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Global Weyl Modules and Maximal Parabolics of Twisted Affine Lie Algebras

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Global Weyl Modules and Maximal Parabolics of Twisted Affine Lie Algebras

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To my grandfather. Rest in peace.

ABSTRACT OF THE DISSERTATION

Global Weyl Modules and Maximal Parabolics of Twisted Affine Lie Algebras

by

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Doctor of Philosophy, Graduate Program in Mathematics
University of California, Riverside, September 2018
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In this paper I will first discuss the structure of non-standard maximal parabolics of twisted affine Lie algebras. Then I will define global Weyl modules in this setting and discuss the associated commutative associative algebra, $\mathbf{A}_\lambda$. These modules are an analog of Verma modules in the affine setting and were defined in 2001 by Chari and Pressley. Global Weyl modules have been studied in other settings such as twisted affine Lie algebras, [3, 8, 10] and non-standard maximal parabolics of untwisted affine Lie algebras, [4].

Contents

Introduction

Integrable representations of affine Lie algebras, $\hat{\mathfrak{g}}$ have been an important family of modules for years. Global Weyl modules, $W(\lambda)$, were introduced in [7] and these modules played a role, for integrable modules, that Verma modules played for modules of simple Lie algebras, $\mathfrak{g}$. By this we mean any highest weight cyclic integrable module for $\hat{\mathfrak{g}}$ is the quotient of a corresponding global Weyl module. We can find a natural commutative algebra, $\mathbf{A}_\lambda$, which acts on $W(\lambda)$ and turns $W(\lambda)$ into a $(\widehat{\mathfrak{g}}, \mathbf{A}_\lambda)$-bimodule. These modules were shown to have a bimodule structure, with the natural left module structure and a right module structure given by the action of a natural commutative algebra, $\mathbf{A}_\lambda$. An additional result in [7] demonstrated that for $\hat{\mathfrak{g}}$, $\mathbf{A}_\lambda$ is isomorphic to a polynomial algebra in finitely many variables.

After their introduction these modules, and their finite dimensional analogue the local Weyl modules, have been further studied in other settings. [2] and [10] addressed the local and global Weyl modules for twisted loop algebras. [8] and [3] studied local Weyl modules for twisted current algebras, with the latter handling the $A_{2n}^{(2)}$ case. These last two papers extended the results for the full affine Lie algebra, twisted and untwisted, to the standard maximal parabolic subalgebras. Chari, Kus, and O'Dell in [4] then extended this approach by studying local and global Weyl modules for non-standard maximal parabolic subalgebras of untwisted affine Lie algebras.

This dissertation will address global Weyl modules for non-standard maximal parabolics of twisted affine Lie algebras. We will investigate how the structure of global Weyl modules differs from the structure it has in other settings.

Chapter 1

Background

1.1 Background

1.1.1 Notation

We denote by $\mathbb{C}, \mathbb{Z}, \mathbb{Z}_+$, and $\mathbb{N}$ the complex numbers, integers, non$-$negative integers, and positive integers, respectively. Unless stated otherwise, all vectors spaces are $\mathbb{C}$-vectors spaces and $\otimes$ stands for $\otimes_{\mathbb{C}}$. Given any Lie algebra $\mathfrak{a}$ we denote by $\mathbf{U}(\mathfrak{a})$ the universal enveloping algebra of $\mathfrak{a}$. We also fix an indeterminate, t, and denote by $\mathbb{C}[t]$ and $\mathbb{C}[t, t^{-1}]$ the corresponding polynomial ring and Laurent polynomial ring with complex coefficients.

1.1.2 Lie algebra Notation

Let $\mathfrak{g}$ be a complex simple finite$-$dimensional Lie algebra of rank n with fixed Cartan subalgebra $\mathfrak{h}$. Let $I = \{1, \ldots, n\}$ and fix a set of simple roots $\{\alpha_i : i \in I\}$ of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Let R, R^+ be the corresponding set of roots, positive roots, respectively. For $i \in I$, let a_i denote the labels of the Dynkin diagram of $\mathfrak{g}$: equivalently the highest root of R^+ is

$\theta = \sum_{i=1}^{n} a_i\alpha_i$. Fix a Chevalley basis $\{x_\alpha^+, h_i : \alpha \in R^+, i \in I\}$ of $\mathfrak{g}$ and set $x_i^\pm = x_{\alpha_i}^\pm$. Let $(,)$ be the non$-$degenerate bilinear form on $\mathfrak{h}^*$ with $(\theta,\theta) = 2$ induced by restricting the Killing form of $\mathfrak{g}$ to $\mathfrak{h}$.

