Three roads to complete lattices: orders, compatibility, polarity

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Abstract

This note aims to clarify the relations between three ways of constructing complete lattices that appear in three different areas: (1) using ordered structures, as in set-theoretic forcing, or doubly ordered structures, as in a recent semantics for intuitionistic logic; (2) using compatibility relations, as in semantics for quantum logic based on ortholattices; (3) using Birkhoff's polarities, as in formal concept analysis.

Keywords: complete lattice, representation, closure operator, doubly ordered structure, compatibility, proximity, polarity, orthoframe, Boolean algebra, Heyting algebra, ortholattice, formal concept analysis

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1 Introduction

Several approaches to the representation of complete lattices appear in applications of lattice theory to logic and computer science. These approaches include:

(1) using ordered structures, as in set-theoretic forcing, or doubly ordered structures, as in a recent semantics for intuitionistic logic;

(2) using compatibility relations, as in semantics for quantum logic based on ortholattices;

(3) using Birkhoff's polarities, as in formal concept analysis.

The aim of this note is to clarify the relations between these three ways of constructing complete lattices. In each case, the relevant structure provides a closure operator c on the lattice of downsets of a preordered set or on the lattice of all subsets of a set. We are interested in the complete lattice of fixpoints of c, taking advantage of the first part of the following classic theorem (see, e.g., [7, Proposition 7.2(ii)], [6, Theorem 5.3]).

Theorem 1.1. The fixpoints of a closure operator on a complete lattice form a complete lattice under the restricted lattice order. Conversely, any complete lattice is isomorphic to the lattice of fixpoints of a closure operator on a powerset lattice (resp. a downset lattice).

In Section 2, we review roads (1), (2), and (3) in turn. Then in Section 3, we connect the roads via direct transformations between each type of structure. We conclude in Section 4 with some questions about the tradeoffs of traveling down one road rather than another.

We first fix some conventions. For an abstract lattice L, we denote its order by $\leq$. When we refer to a lattice of sets, the lattice order is always the inclusion order $\subseteq$. Given a binary relation $\prec$ on a set X, we write $y \prec x$ for $(y, x) \in \prec$, and given an expression φ, we write

$$\forall y \prec x\ \varphi \text{ for } \forall y(y \prec x \Rightarrow \varphi), \text{ and}$$
$$\exists y \prec x\ \varphi \text{ for } \exists y(y \prec x \text{ and } \varphi).$$

2 The three roads

2.1 Orders

The first road begins with the following generalization of a preordered set.

Definition 2.1. A *doubly ordered structure* is a triple $(X, \leq_1, \leq_2)$ where X is a nonempty set and $\leq_1$ and $\leq_2$ are preorders on X. For $Y \subseteq X$, define:

$$\begin{aligned}
\mathsf{Int}_i(Y) &= \{x \in X \mid \forall y \leq_i x\ \ y \in Y\};\\
\mathsf{Cl}_i(Y) &= \{x \in X \mid \exists y \leq_i x\ \ y \in Y\};\\
c_{12}(Y) = \mathsf{Int}_1(\mathsf{Cl}_2(Y)) &= \{x \in X \mid \forall x' \leq_1 x\ \exists x'' \leq_2 x'\ \ x'' \in Y\}.
\end{aligned}$$

Y is a $\leq_i$*-downset* if $Y = \mathsf{Int}_i(Y)$, and $\mathsf{Down}(X, \leq_i)$ is the collection of all $\leq_i$-downsets.

Remark 2.2. Urquhart [18] uses the term 'doubly ordered set' for structures as in Definition 2.1 in which for all $x, y \in X$, if $x \leq_1 y$ and $x \leq_2 y$, then $x = y$. We do *not* assume this condition, for the reason explained before Theorem 2.7 below.

It is straightforward to check the following facts.

Lemma 2.3. Every c_{12}-fixpoint is a $\leq_1$-downset.

Proposition 2.4. For any doubly ordered structure $(X, \leq_1, \leq_2)$, c_{12} is a closure operator on $\mathsf{Down}(X, \leq_1)$.

One can also observe that the functions $\mathsf{Int}_1(X \setminus \cdot)\colon \mathsf{Down}(X, \leq_2) \to \mathsf{Down}(X, \leq_1)$ and $\mathsf{Int}_2(X \setminus \cdot)\colon \mathsf{Down}(X, \leq_1) \to \mathsf{Down}(X, \leq_2)$ form an antitone Galois connection, so their composition $\mathsf{Int}_1(X \setminus \mathsf{Int}_2(X \setminus \cdot)) = \mathsf{Int}_1(\mathsf{Cl}_2(\cdot)) = c_{12}(\cdot)$ is a closure operator on $\mathsf{Down}(X, \leq_1)$.

The following example is well known in the literature on forcing in set theory (see, e.g., [15]).

