and some are capacity preservation results similar to those proven in this chapter (particularly [1]).

The Independence Polynomial

In Chapter 6, we study stability and root location properties of the independence polynomial of a graph, especially as they relate to real-rootedness. Given a graph G, the independence polynomial of G is a polynomial which encodes the independent sets (subsets of non-adjacent vertices) of G. Root information for the independence polynomial is highly studied due to the connection of stability regions to the Lovász local lemma (see, e.g., [19], [58]). Very generally speaking, the Lovász local lemma can be used to construct probabilistic existence proofs in various combinatorial settings, and the stability regions of the independence polynomial tell you when the lemma can be applied.

The matching polynomial of G encodes the matchings (sets of pairs of adjacent vertices) of G. The real-rootedness of the matching polynomial and the Heilmann-Lieb root bound are important results in the theory of undirected simple graphs. In particular, real-rootedness implies log-concavity and unimodality of the matchings of a graph (as discussed above in §1.1), and recently in [52] this root bound was used to show the existence of Ramanujan graphs. Additionally, it is well-known that the matching polynomial of a graph G is equal to the independence polynomial of the line graph of G. With this, one obtains the same results for the independence polynomials of line graphs. This then leads to a natural question: what properties extend to the independence polynomials of all graphs?

Generalization of these results to the independence polynomial has been partially successful. About a decade ago, Chudnovsky and Seymour [15] established the real-rootedness of the independence polynomial for claw-free graphs. A general root bound for the independence polynomial was also given by [23], though it is weaker than that of Heilmann and Lieb. As with the original results, these generalizations are proven using univariate polynomial techniques.

That said, the first part of Chapter 6 is a partial generalization of this stability result to the multivariate independence polynomial of claw-free graphs. In particular, we prove a result related to the real-rootedness of certain weighted independence polynomials. This result was originally proven by Engström in [18] by bootstrapping the Chudnovsky and Seymour result for rational weights and using density arguments. The proof we give here is completely self contained and implies both the original Chudnovsky and Seymour result as well as the weighted generalization. By using a multivariate framework to directly prove the more general result, we obtain a simple inductive proof which we believe better captures the underlying structure.

In addition, the full importance of the claw (3-star) graph is not immediately clear from the univariate framework. Since the result of Chudnovsky and Seymour, there have been attempts to explain more conceptually why the claw-free premise is needed for real-rootedness. In particular, some graphs containing claws actually have real-rooted independence polynomials, disproving the converse to the univariate result. On the other hand, the stronger stability-like property we use here turns out to be equivalent to claw-freeness, yielding a satisfactory converse.

In the second part of Chapter 6, we then extend the Heilmann-Lieb root bound by generalizing some of Godsil's work on the matching polynomial. In [29], Godsil demonstrated the real-rootedness of the matching polynomial of a graph by showing that it divides the matching polynomial of a related tree. (For a tree, root properties are more easily derived.) We prove a similar result for the multivariate matching polynomial, and then we determine conditions for which these divisibility results extend to the multivariate independence polynomial. Further, we prove the Heilmann-Lieb root bound for the independence polynomial of a certain subclass of claw-free graphs. By considering a particular graph called the Schläfli graph, we demonstrate that this root bound does not hold for all claw-free graphs and provide a weaker bound in the general claw-free case.

1.5   Bibliographic Note

Much of this is joint work with Nick Ryder, and all of it can be found in smaller pieces on the arXiv. Specifically, Chapter 3 is joint with Nick from [46], Chapter 4 is joint with Nick from [44], and Chapter 6 is joint with Nick from [45]. Finally, Chapter 5 is from [42].

Chapter 2

Preliminaries

2.1   Notation

We will let $\mathbb{C}$, $\mathcal{H}_+$, $\mathcal{H}_-$, $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}_{++}$, $\mathbb{Z}$, $\mathbb{Z}_+$, and $\mathbb{N}$ denote the complex numbers, the open upper half-plane, the open lower half-plane, the reals, the non-negative reals, the positive reals, the integers, the non-negative integers, and the positive integers respectively.

For vectors, we will utilize a number of shorthands. Given α, β of the same length, we denote $\alpha + \beta$ as the entry-wise sum, $\alpha\beta$ as the entry-wise product, etc.. For the dot product of two vectors, we will usually use inner product notation $\langle \alpha, \beta \rangle$. Similarly, we will write α^β to denote $\prod_i \alpha_i^{\beta_i}$ whenever $\beta \in \mathbb{R}_+^n$. A specific use of this will be denote multivariate monomials via x^α when $\alpha \in \mathbb{Z}_+^n$. Additionally, we will write $\alpha \leq \beta$ whenever the inequality holds entry-wise. Sometimes, we will let 0 or (0^n) and 1 or (1^n) denote the vector of all zeros and ones respectively, depending on the context. Finally, for $\mu, \gamma \in \mathbb{Z}_+^n$ we write $\mu! := \prod_k(\mu_k!)$ and $\binom{\gamma}{\mu} := \frac{\gamma!}{\mu!(\gamma-\mu)!}$.

The space of polynomials in n variables with complex coefficients will be denoted as it usually is: $\mathbb{C}[x] = \mathbb{C}[x_1, ..., x_n]$ (and we will very often use such shorthands for vectors of variables). For any subset $S \subset \mathbb{C}$, we will denote such polynomials with coefficients in S by $S[x]$. Further, we will denote the subspace of polynomials which are of degree at most γ_k in the variable x_k by $S^\gamma[x]$. When $\gamma = (1, ..., 1)$, we will call such polynomials *multiaffine*. Also, we will often denote coefficients of polynomials we are considering via subscripts. E.g.,

$$p(x) \in \mathbb{C}^d[x] \qquad \text{with} \qquad p(x) = \sum_{k=0}^d p_k x^k$$

$$p(x) \in \mathbb{C}^\gamma[x] = \mathbb{C}^\gamma[x_1, ..., x_n] \qquad \text{with} \qquad p(x) = \sum_{0 \leq \mu \leq \gamma} p_\mu x^\mu$$

On occasion we will include binomial or multinomial coefficients in the coefficient expansion, but we will make this explicit. Finally, we will use the notation $\mathrm{Hmg}^\gamma(p)$ to denote the per-variable homogenization of $p \in \mathbb{C}^\gamma[x_1, ..., x_n]$, where x_i is homogenized to degree γ_i via a new variable y_i.

In a number of contexts, we will use λ to denote the non-increasing vector, counting multiplicities, of something like roots. Some examples of this are as follows.

1. $\lambda(f)$ denotes the roots of a univariate real-rooted polynomial f.

2. $\lambda(A)$ denotes the eigenvalues of a Hermitian matrix A.

3. $\lambda(x)$ denotes the roots of $p(te+x)$ where p is hyperbolic with respect to e (see Definition 3.1.4).

Although is does not makes sense to list the roots of a multivariate polynomial in general, we can generalize the notion of "largest root" to the notion of *points above the roots*. Note that while we use notation similar to that of Marcus-Spielman-Srivastava in [49], our definition differs slightly from the usual one in that $a \in \mathrm{Ab}(p)$ does *not* imply $p(a) \neq 0$.

Definition 2.1.1. For real stable $p \in \mathbb{R}[x_1, ..., x_n]$, we say that $a \in \mathbb{R}^n$ is *above the roots of* p if $p(a + y) \neq 0$ for all $y \in \mathbb{R}^n_{++}$. We also let $\mathrm{Ab}(p)$ denote the set of all points above the roots of p. For the sake of simplicity, we say $\mathrm{Ab}(p) = \mathbb{R}^n$ for $p \equiv 0$.

Note that in the univariate case, $\mathrm{Ab}(p)$ is the interval $[\lambda_1(p), \infty)$. It is in this way that $\mathrm{Ab}(p)$ generalizes the largest root of a polynomial. For more discussion on the relation between points above the roots and differential operators, see §3.3.

Derivative-like Operators

The derivative with respect to a variable x_i will be denoted ∂_{x_i}. We will also apply the exponent shorthand above to derivatives, by writing $\partial_x^\alpha := \prod_i \partial_{x_i}^{\alpha_i}$ whenever $\alpha \in \mathbb{Z}_+^n$. Further, given $v \in \mathbb{R}^n$ we will let $\nabla_v := \sum_i v_i \partial_{x_i}$ denote the directional derivative in the direction v.

For univariate p, there will also be a number of special derivative-like operators we will consider. Often these "derivatives" will be with respect to some parameters, and reference to the variable of the polynomial will sometimes be suppressed. As a first important example of this, we consider a polynomial $p \in \mathbb{C}^d[x]$. We write:

$$\partial_x p(x) = \frac{d}{dx} p(x) = \sum_{k=0}^{d-1} (k+1) p_{k+1} x^k$$

$$\partial_x^* p(x) = d \cdot p(x) - x \cdot \frac{d}{dx} p(x) = \sum_{k=0}^{d-1} (d-k) p_k x^k$$

The operator ∂_x^* is classically called the polar derivative with respect to 0, and can be considered as the derivative (with respect to the new variable) of the homogenization of p. Note that the operator ∂_x^* actually depends on the degree d. In practice this d would need to be specified, or at least understood in context. This will not be an issue here as we will

never actually use the polar derivative. We only define it here to give context to the operator $\partial_{q,d}^*$ defined below, which should be seen as a q-generalization of the polar derivative.

In a similar vein, we define for $q > 1$:

$$(\partial_{q,d}p)(x) := \frac{p(qx) - p(x)}{q^{1-d}(q^d - 1)x} \qquad (\partial_{q,d}^*p)(x) := \frac{p(qx) - q^d p(x)}{q^d - 1}$$

These operators are related to q-binomial coefficients and the notion of logarithmic mesh, and we will explore these connections in Chapter 4. Morally, these operators limit to ∂_x and ∂_x^* as $q \to 1$, but this is not technically true. In fact, $\partial_{q,d} \to \frac{1}{d}\partial_x$ and $\partial_{q,d}^* \to -\frac{1}{d}\partial_d^*$.

Finally, we define for $b > 0$:

$$(\Delta_{b,d}p)(x) := \frac{p(x) - p(x - b)}{b} \qquad (\Delta_{b,d}^*p)(x) := d \cdot p(x - b) - (x - b)\Delta_b p(x)$$

For $b = 1$ this is a standard finite difference operator. These operators are related to the distance between consecutive roots of polynomials (mesh), and we will discuss this in Chapter 4. Notice that these finite difference operators literally limit to ∂_x and ∂_x^* as $b \to 0$, in contrast to the case of $\partial_{q,d}$ and $\partial_{q,d}^*$ above.

Graphs

Let $G = (V, E)$ be an undirected graph, which is simple unless otherwise specified. As usual, V is the set of vertices and E is the set of edges. We employ standard notation surrounding these first objects:

- $\{u, v\} \in E$ iff there is an edge between vertices u and v.

- $u \in e$ for $e \in E$ iff u is a vertex of the edge e.

- $N_G[v]$ (resp. $N_G(v)$) denotes the closed (resp. open) neighborhood of v. (Recall: closed means it contains v itself, open means it does not.)

- $H \subseteq G$ (resp. $H \leq G$) iff H is a subgraph (resp. induced subgraph) of G.

We also generalize the definition of "claw" in the following standard way. As usual, let $K_{m,n}$ denote the complete bipartite graph with $m + n$ vertices. So, we refer to $K_{1,3}$ as a *claw* or as a *3-star*. Generalizing, we refer to $K_{1,n}$ as an *n-star*. For any graph H, we say that G is *H-free* if it does not contain H as an induced subgraph.

Finally, we denote the line graph of G by $L(G)$. This is the graph formed by considering the edges of G to be the vertices of $L(G)$, with adjacency in $L(G)$ determined by whether or not the corresponding edges of G share a vertex in G.

2.2   Stable Polynomials

In this section, we discuss all o the basic properties of stable polynomials which we will use throughout. First, the definition.

Definition 2.2.1 (Stability and Bistability). Given a polynomial $p \in \mathbb{C}[x_1, ..., x_n]$, we say that it is *stable* if it is $\mathcal{H}_+^n$-stable as defined above. If p further has real coefficients, we say that p is *real stable*. Additionally, given $p \in \mathbb{C}[x_1, ..., x_n, z_1, ..., z_m] = \mathbb{C}[x, z]$, we say that it is *bistable* if it is $(\mathcal{H}_+^n \times \mathcal{H}_-^m)$-stable.

The motivation for this definition comes from the univariate case, where real stability is equivalent to real-rootedness (since complex roots are symmetric about the real axis in the case of real coefficients). The intuition that stability generalizes real-rootedness can also be seen in the following.

Proposition 2.2.2. *A polynomial* $p \in \mathbb{C}[x_1, ..., x_n]$ *is real stable iff for every* $e \in \mathbb{R}_+^n$ *and* $x \in \mathbb{R}^n$ *the univariate polynomial* $p(et + x)$ *is real-rooted.*

The previous proposition also demonstrates a connection between stable polynomials and hyperbolic polynomials. Hyperbolic polynomials have their origin in the PDE literature ([26]), but more recently have found connections to optimization ([30], [56]). We will mainly make use of these polynomials in Chapter 3, where we define them explicitly (Definition 3.1.4).

The following classical result we will use often, often without even mentioning it.

Proposition 2.2.3 (Hurwitz). *Fix a connected open set* $S \subset \mathbb{C}^n$. *If* $p_n \in \mathbb{C}[x]$ *is a sequence of S-stable polynomials which limits to* p, *then* p *is also S-stable. (Recall that the 0 polynomial is S-stable for any S.)*

Basic Closure Properties

To make their combinatorial nature clear, we now give some basic operators which preserve the class of stable polynomials. Afterwards, we will discuss how these operators can have conceptual interpretations in certain contexts.

Proposition 2.2.4 (Closure Properties). *Let* $p, q \in \mathbb{C}^\gamma[x_1, ..., x_n]$ *be stable (resp. real stable) polynomials, and fix* $k \in [n]$. *Then the following are also stable (resp. real stable).*

1. $p \cdot q$ *(product)*

2. $\partial_{x_k} p$ *(differentiation)*

3. $x_k \partial_{x_k} p$ *(degree-preserving differentiation)*

4. $p(x_1, ..., x_{k-1}, r, x_{k+1}, ..., x_n)$, *for* $r \in \mathbb{R}$ *(real evaluation)*

    5. $p(x_1, ..., x_{k-1}, x_1, x_{k+1}, ..., x_n)$ *(projection)*

    6. $x_k^{\gamma_k} p(x_1, ..., x_{k-1}, -x_k^{-1}, x_{k+1}, ..., x_n)$ *(inversion)*

Proof. $(1), (5), (6)$ are obvious, and $(2), (3)$ follow from Gauss-Lucas. (4) then follows by plugging in $r + \epsilon i$ for $\epsilon > 0$, limiting $\epsilon \to 0$, and applying Hurwitz' theorem. $\qquad\square$

    A classical but more interesting real stability preserver is *polarization*. Polarization plays a crucial role in the theory of real stability preservers, as it allows one to restrict to polynomials of degree at most 1 in every variable. We will see later that polarization also plays a crucial role in the theory of capacity preservers (Chapter 5).

Definition 2.2.5. Given $q \in \mathbb{R}^d[x]$, we define $\mathrm{Pol}^d(q)$ to be the unique symmetric $f \in \mathbb{R}^{(1^d)}[x_1, ..., x_d]$ such that $f(x, ..., x) = q(x)$. Given $p \in \mathbb{R}^\lambda[x_1, ..., x_n]$, we define $\mathrm{Pol}^\lambda(p) := (\mathrm{Pol}^{\lambda_1} \circ \cdots \circ \mathrm{Pol}^{\lambda_n})(p)$, where Pol^{λ_k} acts on the variable x_k for each k. Note that $\mathrm{Pol}^\lambda(p) \in \mathbb{R}^{(1^{\lambda_1 + \cdots + \lambda_n})}[x_{1,1}, ..., x_{n,\lambda_n}]$.

Proposition 2.2.6 ([64]). *Given $p \in \mathbb{R}^\lambda[x_1, ..., x_n]$, we have that p is real-stable iff $\mathrm{Pol}^\lambda(p)$ is real stable.*

    As an example of the combinatorial content of some basic stability preservers, let $G = (V, E)$ be a graph on edges labeled $\{1, ..., n\}$. To this graph, we can associate the *spanning tree polynomial* in $\mathbb{R}_+^{(1,...,1)}[x_1, ..., x_n] = \mathbb{R}_+^1[x]$:

$$f_G(x) := \sum_T x^T = \sum_{T \in \{\text{spanning trees}\}} \prod_{i \in T} x_i$$

It turns out that this polynomial is real stable for any graph (via, e.g., the matrix-tree theorem). So, we can apply the operations of the previous proposition. In particular, consider differentiation and evaluation:

$$\partial_{x_i} f_G(x) = \sum_{T, \; x_i \in T} x^{T-i} \qquad\qquad f_G(x)|_{x_i=0} = \sum_{T, \; x_i \notin T} x^T$$

That is, $\partial_{x_i} f_G(x)$ is precisely the spanning tree polynomial of the graph G/i where the edge i has been contracted, and $f_G(x)|_{x_i=0}$ is precisely the spanning tree polynomial of the graph $G \backslash i$ where the edge i has been deleted. And from this we recover the standard deletion-contraction relation:

$$f_G(x) = x_i f_{G/i}(x) + f_{G \backslash i}(x)$$

The purpose of this is to demonstrate that certain combinatorial operations preserve real stability, opening the possibility to various potential induction strategies. Notice though that the other operators from 2.2.4 may not have such clean combinatorial interpretations, and yet they still preserve stability. And the Borcea-Brändén characterization shows even more stability preservers which are even less combinatorial. This will open up new avenues for induction, beyond the realm of graphs.

This is one of the big reasons why stability theory is so important. One maps a whole class of interesting combinatorial objects (e.g. graphs) into some space of polynomials (perhaps real stable polynomials), but this map is usually very far from being surjective. This allows one to consider the space of polynomials to be a continuous generalization of the space of graphs. What is crucial about this map is mainly the fact that various graph operations translate to polynomial operations which preserve certain nice properties, like stability. The Borcea-Brändén characterization then allows us to generalize even the notion of "graph operation" to that of "polynomial operation". From there we try to answer our questions about graphs by forgetting the original object and working solely with polynomials. We then utilize the analytic intuitions and results we have for polynomials to say something desirable about graphs.

Log-concavity via the Strongly Rayleigh Inequalities

One of the most important features of real stable polynomials is their connection to log-concavity. This is summed up in the following.

Theorem 2.2.7 (Brändén, [9])**.** *Fix $p \in \mathbb{R}^{(1,...,1)}[x_1, ..., x_n]$. Then p is real stable iff for all $i \neq j$ we have*

$$\partial_{x_i} p \cdot \partial_{x_j} p - p \cdot \partial_{x_i} \partial_{x_j} p \geq 0$$

everywhere in $\mathbb{R}^n$. That is, iff the Hessian of $\log(p)$ is entrywise non-positive.

Any combinatorial bound which can be proven via the theory of stable polynomials eventually runs through this result. We will see this at play in particular in the results of Chapter 5 regarding polynomial capacity. This result also gives a good way of proving or disproving the stability properties of specific polynomials.

This result is also easily extended beyond multiaffine polynomials, but an extra condition is acquired a long the way.

Corollary 2.2.8. *Fix $p \in \mathbb{R}^\gamma[x]$. Then p is real stable iff for all $i \neq j$ we have*

$$\partial_{x_i} p \cdot \partial_{x_j} p - p \cdot \partial_{x_i} \partial_{x_j} p \geq 0$$

and for all i we have

$$(1 - \gamma_i^{-1})(\partial_{x_i} p)^2 - p \cdot \partial_{x_i}^2 p \geq 0$$

everywhere in $\mathbb{R}^n$.

Proof. The proof of this essentially comes from the fact that polarization preserves real stability. One can then pass Brändén's result through the polarization operator. Another similar proof could be given using certain invariance properties of the Wronskian. □

Finally, this gives an interesting characterization of univariate real-rooted polynomials.

Corollary 2.2.9. *Fix univariate* $p \in \mathbb{R}^d[x]$. *Then* p *is real-rooted iff*

$$(1 - d^{-1})(p')^2 - p \cdot p'' \geq 0$$

everywhere in $\mathbb{R}$.

Interlacing Roots and Proper Position

The connection between the symbol of an operator and its stability preservation properties is algebraic, or perhaps even linear algebraic (see [41] for a more detailed discussion). Because of this, the BB characterization is easier to prove for stability, and then the result for real stability follows as a corollary. To make this method of proof work, one needs a linear algebraic way to connect stable and real stable polynomials. (Of course we have the analytic connection via limits of stable polynomials, but this does not always help us as much as we'd like.)

This connection can be found in the notion of *interlacing polynomials*, which sometimes goes by the name *proper position* for multivariate polynomials. First we give the univariate definition.

Definition 2.2.10. Let $p, q \in \mathbb{R}[x]$ be univariate real-rooted monic polynomials with roots $\lambda(p), \lambda(q)$ (recall that $\lambda(p)$ is in non-increasing order) and leading coefficients c_p, c_q respectively. Then we say that q *interlaces* p and write $q \ll p$ if $c_p \cdot c_q > 0$ and:

$$\lambda_1(p) \geq \lambda_1(q) \geq \lambda_2(p) \geq \lambda_2(q) \geq \cdots$$

If the inequalities are all strict, we say that q *strictly interlaces* p. And if $c_p \cdot c_q < 0$, then we reverse the relation (i.e. $p \ll q$).

Notice in the above definition that the connotation of "$\ll$" as an order symbol presents itself in the fact that the "larger" polynomial has a larger maximum root (when $c_1 \cdot c_2 > 0$). However, $\ll$ is not a partial order.

Remark 2.2.11. For a real-rooted polynomial f, we write $\mathrm{mesh}(f) \geq b$ if the distance between any pair of roots is at least b. For a positive-rooted polynomial f, we write $\mathrm{lmesh}(f) \geq q \geq 1$ (for "log-mesh") if the ratio of any pair of roots is at least q. Notice that $\mathrm{mesh}(f) \geq b$ iff $f \ll f(x - b)$, and $\mathrm{lmesh}(f) \geq q$ iff $f \ll f(q^{-1}x)$. This gives a strong relationship between the mesh/log-mesh of a polynomial and interlacing, and we will make use of this in Chapter 4.

We now state a few important properties of interlacing polynomials, which demonstrate the connection between interlacing, (complex) stability, linear algebra, and convexity. Note that much of the notation for these results was taken from [63]. In what follows we define the Wronskian of p, q as usual: $W[p, q] := p'q - pq'$.

Proposition 2.2.12 (Hermite-Biehler)**.** *For real-rooted $p, q \in \mathbb{R}[x]$, we have that $q \ll p$ iff $p + iq$ is stable.*

Proposition 2.2.13 (Hermite-Kakeya-Obreschkoff)**.** *For real-rooted $p, q \in \mathbb{R}[x]$, we have that $q \ll p$ or $p \ll q$ iff p, q span a real subspace of real-rooted polynomials. When these equivalent conditions hold, we further have either $W[p, q] \geq 0$ (> 0), in which case $q \ll p$ (strictly) or $W[p, q] \leq 0$ (< 0), in which case $p \ll q$ (strictly).*

The following result requires the notion of *common interlacer* of p, q, which is a polynomial f such that $f \ll p$ and $f \ll q$. It is a statement that has been proven many times, and we refer the reader to [52] for discussion of this.

Proposition 2.2.14. *For real-rooted $p, q \in \mathbb{R}[x]$ with positive leading coefficients, p, q have a common interlacer iff every convex combination of p, q is real-rooted.*

The previous theorem can in fact be generalized to many polynomials with a single common interlacer. We quote here the following definition and result of Chudnovsky and Seymour, who use it in their proof that the independence polynomial of a claw-free graph is real-rooted (see Chapter 6).

Definition 2.2.15. We say that $p_1, ..., p_m \in \mathbb{R}[x]$ are *compatible* if all convex combinations are real rooted.

Theorem 2.2.16 ([15])**.** *Let $p_1, ..., p_m \in \mathbb{R}[x]$ be univariate polynomials with positive leading coefficients. The following are equivalent.*

1. *p_i and p_j are compatible for all $i \neq j$.*

2. *p_i and p_j have a common interlacer for all $i \neq j$.*

3. *$p_1, ..., p_m$ are compatible.*

4. *$p_1, ..., p_m$ have a common interlacer.*

These last few results lead us to a sort of useful interlacing calculus, which we describe a bit here. If $f \ll g$ and $f \ll h$, then $f \ll ag + bh$ for any $a, b \in \mathbb{R}_+$. A similar result holds if $g \ll f$ and $h \ll f$. Note also that $f \ll g$ iff $g \ll -f$, and that $af \ll bf$ for all $a, b \in \mathbb{R}$.

Borcea and Brändén further generalize most of this to multivariate polynomials. Multivariate interlacing can be defined in the obvious way (recall the equivalent definition of real stability in terms of linear restrictions, Proposition 2.2.2).

Definition 2.2.17. Let $p, q \in \mathbb{R}[x_1, ..., x_n]$ be real stable polynomials. We say that p, q are in *proper position* and write $q \ll p$ if for all $e \in \mathbb{R}_+^n$ and $x \in \mathbb{R}^n$ we have that $q(te+x) \ll p(te+x)$. We also sometimes say that q *interlaces* p. The notions of *common interlacer* and *compatible* can also be defined analogously to the univariate case.

Proposition 2.2.18 ([7],[63])**.** *Let $p, q \in \mathbb{R}[x_1, ..., x_n]$ be real stable polynomials. We have:*

1. *(Hermite-Biehler) $q \ll p$ iff $p + iq$ is stable.*

2. *(Hermite-Kakeya-Obreschkoff) $q \ll p$ or $p \ll q$ iff p, q span a real subspace of real stable polynomials.*

3. *p, q have a common interlacer iff every convex combination of p, q is real stable.*

Borcea and Brändén actually prove a few more equivalences, but we omit these here.

2.3 Stability Preservers

We saw in Proposition 2.2.4 a number of basic linear operations that preserve the property of being stable or real stable. We also saw how some of these basic operations have clean combinatorial interpretations, which makes stability an interesting property to study when it comes to polynomials associated to graphs and other combinatorial objects. If we could find stability-preserving operators which do not have such a clean interpretation, then perhaps we could use such operators to say more interesting and/or analytic things about our favorite objects of study.

The Borcea-Brändén Characterization

The Borcea-Brändén characterization gives a complete classification of linear operators which preserve stability and real stability. Their results break into two categories: operators which depend on the degree of the input polynomial, and operators which have no such dependence. The reason for this is that there are some linear operators which only preserve stability when restricted to input polynomials of some fixed bounded degree.

We state their results here, leaving out a degeneracy case in order to make the results more readable. One can also look at [7] and [8] to see the results in full. To state these results we also need the concept of the symbol of a linear operator, which we define now.

Definition 2.3.1. Let $T : \mathbb{C}^\gamma[x_1, ..., x_n] \to \mathbb{C}[x_1, ..., x_m]$ be a linear operator on polynomials of bounded degree. Then the *(Borcea-Brändén) symbol* of T is defined as

$$\mathrm{Symb}(T)(x, z) := T[(z + x)^\gamma] = \sum_{0 \le \mu \le \gamma} \binom{\gamma}{\mu} z^{\gamma - \mu} T(x^\mu)$$

where T only acts on the x variables. Similarly if $T : \mathbb{C}[x_1, ..., x_n] \to \mathbb{C}[x_1, ..., x_m]$ is a linear operator on polynomials of any degree, we write:

$$\mathrm{Symb}(T)(x, z) := T[\exp(-xz)] = \sum_{0 \le \mu} \frac{1}{\mu!}(-1)^\mu z^\mu T(x^\mu)$$

Remark 2.3.2. When discussing the notion of polynomial capacity in Chapter 5, we will need to use a different symbol in our computations. Because T only acts on the x variables in the above expressions, these different symbols can be expressed as a certain twist of the ones defined above. First, for bounded degree inputs:

$$\mathrm{Symb}_+(T)(x, z) := T[(1 + xz)^\gamma] = z^\gamma \, \mathrm{Symb}(T)(x, z^{-1})$$

And then, for unbounded degree inputs:

$$\mathrm{Symb}_+(T)(x, z) := T[\exp(xz)] = \mathrm{Symb}(T)(x, -z)$$

The use of the $+$ subscript is meant to suggest the connection of this symbol to a positive definite inner product on polynomials. We will discuss this further in Chapter 5.

We are now ready to state the BB characterization, with one caveat. Note that in the unbounded degree case, $\mathrm{Symb}(T)$ is defined to be an entire function rather than a polynomial. In order to handle this, we will need a definition of (real) stability for entrie functions. This is where the *Laguerre-Pólya class* comes in, and such fucntions have been classically studied in the univariate case.

Definition 2.3.3. A function f is said to be in the $\mathcal{LP}$ *(Laguerre-Pólya) class* in the variables $x_1, ..., x_n$, if f is the limit (uniformly on compact sets) of real stable polynomials in $\mathbb{R}[x_1, ..., x_n]$. If f is the limit of real stable polynomials in $\mathbb{R}_+[x_1, ..., x_n]$, then we say f is in the $\mathcal{LP}_+$ class. In these cases, we write $f \in \mathcal{LP}[x_1, ..., x_n]$ and $f \in \mathcal{LP}_+[x_1, ..., x_n]$ respectively. Similar definitions can also be given for a *bistable Laguerre-Pólya class* (see Definition 2.2.1).

There are interesting equivalent definitions for this class of entire functions (e.g., see [16]), but we omit them here. Additionally, we refer the reader to [7] for more information in the multivariate case. We now state the BB characterization.

Theorem 2.3.4 (Borcea-Brändén, [7])**.** *Excluding a degeneracy case, a linear operator T on polynomials preserves stability (i.e., $\mathcal{H}_+^n$-stability) iff $\mathrm{Symb}(T)$ is stable. As a note, all of the degenerate operators T are of rank at most 1.*

Theorem 2.3.5 (Borcea-Brändén, [7])**.** *Excluding a degeneracy case, a linear operator T on polynomials preserves real stability iff $\mathrm{Symb}(T)(x, z)$ is either real stable or real bistable. As a note, all of the degenerate operators T are of rank at most 2.*

One important example of such an operator is the *additive convolution*. This convolution was originally studied by Walsh in the univariate case, and also goes by the names Walsh convolution and finite free convolution (see [47] and [48] for recent developments). We will discuss this convolution in further detail throughout this paper (e.g., in Chapter 3), and so

we now demonstrate its stability-preservation properties. The multivariate convolution is defined as follows for $p, q \in \mathbb{R}^\gamma[x]$:

$$(p \boxplus^\gamma q)(x) := \frac{1}{\gamma!} \sum_{0 \leq \mu \leq \gamma} (\partial_x^\mu p)(x) \cdot (\partial_x^{\gamma - \mu} p)(0)$$

To see that this operator preserves stability, we first prove something more general. Let $T : \mathbb{R}^{(\gamma, \gamma)}[x, y] \to \mathbb{R}^\gamma[x]$ be a linear operator defined as follows on the monomial basis:

$$T(x^\mu y^\nu) := x^\mu \boxplus^\gamma x^\nu$$

That is, $T(p(x)q(y)) = (p \boxplus^\gamma q)(x)$. In this case $\text{Symb}(T)$ will be a polynomial in three sets of variables z, w, x, and can be computed as:

$$\text{Symb}(T)(x, z, y) = T((x + z)^\gamma (y + w)^\gamma) = (x + z + w)^\gamma$$

This is obviously real stable, and so the BB characterization applies. Therefore $p \boxplus^\gamma q$ is real stable whenever p, q are.

Interlacing Preserving Operators

Fix univariate $f \in \mathbb{R}^d[x]$ with d simple real roots, $r_1, ..., r_d$, and fix univariate $g \in \mathbb{R}^{d+1}[x]$. By partial fraction decomposition, we have:

$$\frac{g(x)}{f(x)} = (bx + a) + \sum_{k=1}^d \frac{c_{r_k}}{x - r_k}$$

Denoting $f_{r_k}(x) := \frac{f(x)}{x - r_k}$, this implies:

$$g(x) = (bx + a)f(x) + \sum_{k=1}^d c_{r_k} f_{r_k}(x)$$

If $g(r_k) = 0$, then $c_{r_k} = 0$. Otherwise we compute:

$$c_{\alpha_k} = \lim_{x \to \alpha_k} \frac{(x - \alpha_k)g(x)}{f(x)} = \left[\frac{f'(\alpha_k)}{g(\alpha_k)} \right]^{-1} = \left[\left(\frac{f}{g} \right)'(\alpha_k) \right]^{-1}$$

For monic f, we then further compute:

$$b = \lim_{x \to \infty} \frac{f(x)g'(x) - f'(x)g(x)}{f(x)^2} = \lim_{x \to \infty} \left(\frac{g}{f} \right)'(x)$$

This leads to the following classical result.

Proposition 2.3.6 (see [63]). *Fix $f \in \mathbb{R}^d[x]$ and $g \in \mathbb{R}^{d+1}[x]$. Suppose f is monic and has d simple real roots, $r_1, ..., r_d$, and suppose g is of degree at most $d + 1$. Consider the decomposition:*

$$g(x) = (bx + a)f(x) + \sum_{k=1}^{d} c_{r_k} f_{r_k}(x)$$

Then, $g \ll f$ iff $b \leq 0$ and $c_{r_k} \geq 0$ for all k, and $f \ll g$ iff $b \geq 0$ and $c_{r_k} \leq 0$ for all k.

Proof. ($\Rightarrow$). If $f \ll g$, then $\left(\frac{f}{g}\right)' \leq 0$ and $\left(\frac{g}{f}\right)' \geq 0$. This implies $c_{r_k} \leq 0$ for all k and $b \geq 0$. If $g \ll f$, then $f \ll -g$. The same argument implies $c_{r_k} \geq 0$ for all k and $b \leq 0$.

($\Leftarrow$). Supposing $b \geq 0$ and $c_k \leq 0$ for all k, we write:

$$R(x) := \frac{g(x)}{f(x)} = (bx + a) + \sum_{k=1}^{d} \frac{c_{r_k}}{x - r_k}$$

Note that this implies $R(r_k + \epsilon) < 0$ and $R(r_k - \epsilon) > 0$ for small enough $\epsilon > 0$ and for all k. This implies that g has at least one root between each pair of adjacent asymptotes of f. Note that this demonstrates interlacing up to a few missing roots of g.

To show $f \ll g$ we just need to prove that the remaining roots of g do not disrupt this interlacing property, and that the leading coefficient of g is the correct sign. Casework on the possible values of a, b then implies the result. Finally, the result for $b \leq 0$ and $c_k \geq 0$ for all k follows similarly. $\square$

There is actually another way to state this result, in terms of cones of polynomials. Let $\text{cone}(f_1, ..., f_m)$ denote the closure of the positive cone generated by the polynomials $f_1, ..., f_m$.

Corollary 2.3.7. *Let $f \in \mathbb{R}[x]$ be a monic polynomial with d distinct (not necessarily simple) roots, $r_1, ..., r_d$. Then:*

$$\{g \in \mathbb{R}[x] : g \ll f\} = \text{cone}(-xf, -f, f, f_{r_1}, ..., f_{r_d})$$

$$\{g \in \mathbb{R}[x] : f \ll g\} = \text{cone}(xf, f, -f, -f_{r_1}, ..., -f_{r_d})$$

Here we define $f_{r_k} := \frac{f(x)}{x - r_k}$ even when r_k is not a simple root of f.

Proof. First suppose that f has degree exactly d (all roots simple). If the roots of f, g interlace, then g is of degree at most $d + 1$. Since any such g can be written as a linear combination of $xf, f, f_{r_1}, ..., f_{r_d}$, the result follows from the previous proposition.

Now if f has a root r_k of multiplicity $m \geq 2$, then any polynomial g for which $f \ll g$ or $g \ll f$ must have a root at r_k of multiplicity at least $m - 1$. In this case we have:

$$\{g \in \mathbb{R}[x] : g \ll f\} = (x - r_k)^{m-1} \cdot \left\{h \in \mathbb{R}[x] : h \ll \frac{f(x)}{(x - r_k)^{m-1}}\right\}$$

Inducting on this idea then implies the result. $\square$

This immediately yields the following result concerning linear operators preserving certain interlacing relations. Notice that here we restrict to only considering polynomials g of degree at most d, where d is the degree of f.

Definition 2.3.8. Given a real linear operator $T : \mathbb{R}^d[x] \to \mathbb{R}[x]$, and a real-rooted polynomial f, we say that *T preserves interlacing with respect to f* if $g \ll f$ implies $T[g] \ll T[f]$ and $f \ll g$ implies $T[f] \ll T[g]$ for all $g \in \mathbb{R}^d[x]$

Corollary 2.3.9. *Fix a real linear operator $T : \mathbb{R}^d[x] \to \mathbb{R}[x]$, and fix $f \in \mathbb{R}^d[x]$. Suppose f is monic with d simple roots, $r_1, ..., r_n$, and that $T[f_{r_k}] \ll T[f]$ for all k. Then, T preserves interlacing with respect to f.*

This is a generalization of the notion of real-rootedness preserver, as some operators which preserve interlacing with respect to certain polynomials will not preserve real-rootedness (see Chapter 4 for some interesting examples). On the other hand, every linear operator which preserves real-rootedness will also have certain interlacing-preservations properties by Hermite-Kakeya-Obreschkoff (Proposition 2.2.13).

Chapter 3

Classical Additive Convolution

The Walsh additive ([64]) and Grace-Szegö multiplicative ([62]) polynomial convolutions on $f, g \in \mathbb{C}^d[x]$ have been denoted $\boxplus^d$ and $\boxtimes^d$ respectively (e.g., in [47]):

$$f \boxplus^d g := \frac{1}{d!} \sum_{k=0}^{d} \partial_x^k f \cdot (\partial_x^{d-k} g)(0)$$

$$f \boxtimes^d g := \sum_{k=0}^{d} \binom{d}{k}^{-1} (-1)^k f_k g_k x^k$$

These convolutions can be thought of producing polynomials whose roots are contained in the (Minkowski) sum and product of complex discs containing the roots of the input polynomials. When the inputs have real roots (additive) or non-negative roots (multiplicative), this fact also holds in terms of real intervals containing the roots.

Recent interest in understanding how certain differential operators preserving the set of real-rooted polynomials affect the roots of the input polynomial derives from the notion of interlacing families (see the papers of Marcus-Spielman-Srivastava on Ramanujan graphs and the Kadison-Singer problem: [52],[49],[51]). Most uses of interlacing families share the same loose goal: to study spectral properties of random combinatorial objects. To do this, one equates random combinatorial operations on the objects to differential operators on associated characteristic polynomials. Then, understanding the spectrum of the random objects is reduced to understanding how the roots of certain polynomials are affected by differential operators preserving real-rootedness.

The most robust way to study the effects of a differential operator on roots comes from the framework of Marcus, Spielman, and Srivastava. It gives tight largest root bounds for a general class of differential operator preserving real-rootedness, replacing many of the ad hoc methods used before to study specific desired operators. That said, some combinatorial objects require the use of multivariate methods to analyze. Here the associated polynomials are real stable, and for these methods there is no general framework in place to study the analytic effects of the linear operators on the roots. We consider the following to be one of the

big open conceptual questions in this area: How does the multivariate additive convolution affect root information?