Let Q be the root lattice with basis α_i, $i \in I$. Define $\mathbf{a}_i : Q \to \mathbb{Z}$, $i \in I$ by requiring $\eta = \sum_{i=1}^{n} \mathbf{a}_i(\eta)\alpha_i$, and set $ht(\eta) = \sum_{i=1}^{n} \mathbf{a}_i(\eta)$. For $\alpha \in R$ set $d_\alpha = \dfrac{2}{(\alpha,\alpha)}$, and $h_\alpha = \sum_{i=1}^{n} \check{\mathbf{a}}_i(\alpha) h_i$. Let W be the Weyl group of $\mathfrak{g}$ and fix a set of simple reflections s_i, $i \in I$.

1.1.3 Diagram Automorphism

For a simply-laced Lie algebra $\mathfrak{g}$ we let σ denote a diagram automorphism of $\mathfrak{g}$ of order k. For a fixed primitive k^{th} root of unity, ξ, $\mathfrak{g}$ decomposes, as a vector space,

$$\mathfrak{g} = \bigoplus_{s=0}^{k-1} \mathfrak{g}_s^\sigma$$

where

$$\mathfrak{g}_s^\sigma = \{x \in \mathfrak{g} : \sigma(x) = \xi^s x\}.$$

For any subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ that is preserved by σ, set $\mathfrak{a}_m = \mathfrak{g}_m \cap \mathfrak{a}$. It is known that $\mathfrak{g}_0^\sigma$ is a simple Lie algebra, $\mathfrak{h}_0^\sigma = \mathfrak{g}_0^\sigma \cap \mathfrak{h}$ is a Cartan subalgebra and that $\mathfrak{g}_m^\sigma$ is an irreducible representation of $\mathfrak{g}_0^\sigma$ for all $0 \le m \le k-1$. Moreover,

$$\mathfrak{n}^\pm \cap \mathfrak{g}_0^\sigma = \mathfrak{n}_0^\pm = \bigoplus_{\alpha \in R^+_{\mathfrak{g}_0^\sigma}} (\mathfrak{g}_0^\sigma)_{\pm\alpha}.$$

The following table [2] describes the various possibilities for $\mathfrak{g}, \mathfrak{g}_0^\sigma$ and the structures of $\mathfrak{g}_m^\sigma$ as a $\mathfrak{g}_0^\sigma$-module. Here θ_0^s is the highest short root of $\mathfrak{g}_0^\sigma$.

m	$\mathfrak{g}$	$\mathfrak{g}_0$	$\mathfrak{g}_m$
2	A_{2n}	B_n	$V_{\mathfrak{g}_0}(2\theta_0^s)$
2	$A_{2n-1}, \quad n \geq 2$	C_n	$V_{\mathfrak{g}_0}(\theta_0^s)$
2	$D_{n+1}, \quad n \geq 3$	B_n	$V_{\mathfrak{g}_0}(\theta_0^s)$
2	E_6	F_4	$V_{\mathfrak{g}_0}(\theta_0^s)$
3	D_4	G_2	$V_{\mathfrak{g}_0}(\theta_0^s)$

From now on we will denote by $\mathfrak{g}^\sigma$ the eigenspace decomposition of $\mathfrak{g}$ with respect to the diagram automorphism σ. It is noted in [Helgason] that $\mathfrak{g}^\sigma$ is generated by $\{x_i^\pm : i \in I(\mathfrak{g}_0^\sigma), \} \cup \{x_{\theta_0^s}^\pm\}$ where the latter set is the highest, and lowest weight vectors of $\mathfrak{g}_1^\sigma$. This $\{x_i^\pm\}$ notation will be used for the rest of the paper and will refer to the simple roots of $\mathfrak{g}^\sigma$.

1.1.4 Scaling Automorphism

Set $I(j) = I \backslash \{j\}$ and let η be a primitive $\mathbf{a}_j(\theta_0^s) \cdot k^{th}$- root of unity, where k is the order of σ. The following holds by direct computation of the defining relations.

Proposition 1.1.1. *The assignment*

$$X_i^\pm \to X_i^\pm, \quad i \in I(j) \cup \{0\}, \quad X_j^\pm \to \eta^{\pm 1} X_j^\pm,$$

defines an automorphism $\tau : \mathfrak{g}^\sigma \to \mathfrak{g}^\sigma$ of order a_j. Moreover, the set of fixed points $(\mathfrak{g}_0^\sigma)^\tau$ is a semismple subalgebra with Cartan subalgebra $\mathfrak{h}^{\tau\sigma}$ and

$$R_0 = \{\alpha \in R : \mathbf{a}_j(\alpha) \in \{0, \pm a_j\}\},$$

is the set of roots of the pair $((\mathfrak{g}_0^\sigma)^\tau, \mathfrak{h}^{\tau\sigma})$. The set $\{\alpha_i : i \in I(j)\} \cup \{-\theta_0^s\}$ is a simple system for R_0.