Theorem 2.5. L is a complete Boolean algebra if and only if L is isomorphic to the lattice of c_{12}-fixpoints of a preordered set, i.e., a doubly ordered structure in which $\leq_1 = \leq_2$.

When $\leq_1 = \leq_2$, the c_{12}-fixpoints are exactly the *regular open sets* in the topology on X whose open sets are all the $\leq_1$-downsets. As observed by Tarski [16], the regular open sets of any space form a complete Boolean algebra (see, e.g., [11, Ch. 10]), which gives us the right-to-left direction of Theorem 2.5. In the left-to-right direction, one may in fact take the preorder to be a partial order. The poset is constructed by deleting the bottom element of L and restricting the lattice order; the regular open downsets are then exactly the principle downsets plus $\varnothing$, yielding an isomorphic copy of L.

For Heyting algebras, we have the following analogue of Theorem 2.5 from [3] and [14], where it is used to give a semantics for intuitionistic logic based on doubly ordered structures.

Theorem 2.6. *L is a complete Heyting algebra if and only if L is isomorphic to the lattice of c_{12}-fixpoints of a doubly ordered structure in which $\leq_2 \subseteq \leq_1$.*

Doubly ordered structures in which $\leq_2 \subseteq \leq_1$ were introduced by Fairtlough and Mendler [10], who observed that in this case c_{12} is a nucleus (inflationary, idempotent, and multiplicative operation) on the complete Heyting algebra of $\leq_1$-downsets (in fact, Fairtlough and Mendler worked with upsets and therefore defined c_{12} in terms of $\geq_1$ and $\geq_2$ instead of $\leq_1$ and $\leq_2$ as in Definition 2.1). Since the fixpoints of a nucleus on a complete Heyting algebra again form a complete Heyting algebra (see, e.g., [13] or [9, p. 71]), this gives us the right-to-left direction of Theorem 2.6. Also note that in the left-to-right direction of Theorem 2.6, we may assume that $\leq_1$ and $\leq_2$ are partial orders (see [3, Proposition 4.5]).

On the other hand, even the assumption that $\leq_1$ and $\leq_2$ are preorders is not necessary for Proposition 2.4, as discussed in Remark 2.12.

Going beyond Boolean and Heyting algebras, Allwein and MacCaull [1] observed that by moving from Urquhart's [18] notion of 'doubly ordered set' to the more general notion in Definition 2.1, one can represent arbitrary complete lattices. For comparison with later constructions, we include a proof of the following.

Theorem 2.7. *L is a complete lattice if and only if L is isomorphic to the lattice of c_{12}-fixpoints of a doubly ordered structure.*

Proof. The right-to-left direction follows from Proposition 2.4 and Theorem 1.1. From left to right, define $(X, \leq_1, \leq_2)$ as follows:

1. $X = \{(a, b) \in L^2 \mid a \not\leq b\}$;
2. $(a, b) \leq_1 (c, d) \Leftrightarrow a \leq c$;
3. $(a, b) \leq_2 (c, d) \Leftrightarrow b \geq d$.

The elements of the form $(a, 0)$ ordered by $\leq_1$ form a lattice isomorphic to $L \setminus \{0\}$. Thus, the principal $\leq_1$-downsets of elements of the form $(a, 0)$, plus $\varnothing$, ordered by $\subseteq$, form a lattice isomorphic to L. Therefore, to prove the theorem it suffices to show that the c_{12}-fixpoints are exactly the principal $\leq_1$-downsets of elements of the form $(a, 0)$, plus $\varnothing$.

First, we show that each principal $\leq_1$-downset $\downarrow_1(a, 0)$ is a c_{12}-fixpoint. Suppose that $(c, d) \notin \downarrow_1(a, 0)$, so $c \not\leq a$. Then $(c, a) \in X$ and $(c, a) \leq_1 (c, d)$. Now consider any $(c', a') \leq_2 (c, a)$, so $c' \not\leq a'$ and $a' \geq a$. Then $c' \not\leq a$, so $(c', a') \not\leq_1 (a, 0)$. Hence $(c, a) \notin \mathsf{Cl}_2(\downarrow_1(a, 0))$, which with $(c, a) \leq_1 (c, d)$ implies $(c, d) \notin c_{12}\downarrow_1(a, 0)$.