In an attempt to better understand the multivariate case, we expand upon the previous results of Marcus-Spielman-Srivastava and provide more general results about how *all* roots are of a given polynomial are affected by finite free convolutions. To do this, we first expand their bound on the movement of the largest root to all differential operators preserving real-rootedness. Further, we utilize the theory of hyperbolic polynomials to give more interesting root bounds on interior roots (other roots besides the largest).

We then state a number of conjectures (and some counterexamples) in the direction of stronger univariate results on interior roots and of analogous multivariate results. Proving similar multivariate results seems to be a hard problem in general. But the hope is that by better fleshing out the details of the additive convolution in the univariate case, one can better abstract to the multivariate case to handle problems such as Kadison-Singer, the Paving conjecture, and Heilman-Lieb root bounds.

3.1 The Univariate Additive Convolution

Recall the definition of the additive convolution for $p, q \in \mathbb{R}^d[t]$:

$$p \boxplus^d q = \frac{1}{d!} \sum_{k=0}^{d} \partial_t^k p(t) \partial_t^{d-k} q(0)$$

Notice we get a differential operator if we fix a polynomial q and view the additive convolution as a linear operator $p \mapsto p \boxplus^d q$, and we can obtain all constant coefficient differential operators in this fashion. Some well known properties of the additive convolution are given as follows. Recall that we let $\lambda(p)$ denote the non-increasing vector of roots of p counting multiplicities.

Proposition 3.1.1. *For $p, q \in \mathbb{R}^d[t]$, we have the following:*

1. *(Symmetry) $p \boxplus^d q = q \boxplus^d p$*

2. *(Shift-invariance) $(p(t+a) \boxplus^d q)(t) = (p \boxplus^d q)(t+a) = (p \boxplus^d q(t+a))(t)$ for $a \in \mathbb{R}$*

3. *(Scale-invariance) $(p(at) \boxplus^d q(at)) = a^d \cdot (p \boxplus^d q)(at)$ for $a \in \mathbb{R}$*

4. *(Derivative-invariance) $(\partial_t p) \boxplus^d q = \partial_t(p \boxplus^d q) = p \boxplus^d (\partial_t q)$ for all $k \in [d]$*

5. *(Stability-preserving) $p \boxplus^d q$ is real rooted*

6. *(Triangle inequality) $\lambda_1(p \boxplus^d q) \leq \lambda_1(p) + \lambda_1(q)$*

Finally, the additive convolution can be used to characterize differential operators which preserve real-rootedness.

Proposition 3.1.2. *A linear operator $T : \mathbb{R}^d[t] \to \mathbb{R}^d[t]$ is a differential operator which preserves real-rootedness if and only if it can be written in the form $T(p) = p \boxplus^d q$ for some real-rooted $q \in \mathbb{R}^d[t]$.*

We now discuss a number of stronger properties one can achieve for the additive convolution, using hyperbolicity. Then in §3.2, we state and discuss the main result of this chapter: a generalization of the main result from [47] with a simplified and more intuitive proof.

Interior Roots

The triangle inequality above gives the most basic bound on the largest root of the convolution to two polynomials. The first main collection of interior root bounds can be stated in terms of *majorization*. The majorization order is a partial order on vectors in $\mathbb{R}^d$ which can be thought of morally as saying that the coordinates of one vector are more spread out than the coordinates of the other. Formally, majorization is defined as follows. We refer the reader to [53] for more discussion on the following equivalent definitions.

Definition 3.1.3. Given $x, y \in \mathbb{R}^d$, we say that x *majorizes* y and write $y \prec x$ if one of the following equivalent conditions holds. We let $x^{\downarrow} = (x_1^{\downarrow}, ..., x_d^{\downarrow})$ denote the ordering of the entries x in non-increasing order.

1. $\sum_{i=1}^{k} y_i^{\downarrow} \leq \sum_{i=1}^{k} x_i^{\downarrow}$ for all k, with equality for $k = d$.

2. y is contained in the convex hull of $\{(x_{\sigma(1)}, ..., x_{\sigma(d)}) \mid \sigma \in S_d\} \subset \mathbb{R}^d$.

3. There exists a doubly stochastic matrix D (each row and column sum is 1) such that $Dx = y$.

4. There is a sequence of pinches, of the form $x \mapsto (x_1, ..., x_j + \alpha, ..., x_k - \alpha, ..., x_d)$ such that the jth and k^{th} coordinates are getting closer together (without crossing), which takes x to y.

This makes $\prec$ a partial order on $\mathbb{R}^d$ for all d.

Note that condition (1) applied to the vectors of roots of two polynomials can be interpreted as root bounds involving interior roots. What we need then is some way to prove majorization results about the additive convolution. One way to do this is via hyperbolic polynomials, which enables us to convert inequalities regarding matrix eigenvalues into inequalities regarding roots of polynomials.

Definition 3.1.4. Given a homogeneous polynomial $p \in \mathbb{R}[x_1, ..., x_n]$ and a vector $e \in \mathbb{R}^n$, we say that p is *hyperbolic with respect to e* if $p(e) > 0$ and $p(et + x) \in \mathbb{R}[t]$ is real-rooted for all $x \in \mathbb{R}^n$. Whenever p and e are assumed, we let $\lambda(x)$ to denote the vector of roots of the polynomial $p(et + x)$ in nonincreasing order.

Hyperbolic polynomials have been heavily studied over the past few decades, starting with [26]. There are a number of standard results regarding certain convexity properties of such polynomials, but we omit these here (a good reference is [56]).

The intuition that one should have when considering hyperbolic polynomials and $\lambda(x)$ is that of the determinant of a matrix and its eigenvalues. This is formalized in the fact that $\det(X)$ (where X is a symmetric matrix of variables) is hyperbolic with respect to the identity matrix, and in this case $\lambda(X)$ is the vector of eigenvalues of X. This intuition is further justified by the fact that many properties of the eigenvalues of real symmetric matrices seamlessly transfer over to properties about $\lambda(x)$ for any hyperbolic polynomial p. In fact, by exploiting the Helton-Vinnikov theorem (which says that all 3-variable hyperbolic polynomials are determinants involving real symmetric matrices; see [37], [37]) one can obtain *all* of Horn's inequalities (see [38]) for any hyperbolic polynomial. That is, any inequality that holds between $\lambda(X)$, $\lambda(Y)$, and $\lambda(X+Y)$ for any X, Y real symmetric with $p(X) = \det(X)$ (see Definition 3.2.3) will also hold for any hyperbolic $p(x)$ and any vectors x, y. We state this formally as follows.

Theorem 3.1.5 ([4], [31]). *Fix a hyperbolic polynomial p with respect to e. For $v, w \in \mathbb{R}^n$, Horn's inequalities hold for $\lambda(v+w)$, $\lambda(v)$, and $\lambda(w)$. In particular, the following majorization relation holds:*

$$\lambda(v+w) \prec \lambda(v) + \lambda(w)$$

To apply this result, we need to view the additive convolution as a hyperbolic polynomial. We do this in the following.

Proposition 3.1.6. *Consider the following, where $\boxplus$ only acts on the x variables.*

$$p(x, a_1, ..., a_d, b_1, ..., b_d) := \left(\prod_{k=1}^{d} (x - a_k) \right) \boxplus^d \left(\prod_{k=1}^{d} (x - b_k) \right)$$

Then p is hyperbolic with respect to $e = (1, 0, ..., 0)$.

Proof. For $q(t) := \prod_k (t - a_k)$, $r(t) := \prod_k (t - b_k)$, and $y = (c, a_1, ..., a_d, b_1, ..., b_d)$ we compute:

$$p(et + y) = p(t + c, a_1, ..., a_d, b_1, ..., b_d) = (q \boxplus^d r)(t + 2c)$$

So $p(et + y)$ is real-rooted since q and r are and $\boxplus$ preserves real-rootedness. Also, $p(e) = 1 > 0$. $\square$

This fact allows us to immediately apply the previous theorem to the additive convolution as follows.

Corollary 3.1.7. *Let $p, q \in \mathbb{R}^d[x]$ be real-rooted and of degree exactly d. Then:*

$$\lambda(p \boxplus^d q) \prec \lambda(p) + \lambda(q)$$

Proof. Let $v = (0, a_1, ..., a_d, 0, ..., 0)$ and $w = (0, 0, ..., 0, b_1, ..., b_d)$, where the a_k and b_k are the roots of p and q, respectively. Then $\lambda(v) = \lambda(p)$ and $\lambda(w) = \lambda(q)$. Further, $\lambda(v + w) = \lambda(p \boxplus^d q)$. The result then follows from the previous theorem. □

Note that this immediately gives us interior root inequalities of the following form:

$$\sum_{i=1}^{k} \lambda_i(p \boxplus^d q) \leq \sum_{i=1}^{k} \lambda_i(p) + \sum_{i=1}^{k} \lambda_i(q)$$

Using the same proof as in the corollary, we can similarly obtain all of Horn's inequalities for the roots of p, q, and $p \boxplus^d q$. For instance, we obtain the Weyl inequalities (for all i, j) which more directly bound the interior roots:

$$\lambda_{i+j-1}(p \boxplus^d q) \leq \lambda_i(p) + \lambda_j(q)$$

Whenever $i = j = 1$, this boils down to the triangle inequality:

$$\lambda_1(p \boxplus^d q) \leq \lambda_1(p) + \lambda_1(q)$$

As it turns out, Theorem 3.1.5 above also yields an important majorization preservation result regarding the additive convolution. In [6], Borcea and Brändén give a complete characterization of linear operators which preserve majorization of roots. Roughly speaking, the result says that a linear operator T (with certain degree restrictions) which preserves real-rootedness has the following property:

$$\lambda(p) \prec \lambda(q) \implies \lambda(T(p)) \prec \lambda(T(q))$$

Their result then applies to the operator $T_q(p) := p \boxplus^d q$ for any fixed real-rooted q. This result also has a nice proof via hyperbolicity, and we demonstrate this now. As a note, the following proof immediately generalizes to any degree-preserving linear operator preserving real-rootedness. It is likely that one could generalize it further to the full Borcea-Brändén result, using some of the results regarding polynomial degree from [6].

Corollary 3.1.8. *Let $p, q, r \in \mathbb{R}^d[x]$ be real-rooted polynomials of degree exactly d such that $\lambda(p) \prec \lambda(q)$. Then:*

$$\lambda(p \boxplus^d r) \prec \lambda(q \boxplus^d r)$$

Proof. Let a_k, b_k, and c_k be the roots of p, q, and r, respectively. By Definition 3.1.3 and the fact that $\lambda(p) \prec \lambda(q)$, we have that (a_k) is in the convex hull of the permutations of (b_k). That is,

$$(a_1, ..., a_d) = \sum_{\sigma \in S_d} \beta_\sigma \cdot (b_{\sigma(1)}, ..., b_{\sigma(d)})$$

where $\beta_\sigma \geq 0$ and $\sum_\sigma \beta_\sigma = 1$. With this, we use the following notation:

$$v := (0, a_1, ..., a_d, c_1, ..., c_d) \qquad w_\sigma := (0, b_{\sigma(1)}, ..., b_{\sigma(d)}, c_1, ..., c_d)$$

And so we also have that $v = \sum_{\sigma \in S_d} \beta_\sigma w_\sigma$.

Since $\prec$ is a partial order, we can induct on the majorization relation of Theorem 3.1.5, using the hyperbolic polynomial from Proposition 3.1.6:

$$\lambda(v) = \lambda \left(\sum_{\sigma \in S_d} \beta_\sigma w_\sigma \right) \prec \sum_{\sigma \in S_d} \lambda(\beta_\sigma w_\sigma)$$

By the scale-invariance property of $\boxplus$ (see Proposition 3.1.1), we have that $\lambda(\beta_\sigma w_\sigma) = \beta_\sigma \cdot \lambda(w_\sigma)$. This implies:

$$\lambda(p \boxplus^d r) = \lambda(v) \prec \sum_{\sigma \in S_d} \beta_\sigma \cdot \lambda(w_\sigma) = \sum_{\sigma \in S_d} \beta_\sigma \cdot \lambda(q \boxplus^d r) = \lambda(q \boxplus^d r)$$

$\square$

3.2 Submodularity of the Largest Root

In [47], the authors consider the effects of a certain class of differential operators on the largest root of a given real-rooted polynomial:

$$U_\alpha := 1 - \alpha \partial_t$$

This differential operator is inspired by the Cauchy transform, via the following equivalence:

$$U_\alpha p(t) = 0 \iff p(t) - \alpha p'(t) = 0 \iff \frac{p'(t)}{p(t)} = \frac{1}{\alpha} =: \omega$$

Restricting to points larger than the largest root of p, we have that $\frac{p'}{p}$ is a bijection between $(\lambda_1(p), \infty)$ and $(0, \infty)$. Let $\mathcal{K}_\omega(p)$ denote the inverse of ω. Note that as $\omega \to 0$ our inverse tends to infinity, while as $\omega \to \infty$ our inverse tends to $\lambda_1(p)$. Furthermore, $\lambda_1(U_\alpha p) = \mathcal{K}_\omega(p)$. This definition is inspired by similar objects from free probability, as discussed in [47]. The main result from [47] regarding these U_α is given as follows:

Theorem 3.2.1 ([47]). *Let $p, q \in \mathbb{R}^d[t]$ be real-rooted polynomials of degree d. For any $\alpha > 0$ we have:*

$$\lambda_1(U_\alpha(p \boxplus^d q)) + d\alpha \leq \lambda_1(U_\alpha(p)) + \lambda_1(U_\alpha(q))$$

As discussed above, every differential operator on polynomials in $\mathbb{R}^d[t]$ can be represented as $T(p) = p \boxplus^d q$ for some polynomial $q \in \mathbb{R}^d[t]$. In particular we can represent U_α via

$$U_\alpha(p) = p \boxplus^d u_\alpha$$

where $u_\alpha(t) := t^d - d\alpha \cdot t^{d-1}$. Notice that here we have $\lambda_1(u_\alpha) = d\alpha$, which means that the above result can be restated as follows:

$$\lambda_1(p \boxplus^d q \boxplus^d u_\alpha) + \lambda_1(u_\alpha) \leq \lambda_1(p \boxplus^d u_\alpha) + \lambda_1(q \boxplus^d u_\alpha)$$

This is a submodularity relation for the additive convolution. Further, by rearranging this result, it can also be seen as a diminishing returns property of the convolution:

$$\lambda_1(p \boxplus^d q \boxplus^d u_\alpha) - \lambda_1(q \boxplus^d u_\alpha) \leq \lambda_1(p \boxplus^d u_\alpha) - \lambda_1(u_\alpha)$$

The operation $p \mapsto p \boxplus^d q$ can be interpreted as spreading out the roots of p (see the discussion at the beginning of §3.1). The above expression then says that, as the roots of a polynomial become more spread out, the operation of convolving by p has less of an effect on the largest root.

The natural next question is: can u_α be replaced by a larger class of real-rooted polynomials in the above expression? The answer is encapsulated in our main result, which says that it can be replaced by any real-rooted polynomial.

Theorem 3.2.2. *Let $p, q, r \in \mathbb{R}^d[t]$ be real-rooted polynomials of degree d. We have:*

$$\lambda_1(p \boxplus^d q \boxplus^d r) + \lambda_1(r) \leq \lambda_1(p \boxplus^d r) + \lambda_1(q \boxplus^d r)$$

To prove this, we adapt and simplify the proof of the original MSS result above. We leave this proof to §3.2, where we actually prove slightly more general results.

It is important to note that we were unable to prove this result using the hyperbolicity properties of the additive convolution. This should not be surprising, as morally anything provable for hyperbolic polynomials should come from properties of the eigenvalues of a Hermitian matrix (and this submodularity relation does not hold for matrices in general). We discuss this further in the next section.

Submodularity Conjectures

Our main result gives an inequality relating the largest (or smallest) roots of additive convolutions of three polynomials, as is done in the MSS paper. The root bound achieved by MSS is crucial to their proof of the paving conjecture, but it is not strong enough to obtain optimal bounds for the paving conjecture. That said, it is believed that root bounds for the interior roots will help to obtain optimal paving bounds. More generally, such root bounds would further clarify how differential operators affect the roots of polynomials.

As we saw in §3.2, their result can be extended to 3 polynomials in the form of Theorem 3.2.1:

$$\lambda_1(p \boxplus^d q \boxplus^d r) + \lambda_1(r) \leq \lambda_1(p \boxplus^d r) + \lambda_1(q \boxplus^d r)$$

A natural next question becomes: what other inequalities on roots can we achieve in the 3 polynomial case. With this, we arrive at our submodularity conjectures. To simplify the notation, we first make the following definitions.

Definition 3.2.3. Fix $d \in \mathbb{N}$ and let $I, J, K \subset [d]$. We call (I, J, K) a *Horn's triple* if for all Hermitian $d \times d$ matrices A, B we have:

$$\sum_{i \in I} \lambda_i(A + B) \leq \sum_{j \in J} \lambda_j(A) + \sum_{k \in K} \lambda_k(B)$$

That is, if (I, J, K) give rise to one of Horn's inequalties.

Definition 3.2.4. Fix $d \in \mathbb{N}$ and let $I, L, J, K \subset [d]$. We call (I, L, J, K) a *valid 4-tuple* if for all real-rooted p, q, r of degree d we have:

$$\sum_{i \in I} \lambda_i(p \boxplus^d q \boxplus^d r) + \sum_{l \in L} \lambda_l(r) \leq \sum_{j \in J} \lambda_j(p \boxplus^d r) + \sum_{k \in K} \lambda_k(q \boxplus^d r)$$

We want to determine all of the valid 4-tuples. It is worth noting that the method of hyperbolic polynomials (which worked for inequalities relating the roots of 2 polynomials) does not work for determining valid 4-tuples. In fact we have the following, even for diagonal matrices:

$$\lambda_1(A + B + C) + \lambda_1(C) \not\leq \lambda_1(A + C) + \lambda_1(B + C)$$

For example, let $A = B = \mathrm{diag}(2, 0)$ and $C = \mathrm{diag}(0, 2)$.

With these notions in hand, we can now succinctly state our conjectures. The first is a natural generalization of Horn's inequalities for two polynomials.

Conjecture 3.2.5. *Let $p, q, r \in \mathbb{R}^d[x]$ be real-rooted and of degree exactly d, and let (I, J, K) be a Horn's triple. Then (I, I, J, K) is a valid 4-tuple.*

Note that the indices of the left-hand side of the inequality are the same for both polynomials. But perhaps this does not have to be the case here? That is, can we pick $L \neq I$ such that the inequality for (I, L, J, K) is stronger than the inequality for (I, I, J, K), and yet it is still a valid 4-tuple? This turns out to be a difficult question in general.

Of course you can make the set L a "weaker" set of indices than I (meaning that the inequality for (I, L, J, K) is logically weaker than the inequality for (I, I, J, K)) to get a new valid 4-tuple. Since such inequalities follow from the conjecture given above, we will ignore these 4-tuples. That said, the only question left is just how much "stronger" the set L can be. We give yet another conjecture regarding this question, albeit only in the case where $|I| = |L| = |J| = |K| = 1$. To ease notation, we say that (i, j, k) and (i, l, j, k) are a Horn's triple and a valid 4-tuple, respectively (replace singleton sets with the single index).

Conjecture 3.2.6. *Let $p, q, r \in \mathbb{R}^d[x]$ be real-rooted and of degree exactly d, and let (i, j, k) be a Horn's triple. Note that this is equivalent to $i \geq j + k - 1$ (see the Weyl inequalities above, which are strongest Horn's triples of this form). Then $(i, \max(j, k), j, k)$ and $(i, d + 1 - \max(j, k), j, k)$ are valid 4-tuples.*

Notice that for small j, k the first 4-tuple given in the above conjecture is stronger, and for large j, k the second 4-tuple given in above conjecture is stronger.

Submodularity Proof

We now set out to prove Theorem 3.2.1. First we need a basic lemma, which we will use throughout this section. It essentially follows from Hermite-Kakeya-Obreschkoff (Proposition

2.2.13), with a little bit of care about the direction of interlacing. One might also be able to use Hermite-Biehler (Proposition 2.2.12).

Lemma 3.2.7. *Fix real-rooted $p, q, r \in \mathbb{R}^n[x]$ with positive leading coefficients. If $q \ll p$, then:*

$$q \boxplus^n r \ll p \boxplus^n r$$

Now we introduce the notation used in [47]. Given a monic polynomial p of degree d with at least 2 distinct roots, we write:

$$p(x) = \prod_{i=1}^{d}(x - \lambda_i)$$

Order the roots $\lambda_1 \geq \cdots \geq \lambda_d$, and let k be minimal such that $\lambda_1 \neq \lambda_k$. Define $\mu_0 := \frac{\lambda_1 + \lambda_k}{2}$ and $\mu_1 := \lambda_1$. Further, for $\mu \in [\mu_0, \mu_1]$ we define:

$$\tilde{p}_\mu(x) := (x - \mu)^2 \prod_{i \neq 1, k}(x - \lambda_i)$$

We then define:

$$\hat{p}_\mu(x) := p(x) - \tilde{p}_\mu(x) = ((2\mu - (\lambda_1 + \lambda_k))x - (\mu^2 - \lambda_1\lambda_k)) \prod_{i \neq 1, k}(x - \lambda_i)$$

For $\mu > \mu_0$, we have that $\hat{p}_\mu$ is of degree $d - 1$ with positive leading coefficient and the extra root is at least λ_1. (Note that when $\mu = \mu_0$, we have that $\hat{p}_\mu$ is of degree $d - 2$ with negative leading coefficient.) To see this, notice:

$$\rho := \frac{\mu^2 - \lambda_1\lambda_k}{2\mu - (\lambda_1 + \lambda_k)} \geq \lambda_1 \iff \mu^2 - 2\mu\lambda_1 + \lambda_1^2 = (\mu - \lambda_1)^2 \geq 0$$

This then implies that for $f_\mu(x) := (x - \mu)\prod_{i \neq 1, k}(x - \lambda_i)$, we have $f_\mu \ll \tilde{p}_\mu$, $f_\mu \ll \hat{p}_\mu$, and $f_\mu \ll p$. In Figure **??**, we illustrate one possibility for the largest roots of these polynomials.

In what follows, we additionally fix a real-rooted $r \in \mathbb{R}[x]$ of degree d.

Lemma 3.2.8. *Fix any μ, μ' with $\mu_0 \leq \mu \leq \mu' \leq \mu_1$ where μ_0, μ_1 are defined as above. We have:*

$$\lambda_1(\tilde{p}_{\mu_0} \boxplus^d r) \leq \lambda_1(p \boxplus^d r) \leq \lambda_1(\tilde{p}_{\mu_1} \boxplus^d r)$$

$$\lambda_1(\tilde{p}_\mu \boxplus^d r) \leq \lambda_1(\tilde{p}_{\mu'} \boxplus^d r)$$

Proof. The first inequality of the first line follows from the fact that the roots of $\tilde{p}_{\mu_0}$ are majorized by that of p (this is because $\tilde{p}_{\mu_0}$ can obtained via a "pinch" of the roots of p; see property 4 of Definition 3.1.3). The second inequality of the first line follows from Lemma 3.2.7 and the fact that $p \ll \tilde{p}_{\mu_1}$. The second line follows from Lemma 3.2.7 and the fact that $\tilde{p}_\mu \ll g \ll \tilde{p}_{\mu'}$ for $g(x) := (x - \mu)(x - \mu')\prod_{i \neq 1, k}(x - \lambda_i)$. $\square$
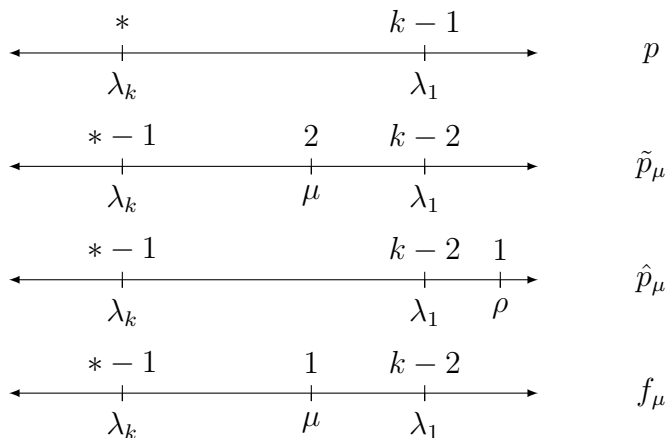
Figure 3.1: Illustration of larger roots of pinched polynomials, with multiplicities.

Corollary 3.2.9. *There exists $\mu \in [\mu_0, \mu_1]$ such that $\lambda_1(\tilde{p}_\mu \boxplus^d r) = \lambda_1(p \boxplus^d r)$.*

Proof. The above lemma and continuity. $\square$

Now, let μ_* denote the maximal $\mu \in [\mu_0, \mu_1]$ such that the previous corollary holds. For simplicity, we will denote $\tilde{p} := \tilde{p}_{\mu_*}$ and $\hat{p} := \hat{p}_{\mu_*}$.

Proposition 3.2.10. *For μ_* defined as above, we have that $\mu_* > \mu_0$ and:*

$$\lambda_1(\hat{p} \boxplus^d r) = \lambda_1(p \boxplus^d r) = \lambda_1(\tilde{p} \boxplus^d r)$$

Proof. The second equality follows from the definition of μ_*. So we only need to prove the first equality. By linearity $\hat{p} \boxplus^d r$ has a root at $\lambda_1(p \boxplus^d r)$, and so $\lambda_1(\hat{p} \boxplus^d r) \geq \lambda_1(p \boxplus^d r)$. So in fact we only need to show that $\lambda_1(\hat{p} \boxplus^d r) \leq \lambda_1(p \boxplus^d r)$.

If $\mu_* = \mu_1$, then $\lambda_1(\hat{p}) = \lambda_1(p)$ and $\hat{p} \ll p$. This implies $\lambda_1(\hat{p} \boxplus^d r) \leq \lambda_1(p \boxplus^d r)$. Otherwise $\mu_0 \leq \mu_* < \mu_1$. Then for $\mu > \mu_*$, we have $\lambda_1(\tilde{p}_\mu \boxplus^d r) > \lambda_1(p \boxplus^d r)$ by Lemma 3.2.8 which implies $\hat{p}_\mu \boxplus^d r > 0$ at $\lambda_1(\tilde{p}_\mu \boxplus^d r)$. Recalling the definition of f_μ above, $f_\mu \ll \tilde{p}_\mu$ implies $\lambda_1(f_\mu \boxplus^d r) \leq \lambda_1(\tilde{p}_\mu \boxplus^d r)$, and $f_\mu \ll \hat{p}_\mu$ implies $\hat{p}_\mu \boxplus^d r$ has at most one root greater than $\lambda_1(f_\mu \boxplus^d r)$. Combining all this with the fact that $\hat{p}_\mu \boxplus^d r$ has positive leading coefficient gives $\lambda_1(\hat{p}_\mu \boxplus^d r) < \lambda_1(\tilde{p}_\mu \boxplus^d r)$. Limiting $\mu \to \mu_*$ from above then implies $\lambda_1(\hat{p} \boxplus^d r) \leq \lambda_1(\tilde{p} \boxplus^d r) = \lambda_1(p \boxplus^d r)$.

Now suppose that $\mu_* = \mu_0$, so as to get a contradiction. As $\mu \to \mu_*$ from above, $\hat{p}_\mu \boxplus^d r$ has positive leading coefficient limiting to zero. So $\hat{p} \boxplus^d r$ then has one less root, and has negative leading coefficient as discussed above. However, since $\lambda_1(\hat{p}_\mu \boxplus^d r) < \lambda_1(\tilde{p}_\mu \boxplus^d r) \leq \lambda_1(\tilde{p}_{\mu_1} \boxplus^d r)$ for all $\mu > \mu_*$ (as noted earlier in this proof), $\hat{p}_\mu \boxplus^d r$ must have a root limiting to $-\infty$ as $\mu \to \mu_*$. Therefore the second-from-leading coefficient of $\hat{p}_\mu \boxplus^d r$ (the sum of negated roots scaled by the leading coefficient) is eventually non-negative as $\mu \to \mu_*$. This contradicts the fact that $\hat{p} \boxplus^d r$ has negative leading coefficient. (Note that this crucially uses the fact that μ_* is maximal.) $\square$

The next lemma provides the base case to a more streamlined induction for the proof. In fact, it may even lead to a proof of some sort of majorization relation.

Definition 3.2.11. For real-rooted $p \in \mathbb{R}^d[x]$ not necessarily of degree d, let $\lambda^d(p) \in \mathbb{R}^d$ be the list of roots of p, padded with the mean of the roots, and then ordered in non-increasing order.

Lemma 3.2.12. *Fix real-rooted $p, q, r \in \mathbb{R}^d[x]$ such that $\deg(q) = \deg(r) = d$ and $\deg(p) = 1$. Then:*

$$\lambda^d(p \boxplus^d q \boxplus^d r) + \lambda^d(r) \prec \lambda^d(p \boxplus^d r) + \lambda^d(q \boxplus^d r)$$

Proof. By shifting, we may assume WLOG that p, q, r all have have roots which sum to 0. Since $\deg(p) = 1$, the result is then equivalent to the following:

$$\lambda^d(\partial_x^{d-1}(q \boxplus^d r)) + \lambda^d(r) \prec \lambda^d(\partial_x^{d-1} r) + \lambda^d(q \boxplus^d r)$$

Since $\boxplus$ preserves the set of polynomials whose roots sum to 0, this is equivalent to:

$$\lambda^d(r) \prec \lambda^d(q \boxplus^d r)$$

Since $r = x^d \boxplus^d r$ and $\lambda(x^d) \prec \lambda(q)$, the result follows from Corollary 3.1.8. $\square$

The following is an immediate corollary of the previous lemma.

Corollary 3.2.13. *Fix real-rooted $p, q, r \in \mathbb{R}^d[x]$ such that $\deg(q) = \deg(r) = d$ and $\deg(p) = 1$. Then:*

$$\lambda_1(p \boxplus^d q \boxplus^d r) + \lambda_1(r) \leq \lambda_1(p \boxplus^d r) + \lambda_1(q \boxplus^d r)$$

We now prove the main result.

Theorem 3.2.14. *Fix real-rooted $p, q, r \in \mathbb{R}^d[x]$ such that $\deg(q) = \deg(r) = d$ and $\deg(p) = k \leq d$. Then:*

$$\lambda_1(p \boxplus^d q \boxplus^d r) + \lambda_1(r) \leq \lambda_1(p \boxplus^d r) + \lambda_1(q \boxplus^d r)$$

Proof. We induct on k, using the previous corollary as the base case. Let p be a polynomial of degree k with roots in $[-R, R]$ (for any fixed R) which maximizes (by compactness):

$$\beta(p) := \lambda_1(p \boxplus^d q \boxplus^d r) + \lambda_1(r) - \lambda_1(p \boxplus^d r) - \lambda_1(q \boxplus^d r)$$

To get a contradiction, we assume $\beta(p) > 0$. In particular this implies p has at least 2 distinct roots, allowing us to apply the above discussion, notation, and results. By induction we have $\beta(\hat{p}) \leq 0$, which implies:

$$\begin{aligned}
\lambda_1(\hat{p} \boxplus^d q \boxplus^d r) &\leq \lambda_1(\hat{p} \boxplus^d r) + \lambda_1(q \boxplus^d r) - \lambda_1(r) \\
&= \lambda_1(p \boxplus^d r) + \lambda_1(q \boxplus^d r) - \lambda_1(r) \\
&= \lambda_1(p \boxplus^d q \boxplus^d r) - \beta(p)
\end{aligned}$$

Since $\mu_* > \mu_0$ by the previous proposition, $\hat{p}$ has positive leading coefficient. This implies $\tilde{p} \boxplus^d q \boxplus^d r = (p - \hat{p}) \boxplus^d q \boxplus^d r < 0$ when evaluated at $\lambda_1(p \boxplus^d q \boxplus^d r)$. Since $\tilde{p}$ has positive leading coefficient, this gives:

$$\beta(\tilde{p}) - \beta(p) = \lambda_1(\tilde{p} \boxplus^d q \boxplus^d r) - \lambda_1(p \boxplus^d q \boxplus^d r) > 0$$

This contradicts the maximality of $\beta(p)$, since all of the roots of $\tilde{p}$ are contained in $[-R, R]$. $\square$

Corollary 3.2.15. *Fix real-rooted $p, q, r \in \mathbb{R}^d[x]$. If all polynomials involved are of degree at least 1, then:*
$$\lambda_1(p \boxplus^d q \boxplus^d r) + \lambda_1(r) \leq \lambda_1(p \boxplus^d r) + \lambda_1(q \boxplus^d r)$$

Note that the following condition is equivalent to the degree restriction:

$$2d < \deg(p) + \deg(q) + \deg(r) \iff (d - \deg(p)) + (d - \deg(q)) + (d - \deg(r)) < d$$

Proof. Consider polynomials of degree d whose roots limit to the roots of p, q, r and extra roots limit to $-\infty$. The previous theorem and continuity (and use of Lemma 3.2.7 to bound the largest roots away from $+\infty$) then imply the result. $\square$

3.3 The Multivariate Additive Convolution

All of the root bounds and conjectures discussed in this chapter thus far have been for univariate polynomials. However, in their resolution of Kadison-Singer, Marcus-Spielman-Srivastava give bounds on how the *points above the roots* of a given multivariate polynomial change under the action of differential operators. This prompts an obvious question: are there multivariate generalizations of the root bounds discussed in this chapter?

To attempt to answer this, we will give the natural multivariate generalization of the additive convolution, along with some basic analogous results. Recall the definition of $\mathrm{Ab}(p)$, the points above the roots of p (Definition 2.1.1), as well as the following classic fact coming from the theory of hyperbolic polynomials.

Proposition 3.3.1. *Let $p \in \mathbb{R}[x_1, ..., x_n]$ be real stable. Then $\mathrm{Ab}(p)$ is convex and is the closure of a connected component of the non-vanishing set of p.*

Now the multivariate convolution and its basic properties.

Definition 3.3.2. *For $p, q \in \mathbb{R}^\gamma[x_1, ..., x_n]$ we define the bilinear function:*

$$(p \boxplus^\gamma q)(x) := \sum_{0 \leq \mu \leq \gamma} \partial_x^\mu p(x) \cdot \partial_x^{\gamma - \mu} q(0)$$

Proposition 3.3.3. *Let $p, q \in \mathbb{R}^\gamma[x_1, ..., x_n]$ be real stable polynomials. We have the following:*

1. *(Symmetry)* $p \boxplus^\gamma q = q \boxplus^\gamma p$

2. *(Shift-invariance)* $(p(x+a) \boxplus^\gamma q)(x) = (p \boxplus^\gamma q)(x+a) = (p \boxplus^\gamma q(x+a))(x)$ *for* $a \in \mathbb{R}^n$

3. *(Scale-invariance)* $(p(ax) \boxplus^\gamma q(ax))(x) = a^\gamma \cdot (p \boxplus^\gamma q)(ax)$ *for* $a \in \mathbb{R}^n$

4. *(Derivative-invariance)* $(\partial_{x_k} p) \boxplus^\gamma q = \partial_{x_k}(p \boxplus^\gamma q) = p \boxplus^\gamma (\partial_{x_k} q)$ *for all* $k \in [n]$

5. *(Stability-preserving)* $p \boxplus^\gamma q$ *is real stable*

6. *(Triangle inequaltiy)* $\mathrm{Ab}(p \boxplus^\gamma q) \supseteq \mathrm{Ab}(p) + \mathrm{Ab}(q)$, *where* $+$ *is Minkowski sum*

Proof. (1), (2), (3) and (4) are straightforward. To prove (5), one can consider the Borcea-Brändén symbol (see [7]) of the operator

$$\boxplus^\gamma : \mathbb{R}^{(\gamma,\gamma)}[x_1, ..., x_n, z_1, ..., z_n] \to \mathbb{R}^\gamma[x_1, ..., x_n]$$

which is defined on products of polynomials (i.e., simple tensors) via

$$\boxplus^\gamma(p(x)q(z)) := (p \boxplus^\gamma q)(x)$$

and linearly extended. Note that if $\boxplus^\gamma$ preserves stability, then (5) follows as a corollary. That said, the symbol of $\boxplus^\gamma$ takes on a very nice form, using property (2):

$$\mathrm{Symb}(\boxplus^\gamma) = \boxplus^\gamma((x+y)^\gamma(z+w)^\gamma) = (x+y)^\gamma \boxplus^\gamma (x+w)^\gamma = (x+y+w)^\gamma$$

This polynomial is obviously real stable, and (5) follows.

To prove (6), we first assume $0 \in \mathrm{Ab}(p) \cap \mathrm{Ab}(q)$ by shifting, since $\mathrm{Ab}(p(x+a)) = \mathrm{Ab}(p) + \{-a\}$. Note also that $0 \in \mathrm{Ab}(p)$ if and only if p has coefficients all of the same sign. (One direction is easy, the other follows by induction and the fact that $\mathrm{Ab}(p) \subseteq \mathrm{Ab}(\partial_{x_i} p)$ by a standard argument.) In this case, p and q have coefficients all of the same sign, and therefore so does $p \boxplus^\gamma q$. That is, in this case $0 \in \mathrm{Ab}(p \boxplus^\gamma q)$.

To complete the proof, we utilize this case to show that $a \in \mathrm{Ab}(p)$ and $b \in \mathrm{Ab}(q)$ implies $a + b \in \mathrm{Ab}(p \boxplus^\gamma q)$. Note that by shifting we have that $0 \in \mathrm{Ab}(p(x+a)) \cap \mathrm{Ab}(q(x+b))$, which implies $0 \in \mathrm{Ab}((p \boxplus^\gamma q)(x+a+b))$ by the previous paragraph. This in turn implies $a + b \in \mathrm{Ab}(p \boxplus^\gamma q)$. $\qquad\square$

In the univariate case $\mathrm{Ab}(p)$ is literally the interval $[\lambda_1(p), \infty)$. The triangle inequality stated above then is equivalent to the classical version: $\lambda_1(p \boxplus^d q) \le \lambda_1(p) + \lambda_1(q)$. This is what justifies our calling it "the triangle inequality".

The upshot of the previous proposition is that many of the nice classical properties of the univariate convolution are shared with the multivariate additive convolution. That said, it becomes natural to ask a similar question for the stronger results discussed in this chapter; that is: what more can we say about how the multivariate additive convolution relates to points above the roots?

Our first conjecture in this direction is a combining of the main theorem (3.2.1) and the multivariate triangle inequality.

Conjecture 3.3.4. *Let $p, q, r \in \mathbb{R}^\gamma[x_1, ..., x_n]$ be real stable. Then:*

$$\mathrm{Ab}(p \boxplus^\gamma q \boxplus^\gamma r) + \mathrm{Ab}(r) \supseteq \mathrm{Ab}(p \boxplus^\gamma r) + \mathrm{Ab}(q \boxplus^\gamma r)$$

In a (as of yet unpublished) paper of Brändén and Marcus, a multivariate analogue of the Marcus-Spielman-Srivastava root bound is given. We believe that this result should follow from the previous conjecture, but it is currently unclear whether or not the methods of Brändén-Marcus can be adapted to prove the conjecture itself.

A Natural (But False) Conjecture

It can be shown that the previous conjecture is not enough to prove optimal bounds for the paving conjecture. For this we need something a bit more refined, which we give in the following. This conjecture represents the most natural generalization of the univariate root bound, and the fact that it precisely implies optimal paving bounds only increases its importance. In addition it has been considered independently of the authors by Mohan Ravichandran (personal correspondence; also see [43]) in attempt to prove optimal paving bounds, and this even further suggests its centrality.