Proof. The key observation is that $\tau\sigma : \mathfrak{g} \to \mathfrak{g}$ is an automorphism given by

$$X_i^\pm \to X_{\sigma(i)}^\pm, \quad i \in I(j), \quad X_j^\pm \to \eta^{\pm 1} X_{\sigma(j)}^\pm,$$

where $\{X_i^{\pm} : i \in I_{\mathfrak{g}}\}$ represents the set of simple generators of $\mathfrak{g}$. A simple checking of the defining relations verifies that τ is an automorphism, $\mathfrak{g}^{\sigma} \to \mathfrak{g}^{\sigma}$. $\square$

Remark: Since $\sigma : \mathfrak{g} \to \mathfrak{g}$ is defined by $\sigma(X_i^{\pm}) = X_{\sigma(i)}^{\pm}$ for simple generators $X_i \in \mathfrak{g}$ and $\tau : \mathfrak{g} \to \mathfrak{g}$ is defined by

$$\tau(X_i^{\pm}) = \begin{cases} X_i^{\pm} & \text{if } i \neq j, \sigma(j) \\ \\ \eta^{\mp 1} X_i & \text{if } i = j, \sigma(j) \end{cases}$$

the two automorphisms commute.

We now set some notation that we will be using for the rest of the dissertation when it is not explicitly stated or ambiguous. Any symbol, such as $\Delta, I, \mathfrak{g}$, etc, with a superscript σ refers to the corresponding structure in $\mathfrak{g}^{\sigma}$. Similarly for a superscript $\tau\sigma$. No superscript refers to the corresponding structure in $\mathfrak{g}$.

1.1.5 New Simple System

Lemma 1.1.1. *Let $w_0^{\tau\sigma}$ be the longest element of the subgroup of $W(\mathfrak{g}_0^{\tau\sigma})$, the Weyl group, generated by $\{s_i : i \in I(j)\}$. The set*

$$\Delta_0 = \{\alpha_i : i \in I^{\sigma}(j)\} \cup \{(w_0^{\tau\sigma})^{-1}\theta_0^s\}$$

is a set of simple roots for $(\mathfrak{g}_0^{\tau\sigma}, \mathfrak{h}_0^{\tau\sigma})$ and the corresponding set $(R_0^{\tau\sigma})^+$ of positive roots is contained in $(R^{\sigma})^+$.

Proof. Since w_0 is the longest element of the Weyl group generated by $\{s_i : i \in I(j)\}$, for $i \in I^{\sigma}(j)$:

$$w_0^{\tau\sigma}(\alpha_i) \in \{-\alpha_i : i \in I^{\sigma}(j)\}.$$

Thus,

$$\Delta_0 = -(w_0^{\tau\sigma})^{-1}(\{\alpha_i : i \in I^\sigma(j)\} \cup \{-\theta_0^s\}).$$

Since $w_0^{\tau\sigma}$ is an element of the Weyl group for $\mathfrak{g}^\sigma$, it follows that $\Delta_0^{\tau\sigma}$ is a simple system for $\mathfrak{g}^{\tau\sigma}$. Moreover, since $\mathbf{a}_j(\theta_0^s) = \mathbf{a}_j(w_0^{\tau\sigma}\theta_0^s)$ we have $\theta_0^s \in (R^\sigma)^+$. Thus, $\Delta_0 \subset (R^\sigma)^+$ and the lemma is proved. $\square$

The $(w_0^{\tau\sigma})^{-1}\theta_0^s$ are listed below separated based on the choice of j, when it matters.

For $A_{2n-1}^{(2)}$ we get $\alpha_0^{\tau\sigma} = \alpha_{j-1} + 2(\alpha_j + \ldots + \alpha_{n-1}) + \alpha_n$.

For $D_n^{(2)}$ we get $\alpha_0^{\tau\sigma} = \alpha_j + \ldots + \alpha_n$

For $D_4^{(3)}$ we get $\alpha_0^{\tau\sigma} = \alpha_1 + \alpha_2$ for $j = 1$

For $E_6^{(2)}$ we get $\alpha_0^{\tau\sigma} = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4$ when $j = 2$

For $E_6^{(2)}$ we get $\alpha_0^{\tau\sigma} = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$ when $j = 3$

For $E_6^{(2)}$ we get $\alpha_0^{\tau\sigma} = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$ when $j = 4$

For $A_{2n}^{(2)}$, we get $\alpha_0^{\tau\sigma} = 2\alpha_j + \ldots + 2\alpha_n$.

Remark: We can now note some immediate consequences of 1.1.1 using the following notation for the rest of the paper, $\alpha_0^{\tau\sigma} = (\omega_0^{\tau\sigma})^{-1}(\theta_0^s)^\sigma, x_0^{\tau\sigma} = x^{\pm}_{\alpha_0^{\tau\sigma}}$, $a_j = \mathbf{a}_j(\alpha_0^{\tau\sigma})$ and $h_0 = h_{\alpha_0}$

(i) $\alpha_0^{\tau\sigma}$ is a short root

(ii) $(\alpha_0^{\tau\sigma}, \alpha_i^{\tau\sigma}) \leq 0$ if $i \in I(j)$ and $(\alpha_0^{\tau\sigma}, \alpha_j^{\tau\sigma}) > 0$