Suppose $U = \{(a_i, b_i) \mid i \in I\}$ is a c_{12}-fixpoint. Where $e = \bigvee\{a_i \mid i \in I\}$, we claim that $U = \downarrow_1(e, 0)$. Clearly $U \subseteq \downarrow_1(e, 0)$. Since U is a $\leq_1$-downset, to show $U \supseteq \downarrow_1(e, 0)$ it suffices to show that $(e, 0) \in U$. Since U is a c_{12}-fixpoint, it suffices to show that for all $(a, b) \leq_1 (e, 0)$ there is a $(c, d) \leq_2 (a, b)$ such that $(c, d) \in U$. Thus, suppose $(a, b) \leq_1 (e, 0)$, so $a \not\leq b$ and $a \leq e$. It follows that for some $i \in I$, $a_i \not\leq b$. For otherwise $e \leq b$, which with $a \leq e$ implies $a \leq b$, contradicting the fact that $a \not\leq b$. Hence $(a_i, b) \in X$ and $(a_i, b) \leq_2 (a, b)$. Finally, since $U = \{(a_i, b_i) \mid i \in I\}$ is a $\leq_1$-downset, we have $(a_i, b) \in U$, which completes the proof that $(e, 0) \in U$. □

Theorems 2.5 and 2.6 can be viewed as showing that the doubly ordered structure used in the proof of Theorem 2.7 can be cut down in the Boolean and Heyting cases as follows:

- if L is a Boolean algebra, then one may restrict X to the pairs $(a, \neg a)$, where $\neg a$ is the complement of a, in which case the restricted relations satisfy $\leq_1 = \leq_2$.
- if L is a Heyting algebra, then one may define $(a, b) \leq_2 (c, d)$ if *both* $a \leq c$ and $b \geq d$, in which case this modified relation satisfies $\leq_2 \subseteq \leq_1$.

For a proof in the Heyting case, see [4, Theorem 4.33].

2.2 Compatibility

The second road is ostensibly the simplest of the three, involving a single set and single binary relation.

Definition 2.8. A *compatibility structure* is a pair $(X, \between)$ where X is a nonempty set and $\between$ is a reflexive relation on X. For $Y \subseteq X$, define

$$c_{\between}(Y) = \{x \in X \mid \forall x' \between x \; \exists x'' \between^{-1} x' \;\; x'' \in Y\},$$

where as usual $x'' \between^{-1} x'$ means $x' \between x''$.

Proposition 2.9. For any compatibility structure $(X, \between)$, $c_{\between}$ is a closure operator on $\wp(X)$.

Proof. That $Y \subseteq Z$ implies $c_{\between}(Y) \subseteq c_{\between}(Z)$ and that $Y \subseteq c_{\between}(Y)$ are obvious. To see that $c_{\between}(c_{\between}(Y)) \subseteq c_{\between}(Y)$, suppose $x \in c_{\between}(c_{\between}(Y))$ and $x' \between x$. Hence there is an $x'' \between^{-1} x'$ such that $x'' \in c_{\between}(Y)$. Since $x' \between x'' \in c_{\between}(Y)$, there is an $x''' \between^{-1} x'$ such that $x''' \in Y$. Thus, for any $x' \between x$ there is an $x''' \between^{-1} x'$ such that $x''' \in Y$. Therefore, $x \in c_{\between}(Y)$. □

Although the reflexivity of $\between$ is not used in the proof of Proposition 2.9, it can be assumed without loss of generality by the proof of Theorem 2.11 below.

As far as we are aware, Definition 2.8 and Proposition 2.9 have not been considered before in the literature, but special cases have. In particular, one of the two kinds of structures used in Goldblatt's [12] semantics for orthologic is a compatibility structure as in Definition 2.8 in which $\between$ is also symmetric, in which case he calls $\between$ a *proximity relation* (for the other kind of structure, using instead the complement of $\between$, see Remark 2.17). In this special symmetric case, we may define $c_{\between}$ by

$$c_{\between}(Y) = \{x \in X \mid \forall x' \between x \; \exists x'' \between x' \;\; x'' \in Y\}.$$

These structures also appear in Dishkant's [8] semantics for quantum logic. Although Dishkant starts with a larger class of structures (see Remark 2.12 below), his representation theorem shows that every complete ortholattice comes from a proximity structure. We recall that an ortholattice is a bounded lattice equipped with a unary orthocomplementation operation $(\cdot)^{\perp}$ satisfying the equations $a \vee a^{\perp} = 1$, $a \wedge a^{\perp} = 0$, $a^{\perp\perp} = a$, $(a \wedge b)^{\perp} = a^{\perp} \vee b^{\perp}$, and $(a \vee b)^{\perp} = a^{\perp} \wedge b^{\perp}$.

Theorem 2.10 ([8])**.** *L is a complete ortholattice if and only if L is isomorphic to the lattice of $c_{\between}$-fixpoints of a compatibility structure in which $\between$ is symmetric, where the orthocomplement of a $c_{\between}$-fixpoint Y is $Y^{\perp} = \{x \in X \mid \forall x' \between x \;\; x' \notin Y\}$.*

Proof. (sketch of $\Rightarrow$) Given L, define $(X, \between)$ by $X = L \setminus \{0\}$ and $x \between y$ if $y \nleq x^{\perp}$, where $x^{\perp}$ is the orthocomplement of x in L. □

As with doubly ordered structures, so too with compatibility structures, every complete lattice can be represented.