Unfortunately though, the conjecture is false in general. We will state it in two equivalent forms, and provide a counterexample.

To do this, we first must relate the notion of $\mathrm{Ab}(p)$ to the notion of *potential* in the multiaffine case. Potential was used by Marcus-Spielman-Srivastava to delicately keep track of root bounds, and so this connection comes at no surprise. We will use the standard definition of potential in what follows:

$$\Phi_p^i(a) := \frac{\partial_{x_i} p}{p}(a)$$

Corollary 3.3.5. *Let $p \in \mathbb{R}^{(1^n)}[x_1, ..., x_n]$ be real stable and multiaffine with $p(0) > 0$ and $0 \in \mathrm{Ab}(p)$. Then:*

$$\Phi_p^i(0) \leq 1 \iff -e_i \in \mathrm{Ab}(p)$$

Proof. Since $p(x) > 0$ for $x \in \mathbb{R}_+^n$ and p is multiaffine we have:

$$\Phi_p^i(c \cdot e_i) < 1 \iff 0 < p(c \cdot e_i) - \partial_{x_i} p(c \cdot e_i) = p(0) + (c-1)\partial_{x_i} p(0) = p((c-1) \cdot e_i)$$

It is straightforward that $\Phi_p^i(c \cdot e_i)$ is strictly decreasing in c (or else identically zero) for $c \geq 0$, and therefore:

$$\Phi_p^i(0) \leq 1 \iff \Phi_p^i(c \cdot e_i) < 1 \text{ for all } c > 0$$

Combining these gives:

$$\Phi_p^i(0) \leq 1 \iff p((c-1) \cdot e_i) > 0 \text{ for all } c > 0$$

Note now that $p((c-1) \cdot e_i)$ is linear in c, and that $(c-1) \cdot e_i = 0 \in \mathrm{Ab}(p)$ for $c = 1$. Therefore Proposition 3.3.1 implies $\Phi_p^i(0) \leq 1$ iff $-e_i \in \mathrm{Ab}(p)$. $\qquad\square$

We now state the false conjecture, once in terms of potential and once in terms of points above the roots.

Conjecture 3.3.6 (Strong conjecture, first form (see [43])). *Let $p, q \in \mathbb{R}^{(1^n)}[x_1, ..., x_n]$ be real stable multiaffine polynomials, and let a and b be above the roots of p and q respectively. Suppose for some $\varphi \in \mathbb{R}_{++}^n$, we have the following for all $i \in [n]$:*

$$\Phi_p^i(a) \leq \varphi_i, \qquad \Phi_q^i(b) \leq \varphi_i$$

Then for all $i \in [n]$ we have:

$$\Phi_{p \boxplus q}^i \left(a + b - \frac{1}{\varphi} \right) \leq \varphi_i$$

Conjecture 3.3.7 (Strong conjecture, second form). *Let $p, q \in \mathbb{R}^{(1^n)}[x_1, ..., x_n]$ be real stable multiaffine polynomials. Suppose for all $i \in [n]$ we have:*

$$-e_i \in \mathrm{Ab}(p), \qquad -e_i \in \mathrm{Ab}(q)$$

The for all $i \in [n]$ we have:

$$-1 - e_i \in \mathrm{Ab}\left(p \boxplus^{(1^n)} q \right)$$

Proof of equivalence. By the previous corollary $-e_i \in \mathrm{Ab}(p)$ is equivalent to $\Phi_p^i(0) \leq 1$. The conclusion of the above conjecture is that $\Phi_{p \boxplus q}^i(-1) \leq 1$. Again by the previous corollary, this is equivalent to $-e_i \in \mathrm{Ab}((p \boxplus q)(x-1))$. This in turn is equivalent to $-1-e_i \in \mathrm{Ab}(p \boxplus q)$.

As a final note, we can restrict to this seemingly less general case (i.e., $\Phi_p^i(0) \leq 1$ instead of $\Phi_p^i(a) \leq \varphi_i$) via shifting and scaling, which completes the proof. $\square$

To disprove this, we give a counterexample to the second formulation. The key idea is to use a polynomial which is extremal with respect to the strongly Rayleigh conditions, see Theorem 2.2.7. We recall the conditions for real stable multiaffine $p \in \mathbb{R}^{(1^n)}[x_1, ..., x_n]$ here:

$$\partial_{x_i} p(x) \cdot \partial_{x_j} p(x) - p(x) \cdot \partial_{x_i} \partial_{x_j} p(x) \geq 0$$

Note that the left-hand side of the above inequality does not depend on x_i or x_j. One can see this by taking the partial derivative of the above expression with respect to x_i or x_j, recalling that p is multiaffine (this expression will be 0). This makes it relatively easy to determine whether or not 3-variable multiaffine polynomials are real stable, as in the following example.

Counterexample 3.3.8. *The polynomial*

$$p = q = \frac{8}{21} x_1 x_2 x_3 + \frac{80}{21} x_1 x_2 + \frac{27}{7} x_1 x_3 + x_2 x_3 + 4x_1 + 4x_2 + 4x_3 + 4$$

provides a counterexample to the above conjectures.

Proof. First we prove that $p = q$ is real stable. By the above comment, we obtain simple expressions for the strongly Rayleigh conditions:

$$\partial_{x_1} p(x) \cdot \partial_{x_2} p(x) - p(x) \cdot \partial_{x_1} \partial_{x_2} p(x) = \frac{1}{21}(7x_3 + 4)^2$$

$$\partial_{x_1} p(x) \cdot \partial_{x_3} p(x) - p(x) \cdot \partial_{x_1} \partial_{x_3} p(x) = \frac{4}{7}(2x_2 + 1)^2$$

$$\partial_{x_2} p(x) \cdot \partial_{x_3} p(x) - p(x) \cdot \partial_{x_2} \partial_{x_3} p(x) = \frac{4}{147}(22x_1 + 21)^2$$

Notice that all of these expressions are nonnegative for all x, which means that $p = q$ is real stable. Also, notice that these polynomials are on the boundary of the set of nonnegative polynomials, and so in some sense $p = q$ is on the boundary of the set of real stable polynomials. Note that this polynomial has 0 above its roots (with $p(0) > 0$), and it is easy to see that $-e_i \in \mathrm{Ab}(p)$ for all $i \in [n]$.

We now compute $p \boxplus q = p \boxplus p$ as follows:

$$p \boxplus q = \frac{64}{441}x_1 x_2 x_3 + \frac{1280}{441}x_1 x_2 + \frac{144}{49}x_1 x_3 + \frac{16}{21}x_2 x_3 + \frac{4768}{147}x_1 + \frac{32}{3}x_2 + \frac{226}{21}x_3 + \frac{1520}{21}$$

Since $\boxplus$ preserves real stability, this polynomial is real stable. Further, we have $0 \in \mathrm{Ab}(p \boxplus q)$ and $(p \boxplus q)(0) > 0$, and so $(p \boxplus q)(x) \geq 0$ for all $x \in \mathrm{Ab}(p \boxplus q)$. With this, we show that $(p \boxplus q)(-1 - e_1) < 0$ which contradicts the above conjecture:

$$(p \boxplus q)(-1 - e_1) = -\frac{1450}{441}$$

$\square$

3.4 Concluding Remarks

Despite its connections to important problems like the paving conjecture and the entropy conjecture, it is still not fully understood how the additive convolution affects the roots of real-rooted polynomials. In [47], Marcus, Spielman, and Srivastava began the study of root movement by investigating the effect of differential operators of the form $1 - \alpha \partial_t$ on the largest root. In this chapter, we extended their result to all differential operators which preserve real-rootedness. This extension alone doesn't have any immediate applications we are aware of.

The resolution of Horn's conjecture by Knutson and Tao (see [38]) gave a full characterization of the eigenvalues of the sum of two Hermitian matrices. We were able to obtain Horn's inequalities for the additive convolution as well via hyperbolicity, but understanding the full effect of the additive convolution on roots remains a mystery. The entropy conjecture, which quantifies the effect of the additive convolution on the discriminant of a polynomial,

is one approach to understanding the effect of the roots holistically. Our submodular majorization (and generalized Horn's inequalities) conjectures provide another insight into the workings of the inner roots. Because submodularity is unique to the additive convolution, we believe it will require a new framework (beyond traditional hyperbolicity tools) to tackle these conjectures.

Another possible future direction is extending submodularity results to the b-additive convolution, in which derivatives are replaced by certain finite differences. Such convolutions have an intimate connection to the *mesh* of a real-rooted polynomial, which is the minimal distance between any two roots (e.g., see [10] and [44]). In our testing we found several submodularity relations among such b-additive convolutions. The additive convolution can be obtained by limiting $b \to 0$, and so any results for the b-additive convolution are strictly stronger than the conjectures in this chapter. The advantage of trying to prove these statements in the finite difference case comes in the limited structures available: fewer operations interact nicely with the mesh of a polynomial compared to those operations which preserve real-rootedness, and this may better direct the study of the roots. We discuss this generalized convolution and root mesh properties in the next chapter.

Finally in the multivariate realm, little is known. And, many of the natural extensions of these results seem to fail in the multivariate case. The state of the art in this direction is currently the ad hoc barrier function arguments used by MSS in their resolution of Kadison-Singer. That said, an important next step for their work is to encapsulate their techniques in a more coherent theory. We believe that our results and conjectures are a step in the right direction.

Chapter 4

Generalized Additive Convolution

In [10], Brändén, Krasikov, and Shapiro show that the classical additive convolution (see the definition at the beginning of Chapter 3) can only increase root mesh, which is defined as the minimum absolute difference between any pair of roots of a given polynomial (see Remark 2.2.11 above, Definition 4.1.1 below). That is, the mesh of the output polynomial is at least as large as the mesh of either of the input polynomials. They use similar arguments to show that, for polynomials with non-negative roots, the multiplicative convolution can only increase logarithmic root mesh. This is similarly defined as the minimum ratio (greater than 1) between any pair of positive roots of a given polynomial.

Regarding mesh and logarithmic mesh, there are natural generalized convolution operators which also preserve such properties. The first is be called the q-multiplicative convolution, and it was shown to preserve logarithmic root mesh of at least q in [39]. This convolution is defined as follows, where p_k an r_k are the coefficients of p and r, respectively. (Note that as $q \to 1$ this limits to the classical multiplicative convolution.)

$$ p \boxtimes_q^d r := \sum_{k=0}^{d} \binom{d}{k}_q^{-1} q^{-\binom{k}{2}} (-1)^k p_k r_k x^k $$

Here, $\binom{d}{k}_q$ denotes the q-binomial coefficients, seen in Definition 4.2.9.

The second generalized convolution is called the b-additive convolution (or finite difference convolution), and the main concern of this chapter is to demonstrate that it preserves root mesh of at least b. This convolution is defined as follows. (Note that as $b \to 0$ this limits to the classical additive convolution.)

$$ p \boxplus_b^d r := \frac{1}{d!} \sum_{k=0}^{d} \Delta_b^k p \cdot (\Delta_b^{d-k} r)(0) $$

Here, Δ_b is a finite b-difference operator, defined as:

$$ \Delta_b : p \mapsto \frac{p(x) - p(x - b)}{b} $$

Note that this generalizes the classical additive convolution in the following conceptual sense: all constant coefficient differential operators on polynomials of degree at most d can be written as convolutions with one fixed input polynomial.

The main goal of this chapter is to settle a conjecture of Brändén, Krasikov, and Shapiro regarding mesh and the b-additive convolution, using two different proof methods. The first method (see §4.1) uses a general result which allows one to transfer results on roots of the q-multiplicative convolution to the b-additive convolution (via Hurwitz's theorem; see Proposition 2.2.3). The second method (see §4.1) simply adapts Lamprecht's proof in the q-multiplicative case to the b-additive case.

4.1 Polynomial Mesh Results

Recall from Remark 2.2.11 the definitions of mesh and log mesh, and the fact that mesh properties are equivalent to certain interlacing properties.

Definition 4.1.1. For a real-rooted polynomial f, we write $\mathrm{mesh}(f) \geq b$ if the distance between any pair of roots is at least b. For a positive-rooted polynomial f, we write $\mathrm{lmesh}(f) \geq q \geq 1$ if the ratio of any pair of roots is at least q. We say that f is *b-mesh* and is *q-log mesh* respectively, and we say *strictly* here if the inequalities are strict.

The q-Multiplicative Convolution

In [39], Lamprecht proves logarithmic mesh preservation properties of the q-multiplicative convolution. We state his result formally as follows.

Theorem 4.1.2 (Lamprecht)**.** *Let p and r be polynomials of degree at most n such that* $\mathrm{lmesh}(p) \geq q$ *and* $\mathrm{lmesh}(r) \geq q$*, for some* $q \in (1, \infty)$*. Then,* $\mathrm{lmesh}(p \boxtimes_q^n r) \geq q$*.*

This result is actually an analogue to an earlier result of Suffridge [61] regarding polynomials with roots on the unit circle. In Suffridge's result, q is taken to be an element of the unit circle, and log mesh translates to mean that the roots are pairwise separated by at least the argument of q. Roughly speaking, he obtains the same result for the corresponding q-multiplicative convolution. Remarkably, the known proofs of his result (even a proof of Lamprecht) differ fairly substantially from Lamprecht's proof of the above theorem.

Additionally, we note here that Lamprecht uses different notation and conventions in [39]. In particular, he uses $q \in (0, 1)$, considers polynomials p with all non-positive roots, and his definition of $\boxtimes_q^n$ does not include the $(-1)^k$ factor. These differences are generally speaking unsubstantial, but it is worth noting that the arguments of §4.2 seem to require the $(-1)^k$ factor.

The b-Additive Convolution

In this chapter, we show the b-additive convolution (or, finite difference convolution) preserves the space of polynomials with root mesh at least b. Our result then solves the first (and second) conjecture stated in [10]. We state it formally here.

Theorem 4.1.3. *Let p and r be polynomials of degree at most n such that* $\mathrm{mesh}(p) \geq b$ *and* $\mathrm{mesh}(r) \geq b$, *for some $b \in (0, \infty)$. Then,* $\mathrm{mesh}(p \boxplus_b^n r) \geq b$.

As a note, Brändén, Krasikov, and Shapiro actually use the *forward* finite difference operator in their definition of the convolution. This is not a problem as our result then differs from their conjecture by a shift of the input polynomials.

Remark 4.1.4. Although the q-multiplicative and b-additive convolutions preserve q-log mesh and b-mesh respectively, they do not preserve real-rootedness. To see this we compute the following example: $x^2 \boxplus_1^2 x^2 = \frac{1}{2}(2x^2 - 2x + 1)$. This polynomial has discriminant -4 and hence is not real rooted. Similarly simple examples demonstrate this for the q-multiplicative convolution.

Proof Methods: An Analytic Connection

Our first method of proof of Theorem 4.1.3 will demonstrate a way to pass root properties of the q-multiplicative convolution to the b-additive convolution. As this is interesting in its own right, we state the most general version of this result here.

Theorem 4.1.5. *Fix $b \geq 0$ and let p, r be polynomials of degree d. We have the following, where convergence is uniform on compact sets.*

$$\lim_{q \to 1}(1 - q)^d \left[E_{q,b}(p) \boxtimes_{q^b}^d E_{q,b}(r) \right](q^x) = p \boxplus_b^d r$$

Where $E_{q,b}$ is defined in 4.2.11. Note that for $b = 0$, this result pertains to the classical convolutions.

Here, the $E_{q,b}$ are certain linear isomorphisms of $\mathbb{C}[x]$ (univariate polynomials), to be explicitly defined below. Notice that uniform convergence allows us to use Hurwitz' theorem to obtain root properties in the limit of the left-hand side. That is, any information about how the q-multiplicative convolution acts on roots will transfer to some statement about how the b-additive convolution acts on roots. As it turns out, a special case of Lamprecht's result (Theorem 3 from [39]) will become our result (Theorem 4.1.3) in the limit. We discuss this transfer process in more detail in §4.2.

Proof Methods: Extending Lamprecht's Method

Our second method of proof of Theorem 4.1.3 is an extension of the method used by Lamprecht to prove the log mesh result for the q-multiplicative convolution. Specifically, he demonstrates that the q-multiplicative convolution preserves a root-interlacing property for q-log mesh polynomials. More formally he proves the following result which gives Theorem 4.1.3 as a corollary.

Theorem 4.1.6 (Lamprecht Interlacing-Preserving). *Let $f, g \in \mathbb{R}^d[x]$ be q-log mesh polynomials of degree d with only negative roots. Let $T_g : \mathbb{R}^d[x] \to \mathbb{R}^d[x]$ be the real linear operator defined by $T_g : r \mapsto r \boxtimes_q^d g$. Then, T_g preserves the set of polynomials whose roots interlace the roots of f.*

We achieve an analogous result for the b-additive convolution using techniques similar to those found in Lamprecht's paper. We state it formally here.

Theorem 4.1.7. *Let $f, g \in \mathbb{R}^d[x]$ be b-mesh polynomials of degree d. Let $T_g : \mathbb{R}^d[x] \to \mathbb{R}^d[x]$ be the real linear operator defined by $T_g : r \mapsto r \boxplus_b^d g$. Then, T_g preserves the set of polynomials whose roots interlace the roots of f.*

In both cases, mesh and log mesh properties can be shown to be equivalent to root interlacing properties ($f(x)$ interlaces $f(x - b)$ for b-mesh, and $f(x)$ interlaces $f(q^{-1}x)$ for q-log mesh). The above theorems then immediately imply the desired mesh preservation properties for the respective convolutions. We discuss this further in §4.3.

4.2   First Proof Method: As a Limit of Generalized Multiplicative Convolutions

In what follows we establish a general analytic connection between the multiplicative and additive convolutions on polynomials of degree at most d. We then extend this connection to the q-multiplicative convolution and the b-additive convolution (Theorem 4.1.5). Using this connection, we transfer root information results of the multiplicative convolution (q or classical) to the additive convolution (b or classical). Specifically, we use this connection to prove Theorem 4.1.3, which is the conjecture of Brändén, Krasikov, and Shapiro discussed above.

To begin we state an observation of Vadim Gorin demonstrating an analytic connection in the classical case using matrix formulations of the classical convolutions given in [47]:

$$\chi(A) \boxtimes^d \chi(B) = \mathbb{E}_P \left[\chi(APBP^T) \right]$$

$$\chi(A) \boxplus^d \chi(B) = \mathbb{E}_P \left[\chi(A + PBP^T) \right]$$

Here, A and B are real symmetric matrices, χ denotes the characteristic polynomial, and the expectations are taken over all permutation matrices. We then write:

$$\begin{aligned}
\lim_{t \to 0} t^{-d} \left[\chi(e^{tA}) \boxtimes^n \chi(e^{tB}) \right] (tx + 1) &= \lim_{t \to 0} t^{-d} \mathbb{E}_P \left[\det \left(txI + I - e^{tA} P e^{tB} P^T \right) \right] \\
&= \lim_{t \to 0} t^{-d} \mathbb{E}_P \left[\det \left(txI - t(APP^T + PBP^T) + O(t^2) \right) \right] \\
&= \lim_{t \to 0} \mathbb{E}_P \left[\chi \left(A + PBP^T + O(t) \right) \right] \\
&= \chi(A) \boxplus^d \chi(B)
\end{aligned}$$

This connection is suggestive and straightforward, but seemingly confined to the classical case. Therefore, we instead state below a slightly modified (but equivalent) version of this observation for the classical convolutions (Theorem 4.2.2) which we are able to then generalize to the q-multiplicative and b-additive convolutions.

The Classical Convolutions

We begin by sketching the proof of the connection between the classical additive and multiplicative convolutions. We then state rigorously the more general result for the q-multiplicative and b-additive convolutions. In this section, many quantities will be defined with $b = 0$ in mind (this corresponds to the classical additive convolution), with the more general quantities given in subsequent sections. Further, we will leave the proofs of the lemmas to the generic b case, omitting them here.

To go from the multiplicative world to the additive world, we use a linear map which acts as an exponentiation on roots, and a limiting process which acts as a logarithm. In particular, we will refer to the following algebra endomorphism on $\mathbb{C}[x]$ as our "exponential map":

$$E_{q,0} : x \mapsto \frac{1 - x}{1 - q}$$

In what follows, any limiting process will mean uniform convergence on compact sets in $\mathbb{C}$, unless otherwise specified. This will allow us to extract analytic information about roots using the classical Hurwitz' theorem. In particular, the following result hints at the analytic information provided by the exponential map $E_{q,0}$.

Proposition 4.2.1. *We have the following for any $p \in \mathbb{C}[x]$.*

$$\lim_{q \to 1} [E_{q,0}(p)](q^x) = p$$

Proof. We first consider $[E_{q,0}(x)](q^x) = \frac{1 - q^x}{1 - q}$, for which we obtain the following by the generalized binomial theorem:

$$\lim_{q \to 1} [E_{q,0}(x)](q^x) = \lim_{q \to 1} \frac{1 - q^x}{1 - q} = \lim_{q \to 1} \sum_{m=1}^{\infty} \binom{x}{m} (q - 1)^{m-1} = \binom{x}{1} = x$$

To show that convergence here is uniform on compact sets, consider the tail for $|x| \leq M$:

$$\left| \sum_{m=2}^{\infty} \binom{x}{m} (q-1)^{m-1} \right| \leq \sum_{m=0}^{\infty} \left| (q-1)^{m+1} \cdot \frac{x(x-1)\cdots(x-m-1)}{(m+2)!} \right|$$

$$\leq |q-1| \sum_{m=0}^{\infty} |q-1|^m \prod_{k=1}^{m+2} \left(1 + \frac{|x|}{k} \right)$$

$$\leq |q-1|(1+M)^2 \sum_{m=0}^{\infty} (|q-1|(1+M))^m$$

$$= \frac{|q-1|(1+M)^2}{1 - |q-1|(1+M)}$$

This limits to zero as $q \to 1$, which proves the desired convergence.

Since $E_{q,0}$ is an algebra morphism, we can use the fundamental theorem of algebra to complete the proof. Specifically, letting $p(x) = c_0 \prod_k (x - \alpha_k)$ we have:

$$\lim_{q \to 1} \left[E_{q,0} \left(c_0 \prod_k (x - \alpha_k) \right) \right] (q^x) = c_0 \prod_k \left(\lim_{q \to 1} [E_{q,0}(x)](q^x) - \alpha_k \right) = c_0 \prod_k (x - \alpha_k)$$

$\square$

We now state our result in the classical case, which gives an analytic connection between the additive and multiplicative convolutions. As a note, many of the analytic arguments used in the proof of this result will have a flavor similar to that of the proof of Proposition 4.2.1.

Theorem 4.2.2. *For $p, r \in \mathbb{C}^d[x]$ we have the following.*

$$\lim_{q \to 1} (1-q)^d \left[E_{q,0}(p) \boxtimes^d E_{q,0}(r) \right] (q^x) = p \boxplus^d r$$

Proof Sketch

We will establish the above identity by calculating it on basis elements. Specifically, we will expand everything into powers of $(1 - q)$. To prove the theorem, it then suffices to show that: (1) the negative degree coefficients are all zero, (2) the series has the desired constant term, and (3) the tail of the series converges to zero uniformly on compact sets. Our first step towards establishing this is expanding q^{kx} in terms of powers of $(1 - q)$.

Remark 4.2.3. Since we will only be considered with behavior for q near 1, we will use the notation $q \approx 1$ to indicate there exists some $\epsilon > 0$ such that the statement holds for $q \in (1 - \epsilon, 1 + \epsilon)$.

Definition 4.2.4. We define a constant, which will help us to simplify the following computations.

$$\alpha_{q,0} := \frac{\ln q}{1 - q}$$

Note that $\lim_{q \to 1} \alpha_{q,0} = -1$.

Lemma 4.2.5. *Fix $k \in \mathbb{N}_0$. For $q \approx 1$, we have the following.*

$$q^{kx} = \sum_{m=0}^{\infty} \frac{x^m}{m!} \alpha_{q,0}^m k^m (1-q)^m$$

For fixed $q \approx 1$, this series has a finite radius of convergence, and this radius approaches infinity as $q \to 1$.

Notice that this is not a true power series in $(1 - q)$, as $\alpha_{q,0}$ depends on q. Using this, we can calculate the series obtained after plugging in specific basis elements.

Lemma 4.2.6. *Fix $q \approx 1$ in $\mathbb{R}_+$ and $j, k, d \in \mathbb{N}_0$ such that $0 \le j \le k \le d$. We have the following.*

$$(1-q)^d \left[(1-x)^j \boxtimes^d (1-x)^k \right] (q^x) = \sum_{m=0}^{\infty} \frac{x^m}{m!} \alpha_{q,0}^m (1-q)^{d+m-j-k} \sum_{i=0}^{j} \frac{\binom{j}{i}\binom{k}{i}}{\binom{d}{i}} (-1)^i i^m$$

We use interpolation arguments to handle the terms of this series, which are combinatorial in nature. In particular we show that this series has no nonzero negative degree terms, as seen in the following.

Proposition 4.2.7. *Fix $j, k, m, d \in \mathbb{N}_0$ such that $j \le k$ and $d + m - j - k \le 0$. We have the following identity.*

$$\sum_{i=0}^{j} \frac{\binom{j}{i}\binom{k}{i}}{\binom{d}{i}} (-1)^i i^m = \begin{cases} (-1)^{d-j-k} \frac{j!k!}{d!} & m = j + k - d \\ 0 & m < j + k - d \end{cases}$$

To deal with the tail of the series, we then use crude bounds to get uniform convergence on compact sets.

Lemma 4.2.8. *Fix $M > 0$, and $j, k, d \in \mathbb{N}_0$ such that $j \le k \le d$. For $|x| \le M$, there exists $\gamma > 0$ such that the following bound holds for $q \in (1 - \gamma, 1 + \gamma)$.*

$$\left| \sum_{m>j+k-d} \frac{x^m}{m!} \alpha_{q,0}^m (1-q)^{d+m-j-k} \sum_{i=0}^{j} \frac{\binom{j}{i}\binom{k}{i}}{\binom{d}{i}} (-1)^i i^m \right| \le c_0 c_1 \sum_{m=1}^{\infty} c_2^m |1 - q|^m$$

Here, c_0, c_1, c_2 are independent of q.

With this, we can now complete the proof of the theorem by comparing the desired quantity to the constant term in our series in $(1 - q)$.

Proof of Theorem 4.2.2. For $j, k, d \in \mathbb{N}_0$ such that $0 \leq j \leq k \leq d$, we can combine the above results to obtain the following. Recall that $\lim_{q \to 1} \alpha_{q,0} = -1$.

$$\lim_{q \to 1} (1 - q)^d \left[\frac{(1 - x)^j}{(1 - q)^j} \boxtimes^d \frac{(1 - x)^k}{(1 - q)^k} \right] (q^x) = \frac{j! k!}{d!(j + k - d)!} x^{j+k-d} = x^j \boxplus^d x^k$$

By symmetry, this demonstrates the desired result on a basis. Therefore, the proof is complete. $\square$

General Connection Preliminaries

We now prove the previous results in more generality, which allows for extension to these generalized convolutions. First though, we give some preliminary notation.

Definition 4.2.9. Fix $q \in \mathbb{R}_+$ and $x \in \mathbb{C}$. We define $(x)_q := \frac{1 - q^x}{1 - q}$. Note that $\lim_{q \to 1} (x)_q = x$, using the generalized binomial theorem on $q^x = (1 + (q - 1))^x$.

Specifically, for any $d \in \mathbb{Z}$, we have:

$$(d)_q := \frac{1 - q^d}{1 - q} = 1 + q + q^2 + \cdots + q^{d-1}$$

We then extend this notation to $(d)_q! := (d)_q (d - 1)_q \cdots (2)_q (1)_q$ and $\binom{d}{k}_q := \frac{(d)_q!}{(k)_q!(d-k)_q!}$. We also define a system of bases of $\mathbb{C}[x]$ which will help us to understand the mesh convolutions.

Definition 4.2.10. For $b \geq 0$ and $q \in \mathbb{R}_+$, we define the following bases of $\mathbb{C}[x]$.

$$v_{q,b}^k := \frac{(1 - x)(1 - q^b x) \cdots (1 - q^{(k-1)b} x)}{(1 - q)^k}$$

$$\nu_b^k := x(x + b)(x + 2b) \cdots (x + (k - 1)b)$$

We demonstrate the relevance of these bases to the generalized convolutions by giving alternate definitions. Consider a linear map A_b on $\mathbb{C}[x]$ defined via $A_b : \nu_0^k \mapsto \nu_b^k$. That is, $A_b : x^k \mapsto x(x + b) \dots (x + (k - 1)b)$. We can then define the *b*-additive convolution as follows:

$$p \boxplus_b^d r := A_b(A_b^{-1}(p) \boxplus^n A_b^{-1}(r))$$

That is, the *b*-additive convolution is essentially a change of basis of the classical additive convolution. Note that equivalently, one can conjugate ∂_x by A_b to obtain $\Delta_b : p \mapsto \frac{p(x) - p(x-b)}{b}$ which demonstrates the definition of $\boxplus_b^d$ in terms of finite difference operators.

Similarly, the q-multiplicative convolution can be seen as a change of basis of the classical multiplicative convolution. Consider a linear map $M_q^{(d)}$ on $\mathbb{C}[x]$ defined via $M_q^{(d)} : \binom{d}{k} x^k \mapsto \binom{d}{k}_q q^{\binom{k}{2}} x^k$, which has the property that $M_{q^b}^{(d)} : (1-x)^d \mapsto (1-q)^d v_{q,b}^d$. We can then define the q-multiplicative convolution as follows:

$$p \boxtimes_q^d r := M_q^{(d)} \left((M_q^{(d)})^{-1}(p) \boxtimes^d (M_q^{(d)})^{-1}(r) \right)$$

These bases will be used to simplify the proof of the general analytic connection for the mesh (non-classical) convolutions. In what follows they will play the role that the basis elements x^k and $(1-x)^k$ did in the classical proof sketch above.

Analytic Connection for Generalized Convolutions

We now generalize the results from the classical ($b = 0$) setting.

Definition 4.2.11. Consider the following generalized "exponential map" using the basis elements defined above:

$$E_{q,b} : \nu_b^k \mapsto v_{q,b}^k$$

Note for $b > 0$ these are no longer algebra morphisms. Specialization to $b = 0$ recovers the original "exponential map". Also notice that for any p, the roots of $E_{q,b}(p)$ approach 1 as $q \to 1$ (multiply the output polynomial by $(1-q)^{\deg(p)}$). In all that follows, previously stated results can be immediately recovered by setting $b = 0$.

Proposition 4.2.12. *We obtain the same key relation for the generalized exponential maps:*

$$\lim_{q \to 1} [E_{q,b}(p)](q^x) = p$$

Proof. We compute on basis elements, using Proposition 4.2.1 in the process:

$$\lim_{q \to 1} [E_{q,b}(\nu_b^k)](q^x) = \lim_{q \to 1} [v_{q,b}^k](q^x)$$

$$= \lim_{q \to 1} \frac{(1 - q^x)(1 - q^{x+b}) \cdots (1 - q^{x+(k-1)})}{(1 - q)^k}$$

$$= \prod_{j=0}^{k-1} \lim_{q \to 1} \frac{1 - q^{x+jb}}{1 - q} = \prod_{j=0}^{k-1} (x + jb) = \nu_b^k$$

$\square$

As in Proposition 4.2.1, one can interpret the $E_{q,b}$ maps as a way to exponentiate the roots of a polynomial. The inverse to these maps is given in the previous proposition by plugging in an exponential and limiting, which corresponds to taking the logarithm of the roots. This discussion will be made more precise in §4.2.

We now state and prove the main result, which gives an analytic link between the b-additive and q-multiplicative convolutions. We follow the proof sketch of the classical result given above, breaking the following full proof up into more manageable sections.

Theorem 4.1.5. *Fix $b \geq 0$ and let p, r be polynomials of degree d. We have the following.*

$$\lim_{q \to 1} (1 - q)^d \left[E_{q,b}(p) \boxtimes_{q^b}^d E_{q,b}(r) \right] (q^x) = p \boxplus_b^d r$$

Series Expansion

In order to prove this theorem, we first expand the left-hand side of the expression in a series in $(1 - q)^m$. As above, this is not quite a power series in $(1 - q)^m$ as $\alpha_{q,b}$ (which we now define) depends on q.

Definition 4.2.13. We define the b-version of the $\alpha_{q,0}$ constants as follows.

$$\alpha_{q,b} := \begin{cases} \frac{-(b)_{q^{-1}}}{q^b} & b > 0 \\ \frac{\ln q}{1 - q} & b = 0 \end{cases}$$

Note that $\lim_{b \to 0} \alpha_{q,b} = \alpha_{q,0}$ for $q \in \mathbb{R}_+$, and $\lim_{q \to 1} \alpha_{q,b} = -1$ for fixed $b \geq 0$.

We now need to understand how exponential polynomials in q relate to our basis elements.

Lemma 4.2.14. *Fix $b \geq 0$, and $k \in \mathbb{N}_0$. For $q \approx 1$ in $\mathbb{R}_+$, we have the following.*

$$q^{kx} = \sum_{m=0}^{\infty} \frac{\nu_b^m}{m!} \alpha_{q,b}^m (k)_{q^{-b}}^m (1 - q)^m$$

For fixed $q \approx 1$, this series has a finite radius of convergence, and this radius approaches infinity as $q \to 1$.

Proof. For $b > 0$, we use the generalized binomial theorem to compute:

$$q^{kx} = (q^{-bk} - 1 + 1)^{-x/b} = \sum_{m=0}^{\infty} \frac{(-b)^{-m}}{m!} x(x+b) \cdots (x + b(m-1))(q^{-bk} - 1)^m$$

$$= \sum_{m=0}^{\infty} \frac{(-b)^{-m} \nu_b^m}{m!} (k)_{q^{-b}}^m (b)_{q^{-1}}^m (q^{-1} - 1)^m$$

$$= \sum_{m=0}^{\infty} \frac{\nu_b^m}{m!} \left(-\frac{(b)_{q^{-1}}}{q^b} \right)^m (k)_{q^{-b}}^m (1 - q)^m$$

For $b = 0$, manipulating the Taylor series of $q^{kx} = e^{kx \ln q}$ gives the result.

For fixed $q \approx 1$, let $\delta > 0$ be small enough such that $|\alpha_{q,b}| < \frac{1 + \delta}{q}$ and $(k)_{q^{-b}} < k + \delta$. Consider:

$$|\nu_b^m| = |x(x+b) \cdots (x + (m-1)b)| \leq m!(|x| + b)^m$$

From this, we obtain:

$$|q^{kx}| \leq \sum_{m=0}^{\infty} \left(\frac{(|x| + b)(1 + \delta)(k + \delta)}{q} \right)^m |1 - q|^m$$

It is then easy to see that the radius of convergence of this series limits to infinity as $q \to 1$. $\qquad\square$

We will now proceed by proving the main result on a basis. To that end, we will prove a number of results related to basis element computations. Most of these are rather tedious and not very illuminating. Perhaps this can be simplified through some more detailed and generalized theory of q- and b-polynomial operators.

Lemma 4.2.15. *Fix $q \approx 1$ in $\mathbb{R}_+$, $b \geq 0$, and $j, k, d \in \mathbb{N}_0$ such that $0 \leq j \leq k \leq d$. We have the following.*

$$(1-q)^d \left[v_{q,b}^j \boxtimes_{q^b}^d v_{q,b}^k \right] (q^x) = \sum_{m=0}^{\infty} \frac{\nu_b^m}{m!} \alpha_{q,b}^m (1-q)^{d+m-j-k} \sum_{i=0}^{j} \frac{\binom{j}{i}_{q^b} \binom{k}{i}_{q^b}}{\binom{d}{i}_{q^b}} q^{bi(i-1)/2} (-1)^i (i)_{q^{-b}}^m$$

Proof. We compute:

$$(1-q)^d \left[v_{q,b}^j \boxtimes_{q^b}^d v_{q,b}^k \right] (q^x) = (1-q)^{d-j-k} \sum_{i=0}^{j} \frac{\binom{j}{i}_{q^b} \binom{k}{i}_{q^b}}{\binom{d}{i}_{q^b}} q^{bi(i-1)/2} (-1)^i q^{ix}$$

$$= \sum_{i=0}^{j} \frac{\binom{j}{i}_{q^b} \binom{k}{i}_{q^b}}{\binom{d}{i}_{q^b}} q^{bi(i-1)/2} (-1)^i \sum_{m=0}^{\infty} \frac{\nu_b^m}{m!} \alpha_{q,b}^m (1-q)^{d+m-j-k} (i)_{q^{-b}}^m$$

$$= \sum_{m=0}^{\infty} \frac{\nu_b^m}{m!} \alpha_{q,b}^m (1-q)^{d+m-j-k} \sum_{i=0}^{j} \frac{\binom{j}{i}_{q^b} \binom{k}{i}_{q^b}}{\binom{d}{i}_{q^b}} q^{bi(i-1)/2} (-1)^i (i)_{q^{-b}}^m$$

$\qquad\square$

Q-Lagrange Interpolation

To prove convergence in Theorem 4.1.5, we break up the infinite sum of Lemma 4.2.15 into two pieces. For $d+m-j-k \leq 0$, we use an interpolation argument to obtain the following identity. Note that this generalizes a similar identity (for $q = 1$) found in [55].

Proposition 4.2.16. *Fix $q \approx 1$, $b \geq 0$, and $j, k, m, d \in \mathbb{N}_0$ such that $j \leq k$ and $d+m-k \leq j$. We have the following identity.*

$$\sum_{i=0}^{j} \frac{\binom{j}{i}_{q^b} \binom{k}{i}_{q^b}}{\binom{d}{i}_{q^b}} q^{bi(i-1)/2} (-1)^i (i)_{q^{-b}}^m = \begin{cases} (-1)^{d-j-k} q^{b\binom{d}{2} - b\binom{j}{2} - b\binom{k}{2}} \frac{(j)_{q^b}! (k)_{q^b}!}{(d)_{q^b}!} & m = j+k-d \\ 0 & m < j+k-d \end{cases}$$

We first give a lemma (see Lemma 4.5.1 for a proof), and then the proof of the proposition will follow. Let $[t^j] p(t)$ denote the coefficient of p corresponding to the monomial t^j.

Lemma 4.2.17. *Fix $p \in \mathbb{C}_j[x]$. We have the following identity:*

$$(-1)^j q^{-\binom{j}{2}} \cdot [t^j]p(t) = \sum_{i=0}^{j} p((i)_{q^{-1}}) \frac{(-1)^i}{(i)_q!(j-i)_q!} q^{\binom{i}{2}}$$

Proof of Proposition 4.2.16. Consider the polynomial $p(t) = t^m((d)_{q^{-b}} - t)((d-1)_{q^{-b}} - t) \cdots ((k+1)_{q^{-b}} - t)$, which is of degree $m + d - k \leq j$. So, $[t^j]p(t) = (-1)^{d-k}\delta_{m=j+k-d}$. Also, recall the identity $(i)_{q^{-b}}! = q^{-b\binom{i}{2}}(i)_{q^b}!$. Using the previous lemma and replacing q by q^b, we obtain:

$$(-1)^j q^{-b\binom{j}{2}} \cdot (-1)^{d-k}\delta_{m=j+k-d} = \sum_{i=0}^{j} p((i)_{q^{-b}}) \frac{(-1)^i}{(i)_{q^b}!(j-i)_{q^b}!} q^{b\binom{i}{2}}$$

$$= \sum_{i=0}^{j} q^{-bi(d-k)} \frac{(d-i)_{q^{-b}}!}{(k-i)_{q^{-b}}!} \frac{(-1)^i}{(i)_{q^b}!(j-i)_{q^b}!} q^{b\binom{i}{2}}(i)_{q^{-b}}^m$$

$$= \sum_{i=0}^{j} q^{b\binom{k}{2}} q^{-b\binom{d}{2}} \frac{(d-i)_{q^b}!}{(k-i)_{q^b}!} \frac{(-1)^i}{(i)_{q^b}!(j-i)_{q^b}!} q^{b\binom{i}{2}}(i)_{q^{-b}}^m$$

$$= \sum_{i=0}^{j} q^{b\binom{k}{2}} q^{-b\binom{d}{2}} \frac{(d)_{q^b}}{(j)_{q^b}(k)_{q^b}} \cdot \frac{\binom{j}{i}_{q^b}\binom{k}{i}_{q^b}}{\binom{d}{i}_{q^b}} q^{b\binom{i}{2}}(-1)^i(i)_{q^{-b}}^m$$

The result follows. □

Tail of the Series

For $d + m - j - k > 0$, we show that the tail of the infinite series in Lemma 4.2.15 is bounded by a geometric series in $\epsilon \to 0$ as $q \to 1$. The proof, is somewhat similar to the discussion of convergence in the proof of Lemma 4.2.14.