(iii) $\mathbf{a}_j(\alpha_0^{\tau\sigma}) = a_j$, and

(iv) ht $\alpha^{\tau\sigma} \geq$ ht $\alpha_0^{\tau\sigma}$ for all $\alpha^{\tau\sigma} \in R_0^+$ with $\mathbf{a}_j(\alpha^{\tau\sigma}) = a_j^{\tau\sigma}$

For $1 \leq m < ka_j$ we set

$$(R_m)^{\tau\sigma} = \{\alpha \in R^\sigma : \mathbf{a}_j(\alpha) \in \{m, -a_j + m\}\},$$

$$\mathfrak{g}_m^{\tau\sigma} = \bigoplus_{\alpha \in R_m} \mathfrak{g}_\alpha^\sigma.$$

Equivalently

$$\mathfrak{g}_m^{\tau\sigma} = \{x \in \mathfrak{g}^\sigma : \tau\sigma(x) = \eta^m x\}.$$

Setting $(R_m^{\tau\sigma})^+ = R_m^{\tau\sigma} \cap (R^\sigma)^+$, we observe that

$$[x_0^+, R_k^+] = 0, \quad 1 \leq m < ka_j. \tag{1.1}$$

1.1.6 Explicit Description of $\mathfrak{g}^{\tau\sigma}$

Using the previous sections we now give an explicit description of $(\mathfrak{g}_m)^{\tau\sigma}$ when $\mathbf{a}_j(\theta_0^s) \neq 1$. In each table below the $\mathfrak{g}_m = \{x \in \mathfrak{g}^\sigma : \tau\sigma(x) = \eta^m x\}$ where η is a ka_j-primitive root of unity. Each row in the table corresponds to $\mathfrak{g}_0^\sigma, \mathfrak{g}_1^\sigma$, and $\mathfrak{g}_2^\sigma$ respectively.

$A_{2n-1}^{(2)}$

$\mathfrak{g}_0$

$\pm\alpha_1, \ldots, \alpha_{j-1}, \alpha_{j+1}, \ldots, \alpha_n$, $\ldots$ $\alpha_{j+1} + \ldots, \alpha_n$, $\alpha_1 + \ldots + \alpha_{j-1}$
$\pm\alpha_0 = \pm\alpha_{j-1} + 2(\alpha_j + \ldots + \alpha_{n-1}) + \alpha_n$, θ_0^s

$\mathfrak{g}_1$

$\alpha_j, \ldots \alpha_1 + \ldots + \alpha_j + 2(\alpha_{j+1} + \ldots + \alpha_{n-1}) + \alpha_n$
$-(\alpha_1 + \ldots + \alpha_j + 2(\alpha_{j+1} + \ldots + \alpha_{n-1}) + \alpha_n), \ldots, -(\alpha_1 + \ldots + \alpha_n), -(\alpha_j + \ldots + \alpha_n)$

$\mathfrak{g}_2$

$-(2\alpha_j + \ldots + 2\alpha_{n-1} + \alpha_n), \ldots, -\theta, (2\alpha_j + \ldots + 2\alpha_{n-1} + \alpha_n), \ldots, \theta$
$h_1, \ldots, h_{j-1}, h_{j+1}, \ldots, h_{n-1}, \alpha_1 + \ldots + \alpha_{j-1}, \alpha_{j+1} + \ldots + \alpha_n$

$\mathfrak{g}_3$

$-\alpha_1+\ldots+\alpha_j,\ldots,\alpha_j,\ldots,-(\alpha_j+\ldots+\alpha_n),\ldots$
$\alpha_j,\ldots\alpha_1+\ldots+\alpha_j+2\alpha_{j+1}+2\alpha_{n-1}+\alpha_n$

$\mathfrak{g}_3$ ctd.

$-(\alpha_1+\ldots+\alpha_j+2(\alpha_{j+1}+\ldots+\alpha_{n-1})+\alpha_n)$

$D_4^{(3)}$

$\mathfrak{g}_0$

$h_1,\pm\alpha_1,h_2$
$-\alpha_1-2\alpha_2$
$\alpha_1+2\alpha_2$

$\mathfrak{g}_1$

$\alpha_2,\alpha_1+\alpha_2$
$-\alpha_1-\alpha_2,-\alpha_2$

$\mathfrak{g}_2$

$\alpha_1+2\alpha_2$
h_2
$-\alpha_1-2\alpha_2$

$\mathfrak{g}_3$

$-\alpha_1-3\alpha_2,\ -2\alpha_1-3\alpha_2,\ 2\alpha_1+3\alpha_2,\ \alpha_1+3\alpha_2$
$\alpha_2,\alpha_1+\alpha_2$
$-\alpha_2,-\alpha_1-\alpha_2$

$\mathfrak{g}_4$

$-\alpha_1-2\alpha_2$
$\alpha_1+2\alpha_2$
h_2

$\mathfrak{g}_5$

$-\alpha_1-\alpha_2$
$\alpha_2,\alpha_1+\alpha_2$

$A_{2n}^{(2)}$

$\mathfrak{g}_0$

$\pm\{\alpha_i : 1 \leq i \neq j \leq n\}$, $\{h_i : 1 \leq i \leq n\}$, $\pm\{\alpha_1 + \ldots + \alpha_{j-1}\}$
$\pm\theta_0^s, \ldots \pm (2\alpha_j + \ldots + 2\alpha_n)$

$\mathfrak{g}_0$ ctd.