Theorem 2.11. *L is a complete lattice if and only if L is isomorphic to the lattice of $c_{\between}$-fixpoints of a compatibility structure.*

Proof. (sketch of $\Rightarrow$) Given L, define $(X, \between)$ as follows:

1. $X = \{(a, b) \in L^2 \mid a \nleq b\}$;
2. $(a, b) \between (c, d) \Leftrightarrow c \nleq b$.

This is exactly the result of applying to the doubly ordered structure in the proof of Theorem 2.7 the method of Section 3.1 below for turning doubly ordered structures into compatibility structures (see Remark 3.4). Thus, the fact that the lattice of $c_{\between}$-fixpoints is isomorphic to L follows from the proof of Theorem 2.7 together with Theorem 3.3 below. □

The construction in the proof of Theorem 2.10 shows that in the case of a complete ortholattice L, we may restrict X in the proof of Theorem 2.11 to the pairs $(a, a^{\perp})$. By contrast, it is noteworthy that when representing the same L using a doubly ordered structure, we cannot restrict the doubly ordered structure in the proof of Theorem 2.7 to the pairs $(a, a^{\perp})$, for this would imply $\leq_1 = \leq_2$, making the lattice of c_{12}-fixpoints Boolean.

Remark 2.12. Some of the results of Sections 2.1-2.2 can be generalized using ideas of Dishkant [8]. First, we generalize both the structures of Definition 2.1 and Dishkant's original structures (defined below) as follows. Let a *double Dishkant structure* be a triple $(X, \leq_1, \leq_2)$ where X is a nonempty set and $\leq_1$ and $\leq_2$ are reflexive binary relations on X such that for all $x, y \in X$:

$$y \leq_2 x \Rightarrow \exists z \leq_1 y \; \forall w \leq_2 z \;\; w \leq_2 x.$$

Defining c_{12} on $\wp(X)$ as in Definition 2.1, we obtain a generalization of Proposition 2.4: for any double Dishkant structure $(X, \leq_1, \leq_2)$, c_{12} is a closure operator on $\mathsf{Down}(X, \leq_1)$. The structures with which Dishkant begins his paper are a special case. Let a *Dishkant structure* be a pair $(X, \leq)$ such that $(X, \leq, \leq)$ is a double Dishkant structure. Define $c_{\leq}$ on $\wp(X)$ by

$$c_{\leq}(Y) = \{x \in X \mid \forall x' \leq x\ \exists x'' \leq x'\ \ x'' \in Y\}.$$

(Note we use $x'' \leq x'$, as in the definition of c_{12} in Definition 2.1, rather than $x'' \leq^{-1} x'$, as in the Definition of $c_{\between}$ in Definition 2.8.) Every compatibility structure in which $\between$ is symmetric is a Dishkant structure, but not vice versa. Generalizing the right-to-left direction of Theorem 2.10, Dishkant proves that the $c_{\leq}$-fixpoints of a Dishkant structure form a complete ortholattice, where the orthocomplement of a $c_{\leq}$-fixpoint Y is $Y^{\perp} = \{x \in X \mid \forall x' \leq x\ \ x' \notin Y\}$. On the other hand, the left-to-right direction of Theorem 2.10 shows that the more restrictive class of symmetric compatibility structures suffices for the representation of all complete ortholattices.

2.3 Polarity

The third road, introduced by Birkhoff [5], is perhaps the oldest and best known, forming the basis of formal concept analysis (see, e.g., [7, Ch. 3]). In formal concept analysis, the following structures are called "formal contexts."

Definition 2.13. A *polarity structure* is a triple (X, A, I) where X and A are nonempty sets and $I \subseteq X \times A$. For $Y \subseteq X$ and $B \subseteq A$, define:

$$\begin{aligned} Y^* &= \{a \in A \mid \forall y(y \in Y \Rightarrow yIa)\}; \\ B^\dagger &= \{x \in X \mid \forall a(a \in B \Rightarrow xIa)\}; \\ c_I(Y) &= (Y^*)^\dagger = \{x \in X \mid \forall a(\forall y(y \in Y \Rightarrow yIa) \Rightarrow xIa)\}. \end{aligned}$$

Then $(\cdot)^*$ and $(\cdot)^\dagger$ form an antitone Galois connection between $\wp(X)$ and $\wp(A)$, so their composition c_I is a closure operator on $\wp(X)$.

Theorem 2.14 ([5], § 32)**.** For any polarity structure (X, A, I), c_I is a closure operator on $\wp(X)$.

A proof of the following may be found in, e.g., [7, Theorem 3.9].