Lemma 4.2.18. *Fix $b \geq 0$, $M > 0$, and $j, k, d \in \mathbb{N}_0$ such that $j \leq k \leq d$. For $|x| \leq M$, there exists $\gamma > 0$ such that the following bound holds for $q \in (1 - \gamma, 1 + \gamma)$.*

$$\left| \sum_{m>j+k-d} \frac{\nu_b^m}{m!} \alpha_{q,b}^m (1-q)^{d+m-j-k} \sum_{i=0}^{j} \frac{\binom{j}{i}_{q^b}\binom{k}{i}_{q^b}}{\binom{d}{i}_{q^b}} q^{bi(i-1)/2}(-1)^i(i)_{q^{-b}}^m \right| \leq c_0 c_1 \sum_{m=1}^{\infty} c_2^m |1-q|^m$$

Here, c_0, c_1, c_2 are independent of q.

Proof. Fix $d + m - j - k > 0$ with $j \leq k \leq d$ and $q \approx 1$. We have the following bound, where c_0 is some positive constant independent of q:

$$\left| \sum_{i=0}^{j} \frac{\binom{j}{i}_{q^b}\binom{k}{i}_{q^b}}{\binom{d}{i}_{q^b}} q^{bi(i-1)/2}(-1)^i(i)_{q^{-b}}^m \right| \leq \sum_{i=0}^{j} c_0(i+\delta)^m \leq c_0(d+\delta)^{m+1}$$

For $|x| \leq M$, we have:

$$|\nu_b^m| = |x(x+b) \cdots (x + (m-1)b)| \leq |M(M+b) \cdots (M + (m-1)b)| \leq m!(M+b)^m$$

This then implies the following bound on the tail. Let $c_1 := (d+\delta)\left[(1+\delta)(M+b)(d+\delta)\right]^{j+k-d}$ and $c_2 := (1+\delta)(M+b)(d+\delta)$, where small $\delta > 0$ is needed to deal with limiting details.

$$\left| \sum_{m=j+k+1-d}^{\infty} \frac{\nu_b^m}{m!} \alpha_{q,b}^m (1-q)^{d+m-j-k} \sum_{i=0}^{j} \frac{\binom{j}{i}_{q^b} \binom{k}{i}_{q^b}}{\binom{d}{i}_{q^b}} q^{bi(i-1)/2} (-1)^i (i)_{q^{-b}}^m \right|$$

$$\leq \sum_{m=j+k+1-d}^{\infty} \left| \frac{\nu_b^m}{m!} \alpha_{q,b}^m (1-q)^{d+m-j-k} \right| c_0 (d+\delta)^{m+1}$$

$$\leq c_0 \sum_{m=j+k+1-d}^{\infty} (1+\delta)^m (M+b)^m |1-q|^{d+m-j-k} (d+\delta)^{m+1}$$

$$\leq c_0 c_1 \sum_{m=j+k+1-d}^{\infty} \left[(1+\delta)(M+b)(d+\delta)|1-q| \right]^{d+m-j-k}$$

$$= c_0 c_1 \sum_{m=1}^{\infty} c_2^m |1-q|^m$$

So, for any $\epsilon > 0$ we can select q close enough to 1 such that $|1 - q| < \frac{\epsilon}{c_2}$. This implies the above series is geometric with terms bounded by ϵ^m.

$\square$

The above lemma in particular demonstrates that the tail of the series in Lemma 4.2.15 converges to 0 uniformly on compact sets. With this, we can now complete the proof of the theorem.

Proof of Theorem 4.1.5. For $j, k, d \in \mathbb{N}_0$ such that $0 \leq j \leq k \leq d$, we can combine the above results. When we expand our limit as a sum of powers of $(1-q)$, we have shown that everything limits to zero except for the constant term. Recall that $\lim_{q \to 1} \alpha_{q,b} = -1$.

$$\lim_{q \to 1} (1-q)^d \left[v_{q,b}^j \boxtimes_{q^b}^d v_{q,b}^k \right] (q^x) = \lim_{q \to 1} \frac{\nu_b^{j+k-d} \alpha_{q,b}^{j+k-d}}{(j+k-d)!} (-1)^{j+k-d} q^{b\binom{d}{2} - b\binom{j}{2} - b\binom{k}{2}} \frac{(j)_{q^b}! (k)_{q^b}!}{(d)_{q^b}!}$$

$$= \frac{j! k!}{d!(j+k-n)!} \nu_b^{j+k-d}$$

$$= \nu_b^j \boxplus_b^d \nu_b^k$$

By symmetry, this demonstrates the desired result on a basis. Therefore, the proof is complete.

$\square$

Applications To Previous Results

The main motivation for the multiplicative to additive convolution connection was to be able to relate seemingly analogous root information results. The following table outlines the results we proceed to connect.

Additive Convolution	Multiplicative Convolution
Preserves Real Rooted Polynomials	Preserves Positive Rooted Polynomials
Additive Max Root Triangle Inequality	Multiplicative Max Root Triangle Inequality
Preserves $b-$Mesh	Preserves $q-$Logarithmic Mesh

All of these connections have a similar flavor, and rely on the following elementary facts about exponential polynomials. We say $f(x) = \sum_{k=0}^{d} c_k q^{kx}$ is an *exponential polynomial of degree d with base q*. A real number x is a root of f if and only if q^x is a root of $\sum_{k=0}^{d} c_k x^k$. Because of this we can bootstrap the fundamental theorem of algebra.

Definition 4.2.19. We call $\left\{ x \in \mathbb{C} : \frac{-\pi}{|\ln(q)|} < \Im(x) < \frac{\pi}{|\ln(q)|} \right\}$ the *principal strip* (with respect to q). Let $p(q^x)$ be an exponential polynomial of degree n with base q. The number of roots of $p(q^x)$ in the principal strip is the same as the number of roots of p in $\mathbb{C} \setminus (-\infty, 0]$. We call the roots in the principal strip the *principal roots*.

Lemma 4.2.20. *The principal roots of $E_{q,b}(p)[q^x]$ converge to the roots of p as $q \to 1$. In particular, $E_{q,b}(p)[q^x]$ has $\deg(p)$ principal roots for $q \approx 1$.*

Proof. This follows from the fact that, as $q \to 1$, $E_{q,b}(p)[q^x]$ converges uniformly on compact sets to p and the principal strip grows towards the whole plane. $\qquad \square$

We can analyze the behavior of this convergence when p is real rooted with distinct roots.

Lemma 4.2.21. *Suppose p is real with real distinct roots. For $q \approx 1$, we have that $E_{q,b}(p)[q^x]$ has principal roots which are real and distinct (and converging to the roots of p).*

Proof. Since p has real coefficients, the roots of $E_{q,b}(p)[q^x]$ are either real or come in conjugate pairs. (Consider the fact that $q^{\bar{x}} = \overline{q^x}$.) If p has real distinct roots, the previous lemma implies the principal roots of $E_{q,b}(p)[q^x]$ have distinct real part for q close enough to 1. Therefore, the principal roots of $E_{q,b}(p)[q^x]$ must all be real. $\qquad \square$

If we exponentiate (with base q) the principal roots of $E_{q,b}(p)[q^x]$, we get the roots of $E_{q,b}(p)$. So if the principal roots of $E_{q,b}(p)[q^x]$ are real, then the roots of $E_{q,b}(p)$ are positive. Considering the above results, this means that $E_{q,b}$ maps polynomials with distinct real roots to polynomials with distinct positive roots for $q \approx 1$. (In fact, the roots will be near 1.)

Root Preservation

The most classical results about the roots are the following:

Theorem (Root Preservation)**.**

- *If $p, r \in \mathbb{R}^d[x]$ have positive roots, then $p \boxtimes^d r$ has positive roots.*

- *If $p, r \in \mathbb{R}^d[x]$ have real roots, then $p \boxplus^d r$ has real roots.*

Neither of these results are particularly hard to prove, but showing how the additive result follows from the multiplicative serves as a prime example of how our theorem connects results on the roots.

Proof of Additive from Multiplicative. We can reduce to showing that the additive convolution preserves real rooted polynomials with distinct roots since the closure of polynomials with distinct real roots is all real rooted polynomials.

By Lemma 4.2.21, the roots of $E_{q,0}(p)$ are real, distinct, and exponentials of the principal roots of $E_{q,0}[p](q^x)$ for $q \approx 1$. This implies that $E_{q,0}(p)$ has positive real roots. By the multiplicative result, $E_{q,0}(p) \boxtimes^d E_{q,0}(r)$ has positive real roots, and therefore $[E_{q,0}(p) \boxtimes^d E_{q,0}(r)](q^x)$ has real principal roots. By the main result of this section, $(1-q)^d[E_{q,0}(p) \boxtimes^d E_{q,0}(r)](q^x)$ converges to $p \boxplus^d r$. The real-rootedness of $[E_{q,0}(p) \boxtimes^d E_{q,0}(r)](q^x)$ for $q \approx 1$ then implies $p \boxplus^d r$ is real-rooted. $\qquad \square$

Triangle Inequality

The next classical theorem relates to the max root of a given polynomial. Given an exponential polynomial f with principal roots all real, let $\lambda_1(f)$ denote the largest principal root of f. Also, denote $\exp_q(\alpha) := q^\alpha$.

Theorem (Triangle Inequalities)**.**

- *Given positive-rooted polynomials p, r we have $\lambda_1(p \boxtimes^d r) \leq \lambda_1(p) \cdot \lambda_1(r)$*

- *Given real-rooted polynomials p, r we have $\lambda_1(p \boxplus^d r) \leq \lambda_1(p) + \lambda_1(r)$*

As before, neither of these have particularly complicated proofs, but we can use the multiplicative result to deduce the additive result in the following.

Proof of Additive from Multiplicative. As in the previous proof, we can reduce to showing that the result holds for p, r with distinct roots. For this proof, we only consider $q > 1$.

By Lemma 4.2.21, we have that the roots of $E_{q,0}(p)$ are real, distinct, and exponentials of the principal roots of $E_{q,0}[p](q^x)$ for $q \approx 1$. This implies the roots of $E_{q,0}(p)$ are positive for $q \approx 1$. Additionally, notice that $\exp_q(\lambda_1(f(q^x))) = \lambda_1(f(p))$ whenever f is positive-rooted.

From the multiplicative result and the fact that $\boxtimes^d$ preserves positive-rootednes, we have the following for $q \approx 1$:

$$
\begin{aligned}
\exp_q(\lambda([E_{q,0}(p) \boxtimes^d E_{q,0}(r)](q^x))) &= \lambda(E_{q,0}(p) \boxtimes^d E_{q,0}(r)) \\
&\leq \lambda(E_{q,0}(p)) \cdot \lambda(E_{q,0}(r)) \\
&= \exp_q(\lambda(E_{q,0}[p](q^x)) + \lambda(E_{q,0}[r](q^x)))
\end{aligned}
$$

Therefore, $\lambda([E_{q,0}(p) \boxtimes^d E_{q,0}(r)](q^x)) \leq \lambda(E_{q,0}[p](q^x)) + \lambda(E_{q,0}[r](q^x))$. By the main result of this section, $(1-q)^d[E_{q,0}(p) \boxtimes^d E_{q,0}(r)](q^x)$ converges to $p \boxplus^d r$, and therefore $\lambda([E_{q,0}(p) \boxtimes^d E_{q,0}(r)](q^x))$ converges to $\lambda(p \boxplus^d r)$. Similarly $\lambda(E_{q,0}[p](q^x))$ converges to $\lambda(p)$, and the result follows. $\qquad\square$

Application to Mesh Preservation Conjecture

Recall the log mesh result of Lamprecht in [39] regarding the q-multiplicative convolution.

Theorem 4.1.2. *Fix $q > 1$. Given positive-rooted polynomials $p, r \in \mathbb{R}^d[x]$ with $\mathrm{lmesh}(p), \mathrm{lmesh}(r) \geq q$, we have:*

$$
\mathrm{lmesh}(p \boxtimes_q^d r) \geq q
$$

In [10], Brändén, Krasikov, and Shapiro conjectured the analogous result for the b-additive convolution (for $b = 1$). Using our connection we will confirm this conjecture:

Theorem 4.1.3. *Given real-rooted polynomials $p, r \in \mathbb{R}^d[x]$ with $\mathrm{mesh}(p), \mathrm{mesh}(r) \geq b$, we have:*

$$
mesh(p \boxplus_b^d r) \geq b
$$

Proof. We will prove this claim for polynomials p, r with $\mathrm{mesh}(p), \mathrm{mesh}(r) > b$. Since we can approximate any polynomial with $\mathrm{mesh}(p) = b$ by polynomials with larger mesh, the result then follows.

By Lemma 4.2.21, $E_{q,b}[p](q^x)$ has real roots which converge to the roots of p for $q \approx 1$. Since the roots of p satisfy $\mathrm{mesh}(p) > b$, the principal roots of $E_{q,b}[p](q^x)$ will have mesh greater than b for $q \approx 1$. Further, $\mathrm{lmesh}(E_{q,b}(p)) = \exp_q(\mathrm{mesh}(E_{q,b}[p](q^x))) > q^b$. (All of this discussion holds for r as well.) By the main result of this section, we have:

$$
\lim_{q \to 1}(1-q)^d \left[E_{q,b}(p) \boxtimes_{q^b}^d E_{q,b}(r) \right](q^x) = p \boxplus_b^d r
$$

By the previous theorem, the q^b-multiplicative convolution of $E_{q,b}(p)$ and $E_{q,b}(r)$ has logarithmic mesh at least q^b. Precomposition by q^x then yields an exponential polynomial with mesh (of the principal roots) at least b. The principal roots of this exponential polynomial then converge to $p \boxplus_b^d r$, and hence $p \boxplus_b^d r$ has mesh at least b. $\qquad\square$

4.3 Second Proof Method: A Direct Proof using Interlacing

While the previous framework generically transferred Lamprecht's multiplicative result to prove the conjectured result in the additive realm, one might desire a direct proof to gain insight on the underlying structure of the convolution. In what follows, we first outline the preliminary knowledge required to understand a special case of Lamprecht's argument. Then we outline his approach in the multiplicative case and extend this approach to the additive realm to prove the desired conjecture. Recall the equivalence between mesh and interlacing, as discussed in Remark 2.2.11.

Remark 4.3.1. A polynomial f with non-negative roots is q-log mesh if and only if $f \ll f(q^{-1}x)$ and strictly q-log mesh if and only if $f \ll f(q^{-1}x)$ strictly (for $q > 1$). Similarly, a polynomial f with real roots is b-mesh if and only if $f \ll f(x - b)$ and strictly b-mesh if and only if $f \ll f(x - b)$ strictly (for $b > 0$).

Lamprecht's Approach

In what follows, we follow Lamprecht's approach to proving that the space of q-log mesh polynomials is preserved by the q-mutliplicative convolution. Here, we are only interested in proving this result for q-log mesh polynomials with non-negative roots, which simplifies the proof. (Lamprecht demonstrates this result for a more general class of polynomials.) The main structure of the proof is: (1) establish properties of two distinguished polar derivatives, (2) show how these derivatives relate to the q-multiplicative convolution, and (3) use this to prove that the q-multiplicative convolution preserves certain interlacing properties. In the next section, we will emulate this method for b-mesh polynomials and the b-additive convolution.

q-Polar Derivatives

In [39], Lamprecht defines q-derivative operators, which generalize the operators $\frac{1}{d}\partial_x$ and $-\frac{1}{d}\partial_x^*$ as discussed in §2.1. (As a note, Lamprecht uses the Δ symbol for these derivatives, and actually gives different definitions as his convention is $q \in (0, 1)$.) Recall these operations from §2.1, where $q > 1$ is always assumed.

$$(\partial_{q,d}f)(x) := \frac{f(qx) - f(x)}{q^{1-d}(q^d - 1)x} \qquad (\partial_{q,d}^*f)(x) := \frac{f(qx) - q^d f(x)}{q^d - 1}$$

He then goes on to show that these "derivative" operators have similar preservation properties to that of the usual derivatives. In particular, he obtains the following.

Proposition 4.3.2. *The operators $\partial_{q,d} : \mathbb{R}^d[x] \to \mathbb{R}^{d-1}[x]$ and $\partial_{q,d}^* : \mathbb{R}^d[x] \to \mathbb{R}^{d-1}[x]$ preserve the space of q-log mesh polynomials and the space of strictly q-log mesh polynomials. Further, we have that $\partial_{q,d}f \ll f$ and $\partial_{q,d}^*f \ll \partial_{q,d}f$.*

The above result is actually spread across a number of results in Lamprecht's paper. We omit the proof for now, referring the reader to Section 6 in [39], mainly Theorems 25 and 28 (in the arXiv version). Note also that our definitions of $\partial_{q,d}$ and $\partial_{q,d}^*$ are slightly different than that of Lamprecht.

Recursive Identities

Lamprecht then determines the following identities, which are crucial to his inductive proof of the main result of this section. Fix $f \in \mathbb{R}^{d-1}[x]$ and $g \in \mathbb{R}^d[x]$.

$$f \boxtimes_q^d g = f \boxtimes_q^{d-1} \partial_{q,d}^* g \qquad\qquad (xf) \boxtimes_q^d g = x(f \boxtimes_q^{d-1} \partial_{q,d} g)$$

Lamprecht's Proof

With this, we now state an interesting result about interlacing preservation of the q-convolution operator. We will then derive the main result as a corollary.

Theorem 4.1.6 (Lamprecht Interlacing-Preserving)**.** *Let $f, g \in \mathbb{R}^d[x]$ be q-log mesh polynomials of degree d with only positive roots. Let $T_g : \mathbb{R}^d[x] \to \mathbb{R}^d[x]$ be the real linear operator defined by $T_g : r \mapsto r \boxtimes_q^d g$. Then, T_g preserves interlacing with respect to f.*

Proof. We prove the theorem by induction. For $d = 1$ the result is straightforward, as $\boxtimes_q^1 \equiv \boxtimes^1$. For $m > 1$, we inductively assume that the result holds for $d = m-1$. By Corollary 2.3.9 and the fact that f has d simple roots, we only need to show that $T_g[f_{\alpha_k}] \ll T_g[f]$ for all roots α_k of f. That is, we want to show $f_{\alpha_k} \boxtimes_q^m g \ll f \boxtimes_q^m g$ for all k.

By Proposition 4.3.2, we have that $\partial_{q,m} g$ and $\partial_{q,m}^* g$ are q-log mesh and $\partial_{q,m}^* g \ll \partial_{q,m} g$. Further, $\partial_{q,m} g$ and $\partial_{q,m}^* g$ are of degree $m-1$ and have no roots at 0. The inductive hypothesis and symmetry of $\boxtimes_q^d$ then imply:

$$f_{\alpha_k} \boxtimes_q^{m-1} \partial_{q,m}^* g \ll f_{\alpha_k} \boxtimes_q^{m-1} \partial_{q,m} g$$

The fact that these polynomials have leading coefficients with the same sign means that the max root of $f_{\alpha_k} \boxtimes_q^{m-1} \partial_{q,m} g$ is larger than that of $f_{\alpha_k} \boxtimes_q^{m-1} \partial_{q,m}^* g$. Further, since all roots are positive we obtain:

$$f_{\alpha_k} \boxtimes_q^{m-1} \partial_{q,m}^* g \ll x(f_{\alpha_k} \boxtimes_q^{m-1} \partial_{q,m} g)$$

By properties of $\ll$, this gives:

$$f_{\alpha_k} \boxtimes_q^{m-1} \partial_{q,m}^* g \ll x(f_{\alpha_k} \boxtimes_q^{m-1} \partial_{q,m} g) - \alpha_k(f_{\alpha_k} \boxtimes_q^{m-1} \partial_{q,m}^* g)$$

By the above identities and the fact that $f(x) = (x - \alpha_k)f_{\alpha_k}(x)$, this is equivalent to $f_{\alpha_k} \boxtimes_q^m g \ll f \boxtimes_q^m g$. $\square$

Corollary 4.1.2. *Let $f, g \in \mathbb{R}^d[x]$ be q-log mesh polynomials (with non-negative roots), not necessarily of degree d. Then, $f \boxtimes_q^d g$ is q-log mesh.*

Proof. First suppose f, g are of degree d with only positive roots. Since $f \ll f(q^{-1}x)$, the previous theorem implies:

$$f \boxtimes_q^d g \ll f(q^{-1}x) \boxtimes_q^d g = (f \boxtimes_q^d g)(q^{-1}x)$$

That is, $f \boxtimes_q^d g$ is q-log mesh.

Otherwise, suppose f is of degree $m_f \leq d$ with z_f roots at 0 and g is of degree $m_g \leq d$ with z_g roots at zero. Intuitively, we now add roots "near 0 and ∞" and limit. Let new polynomials F and G be given as follows:

$$F(x) := f(x) \cdot x^{-z_f} \prod_{j=1}^{z_f} \left(x - \frac{1}{\alpha_j} \right) \cdot \prod_{j=m_f+1}^{d} \left(\frac{x}{\alpha_j} - 1 \right)$$

$$G(x) := g(x) \cdot x^{-z_g} \prod_{j=1}^{z_g} \left(x - \frac{1}{\beta_j} \right) \cdot \prod_{j=m_g+1}^{d} \left(\frac{x}{\beta_j} - 1 \right)$$

Here, α_j and β_j are any large positive numbers such that F and G are q-log mesh polynomials of degree d. By the previous argument, $F \boxtimes_q^d G$ is q-log mesh. Letting α_j and β_j limit to ∞ (while preserving q-log mesh) implies $F \boxtimes_q^d G \to f \boxtimes_q^d g$ root-wise, which implies $f \boxtimes_q^d g$ is q-log mesh. $\square$

Lamprecht is actually able to remove the degree d with positive roots restriction earlier in the line of argument, albeit at the cost of a more complicated proof. We have elected here to take the simpler route. He also proves similar results for a class of q-log mesh polynomials with possibly negative roots, which we omit here.

b-Additive Convolution

The main structure of Lamprecht's argument revolves around the two "polar" q-derivatives, $\partial_{q,d}$ and $\partial_{q,d}^*$. The key properties of these derivatives are: (1) they preserve the space of q-log mesh polynomials, and (2) they recursively work well with the definition of the q-multiplicative convolution. So, when extending this argument to the b-additive convolution we face an immediate problem: there is only one natural derivative which preserves the space of b-mesh polynomials. This stems from the fact that 0 and ∞ have special roles in the q-multiplicative world, whereas only ∞ is special in the b-additive world. The key idea we introduce then is that given a fixed b-mesh polynomial f, we can pick a polar derivative with pole "close enough to ∞" so that it maps f to a b-mesh polynomial. The fact that we use a different polar derivative for each fixed input f does not affect the proof method.

We now give a few facts about the finite difference operator Δ_b, which plays a crucial role in the definition of the b-additive convolution. Recall its definition:

$$(\Delta_{b,d}f)(x) \equiv (\Delta_b f)(x) := \frac{f(x) - f(x - b)}{b}$$

(We use the notation $\Delta_{b,d}$ when we want to restrict the domain to $\mathbb{R}^d[x]$, as in Proposition 4.3.3 below.) This operator acts on rising factorial polynomials as the usual derivative acts on monomials. That is, for all k:

$$\Delta_b x(x+b)\cdots(x+(k-1)b) = kx(x+b)\cdots(x+(k-2)b)$$

This operator has preservation properties similar to that of the usual derivative and the q-derivatives. The following result, along with many others regarding mesh and log-mesh polynomials, can be found in Chapter 8 of [24].

Proposition 4.3.3 ([24], Lemma 8.8). *The operator $\Delta_{b,d} : \mathbb{R}^d[x] \to \mathbb{R}^{d-1}[x]$ preserves the space of b-mesh polynomials, and the space of strictly b-mesh polynomials. Further, we have $\Delta_b f \ll f$ and $\Delta_b f \ll f(x-b)$. If f is strictly b-mesh, then these interlacings are strict.*

Finding Another Polar Derivative

We now define another "derivative-like" operator that is meant to generalize ∂_d^* and $\partial_{q,d}^*$ (see §2.1). Notice that unlike Δ_b, this operation depends on d.

$$(\Delta_{b,d}^* f)(x) := df(x-b) - (x-b)\Delta_b f(x)$$

Unfortunately, this operator does *not* preserve b-mesh. However, it does generalize other important properties of $\partial_d^* = \partial_{1,d}^*$. In particular, it maps $\mathbb{R}^d[x]$ to $\mathbb{R}^{d-1}[x]$, and as $b \to 0$ it limits to $\partial_d^* f$, the polar derivative of f with respect to 0. Further, we have the following results.

Lemma 4.3.4. *Fix $f \in \mathbb{R}^d[x]$ and write $f = \sum_{k=0}^d a_k x(x+b)\cdots(x+(k-1)b)$. Then:*

$$(\Delta_{b,d}^* f)(x+b) = \sum_{k=0}^{d-1}(d-k)a_k x(x+b)\cdots(x+(k-1)b)$$

This next lemma is a generalization of the corollary following it.

Lemma 4.3.5. *Fix monic polynomials $f, g \in \mathbb{R}[x]$ of degree m and $m-1$, respectively, such that g is strictly b-mesh and $g \ll f$ strictly. Denote $h_{a,t}(x) := af(x) - (x-t)g(x)$ for $a \geq 1$ and $t > 0$. For all t large enough, we have $g \ll h_{a,t}$ strictly, $h_{a,t} \ll f$ strictly, and $h_{a,t}$ is strictly b-mesh.*

Proof. Denote $h_{a,t}(x) := af(x) - (x-t)g(x)$. Since f, g are monic, we have that $h_{a,t}$ is of degree at most m with positive leading coefficient (for large t if $a = 1$). Further, if $\alpha_1 < \cdots < \alpha_{m-1}$ are the roots of g and $\beta_1 < \cdots < \beta_m$ are the roots of f, then $g \ll f$ strictly and t large implies:

$$
\begin{array}{ll}
h_{a,t}(\alpha_{m-1}) = af(\alpha_{m-1}) < 0 & \qquad h_{a,t}(\beta_m) = -(\beta_m - t)g(\beta_m) > 0 \\
h_{a,t}(\alpha_{m-2}) = af(\alpha_{m-2}) > 0 & \qquad h_{a,t}(\beta_{m-1}) = -(\beta_{m-1} - t)g(\beta_{m-1}) < 0 \\
h_{a,t}(\alpha_{m-3}) = af(\alpha_{m-3}) < 0 & \qquad h_{a,t}(\beta_{m-2}) = -(\beta_{m-2} - t)g(\beta_{m-2}) > 0 \\
\qquad\qquad \vdots & \qquad\qquad\qquad \vdots
\end{array}
$$

The alternating signs imply $h_{a,t}$ has an odd number of roots in the interval (α_k, β_{k+1}) and an even number of roots in the interval (β_k, α_k) for all $1 \le k \le m-1$. Since the degree of $h_{a,t}$ is at most m, each of these intervals must contain exactly one root and zero roots, respectively. If $h_{a,t}$ is of degree m, then it has one more root which must be real since $h_{a,t} \in \mathbb{R}[x]$. Additionally, since $h_{a,t}$ has positive leading coefficient, this last root must lie in the interval $(-\infty, \beta_1)$ (and not in (β_m, ∞)). Therefore, $g \ll h_{a,t}$ strictly and $h_{a,t} \ll f$ strictly.

Finally, $h_{a,t} \to g$ as $t \to \infty$ coefficient-wise, and so therefore also in terms of the zeros. This means that the root in the interval (α_k, β_{k+1}) will limit to α_k from above (for all k). Further, the possible root in the interval $(-\infty, \beta_1)$ will then limit to $-\infty$, as g is of degree $m-1$. Since g is strictly b-mesh, this implies $h_{a,t}$ is also strictly b-mesh for large enough t. $\qquad\square$

Corollary 4.3.6. *Let $f \in \mathbb{R}^d[x]$ be strictly b-mesh. Then for all $t > 0$ large enough, we have that $(t\Delta_{b,d} + \Delta_{b,d}^*)f$ is strictly b-mesh and $\Delta_{b,d}f \ll (t\Delta_{b,d} + \Delta_{b,d}^*)f$ strictly.*

Proof. Consider $(t\Delta_{b,d} + \Delta_{b,d}^*)f = df(x - b) - (x - b - t)\Delta_{b,d}f$. Note that $\Delta_{b,d}f \in \mathbb{R}^{d-1}[x]$ is strictly b-mesh and of degree one less than f, and $\Delta_{b,d}f \ll f(x - b)$ strictly by Proposition 4.3.3. Now assume WLOG that f is monic and of degree at least 1. Letting c denote the leading coefficient of $\Delta_{b,d}f$, we have $1 \le c \le d$. We can then write:

$$\frac{1}{c}(t\Delta_{b,d} + \Delta_{b,d}^*)f = \frac{d}{c}f(x - b) - (x - b - t)\frac{\Delta_{b,d}f}{c}$$

Applying the previous lemma to $f(x - b)$ and $\frac{\Delta_{b,d}f}{c}$ with $a = \frac{d}{c}$ gives the result. $\qquad\square$

This corollary says that $t\Delta_{b,d} + \Delta_{b,d}^*$ preserves b-mesh, even though $\Delta_{b,d}^*$ does not. The operator $t\Delta_{b,d} + \Delta_{b,d}^*$ can be thought of as the polar derivative with respect to t, since by limiting $b \to 0$ we obtain the classical polar derivative.

Recursive Identities

The $\Delta_{b,d}^*$ operator is also required to obtain b-additive convolution identities similar to Lamprecht's given above.

Lemma 4.3.7. *Fix $f \in \mathbb{R}^{d-1}[x]$ and $g \in \mathbb{R}^d[x]$. We have:*

$$f \boxplus_b^d g = f \boxplus_b^{d-1} \Delta_{b,d}g \qquad\qquad (xf) \boxplus_b^d g = x(f \boxplus_b^{d-1} \Delta_{b,d}g) + f \boxplus_b^{d-1} \Delta_{b,d}^*g$$

Proof. The first identity is straightforward from the definition of $\boxplus_b^d$. As for the second, we compute:

$$\Delta_b^k(xf) = \Delta_b^{k-1}(x\Delta_b f + f(x - b)) = \cdots = x\Delta_b^k f + k\Delta_b^{k-1}f(x - b)$$

Notice that Δ_b commutes with shifting, so this is unambiguous. This implies:

$$(xf) \boxplus_b^d g = \sum_{k=0}^{d} (x\Delta_b^k f + k\Delta_b^{k-1} f(x-b)) \cdot (\Delta_b^{d-k} g)(0)$$

$$= x(f \boxplus_b^{d-1} \Delta_{b,d} g) + \sum_{k=1}^{d} k\Delta_b^{k-1} f(x-b) \cdot (\Delta_b^{d-k} g)(0)$$

$$= x(f \boxplus_b^{d-1} \Delta_{b,d} g) + \sum_{k=0}^{d-1} \Delta_b^{d-1-k} f(x-b) \cdot ((d-k)\Delta_b^k g)(0)$$

$$= x(f \boxplus_b^{d-1} \Delta_{b,d} g) + f(x-b) \boxplus_b^{d-1} (\Delta_{b,d}^* g)(x+b)$$

The last step of the above computation uses Lemma 4.3.4 and the fact that $(\Delta_b^k g)(0)$ picks out the coefficient corresponding to the kth rising factorial term. Finally:

$$f(x-b) \boxplus_b^{d-1} (\Delta_{b,d}^* g)(x+b) = (f \boxplus_b^{d-1} (\Delta_{b,d}^* g)(x+b))(x-b) = f \boxplus_b^{d-1} \Delta_{b,d}^* g$$

This implies the second identity. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

With this we can now emulate Lamprecht's proof to prove interlacing preserving properties of the b-additive convolution.

Lamprecht-Style Proof

Theorem 4.1.7. *Let $f, g \in \mathbb{R}^d[x]$ be strictly b-mesh polynomials of degree d. Let $T_g : \mathbb{R}^d[x] \to \mathbb{R}^d[x]$ be the real linear operator defined by $T_g : r \mapsto r \boxplus_b^d g$. Then, T_g preserves interlacing with respect to f.*

Proof. We prove the theorem by induction. For $d = 1$ the result is straightforward, as $\boxplus_b^1 \equiv \boxplus^1$. For $m > 1$, we inductively assume that the result holds for $d = m - 1$. By Corollary 2.3.9, we only need to show that $T_g[f_{\alpha_k}] \ll T_g[f]$ for all roots α_k of f. That is, we want to show $f_{\alpha_k} \boxplus_b^m g \ll f \boxplus_b^m g$ for all k.

By Proposition 4.3.3 and Corollary 4.3.6, we have that $\Delta_{b,m} g$ and $(t\Delta_{b,m} + \Delta_{b,m}^*) g$ are strictly b-mesh and $\Delta_{b,m} g \ll (t\Delta_{b,m} + \Delta_{b,m}^*) g$ strictly for large enough t. Further, $\Delta_{b,m} g$ and $(t\Delta_{b,m} + \Delta_{b,m}^*) g$ are of degree $m - 1$. The inductive hypothesis and symmetry of $\boxplus_b^d$ then imply:

$$f_{\alpha_k} \boxplus_b^{m-1} \Delta_{b,m} g \ll f_{\alpha_k} \boxplus_b^{m-1} (t\Delta_{b,m} + \Delta_{b,m}^*) g$$

It is easy to see, (e.g. from 2.3.7)

$$f_{\alpha_k} \boxplus_b^{m-1} \Delta_{b,m} g \ll (x - \alpha_k - t)(f_{\alpha_k} \boxplus_b^{m-1} \Delta_{b,m} g)$$

By properties of $\ll$, this gives:

$$f_{\alpha_k} \boxplus_b^{m-1} \Delta_{b,m} g \ll (x - \alpha_k - t)(f_{\alpha_k} \boxplus_b^{m-1} \Delta_{b,m} g) + f_{\alpha_k} \boxplus_b^{m-1} (t\Delta_{b,m} + \Delta_{b,m}^*) g$$

$$= (x - \alpha_k)(f_{\alpha_k} \boxplus_b^{m-1} \Delta_{b,m} g) + f_{\alpha_k} \boxplus_b^{m-1} \Delta_{b,m}^* g$$

By the above identities and the fact that $f(x) = (x - \alpha_k)f_{\alpha_k}(x)$, this is equivalent to $f_{\alpha_k} \boxplus_b^m g \ll f \boxplus_b^m g$. $\qquad\square$

Corollary 4.1.3. *Let $f, g \in \mathbb{R}^d[x]$ be strictly b-mesh polynomials. Then, $f \boxplus_b^d g$ is b-mesh.*

Proof. First suppose f, g are of degree d. Since $f \ll f(x - b)$ strictly, the previous theorem implies the following:

$$f \boxplus_b^d g \ll f(x - b) \boxplus_b^d g = (f \boxplus_b^d g)(x - b)$$

That is, $f \boxplus_b^d g$ is b-mesh.

If f, g are not of degree d, then the result follows by adding new roots and limiting them to ∞, in a fashion similar to the proof of Corollary 4.1.2 given above. $\qquad\square$

4.4   Concluding Remarks

Extensions of other classical convolution results

In this chapter we investigated connections between the additive and multiplicative convolutions and their mesh generalizations. Looking forward, it is natural to look at other results in the classical case and ask for mesh generalizations. To our knowledge, there are two classical results which have been extended to mesh analogues: in [10], the authors explore extensions of the Hermite-Poulain theorem to the 1-mesh world, and in [39], Lamprecht extends classical results for the multiplicative convolution to the q-log mesh world.

An important related result in the classical case is the triangle inequality, which we discuss in §4.2 and in the previous chapter. To our knowledge, there is not a known generalization of the triangle inequality to the mesh and log mesh cases. If one could establish such a result for the q-multiplicative convolution, it would automatically extend to the b-additive convolution using our analytic connection. Establishing this is the first step towards potentially getting a grasp on b-additive submodularity.

Extensions of other q-multiplicative convolution results

In addition to log mesh preservation, Lamprecht proves other results about the q-multiplicative convolution. Here we comment on these and their relation to the mesh world.

Beyond the finite degree case, Lamprecht discusses the extension of Laguerre-Polya functions to the q-multiplicative world, and then establishes a q-version of Polya-Schur multiplier sequences via a power series convolution. Since we are not aware of analogous power series results for the classical additive convolution, we have not explored the connections to the b-additive case.

Additionally, Lamprecht classifies log-concave sequences in terms of q-log mesh polynomials using the Hadamard product and a limiting argument. There might be an analogue

result in the mesh world for concave sequences, but it is unclear what would take the place of the Hadamard product.

Lamprecht details the classes of polynomials that the q-multiplicative convolution preserves. Most of these results come from the presence of two poles in the q-multiplicative case, yielding derivative operators which preserve negative- and positive-rootedness respectively. The b-additive case does not have such complications. In our simplification of Lamprecht's argument, we assume the input polynomials to be generic (strictly b-mesh), and then limit to obtain the result for all b-mesh polynomials. By keeping track of boundary case information, Lamprecht is able to get more precise results about boundary elements of the space of b-mesh polynomials. We believe it is likely possible to emulate this in the above proof with more bookkeeping.

The analytic connection applied to other known classical results

There are other results known about the classical multiplicative convolution which we believe could be transferred to the additive convolution using our generic framework. Specifically in [47], Marcus, Spielman, and Srivastava establish a refinement of the triangle inequality for both the additive and multiplicative convolutions. These refinements parallel the well studied transforms from free probability theory. We have not yet worked out the details of this connection.

Further directions for the generic analytic connection

Finally, it is worth nothing that our analytic connection can only transfer results about the multiplicative convolution to the additive convolution. The main obstruction is finding the appropriate analogue to the exponential map. The following limiting connection between exponential polynomials and polynomials motivated our investigation:

$$\lim_{q \to 1} \frac{1 - q^x}{1 - q} = x$$

Finding the appropriate "logarithmic analogue" could yield a way to pass results from the additive convolution to the multiplicative convolution. That said, some heuristic evidence suggests that such an analogue might not exist.

Above all, our analytic connection still remains rather mysterious. We suspect that there exists a more general theory which provides better intuition for this multiplicative-to-additive connection. While developing this connection, we found multiple candidate exponential maps which experimentally worked. We settled on the ones introduced in this chapter due to their relatively nice combinatorial properties. Ideally, an alternative approach would avoid proving the result on a basis and better explain the role of these "exponential maps".