$\pm\{\alpha_{j+1} + 2\alpha_{j+2} + \ldots + 2\alpha_n\}$

$\mathfrak{g}_1$

$\alpha_j, \ldots, \alpha_1 + \ldots + \alpha_j + 2\alpha_{j+1} + 2\alpha_n$
$-\alpha_j - 2\alpha_{j+1} - \ldots - 2\alpha_n, \ldots$

$\mathfrak{g}_1$ ctd.

$-\alpha_1 - \ldots - \alpha_j - 2\alpha_{j+1} - \ldots - 2\alpha_n, \ldots, -\alpha_j - \ldots - \alpha_n, \ldots, -\alpha_j$

$\mathfrak{g}_2$

$\theta, \ldots, \alpha_{j-1} + 2(\alpha_j + \ldots + \alpha_n)$, $-\theta, \ldots, -\alpha_{j-1} - 2(\alpha_j + \ldots + 2\alpha_n)$
$h_j, \alpha_1, \ldots, \alpha_{j-1}, \ldots, \alpha_1 + \alpha_{j-1}, \alpha_1 + \ldots + \alpha_{j-1}, \alpha_{j+1} + 2(\alpha_{j+1} + \ldots + \alpha_n)$

$\mathfrak{g}_3$

$-\alpha_1 - \ldots - \alpha_j, \ldots, -2(\alpha_j + \ldots + \alpha_n),$
$\alpha_j + 2(\alpha_{j+1} + \ldots + \alpha_n), \ldots \alpha_1 + \ldots + \alpha_j + 2(\alpha_{j+1} + \ldots + \alpha_n)$

$\mathfrak{g}_3$ ctd.

$-\alpha_{j-1} - \alpha_j - 2(\alpha_{j+1} + \ldots + \alpha_n), \ldots - \alpha_j - \ldots - \alpha_n$

$E_6^{(2)}$ with $j = 2$

$\mathfrak{g}_0$

$\pm\alpha_1, \alpha_3, \alpha_4, \alpha_3 + \alpha_4$
$\pm\alpha_0 = \pm(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4), \ldots, \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4$

$\mathfrak{g}_1$

$\alpha_2, \ldots, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4, -\theta, -\alpha_1 - 3\alpha_2 - 4\alpha_3 - 2\alpha_4$
$-(\alpha_2 + \alpha_3), \ldots, -\alpha_1 - \alpha_2 - 2\alpha_3 - 2\alpha_4$

$\mathfrak{g}_2$

$\alpha_1+2\alpha_2+2\alpha_3+\alpha_4,\dots,\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4$
$-\alpha_3-\alpha_4,-\alpha_3,-\alpha_4,h_3,h_4,\alpha_3,\alpha_4,\alpha_3+\alpha_4$

$\mathfrak{g}_2$ ctd.

$-\alpha_1-2\alpha_2-2\alpha_3,\dots,-\alpha_1-2\alpha_2-4\alpha_3-2\alpha_4$

$\mathfrak{g}_3$

$\alpha_1+3\alpha_2+4\alpha_3+2\alpha_2,\theta,-\alpha_2,\dots,\alpha_1-\alpha_2-2\alpha_3-\alpha_4$
$\alpha_2,\dots,\alpha_1+\alpha_2+2\alpha_3+\alpha_4$

$E_6^{(2)}$ with $j=3$, $\alpha_0=\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4$

$\mathfrak{g}_0$

$\pm\alpha_1,\alpha_2,\alpha_4,\alpha_1+\alpha_2$
$\pm\alpha_0=\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4,\alpha_1+2\alpha_2+3\alpha_3+\alpha_4$

$\mathfrak{g}_1$

$\alpha_3,\dots,\alpha_1+\alpha_2+\alpha_3+\alpha_4$
$-\alpha_2-2\alpha_3-\alpha_4,\dots,-\alpha_1-2\alpha_2-2\alpha_3-\alpha_4$

$\mathfrak{g}_2$

$\alpha_2+2\alpha_3,\dots,\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4,-\theta,\dots,-\alpha_1-2\alpha_2-4\alpha_3-2\alpha_4$
$-\alpha_3,\dots,-\alpha_1-\alpha_2-\alpha_3-\alpha_4$

$\mathfrak{g}_3$

$\alpha_1+2\alpha_2+3\alpha_3+\alpha_4,\theta_0^s,-\alpha_1-2\alpha_2-3\alpha_3-\alpha_4,-\alpha_1-2\alpha_2-3\alpha_3-2\alpha_4$
h_3,α_4