Theorem 2.15 ([5], §§ 32-4)**.** L is a complete lattice if and only if L is isomorphic to the lattice of c_I-fixpoints of a polarity structure.

Proof. (sketch of $\Rightarrow$) Given L, one takes the polarity structure $(L, L, \leq)$, where $\leq$ is the order in L. □

Birkhoff related polarity structures to ortholattices as follows.

Theorem 2.16 ([5], §§ 32-4)**.** L is a complete ortholattice if and only if L is isomorphic to the lattice of c_I-fixpoints of a polarity structure (X, A, I) in which $X = A$ and I is symmetric and irreflexive, where the orthocomplement of a c_I-fixpoint Y is Y^*.

Remark 2.17. In his semantics for orthologic, Goldblatt [12] calls polarity structures as in Theorem 2.16 *orthoframes*.

Figure 1 below summarizes the constructions of structures from complete lattices that we have discussed in this section.

orders	**compatibility**	**polarity**
$(X, \leq_1, \leq_2)$	$(X, \between)$	(X, A, I)
$X = \{(a, b) \in L^2 \mid a \not\leq b\}$	$X = \{(a, b) \in L^2 \mid a \not\leq b\}$	$X = A = L$
$(a, b) \leq_1 (c, d) \Leftrightarrow a \leq c$	$(a, b) \between (c, d) \Leftrightarrow c \not\leq b$	$I = \leq$
$(a, b) \leq_2 (c, d) \Leftrightarrow b \geq d$		

Figure 1: constructions of structures from a complete lattice L with order $\leq$.

3 Connecting the roads

We will connect the three roads of Section 2 by defining transformations between structures in the directions shown in Figure 2 (full compatibility structures are defined in Section 3.3).

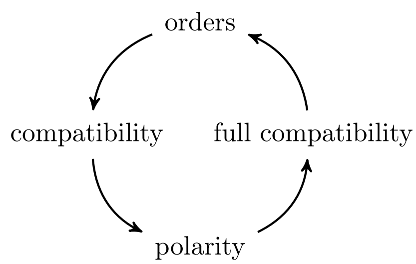

Figure 2: directions of transformation between structures.

3.1 From orders to compatibility

Given $(X, \leq_1, \leq_2)$, define $(X, \between)$ by

$$x \between y \Leftrightarrow \exists z \in X \colon z \leq_2 x \text{ and } z \leq_1 y.$$

Since $\leq_1$ and $\leq_2$ are reflexive, so is $\between$. That c_{12} and the derived $c_{\between}$ have the same fixpoints—and hence induce the same complete lattice—follows from the next two lemmas.

Lemma 3.1. Every $c_{\between}$-fixpoint is a $\leq_1$-downset.

Proof. Suppose Y is a $c_{\between}$-fixpoint, $x \in Y$, and $y \leq_1 x$. Since Y is a $c_{\between}$-fixpoint, to show $y \in Y$, it suffices to show that for all $y' \between y$ there is a $y'' \between^{-1} y'$ such that $y'' \in Y$. Suppose $y' \between y$, so there is a z such that $z \leq_2 y'$ and $z \leq_1 y$. From $z \leq_1 y \leq_1 x$ and the transitivity of $\leq_1$, we have $z \leq_1 x$. Then from $z \leq_2 y'$ and $z \leq_1 x$, we have $y' \between x$. Then since $x \in Y$ and Y is a $c_{\between}$-fixpoint, there is a $y'' \between^{-1} y'$ such that $y'' \in Y$, which completes the proof. □

Lemma 3.2. If Y is a $\leq_1$-downset, then $c_{12}(Y) = c_{\between}(Y)$.

Proof. Let Y be a $\leq_1$-downset. We must show:

$$\{x \in X \mid \forall x' \leq_1 x \; \exists x'' \leq_2 x' \;\; x'' \in Y\} = \{x \in X \mid \forall x' \between x \; \exists x'' \between^{-1} x' \;\; x'' \in Y\}. \tag{1}$$

Suppose x is in the lhs of (1). To show that x is in the rhs, suppose $x' \between x$, so there is a z such that $z \leq_2 x'$ and $z \leq_1 x$. Then since x is in the lhs and $z \leq_1 x$, there is a $z' \leq_2 z$ such that $z' \in Y$. From $z' \leq_2 z \leq_2 x'$ and the transitivity of $\leq_2$, we have $z' \leq_2 x'$, which with $z' \leq_1 z'$ implies $x' \between z'$, so $z' \between^{-1} x'$. Thus, we have shown that for all $x' \between x$ there is a $z' \between^{-1} x'$ such that $z' \in Y$, which shows that x is in the rhs of (1).

