Title
Model-theoretic Elekes-Szabó in the strongly minimal case

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A MODEL-THEORETIC GENERALIZATION OF THE ELEKES-SZABÓ THEOREM

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Abstract. We prove a generalizations of the Elekes–Szabó theorem [5] for relations definable in strongly minimal structures that are interpretable in distal structures.

1. Introduction and preliminaries

Our notation is mostly standard. For $n \in \mathbb{N}$ we denote by $[n]$ the set $[n] = \{1, \ldots, n\}$. If X is a set and $n \in \mathbb{N}$, then we write $A \subseteq_n X$ to denote that A is a subset of X with $|A| \leq n$. Given a binary relation $E \subseteq X \times Y$ and $a \in X$, we write $E_a = \{b \in Y : (a, b) \in E\}$ to denote the fiber of E at a. Similarly, given $b \in Y$, we write $E_b = \{a \in X : (a, b) \in E\}$ to denote the fiber of E at b.

As usual, for functions $f, g\colon \mathbb{N} \to \mathbb{R}$ we write

- $f(n) = O(g(n))$ if there is positive $C \in \mathbb{R}$ and $n_0 \in \mathbb{N}$ such that $f(n) \leq Cg(n)$ for all $n > n_0$;
- $f(n) = \Omega(g(n))$ if there is positive $C \in \mathbb{R}$ and $n_0 \in \mathbb{N}$ such that $f(n) \geq Cg(n)$ for all $n > n_0$.

We will use freely some standard model-theoretic notions such as saturated models, algebraic closure (by which we will always mean the algebraic closure in $\mathcal{M}^{\mathrm{eq}}$) and Morley rank (see e.g. [9, 15]).

Definition 1.1. (1) We say that a subset $F \subseteq X \times Y$ is *cartesian* if $I \times J \subseteq F$ for some infinite $I \subseteq X, J \subseteq Y$.
(2) We say that a subset $F \subseteq S_1 \times S_2 \times \cdots \times S_k$ is *cylindrical* if it is cartesian as a subset of $S_i \times \hat{S}_i$ for some $i \in [k]$, where $\hat{S}_i = \prod_{j \neq i} S_j$.

Let $\mathcal{M}$ be a sufficiently saturated first order structure and let X, Y, Z be strongly minimal sets definable in $\mathcal{M}$. Let $F \subseteq X \times Y \times Z$ be a definable set of Morley rank 2. As usual, we say that $\bar{a} = (a_1, a_2, a_3) \in$

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F is *generic in* F over a set of parameters $C \subseteq X \times Y \times Z$ if $\mathrm{RM}(\bar{a}/C) = \mathrm{RM}(F) = 2$.

Definition 1.2. We say that a relation F as above is *group-like* if there is a group G of Morley rank 1 and degree 1 (hence, abelian) definable in $\mathcal{M}$ over a small set C, elements $g_1, g_2, g_3 \in G$, and $\alpha_1 \in X$, $\alpha_2 \in Y$, $\alpha_3 \in Z$ such that α_i and g_i are inter-algebraic over C for all $i \in [3]$ (i.e. $\alpha_i \in \mathrm{acl}(g_i C)$ and $g_i \in \mathrm{acl}(a_i C)$), $\bar{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \in F$ is generic in F over C and $g_1 \cdot g_2 \cdot g_3 = 1$ in G.

We can now state our main result.

Theorem 1.3 (Main Theorem)**.** *Let X, Y, Z be strongly minimal sets definable in a sufficiently saturated structure $\mathcal{M}$ and let $F \subseteq X \times Y \times Z$ be a definable set of Morley rank 2. Assume in addition that $\mathcal{M}$ is interpretable in a distal structure. Then one of the following holds.*

(a) There is $\varepsilon > 0$ such that for all $A \subseteq_n X, B \subseteq_n Y, C \subseteq_n Z$ we have
$$|F \cap A \times B \times C| = O(n^{2-\varepsilon}).$$
(b) F is group-like.
(c) F is cylindrical.

Remark 1.4. Theorem 1.3 can be viewed as a generalization of the Elekes-Szabó Theorem [5] which established it for $\mathcal{M}$ the field of complex numbers (a strongly minimal structure), which is interpretable in the field of reals — a distal structure.

Remark 1.5. Various improvements of the Elekes-Szabó theorem, including explicit bounds on ε, have been obtained [4, 12, 17]. In our general situation we don't optimize the bounds, even though they can be calculated explicitly in terms of the size of the available distal cell decomposition, see Section 2.1.

Remark 1.6. It can be shown that if the Morley degree of F is 1, then the three cases in Theorem 1.3 are mutually exclusive.

The proof of Theorem 1.3 consists of three main ingredients: a bound on the number of edges for non-cartesian relations in our context (i.e. Theorem 2.15 established in Section 2 using local stability and the distal cutting lemma from [1]); Hrushovski's group configuration theorem in stable theories; and the construction of the group configuration in the cartesian case connecting the two aforementioned results. In Section 3 this last part is reduced to a certain dichotomy for binary relations between sets of rank 2, which this dichotomy is proved in Section 4).

2. Bounds for non-cartesian relations

2.1. **Zarankiewicz for distal relations.** In this section we demonstrate that the proof of the incidence bound due to Elekes and Szabó in [5] generalizes to arbitrary graphs definable in *distal* structures. Distal structures constitute a subclass of purely unstable NIP structures [14] that contains all o-minimal structures, various expansions of the field $\mathbb{Q}_p$ and the valued differential field of transseries (we refer to the introduction of [3] for a general discussion of distality and references). It is demonstrated in [1, 3] that many of the results in semialgebraic incidence combinatorics generalize to relations definable in distal structures.

The following definition abstracts the conclusion of the cutting lemma in incidence geometry.

Definition 2.1. Let U, V be infinite sets, and $E \subseteq U \times V$. We say that E is *admits cuttings with exponent* D if there is some constant $c \in \mathbb{R}$ satisfying the following. For any $A \subseteq_n U$ and any $r \in \mathbb{R}$ with $1 < r < n$ there are some sets $V_1, \ldots, V_t \subseteq V$ covering V with $t \leq cr^D$ and such that each V_i is crossed by at most $\frac{n}{r}$ of the fibers $\{E_a : a \in A\}$ (as usual, we say that E_a *crosses* a set $V' \subseteq V$ if both $E_a \cap V' \neq \emptyset$ and $V' \not\subseteq E_a$).