$\mathfrak{g}_4$

$\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4,\dots,\theta,-\alpha_2-2\alpha_3,\dots,-\alpha_1-2\alpha_2-2\alpha_3-2\alpha_4$
$\alpha_3,\dots,\alpha_1+\alpha_2+\alpha_3+\alpha_4$

$\mathfrak{g}_5$

$-\alpha_3, \ldots, -\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4$
$\alpha_2 + 2\alpha - 3 + \alpha_4, \ldots, \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4$

$E_6^{(2)}$ with $j = 4, \alpha_0 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$

$\mathfrak{g}_0$

$\pm\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_1 + \alpha_2 + 2\alpha_3$
$\pm\alpha_0 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$

$\mathfrak{g}_1$

$\alpha_4, \ldots, \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4$
$-\alpha_4, \ldots, -\alpha_1 - 2\alpha_2 - 3\alpha_3 - \alpha_4$

$\mathfrak{g}_2$

$\alpha_2 + 2\alpha_3 + 2\alpha_4, \ldots, \theta, -\alpha_2 - 2\alpha_3 - 2\alpha_4, \ldots, -\theta$
h_4, α_3

$\mathfrak{g}_3$

$-\alpha_4, \ldots, -\alpha_1 - 2\alpha_2 - 3\alpha_3 - \alpha_4$
$\alpha_4, \ldots, \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4$

Proposition 1.1.2. *Using the tables above we can observe that*

(i) $\mathfrak{g}_0^{\tau\sigma} = [\mathfrak{g}_1^{\tau\sigma}, \mathfrak{g}_{ka_j-1}^{\tau\sigma}]$

(ii) For all $1 \leq m < ka_j$ *the subspace* $\mathfrak{g}_m^{\tau\sigma}$ *is an irreducible* $\mathfrak{g}_0^{\tau\sigma}$*-module*

(iii) For all $0 \leq m < \ell < ka_j$*, we have* $\mathfrak{g}_\ell^{\tau\sigma} = [\mathfrak{g}_{\ell-m}^{\tau\sigma}, \mathfrak{g}_m^{\tau\sigma}]$.

Proof. Since $\mathfrak{g}_0^{\tau\sigma}$ is a semisimple, write it as the direct sum of ideals, each of which has a simple root, $\alpha_i^{\tau\sigma}$, such that $\alpha_i^{\tau\sigma}(h_j) \neq 0$. Since $h_j^\sigma = [(x_j^\sigma)^+, (x_j^\sigma)^-] \in [\mathfrak{g}_1^{\tau\sigma}, \mathfrak{g}_{ka_j-1}^{\tau\sigma}]$, $[\mathfrak{g}_1^{\tau\sigma}, \mathfrak{g}_{ka_j-1}^{\tau\sigma}]$ intersects every simple ideal of $\mathfrak{g}_0^{\tau\sigma}$. Since we wrote $\mathfrak{g}_0^{\tau\sigma}$ as a direct sum of ideals, this means there is only one ideal in the sum. Part(ii) is proven by a case by case analysis. Part (iii) is also proved by inspection and the observation $[\mathfrak{g}^{\tau\sigma}_{\ell-m}, \mathfrak{g}^{\tau\sigma}_m]$ is a non-zero $\mathfrak{g}^{\tau\sigma}_0$-module. $\square$

Remark: Part (ii) of the proposition implies $\exists\, \theta_m \in (R^{\tau\sigma}_m)^+$ such that

$$(\alpha_i^{\tau\sigma}, \theta_m) \geq 0 \text{ and } [(x_i^{\tau\sigma})^+, x^+_{\theta_m}] = 0, \quad i \in I(j) \cup \{0\}. \tag{1.2}$$

Since either $\theta_m \neq \theta^\sigma$ or $\theta_m \neq \theta_0^s$, we can observe that

$$[x_j^+, (\mathfrak{g}_m^{\tau\sigma})^+] \neq 0$$

We can also observe that $x^-_{\theta_m} \in \mathfrak{g}_{ka_j-m}$ and that it is a lowest weight with respect to $\mathfrak{g}^{\tau\sigma}_0$ and

$$\mathbf{a}_i(\theta_m) > 0 \text{ for } i \in I, 1 \leq m < ka_j. \tag{1.3}$$

To see this note that the set $\{i \in I^\sigma : \mathbf{a}_i(\theta_m) = 0\}$ is contained in $I(j)$. Suppose $\exists i \in \{i \in I^\sigma : \mathbf{a}_i(\theta_m) = 0\}$. Since R^σ is irreducible there must exist $p \in I$ with $\mathbf{a}_p(\theta_m) > 0$ and $(\alpha_i, \alpha_p) < 0$. It follows that $(\theta_k, \alpha_i) < 0$ which contradicts 1.2. As a consequence of 1.3 we get

$$(\theta, \theta_m) > 0, \quad 1 \leq m < ka_j, \quad \text{ and hence } \theta - \theta_m \in R^+_{ka_j - m}. \tag{1.4}$$