Suppose x is in the rhs of (1). To show that x is in the lhs, suppose $x' \leq_1 x$, which with $x' \leq_2 x'$ implies $x' \between x$. Then since x is in the rhs of (1), there is an x'' such that $x' \between x''$ and $x'' \in Y$. By definition of $\between$, it follows that there is a $z \leq_2 x'$ such that $z \leq_1 x''$. Then since Y is a $\leq_1$-downset, we have $z \in Y$. Thus, we have shown that for all $x' \leq_1 x$ there is a $z \leq_2 x'$ such that $z \in Y$, which shows that x is in the lhs of (1). □

Theorem 3.3. The fixpoints of c_{12} are exactly the fixpoints of $c_{\between}$.

Proof. Immediate from Lemmas 2.3, 3.1, and 3.2. □

Remark 3.4. Recall the doubly ordered structure in the proof of Theorem 2.7, for a given complete lattice L: $X = \{(a,b) \in L^2 \mid a \not\leq b\}$; $(a,b) \leq_1 (c,d) \Leftrightarrow a \leq c$; and $(a,b) \leq_2 (c,d) \Leftrightarrow b \geq d$. Now if we construct $(X, \between)$ from $(X, \leq_1, \leq_2)$ as above, then we have:

$$\begin{aligned}(a,b) \between (c,d) &\Leftrightarrow \exists (e,f) \in X \colon (e,f) \leq_2 (a,b) \text{ and } (e,f) \leq_1 (c,d) \\ &\Leftrightarrow \exists e, f \in L \colon e \not\leq f, f \geq b, \text{ and } e \leq c \\ &\Leftrightarrow c \not\leq b,\end{aligned}$$

which agrees with the direct construction of $(X, \between)$ from L in the proof of Theorem 2.11.

3.2 From compatibility to polarity

Given $(X, \between)$, define (X, X, I) by

$$xIy \Leftrightarrow \text{not } y \between x.$$

In this case, it is even more direct that $c_{\between}$ and the derived c_I have the same fixpoints and hence induce the same complete lattice.

Proposition 3.5. For any $Y \subseteq X$, $c_{\between}(Y) = c_I(Y)$.

Proof. We have:

$$\begin{aligned}
c_I(Y) &= \{x \in X \mid \forall x'(\forall x''(x'' \in Y \Rightarrow x'' I x') \Rightarrow x I x')\} \\
&= \{x \in X \mid \forall x'(\forall x''(x'' \in Y \Rightarrow \text{not } x' \between x'') \Rightarrow \text{not } x' \between x)\} \\
&= \{x \in X \mid \forall x'(x' \between x \Rightarrow \text{not } \forall x''(x'' \in Y \Rightarrow \text{not } x' \between x''))\} \\
&= \{x \in X \mid \forall x'(x' \between x \Rightarrow \exists x''(x'' \in Y \text{ and } x' \between x''))\} \\
&= \{x \in X \mid \forall x' \between x \, \exists x'' \between^{-1} x' \;\; x'' \in Y\} \\
&= c_{\between}(Y).
\end{aligned}$$

□

3.3 From polarity to full compatibility

To move from polarity structures to compatibility structures, we need to rule out trivial polarity structures in the following sense.

Definition 3.6. A polarity structure (X, A, I) is *trivial* if $I = X \times A$. Otherwise it is *nontrivial*.

This terminology is justified by the following facts.

Lemma 3.7. For any polarity structure (X, A, I) and $x \in X$, if xIa for all $a \in A$, then $x \in Y$ for every c_I-fixpoint $Y \subseteq X$.

Proof. Immediate from Definition 2.13. □

Lemma 3.8. If (X, A, I) is trivial, then the only c_I-fixpoints are $\varnothing$ and X.

Proof. Immediate from Lemma 3.7. □

Given a nontrivial (X, A, I), define $(P, \between)$ by $P = \{(x, a) \in X \times A \mid \text{not } xIa\}$ and

$$(x, a) \between (y, b) \Leftrightarrow \text{not } yIa.$$

By nontriviality, P is nonempty. Also note that $\between$ is reflexive.

Definition 3.9. A compatibility structure $(X, \between)$ is *full* if it satisfies:

$$x \between y \Rightarrow \exists w \forall z \, ((w \between z \Rightarrow x \between z) \text{ and } (z \between w \Rightarrow z \between y)).$$

Proposition 3.10. $(P, \between)$ is a full compatibility structure.

Proof. Given $(x, a) \between (y, b)$, consider (y, a). For any (z, c), if $(y, a) \between (z, c)$, so *not* zIa, then $(x, a) \between (z, c)$. In addition, for any (z, c), if $(z, c) \between (y, a)$, so *not* yIc, then $(z, c) \between (y, b)$. □

Remark 3.11. By the same reasoning as in the proof of Proposition 3.10, the compatibility structure constructed in the proof of Theorem 2.11 is also full, showing immediately that any complete lattice L can be represented by a full compatibility structure.