For our application here, we will only need the following result about distality.

Fact 2.2. *(1)* [1, 2] *Assume that E is definable in a distal structure. Then it admits a distal cell decomposition with exponent D for some $D \in \mathbb{N}$ (in the sense of* [1, Definition 2.6]*).*

(2) [1, Theorem 3.1] *Assume that E admits a distal cell decomposition with exponent D. Then E admits cuttings with exponent D.*

As usual, given $s, t \in \mathbb{N}$ we say that a bipartite graph $E \subseteq U \times V$ is $K_{s,t}$*-free* if it does not contain a copy of the complete bipartite graph $K_{s,t}$ with its parts of size s and t, respectively.

Fact 2.3. *Let $G = (U, V, E)$ be a finite $K_{s,t}$-free bipartite graph. Then:*

(1) [8] $|E| \leq s^{\frac{1}{t}}|U|^{1-\frac{1}{t}}|V| + t|U| = O(|U|^{\frac{t-1}{t}}|V| + |U|)$*;*

(2) every complete bipartite subgraph of E has at most $O(|U| + |V|)$ edges.

The following theorem is similar to [5, Theorem 9].

Theorem 2.4. *Let $E \subseteq U \times V$ be $K_{s,t}$-free and assume that E admits cuttings with exponent D. Then for any $0 < \varepsilon < \frac{t-1}{t(Dt-1)}$ and $\alpha :=$*

$\frac{D(t-1)}{Dt-1} - \varepsilon, \beta := t(1-\alpha) = \frac{t(D-1)}{Dt-1} + t\varepsilon,$ *we have*

$$|E \ \cap A \times B| \leq c\left(|A|^{\alpha}|B|^{\beta} + |A| + |B|\log(2|A|)\right)$$

for all finite $A \subseteq U, B \subseteq V$ *and some* $c = c(s,t,\varepsilon,D)$.

Proof. We follow closely the proof of [5, Theorem 9].

Let s, t, D, ε be given by assumption. We fix a sufficiently large r to be determined later. Let $A \subseteq_m U, B \subseteq_n V$ be given.

Case 1. If $m \leq r$, then $|E \cap A \times B| \leq mn \leq rn$, so the bound holds with any $c \geq r$.

Case 2. Assume $r^{\frac{D}{1-\alpha}} m \geq n^t$. By Fact 2.3(1),

$$|E \cap A \times B| \leq c_1\left(|A| + |A|^{1-\frac{1}{t}}|B|\right)$$

for some $c_1 = c_1(s,t)$. Hence in this case

$$|E \cap A \times B| \leq c_1\left(m + m^{1-\frac{1}{t}}\left(r^{\frac{D}{\beta}} m^{\frac{1}{t}}\right)\right) = c_1\left(1 + r^{\frac{D}{\beta}}\right)m$$

— which satisfies the bound with c large enough compared to r.

Case 3. In the remaining case we have $r < m < r^{\frac{-D}{1-\alpha}} n^t$.

As E admits cuttings with exponent D, there is some constant $c_2 = c_2(E)$ and a covering $V_1, \ldots, V_M$ of V by $M \leq c_2 r^D$ parts, such that each part is cut by at most $\frac{m}{r}$ of the sets from $\{E_a : a \in A\}$.

We have

$$(*)\ m < r^{-D} m^{\alpha} n^{t(1-\alpha)} = r^{-D} m^{\alpha} n^{\beta}.$$

Let $B_i := B \cap V_i,\ A_i := \{a \in A : E_a \text{ cuts } V_i\},\ n_i := |B_i|, m_i := |A_i|$. Then $M \leq c_2 r^D, \sum n_i = n$ and $m_i \leq \frac{m}{r}$ for all i.

Assume that $b \in E_a$ for some $b \in B_i$ and $a \notin A_i$. Then E_a doesn't cut B_i, hence $B_i \subseteq E_a$. Hence, by Fact 2.3(2), for every $i \leq M$ we have $|E \cap (A \setminus A_i) \times B_i| \leq c_3(|A \setminus A_i| + |B_i|) \leq c_3(m+n)$. Besides, for every i, we may estimate $|E \cap A_i \times B_i|$ by the inductive assumption. Putting it all together and using the inductive assumption, we have:

$$|E \cap A \times B| = \sum_{i=1}^{M} |E \cap A_i \times B_i| + \sum_{i=1}^{M} |E \cap (A \setminus A_i) \times B_i|$$

$$\leq \sum_{i=1}^{M} c\left(m_i^{\alpha} n_i^{\beta} + m_i + n_i\right) + M c_3(m+n),$$

and the proof can be concluded following the final estimate in the proof of [5, Theorem 9] verbatim. $\square$

Remark 2.5. The proof goes through under the weaker assumption that the *combinatorial dimension* of E is $\leq t$, as in [5], but we won't need it here.

Corollary 2.6. *If E admits cuttings with exponent D for some $D \in \mathbb{N}$ and E is $K_{s,2}$-free for some $s \in \mathbb{N}$, then there is some $\delta = \delta(E) > 0$ such that for all $A \subseteq_n U, B \subseteq_n V$ we have $|E(A, B)| = O(n^{\frac{3}{2}-\delta})$.*

Proof. By Theorem 2.4 with $t = 2$, we get $\frac{3}{2} - (\alpha + \beta) = \frac{1}{2(2D-1)} - \varepsilon := \delta$, hence taking $0 < \varepsilon < \frac{1}{2(2D-1)}$ we can choose a sufficiently large constant c that works. $\square$

This applies in particular to relations definable in $\mathbb{C}$, giving a version of the theorem of Tóth [16] with a weaker bound.

2.2. **Local stability.** For the rest of Section 2 we assume that $\mathcal{M}$ is a sufficiently saturated structure, $\tilde{Y}, \tilde{Z}$ are definable subsets, and that $\Phi \subseteq \tilde{Y} \times \tilde{Z}$ is a stable relation.