Finally, we note that since $(\theta_m + a_j, \alpha_0) = (\theta_k, \alpha_0) + (\alpha_j, \alpha_0) > 0$ we now have

$$\theta_m + \alpha_j - \alpha_0 \in R^\sigma, \quad m \neq ka_j - 1, \quad \theta_{a_j - 1} + \alpha_j - \alpha_0 \in (R^\sigma_0)^+ \cup \{0\}. \tag{1.5}$$

Chapter 2

$\mathfrak{g}_0[t]^{\tau\sigma}$ and Maximal parabolics

2.1 $\mathfrak{g}[t]^{\tau\sigma}$ and Maximal Parabolics

2.1.1 $\mathfrak{g}[t]^{\tau\sigma}$

For a semisimple Lie algebra, $\mathfrak{g}$, let $\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t]$ be the Lie algebra with Lie bracket given by extending scalars. We can extend the diagram automorphism σ to the current algebra, $\mathfrak{g}[t] := \mathfrak{g} \otimes \mathbb{C}[t]$ in the following manner:

$$\sigma(x \otimes t^r) = \sigma(x) \otimes \xi^{-r} t^r, \quad x \in \mathfrak{g}, \quad r \in \mathbb{Z}_+,$$

where ξ is a primitive k^{th}-root of unity. The set of fixed points of σ is denoted by $\mathfrak{g}[t]^\sigma$ and is called the twisted current algebra. We can then extend the automorphism τ from $\mathfrak{g}^\sigma$ to $\mathfrak{g}[t]^\sigma$ is a similar manner:

$$\tau(x \otimes t^r) = \tau(x) \otimes \eta^{-r} t^r, \quad (x \otimes t) \in \mathfrak{g}[t]^\sigma$$

where η is an ka_j^{th} root of unity.

We will let $\mathfrak{g}[t]^{\tau\sigma}$ denote the subalgebra of fixed points of $\tau\sigma$ and carry the notation from the previous section when referring to eigenspaces with respect to $\sigma, \tau\sigma$. It is immediately clear that

$$\mathfrak{g}[t]^{\tau\sigma} = \bigoplus_{r=0}^{a_j k-1} \mathfrak{g}_r \otimes t^r \mathbb{C}[t^{ka_j}].$$

Further if we regard $\mathfrak{g}[t]^{\tau\sigma}$ as a $\mathbb{Z}_+$-graded Lie algebra, by requiring the grade of $x \otimes t^s$ to be s, then $\mathfrak{g}[t]^{\tau\sigma}$ is also a $\mathbb{Z}_+-$graded Lie algebra, i.e.,

$$\mathfrak{g}[t]^{\tau\sigma} = \bigoplus_{s\in\mathbb{Z}_+} \mathfrak{g}[t]^{\tau\sigma}[s].$$

A graded representation of $\mathfrak{g}[t]^{\tau\sigma}$ is a $\mathbb{Z}_+-$graded vector space V which admits a compatible Lie algebra action of $\mathfrak{g}[t]^{\tau\sigma}$, i.e.,

$$V = \bigoplus_{s\in\mathbb{Z}_+} V[s], \quad \mathfrak{g}[t]^{\tau\sigma}[q]V[s] \subset V[q+s],\ q, s \in \mathbb{Z}_+.$$

2.1.2 Evaluation modules and ideals of $\mathfrak{g}[t]^{\tau\sigma}$

Given $z \in \mathbb{C}$, let $ev_z : \mathfrak{g}[t]^{\tau\sigma} \to \mathfrak{g}$ be defined by $ev_z(x \otimes t^r) = z^r v$. It is easy to see that $ev_0(\mathfrak{g}[t]^{\tau\sigma})$ is a semi-simple Lie algebra associated to dropping node j in the Dynkin diagram for the Twisted Affine Lie algebra, and

$$ev_0(\mathfrak{g}[t]^{\tau\sigma}) = \mathfrak{g}_0^{\tau\sigma}, \quad ev_z(\mathfrak{g}[t]^{\tau\sigma}) = \mathfrak{g}, \quad z \neq 0. \tag{2.1}$$

The kernels of these evaluation maps are ideals of finite codimension and one can construct ideals of finite codimension in $\mathfrak{g}[t]^{\tau\sigma}$ in a more general fashion as follows. Let $f \in \mathbb{C}[t^{ka_j}]$ and $0 \leq m < ka_j$. The ideal $(\mathfrak{g} \otimes t^m f\mathbb{C}[t])^{\tau\sigma}$ of $\mathfrak{g}[t]^{\tau\sigma}$ is of finite codimension since there are finitely many vectors with grade less than $\deg(t^m f)$. Let $I_{m,f} = (\mathfrak{g} \otimes t^m f\mathbb{C}[t^{a_j}])^{\tau\sigma}$, and observe that

$$\ker \mathrm{ev}_0 \cap \mathfrak{g}[t]^{\tau\sigma} = I_{1,1}, \quad \ker \mathrm{ev}_z \cap \mathfrak{g}[t]^{\tau\sigma} = I_{(0,t^{ka_j}-z^{ka_j})}$$

We can use $\{I_{m,f} : m \in \mathbb{Z}_+, f \in \mathbb{C}[t]\}$ to show that any non-zero ideal of $\mathfrak{g}[t]^{\tau\sigma}$ is of finite codimension.