Define a function f from the lattice of c_I-fixpoints of (X, A, I) to $\wp(P)$ by

$$f(Y) = \{(y, a) \in P \mid y \in Y\}.$$

Theorem 3.12. The map f is an isomorphism between the lattice of c_I-fixpoints of (X, A, I) and the lattice of $c_{\between}$-fixpoints of $(P, \between)$.

Proof. It suffices to show that for all c_I-fixpoints $Y, Y' \subseteq X$ and $c_{\between}$-fixpoints $S \subseteq P$:

1. $f(Y)$ is a $c_{\between}$-fixpoint;

2. $Y \subseteq Y' \Leftrightarrow f(Y) \subseteq f(Y')$;

3. $S = f(\{x \in X \mid \exists a \in A \;\; (x, a) \in S\})$.

For part 1, to show $c_{\between}(f(Y)) \subseteq f(Y)$, suppose $(x, a) \in c_{\between}(f(Y))$. Since $(x, a) \in P$, to show $(x, a) \in f(Y)$ it suffices to show $x \in Y$. For this, we use that Y is a c_I-fixpoint. Suppose $b \in A$ is such that for all $y \in Y$, yIb. Then we claim xIb. If instead *not* xIb, then $(x, b) \in P$ and $(x, b) \between (x, a)$. Then since $(x, a) \in c_{\between}(f(Y))$, there is a $(x, b) \in P$ such that $(x, b) \between (z, c)$ and $(z, c) \in f(Y)$. It follows that $z \in Y$ and *not* zIb, which contradicts our assumption that yIb for all $y \in Y$. Thus, xIb. Then since Y is a c_I-fixpoint, we have $x \in Y$, as desired.

For part 2, suppose $Y \subseteq Y'$ and $(y, a) \in f(Y)$. From $(y, a) \in f(Y)$ we have $y \in Y$ and hence $y \in Y'$, which with $(y, a) \in P$ yields $(y, a) \in f(Y')$. Conversely, suppose $f(Y) \subseteq f(Y')$ and $y \in Y$. Case 1: there is an $a \in A$ such that *not* yIa, so $(y, a) \in P$. Then $(y, a) \in f(Y)$ and hence $(y, a) \in f(Y')$, so $y \in Y'$. Case 2: yIa for all $a \in A$. Then since Y' is a c_I-fixpoint, we have $y \in Y'$ by Lemma 3.7.

For part 3, if $(x, a) \in S$, then $x \in \{x \in X \mid \exists a \in A\colon (x, a) \in S\}$. Then since $(x, a) \in P$, $(x, a) \in f(\{x \in X \mid \exists a \in A\colon (x, a) \in S\})$. Conversely, if $(x, b) \in f(\{x \in X \mid \exists a \in A\colon (x, a) \in S\})$, then there is an $a \in A$ with $(x, a) \in S$. Suppose $(y, c) \between (x, b)$, so *not* xIc. Then $(y, c) \between (x, a)$, which with $(x, a) \in S$ and the fact that S is a $c_{\between}$-fixpoint implies that there is a $(z, d) \between^{-1} (y, c)$ such that $(z, d) \in S$. Thus, for any $(y, c) \between (x, b)$ there is a $(z, d) \between^{-1} (y, c)$ such that $(z, d) \in S$. Then since S is a $c_{\between}$-fixpoint, $(x, b) \in S$. ☐

Remark 3.13. If we take the polarity structure representing L in the proof of Theorem 2.15 and transform it into a compatibility structure as above, the result is exactly the compatibility structure representing L in the proof of Theorem 2.11.

Remark 3.14.
1. By applying the transformation of Section 3.2 followed by that of this subsection, any compatibility structure can be turned into a full compatibility structure such that their lattices of $c_{\between}$-fixpoints are isomorphic. We send $(X, \between)$ to $(P, \between')$ where $P = \{(x, y) \in X^2 \mid y \between x\}$ and $(x_1, y_1) \between' (x_2, y_2) \Leftrightarrow y_1 \between x_2$.

2. By applying the transformation of this subsection followed by that of Section 3.2, any nontrivial polarity structure can be turned into a polarity structure (X, A, I) in which $X = A$ such that their lattices of c_I-fixpoints are isomorphic. We send (X, A, I) to (P, P, I') where $P = \{(x, a) \in X \times A \mid \text{not } xIa\}$ and $(x, a)I'(y, b) \Leftrightarrow xIb$.