As usual, by a Φ-definable set we mean a subset $B \subseteq \tilde{Y}$ that is a finite Boolean combination of sets defined by $\Phi(y, c)$, $c \in \tilde{Z}$. We write $\Phi^* \subseteq \tilde{Z} \times \tilde{Y}$ for the relation obtained from Φ by exchanging the roles of the variables. Similarly we have a notion of Φ^*-definable subsets of $\tilde{Z}$. We denote by $S_\Phi(M)$ the set of all complete Φ-types on $\tilde{Y}$ over M (equivalently, the set of all ultrafilters on the Boolean algebra of all Φ-definable subsets of $\tilde{Y}$), and similarly we denote by $S_{\Phi^*}(M)$ the set of all complete Φ^* types on $\tilde{Z}$. If $\mathbb{U}$ is an elementary extension of $\mathcal{M}$, then for an M-definable set V we will denote by $V(\mathbb{U})$ the set of elements of $\mathbb{U}$ realizing a formula defining V. We say that a Φ-type $p(y)$ is *non-algebraic* if in some elementary extension of $\mathcal{M}$ it has a realization outside of M.

The following are some basic facts from local stability, all of which can be found in e.g. [11, Chapter 1, Sections 1–3].

Fact 2.7. *For $p(y) \in S_\Phi(M)$, the set $\{c \in \tilde{Z} : \Phi(y, c) \in p\}$ is uniformly Φ^*-definable.*

Similarly, for $q \in S_{\Phi^*}(M)$, the set $\{b \in \tilde{Y} : \Phi(b, z) \in q\}$ is uniformly Φ-definable.

Fact 2.8. *Let $\mathbb{U}$ be an elementary extension of $\mathcal{M}$, $\beta \in \tilde{Y}(\mathbb{U})$ and $\gamma \in \tilde{Z}(\mathbb{U})$. Then $\mathrm{tp}_\Phi(\beta/M\gamma)$ is finitely satisfiable in M if and only if $\mathrm{tp}_{\Phi^*}(\gamma/M\beta)$ is finitely satisfiable in M.*

Definition 2.9. For an elementary extension $\mathbb{U}$ of $\mathcal{M}$, $\beta \in \tilde{Y}(\mathbb{U})$ and $\gamma \in \tilde{Z}(\mathbb{U})$ we say that β and γ are Φ-*independent over* $\mathcal{M}$ if $\mathrm{tp}_\Phi(\beta/M\gamma)$ is finitely satisfiable in M.

The following is a consequence of the fundamental theorem of local stability (it was also used in e.g. [6]).

Fact 2.10. *For types $p(y) \in S_\Phi(M)$, $q \in S_{\Phi^*}(M)$ the following conditions are equivalent.*
(1) There are realizations $\beta \models p(y)$ and $\gamma \models q(z)$ that are Φ-independent over M and such that $\models \Phi(\beta, \gamma)$.
(2) For any realizations $\beta \models p(y)$ and $\gamma \models q(z)$ that are Φ-independent over M we have $\models \Phi(\beta, \gamma)$.
(3) $d_p^\Phi(z) \in q(z)$, where $d_p^\Phi(z)$ is a formula that defines the set $\{c \in \tilde{Z} \colon \Phi(y, c) \in p(y)\}$.
(4) $d_q^{\Phi^*}(y) \in p(y)$.

For types $p(y) \in S_\Phi(M)$ and $q(z) \in S_{\Phi^*}(M)$ we write $\Phi(p, q)$ if one of the equivalent conditions of Fact 2.10 holds.

2.3. Cartesian relations and populated types.

Proposition 2.11. *The relation Φ is cartesian if and only if $\models \Phi(p, q)$ for some non-algebraic types $p(y) \in S_\Phi(M)$ and $q(z) \in S_{\Phi^*}(M)$.*

Proof. Let $B \subseteq \tilde{Y}$, $C \subseteq \tilde{Z}$ be infinite sets with $B \times C \subseteq \Phi$. By compactness, there is a non-algebraic type $p(y) \in S_\Phi(M)$ with $\Phi(y, c) \in p$ for all $c \in C$. Hence the set $d_p^\Phi(z)$ is infinite, and we can take $q(z)$ to be any non-algebraic type containing this formula.

The converse is easy since for Φ-independent realizations we have finite satisfiability in $\mathcal{M}$ in both variables. □

The following definition is inspired by [13].

Definition 2.12. A non-algebraic type $p(y) \in S_\Phi(M)$ is called *popular*, or *pop*, if the set $\{c \in \tilde{Z} \colon \Phi(y; c) \in p(y)\}$ is infinite.

Similarly, a non-algebraic type $q(z) \in S_{\Phi^*}(M)$ is popular if the set $\{b \in \tilde{Y} \colon \Phi(b; z) \in q(z)\}$ is infinite.

Lemma 2.13. *(1) A non-algebraic type $p(y) \in S_\Phi(M)$ is pop if and only if there is a non-algebraic type $q(z) \in S_{\Phi^*}(M)$ with $\models \Phi(p, q)$.*
(2) A non-algebraic type $q(z) \in S_{\Phi^*}(M)$ is pop if and only if there is a non-algebraic type $p(y) \in S_\Phi(M)$ with $\models \Phi(p, q)$.

Proof. Assume that $p(y)$ is pop, then the definable set $d_p^\Phi(z)$ is infinite and we can take q to be any non-algebraic type containing this set.

Assume $\models \Phi(p, q)$ for some non-algebraic $q(z) \in S_{\Phi^*}(M)$. Since $q(z)$ contains $d_p^\Phi(z)$, the set defined by $d_p^\Phi(z)$ must be infinite. □

Combining, we have the following equivalence.

Proposition 2.14. *The following conditions are equivalent.*

(1) The relation Φ is cartesian.
(2) There is a pop type $p(y) \in S_\Phi(M)$.
(3) There is a pop type $q(z) \in S_{\Phi^*}(M)$.

2.4. **Bounds on the number of edges in non-cartesian relations.** The following theorem is a generalization of Theorem 1.3 in [10].