4.5 Computations

Bounding

Fix $n + m - j - k > 0$ with $j \leq k \leq n$ and $q \approx 1$. We have the following bound, where c_0 is some positive constant independent of q:

$$\left| \sum_{i=0}^{j} \frac{\binom{j}{i}_{q^b} \binom{k}{i}_{q^b}}{\binom{n}{i}_{q^b}} q^{bi(i-1)/2} (-1)^i (i)_{q^{-b}}^m \right| \leq \sum_{i=0}^{j} c_0 (i + \delta)^m \leq c_0 (n + \delta)^{m+1}$$

For $|x| \leq M$, we have:

$$|\nu_b^m| = |x(x + b) \cdots (x + (m-1)b)| \leq |M(M+b) \cdots (M + (m-1)b)| \leq m!(M+b)^m$$

This then implies the following bound on the tail. Let $c_1 := (n+\delta)\big[(1+\delta)(M+b)(n+\delta)\big]^{j+k-n}$ and $c_2 := (1+\delta)(M+b)(n+\delta)$, where small $\delta > 0$ is needed to deal with limiting details.

$$\left| \sum_{m=j+k+1-n}^{\infty} \frac{\nu_b^m}{m!} \alpha_{q,b}^m (1-q)^{n+m-j-k} \sum_{i=0}^{j} \frac{\binom{j}{i}_{q^b} \binom{k}{i}_{q^b}}{\binom{n}{i}_{q^b}} q^{bi(i-1)/2} (-1)^i (i)_{q^{-b}}^m \right|$$

$$\leq \sum_{m=j+k+1-n}^{\infty} \left| \frac{\nu_b^m}{m!} \alpha_{q,b}^m (1-q)^{n+m-j-k} \right| c_0 (n+\delta)^{m+1}$$

$$\leq c_0 \sum_{m=j+k+1-n}^{\infty} (1+\delta)^m (M+b)^m |1-q|^{n+m-j-k} (n+\delta)^{m+1}$$

$$\leq c_0 c_1 \sum_{m=j+k+1-n}^{\infty} \big[(1+\delta)(M+b)(n+\delta)|1-q|\big]^{n+m-j-k}$$

$$= c_0 c_1 \sum_{m=1}^{\infty} c_2^m |1-q|^m$$

So, for any $\epsilon > 0$ we can select q close enough to 1 such that $|1 - q| < \frac{\epsilon}{c_2}$. This implies the above series is geometric with max term ϵ^m.

Q-Lagrange Interpolation

Let $[t^j]p(t)$ denote the coefficient of p corresponding to the monomial t^j.

Lemma 4.5.1. *Fix $p \in \mathbb{C}_j[x]$. We have the following identity:*

$$(-1)^j q^{-\binom{j}{2}} \cdot [t^j]p(t) = \sum_{i=0}^{j} p((i)_{q^{-1}}) \frac{(-1)^i}{(i)_q!(j-i)_q!} q^{\binom{i}{2}}$$

Proof. Using Lagrange interpolation, the following holds for any polynomial of degree at most j:

$$[t^j]p(t) = \sum_{i=0}^{j} p((i)_q) \frac{(-1)^{j-i}}{(i)_q!(j-i)_q!} q^{-\binom{i}{2}} q^{-i(j-i)}$$

Using the identity $(i)_{q^{-1}}! = q^{-\binom{i}{2}}(i)_q!$ (via $(i)_{q^{-1}} = q^{-i+1}(i)_q$) and replacing q by q^{-1} gives:

$$[t^j]p(t) = \sum_{i=0}^{j} p((i)_{q^{-1}}) \frac{(-1)^{j-i}}{(i)_q!(j-i)_q!} q^{2\binom{i}{2}} q^{i(j-i)} q^{\binom{j-i}{2}}$$

$$= \sum_{i=0}^{j} p((i)_{q^{-1}}) \frac{(-1)^{j-i}}{(i)_q!(j-i)_q!} q^{\binom{i}{2}} q^{\binom{j}{2}}$$

The result follows.   □

Corollary 4.5.2. *Fix $j, k, m \in \{0, 1, ..., n\}$ such that $j \leq k$ and $m + n - k \leq j$. We have the following identity:*

$$(-1)^{n-j-k} q^{b\binom{n}{2} - b\binom{j}{2} - b\binom{k}{2}} \delta_{m+n-k=j} \frac{(j)_{q^b}(k)_{q^b}}{(n)_{q^b}} = \sum_{i=0}^{j} \frac{\binom{j}{i}_{q^b} \binom{k}{i}_{q^b}}{\binom{n}{i}_{q^b}} q^{b\binom{i}{2}} (-1)^i (i)_{q^{-b}}^m$$

Proof. Consider the polynomial $p(t) = t^m((n)_{q^{-b}} - t)((n-1)_{q^{-b}} - t) \cdots ((k+1)_{q^{-b}} - t)$, which is of degree $m + n - k$. So, $[t^j]p(t) = (-1)^{n-k}\delta_{m+n-k=j}$. Also, recall the identity $(i)_{q^{-b}}! = q^{-b\binom{i}{2}}(i)_{q^b}!$. Using the previous lemma and replacing q by q^b, we obtain:

$$(-1)^j q^{-b\binom{j}{2}} \cdot (-1)^{n-k}\delta_{m+n-k=j} = \sum_{i=0}^{j} p((i)_{q^{-b}}) \frac{(-1)^i}{(i)_{q^b}!(j-i)_{q^b}!} q^{b\binom{i}{2}}$$

$$= \sum_{i=0}^{j} q^{-bi(n-k)} \frac{(n-i)_{q^{-b}}!}{(k-i)_{q^{-b}}!} \frac{(-1)^i}{(i)_{q^b}!(j-i)_{q^b}!} q^{b\binom{i}{2}} (i)_{q^{-b}}^m$$

$$= \sum_{i=0}^{j} q^{b\binom{k}{2}} q^{-b\binom{n}{2}} \frac{(n-i)_{q^b}!}{(k-i)_{q^b}!} \frac{(-1)^i}{(i)_{q^b}!(j-i)_{q^b}!} q^{b\binom{i}{2}} (i)_{q^{-b}}^m$$

$$= \sum_{i=0}^{j} q^{b\binom{k}{2}} q^{-b\binom{n}{2}} \frac{(n)_{q^b}}{(j)_{q^b}(k)_{q^b}} \cdot \frac{\binom{j}{i}_{q^b} \binom{k}{i}_{q^b}}{\binom{n}{i}_{q^b}} q^{b\binom{i}{2}} (-1)^i (i)_{q^{-b}}^m$$

The result follows.   □

Chapter 5

Polynomial Capacity and Bipartite Graphs

In a series of papers (e.g., see [35]), Gurvits gave a vast generalization of the Van der Waerden lower bound for permanents of doubly stochastic matrices and the Schrijver lower bound on the number of perfect matchings of regular graphs. In particular, he proved an inequality on how much the derivative can affect a particular analytic quantity called the capacity of a polynomial, and we now state it formally. Recall the notation $x^\alpha := \prod_k x_k^{\alpha_k}$, which will be heavily used here.

Definition 5.0.1 (Gurvits). Given a polynomial $p \in \mathbb{R}[x_1, ..., x_n]$ with non-negative coefficients and a vector $\alpha \in \mathbb{R}^n$ with non-negative entries, we define the α-*capacity* of p as:

$$\text{Cap}_\alpha(p) := \inf_{x>0} \frac{p(x)}{x^\alpha}$$

Theorem 5.2.1 (Gurvits). *For real stable $p \in \mathbb{R}_+^\gamma[x_1, ..., x_n]$ we have:*

$$\frac{\text{Cap}_{(1^{n-1})}\left(\partial_{x_k} p\big|_{x_k=0}\right)}{\text{Cap}_{(1^n)}(p)} \geq \left(\frac{\gamma_k - 1}{\gamma_k}\right)^{\gamma_k - 1}$$

Recall (1^j) denotes the all-ones vector of length j.

This is as a statement about the capacity preservation properties of the derivative. That is, taking a partial derivative of a real stable polynomial (and then evaluating to 0) can only decrease the capacity of that polynomial by at most the stated multiplicative factor.

To generalize such capacity preservation properties to other real stability preservers, we combine Gurvits' ideas with the Borcea-Brändén characterization (see §2.3). Specifically, we show that the symbol of an operator holds not only the stability preservation information of the operator, but also the capacity preservation information of T. Our main results in this direction are stated as follows. Recall the definitions of Symb and Symb$_+$ (Definition 2.3.1), and of the Laguerre-Pólya class (Definition 2.3.3).

Theorem 5.3.10 (Bounded degree)**.** *Let T be a linear operator taking input in $\mathbb{R}_+^\gamma[x_1, ..., x_n]$, such that $\mathrm{Symb}_+(T)$ is real stable or bistable (recall Definition 2.2.1) with non-negative coefficients. For any real stable $p \in \mathbb{R}_+^\gamma[x]$ and any sensible $\alpha, \beta \in \mathbb{R}_+^n$, we have:*

$$\frac{\mathrm{Cap}_\beta(T(p))}{\mathrm{Cap}_\alpha(p)} \geq \frac{\alpha^\alpha(\lambda - \alpha)^{\lambda - \alpha}}{\lambda^\lambda} \, \mathrm{Cap}_{(\alpha, \beta)}(\mathrm{Symb}_+(T))$$

Further, this bound is tight for fixed T, α, β.

Theorem 5.3.11 (Unbounded degree)**.** *Let T be a linear operator on polynomials of any degree, such that $\mathrm{Symb}_+(T)$ is in the Laguerre-Pólya class or the bistable equivalent (see Definition 2.3.3) with non-negative coefficients. For any real stable $p \in \mathbb{R}_+[x_1, ..., x_n]$ and any sensible $\alpha, \beta \in \mathbb{R}_+^n$, we have:*

$$\frac{\mathrm{Cap}_\beta(T(p))}{\mathrm{Cap}_\alpha(p)} \geq e^{-\alpha}\alpha^\alpha \, \mathrm{Cap}_{(\alpha, \beta)}(\mathrm{Symb}_+(T))$$

Further, this bound is tight for fixed T, α, β.

Our main application of the theory is a new proof of Csikvári's bound on the number of k-matchings of a biregular bipartite graph [17]. This result generalizes Schrijver's inequality and is actually stronger than Friedland's lower matching conjecture (see [25]). The computations involved in this new proof never exceed the level of basic calculus. This was one of the most remarkable features of Gurvits' original result, and this theme continues to play out here. We state Csikvári's result now.

Theorem 5.2.6 (Csikvári)**.** *Let G be an (a, b)-biregular bipartite graph with vertices which are (m, n)-bipartitioned (so that $am = bn$ is the number of edges of G). Then the number of size-k matchings of G is bounded as follows:*

$$\mu_k(G) \geq \binom{n}{k}(ab)^k \frac{m^m(ma - k)^{ma-k}}{(ma)^{ma}(m - k)^{m-k}}$$

Beyond these specific applications, one of the main purposes of this chapter is to unify the various results that fit into the lineage of the concept of capacity. Some of these are inequalities for specific combinatorial quantities ([35], [33], [34]), some are approximation algorithms for those quantities ([2], [60]), and some are capacity preservation results similar to those in this chapter (particularly [1]).

The rest of this chapter is outlined as follows. In §5.2, we discuss applications of the capacity preservation theory. In §5.3, we prove the main inequalities. In §5.4, we discuss some continuity properties of capacity.

5.1 Capacity Basics

Recall the definition of capacity:

$$\mathrm{Cap}_\alpha(p) := \inf_{x>0} \frac{p(x)}{x^\alpha}$$

In general, the conceptual meaning of capacity is not completely understood. However, in this section we hope to illuminate some of its basic features. This will include its connections to the coefficients of a polynomial, to probabilistic interpretations of polynomials, to the AM-GM inequality, and to the Legendre (Fenchel) transformation.

As discussed at the beginning of this chapter, the sort of capacity results we will be interested in are those of capacity preservation (that is, bounds on how much the capacity can change under various operations). In fact, our use of the Borcea-Brändén characterization consists in combining it with capacity bounds in order to give something like a characterization of capacity preservers. In a sense, this can be seen as an analytic refinement of the characterization: not only do such operators preserve stability, but they also preserve capacity. That said, we now state a few basic properties and interpretations of capacity that will be needed to state and discuss this analytic refinement. First recall the definitions of the Newton polytope and the support of a polynomial.

Definition 5.1.1. Given $p \in \mathbb{R}[x_1, ..., x_n]$, the *Newton polytope* of p, denoted $\mathrm{Newt}(p)$, is the convex hull of the support of p. The *support* of p, denoted $\mathrm{supp}(p)$, is the set of all $\mu \in \mathbb{Z}_+^n$ such that x^μ has a non-zero coefficient in p.

Capacity is perhaps most basically understood as a quantity which mediates between the coefficients of p and the evaluations of p. For example, if $\mu \in \mathrm{supp}(p)$ then:

$$p_\mu \leq \mathrm{Cap}_\mu(p) \leq p(1, ..., 1)$$

Capacity can also be understood probabilistically. If $p \in \mathbb{R}_+^{(1^n)}[x_1, ..., x_n]$ and $p(1, ..., 1) = 1$, then p can be considered as the probability generating function for some discrete distribution on $\mathrm{supp}(p)$. In this case, a simple proof demonstrates:

Fact 5.1.2. *Let $p \in \mathbb{R}_+^{(1^n)}[x_1, ..., x_n]$ be the probability generating function for some distribution ν. Then:*

1. *$0 \leq \mathrm{Cap}_\alpha(p) \leq 1$ for all $\alpha \in \mathbb{R}_+^n$.*

2. *$\mathrm{Cap}_\alpha(p) = 1$ if and only if α is the vector of marginal probabilities of ν.*

Proof. (1) is straightforward, and (2) follows from concavity of log (e.g., see [35], Fact 2.2) and the fact that $\mathrm{Cap}_\alpha(p) = 1$ implies $\frac{p(x)}{x^\alpha}$ is minimized at the all-ones vector. $\square$

The following "log-exponential polynomial" associated to p has some nice properties which often makes it convenient to use in the context of capacity. These properties also shed light on the potential connection between capacity, convexity, and the Legendre/Fenchel transformation (consider the expressions which show up in Fact 5.1.4 below).

Definition 5.1.3. Given a polynomial $p \in \mathbb{R}_+[x_1, ..., x_n]$, we let capitalized P denote the following function:
$$P(x) := \log(p(\exp(x))) = \log \sum_\mu p_\mu e^{\mu \cdot x}$$

Fact 5.1.4. *Given $p \in \mathbb{R}_+[x_1, ..., x_n]$, consider P as defined above. We have:*

1. $\text{Cap}_\alpha(p) = \exp \inf_{x \in \mathbb{R}^n} (P(x) - \alpha \cdot x)$

2. $P(x) - \alpha \cdot x$ *is convex in $\mathbb{R}^n$ for any $\alpha \in \mathbb{R}^n$.*

The next result is essentially a corollary of the (weighted) AM-GM inequality. In a sense, this inequality is the foundational result that makes the notion of capacity so useful. Because of this we provide a partial proof of the following result, taken from [1].

Fact 5.1.5. *For $p \in \mathbb{R}_+[x_1, ..., x_n]$, P defined as above, and $\alpha \in \mathbb{R}_+^n$, the following are equivalent.*

1. $\alpha \in \text{Newt}(p)$

2. $\text{Cap}_\alpha(p) > 0$

3. $P(x) - \alpha \cdot x$ *is bounded below.*

Proof. That (2) $\Leftrightarrow$ (3) follows from the previous fact. We now prove (1) $\Rightarrow$ (2). The (2) $\Rightarrow$ (1) direction also has a short proof, based on a separating hyperplane for α and $\text{Newt}(p)$ whenever $\alpha \notin \text{Newt}(p)$. The details can be found in Fact 2.18 of [1].

Suppose that $\alpha \in \text{Newt}(p)$. So, $\alpha = \sum_{\mu \in S} c_\mu \mu$, where $S \subset \text{supp}(p)$, $c_\mu > 0$, and $\sum_{\mu \in S} c_\mu = 1$. Using the AM-GM inequality and the fact that the coefficients of p are non-negative, we have the following for $x \in \mathbb{R}_+^n$:

$$p(x) \geq \sum_{\mu \in S} p_\mu x^\mu = \sum_{\mu \in S} c_\mu \frac{p_\mu x^\mu}{c_\mu} \geq \prod_{\mu \in S} \left(\frac{p_\mu x^\mu}{c_\mu} \right)^{c_\mu} = x^\alpha \prod_{\mu \in S} \left(\frac{p_\mu}{c_\mu} \right)^{c_\mu}$$

This then implies:
$$\text{Cap}_\alpha(p) = \inf_{x > 0} \frac{p(x)}{x^\alpha} \geq \prod_{\mu \in S} \left(\frac{p_\mu}{c_\mu} \right)^{c_\mu} > 0$$

$\square$

Due to the previous result, we will only ever consider values of α which are in the Newton polytope of the relevant polynomials. In a sense, other α can be considered (even negative α) as most results will then become trivial. That said, we will often make this assumption about α without explicitly stating it.

The next result emulates Proposition 2.2.4 by giving a collection of basic capacity preserving operators. Note that these results are either equalities, or give something of the form $\mathrm{Cap}(T(p)) \geq c_T \cdot \mathrm{Cap}(p)$ for various operators T.

Proposition 5.1.6 (Basic capacity preservers). *For $p, q \in \mathbb{R}_+^\gamma[x]$ and $\alpha, \beta \in \mathbb{R}_+^n$, we have:*

1. *Scaling:* $\mathrm{Cap}_\alpha(bp) = b \cdot \mathrm{Cap}_\alpha(p)$ *for* $b \in \mathbb{R}_+$

2. *Product:* $\mathrm{Cap}_{\alpha+\beta}(pq) \geq \mathrm{Cap}_\alpha(p) \mathrm{Cap}_\beta(q)$

3. *Disjoint product:* $\mathrm{Cap}_{(\alpha,\beta)}(p(x)q(z)) = \mathrm{Cap}_\alpha(p) \mathrm{Cap}_\beta(q)$

4. *Evaluation:* $\mathrm{Cap}_{(\alpha_1,...,\alpha_{n-1})}(p(x_1, ..., x_{n-1}, y_n)) \geq y_n^{\alpha_n} \mathrm{Cap}_\alpha(p)$ *for* $y_n \in \mathbb{R}_+$

5. *External field:* $\mathrm{Cap}_\alpha(p(cx)) = c^\alpha \mathrm{Cap}_\alpha(p)$ *for* $c \in \mathbb{R}_+^n$

6. *Inversion:* $\mathrm{Cap}_{(\gamma-\alpha)}(x^\gamma p(x_1^{-1}, ..., x_n^{-1})) = \mathrm{Cap}_\alpha(p)$

7. *Concavity:* $\mathrm{Cap}_\alpha(bp + cq) \geq b \cdot \mathrm{Cap}_\alpha(p) + c \cdot \mathrm{Cap}_\alpha(q)$ *for* $b, c \in \mathbb{R}_+$

8. *Diagonalization:* $\mathrm{Cap}_{\sum \alpha_k}(p(x, ..., x)) \geq \mathrm{Cap}_\alpha(p)$

9. *Symmetric diagonalization:* $\mathrm{Cap}_{n \cdot \alpha_0}(p(x, ..., x)) = \mathrm{Cap}_\alpha(p)$ *if* $\alpha = (\alpha_0, ..., \alpha_0)$ *and* p *is symmetric*

10. *Homogenization:* $\mathrm{Cap}_{(\alpha,\gamma-\alpha)}(\mathrm{Hmg}^\gamma(p)) = \mathrm{Cap}_\alpha(p)$

Proof. Symmetric diagonalization is the only nontrivial property, and it is a consequence of the AM-GM inequality. First of all, we automatically have (the diagonalization inequality):

$$\mathrm{Cap}_{n \cdot \alpha_0}(p(x, ..., x)) = \inf_{x>0} \frac{p(x, ..., x)}{x^{\alpha_0} \cdots x^{\alpha_0}} \geq \inf_{x>0} \frac{p(x_1, ..., x_n)}{x_1^{\alpha_0} \cdots x_n^{\alpha_0}} = \mathrm{Cap}_\alpha(p)$$

For the other direction, fix $x \in \mathbb{R}_+^n$ and let $y := (x_1 \cdots x_n)^{1/n}$. Further, let $S(p)$ denote the symmetrization of p. For any $\mu \in \mathbb{Z}_+^n$, the AM-GM inequality gives:

$$S(x^\mu) = \frac{1}{n!} \sum_{\sigma \in S_n} x_{\sigma(1)}^{\mu_1} \cdots x_{\sigma(n)}^{\mu_n} \geq \left(\prod_{\sigma \in S_n} x_{\sigma(1)}^{\mu_1} \cdots x_{\sigma(n)}^{\mu_n} \right)^{1/n!} = \left(\prod_{j,k} x_j^{\mu_k} \right)^{1/n} = y^{\mu_1} \cdots y^{\mu_n}$$

Additionally, $x^\alpha = x_1^{\alpha_0} \cdots x_n^{\alpha_0} = y^{n \cdot \alpha_0}$. Since p is symmetric, we then have the following:

$$\frac{p(x)}{x^\alpha} = \frac{S(p)(x)}{x^\alpha} = \sum_{\mu \in \mathrm{supp}(p)} p_\mu \frac{S(x^\mu)}{x^\alpha} \geq \sum_{\mu \in \mathrm{supp}(p)} p_\mu \frac{y^{\mu_1} \cdots y^{\mu_n}}{y^{n \cdot \alpha_0}} = \frac{p(y, ..., y)}{y^{n \cdot \alpha_0}}$$

That is, for any $x \in \mathbb{R}_+^n$, there is a $y \in \mathbb{R}_+$ such that $\frac{p(x)}{x^\alpha} \geq \frac{p(y,...,y)}{y^{n \cdot \alpha_0}}$. Therefore:

$$\mathrm{Cap}_\alpha(p) \geq \mathrm{Cap}_{n \cdot \alpha_0}(p(x,...,x))$$

This completes the proof. $\square$

Many of these operations are similar to those that preserve real stability. This is to be expected, as we hope to combine the two theories. In this vein, we now discuss the capacity preservation properties of the polarization operator. As it does for real stability preservers, polarization will play a crucial role in working out the theory of capacity preservers. To state this result, we define a sort of polarization of the vector α as follows, where each value $\frac{\alpha_k}{\gamma_k}$ shows up γ_k times:

$$\mathrm{Pol}^\gamma(\alpha) := \left(\frac{\alpha_1}{\gamma_1}, ..., \frac{\alpha_1}{\gamma_1}, \frac{\alpha_2}{\gamma_2}, ..., \frac{\alpha_2}{\gamma_2}, ..., \frac{\alpha_n}{\gamma_n}, ..., \frac{\alpha_n}{\gamma_n} \right)$$

Proposition 5.1.7. *Given $p \in \mathbb{R}_+^\gamma[x_1, ..., x_n]$ and $\alpha \in \mathbb{R}_+^n$, we have that $\mathrm{Cap}_{\mathrm{Pol}^\gamma(\alpha)}(\mathrm{Pol}^\gamma(p)) = \mathrm{Cap}_\alpha(p)$.*

Proof. We essentially apply the diagonalization property to each variable in succession. Specifically, we have:

$$\mathrm{Cap}_\alpha(p) = \inf_{y_1,...,y_{n-1}>0} \frac{1}{y_1^{\alpha_1} \cdots y_{n-1}^{\alpha_{n-1}}} \inf_{x_n>0} \frac{p(y_1, ..., y_{n-1}, x_n)}{x_n^{\alpha_n}}$$

$$= \inf_{y_1,...,y_{n-1}>0} \frac{1}{y_1^{\alpha_1} \cdots y_{n-1}^{\alpha_{n-1}}} \mathrm{Cap}_{\alpha_n}(p(y_1, ..., y_{n-1}, x_n))$$

$$= \inf_{y_1,...,y_{n-1}>0} \frac{1}{y_1^{\alpha_1} \cdots y_{n-1}^{\alpha_{n-1}}} \mathrm{Cap}_{\mathrm{Pol}^{\gamma_n}(\alpha_n)}(\mathrm{Pol}^{\gamma_n}(p(y_1, ..., y_{n-1}, \cdot)))$$

By now rearranging the inf's in the last expression above, we can let $\inf_{y_{n-1}>0}$ be the innermost inf. We can then apply the above argument again, and this will work for every y_k in succession. At the end of this process, we obtain:

$$\mathrm{Cap}_\alpha(p) = \mathrm{Cap}_{(\mathrm{Pol}^{\gamma_1}(\alpha_1),...,\mathrm{Pol}^{\gamma_n}(\alpha_n))}(\mathrm{Pol}^{\gamma_1} \circ \cdots \circ \mathrm{Pol}^{\gamma_n}(p)) = \mathrm{Cap}_{\mathrm{Pol}^\gamma(\alpha)}(\mathrm{Pol}^\gamma(p))$$

$\square$

Note that the two main results on polarization—capacity preservation and real stability preservation—imply that we only really need to prove our results in the multiaffine case (i.e., where polynomials are of degree at most 1 in each variable). We will make use of this reduction when we prove our technical results in §5.3.

Finally before moving on, we give one basic capacity calculation which will prove extremely useful to us almost every time we want to compute capacity.

Lemma 5.1.8. *For $c, \alpha \in \mathbb{R}_+^n$ and $m := \sum_k \alpha_k$, we have the following:*

$$\mathrm{Cap}_\alpha((c \cdot x)^m) = \left(\frac{mc}{\alpha}\right)^\alpha$$

Proof. Note first that:

$$\mathrm{Cap}_\alpha((c \cdot x)^m) = \left(\mathrm{Cap}_{\frac{\alpha}{m}}(c \cdot x)\right)^m$$

To compute $\mathrm{Cap}_{\frac{\alpha}{m}}(c \cdot x)$, we use calculus. Let $\beta := \frac{\alpha}{m}$, and for now we assume that $\beta > 0$ and $c > 0$ strictly.

$$\partial_{x_k}\left(\frac{c \cdot x}{x^\beta}\right) = \frac{x^\beta c_k - \beta_k x^{\beta - \delta_k}(c \cdot x)}{x^{2\beta}} = \frac{x_k c_k - \beta_k(c \cdot x)}{x^{\beta + \delta_k}}$$

That is, the gradient of $\frac{c \cdot x}{x^\beta}$ is the 0 vector whenever $\frac{c_k}{\beta_k} x_k = c \cdot x$ for all k. And in fact, any vector satifying those conditions should minimize $\frac{c \cdot x}{x^\beta}$, by homogeneity. Since $\sum_k \beta_k = 1$, the vector $x_k := \frac{\beta_k}{c_k}$ satisfies the conditions. This implies:

$$\mathrm{Cap}_\beta(c \cdot x) = \frac{c \cdot (\beta/c)}{(\beta/c)^\beta} = \left(\frac{c}{\beta}\right)^\beta$$

Therefore:

$$\mathrm{Cap}_\alpha((c \cdot x)^m) = \left(\frac{c}{\beta}\right)^{m\beta} = \left(\frac{mc}{\alpha}\right)^\alpha$$

$\square$

5.2 Bipartite Mathcings and Other Applications

We now formally state and discuss our main results and their applications. As mentioned above, we will emulate the Borcea-Brändén characterization for capacity preservers. Further, we will also demonstrate how our results encapsulate many of the previous results regarding capacity. With this in mind, we first give our main capacity preservation results: one for bounded degree operators and one for unbounded degree operators. Notice that the unbounded degree case is something like a limit of the bounded degree case: the scalar $\frac{\alpha^\alpha(\gamma-\alpha)^{\gamma-\alpha}}{\gamma^\gamma}$ is approximately $\left(\frac{\alpha}{\gamma}\right)^\alpha e^{-\alpha}$ as $\gamma \to \infty$. (The proof of Theorem 5.3.8 shows why the extra $\gamma^{-\alpha}$ factor disappears.)

Theorem 5.3.10 (Bounded degree). *Let $T : \mathbb{R}_+^\gamma[x_1, ..., x_n] \to \mathbb{R}_+[x_1, ..., x_m]$ be a linear operator with real stable (or bistable) symbol. Then for any $\alpha \in \mathbb{R}_+^n$, any $\beta \in \mathbb{R}_+^m$, and any real stable $p \in \mathbb{R}_+^\gamma[x_1, ..., x_n]$ we have:*

$$\frac{\mathrm{Cap}_\beta(T(p))}{\mathrm{Cap}_\alpha(p)} \geq \frac{\alpha^\alpha(\gamma-\alpha)^{\gamma-\alpha}}{\gamma^\gamma} \mathrm{Cap}_{(\alpha,\beta)}(\mathrm{Symb}_+(T))$$

Further, this bound is tight for fixed T, α, and β.

Theorem 5.3.11 (Unbounded degree). *Let $T : \mathbb{R}_+[x_1, ..., x_n] \to \mathbb{R}_+[x_1, ..., x_m]$ be a linear operator with real stable (or bistable) symbol. Then for any $\alpha \in \mathbb{R}_+^n$, any $\beta \in \mathbb{R}_+^m$, and any real stable $p \in \mathbb{R}_+[x_1, ..., x_n]$ we have:*

$$\frac{\mathrm{Cap}_\beta(T(p))}{\mathrm{Cap}_\alpha(p)} \geq \alpha^\alpha e^{-\alpha} \mathrm{Cap}_{(\alpha,\beta)}(\mathrm{Symb}_+(T))$$

Further, this bound is tight for fixed T, α, and β.

Note that by Theorem 2.3.5, the above theorems apply to real stability preservers of rank greater than 2 (see Corollaries 5.3.12 and 5.3.13).

Gurvits' Theorem and its Corollaries

With these results in hand, we now reprove Gurvits' theorem and discuss its importance. Gurvits' original proof of this fact was not very complicated, and our proof will be similar in this regard. This is of course what makes capacity and real stability more generally so intriguing: answers to seemingly hard questions follow from a few basic computations on polynomials.

Theorem 5.2.1 (Gurvits). *For real stable $p \in \mathbb{R}_+^\gamma[x_1, ..., x_n]$ we have:*

$$\frac{\mathrm{Cap}_{(1^{n-1})}\left(\partial_{x_k} p|_{x_k=0}\right)}{\mathrm{Cap}_{(1^n)}(p)} \geq \left(\frac{\gamma_k - 1}{\gamma_k}\right)^{\gamma_k - 1}$$

Proof. We apply Theorem 5.3.10 above for $T := \partial_{x_k}|_{x_k=0}$, $\alpha := (1^n)$, and $\beta := (1^{n-1})$. To do this we need to compute the right-hand side of the expression in Theorem 5.3.10, making use of properties from Proposition 5.1.6.

$$\begin{aligned}
\frac{\alpha^\alpha(\gamma - \alpha)^{\gamma - \alpha}}{\gamma^\gamma} \mathrm{Cap}_{(\alpha,\beta)}(\mathrm{Symb}_+(T)) &= \frac{(\gamma - 1)^{\gamma - 1}}{\gamma^\gamma} \mathrm{Cap}_{(1^n, 1^{n-1})}\left(\partial_{x_k}(1 + xz)^\gamma|_{x_k=0}\right) \\
&= \frac{(\gamma - 1)^{\gamma - 1}}{\gamma^\gamma} \mathrm{Cap}_{(1^n, 1^{n-1})}\left(\gamma_k z_k \prod_{j \neq k}(1 + x_j z_j)^{\gamma_j}\right) \\
&= \frac{(\gamma - 1)^{\gamma - 1}}{\gamma^\gamma} \gamma_k \prod_{j \neq k} \mathrm{Cap}_{(1,1)}\left((1 + x_j z_j)^{\gamma_j}\right)
\end{aligned}$$

Note that $\mathrm{Cap}_{(1,1)}((1 + x_j z_j)^{\gamma_j}) = \mathrm{Cap}_1((1 + x_j)^{\gamma_j})$. Using the homogenization property and Lemma 5.1.8, we then have:

$$\mathrm{Cap}_1((1 + x_j)^{\gamma_j}) = \mathrm{Cap}_{(1, \gamma_j - 1)}((x_j + y_j)^{\gamma_j}) = \gamma_j \left(\frac{\gamma_j}{\gamma_j - 1}\right)^{\gamma_j - 1}$$

Therefore:

$$\frac{\alpha^\alpha(\gamma-\alpha)^{\gamma-\alpha}}{\gamma^\gamma}\operatorname{Cap}_{(\alpha,\beta)}(\operatorname{Symb}^\gamma_+(T)) = \frac{(\gamma-1)^{\gamma-1}}{\gamma^\gamma}\gamma_k\prod_{j\neq k}\operatorname{Cap}_{(1,1)}\left((1+x_jz_j)^{\gamma_j}\right)$$

$$= \frac{(\gamma-1)^{\gamma-1}}{\gamma^\gamma}\gamma_k\prod_{j\neq k}\gamma_j\left(\frac{\gamma_j}{\gamma_j-1}\right)^{\gamma_j-1}$$

$$= \left(\frac{\gamma_k-1}{\gamma_k}\right)^{\gamma_k-1}$$

$\square$

This proof will serve as a good baseline for other applications of our main theorems. Roughly speaking, most applications will make use of Lemma 5.1.8 and the properties of Proposition 5.1.6 in interesting ways. And often, the inequalities obtained will directly translate to various combinatorial statements.

Specifically, what sorts of combinatorial statements can be derived from Gurvits' theorem? The most well known are perhaps Schrijver's theorem and the Van der Waerden bound on the permanent. We explicitly go through these and other examples now, noting that this material has essentially been taken from [35]. Our purpose for doing this is to give a few simpler examples of the use of Theorem 5.3.10 before using it to prove Csikvári's results.

To make the link between capacity and the combinatorial objects we care about, we define the following for a given matrix M:

$$p_M(x) := \prod_i\sum_j m_{ij}x_j$$

Note that this polynomial is real stable whenever the entries of M are nonnegative. The following is then quite suggestive.

Lemma 5.2.2 (Gurvits). *If M is a doubly stochastic matrix, then $\operatorname{Cap}_{(1^n)}(p_M) = 1$.*

With this, we are ready to prove the corollaries of Gurvits' thoerem.

Corollary 5.2.3 (Schrijver). *Let G be a d-regular bipartite graph with $2n$ vertices. Then the number of perfect matchings of G is bounded below by:*

$$\mu_n(G) \geq \left(\frac{(d-1)^{d-1}}{d^{d-2}}\right)^n$$

Proof. We apply Theorem 5.3.10 directly, but one could iteratively use Gurvits' theorem as well. Let M be the bipartite adjacency matrix for G. Note that d-regularity then implies that $\frac{1}{d}M$ is doubly stochastic. Further, the number of perfect matchings of G can be computed via:

$$\mu_n(G) = \partial_{x_1}\cdots\partial_{x_n}|_{x_1=\cdots=x_n=0}\,p_M = \operatorname{Cap}_\varnothing(\partial_{x_1}\cdots\partial_{x_n}|_{x_1=\cdots=x_n=0}\,p_M)$$

To apply the main theorem to the linear operator $T := \partial_{x_1} \cdots \partial_{x_n}|_{x_1 = \cdots = x_n = 0}$, we compute:

$$\mathrm{Cap}_{(1^n, \varnothing)}(\mathrm{Symb}_+(T)) = \inf_{z > 0} \frac{T[\prod_k (1 + x_k z_k)^{\gamma_k}]}{\prod_k z_k} = \inf_{z > 0} \frac{\prod_k \gamma_k z_k}{\prod_k z_k} = \prod_k \gamma_k = \gamma^1$$

Since $p_M \in \mathbb{R}_+^{(d,\dots,d)}[x_1, \dots, x_n]$, we apply the main theorem to obtain:

$$\frac{1}{d^n} \mu_n(G) = \mathrm{Cap}_\varnothing(T[p_{M/d}]) \geq \frac{1^1 (\gamma - 1)^{\gamma - 1}}{\gamma^\gamma} \cdot \mathrm{Cap}_{(1^n)}(p_{M/d}) \cdot \mathrm{Cap}_{(1^n, \varnothing)}(\mathrm{Symb}_+(T))$$

$$= \frac{(d - 1)^{n(d-1)}}{d^{nd}} \cdot 1 \cdot d^n$$

Rearranging implies the result. $\square$

We were able to apply Theorem 5.3.10 directly to the operator $T = \partial_{x_1} \cdots \partial_{x_n}|_{x_1 = \cdots = x_n = 0}$ because we did not try to take into account the effect of each operator $\partial_{x_i}|_{x_i = 0}$ individually. That said, we will need to be more careful for the next corollary.

Corollary 5.2.4 (Falikman, Erorychev). *The permanent of a doubly stochastic $n \times n$ matrix is at least $\frac{n!}{n^n}$.*

Proof. We consider $p_M \in \mathbb{R}_+^{(n,\dots,n)}[x_1, \dots, x_n]$ as before. Note that the permanent of M can be computed via:

$$\mathrm{per}(M) = \partial_{x_1}|_{x_1 = 0} \cdots \partial_{x_n}|_{x_n = 0} \, p_M = \mathrm{Cap}_\varnothing(\partial_{x_1}|_{x_1 = 0} \cdots \partial_{x_n}|_{x_n = 0} \, p_M)$$

A direct application of Gurvits' theorem gives:

$$\mathrm{Cap}_{(1^{n-1})}(\partial_{x_n}|_{x_n = 0} \, p_M) \geq \mathrm{Cap}_{(1^n)}(p_M) \left(\frac{n - 1}{n} \right)^{n-1} = \left(\frac{n - 1}{n} \right)^{n-1}$$

The key observation now is that $\partial_{x_n}|_{x_n = 0} \, p_M$ is of degree at most $n - 1$ in each of its variables, by homogeneity. We inductively combine this with Gurvits' theorem to obtain (here $0^0 = 1$):

$$\mathrm{per}(M) = \mathrm{Cap}_\varnothing(\partial_{x_1}|_{x_1 = 0} \cdots \partial_{x_n}|_{x_n = 0} \, p_M) \geq \prod_{k=1}^{n} \left(\frac{k - 1}{k} \right)^{k-1} = \frac{n!}{n^n}$$

$\square$

Notice that if we had more information about the number and location of nonzero entries of M, we could potentially strengthen the previous corollary to account for this. In fact, this can be done and the result is a strengthening of another theorem of Schrijver [57]. Again, this corollary is already proven by Gurvits in [35].