3.4 From full compatibility to orders

Given a full compatibility structure $(X, \between)$, define $(X, \leq_1, \leq_2)$ by:

$$x \leq_1 y \Leftrightarrow \forall z \in X \, (z \between x \Rightarrow z \between y);$$
$$x \leq_2 y \Leftrightarrow \forall z \in X \, (x \between z \Rightarrow y \between z).$$

Then clearly $\leq_1$ and $\leq_2$ are preorders, and since $(X, \between)$ is full, we have:

$$x \between y \Leftrightarrow \exists w \in X \colon w \leq_2 x \text{ and } w \leq_1 y.$$

Thus, $(X, \between)$ and $(X, \leq_1, \leq_2)$ are related exactly as in Section 3.1, so we have the following.

Theorem 3.15. The fixpoints of $c_\between$ are exactly the fixpoints of c_{12}.

We have now come full circle (recall Figure 2). Transformations between any of the four types of structures (doubly ordered, compatibility, full compatibility, polarity) may be obtained by compositions from the transformations we have explicitly defined. For convenience, we summarize the transformations in Figure 3.

$(X, \leq_1, \leq_2)$ to $(X, \between)$	$(X, \between)$ to (X, X, I)
$x \between y \Leftrightarrow \exists z \in X \colon z \leq_2 x$ and $z \leq_1 y$	$xIy \Leftrightarrow$ not $y \between x$
nontrivial (X, A, I) to full $(P, \between)$	full $(X, \between)$ to $(X, \leq_1, \leq_2)$
$P = \{(x, a) \in X \times A \mid \text{not } xIa\}$ $(x, a) \between (y, b) \Leftrightarrow$ not yIa	$x \leq_1 y \Leftrightarrow \forall z \in X\,(z \between x \Rightarrow z \between y)$ $x \leq_2 y \Leftrightarrow \forall z \in X\,(x \between z \Rightarrow y \between z)$

Figure 3: the four transformations.

We finish this section with an example of how our transformations may be used to easily convert representation theorems from one setting to another.

Example 3.16. We claim that L is a complete Boolean algebra if and only if L is isomorphic to the lattice of $c_\between$-fixpoints of a compatibility structure $(X, \between)$ in which $\between$ is symmetric and for all $x, y \in X$, if $x \between y$, then $\exists w \forall z (z \between w \Rightarrow (z \between x \text{ and } z \between y))$.

From right to left, if $(X, \between)$ is a compatibility structure satisfying the stated condition, then $(X, \between)$ is full, so we can transform it into a doubly ordered structure $(X, \leq_1, \leq_2)$ as above, and the assumed symmetry of $\between$ implies that $\leq_1 = \leq_2$. Then apply Theorem 3.15 and the right-to-left direction of Theorem 2.5. From left to right, by the left-to-right direction of Theorem 2.5, L is isomorphic to the lattice of c_{12}-fixpoints of a doubly ordered structure $(X, \leq_1, \leq_2)$ in which $\leq_1 = \leq_2$. Transforming $(X, \leq_1, \leq_2)$ into $(X, \between)$ as in Section 3.1, one easily sees that $(X, \between)$ satisfies the stated condition. Now apply Theorem 3.3.

4 Conclusion

We have seen how to directly transfer between three roads to complete lattices: doubly ordered structures, compatibility structures, and polarity structures. However, there may be advantages to staying on a particular road depending on the kind of complete lattices one wishes to produce. As we have seen, there are classes of doubly ordered structures giving rise to all complete Boolean algebras and complete Heyting algebras that are very simply described, namely by the conditions $\leq_1 = \leq_2$ and $\leq_2 \subseteq \leq_1$, respectively; and there is a class of compatibility structures (resp. polarity structures) giving rise to all complete ortholattices that is very simply described, namely by the conditions of reflexivity and symmetry for $\between$

(resp. irreflexivity and symmetry for I). Describing the compatibility structures (resp. polarity structures) giving rise to all complete Boolean or Heyting algebras is more complicated, as is describing the doubly ordered structures giving rise to all complete ortholattices. It would be desirable to explain this systematically in terms of a theory that relates the equations one wants to hold of a complete lattice to the conditions (preferably first-order) that one may then assume for one's doubly ordered/compatibility/polarity structures. First of all one may seek out the conditions on structures that *correspond* to the lattice of fixpoints satisfying certain equations, in the sense of modal correspondence theory [2]. However, it is noteworthy that the conditions we have mentioned are stronger than the conditions that are correspondents. For example, a weaker condition than $\leq_2 \subseteq \leq_1$ (though still not the weakest) that suffices for the lattice of c_{12}-fixpoints of a doubly ordered structure to be a Heyting algebra is the following more complicated condition:

$$\text{if } y \leq_2 x, \text{ then } \exists z \in X\colon\ z \leq_1 y,\ z \leq_1 x, \text{ and } z \leq_2 x.$$

Thus, we would like to understand not only the structural conditions that correspond to the lattice of fixpoints satisfying given equations, but also the strongest or simplest structural conditions that allow for the representation of all complete lattices satisfying given equations.

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