Theorem 2.15. *Let $\mathcal{M}$ be a sufficiently saturated structure eliminating $\exists^\infty$, and let $\tilde{Y}$ and $\tilde{Z}$ be definable sets in $\mathcal{M}$, both of Morley rank 2 and degree 1. Let $\Phi \subseteq \tilde{Y} \times \tilde{Z}$ be an $\mathcal{M}$-definable set such that for every $\beta \in \tilde{Y}$ the fiber $\Phi_\beta = \{z \in \tilde{Z} \colon (\beta, z) \in \Phi\}$ has Morley rank at most 1. Then the following conditions are equivalent.*

(1) Φ is not cartesian.
(2) Φ is $K_{k,k}$-free for some $k \in \mathbb{N}$.
(3) For all $B \subseteq_n \tilde{Y}, C \subseteq_n \tilde{Z}$ we have

$$|\Phi \cap (B \times C)| = O(n^{3/2}).$$

If, in addition, Φ admits cuttings (with some exponent D), then we also have

(4) There is some $\delta > 0$ such that for all $B \subseteq_n \tilde{Y}, C \subseteq_n \tilde{Z}$ we have

$$|\Phi \cap (B \times C)| = O(n^{3/2-\delta}).$$

Proof. (1) implies (3). Assume that Φ is not cartesian.

Assume first that that there is some $b \in \tilde{Y}$ for which there are some pairwise distinct $(b_i : i \in \mathbb{N})$ in $\tilde{Y}$ such that $\mathrm{RM}(\Phi_b \cap \Phi_{b_i}) \geq 1$ (hence $= 1$) for all $i \in \mathbb{N}$. Then each of these sets contains a complete Φ^*-type of Morley rank 1. By the definition of Morley rank, there are only finitely many complete Φ^*-types $q_1, \ldots, q_s \in S_{\Phi^*}(M)$ with $\Phi_b \in q_i$ and $\mathrm{RM}(q_i) = 1$. But then one of these types must contain Φ_{b_i} for infinitely many different i, hence it is a pop type — contradicting the assumption by Proposition 2.14.

Thus there is no $b \in \tilde{Y}$ as above. Using that T eliminates $\exists^\infty$, there is some $r \in \mathbb{N}$ such that for every $b \in \tilde{Y}$, there are at most r many $b' \in \tilde{Y}$ such that $\Phi_b \cap \Phi_{b'}$ is infinite.

Given a finite set $B \subseteq \tilde{Y}$, consider the graph with the vertex set B and the edge relation E defined by $bEb' \iff \Phi_b \cap \Phi_{b'}$ is infinite. Then the graph (B, E) has degree at most r by the previous paragraph, and so it is $r + 1$ colorable by a standard result in graph theory. Let $B_i \subseteq B$ be the set of vertices corresponding to the ith color. Then $B = \bigsqcup_{1 \leq i \leq r+1} B_i$ and, by elimination of $\exists^\infty$ again, there is some $t \in \mathbb{N}$

depending only of Φ such that $|\Phi_b \cap \Phi_{b'}| \leq t$ for any $b, b' \in B_i$ and $1 \leq i \leq r+1$.

Now if C is a finite subset of $\tilde{Z}$, we have that $\Phi \restriction (B_i \times C)$ is $K_{2,t}$-free for each $1 \leq i \leq r+1$. Then, using Fact 2.3 with $s = 2$, we have

$$|\Phi \cap (B \times C)| \leq \sum_{i=1}^{r+1} |\Phi \cap (B_i \times C)| \leq (r+1)cn^{\frac{3}{2}}.$$

Hence taking $c' := (r+1)c$ depending only on Φ does the job. When Φ admits cuttings, we use Corollary 2.6 instead of Fact 2.3.

Finally, (3) implies (2) and (2) implies (1) are straightforward. □

Remark 2.16. Theorem 2.15(4) is the only palce where the assumption of the existence of a distal expansion is used. It is necessary to get a bound strictly less than $n^{\frac{3}{2}}$, as the points-lines incidence relation on the plane in an algebraically closed field of characteristic p (a strongly minimal structure) demonstrates. However, this $\delta > 0$ improvement is crucial for our proof to give a non-trivial statement.

3. Reducing Main Theorem to a dichotomy for binary relations

To prove the main theorem we introduce some notions and make some reductions. Since we are only interested in definable subsets of products of strongly minimal sets, we may and will assume that $\mathcal{M}$ has finite Morley rank and eliminates the quantifier $\exists^\infty$.

Assumption 1. *For the rest of this section (Section 3) we assume that $\mathcal{M}$ is a sufficiently saturated structure of finite Morley rank that eliminates the quantifier $\exists^\infty$.*

We fix strongly minimal sets X, Y, Z definable in $\mathcal{M}$.

We also fix an M-definable set $F \subseteq X \times Y \times Z$ of Morley rank 2.

We assume that X, Y, Z and F are definable over the empty set.

Notice that writing F as a union $F = \cup_{i=1}^k F_i$ and applying Theorem 1.3 to each F_i, it is sufficient to consider only the case when F has Morley degree 1.

Assumption 2. *In addition, for the rest of Section 3, we assume that F has Morley degree 1.*

3.1. Δ-algebraic relations.

Definition 3.1. Let S_1, S_2, S_3 be sets and $G \subseteq S_1 \times S_2 \times S_3$ be a subset. We say that G is Δ-*algebraic* if there is some $d \in \mathbb{N}$ such that for $\{i,j,k\} = \{1,2,3\}$ we have

$$\models \forall y_i \in S_i \, \forall y_j \in S_j \, \exists^{\leq d} y_k \in S_k \, G(y_1, y_2, y_3).$$

Assume, in addition to Assumptions 1 and 2, that $F \subseteq X \times Y \times Z$ is not cylindrical. Then, since Z is strongly minimal, the set $\{(a,b) \in X \times Y : \exists^{\infty} z F(a,b,z)\}$ is finite (otherwise we can find infinite sequences $a_i \in X, b_i \in Y$ such that $F(a_i, b_i, Z)$ is cofinite, hence $\bigcap_{i \in \mathbb{N}} F(a_i, b_i, Z)$ is infinite using saturation of $\mathcal{M}$ — so F is cylindrical). Thus there are co-finite $X_0 \subseteq X$, $Y_0 \subseteq Y$ such that

$$\models \forall x \in X_0 \, \forall y \in Y_0 \, \exists^{<\infty} z F(x,y,z).$$

Applying the same argument for every partition of the coordinates of F we conclude that if F is not cylindrical then there are co-finite $X_0 \subseteq X$, $Y_0 \subseteq Y$, $Z_0 \subseteq Z$ such that the restriction of F to $X_0 \times Y_0 \times Z_0$ is Δ-algebraic.