Corollary 5.2.5 (Schrijver, Gurvits). *Let M be a matrix with nonnegative integer entries and row sums and column sums all equal to m. We have:*

$$\text{per}(M) \geq m! \left(\frac{(m-1)^{m-1}}{m^{m-2}} \right)^{n-m} > \left(\frac{(m-1)^{m-1}}{m^{m-2}} \right)^n$$

Proof. We apply a similar argument to that of the previous corollary to obtain the result. As above:

$$\text{per}(M) = \partial_{x_1}|_{x_1=0} \cdots \partial_{x_n}|_{x_n=0} \, p_M = \text{Cap}_\varnothing(\partial_{x_1}|_{x_1=0} \cdots \partial_{x_n}|_{x_n=0} \, p_M)$$

Note also that the restriction on the row and column sums implies $p_M \in \mathbb{R}_+^{(m,\dots,m)}[x_1, \dots, x_n]$ and that $\frac{1}{m}M$ is doubly stochastic. For the first $n-m$ applications of Gurvits' theorem, we do not consider how $\partial_{x_i}|_{x_i=0}$ affects the degree:

$$\text{Cap}_{(1^m)}(\partial_{x_{m+1}}|_{x_{m+1}=0} \cdots \partial_{x_n}|_{x_n=0} \, p_{M/m}) \geq \text{Cap}_{(1^n)}(p_{M/m}) \prod_{k=m+1}^{n} \left(\frac{m-1}{m} \right)^{m-1}$$

$$= \left(\frac{m-1}{m} \right)^{(n-m)(m-1)}$$

Since $\partial_{x_{m+1}}|_{x_{m+1}=0} \cdots \partial_{x_n}|_{x_n=0} \, p_{M/m} \in \mathbb{R}_+^{(m,\dots,m)}[x_1, \dots, x_n]$, we can then apply the arguments to Corollary 5.2.4 to obtain:

$$\frac{1}{m^n} \text{per}(M) = \text{Cap}_\varnothing(\partial_{x_1}|_{x_1=0} \cdots \partial_{x_n}|_{x_n=0} \, p_{M/m})$$

$$\geq \text{Cap}_{(1^m)}(\partial_{x_{m+1}}|_{x_{m+1}=0} \cdots \partial_{x_n}|_{x_n=0} \, p_{M/m}) \frac{m!}{m^m}$$

Combining these inequalities gives:

$$\frac{1}{m^n} \text{per}(M) \geq \frac{m!}{m^m} \left(\frac{m-1}{m} \right)^{(n-m)(m-1)} = \frac{m!}{m^m} \left(\frac{(m-1)^{m-1}}{m^{m-1}} \right)^{n-m}$$

Rearranging implies the result. $\qquad\square$

Notice that this result immediately strengthens Corollary 5.2.3 when M is the bipartite adjacency matrix of G and $m = d$. Considering the ideas used here, it is also apparent that even more specific information about M could lead to further strengthenings of these results.

In addition to these types of inequalities, Gurvits also demonstrates how his theorem implies similar results for "doubly stochastic" n-tuples of matrices (a conjecture due to Bapat [3]). In fact, this notion of doubly stochastic aligns with a generalized notion used recently in [27],[13]. In those papers, doubly stochastic matrices and other similar objects play a crucial role in defining certain important orbits of actions on tuples of matrices. Specifically in [27] (or [28]), a version of this idea was used to produce a polynomial time algorithm for the noncommutative polynomial identity testing problem. A certain notion of capacity for matrices was quite important in the analysis of their algorithms.

Imperfect Matchings and Biregular Graphs

The most important application of our results is a new proof of a bound on size-k matchings of a biregular bipartite graph, due to Csikvári [17]. This result is a generalization of Schrijver's bound (Corollary 5.2.3 above), and it also settled and strengthened the Friedland macthing conjecture [25]. We first state Csikvári's results, in a form more amenable to the notation of this thesis.

Theorem 5.2.6 (Csikvári). *Let G be an (a, b)-biregular bipartite graph with vertices which are (m, n)-bipartitioned (so that $am = bn$ is the number of edges of G). Then the number of size-k matchings of G is bounded as follows:*

$$\mu_k(G) \geq \binom{n}{k}(ab)^k \frac{m^m(ma - k)^{ma-k}}{(ma)^{ma}(m - k)^{m-k}}$$

Notice that this immediately implies the following bound for regular bipartite graphs.

Corollary 5.2.7 (Csikvári). *Let G be a d-regular bipartite graph with $2n$ vertices. Then:*

$$\mu_k(G) \geq \binom{n}{k}d^k \left(\frac{nd - k}{nd}\right)^{nd-k} \left(\frac{n}{n - k}\right)^{n-k}$$

To prove these results, we first need to generalize Gurvits' capacity lemma for doubly stochastic matrices. Specifically we want to be able to handle (a, b)-*stochastic matrices*, which are matrices with row sums equal to a and columns sums equal to b. We care about such matrices, because the bipartite adjacency matrix of a (a, b)-biregular graph is (a, b)-stochastic. Note that if M is an (a, b)-stochastic matrix which is of size $m \times n$, then $am = bn$.

Lemma 5.2.8. *If M is an (a, b)-stochastic matrix, then $\text{Cap}_{(\frac{m}{n}, \ldots, \frac{m}{n})}(p_M) = a^m$.*

We also need a linear operator which computes the number of size-k matchings of an (a, b)-biregular bipartite graph. In fact when M is the bipartite adjacency matrix of G, we have the following:

$$a^{m-k}\mu_k(G) = \sum_{S \in \binom{[n]}{k}} \partial_x^S p_M(1) = \text{Cap}_\varnothing \left(\sum_{S \in \binom{[n]}{k}} \partial_x^S p_M(1)\right)$$

Note that each differential operator in the sum picks out a disjoint collection of $k \times k$ subpermutations of the matrix M. After applying each differential operator, we are left with terms which are products of $m - k$ remaining linear forms from p_M. Plugging in 1 then gives a^{m-k} (since row sums are a), and this is why that factor appears above.

Next, we need to prove that we can apply Theorem 5.3.10 to the linear operator $T := \sum_{S \in \binom{[n]}{k}} \partial_x^S \big|_{x=1}$.

Lemma 5.2.9. *The operator* $T := \sum_{S \in \binom{[n]}{k}} \partial_x^S \big|_{x=1}$ *acting on* $\mathbb{R}^\gamma[x_1, ..., x_n]$ *has real stable symbol.*

Proof. Here the input polynomial space is $R_+^{(b,...,b)}[x_1, ..., x_n]$, since degree is determined by the column sums. Denoting $\gamma := (b, ..., b)$, we compute $\text{Symb}_+(T)$:

$$
\begin{aligned}
T[(1+xz)^\gamma] &= \sum_{S \in \binom{[n]}{k}} \partial_x^S \big|_{x=1} (1+xz)^\gamma \\
&= \sum_{S \in \binom{[n]}{k}} b^k z^S (1+z)^{\gamma-S} \\
&= b^k (1+z)^{\gamma-1} \sum_{S \in \binom{[n]}{k}} z^S (1+z)^{1-S}
\end{aligned}
$$

Notice that $\sum_{S \in \binom{[n]}{k}} z^S (1+z)^{1-S} = \binom{n}{k} \text{Pol}^n(x^k(1+x)^{n-k})$, which is real stable by Proposition 2.2.6. $\qquad \square$

Applying Theorem 5.3.10 now shows us the way toward the rest of the proof. Denoting $\gamma := (b, ..., b)$ and $\alpha := (\frac{m}{n}, ..., \frac{m}{n})$, we now have:

$$
\begin{aligned}
a^{m-k} \mu_k(G) = \sum_{S \in \binom{[n]}{k}} \partial_x^S p_M(1) &\geq \frac{\alpha^\alpha (\gamma - \alpha)^{\gamma - \alpha}}{\gamma^\gamma} \text{Cap}_\alpha(p_M) \text{Cap}_{(\alpha, \varnothing)}(\text{Symb}_+(T)) \\
&= \left(\frac{(\frac{m}{n})^{\frac{m}{n}} (b - \frac{m}{n})^{b - \frac{m}{n}}}{b^b} \right)^n a^m \text{Cap}_\alpha(\text{Symb}_+(T)) \\
&= \frac{(ma)^m (nb - m)^{nb-m}}{(nb)^{nb}} \text{Cap}_\alpha(\text{Symb}_+(T))
\end{aligned}
$$

So the last computation we need to make is that of $\text{Cap}_\alpha(\text{Symb}_+(T))$. Fortunately since $\text{Symb}_+(T)$ is symmetric and $\alpha = (\frac{m}{n}, ..., \frac{m}{n})$, we can apply the symmetric diagonalization property to simplify this computation. Using our previous computation of $\text{Symb}_+(T)$, this gives:

$$
\text{Cap}_{(\frac{m}{n}, ..., \frac{m}{n})}(\text{Symb}_+(T)) = \text{Cap}_m \left(b^k \binom{n}{k} z^k (1+z)^{nb-k} \right) = b^k \binom{n}{k} \text{Cap}_m(z^k (1+z)^{nb-k})
$$

The remaining capacity computation then follows from homogenization and Lemma 5.1.8:

$$
\begin{aligned}
\text{Cap}_m(z^k (1+z)^{nb-k}) &= \inf_{z>0} \frac{z^k (1+z)^{nb-k}}{z^m} \\
&= \text{Cap}_{m-k}((1+z)^{nb-k}) \\
&= \left(\frac{nb-k}{m-k} \right)^{m-k} \left(\frac{nb-k}{nb-m} \right)^{nb-m}
\end{aligned}
$$

Putting all of these computations together and recalling $ma = nb$ gives:

$$\mu_k(G) \geq a^{k-m} \frac{(ma)^m(nb-m)^{nb-m}}{(nb)^{nb}} b^k \binom{n}{k} \left(\frac{nb-k}{m-k}\right)^{m-k} \left(\frac{nb-k}{nb-m}\right)^{nb-m}$$

$$= \binom{n}{k} a^{k-m} b^k \frac{(ma)^m(nb-k)^{nb-k}}{(nb)^{nb}(m-k)^{m-k}}$$

$$= \binom{n}{k} (ab)^k \frac{m^m(ma-k)^{ma-k}}{(ma)^{ma}(m-k)^{m-k}}$$

This is precisely the desired inequality.

Differential Operators in General

We now give general capacity preservation bounds for stability preservers which are differential operators. This was done in [1] for differential operators which preserve real stability on input polynomials of all degrees. Here, we restrict to those operators which only preserve real stability for polynomials of some fixed bounded degree. That said, consider the following bilinear operator:

$$(p \boxplus^\gamma q)(x) := \sum_{0 \leq \mu \leq \gamma} (\partial_x^\mu p)(x)(\partial_x^{\gamma-\mu} q)(0)$$

It is straightforward to see that by fixing q, one can construct any constant coefficient differential operator on $\mathbb{R}^\gamma[x_1, ..., x_n]$. And it turns out that if q is real stable, then $(\cdot \boxplus^\gamma q)(x)$ is a real stability preserver.

It turns out that more is true, however. The operator $\boxplus^\gamma$ can actually be applied to polynomials in $\mathbb{R}^{(\gamma,\gamma)}[x_1, ..., x_n, y_1, ..., y_n]$ by considering this polynomial space as a tensor product of polynomial spaces. More concretely, we specify how this operator acts on the monomial basis:

$$\boxplus^\gamma : x^\mu y^\nu \mapsto x^\mu \boxplus^\gamma x^\nu$$

We can then compute the symbol of this operator:

$$\mathrm{Symb}_+[\boxplus^\gamma] = (1 + xz)^\gamma \boxplus^\gamma (1 + xw)^\gamma = (z + w + zwx)^\gamma = (zw)^\gamma(x + z^{-1} + w^{-1})^\gamma$$

Note that $\mathrm{Symb}_+[\boxplus^\gamma](z, w, -x)$ is real stable, and so $\boxplus^\gamma$ preserves real stability by Theorem 2.3.5.

With this, we compute the capacity for $\gamma = (1, 0, ..., 0)$:

$$\mathrm{Cap}_{(\alpha,\beta,\delta)}(z + t + ztx) = \inf_{x,z,t>0} \frac{z + t + ztx}{x^\alpha z^\beta t^\delta} = \inf_{x,z,t>0} \frac{t^{-1} + z^{-1} + x}{x^\alpha z^{\beta-1} t^{\delta-1}} = \inf_{x,z,t>0} \frac{t + z + x}{x^\alpha z^{1-\beta} t^{1-\delta}}$$

Note that (α, β, δ) is in the Newton polytope of $(z + t + ztx)$ iff $\alpha = \beta + \delta - 1$. By Lemma 5.1.8, we have:

$$\mathrm{Cap}_{(\alpha,\beta,\delta)}(z + t + ztx) = \frac{1}{\alpha^\alpha(1-\beta)^{1-\beta}(1-\delta)^{1-\delta}}$$

We now generalize this to general γ, supposing $\alpha = \beta + \delta - \gamma$:

$$\mathrm{Cap}_{(\alpha,\beta,\delta)}((z+t+ztx)^\gamma) = \prod_{j=1}^n \left((\alpha_j/\gamma_j)^{\alpha_j/\gamma_j}(1-\beta_j/\gamma_j)^{1-\beta_j/\gamma_j}(1-\delta_j/\gamma_j)^{1-\delta_j/\gamma_j}\right)^{-\gamma_j}$$

$$= \prod_{j=1}^n (\alpha_j/\gamma_j)^{-\alpha_j}(1-\beta_j/\gamma_j)^{\beta_j-\gamma_j}(1-\delta_j/\gamma_j)^{\delta_j-\gamma_j}$$

$$= \alpha^{-\alpha}(\gamma-\beta)^{\beta-\gamma}(\gamma-\delta)^{\delta-\gamma}\gamma^{\alpha-\beta-\delta+2\gamma}$$

$$= \frac{\gamma^\gamma}{\alpha^\alpha(\gamma-\beta)^{\gamma-\beta}(\gamma-\delta)^{\gamma-\delta}}$$

Applying Theorem 5.3.10, we get:

$$\mathrm{Cap}_\alpha(p \boxplus^\gamma q) \geq \frac{\beta^\beta\delta^\delta(\gamma-\beta)^{\gamma-\beta}(\gamma-\delta)^{\gamma-\delta}}{\gamma^\gamma\gamma^\gamma} \cdot \frac{\gamma^\gamma}{\alpha^\alpha(\gamma-\beta)^{\gamma-\beta}(\gamma-\delta)^{\gamma-\delta}} \cdot \mathrm{Cap}_\beta(p)\,\mathrm{Cap}_\delta(q)$$

$$= \frac{\beta^\beta\delta^\delta}{\alpha^\alpha\gamma^\gamma}\,\mathrm{Cap}_\beta(p)\,\mathrm{Cap}_\delta(q)$$

Again, this is all under the assumption that $\alpha = \beta + \delta - \gamma$. (We will be outside the Newton polytope otherwise, and so the result in that case will be trivial.) We state the result of this discussion as follows. Note that is can be seen as a sort of multiplicative reverse triangle inequality for capacity of differential operators.

Corollary 5.2.10. *Let p, r be two real stable polynomials of degree γ with positive coefficients. We have:*

$$(\alpha^\alpha \mathrm{Cap}_\alpha(p \boxplus^\gamma q)) \geq \frac{1}{\gamma^\gamma}(\beta^\beta \mathrm{Cap}_\beta(p))(\delta^\delta \mathrm{Cap}_\delta(q))$$

With this, we have given tight capacity bounds for all differential operators on polynomials of at most some fixed bounded degree. Note that root bounds of this form are given in [47] by Marcus, Spielman, and Srivastava, and these bounds are related to those obtained in their proof of Kadison-Singer in [50]. It is an open and interesting question whether or not capacity can be utilized to bound the roots of polynomials.

5.3 The Main Inequalities

We now discuss our main results and the inequalities we use to obtain them. These inequalities are bounds on certain inner products applied to polynomials. The most basic of these is the main result from [1], which applies to multiaffine polynomials. We extend their methods to obtain bounds on polynomials of all degrees. Finally, a limiting argument implies bounds for the $\mathcal{LP}_+$ class. This last bound can also be found in [1], but the proof we give here is simpler and makes clearer the connection between these inequalities and the Borcea-Brändén characterization.

Inner Product Bounds, Bounded Degree

For polynomials of some fixed bounded degree, we consider the following inner product.

Definition 5.3.1. For fixed $\gamma \in \mathbb{Z}_+^n$ and $p, q \in \mathbb{R}^\gamma[x_1, ..., x_n]$, define:

$$\langle p, q \rangle^\gamma := \sum_{0 \le \mu \le \gamma} \binom{\gamma}{\mu}^{-1} p_\mu q_\mu$$

As mentioned above, Anari and Gharan prove a bound on the above inner product for multiaffine polynomials in [1], and we state their result here without proof. We note though that the proof is essentially a consequence of the strongly Rayleigh inequalities for real stable polynomials, which we now state. These fundamental inequalities (due to Brändén) should be seen as log-concavity conditions, and this intuition extends to all the inner product bounds we state here. And this intuition is not without evidence: the connection of capacity to the Alexandrov-Fenchel inequalities (see [32]), as well as to matroids and log-concave polynomials (see [2]), has been previously noted.

Proposition 5.3.2 (Strongly Rayleigh inequalities [9])**.** *For any real stable $p \in \mathbb{R}^{(1^n)}$ and any $i, j \in [n]$, we have the following inequality pointwise on all of $\mathbb{R}^n$:*

$$(\partial_{x_i} p) \cdot (\partial_{x_j} p) \ge p \cdot (\partial_{x_i} \partial_{x_j} p)$$

We now state the Anari-Gharan bound for multiaffine polynomials. They also prove a weaker bound on polynomials of any degree, but we will discuss this later.

Theorem 5.3.3 (Anari-Gharan)**.** *Let $p, q \in \mathbb{R}_+^{(1^n)}[x_1, ..., x_n]$ be real stable. Then for any $\alpha \in \mathbb{R}_+^n$ we have:*

$$\langle p, q \rangle^{(1^n)} \ge \alpha^\alpha (1 - \alpha)^{1-\alpha} \operatorname{Cap}_\alpha(p) \operatorname{Cap}_\alpha(q)$$

In this chapter, we generalize this to polynomials of degree γ as follows. Note that this result is strictly stronger than the bound obtained in [1] for the non-multiaffine case.

Theorem 5.3.4. *Let $p, q \in \mathbb{R}_+^\gamma[x_1, ..., x_n]$ be real stable. Then for any $\alpha \in \mathbb{R}_+^n$ we have:*

$$\langle p, q \rangle^\gamma \ge \frac{\alpha^\alpha (\gamma - \alpha)^{\gamma - \alpha}}{\gamma^\gamma} \operatorname{Cap}_\alpha(p) \operatorname{Cap}_\alpha(q)$$

The proof of this fact is essentially due to the fact that both $\langle \cdot, \cdot \rangle^\gamma$ and capacity interact nicely with polarization. We have already explicated the connection between capacity and polarization (see Proposition 5.1.7), and we now demonstrate how these inner products fit in.

Lemma 5.3.5. *Given $p, q \in \mathbb{R}^\gamma[x_1, ..., x_n]$, we have:*

$$\langle p, q \rangle^\gamma = \langle \operatorname{Pol}^\gamma(p), \operatorname{Pol}^\gamma(q) \rangle^{(1^\gamma)}$$

Proof. We compute this equality on a basis in the univariate case. The result then follows since Pol^γ is a composition of polarizations on each variable of p. For $0 \le k \le m$ we have:

$$\left\langle \text{Pol}^m(x^k), \text{Pol}^m(x^k) \right\rangle^{(1^m)} = \binom{m}{k}^{-2} \sum_{S \in \binom{[m]}{k}} \langle x^S, x^S \rangle^{(1^m)} = \binom{m}{k}^{-1} = \langle x^k, x^k \rangle^m$$

$\square$

The proof of Theorem 5.3.4 then essentially follows from this algebraic identity.

Proof of Theorem 5.3.4. Suppose that $p, q \in \mathbb{R}_+^\gamma[x_1, ..., x_n]$ are real stable polynomials. Then $\text{Pol}^\gamma(p)$ and $\text{Pol}^\gamma(q)$ are real stable multiaffine polynomials by Proposition 2.2.6. We now use the multiaffine bound to prove the result for any $\alpha \in \mathbb{R}_+^n$. For simplicity, let $\beta := \text{Pol}^\gamma(\alpha)$, where $\text{Pol}^\gamma(\alpha)$ is originally defined in §5.1.

$$\langle p, q \rangle^\gamma = \langle \text{Pol}^\gamma(p), \text{Pol}^\gamma(q) \rangle^{(1^\gamma)} \ge \beta^\beta (1 - \beta)^{1-\beta} \, \text{Cap}_\beta(\text{Pol}^\gamma(p)) \, \text{Cap}_\beta(\text{Pol}^\gamma(q))$$

By Proposition 5.1.7, we have that $\text{Cap}_\beta(\text{Pol}^\gamma(p)) = \text{Cap}_\alpha(p)$. So to complete the proof, we compute:

$$\beta^\beta (1 - \beta)^{1-\beta} = \prod_{k=1}^n \prod_{j=1}^{\gamma_k} \left(\frac{\alpha_k}{\gamma_k} \right)^{\alpha_k/\gamma_k} \left(1 - \frac{\alpha_k}{\gamma_k} \right)^{1-\alpha_k/\gamma_k} = \prod_{k=1}^n \left(\frac{\alpha_k}{\gamma_k} \right)^{\alpha_k} \left(\frac{\gamma_k - \alpha_k}{\gamma_k} \right)^{\gamma_k - \alpha_k}$$

This is precisely $\frac{\alpha^\alpha (\gamma - \alpha)^{\gamma - \alpha}}{\gamma^\gamma}$, which is what was claimed. $\square$

Inner Product Bounds, Unbounded Degree

For general polynomials and power series in the $\mathcal{LP}_+$ class, we consider the following inner product.

Definition 5.3.6. For $p, q \in \mathbb{R}[x_1, ..., x_n]$ or power series in $x_1, ..., x_n$, define:

$$\langle p, q \rangle^\infty := \sum_{0 \le \mu} \mu! p_\mu q_\mu$$

Note that this may not be well-defined for some power series.

Consider the following power series in $x_1, ..., x_n$, where $c_\mu \ge 0$:

$$f(x_1, ..., x_n) = \sum_{0 \le \mu} \frac{1}{\mu!} c_\mu x^\mu$$

Next consider the following weighted truncations of f:

$$f_\gamma(x) := \sum_{0 \le \mu \le \gamma} \binom{\gamma}{\mu} c_\mu x^\mu$$

If $f \in \mathcal{LP}_+[x_1, ..., x_n]$, then f_γ is real stable for all γ and $f_\gamma(x/\gamma) \to f(x)$ uniformly on compact sets in $\mathbb{C}^n$ (see Theorem 5.1 in [7]). The idea then is to limit capacity bounds for polynomials of some bounded degree to capacity bounds for general polynomials and functions in the $\mathcal{LP}_+$ class.

To do this, we need some kind of continuity result for capacity. Note that Fact 5.1.5 implies $\mathrm{Cap}_\alpha(p)$ is *not* continuous in α at the boundary of the Newton polytope of p. However, it turns out $\mathrm{Cap}_\alpha(p)$ is continuous in p, for the topology of uniform convergence on compact sets. This is discussed in §5.4 more thoroughly, and we now state the main result from that section.

Corollary 5.4.8. *Let p_n be polynomials with nonnegative coefficients and p analytic such that $p_n \to p$ uniformly on compact sets. For $\alpha \in \mathrm{Newt}(p)$, we have:*

$$\lim_{n \to \infty} \mathrm{Cap}_\alpha p_n = \mathrm{Cap}_\alpha p$$

We now demonstrate the link between the bounded and unbounded degree inner products, and we will use this to obtain bounds on the latter via limiting.

Lemma 5.3.7. *Let f and f_γ be defined as above. For any $p \in \mathbb{R}_+[x_1, ..., x_n]$ we have:*

$$\lim_{\gamma \to \infty} \langle f_\gamma, p \rangle^\gamma = \langle f, p \rangle^\infty$$

Proof. Letting c_μ denote the weighted coefficients of f and f_γ as above, we compute:

$$\lim_{\gamma \to \infty} \langle f_\gamma, p \rangle^\gamma = \lim_{\gamma \to \infty} \sum_{0 \le \mu \le \gamma} c_\mu p_\mu = \sum_{0 \le \mu} c_\mu p_\mu = \langle f, p \rangle^\infty$$

Notice that the limit here is guaranteed to exist, since p has finite support. $\square$

With this, we can bootstrap our capacity bound on $\langle \cdot, \cdot \rangle^\gamma$ to get a bound on $\langle \cdot, \cdot \rangle^\infty$. Notice here that we achieve the same bound as Anari and Gharan in [1], albeit with a simpler proof.

Theorem 5.3.8 (Anari-Gharan). *Fix $f \in \mathcal{LP}_+[x_1, ..., x_n]$ and any real stable $p \in \mathbb{R}_+[x_1, ..., x_n]$. Then for any $\alpha \in \mathbb{R}_+^n$ we have:*

$$\langle f, p \rangle^\infty \ge \alpha^\alpha e^{-\alpha} \mathrm{Cap}_\alpha(f) \mathrm{Cap}_\alpha(p)$$

Proof. As above, we write:

$$f(x) = \sum_{0 \le \mu} \frac{1}{\mu!} c_\mu x^\mu \qquad f_\gamma(x) = \sum_{0 \le \mu \le \gamma} \binom{\gamma}{\mu} c_\mu x^\mu$$

By the previous lemma, we have:

$$\langle f, p \rangle^\infty = \lim_{\gamma \to \infty} \langle f_\gamma, p \rangle^\gamma \geq \lim_{\gamma \to \infty} \left[\frac{\alpha^\alpha (\gamma - \alpha)^{\gamma - \alpha}}{\gamma^\gamma} \operatorname{Cap}_\alpha(f_\gamma) \operatorname{Cap}_\alpha(p) \right]$$

$$= \alpha^\alpha \operatorname{Cap}_\alpha(p) \cdot \lim_{\gamma \to \infty} \left[\frac{(\gamma - \alpha)^{\gamma - \alpha}}{\gamma^\gamma} \cdot \inf_{x > 0} \frac{f_\gamma(x/\gamma)}{(x/\gamma)^\alpha} \right]$$

$$= \alpha^\alpha \operatorname{Cap}_\alpha(p) \cdot \lim_{\gamma \to \infty} \left[\frac{(\gamma - \alpha)^{\gamma - \alpha}}{\gamma^{\gamma - \alpha}} \cdot \operatorname{Cap}_\alpha(f_\gamma(x/\gamma)) \right]$$

Notice that $\lim_{\gamma \to \infty} \operatorname{Cap}_\alpha(f_\gamma(x/\gamma)) = \operatorname{Cap}_\alpha(f)$ by Corollary 5.4.8. So we just need to compute the limit of the scaling factor:

$$\lim_{\gamma \to \infty} \left(\frac{\gamma - \alpha}{\gamma} \right)^{\gamma - \alpha} = \lim_{\gamma \to \infty} \prod_{k=1}^{n} \left(1 - \frac{\alpha_k}{\gamma_k} \right)^{\gamma_k - \alpha_k} = \prod_{k=1}^{n} e^{-\alpha_k} = e^{-\alpha}$$

This completes the proof. $\qquad \square$

From Inner Products to Linear Operators

The main purpose of this section, aside from proving the main technical result of the chapter, is to demonstrate the power of a certain interpretation of the symbol of a linear operator. We will show that a simple observation regarding the symbol (which is explicated in more detail in [41]) will immediately enable us to transfer inner product bounds to bounds on linear operators. We now state this observation, which could be considered as a more algebraic definition of the symbol.

Lemma 5.3.9. *Let $\langle \cdot, \cdot \rangle$ be either $\langle \cdot, \cdot \rangle^\gamma$ or $\langle \cdot, \cdot \rangle^\infty$, and let Symb_+ denote the respective symbol (see Definition 2.3.1). Let T be a linear operator on polynomials of appropriate degree, and let $p, q \in \mathbb{R}_+[x_1, ..., x_n]$ be polynomials of appropriate degree. Then we have the following, where the inner product acts on the z variables:*

$$T[p](x) = \langle \operatorname{Symb}_+[T](z, x), p(z) \rangle$$

Proof. Straightforward, as the scalars present in the expressions of $\langle \cdot, \cdot \rangle$ and Symb_+ were chosen such that they cancel out in the above expression. One could compute this on the monomial basis, for example. $\qquad \square$

As we will see very shortly, this will make for quick proofs of the main results given the inner product bounds we have already achieved. Before doing this though, let us discuss some of the linear operator bounds that Anari and Gharan achieved in [1]. Note the following differential operator form of $\langle \cdot, \cdot \rangle^\infty$:

$$\langle p, q \rangle^\infty = q(\partial_x) q(x)\big|_{x=0}$$

Anari and Gharan then use use their inner product bound to essentially give capacity preservation results for certain differential operators. Similarly, for multiaffine polynomials $\langle p, q \rangle^{(1^n)} = q(\partial_x)q(x)|_{x=0}$, which gives a better bound in the multiaffine case. We now vastly generalize this idea, with a rather short proof.

Theorem 5.3.10. *Let $T : \mathbb{R}_+^\gamma[x_1, ..., x_n] \to \mathbb{R}_+[x_1, ..., x_m]$ be a linear operator such that $\mathrm{Symb}_+(T)$ is real stable in z for every $x \in \mathbb{R}_+^m$. Then for any real stable $p \in \mathbb{R}_+^\gamma[x_1, ..., x_n]$, any $\alpha \in \mathbb{R}_+^n$, and any $\beta \in \mathbb{R}_+^m$ we have:*

$$\frac{\mathrm{Cap}_\beta(T(p))}{\mathrm{Cap}_\alpha(p)} \geq \frac{\alpha^\alpha(\gamma - \alpha)^{\gamma-\alpha}}{\gamma^\gamma} \mathrm{Cap}_{(\alpha,\beta)}(\mathrm{Symb}_+(T))$$

Further, this bound is tight for fixed T, α, and β.

Proof. In the proof, let $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle^\gamma$. By the previous lemma, we have the following for any fixed $x_0 \in \mathbb{R}_+^n$. (Here, the inner product acts on the z variables.)

$$T(p)(x_0) = \langle \mathrm{Symb}_+(T)(z, x_0), p(z) \rangle$$

Theorem 5.3.4 then implies:

$$T(p)(x_0) = \langle \mathrm{Symb}_+(T)(z, x_0), p(z) \rangle \geq \frac{\alpha^\alpha(\gamma - \alpha)^{\gamma-\alpha}}{\gamma^\gamma} \mathrm{Cap}_\alpha(p) \cdot \mathrm{Cap}_\alpha(\mathrm{Symb}_+(T)(\cdot, x_0))$$

Dividing by x_0^β on both sides and taking inf gives:

$$\inf_{x_0>0} \frac{T(p)(x_0)}{x_0^\beta} \geq \frac{\alpha^\alpha(\gamma - \alpha)^{\gamma-\alpha}}{\gamma^\gamma} \mathrm{Cap}_\alpha(p) \cdot \inf_{x_0>0} \inf_{z>0} \frac{\mathrm{Symb}_+(T)(z, x_0)}{z^\alpha x_0^\beta}$$

This is the desired result. Tightness then follows from considering input polynomials of the form $p(x) = \prod_k (1 + x_k y_k)$ for fixed $y \in \mathbb{R}_+^n$, and then taking inf over y. □

As stated in the introduction, this is our main technical result, and we have already discussed some of its applications in §5.2. We give a similar result for linear operators on polynomials of any degree.

Theorem 5.3.11. *Let $T : \mathbb{R}_+[x_1, ..., x_n] \to \mathbb{R}_+[x_1, ..., x_m]$ be a linear operator such that $\mathrm{Symb}_+(T)$ is in $\mathcal{LP}_+[z_1, ..., z_n]$ for every $x \in \mathbb{R}_+^m$. Then for any $p \in \mathbb{R}_+[x_1, ..., x_n]$, any $\alpha \in \mathbb{R}_+^n$, and any $\beta \in \mathbb{R}_+^m$ we have:*

$$\frac{\mathrm{Cap}_\beta(T(p))}{\mathrm{Cap}_\alpha(p)} \geq e^{-\alpha}\alpha^\alpha \mathrm{Cap}_{(\alpha,\beta)}(\mathrm{Symb}_+(T))$$

Further, this bound is tight for fixed T, α, and β.

Proof. The proof given above for Theorem 5.3.10 can be essentially copied verbatim. □

We now combine these results with the Borcea-Brändén characterization results (Theorem 2.3.5) to give concrete corollaries which directly relate to stability preservers.

Corollary 5.3.12. *Suppose $T : \mathbb{R}_+^\gamma[x_1, ..., x_n] \to \mathbb{R}_+[x_1, ..., x_m]$ is a linear operator of rank greater than 2, such that T preserves real stability. Then Theorem 5.3.10 applies to T.*

Proof. Since the image of T is of dimension greater than 2, Theorem 2.3.5 implies one of two possibilities:

1. $\text{Symb}_+[T]$ is real stable.

2. $\text{Symb}_+[T](z_1, ..., z_n, -x_1, ..., -x_n)$ is real stable.

In either case, we have that $\text{Symb}_+[T]$ is real stable in z for every fixed $x \in \mathbb{R}_+^m$ (see Proposition 2.2.4). Therefore Theorem 5.3.10 applies. □

Corollary 5.3.13. *Suppose $T : \mathbb{R}_+[x_1, ..., x_m] \to \mathbb{R}_+[x_1, ..., x_m]$ is a linear operator of rank greater than 2, such that T preserves real stability. Then Theorem 5.3.11 applies to T.*

Proof. The same proof works, using the BB characterization for linear operators on polynomials of all degrees. □

5.4   Continuity of Capacity

In this section, we discuss the continuity of capacity as a function of the the input polynomial p. The main result of this section allows us to limit inner product bounds from $\langle \cdot, \cdot \rangle^\gamma$ to $\langle \cdot, \cdot \rangle^\infty$, which is exactly how we proved Theorem 5.3.8.

Given a (positive) discrete measure μ on $\mathbb{R}^n$, we define its generating function as:

$$p_\mu(x) := \sum_{\kappa \in \text{supp}(\mu)} \mu(\kappa) x^\kappa$$

We further define the log-generating function of μ as:

$$P_\mu(x) := \log(p_\mu(\exp(x))) = \log \sum_{\kappa \in \text{supp}(\mu)} \mu(\kappa) \exp(x \cdot \kappa)$$

More generally for such a function $p(x)$, we will write:

$$p(x) := \sum_\kappa p_\kappa x^\kappa$$

$$P(x) := \log(p(\exp(x))) = \log \sum_\kappa p_\kappa \exp(x \cdot \kappa)$$

We care about discrete measures (with not necessarily finite support) whose generating functions are convergent and continuous on $\mathbb{R}_+^n$. This is equivalent to the log-generating function being continuous on $\mathbb{R}^n$. Note that an important example of such a measure is one which has finite support entirely in $\mathbb{Z}_+^n$. The generating functions of such measures are polynomials.

From now on we will write $\mathrm{supp}(p) = \mathrm{supp}(P)$ to denote the support of μ and $\mathrm{Newt}(p) = \mathrm{Newt}(P)$ to denote the polytope generated by its support. We first give a few basic results.

Fact 5.1.5. *For p a continuous generating function, the following are equivalent.*

1. $\alpha \in \mathrm{Newt}(p)$

2. $\mathrm{Cap}_\alpha p(x) > 0$

3. $P(x) - \alpha \cdot x$ is bounded below.

Lemma 5.4.1. *Any continuous log-generating function $Q(x)$ is convex in $\mathbb{R}^n$.*

Proof. Hölder's inequality. □

Note that proving statements for p is essentially the same as proving for P, as suggested in the following lemma.

Lemma 5.4.2. *Let p, p_n be continuous generating functions. Then $p_n \to p$ uniformly on compact sets of $\mathbb{R}_+^n$ iff $P_n \to P$ uniformly on compact sets of $\mathbb{R}^n$.*

Proof. Equivalence of $p_n \to p$ and $\exp(P_n) \to \exp(P)$ follows form the fact that $\exp : \mathbb{R}_+^n \to \mathbb{R}^n$ is a homeomorphism (and so gives a bijection of compact sets). The fact that $\exp$ and $\log$ are (uniformly) continuous on every compact set in their domains then completes the proof. □

We now get the first half of the desired equality, which is the easier half.

Lemma 5.4.3. *With p, p_n continuous generating functions and $p_n \to p$ uniformly on compact sets, we have:*
$$\lim_{n \to \infty} \inf p_n \leq \inf p$$

Proof. Let $(x_m) \subset \mathbb{R}_+^n$ be a sequence such that $p(x_m) \to \inf p$. For each m we have that $p_n(x_m)$ is eventually near to $p(x_m)$. So for any fixed $\epsilon > 0$, we have the following for $m = m(\epsilon)$ and $n \geq N(\epsilon, m)$:
$$\inf p_n \leq p_n(x_m) \leq p(x_m) + \epsilon \leq \inf p + 2\epsilon$$

The result follows by sending $\epsilon \to 0$. □

We now set out to prove the second half of the desired equality, the difficulty for which arises whenever α is on the boundary of $\mathrm{Newt}(p)$.

Lemma 5.4.4. *Suppose 0 is in the interior of* $\text{Newt}(p)$. *Then* $\inf P$ *is attained precisely on some compact convex subset K of $\mathbb{R}^n$.*

Proof. By a previous lemma, $\inf P$ is finite. Suppose x_n is an unbounded sequence (with monotonically increasing norm) such that $P(x_n)$ limits to $\inf P$. By compactness of the n-dimensional sphere, we can assume by restricting to a subsequence that $\frac{x_n}{||x_n||}$ limits to some u. Pick $\epsilon > 0$ small enough such that $\epsilon u \in \text{Newt}(p)$, and consider $P(x) - \epsilon u \cdot x$. We then have:

$$\lim_{n \to \infty} P(x_n) - \epsilon u \cdot x_n = \lim_{n \to \infty} P(x_n) - \epsilon ||x_n|| \left(u \cdot \frac{x_n}{||x_n||} \right) = -\infty$$

However, since $\epsilon u \in \text{Newt}(p)$ we have that $P(x) - \epsilon u \cdot x$ is bounded below, a contradiction. So, every sequence limiting to $\inf P$ is bounded, and therefore $\inf P$ is attained on a bounded set. By convexity of P, this set is convex. $\qquad \square$

The next few results then finish the proof of continuity of $\text{Cap}_\alpha(\cdot)$ under certain support conditions.

Proposition 5.4.5. *Let p and p_n be continuous generating functions such that $p_n \to p$, with 0 in the interior of* $\text{Newt}(p)$. *Then:*

$$\lim_{n \to \infty} \inf p_n = \inf p$$

Proof. Given the above lemma, we only have the $\geq$ direction left to prove. Since 0 is in the interior of $\text{Newt}(p)$, there is some compact convex $K \subset \mathbb{R}^n$ such that $P(x) = \inf P$ iff $x \in K$. Further, this implies that for any compact set K' whose interior contains K, there exists $c_0 > 0$ such that $P(x) > \inf P + c_0$ on the boundary of K'. For any fixed positive $\epsilon < \frac{c_0}{2}$ and large enough n, we then have:

$$|P_n - P| < \epsilon < \frac{c_0}{2} \text{ in } K' \implies |P_n - \inf P| < \epsilon < \frac{c_0}{2} \text{ in } K$$

$$P_n > \inf P + (c_0 - \epsilon) > \inf P + \frac{c_0}{2} \text{ on the boundary of } K'$$

Convexity of P_n then implies $P_n(x) > \inf P + \frac{c_0}{2}$ outside of K'. Therefore for any ϵ and large enough n:

$$\inf P_n = \inf_{x \in K'} P_n \geq \inf P - \epsilon$$

Letting $\epsilon \to 0$ gives the result. $\qquad \square$

We now set out to prove a similar statement whenever 0 is on the boundary on $\text{Newt}(p)$. This ends up needing a bit more restriction.