It is not hard to see that passing to co-finite subsets does not change the clauses (a), (b) and (c) in Theorem 1.3, hence Theorem 1.3 follows from the following theorem.

Theorem 3.2. *Assume, in addition to Assumptions 1 and 2, that F is Δ-algebraic and also that $\mathcal{M}$ is interpretable in a distal structure. Then one of the following holds.*

(a) There is $\varepsilon > 0$ such that for all $A \subseteq_n X, B \subseteq_n Y, C \subseteq_n Z$ we have

$$|F \cap A \times B \times C| = O(n^{2-\varepsilon}).$$

(b) F is group-like.

Assumption 3. *For the rest of Section 3 we assume in addition that the relation F is Δ-algebraic and $d \in \mathbb{N}$ is as in Definition 3.1.*

3.2. **On acl-diagrams.** We say that three elements p_1, p_2, p_3 of M form an acl-*triangle* if $\mathrm{RM}(p_i/\emptyset) = 1$ for $i \in [3]$, $\mathrm{RM}(p_1 p_2 p_3/\emptyset) = 2$, and for all $\{i,j,k\} = \{1,2,3\}$ we have $p_i \in \mathrm{acl}(p_j p_k)$ (hence also $p_i \mathop{\smile\!\!\!\!\!\!\;\;|}\, p_j$ for all $i \neq j \in \{1,2,3\}$).

Since F is Δ-algebraic of Morley rank 2, we have the following claim.

Claim 3.3. *Let (a,b,c) be generic in F, i.e. $(a,b,c) \in F$ with* $\mathrm{RM}(abc/\emptyset) = 2$*. Then a, b, c form an* acl*-triangle.*

In particular, b and c are independent generics in Y and Z, respectively; and, by stationarity (as $Y \times Z$ has Morley degree 1), for any independent generics $b' \in Y$, $c' \in Z$ we have some $x \in X$ such that $(x, b', c') \in F$ and (x, b', c') is generic in F).

Similarly, for any independent generics $a' \in X$, $b' \in Y$ we have $\mathcal{M} \models \exists z \, F(a', b', z)$.

In this paper we will consider some simple diagrams, where by a *diagram* we mean a collection of elements of M and lines between them (subsets of the given elements).

Definition 3.4. We say that a given diagram is an acl-*diagram* if
(1) $\mathrm{RM}(p/\emptyset) = 1$ for every point p in the diagram;
(2) Every three collinear points form an acl-triangle;
(3) $\mathrm{RM}(pqr/\emptyset) = 3$ for any three non-colinear points p, q, r.

3.3. **The 4-ary relation G.** Our next goal is to restate Theorem 3.2 in term of a 4-ary relation G. (We continue to use Assumptions 1-3.)

Let $G \subseteq Y^2 \times Z^2$ be the definable relation

$G = \{(y, y', z, z') \in Y^2 \times Z^2 \colon \exists x{\in}X((x, y, z){\in} F \,\&\, (x, y', z'){\in}F)\}.$

We first observe some basic properties of G.

Claim 3.5. *For $(b_1, b_2) \in Y^2$ and $c_1 \in Z$ the set*

$$\{z \in Z \colon (b_1, b_2, c_1, z) \in G\}$$

has size at most d^2.

Similarly, For $(c_1, c_2) \in Z^2$ and $b_1 \in Y$ the set

$$\{y \in Y \colon (b_1, y, c_1, c_2) \in G\}$$

has size at most d^2.

Proof. Let b_1, b_2, c_1 be fixed. As F is Δ-algebraic, by the choice of d there are at most d elements $x \in X$ such that $(x, b_1, c_1) \in F$, and for each such x, there are at most d elements $z \in Z$ such that $(x, b_2, z) \in F$. Hence, by definition of G, there are at most d^2 elements $z \in Z$ such that $(b_1, b_2, c_1, z) \in G$. $\square$

Corollary 3.6. *For any $\bar{b} = (b_1, b_2) \in Y^2$ the Morley rank of the fiber $G_{\bar{b}} = \{\bar{z} \in Z^2 \colon (\bar{b}, \bar{z}) \in G\}$ is at most 1.*

Similarly, for any $\bar{c} = (c_1, c_2) \in Z^2$ the Morley rank of the fiber $G^{\bar{c}} = \{\bar{y} \in Y^2 \colon (\bar{y}, \bar{c}) \in G\}$ is at most 1.

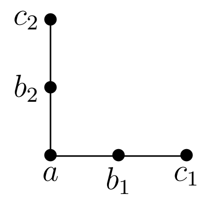
Corollary 3.7. *The Morley rank of G is 3.*

Proof. It follows from the additivity of Morley rank and Corollary 3.6 that $\mathrm{RM}(G) \leq \mathrm{RM}(Y^2) + 1 = 3$.

On the other hand, let $b_1, b_2 \in Y, c_1 \in Z$ be independent generics, hence $\mathrm{RM}(b_1 b_2 c_1/\emptyset) = 3$. By Claim 3.3, there is a generic $a \in X$ with $\mathcal{M} \models F(a, b_1, c_1)$. Applying Claim 3.3 to a and b_2 (as $a \in \mathrm{acl}(b_1 c_1)$ and $b_1 c_1 \mathop{\smile\!\!\!\!\!\!\mid}\, b_2$, we have $a \mathop{\smile\!\!\!\!\!\!\mid}\, b_2$) we get $c_2 \in Z$ with $\mathcal{M} \models F(a, b_2, c_2)$. Since $\mathcal{M} \models G(b_1, b_2, c_1, c_2)$ we have $\mathrm{RM}(G) \geq 3$.

$\square$

Lemma 3.8. *Let (b_1, b_2, c_1, c_2) be generic in G. Then there is $a \in X$ such that the diagram*



is an acl-*diagram with $\mathcal{M} \models F(a, b_i, c_i)$.*

Proof. By the definition of G we can find $a \in X$ with

$$\mathcal{M} \models F(a, b_1, c_1) \,\&\, F(a, b_2, c_2).$$

We claim that for this a the diagram above is an acl-diagram.

First notice that $\mathrm{RM}(p/\emptyset) \leq 1$ for any point p.

Secondly, by Δ-algebraicity of F, if p, q, r are three non-colinear points then every point of the diagram is in $\mathrm{acl}(pqr)$; in particular, since $\mathrm{RM}(b_1 b_2 c_1 c_2/\emptyset) = 3$, we have $\mathrm{RM}(pqr/\emptyset) = 3$.

Finally, given any two distinct non-collinear points (p, q) we can find a point r such that p, q, r are non-collinear (so $\mathrm{RM}(p, q, r/\emptyset) = 3$), hence $\mathrm{RM}(pq/\emptyset) = 2$ by additivity of Morley rank.

It follows then that any three collinear points, after reordering, form a generic realization of F, and hence, by Claim 3.3, an acl-triangle. $\square$

3.4. **The clause (a) in Theorem 3.2.** In this section we state a property of the relation G that implies the clause (a) in Theorem 3.2.

First we need some basic counting properties.

Corollary 3.9. *For any $\bar{b} \in Y^2$ and a finite set $C \subseteq Z$ we have*

$$|G \cap (\{\bar{b}\} \times C^2)| \leq d^2 |C|.$$

Similarly, For any $\bar{c} \in Z^2$ and a finite set $B \subseteq Y$ we have

$$|G \cap (B^2 \times \{\bar{c}\})| \leq d^2 |B|.$$

Proof. Follows from Claim 3.5 $\square$

The following bound is similar to [13, Lemma 2.2].

Proposition 3.10. *Let $A \subseteq X, B \subseteq Y, C \subseteq Z$ be finite. Then for $F' = F \cap (A \times B \times C)$ and $G' = G \cap (B^2 \times C^2)$ we have*

$$|F'| \leq d |A|^{1/2} |G'|^{1/2}.$$

Proof. Let $W \subseteq X \times Y^2 \times Z^2$ be the definable set

$$W = \{(x, y, y', z, z') \in X \times Y^2 \times Z^2 \colon (x, y, z) \in F \,\&\, (x, y'z') \in F\},$$

and let $W' = W \cap (A \times B^2 \times C^2)$.

As usual, for a set $S \subseteq A \times D$ and $a \in A$ we denote by S_a the fiber $S_a = \{u \in D \colon (a,u) \in S\}$.

Notice that $|F'| = \sum_{a \in A} |F'_a|$, and $|W'| = \sum_{a\in A} |F'_a|^2$. By the Cauchy-Schwarz inequality

$$|F'| \leq |A|^{1/2} \Big(\sum_{a \in A} |F'_a|^2 \Big)^{1/2} = |A|^{1/2} |W'|^{1/2}.$$

For a point $g \in G'$, the fiber W'_g has size at most d as F is Δ-algebraic, hence $|W'| \leq d|G'|$ and $|F'| \leq d|A|^{1/2}|G'|^{1/2}$ □

The next proposition shows that the bound $O(n^{3/2-\delta})$ for G translates to the bound $O(n^{2-\varepsilon})$ for F.

Proposition 3.11. *Let $\tilde{Y} = Y^2$, $\tilde{Z} = Z^2$, and we view G as a subset of $\tilde{Y} \times \tilde{Z}$. Assume that there are definable sets $Y_0 \subseteq \tilde{Y}$ and $Z_0 \subseteq \tilde{Z}$ of Morley rank at most 1 such that for some $\delta > 0$ for all $B' \subseteq_m \tilde{Y} \setminus Y_0, C' \subseteq_m \tilde{Z} \setminus Z_0$ we have $|G \cap (A' \times B')| = O(m^{3/2-\delta})$. Then F satisfies the clause (a) of Theorem 3.2.*

Proof. We fix $Y_0 \subseteq \tilde{Y}$ and $Z_0 \subseteq \tilde{Z}$ of Morley rank at most 1, $c_0 \in \mathbb{R}$ and $\delta > 0$ such that for all m large enough and for all $B' \subseteq_m \tilde{Y} \setminus Y_0, C' \subseteq_m \tilde{Z} \setminus Z_0$ we have $|G \cap A' \times B'| \leq c_0 m^{3/2-\delta}$.

Since Y_0 has Morley rank at most 1, using elimination of $\exists^\infty$, it is not hard to see that there is $k_1 \in \mathbb{N}$ such that for any finite $B \subseteq Y$ we have $|B^2 \cap Y_0| \leq k_1 |B|$.

Similarly, there is $k_2 \in \mathbb{N}$ such that for any finite $C \subseteq Z$ we have $|C^2 \cap Z_0| \leq k_2|C|$.

Given $A \subseteq_n X$, $B \subseteq_n Y$, $C \subseteq_n Z$, let $B' = B^2 \cap (\tilde{Y} \setminus Y_0)$ and $C' = C^2 \cap (\tilde{Z} \setminus Z_0)$. Obviously $|B'| \leq n^2$ and $|C'| \leq n^2$.

We have

$$|G \cap (B^2 \times C^2)| \leq \\ |G \cap (B' \times C')| + |(B^2 \cap Y_0) \times C^2| + |(B^2 \times (C^2 \cap Z_0)|.$$

By our assumptions $|G \cap (B' \times C')| \leq c_0 n^{3-\delta}$. Since $|B^2 \cap Y_0| \leq k_1 n$, from Corollary 3.9, we get $|(B^2 \cap Y_0) \times C^2| \leq k_1 d^2 n^2$; and similarly $|(B^2 \times (C^2 \cap Z_0)| \leq k_2 d^2 n^2$.

Thus $|G \cap (B^2 \times C^2)| \leq c_1 n^{3-\varepsilon}$, where $c_1 > 0$ and $\varepsilon > 0$ do not depend on n, A, B, C.

Applying Proposition 3.10, we obtain

$$|F \cap (A \times B \times C)| \leq d(nc_1 n^{3-\varepsilon})^{1/2} = O(n^{2-\varepsilon/2}).$$

□

Combining this with Theorem 2.15, we obtain a property of G that implies the clause (a) in Theorem 3.2.