Lemma 5.4.6. *Suppose 0 is on the boundary on* $\text{Newt}(P)$. *Then there exists $A \in SO_n(\mathbb{R})$ such that:*

$$\text{Newt}(A \cdot P) \subset \{\kappa : \kappa_n \geq 0\}$$

$$\inf (A \cdot P)|_{x_n = -\infty} = \inf P$$

Proof. Since 0 is on the boundary of the convex set $\mathrm{Newt}(P)$, a separating hyperplane gives a unit vector c such that $(c|\mu) \geq 0$ for all $\mu \in \mathrm{Newt}(P)$. Let $A \in SO_n(\mathbb{R})$ be such that $Ac = e_n$. We first have:

$$\inf A \cdot P = \inf P(A^{-1}x) = \inf P$$

Since $\mathrm{Newt}(A \cdot P) = A \cdot \mathrm{Newt}(P)$ and $(e_n|A\mu) = (c|\mu) \geq 0$ for every $\mu \in \mathrm{Newt}(P)$, we have that $\mathrm{Newt}(A \cdot P) \subset \{\kappa : \kappa_n \geq 0\}$. Therefore:

$$\inf (A \cdot P)|_{x_n=-\infty} = \inf A \cdot P = \inf P$$

Note that $(A \cdot P)|_{x_n=-\infty}$ denotes the continuous log-generating function given by the terms κ of the support of $A \cdot P$ for which $\kappa_n = 0$. This is justified, as $\mathrm{Newt}(A \cdot P) \subset \{\kappa : \kappa_n \geq 0\}$ implies that $A \cdot P$ decreases as x_n decreases (and we care about inf). $\qquad\square$

Theorem 5.4.7. *Let p and p_m be continuous generating functions such that $p_m \to p$, with $0 \in \mathrm{Newt}(p)$. Suppose further that eventually $\mathrm{Newt}(p_m) \subseteq \mathrm{Newt}(p)$. Then:*

$$\lim_{m\to\infty} \inf p_m = \inf p$$

Proof. Given the above proposition, we only need to prove this in the case where 0 is on the boundary of $\mathrm{Newt}(p)$. In that case, the previous lemma gives an $A \in SO_n(\mathbb{R})$ such that $\mathrm{Newt}(A \cdot P) \subset \{\kappa : \kappa_n \geq 0\}$ and $\inf (A \cdot P)|_{x_n=-\infty} = \inf P$. Since $P_m \to P$ implies $A \cdot P_m \to A \cdot P$, we now relax to proving $\lim_{m\to\infty} \inf A \cdot P_m = \inf A \cdot P$. By assumption, eventually $\mathrm{Newt}(P_m) \subseteq \mathrm{Newt}(P)$ which implies $\mathrm{Newt}(A \cdot P_m) \subseteq \mathrm{Newt}(A \cdot P) \subset \{\kappa : \kappa_n \geq 0\}$. So, eventually $\mathrm{Newt}((A \cdot P_m)|_{x_n=-\infty}) \subseteq \mathrm{Newt}((A \cdot P)|_{x_n=-\infty})$ and $\inf A \cdot P_m = \inf (A \cdot P_m)|_{x_n=-\infty}$. By induction on the number of variables, we then have:

$$\lim_{m\to\infty} \inf A \cdot P_m = \lim_{m\to\infty} \inf (A \cdot P_m)|_{x_n=-\infty} = \inf (A \cdot P)|_{x_n=-\infty} = \inf A \cdot P$$

For the base case, p_m and p are scalars and the result is trivial. $\qquad\square$

Corollary 5.4.8. *Let p_n be polynomials with nonnegative coefficients and p analytic such that $p_n \to p$, with $\alpha \in \mathrm{Newt}(p)$. Then:*

$$\lim_{n\to\infty} \mathrm{Cap}_\alpha p_n = \mathrm{Cap}_\alpha p$$

Proof. As in the previous proposition, we only have the $\geq$ direction to prove. Let q_n be defined as the sum of the terms of p_n which appear in the support of p. Since the p_n are polynomials with nonnegative coefficients, we have that $q_n \to p$. By the previous theorem, we then have:

$$\lim_{n\to\infty} \mathrm{Cap}_\alpha p_n \geq \lim_{n\to\infty} \mathrm{Cap}_\alpha q_n = \mathrm{Cap}_\alpha p$$

$\qquad\square$

Note that the fact that $q_n \to p$ holds after restricting to the support of p relies on the fact that p_n and q_n are polynomials with positive coefficients. This is the main barrier to generalizing this corollary to all continuous generating functions.

5.5 Concluding Remarks

We have given here tight bounds on capacity preserving operators related to real stable polynomials. These results are essentially corollaries of inner product bounds, extended from bounds of Anari and Gharan, all eventually based on the strong Rayleigh inequalities. That said, there are a number of pieces of this that may be able to be altered or generalized, and this raises new questions.

The first is that of the inner product: are there other inner products for which we can obtain bounds? The main conjecture in this direction is that of Gurvits in [33].

Conjecture 5.5.1 (Gurvits). *Let $p, q \in \mathbb{R}_+[x_1, ..., x_n]$ be homogeneous real stable polynomials of total degree d. Then:*

$$\sum_{|\mu|=d} \binom{d}{\mu}^{-1} p_\mu q_\mu \geq \frac{\alpha^\alpha}{d^d} \operatorname{Cap}_\alpha(p) \operatorname{Cap}_\alpha(q)$$

The main difference here is that we use multinomial coefficients rather than products of binomial coefficients. Note that the symbol operator associated to this inner product is given by $T[(z \cdot x)^d]$ (dot product of z and x). It is not immediately clear how this inner product relates to real stable polynomials, as the link to stability preservers is less clear than in the Borcea-Brändén case.

The next is the class of polynomials: are there more general classes of polynomials for which weaker capacity bounds can be achieved? One such bound is achieved for *completely log-concave polynomials* in [2], and this class contains basis generating polynomials of matroids. This bound relies on a weakened version of the strong Rayleigh inequalities, where a factor of 2 is introduced. It is unclear what applications such a bound has beyond those of [2].

The last is a question about the further applicability of the main results of this chapter. In particular, all of the operators studied here are differential operators. Are there applications of non-differential operators? Also, are there ways to get a handle on the location of the roots of a polynomial via capacity? This second question is of particular interest, as it may lead to a more unified and a direct approach to various root bounding results. For example, the root bounds of [47] are at the heart of the proof of Kadison-Singer in [50]. Can capacity be used to achieve those bounds?

Chapter 6

The Independence Polynomial

Given a graph $G = (V, E)$, the matching polynomial of G and the independence polynomial of G are defined as follows.

$$\mu(G) := \sum_{\substack{M \subset E \\ M, \text{matching}}} \left(-x^2\right)^{|M|} \qquad\qquad I(G) := \sum_{\substack{S \subset V \\ S, \text{independent}}} x^{|S|}$$

The real-rootedness of the matching polynomial and the Heilmann-Lieb root bound are important results in the theory of undirected simple graphs. In particular, real-rootedness implies log-concavity and unimodality of the matchings of a graph, and recently in [52] the root bound was used to show the existence of Ramanujan graphs. Additionally, it is well-known that the matching polynomial of a graph G is equal to the independence polynomial of the line graph of G. With this, one obtains the same results for the independence polynomials of line graphs. This then leads to a natural question: what properties extend to the independence polynomials of all graphs?

About a decade ago, Chudnovsky and Seymour [15] established the real-rootedness of the independence polynomial for claw-free graphs. (The independence polynomial of the claw is not real-rooted.) A general root bound for the independence polynomial was also given by [23], though it is weaker than the Heilmann-Lieb bound. As with the original results, these generalizations are proven using univariate polynomial techniques.

In this chapter, we prove a result related to the real-rootedness of certain weighted independence polynomials. This result was originally proven by Engström in [18] by bootstrapping the Chudnovsky and Seymour result for rational weights and using density arguments. Further, we then extend the Heilmann-Lieb root bound by generalizing some of Godsil's work on the matching polynomial. In particular his arguments extend to the multivariate matching polynomial, and we determine conditions for which these divisibility results extend to the multivariate independence polynomial. We also prove the Heilmann-Lieb bound for the independence polynomial of a certain subclass of claw-free graphs, and also give a claw-free counterexample (Schläfli graph) to this bound.

6.1   Same-phase Stability

We now introduce a new notion of stability. Notice that the connection between the following conditions is somewhat similar to that which is given by Proposition 2.2.2.

Definition 6.1.1. A polynomial $p \in \mathbb{R}[z_1, ..., z_n]$ is said to be *same-phase stable* if one of the following equivalent conditions is satisfied.

  (i) For every $\alpha \in \mathbb{R}_+^n$, the univariate restriction $p(\alpha z)$ is stable (and therefore real rooted).

  (ii) If $\arg(z_1) = \arg(z_2) = \cdots = \arg(z_n)$, then $p(z_1, ..., z_n) = 0$ implies $z_k \notin \mathcal{H}_+$ for some k.

We will primarily make use of condition (i).

    This notion is strictly weaker than that of "stable", and it will serve as the basic concept in what follows (as stability and real stability did in the previous section). Next, we define a notion of compatibility for real same-phase stable polynomials, which is similar to that of Chudnovsky and Seymour in [15].

Definition 6.1.2. Polynomials $p_1, ..., p_m \in \mathbb{R}_+[z_1, ..., z_n]$ with non-negative coefficients are said to be *same-phase compatible* if p_k is same-phase stable for all k, and the polynomials $\{p_k(\alpha z)\}_{k=1}^m$ are compatible for each $\alpha \in \mathbb{R}_+^n$. Note that by Theorem 2.2.16, we could instead require $\{p_k(\alpha z)\}_{k=1}^m$ have a common interlacing for each $\alpha \in \mathbb{R}_+^n$.

Remark 6.1.3. In order to utilize the theory of interlacing and compatible polynomials, we need to assume that the polynomials we are using have nonnegative coefficients. This is because results like Theorem 2.2.16 no longer hold if negative or complex coefficients are allowed. That said, this restriction is not required to define same-phase stable polynomials, and many other properties also hold without it.

    We now can apply Chudnovsky and Seymour's equivalence result (Theorem 2.2.16) to get the following:

Corollary 6.1.4. *Let $p_1, \dots, p_k \in \mathbb{R}_+[z_1, ..., z_n]$ be polynomials with nonnegative coefficients. The following are equivalent.*

  1. p_i and p_j are same-phase compatible for all $i \neq j$.

  2. $p_1, ..., p_k$ are same-phase compatible.

Same-phase Stability for Multi-affine Polynomials

We now begin to develop a general theory of same-phase stability for multi-affine real polynomials. This class of polynomials is of particular importance here, as most multivariate graph polynomials are real and multi-affine. We start by giving some basic closure properties, in the vein of Proposition 2.2.4.

Proposition 6.1.5 (Closure Properties). *Let $p \in \mathbb{R}[z_1, ..., z_n]$ and $q \in \mathbb{R}[w_1, ..., w_m]$ be multi-affine same-phase stable polynomials, and fix $k \in [n]$. Then the following are also multi-affine same-phase stable. Note that if in addition p and q have nonnegative coefficients, then the following do as well.*

(i) $p \cdot q$ (disjoint product)

(ii) $\partial_{z_k} p$ (differentiation)

(iii) $z_k \partial_{z_k} p$ (variable selection)

(iv) $p(z_1, ..., z_{k-1}, 0, z_{k+1}, ..., z_n)$ (variable deselection)

(v) $z_1 z_2 \cdots z_n p(z_1^{-1}, ..., z_n^{-1})$ (selection inversion)

Proof. (i) Straightforward.

(ii) Fix $\alpha \in \mathbb{R}_+^n$, letting α_k vary. Also, define $\alpha' := (\alpha_1, ..., \hat{\alpha}_k, ..., \alpha_n)$. So, $p(\alpha z)$ is real-rooted for any $\alpha_k \in \mathbb{R}_+$. By Hurwitz's theorem,

$$(\partial_{z_k} p)(\alpha' z) = \lim_{\alpha_k \to \infty} \alpha_k^{-1} p(\alpha z)$$

is also real-rooted. So, $\partial_{z_k} p$ is same-phase stable.

(iii) This follows from (i), since $(z_k \partial_{z_k} p)(\alpha z) = \alpha_k z (\partial_{z_k} p)(\alpha' z)$ is real-rooted iff $(\partial_{z_k} p)(\alpha' z)$ is.

(iv) For any $\alpha \in \mathbb{R}_+^n$ with $\alpha_k = 0$, we have that $p(\alpha_1 z, ..., \alpha_{k-1} z, 0, \alpha_{k+1} z, ..., \alpha_n z) = p(\alpha z)$ is real-rooted by definition of same-phase stability.

(v) Given $\alpha \in \mathbb{R}_+^n$ with strictly positive entries, we have that $p(\alpha^{-1} z)$ has real roots, say at $\gamma_1, ..., \gamma_m$. So, $z^n p(\alpha^{-1} z^{-1}) = \alpha_1 z ... \alpha_n z \cdot p((\alpha_1 z)^{-1}, ..., (\alpha_n z)^{-1})$ has real roots at $\gamma_1^{-1}, ..., \gamma_m^{-1}$. Of course, some of these inverse zeros may be missing when some $\gamma_j = 0$, and there may be extra zeros at $z = 0$. However, this will not affect the real-rootedness of the inverted polynomial. Hurwitz's theorem then allows us to limit to all $\alpha \in \mathbb{R}_+^n$. $\qquad\square$

The names given to some of the closure properties are specific to multi-affine polynomials. In particular, "variable selection" (resp. "variable deselection") refers to the fact that the associated actions will pick out the terms of p which contain (resp. do not contain) a particular variable. Then, "selection inversion" inverts which terms contain which variables. The idea here is to give a combinatorial interpretation to these actions. For example, if the variables correspond to vertices on some graph, then variable deselection might correspond to removal of some vertex.

The next definition is inspired by $p + z_{n+1} q$ used in [7], Lemma 1.8. The proposition that follows then relates this definition to multi-affine polynomials.

Definition 6.1.6. Let $p, f_0, f_1, ..., f_m \in \mathbb{R}[z_1, ..., z_n]$ be polynomials, not necessarily multi-affine, such that

$$p = f_0 + z_{i_1} f_1 + \cdots + z_{i_m} f_m.$$

We call such an expression a *proper splitting* of p (with respect to $\{z_{i_j}\}_j$) if none of the f_k's depend on any of the z_{i_j}'s. We also say that $\{z_{i_j}\}_{j=1}^m$ *splits* p.

Proposition 6.1.7. *Let $p \in \mathbb{K}[z_1, ..., z_n]$ be a multi-affine polynomial, and suppose $\{z_{i_j}\}_{j=1}^m$ splits p. Then p has a unique proper splitting with respect to $\{z_{i_j}\}_j$, expressed as*

$$p = p_0 + \sum_{j=1}^m z_{i_j} \partial_{z_{i_j}} p,$$

where p_0 is the polynomial p with the variables $\{z_{i_j}\}_j$ evaluated at 0.

Another way to think about this proposition is as follows. For a multi-affine polynomial $p \in \mathbb{K}[z_1, ..., z_n]$, we have that $\{z_{i_j}\}_{j=1}^m$ splits p iff no term of p contains more than one variable from $\{z_{i_j}\}_{j=1}^m$. This naturally leads to the use of "variable selection" $(z_{i_j} \partial_{z_{i_j}} p)$ and "variable deselection" (p_0) in the decomposition of p into the above sum of polynomials.

We now reach the main theorem of this section. As mentioned before, this can be seen as a loose analogue of the stability equivalence theorem of Borcea and Brändén in [7].

Theorem 6.1.8. *Let $p \in \mathbb{R}_+[z_1, ..., z_n]$ be a multi-affine polynomial with nonnegative coefficients. The following are equivalent.*

(i) *The polynomial p is same-phase stable.*

(ii) *Given any proper splitting*

$$p = f_0 + \sum_{j=1}^m z_{i_j} f_j$$

we have that f_0, $z_{i_1} f_1$, ..., and $z_{i_m} f_m$ are same-phase compatible.

(iii) *There exists some proper splitting*

$$p = f_0 + \sum_{j=1}^m z_{i_j} f_j$$

such that f_0, $z_{i_1} f_1$, ..., and $z_{i_m} f_m$ are same-phase compatible.

Proof. $(i) \Rightarrow (ii)$ Let $p = f_0 + \sum_{j=1}^m z_{i_j} f_j$ be a proper splitting of p. By uniqueness of the proper splitting, f_0 is the polynomial p with variables $\{z_{i_j}\}$ evaluated at 0, and $z_{i_j} f_j = z_{i_j} \partial_{z_{i_j}} p$. So, by closure properties, each of f_0, $z_{i_1} f_1$, ..., and $z_{i_m} f_m$ is same-phase stable. Now, fix $\alpha \in \mathbb{R}_+^n$ and $\beta \in \mathbb{R}_+^m$, and let $\beta\alpha$ be defined as:

$$(\alpha\beta)_i := \begin{cases} \beta_j \alpha_{i_j}, & i = i_j \\ \alpha_i, & i \notin \{i_j\}_{j=1}^m \end{cases}$$

That is, $\beta\alpha$ is obtained by multiplying the i_j'th entry of α by β_j for all $j \in [m]$. With this, same-phase stability of p implies

$$\left(1 + \sum_j \beta_j\right)^{-1} p(\beta\alpha z) = \frac{f_0(\alpha z) + \sum_j \beta_j [\alpha_{i_j} z f_j(\alpha z)]}{1 + \sum_j \beta_j}$$

is real-rooted for every choice of β, which means every convex combination of $f_0(\alpha z)$, $\alpha_{i_1} z f_1(\alpha z)$, ..., and $\alpha_{i_m} z f_m(\alpha z)$ is real-rooted. So, $f_0(\alpha z)$, $\alpha_{i_1} z f_1(\alpha z)$, ..., and $\alpha_{i_m} z f_m(\alpha z)$ have a common interlacing. Since α was arbitrary, this implies f_0, $z_{i_1} f_1$, ..., and $z_{i_m} f_m$ are same-phase compatible.

$(ii) \Rightarrow (iii)$ This is trivial, given the existence of some proper splitting. In particular, $p = p(0, z_2, ..., z_n) + z_1 \partial_{z_1} p$ is always a proper splitting for multi-affine p.

$(iii) \Rightarrow (i)$ Fix $\alpha \in \mathbb{R}^n_+$. Same-phase compatibility of f_0, $z_{i_1} f_1$, ..., and $z_{i_m} f_m$ implies $f_0(\alpha z)$, $z f_1(\alpha z)$, ..., and $z f_m(\alpha z)$ have a common interlacing. So,

$$\left(1 + \sum_j \alpha_{i_j}\right)^{-1} p(\alpha z) = \frac{f_0(\alpha z) + \sum_j \alpha_{i_j} z f_j(\alpha z)}{1 + \sum_j \alpha_{i_j}}$$

is real-rooted. Since α was arbitrary, this implies p is same-phase stable. $\square$

The power of this statement comes from the fact that same-phase compatibility of any particular splitting implies same-phase compatibility of every possible splitting. We will use this to our advantage in an inductive argument to follow.

6.2 Multivariate Graph Polynomials and Stability

In this section, we discuss the multivariate analogues of the independence and matching polynomials. Though somewhat counterintuitive, considering the multivariate versions of these polynomials actually simplifies the situation. In the multivariate world, one can directly manipulate how particular vertices and edges influence the polynomial by manipulating the associated variable. And further, these polynomials are multiaffine: important operations like differentiation and evaluation at 0 have intuitive interpretations.

Notions like real-rootedness and root bounds become trickier in the multivariate world, but real stability and similar notions can often play the analogous parts. This is true for the multivariate matching polynomial and somewhat true for the multivariate independence polynomial, as we will see below. But first, let's set up some notation.

The Matching Polynomial

The univariate and multivariate matching polynomials have been well studied. In 1972, Heilmann and Lieb proved that for any graph the multivariate matching polynomial is real-stable. This implies the real-rootedness of the univariate matching polynomial, and in fact

Heilmann and Lieb gave bounds on its largest root. More recently, Choe, Oxley, Sokal, and Wagner [14] gave a simpler proof of this fact using a special linear operator on polynomials, called the "multi-affine part". We their proof below.

First though, we define and discuss a few multivariate matching polynomials. The reader should be aware that our notation will be slightly different from that which is standard; we do this to emphasize the connection between the matching and independence polynomials. We give examples of all these polynomials in Figure 6.1.

Given any graph G, we define the multi-affine *vertex matching polynomial* of G as follows.

$$\mu_V(G) \equiv \mu_V(G)(x) := \sum_{\substack{M \subset E \\ M,\text{matching}}} \prod_{\{u,v\} \in M} -x_u x_v$$

Notice that the univariate restriction of $\mu_V(G)$ is the univariate matching polynomial used by Godsil and Heilmann-Lieb, but with the degrees inverted. So, for instance, Heilmann and Lieb's upper bound on the absolute value of the roots of the matching polynomial would translate to a bound *away* from zero for this inverted polynomial. We will discuss this further later. We also define the multiaffine *edge matching polynomial* of G as follows.

$$\mu_E(G) \equiv \mu_E(G)(x) := \sum_{\substack{M \subset E \\ M,\text{matching}}} \prod_{e \in M} x_e$$

We now give the proof of real stability of the vertex matching polynomial, and show its connection to the edge matching polynomial.

Theorem 6.2.1 ([36], [14], [8])**.** *For any graph G, the vertex matching polynomial $\mu_V(G)$ is real stable.*

Proof. Let MAP ("Multi-Affine Part") denote the linear operator on multivariate polynomials which removes any terms which are not multi-affine. By [8], this operator preserves real stability. We then have the following.

$$\mu_V(G)(x) = \text{MAP}\left(\prod_{\{u,v\} \in E} (1 - x_u x_v)\right)$$

Since $(1 - x_u x_v)$ is real stable and the product of real stable polynomials is real stable, this implies the result.   □

This then implies real-rootedness of the univariate matching polynomial via univariate restriction. As for the edge matching polynomial, we don't quite have real stability. However, we do have same-phase stability, which still implies real-rootedness of the univariate restriction.

Corollary 6.2.2. *For any graph G, the edge matching polynomial $\mu_E(G)$ is same-phase stable.*

Proof. Let $\Pi^{\downarrow}$ be the projection operator, which sends all variables x_v to a single variable x. Fixing $(\alpha_e)_{e \in E} \in \mathbb{R}_+^{|E|}$, we have the following.

$$\mu_E(G)(-\alpha x^2) = \sum_{\substack{M \subset E \\ M, \text{matching}}} \prod_{e \in M} -\alpha_e x^2 = (\Pi^{\downarrow} \circ \text{MAP}) \left(\prod_{\{u,v\} \in E} (1 - \alpha_e x_u x_v) \right)$$

By closure properties of real stability and the fact that $\alpha_e > 0$ implies $(1 - \alpha_e x_u x_v)$ is real stable, the right-hand side of the above equation is real-rooted. So, $\mu_E(G)(-\alpha x^2)$ is real-rooted, which implies $\mu_E(G)(\alpha x)$ is real-rooted. (In fact, it has all its roots on the negative part of the real line.) Since α was arbitrary, this implies the result. $\qquad\square$

It's well-known that matchings of graphs are related to independent sets of line graphs. This connection is made particularly clear by considering the (multivariate) edge matching polynomial, as we will see in the next section.

The Independence Polynomial

The univariate independence polynomial of a graph is another well-studied graph polynomial. However, consideration of its roots has proven a bit more difficult. For example, the independence polynomial of a graph is *not* real-rooted in general, and it has only been about a decade since the first proof of real-rootedness for claw-free graphs was published in [15]. Since then a number of proofs of real-rootedness have appeared, along with interesting results about location and modulus of certain roots ([22], [12], [40], [5]).

Here, we give another proof of real-rootedness for claw-free graphs by proving something stronger: namely, that the multivariate independence polynomial of a graph is same-phase stable if and only if the graph is claw-free. In their original proof, Chudnovsky and Seymour show real-rootedness using an intricate recursion based on a combinatorial structure known as a "simplicial clique". By encoding the recursive compatibility using our notion of same-phase stability, we are able to avoid the introduction of simplicial cliques and use simpler graph structures in the recursion. Same-phase stability of the edge matching polynomial then serves as the base case.

Before giving this proof, we need to set up the relevant notation. Given any graph G, we define the multi-affine *independence polynomial* of G as follows.

$$I(G) \equiv I(G)(x) := \sum_{\substack{S \subset V \\ S, \text{independent}}} \prod_{v \in S} x_v$$

Stability properties of the multivariate independence polynomial have been previously studied by Scott and Sokal. In [59], they observe this polynomial as a specific case of a more general statistical-mechanical partition function, and generic lower bounds on the modulus of the roots are studied. In particular, the Lovász local lemma is used to give a universal lower bound of $\frac{1}{e \cdot \Delta}$, where Δ is the maximum degree of G.
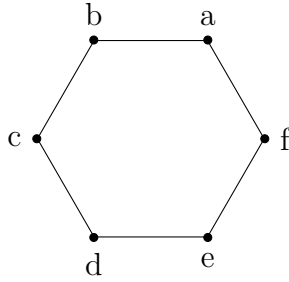
$$\mu_E(C_6, x) = 1 + x_{ab} + x_{bc} + x_{cd} + x_{de} + x_{ef} + x_{fa} +$$
$$x_{ab}x_{cd} + x_{ab}x_{de} + x_{ab}x_{ef} + x_{bc}x_{de} + x_{bc}x_{ef} + x_{bc}x_{fa} +$$
$$x_{cd}x_{ef} + x_{cd}x_{fa} + x_{de}x_{fa} + x_{ab}x_{cd}x_{ef} + x_{bc}x_{de}x_{fa}$$

$$\mu_V(C_6, x) = 1 - x_a x_b - x_b x_c - x_c x_d - x_d x_e - x_e x_f -$$
$$x_f x_a + x_a x_b x_c x_d + x_a x_b x_d x_e + x_a x_b x_e x_f + x_b x_c x_d x_e +$$
$$x_b x_c x_e x_f \ + \ x_b x_c x_f x_a \ + \ x_c x_d x_e x_f \ + \ x_c x_d x_f x_a \ +$$
$$x_d x_e x_f x_a - x_a x_b x_c x_d x_e x_f - x_b x_c x_d x_e x_f x_a$$

$$I(C_6, x) = 1 + x_a + x_b + x_c + x_d + x_e + x_f + x_a x_c +$$
$$x_a x_d + x_a x_e + x_b x_d + x_b x_e + x_b x_f + x_c x_e + x_c x_f +$$
$$x_d x_f + x_a x_c x_e + x_b x_d x_f$$

Figure 6.1: A small graph C_6 with associated independence polynomial, vertex/edge matching polynomials.

As discussed in the notation above, for a given graph G we denote the line graph of G by $L(G)$. Since line graphs are claw-free, we have the following first step toward the desired result.

Corollary 6.2.3. *For any graph G, the independence polynomial $I(L(G))$ of the line graph of G is same-phase stable.*

Proof. By considering the fact that the operator L maps edges to vertices and shared vertices to edges, we actually have the following identity.

$$\mu_E(G) = I(L(G))$$

The previous corollary gives the desired result. □

Of course, this is quite far from the claim that *all* claw-free graphs are same-phase stable. However, as it turns out, line graphs will serve a base case in our induction on general claw-free graphs. To illustrate this, we first give the following lemma.

Lemma 6.2.4. *Let G be a connected claw-free graph which is also triangle-free. Then, G is either a path or a cycle. In particular, G is a line graph.*

Proof. Given a vertex $v \in G$, if the degree of v is greater than 2 then we get either a claw with v as the base or a triangle. We conclude that a graph which is connected, claw-free, and triangle-free is equivalent to being connected and triangle-free with all vertices degree 1 or 2. □

With this, we now give the proof of same-phase stability for claw-free graphs, using the theory of same-phase compatibility developed above. As mentioned in the introduction, this result is a reformulation of a theorem of Engström given in [18].

Theorem 6.2.5 (Engström). *For any claw-free graph G, the independence polynomial $I(G)$ is same-phase stable.*

Proof. We induct on the number of vertices. If G is disconnected, then its independence polynomial is the product of the independence polynomials of its connected components. The inductive hypothesis on components of G (along with the disjoint product closure property for same-phase stable polynomials) then implies the result for G. If G is connected and contains no 3-cliques (triangles), then G is a line graph by the previous lemma. The line graph corollary then implies the result for G. If neither of these conditions is satisfied, then G is a connected graph with at least one 3-clique. Let u, v, w denote the vertices of this 3-clique.

In the independence polynomial $I(G)$, let the variables z_u, z_v, z_w represent the vertices u, v, w, respectively. Consider the following expressions, which are all equal to $I(G)$.

$$I(G)|_{u=v=w=0} + z_u \partial_{z_u} I(G) + z_v \partial_{z_v} I(G) + z_w \partial_{z_w} I(G)$$

$$I(G \setminus \{u, v, w\}) + z_u I(G \setminus N[u]) + z_v I(G \setminus N[v]) + z_w I(G \setminus N[w])$$

$$[I((G \setminus \{u\}) \setminus \{v, w\}) + z_v I((G \setminus \{u\}) \setminus N[v]) + z_w I((G \setminus \{u\}) \setminus N[w])] + z_u I(G \setminus N[u])$$

$$[I((G \setminus \{v\}) \setminus \{u, w\}) + z_u I((G \setminus \{v\}) \setminus N[u]) + z_w I((G \setminus \{v\}) \setminus N[w])] + z_v I(G \setminus N[v])$$

$$[I((G \setminus \{w\}) \setminus \{u, v\}) + z_u I((G \setminus \{w\}) \setminus N[u]) + z_v I((G \setminus \{w\}) \setminus N[v])] + z_w I(G \setminus N[w])$$

The square-bracketed sections of the last three expressions are proper splittings of $I(G \setminus \{u\})$, $I(G \setminus \{v\})$, and $I(G \setminus \{w\})$, respectively. By the inductive hypothesis and the same-phase stability theorem, these proper splittings have terms which are same-phase compatible. So, the terms of the first expression of $I(G)$ are pairwise same-phase compatible. By Corollary 6.1.4, we have that all the terms of the first expression are same-phase compatible. These terms give a proper splitting of $I(G)$, and so Theorem 6.1.8 implies $I(G)$ is same-phase stable. $\square$

An interesting feature of the above proof is the fact that the inductive step did not use the fact that G is claw-free. This suggests that perhaps the theorem can be extended to certain clawed graphs. However, the following corollary shows that this is not the case.

Corollary 6.2.6. *For any graph G, the independence polynomial $I(G)$ is same-phase stable if and only if G is claw-free (3-star-free).*

Proof. By the above theorem, we only need to show that the independence polynomial of a graph with a claw is not same-phase stable. To get a contradiction, let G be a graph such that the vertices u, v, w, x form a claw, and yet $I(G)$ is same-phase stable. Let $p(z_u, z_v, z_w, z_x)$ be the polynomial obtained by evaluating $I(G)$ at zero for all other variables (besides z_u, z_v, z_w, and z_x). By closure properties, p is also same-phase stable. With this we compute $p(\alpha z)$ for $\alpha = (1, 1, 1, 1)$:

$$p(z, z, z, z) = 1 + 4z + 3z^2 + z^3$$

This polynomial is not real-rooted, which gives the desired contradiction. $\square$

With this equivalence in mind, one might wonder for what smaller class of graphs the independence polynomial is actually real stable. A somewhat surprising result is the following.

Proposition 6.2.7. *For any connected graph G, the independence polynomial $I(G)$ is real stable if and only if G is complete (2-star-free).*

Proof. If G is a complete graph, then the independence polynomial of G is $1 + \sum_{v \in V} x_v$, which is real stable. On the other hand, suppose G is some connected incomplete graph such that $I(G)$ is real stable. By incompleteness and connectedness, G contains an induced path P of length at least 2. (E.g., consider the shortest path between two non-adjacent vertices.) In fact, we can assume P is of length exactly 2 by removing all but 3 consecutive vertices. Notice that P is now an induced 2-star. Evaluating $I(G)$ at 0 the variables x_v for which $v \notin P$, we obtain $I(P)$, the independence polynomial of P. Closure properties imply $I(P)$ is real stable.

Labeling the vertices of P as u, v, w, we then have

$$I(P)(x) = 1 + x_u + x_v + x_w + x_u x_w,$$

which, for $x' = (-1, 1, -1)$, gives

$$\partial_{x_u} I(P)(x') \cdot \partial_{x_w} I(P)(x') = 0 < 1 = \partial_{x_u x_w} I(P)(x') \cdot I(P)(x').$$

That is, $I(P)$ is not strongly Rayleigh. So, $I(P)$ is not real stable, which is the desired contradiction. $\square$

6.3 Root Bounds

In addition to proving real rootedness of the matching polynomial, Heilmann and Lieb established bounds on the modulus of roots of the matching polynomial. Since we use the inverted matching polynomial, this result bounds the roots, λ, of $\mu_V(G)$ away from zero:

$$|\lambda| \geq \frac{1}{2\sqrt{\Delta - 1}}$$

Since $\mu_V(G)(x) = I(L(G))(-x^2)$ this result can be stated equivalently as a bound on the root closest to zero, λ_1, for the independence polynomial of line graphs. To do this note that the maximum degree, Δ, of a graph is equal to the clique size, ω, of its line graph.

$$\lambda_1(I(L(G))) \leq \frac{1}{4(\omega - 1)}$$

Since all line graphs are claw-free graphs, we can seek out similar bounds for the independence polynomial of claw-free graphs. In what follows, we adapt the methods of Godsil

to determine such root bounds for a certain subclass of claw-free graphs, namely those which contain a simplicial clique. (Although we were able to avoid simplicial cliques in the proof of real-rootedness, they turn out to be crucial to generalizing the Heilmann-Lieb root bound.) We then discuss how the bound does not extend to all claw-free graphs.

To this end, we first discuss Godsil's original divisibility result which was key to his proof of the Hielmann-Lieb root bound. We do this in the multivariate world, though, so as to provide context for the later results on the independence polynomial.

Path Trees

A basic element of Godsil's proof of the root bound is the notion of a path tree of a graph. We now define this notion as he did, and subsequently discuss what needs to be altered in order to apply it to the multivariate matching polynomial.

Definition 6.3.1. Given a graph G and vertex v, we define the *(labeled) path tree* $T_v(G)$ of G with respect to v recursively as follows. If G is a tree, we define $T_v(G) = G$, and we say that v is the root of $T_v(G)$. We also label the vertices of $T_v(G)$ using the vertices of G. (In the recursive step, we will continue to label using vertices of G.)

For an arbitrary graph G, we first consider the forest which is the disjoint union of the labeled trees $T_w(G \setminus \{v\})$ for each $w \in N(v)$. We then define $T_v(G)$ by appending a vertex (the root) labeled v and connecting it to the roots of each of these trees.

Remark 6.3.2. Figure 6.2 gives an example of a path tree. Note that it is defined in such a way that the paths stemming from v in G and from the root, v in $T_v(G)$, are in order preserving bijection (where the order on paths is the subpath ordering).

In Godsil's proof of the root bound for the matching polynomial, he shows that the univariate vertex matching polynomial of G divides that of $T_v(G)$ for any v. In the multivariate world, this divisibility relation won't be possible, a priori, since there are potentially far more vertices (and hence, variables) in $T_v(G)$ than in G. However, using the labeling of the vertices described above, we can in fact extend this divisibility result. We now formalize this notion of labeling, so as to easily generalize it to all relevant multivariate graph polynomials.

Let G, H be two graphs, and let $\phi : G \to H$ be a graph homomorphism. We call this homomorphism a *labeling of G by H*. For a graph G, we define the *relative vertex matching polynomial* (with respect to ϕ) as follows.

$$\mu_V^\phi(G) \equiv \mu_V^\phi(G)(x) := \sum_{\substack{M \subset E(G) \\ M, \text{matching}}} \prod_{\{u,v\} \in M} -x_{\phi(u)} x_{\phi(v)}$$

We define the *relative edge matching polynomial* and the *relative independence polynomial* (with respect to ϕ) analogously. When unambiguous, we will remove the ϕ superscript from the notation. Notice that the univariate specialization of each of the normal matching and independence polynomials is the same as that of the relative matching and independence

polynomials, for any ϕ. This notion then gives us a way to compare multivariate matching and independence polynomials from different graphs without destroying any univariate information.

Now, consider the labeling of vertices described in the construction of $T_v(G)$ above. This can extended to a graph homomorphism, $\phi_v : T_v(G) \to G$ in a unique way. Specifically, the vertices of $T_v(G)$ are mapped to the vertices of G via the labeling given above (e.g., the root of $T_v(G)$ maps to $v \in G$, the neighbors of the root are mapped to the neighbors of $v \in G$, etc.). An edge $\{u, w\}$ of $T_v(G)$ is then mapped to the edge $\{T_v(u), T_v(w)\}$ in G, which exists by the inductive construction given above.

In what follows, we will consider the graph polynomials $\mu_V^{\phi_v}(T_v(G))$ and $\mu_E^{\phi_v}(T_v(G))$. For simplicity of notation, we will from now on denote these polynomials $\mu_V(T_v(G))$ and $\mu_E(T_v(G))$, respectively. That is, reference to ϕ_v will be dropped.

With this, we now state the generalization of Godsil's divisibility theorem for the vertex matching polynomial. We omit the proof, as this theorem turns out to be a corollary of a more general result related to independence polynomials.

Theorem 6.3.3 (Godsil). *Let v be a vertex of the graph $G = (V, E)$, and let $T \equiv T_v(G)$ be the path tree of G with respect to v. Further, let $\mu_V(T) \equiv \mu_V^{\phi_v}(T)$ denote the relative vertex matching polynomial. We then have the following.*

$$\frac{\mu_V(G)}{\mu_V(G \setminus v)} = \frac{\mu_V(T)}{\mu_V(T \setminus v)}$$

Further, $\mu_V(G)$ divides $\mu_V(T)$.

By univariate specialization, this gives us the first step toward the well-known Heilmann and Lieb root bound (up to inversion of the input variable). We now attempt to generalize this divisibility to independence polynomials. First, however, we will need to develop some path tree analogues.

Path Tree Analogues

Induced Path Trees

Given a graph G and a vertex v, the *induced path tree* $T_v^{\angle}(G)$ of G with respect to v is intuitively defined as follows: it is the path tree that is constructed when only *induced* paths are considered. That is, we use the recursive process of creating the usual path tree, only we forbid traversal of vertices which are neighbors of previously traversed vertices. So, another name that could be used for this tree is the "neighbor-avoiding" path tree.

We now give an explicit definition of the induced path tree. The crucial difference between this definition and the definition of the path tree given above is that neighbors of a vertex are excluded in the recursive step.

Definition 6.3.4. Given a graph G and vertex v, we define the *induced path tree* $T_v^{\angle}(G)$ of G with respect to v recursively as follows. If G is a tree, we define $T_v^{\angle}(G) = G$, and we say that v is the root of $T_v^{\angle}(G)$.

For an arbitrary graph G, we first consider the forest which is the disjoint union of the trees $T_w^{\angle}(G \setminus N[v] \cup \{w\})$ for each $w \in N(v)$. We then define $T_v^{\angle}(G)$ by appending a vertex corresponding to v (the root) and connecting it to the roots of each of these trees.

We also define a slightly different version of the induced path tree. As will be seen, this adjusted definition is more appropriate for our purposes.

Definition 6.3.5. Given a graph G and a clique K, the *induced path tree* $T_K^{\angle}(G)$ of G with respect to K is defined as follows. Construct a new graph G^* by attaching a new vertex $*$ to G, with the property that $\{*, u\} \in E(G^*)$ iff $u \in K$. Then, define $T_K^{\angle}(G) := T_{\{*\}}^{\angle}(G^*)$.