Proposition 3.12. *Let $\tilde{Y} = Y^2$, $\tilde{Z} = Z^2$, and we view G as a subset of $\tilde{Y} \times \tilde{Z}$. Assume in addition that G admits cuttings. Assume also that there are definable sets $Y_0 \subseteq \tilde{Y}$ and $Z_0 \subseteq \tilde{Z}$ of Morley rank at most 1 such that the restriction of G to $(\tilde{Y} \setminus Y_0) \times (\tilde{Z} \setminus Z_0)$ is not cartesian. Then F satisfies the clause (a) of Theorem 3.2.*

3.5. **The clause (b) of Theorem 3.2.** We also fix a saturated elementary extension $\mathbb{U}$ of $\mathcal{M}$.

Proposition 3.13. *Assume there are $\beta = (\beta_1, \beta_2) \in Y^2(\mathbb{U})$ and $\gamma = (\gamma_1, \gamma_2) \in Z^2(\mathbb{U})$ with $(\beta, \gamma) \in G(\mathbb{U})$, such that $\mathrm{RM}(\beta/M) > 0$, $\mathrm{RM}(\gamma/M) > 0$, $\beta \mathop{\smile\!\!\!\!\!\!\mid}_M \gamma$ and $\mathrm{acl}(\beta) \cap \mathrm{acl}(\gamma) \not\subseteq \mathrm{acl}(\emptyset)$ Then F is group-like.*

Proof. Choose $t \in \big(\mathrm{acl}(\beta) \cap \mathrm{acl}(\gamma)\big) \setminus \mathrm{acl}(\emptyset)$. We first list some properties of β, γ and t.

(i) Since $\beta \mathop{\smile\!\!\!\!\!\!\mid}_M \gamma$, and $t \in \big(\mathrm{acl}(\beta) \cap \mathrm{acl}(\gamma)\big)$ we have
$$t \in M.$$

(ii) Since $t \in \mathrm{acl}(\beta) \setminus \mathrm{acl}(\emptyset)$ we have
$$\beta \underset{\emptyset}{\not\mathop{\smile\!\!\!\!\!\!\mid}} t;$$
and, similarly,
$$\gamma \underset{\emptyset}{\not\mathop{\smile\!\!\!\!\!\!\mid}} t.$$

(iii) From (i) and (ii), since $\mathrm{RM}(\beta/\emptyset) \leq 2$ and $\mathrm{RM}(\beta/M) > 0$, we obtain
$$\mathrm{RM}(\beta/\emptyset) = 2 \text{ and } \mathrm{RM}(\beta/t) = \mathrm{RM}(\beta/M) = 1;$$
and, similarly,
$$\mathrm{RM}(\gamma/\emptyset) = 2 \text{ and } \mathrm{RM}(\gamma/t) = \mathrm{RM}(\gamma/M) = 1.$$

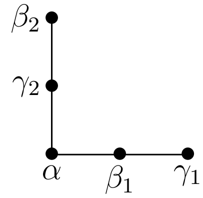
(iv) Since $t \in \mathrm{acl}(\beta) \setminus \mathrm{acl}(\emptyset)$ and $\beta \notin \mathrm{acl}(t)$ we have $\mathrm{RM}(t/\emptyset) = 1$.

(v) Since β and γ are independent over M we have $RM(\beta\gamma/M) = 2$, and since $\beta \not\mathop{\smile\!\!\!\!\!\!\mid}_\emptyset M$, we have $RM(\beta\gamma/\emptyset) = 3$, i.e. (β, γ) is generic in $G(\mathbb{U})$.

(vi) It follows from (v) and Lemma 3.8 that $\beta_i \notin \mathrm{acl}(\gamma)$ and $\gamma_i \notin \mathrm{acl}(\beta)$ for $i \in [2]$.

We also have that both $\{\beta_1, \beta_2, t\}$ and $\{\gamma_1, \gamma_2, t\}$ are acl-triangles. Indeed, for example, since $\mathrm{RM}(\beta_1\beta_2 t/\emptyset) = 2$, to show that $\{\beta_1, \beta_2, t\}$ is a triangle it is sufficient to check that $t \notin \mathrm{acl}(\beta_i)$ for $i = 1, 2$. But if $t \in \mathrm{acl}(\beta_i)$, then $\beta_i \in \mathrm{acl}(t)$, hence $\beta_i \in \mathrm{acl}(\gamma)$ — contradicting (vi).

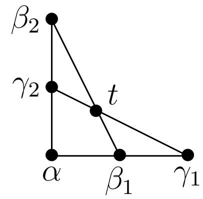
By Lemma 3.8 there is $\alpha \in X(\mathbb{U})$ such that the diagram



(3.1)

is an acl-diagram with $\mathbb{U} \models F(\alpha, \beta_i, \gamma_i)$.

We claim that



(3.2)

is an acl-diagram.

We already have that every three colinear points form an acl-triangle, and it is sufficient to check that for three non-colinear points $\{p, q, r\}$ we have $\mathrm{RM}(pqr/\emptyset) = 3$. If $t \notin \{p, q, r\}$ then it follows from the diagram (3.1). Assume $t \in \{p, q, r\}$, say $\{p, q, r\} = \{t, \alpha, \beta_1\}$. Then $\{t, \beta_1\}$ is inter-algebraic with $\{\beta_1, \beta_2\}$ and hence

$$\mathrm{RM}(t\alpha\beta_1/\emptyset) = \mathrm{RM}(\beta_2\alpha\beta_1/\emptyset) = 3.$$

The same argument works for any three non-collinear points containing t.

Thus the diagram (3.2) is an acl-diagram. It follows from the Group Configuration Theorem in stable theories (see [7, Theorem 6.1] and the discussion in [7, Section 6.2]) that F is group-like. □

4. Dichotomy for binary relations

In this section we prove a dichotomy theorem for binary relations between sets of Morley rank 2. By Propositions 3.12 and 3.13, Theorem 3.2 follows from Theorem 4.1 applied with $\Phi := G, \tilde{Y} := Y^2, \tilde{Z} := Z^2$ as in Section 3.

Theorem 4.1. *Let $\mathcal{M}$ be a sufficiently saturated structure of finite Morley rank that eliminates quantifier $\exists^\infty$.*

Let $\tilde{Y}$ and $\tilde{Z}$ be M-definable sets of Morley rank 2 and Morley degree 1. Let $\Phi \subseteq \tilde{Y} \times \tilde{Z}$ be a definable subset of Morley rank 3. Then one of the following holds.

(a) *There are definable sets $Y_0 \subseteq \tilde{Y}$ and $Z_0 \subseteq \tilde{Z}$ of Morley rank at most* 1 *such that the restriction of Φ to $(\tilde{Y} \setminus Y_0) \times (\tilde{Z} \setminus Z_0)$ is not cartesian.*
(b) *There are $\beta \in \tilde{Y}(\mathbb{U})$ and $\gamma \in \tilde{Z}(\mathbb{U})$ with $(\beta, \gamma) \in \Phi(\mathbb{U})$, such that* $\mathrm{RM}(\beta/M) > 0$, $\mathrm{RM}(\gamma/M) > 0$, $\beta \mathop{\smile\!\!\!\!\!\mid}\nolimits_M \gamma$ *and* $\mathrm{acl}(\beta) \cap \mathrm{acl}(\gamma) \nsubseteq \mathrm{acl}(\emptyset)$.

Proof of Theorem. As usual, for a Φ-type $p(y) \in S_\Phi(M)$ we denote by $\mathrm{RM}(p(y))$ the Morley rank of p as an incomplete type.

We assume that (a) doesn't hold, and show that then (b) must hold. For a definable set $\tilde{Y}' \subseteq \tilde{Y}$ we say that $\tilde{Y}'$ is *large* in $\tilde{Y}$ if $\mathrm{RM}(\tilde{Y} \setminus \tilde{Y}_0) \leq 1$; and the same for a subset $\tilde{Z}' \subseteq \tilde{Z}$. Notice that in the proof we can freely replace $\tilde{Y}$ and $\tilde{Z}$ by their large subsets.

Let $p(y) \in S(M)$ be the generic type on $\tilde{Y}$, it is the unique type on $\tilde{Y}$ of Morley rank 2 (as $\tilde{Y}$ has Morley degree 1 by assumption).

Since Φ has Morley rank 3, the set $\{c \in \tilde{Z} \colon \Phi_c \in p\}$ is definable (by Fact 2.7) and has Morley rank at most 1 by additivity of Morley rank. Thus we can throw away this set and assume that the Morley rank of Φ_c is at most 1 for all $c \in \tilde{Z}$. Similarly, we may assume that the Morley rank of $\Phi_b \subseteq \tilde{Z}$ is at most 1 for all $b \in \tilde{Y}$.

Assume that p is a pop type for Φ (see Definition 2.12). Then $\mathrm{RM}(p) = 1$ ($\mathrm{RM}(p) \geq 1$ as p is non-algebraic, and $\mathrm{RM}(p) \leq 1$ as $\Phi_c \in p$ for some c by definition of pop-types, and $\mathrm{RM}(\Phi_c) = 1$ by the previous paragraph). If there are only finitely many pop types for Φ, then we can throw away finitely many definable sets of Morley rank 1 (one in each of the pop types), and pass to a large subset on which there are no pop types (hence, obtaining (a) using Proposition 2.14). Thus we can assume that there are infinitely many pop types on $\tilde{Y}$.

Let $\mathcal{P}$ be the set of all pop types on $\tilde{Y}$ and $\mathcal{Q}$ be the set of all pop types on $\tilde{Z}$.

For $p \in S_\Phi(M)$ we denote by $[p] \in \mathcal{M}^{\mathrm{eq}}$ the canonical parameter of the Φ^*-definable set $d_p^\Phi = \{c \in \tilde{Z} \colon \Phi_c \in p\}$; and for $q \in S_{\Phi^*}(M)$ we will denote by $[q] \in \mathcal{M}^{\mathrm{eq}}$ the canonical parameter for $d_q^{\Phi^*}$.

Clearly both maps $p \mapsto [p]$ and $q \mapsto [q]$ are injective.

Claim 4.2. *(1) The set $\{[p] \colon p \in \mathcal{P}\}$ is type-definable.*
(2) The set $\{[q] \colon q \in \mathcal{Q}\}$ is type-definable.

Proof. Using that $\mathcal{M}$ eliminates $\exists^\infty$, the desired set $\{[p] : \exists^\infty z d_p^\Phi(z) \wedge \bigwedge_{n \in \mathbb{N}} (\forall z_1 \dots \forall z_n (\bigwedge_{i=1}^n d_p^\Phi(z_i) \rightarrow \exists^\infty y \bigwedge_{i=1}^n \Phi(y, z_i)\}$ is type-definable. □

Claim 4.3. *If $p \in \mathcal{P}$ and $c \in d_p^\Phi$ then $[p] \in \mathrm{acl}(c)$.*

Similarly, if $q \in \mathcal{Q}$ *and* $b \in d_p^{\Phi^*}$ *then* $[q] \in \operatorname{acl}(b)$.

Proof. As for any $c \in \tilde{Z}$ there are only finitely many Φ-types of Morley rank 1 containing Φ_c. ☐

Claim 4.4. *For any* $p \in \mathcal{P}$ *there are only finitely many* $q \in \mathcal{Q}$ *with* $\models \Phi(p, q)$*, and vice versa.*

Proof. Let $\beta \in \mathbb{U}$ realize p. It is not hard to see that, for $q \in \mathcal{Q}$, if $\models \Phi(p, q)$ then the Morley rank of the partial type $\{\Phi(\beta, z)\} \cup q(z)$ is 1. Since the Morley rank of $\Phi(\beta, z)$ is 1, there only finitely many such q. ☐

Since the set $\mathcal{P}$ is infinite, we can find a pop type $p \in \mathcal{P}$ such that $[p] \notin \operatorname{acl}(\varnothing)$. Choose $q \in \mathcal{Q}$ with $\models \Phi(p, q)$.

Choose some $\beta \models p(y)$ and $\gamma \models q(z)$ independent over M.

We have $[p] \in \operatorname{acl}(\gamma)$, $[q] \in \operatorname{acl}(\beta)$ with $[p]$ and $[q]$ inter-algebraic over the empty set. Hence (b) holds. ☐

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