Remark 6.3.6. As with the path tree, we can label the vertices of the induced path tree in a natural way. This gives rise to graph homomorphisms $\phi_v : T_v^{\angle}(G) \to G$ and $\phi_K : T_K^{\angle}(G) \to G^*$.

Simplicial Clique Trees

We need two graph theoretic concepts before defining our final path tree analogue. Given a graph G, let $K \leq G$ be an induced clique. Then, K is called a *simplicial clique* if for all $u \in K$, $N[u] \cap (G \setminus K)$ is a clique as an induced subgraph of G (or equivalently, as an induced subgraph of $G \setminus K$). Intuitively, this means that neighborhoods of each $u \in K$ are two cliques joined at u: one is K itself, and the other consists of the remaining neighbors of u. Simplicial cliques have been studied frequently in relation to the independence polynomial of a graph, and in particular, they were used in Chudnovsky and Seymour's original proof of real-rootedness for claw-free graphs.

We further say that a graph G is *simplicial* if it is claw-free and contains a simplicial clique. It may at first seem strange as to why "claw-free" is included in this definition. The main reason is the useful recursive structure that can be extracted from the following lemma.

Lemma 6.3.7 ([15]). *Let G be claw-free, and let $K \leq G$ be a simplicial clique in G. For any $u \in K$, $N[u] \cap (G \setminus K)$ is a simplicial clique in $G \setminus K$.*

Remark 6.3.8. One can easily check that our definition of a simplicial graph is equivalent to having a recursive structure of simplicial cliques as indicated in the previous lemma.

A *block graph* (or *clique tree*) is a graph in which every maximal 2-connected subgraph is a clique [54]. As it turns out, block graphs are precisely the line graphs of trees. From this observation we note that there is a natural tree-like recursive structure on block graphs. Specifically, let B be a block graph, and let K be a clique in B. Then, $B \setminus K$ is a "forest of block graphs". That is, if we refer to K as the "root clique" in B, then each "root clique" in the forest $B \setminus K$ is connected to some vertex of K in B.

We now define a special kind of clique tree. Notice that while the term "tree" is used, the graphs defined here are not actually trees in the usual sense.

Definition 6.3.9. Given a simplicial graph G and simplicial clique $K \leq G$, we define the *(simplicial) clique tree* $T_K^{\boxtimes}(G)$ of G with respect to K recursively as follows. If $G = K$, we define $T_K^{\boxtimes}(G) = G$, and we say that K is the "root clique" of $T_K^{\boxtimes}(G)$.

For an arbitrary graph G, we first consider the "forest of simplicial clique trees" which is the disjoint union of $T_{J_u}^{\boxtimes}(G \setminus K)$ for each $u \in K$. (Here, we define $J_u := N[u] \cap (G \setminus K)$.) Note that this is valid, since the previous lemma implies J_u is a simplicial clique for all $u \in K$. We then define $T_K^{\boxtimes}(G)$ by appending the clique K (the root clique) and connecting each vertex $u \in K$ to each vertex of the root clique of $T_{J_u}^{\boxtimes}(G \setminus K)$.

Remark 6.3.10. We can label the vertices of the (simplicial) clique tree in the usual way, and this gives rise to a natural graph homomorphism $\phi_K : T_K^{\boxtimes}(G) \to G$.

For examples of the induced path tree and the simplicial clique tree, see Figures 6.2 and 6.3.

Divisibility Relations

Given the above definitions, the main goal of this section is to demonstrate the following theorem. Here, for $v \in G$ we define $K_v \leq L(G)$ via $K_v := L(\{e \in E(G) : v \in e\})$. That is, K_v can be thought of as "the clique in $L(G)$ associated to $N[v]$".

Theorem 6.3.11. *Let L be the line graph operator, T_v the path tree operator with respect to v, $T_K^{\angle}$ the induced path tree operator with respect to K, and $T_K^{\boxtimes}$ the clique tree operator with respect to K. Then the following diagram commutes up to isomorphism.*

$$
\begin{array}{ccc}
\{graphs\} & \xrightarrow{\quad T_v \quad} & \{trees\} \\
\Big\downarrow{\scriptstyle L} & \raisebox{1.5ex}{$\scriptstyle T_K^{\angle}$}\nearrow & \Big\downarrow{\scriptstyle L} \\
\{simplicial\ graphs\} & \xrightarrow{\ T_K^{\boxtimes}\ } & \{simpl.\ block\ graphs\}
\end{array}
$$

In the upper left triangle, commutativity is achieved for $K = K_v$.

This can be broken down into a few results, which we give now.

Lemma 6.3.12. *For any graph G and any $v \in G$, K_v is a simplicial clique of $L(G)$. In particular, $L(G)$ is simplicial.*

Proof. It is easy to see that K_v is a clique. If we consider $w \in K_v$, this corresponds to an edge $e_w \in E(G)$ that has v as an endpoint. Then given any two neighbors of w that are not in K_v, we know they correspond to two edges which share an endpoint with e_w but do not have v as an endpoint. Hence they both share the other endpoint of e_w and are therefore

connected in the line graph. This shows that $N[w] \setminus K_v$ is a clique, so K_v is a simplicial clique.

It is well known that line graphs are claw-free, so all line graphs are simplicial.   □

Proposition 6.3.13. *For any (nonempty) graph G and any $v \in V$, the induced path tree of $L(G)$ with respect to K_v is isomorphic to the path tree of G with respect to v. That is, $T^{\angle}_{K_v}(L(G)) \cong T_v(G)$.*

Proof. First, let G be the graph with one vertex, v. Then, $L(G)$ is the empty graph and $T^{\angle}_{K_v} \circ L(G)$ is also the graph with one vertex (recall that the operator $T^{\angle}_{K_v}$ adds an extra vertex to the input graph). On the other hand, $T_v(G)$ is the graph with one vertex, and the result holds in this case.

Now, let G be a connected graph consisting of two or more vertices, and let v be some vertex of G. (We can assume WLOG that G is connected, since T_v and $T^{\angle}_{K_v}$ only deal with connected components of v and K_v, respectively.) We proceed inductively, adopting the convention that $K_u \le L(G)$ and $K'_u \le L(G \setminus \{v\})$ are the cliques associated to $N[u]$ in the respective line graphs.

We first consider $T_v(G)$. For each $u \in N(v)$, we have that $T_u(G \setminus \{v\})$ is naturally a subtree of $T_v(G)$. In fact, $T_v(G)$ can be viewed as the disjoint union of $T_u(G \setminus \{v\})$ for all $u \in N(v)$, connected to a single vertex corresponding to v.

We next consider $T^{\angle}_{K_v} \circ L(G)$. Notice that $L(G \setminus \{v\}) \cong L(G) \setminus K_v$. For any $u \in N(v)$, this implies $T^{\angle}_{K'_u} \circ L(G \setminus \{v\}) \cong T^{\angle}_{J_u}(L(G) \setminus K_v)$, where $J_u := K_u \cap (L(G) \setminus K_v)$. Recall that the $T^{\angle}_K$ operator adds an extra vertex attached to each vertex of K. So, we can view $T^{\angle}_{K_v} \circ L(G)$ as the disjoint union of $T^{\angle}_{J_u}(L(G) \setminus K_v)$ for all $u \in N(v)$, along with an extra vertex connected to each of the added extra vertices in the disjoint union.

By the induction hypothesis, we have $T_u(G \setminus \{v\}) \cong T^{\angle}_{K'_u} \circ L(G \setminus \{v\})$ for all $u \in N(v)$. This implies that the two descriptions given above of $T_v(G)$ and $T^{\angle}_{K_v} \circ L(G)$, respectively, are equivalent. Therefore, $T_v(G) \cong T^{\angle}_{K_v} \circ L(G)$.   □

Proposition 6.3.14. *For any simplicial graph G and any simplicial clique $K \le G$, the line graph of the induced path tree of G with respect to K is isomorphic to the clique tree of G with respect to K. That is, $L(T^{\angle}_K(G)) \cong T^{\boxtimes}_K(G)$.*

Proof. There is a natural grading on the edges of $T^{\angle}_K(G)$, where the edges from $*$ to vertices in K have grading 1, and edges from vertices $v \in K$ to vertices in $N[v] \setminus K$ have grading 2, and so forth. Then under the line graph operation we get a grading on the vertices of $L \circ T^{\angle}_K(G)$.

Similarly $T^{\boxtimes}_K(G)$ has a natural grading on the vertices by grading K as grade 1, and for every vertex $v \in K$, grading the clique $N[v] \setminus K$ as grade 2, and so forth.

Now we can induct on the number of vertices in G. The result is obviously true for the graph with one vertex. It is then clear that the first grades of $L \circ T^{\angle}_K(G)$ and $T^{\boxtimes}_K(G)$ are isomorphic: they are both cliques of size K. We then label the vertices of the first grade in $L \circ T^{\angle}_K(G)$ by vertices in K as follows. Each vertex of the first grade comes from an edge in
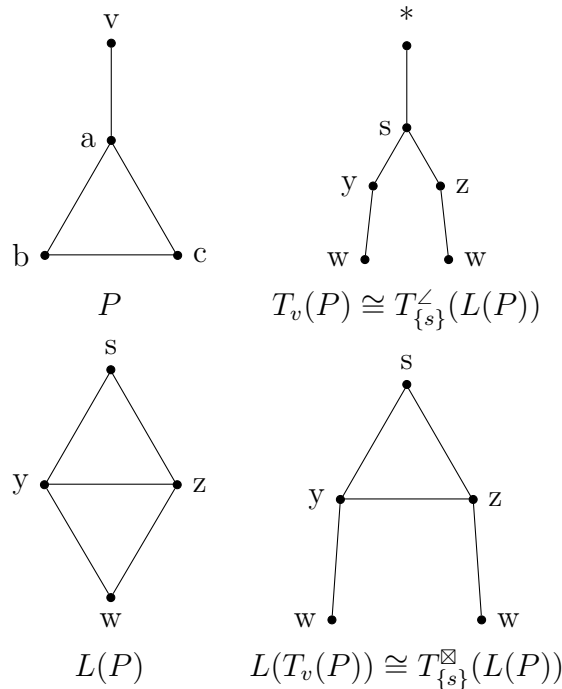
Figure 6.2: An example of a graph and its line graph, induced path tree and simplicial clique tree.

$T_K^{\angle}(G)$ of the form $\{*, v\}$, for some $v \in K$. So, we label this first-grade vertex in $L \circ T_K^{\angle}(G)$ by "v".

In $L \circ T_K^{\angle}(G)$, this vertex labeled "v" connects to edges in G from v to vertices in $N[v] \setminus K$ in $T_K^{\angle}(G)$. In this way we see viewing the sub-clique tree (obtained by looking at v and all of the grades below it) rooted at the vertex labeled v in $L \circ T_K^{\angle}(G)$ is $L \circ T_{N[v] \setminus K}^{\angle}(G)$. Likewise by looking at the vertex labeled v in $T_K^{\boxtimes}(G)$ we see the sub-clique tree obtained by looking at v and all grades below it is exactly $T_{N[v] \setminus K}^{\boxtimes}(G)$, by definition of the simplicial clique tree. By induction our claim is proved. $\qquad\square$

There are two comments to be made about this diagram. First, we can consider the induced path tree operator as some sort of "inverse" or "adjoint" to the line graph operator. In fact, for $G \in \{\text{trees}\}$ (resp. $G \in \{\text{simpl. block graphs}\}$) we have that $T_K^{\angle}$ is the left (resp. right) inverse of L.

Second, consider the outer rectangle of the diagram. We see that the line graph operator "passes" the path tree operator to the clique tree operator. So, if Godsil's divisibility relation can be shown to hold between a simplicial graph and its clique tree, we will be able to derive the same relation between a graph and its path tree as a corollary. (The corollary will actually be for the *edge* matching polynomial. A simple argument then gives the result for the vertex matching polynomial, as we will see below.)
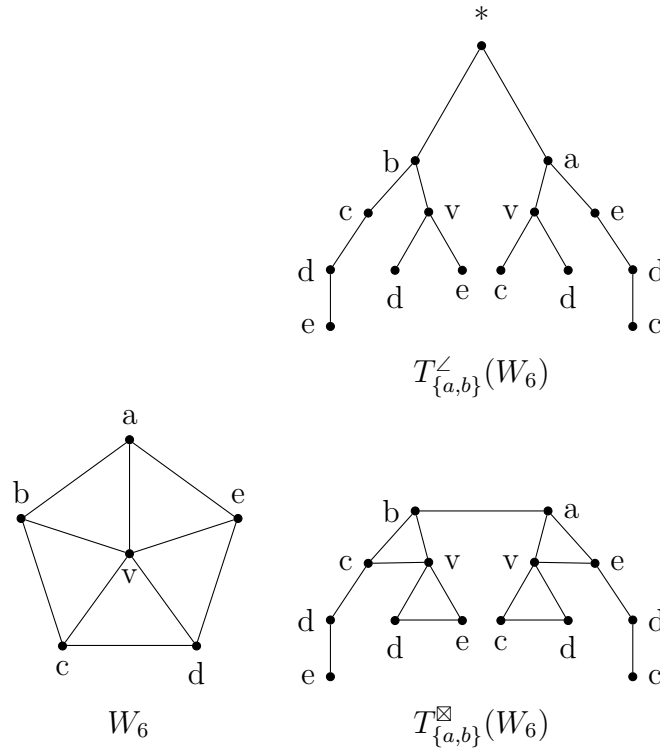
We now generalize Godsil's theorem.

Figure 6.3: An example of a non-line graph W_6, its induced path tree and simplicial clique tree.

Theorem 6.3.15. *Let K be a simplicial clique of the simplicial graph $G = (V, E)$, and let $T \equiv T_K^{\boxtimes}(G)$ be the clique tree of G with respect to K. Further, let $I(T) \equiv I^{\phi_K}(T)$ denote the relative independence polynomial. We then have the following.*

$$\frac{I(G)}{I(G \setminus K)} = \frac{I(T)}{I(T \setminus K)}$$

Proof. We induct on $|V(G)|$. Note that if G is a simplicial block graph, then $T = G$, and so the result is true.

For the general case we get:

$$\frac{I(G)}{I(G \setminus K)} = \frac{I(G \setminus K) + \sum_{v \in K} x_v I(G \setminus N[v])}{I(G \setminus K)}$$

$$= 1 + \sum_{v \in K} \frac{x_v I(T_{N[v]}^{\boxtimes}(G \setminus K) \setminus N[v])}{I(T_{N[v]}^{\boxtimes}(G \setminus K))}$$

$$= 1 + \sum_{v \in K} \frac{x_v I(T_{N[v]}^{\boxtimes}(G \setminus K) \setminus N[v]) \prod_{w \in K, w \neq v} I(T_{N[w]}^{\boxtimes}(G \setminus K))}{I(T_K^{\boxtimes}(G) \setminus K)}$$

$$= 1 + \sum_{v \in K} \frac{x_v I(T_K^{\boxtimes}(G) \setminus N[v])}{I(T_K^{\boxtimes}(G) \setminus K)}$$

$$= \frac{I(T_K^{\boxtimes}(G) \setminus K) + \sum_{v \in K} x_v I(T_K^{\boxtimes}(G) \setminus N[v])}{I(T_K^{\boxtimes}(G) \setminus K)}$$

$$= \frac{I(T_K^{\boxtimes}(G))}{I(T_K^{\boxtimes}(G) \setminus K)}$$

In the above we use the recursion formula for the independence polynomial expanding at a clique and the fact that $N[v]$ is a simplicial clique in $G \setminus K$ when K is a simplicial clique. Notice also that the relative independence polynomial $I \equiv I^{\phi_K}$ is needed in order for the last equality to hold. $\qquad \square$

Remark 6.3.16. We compute the independence polynomials of the appropriate graphs from Figure 6.2 to illustrate the divisibility relations proved in the preceding theorem: $I(L(P), x) = 1 + x_s + x_y + x_z + x_w + x_s x_w$, $I(T_{\{s\}}^{\boxtimes}(L(P))) = (1 + x_s + x_y + x_z + x_w + x_s x_w) \cdot (1 + x_w) = I(L(P), x) \cdot (1 + x_w)$

The proof we gave for the previous theorem is essentially the one Godsil gives for his original theorem, except that we deal with simplicial cliques rather than vertices. The previous theorem now yields the following corollaries.

Corollary 6.3.17. $I(G)$ *divides* $I(T_K^{\boxtimes}(G))$ *for any simplicial graph G with simplicial clique K.*

Proof. We have seen that $G \setminus K$ is a simplicial graph. The previous theorem can be written as:

$$\frac{I(T_K^{\boxtimes}(G))}{I(G)} = \frac{I(T_K^{\boxtimes}(G) \setminus K)}{I(G \setminus K)} = \frac{\prod_{v \in K} I(T_{N[v] \setminus K}^{\boxtimes}(G \setminus K))}{I(G \setminus K)}$$

Then since $N[v] \setminus K$ is a simplicial clique in $G \setminus K$, by induction we have the denominator divides any term in the numerator, so the right hand side is a polynomial, as desired. $\qquad \square$

Corollary 6.3.18. *Given a simplicial graph G, we have that $\lambda_1(G) \leq \frac{-1}{4(\omega - 1)}$.*

Proof. By the previous corollary we have $\lambda_1(G) \leq \lambda_1(T_K^{\boxtimes}(G))$. Then by the commutativity of the diagram, we have seen $T_K^{\boxtimes}(G) = L(T_K^{\angle}(G))$. Hence we have $\lambda_1(T_K^{\boxtimes}(G)) \leq \frac{-1}{4(\omega-1)}$ is equivalent to the identical root bound on $\mu_E(T_K^{\angle}(G))$. Godsil provides bounds on this root by relating the matching polynomial of a tree to its characteristic polynomial, and then bounding the roots of the characteristic polynomial by its maximal degree Δ. Since the maximum degree of the vertices in $T_K^{\angle}(G)$ is ω, we get our desired bound. $\qquad\square$

Remark 6.3.19. In their original paper, Heilmann and Lieb prove a root bound for weighted matching polynomials, where one puts weights on the vertices. Since the previous corollary works in the multivariate case, one could use this framework to derive similar results for weighted independence polynomials.

Other Bound on λ_1

Briefly we mention some easy lower bounds on $\lambda_1(G)$. In what follows we let G be any graph. First we note how modifying our graph by removing edges or removing vertices affects $\lambda_1(G)$.

Proposition 6.3.20. *Let G be any graph, v a vertex in that graph, and $e = \{u, w\}$ an edge in the graph.*

1. $\lambda_1(G \setminus v) \leq \lambda_1(G)$

2. $\lambda_1(G \setminus e) \leq \lambda_1(G)$

Proof. To prove these we need the following recurrences:

$$I(G) = I(G \setminus v) + xI(G \setminus N[v])$$

$$I(G) = I(G \setminus e) - x^2 I(G \setminus (N[u] \cup N[w]))$$

To prove the first statement we prove the following statement by induction: Given any $H \subset V(G)$, we have $I(G \setminus H)$ is nonnegative on the interval $[\lambda_1(G), \infty)$. If $G \setminus H$ is not the empty graph, then $I(G\setminus H)$ is not the zero polynomial so this implies that $\lambda_1(G\setminus H) \leq \lambda_1(G)$. If $G \setminus H$ is the empty graph it is trivially true.

For $|V(G)| = 1$, it is easily checked to be true. Assuming this to be true for $|V(G)| \leq n-1$, let G be a graph with $|V(G)| = n$. Then if $H = G$, we noted this is trivially true. Then it suffices to show that $\lambda_1(G \setminus v) \leq \lambda_1(G)$. By induction we know $I(G \setminus N[v])$ is nonnegative on $[\lambda_1(G \setminus v), \infty)$. Then we know $xI(G \setminus N[v])$ is nonpositive on $[\lambda_1(G \setminus v), 0)$ (all the roots of independence polynomials are negative). By the recurrence relation, $I(G)$ at $\lambda_1(G \setminus v)$ is nonpositive, so by the intermediate value theorem $I(G)$ has a root in $[\lambda_1(G \setminus v), 0)$, as desired.

To prove the second claim, since $G \setminus (N[u] \cup N[w])$ is a induced subgraph of $G \setminus e$, we have $I(G \setminus (N[u] \cup N[w]))$ is nonnegative on $[\lambda_1(G \setminus e), \infty)$. By the recurrence, we have that $I(G)$ evaluated at $\lambda_1(G \setminus e)$ is nonpositive, and so by the intermediate value theorem we see $\lambda_1(G \setminus e) \leq \lambda_1(G)$. $\qquad\square$

Using this we can get the following simple lower bound on λ_1:

Proposition 6.3.21. $\frac{-1}{\omega} \leq \lambda_1(G)$

Proof. Let $K_\omega \leq G$ be the largest clique in G. Then by our previous proposition we have $\lambda_1(K_\omega) \leq \lambda_1(G)$. We have $I(K_\omega) = 1 + \omega x$, so $\lambda_1(K_\omega) = \frac{-1}{\omega}$. $\qquad\qquad\square$

These results hold for all graphs, but combining these with our previous results for simplicial graphs G, we see:
$$\frac{-1}{\omega} \leq \lambda_1(G) \leq \frac{-1}{4(\omega - 1)}$$

6.4   Failure of the Root Bounds

Recall we have the following inclusions of types of graphs:

$$\{\text{Line Graphs}\} \subset \{\text{Simplicial Graphs}\} \subset \{\text{Claw-Free Graphs}\}$$

The root bounds for the matching polynomial carry over to the independence polynomial for line graphs. And by extending the proof method of Godsil, we demonstrated the equivalent root bounds for simplicial graphs. The natural next question is: how general can the graphs get before the root bound fails?

In what follows we provide a claw-free graph (which is not simplicial) for which the root bound fails. We then provide a much weaker root bound for claw-free graphs. It is unknown whether this weaker root bound is tight due to our lack of examples of claw-free graphs which are not simplicial.

Schläfli Graph

The Schläfli graph is the unique strongly regular graph with parameters 27, 16, 10, 8. It is the complement of the Clebsch graph, the intersection graph of the 27 lines on a cubic surface. The Clebsch graph is triangle free, and hence the Schläfli graph is claw-free. We refer the reader to [11] for a comprehensive reference on the Schläfli graph and related graphs.

Keeping in mind that our root bound is equivalent to the statement $\lambda_1(G) \cdot 4 \cdot (\omega - 1) \leq -1$, we calculate the following.

Lemma 6.4.1. *We have the following:*

(i) *The independence polynomial of the Schläfli graph is $45t^3 + 135t^2 + 27t + 1$.*

(ii) *The clique size of the Schläfli graph is 6.*

(iii) $\lambda_1(Schläfli\ graph) \cdot 4 \cdot (\omega - 1) > -1$

Proof. One can calculate the independence polynomial and clique size using any computer algebra system; we used Sage.

To show our graph breaks the root bound it suffices to show that $I(G)(t/20)$ has a root in $(-1, 0)$. In fact we can easily calculate that $I(G)(-1/20) = -29/1600$ while $I(G)(0) = 1$, so there is a root in $(-1, 0)$. □

Weaker Root Bounds for Claw-free Graphs

Given any claw-free graph G, we can introduce a simplicial clique by modifying the graph as follows:

Lemma 6.4.2. *Let G be a claw-free graph. Given any vertex $v \in G$, we can form a new graph $S_v(G)$ by connecting all of $N[v]$ together to form a clique. Then, $S_v(G)$ is claw-free and $\{v\}$ is a simplicial clique in $S_v(G)$.*

Proof. It is clear that $\{v\}$ will be a simplicial clique in $S_v(G)$. To see that $S_v(G)$ is claw-free, suppose one of the added edges creates a claw. Then we have $u, w \in N(v)$ and a claw with some u as the internal node and w as a leaf. Since we have connected all of the neighbors of v together, we must have the other two leaves of the claw outside of $N[v]$. However these two vertices therefore are not connected to v or each other, and hence form a claw with u as the internal node and v as the other leaf. This provides a contradiction since G is claw-free. □

When analyzing the clique tree of $S_v(G)$ starting at the newly formed simplicial clique $\{v\}$, we notice that the first rung of the clique tree is $\{v\}$, the second rung is $N(v)$, and beyond that are clique trees that live in $G \setminus N[v]$. This observation immediately yields the following:

Proposition 6.4.3. *Given any claw-free graph G and a vertex $v \in G$, we have:*

$$\lambda_1(S_v(G)) \leq \frac{-1}{4 \cdot \max\{\omega - 1, deg(v)\}}$$

This yields the following root bound for G:

$$\lambda_1(G) \leq \frac{-1}{4 \cdot \max\{\omega - 1, \delta\}}$$

Proof. By Proposition 6.3.20, we have $\lambda_1(G) \leq \lambda_1(S_v(G))$. To optimize the bound we pick the vertex v which has minimal degree in the graph, δ. □

In the Schläfli graph we have a large gap between the clique size of 6 and minimal degree of 16. We think that other non-simplicial claw-free graphs with a large gap between clique size and minimal degree may provide good candidates for studying this root bound. Further, finding a family of graphs which require this looser bound could assist in showing how optimal this bound is for non-simplicial claw-free graphs.

6.5 Concluding Remarks

Above, we presented independence polynomials analogues to the real-rootedness (subsequently real stability) and the root bounds of the matching polynomial. We expect other results about the matching polynomial to be generalizable to the independence polynomial. In what follows we list a few examples and comment on these.

In [23], Fisher and Solow remark that $I(G)^{-1}$ can be viewed as a generating function which enumerates the number of n letter words, where the letters are the vertices of the graph and two letters commute iff they have an edge between them on the graph. Similarly in [29], Godsil shows that $\frac{x(x^{2n}\mu_V(G,x^{-1}))'}{x^{2n}\mu_V(G,x^{-1})}$ is a generating function in x^{-1} for closed tree-like walks in G. We believe that there is a multivariate generalization of Fisher and Solow's remark by working in the ring $\mathbb{Z}[x_1, \ldots, x_n]$ where variables commute if and only if they correspond to vertices in the graph G which share an edge. Godsil's tree-like result should be a combinatorial consequence of the more general Fisher and Solow result.

In a previous paper of Bencs, Christoffel-Darboux like identities are established for the independence polynomial [5]. One can similarly establish multivariate generalizations of these identities. By generalizing in this way, one can give a single identity that implies all the others through simple multivariate operations.

Another area of interest is studying independent sets in hypergraphs. One can naturally define the multivariate independence polynomial of a hypergraph. Namely given a hyper graph $G = (V, E)$ a set $S \subset V$ is independent if $e \not\subset S$ for all edges $e \in E$. If two edges are comparible in G ($e \subset f$), then we note that by removing f from the edge set we do not change the independent sets of G. If G contains any edges of size one, then that vertex never shows up in the independence polynomial so we can further reduce G by removing that vertex. Thus we can do this to obtain the reduction, $\tilde{G}$, of G which has the same multivariate independence polynomial and has no comparable edges and no edges of size 1.

Proposition 6.5.1. *Given a hypergraph G, $I(G, x)$ is same-phase stable if and only if $\tilde{G}$ is a 2-uniform claw-free graph.*

Proof. As noted, $I(G, x) = I(\tilde{G}, x)$, so if $\tilde{G}$ is 2-uniform and claw-free we see $I(G, x)$ is same-phase stable by previous results. If $\tilde{G}$ is not 2-uniform, then we have some edge e with $|e| > 2$. If $I(\tilde{G}, x)$ were same-phase stable then we could restrict to the subgraph of vertices in e and obtain a same-phase stable independence polynomial. Since no other edges are comparable to e by construction of $\tilde{G}$, we have this subgraph only contains the edge e. Then we can diagonalize to get the independence polynomial $(1 + x)^n - x^n$. If $I(\tilde{G}, x)$ were same-phase stable, this polynomial would be real rooted. However this would imply that its derivatives were real rooted, namely $(1 + x)^3 - x^3 = 1 + 3x + 3x^2$ would be real rooted, a contradiction. $\square$

Bibliography

[1] Nima Anari and Shayan Oveis Gharan. "A generalization of permanent inequalities and applications in counting and optimization". In: *Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing*. ACM. 2017, pp. 384–396.

[2] Nima Anari, Shayan Oveis Gharan, and Cynthia Vinzant. "Log-concave polynomials, entropy, and a deterministic approximation algorithm for counting bases of matroids". In: *arXiv preprint arXiv:1807.00929* (2018).

[3] RB Bapat. "Mixed discriminants of positive semidefinite matrices". In: *Linear Algebra and its Applications* 126 (1989), pp. 107–124.

[4] Heinz H Bauschke et al. "Hyperbolic polynomials and convex analysis". In: *Canadian Journal of Mathematics* 53.3 (2001), pp. 470–488.

[5] Ferenc Bencs. "Christoffel-Darboux type identities for independence polynomial". In: *arXiv preprint arXiv:1409.2527* (2014).

[6] Julius Borcea and Petter Brändén. "Hyperbolicity preservers and majorization". In: *arXiv preprint arXiv:1005.5293* (2010).

[7] Julius Borcea and Petter Brändén. "The Lee-Yang and Pólya-Schur programs. I. linear operators preserving stability". In: *Inventiones mathematicae* 177.3 (2009), pp. 541–569.

[8] Julius Borcea and Petter Brändén. "The Lee-Yang and Pólya-Schur programs. II. Theory of Stable Polynomials and Applications". In: *Communications on Pure and Applied Mathematics* 62.12 (2009), pp. 1595–1631.

[9] Petter Brändén. "Polynomials with the half-plane property and matroid theory". In: *Advances in Mathematics* 216.1 (2007), pp. 302–320.

[10] Petter Brändén, Ilia Krasikov, and Boris Shapiro. "Elements of Pólya-Schur Theory in the Finite Difference Setting". In: *Proceedings of the American Mathematical Society* 144, 11 (2016), pp. 4831–4843.

[11] A. E. Brouwer, A. M. Cohen, and A. Neumaier. *Distance Regular Graphs*. New York: Springer-Verlag, 1989, pp. 103–104, 312.

[12] J. I. Brown and R. Nowakowski. "Bounding the roots of the independence polynomials". In: *Ars Combin.* 58 (2001), pp. 113–120.

[13] Peter Bürgisser et al. "Alternating minimization, scaling algorithms, and the null-cone problem from invariant theory". In: *arXiv preprint arXiv:1711.08039* (2017).

[14] Young-Bin Choe et al. "Homogeneous multivariate polynomials with the half-plane property". In: *Advances in Applied Mathematics* 32.1-2 (2004), pp. 88–187.

[15] Maria Chudnovsky and Paul Seymour. "The roots of the independence polynomial of a clawfree graph". In: *Journal of Combinatorial Theory.* B (87 May 2007), pp. 350–357.

[16] Thomas Craven and George Csordas. "Jensen polynomials and the Turán and Laguerre inequalities". In: *Pacific Journal of Mathematics* 136.2 (1989), pp. 241–260.

[17] Péter Csikvári. "Lower matching conjecture, and a new proof of Schrijver's and Gurvits's theorems". In: *arXiv preprint arXiv:1406.0766* (2014).

[18] Alexander Engström. "Inequalites on Well-Distrubted Points Sets on Circles". In: *Journal of Inequalities in Pure and Applied Mathematics* 8.2 (2007).

[19] Paul Erdős and László Lovász. "Problems and results on 3-chromatic hypergraphs and some related questions". In: *COLLOQUIA MATHEMATICA SOCIETATIS JANOS BOLYAI 10. INFINITE AND FINITE SETS, KESZTHELY (HUNGARY)*. Citeseer. 1973.

[20] GP Erorychev. "Proof of the van der Waerden conjecture for permanents". In: *Siberian Mathematical Journal* 22.6 (1981), pp. 854–859.

[21] Dmitry I Falikman. "Proof of the van der Waerden conjecture regarding the permanent of a doubly stochastic matrix". In: *Mathematical notes of the Academy of Sciences of the USSR* 29.6 (1981), pp. 475–479.

[22] D.C. Fisher and J. Ryan. "Bounds on the largest root of the matching polynomial". In: *Discrete Mathematics* 110 (1992), pp. 275–278.

[23] David C. Fisher and Anita E. Solow. "Dependence Polynomials". In: *Discrete Mathematics* (82 1990), pp. 251–258.

[24] S Fisk. "Polynomials, roots, and interlacing". In: *https://arxiv.org/abs/math/0612833* (2006), pp. xx+700.

[25] Shmuel Friedland, E Krop, and Klas Markström. "On the number of matchings in regular graphs". In: *the electronic journal of combinatorics* 15.1 (2008), p. 110.

[26] Lars Gårding. "An inequality for hyperbolic polynomials". In: *Journal of Mathematics and Mechanics* (1959), pp. 957–965.

[27] Ankit Garg et al. "A deterministic polynomial time algorithm for non-commutative rational identity testing". In: *Foundations of Computer Science (FOCS), 2016 IEEE 57th Annual Symposium on*. IEEE. 2016, pp. 109–117.

[28] Ankit Garg et al. "Operator scaling: theory and applications". In: *arXiv preprint arXiv:1511.03730* (2015).

[29] Chris Godsil. *Algebraic Combinatorics*. CRC Press, 1993.

[30] Osman Güler. "Hyperbolic polynomials and interior point methods for convex programming". In: *Mathematics of Operations Research* 22.2 (1997), pp. 350–377.

[31] Leonid Gurvits. "Combinatorics hidden in hyperbolic polynomials and related topics". In: *arXiv preprint math/0402088* (2004).

[32] Leonid Gurvits. "Hyperbolic polynomials approach to Van der Waerden/Schrijver-Valiant like conjectures: sharper bounds, simpler proofs and algorithmic applications". In: *Proceedings of the thirty-eighth annual ACM symposium on Theory of computing.* ACM. 2006, pp. 417–426.

[33] Leonid Gurvits. "On multivariate Newton-like inequalities". In: *Advances in Combinatorial Mathematics.* Springer, 2009, pp. 61–78.

[34] Leonid Gurvits. "Unleashing the power of Schrijver's permanental inequality with the help of the Bethe approximation". In: *arXiv preprint arXiv:1106.2844* (2011).

[35] Leonid Gurvits. "Van der Waerden/Schrijver-Valiant like conjectures and stable (aka hyperbolic) homogeneous polynomials: one theorem for all". In: *the electronic journal of combinatorics* 15.1 (2008), p. 66.

[36] Ole J. Heilmann and Elliott H. Lieb. "Theory of Monomer-Dimer Systems". In: *Communications in Mathematical Physics* (25.3 1972), pp. 190–232.

[37] J William Helton and Victor Vinnikov. "Linear matrix inequality representation of sets". In: *Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences* 60.5 (2007), pp. 654–674.

[38] Allen Knutson and Terence Tao. "Honeycombs and sums of Hermitian matrices". In: *Notices Amer. Math. Soc* 48.2 (2001).

[39] Martin Lamprecht. "Suffridge's Convolution Theorem for Polynomials and Entire Functions having only Real Zeros". In: *Advances in Mathematics* 288 (2016), pp. 426–463.

[40] Bodo Lass. "Mehler formulae for matching polynomials of graphs and independence polynomials of clawfree graphs". In: *Journal of Combinatorial Theory* (B 102.2 2012), pp. 411–423.

[41] Jonathan Leake. "A Representation Theoretic Explanation of the Borcea-Brändén Characterization, and an Extension to Interval-Rooted Polynomials". In: *arXiv preprint arXiv:1706.06168* (2017).

[42] Jonathan Leake. "Capacity Preserving Operators". In: *arXiv preprint arXiv:1804.04351* (2018).

[43] Jonathan Leake and Mohan Ravichandran. "Mixed determinants and the Kadison-Singer problem". In: *arXiv preprint arXiv:1609.04195* (2016).

[44] Jonathan Leake and Nick Ryder. "Connecting the q-Multiplicative Convolution and the Finite Difference Convolution". In: *arXiv preprint arXiv:1712.02499* (2017).

[45]   Jonathan Leake and Nick Ryder. "Generalizations of the Matching Polynomial to the Multivariate Independence Polynomial". In: *arXiv preprint arXiv:1610.00805* (2016).

[46]   Jonathan Leake and Nick Ryder. "On the Further Structure of the Finite Free Convolutions". In: *arXiv preprint arXiv:1811.06382* (2018).

[47]   A Marcus, D Spielman, and N Srivastava. "Finite Free Convolutions of Polynomials". In: *arXiv preprint arXiv:1504.00350* (2015).

[48]   Adam W Marcus. "Polynomial convolutions and (finite) free probability". In: *preprint* (2016).

[49]   Adam W Marcus, Daniel A Spielman, and Nikhil Srivastava. "Interlacing families II: Mixed characteristic polynomials and the Kadison—Singer problem". In: *Annals of Mathematics* (2015), pp. 327–350.

[50]   Adam W Marcus, Daniel A Spielman, and Nikhil Srivastava. "Interlacing families II: Mixed characteristic polynomials and the Kadison—Singer problem". In: *Annals of Mathematics* (2015), pp. 327–350.

[51]   Adam W Marcus, Daniel A Spielman, and Nikhil Srivastava. "Interlacing families iv: Bipartite ramanujan graphs of all sizes". In: *Foundations of Computer Science (FOCS), 2015 IEEE 56th Annual Symposium on.* IEEE. 2015, pp. 1358–1377.

[52]   Adam Marcus, Daniel Spielman, and Nikhil Srivastava. "Interlacing families I: Bipartite Ramanujan graphs of all degrees". In: *Annals of Mathematics* 182 (2015), pp. 307–325.

[53]   Albert W Marshall, Ingram Olkin, and Barry C Arnold. *Inequalities: theory of majorization and its applications.* Vol. 143. Springer, 1979.

[54]   Henry Mulder. "Distance-Hereditary Graphs". In: *Journal of Combinatorial Theory, Series B* 41.2 (1986), pp. 182–208.

[55]   Fedor (https://mathoverflow.net/users/4312/fedor-petrov) Petrov. *Combinatorial Identity with Connection Coefficients and Falling Factorial.* MathOverflow. URL: `https://mathoverflow.net/q/256199`.

[56]   James Renegar. "Hyperbolic programs, and their derivative relaxations". In: *Foundations of Computational Mathematics* 6.1 (2006), pp. 59–79.

[57]   Alexander Schrijver. "Counting 1-factors in regular bipartite graphs". In: *J. Comb. Theory, Ser. B* 72.1 (1998), pp. 122–135.

[58]   Alexander D Scott and Alan D Sokal. "The repulsive lattice gas, the independent-set polynomial, and the Lovász local lemma". In: *Journal of Statistical Physics* 118.5-6 (2005), pp. 1151–1261.

[59]   Alexander Scott and Alan Sokal. "The repulsive lattice gas, the independent-set polynomial, and the lovasz local lemma". In: *Journal of Statiscial Physics* 118 (2005), pp. 1151–1261.

[60] Damian Straszak and Nisheeth K Vishnoi. "Real stable polynomials and matroids: Optimization and counting". In: *Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing.* ACM. 2017, pp. 370–383.

[61] T Suffridge. "Starlike functions as limits of polynomials". In: *Lecture Notes in Mathematics* 505 (1973), pp. 164–203.

[62] G Szegö. "Bemerkungen zu einem Satz von J H Grace über die Wurzeln Algebraischer Gleichungen". In: *Mathematische Zeitschrift* (1922), pp. 28–55.

[63] David Wagner. "Multivariate stable polynomials: theory and applications". In: *Bulletin of the American Mathematical Society* 48.1 (2011), pp. 53–84.

[64] JL Walsh. "On the Location of the Roots of Certain Types of Polynomials". In: *Transactions of the American Mathematical Society* (24(3) 1922), pp. 163–180.