Proposition 2.1.1. *Let ι be a non−zero ideal in $\mathfrak{g}[t]^{\tau\sigma}$. Then there exists $0 \leq m < ka_j$ and $f \in \mathbb{C}[t^{ka_j}]$ such that $I_{m,f} \subset \iota$. In particular, any non−zero ideal in $\mathfrak{g}[t]^{\tau\sigma}$ is of finite codimension.*

Proof. We start by observing that it suffices to prove $\mathfrak{g}_r \otimes t^r g \subset \iota$ for some $g \in \mathbb{C}[t^{ka_j}]$ and $r > 0$. Since ι is non-zero and is preserved under the adjoint action of $\mathfrak{h}^\sigma$ and one of the following holds: (i) there is a non-zero element $h \in \iota \cap (\mathfrak{h} \otimes \mathbb{C}[t^{ka_j}])^\sigma$ or, (ii) ι contains an element of the form $(x_\alpha^+ \otimes t^s f)^{\tau\sigma}$ for some $f \in \mathbb{C}[t^{ka_j}]$, $0 \leq s < ka_j$, and $\alpha \in (R^\sigma)^+$. If the first case holds we write

$$0 \neq h = \sum_{i \in I(j) \cup \{0\}} h_i \otimes f_i \in \iota \cap (\mathfrak{h} \otimes \mathbb{C}[t^{ka_j}]),$$

and we then have

$$[h, x_p^+] = x_p^+ \otimes \sum_{i \in I(j) \cup \{0\}} \alpha_p(h_i) f_i \in \iota, \quad p \in I(j) \cup \{0\}.$$

Since the Cartan matrix of $\mathfrak{g}_0$ is invertible it follows that $\sum_{i \in I(j) \cup \{0\}} \alpha_p(h_i) f_i$ is non-zero for some $p \in I(j) \cup \{0\}$ and hence we see that ι contains an element of the form $x_\alpha^+ \otimes t^s g$ for some $\mathfrak{g} \in \mathbb{C}[t^{ka_j}]$, $x_\alpha^+ \otimes t^s \in \mathfrak{g}[t]^{\tau\sigma}$, and $\alpha \in (R^{\tau\sigma})_0^+$. Let $\mathfrak{a}$ be the simple summand of $\mathfrak{g}_0^{\tau\sigma}$ containing x_α^+. Taking repeated commutators with $\sum_{i \in I(j) \cup \{0\}} h_i \otimes t^{ka_j}$ and elements of $\mathfrak{a}$ we see that $\mathfrak{a} \otimes g\mathbb{C}[t^{ka_j}] \subset \iota$. Moreover recalling that $\alpha_j(\mathfrak{h}^\sigma \cap \mathfrak{a}) \neq 0$ we choose $h \in \mathfrak{h}^\sigma \cap \mathfrak{a}$

with $\alpha_j(h) \neq 0$ and hence

$$\alpha_j(h)^{-1}[x_j^+ \otimes t, h \otimes g\mathbb{C}[t^{ka_j}]] = x_j^+ \otimes tg\mathbb{C}[t^{ka_j}] \in \iota.$$

Since $\mathfrak{g}_1$ is an irreducible $\mathfrak{g}_0$-module it follows that $\mathfrak{g}_1 \otimes tg\mathbb{C}[t^{ka_j}] \subset \iota$ and the claim is proved in the first case. The preceding argument also proves the claim in case (ii) if $s = 0$ and if $s > 0$, the irreducibility of $\mathfrak{g}_s$ as a $\mathfrak{g}_0$-modules establishes the claim.

As a consequence of the claim, we see that if we set

$$S_m = \{g \in \mathbb{C}[t^{ka_j}] : x \otimes t^m g \in \iota \text{ for all } x \in \mathfrak{g}_m\}, \quad 0 \leq m \leq ka_j - 1,$$

then $S_m \neq 0$ for some $m > 0$. We now prove that S_m is an ideal in $\mathbb{C}[t^{ka_j}]$ and also that

$$t^{ka_j} S_{ka_j-1} \subset S_0 \subset S_1 \subset \cdots \subset S_{ka_j-1}. \tag{2.2}$$

In particular this shows that S_m is non-zero for all $0 \leq m \leq ka_j - 1$. Using 1.1.2 we write an element $x \in \mathfrak{g}_s$ as a sum $x = \displaystyle\sum_{p=1}^{r} [z_p, y_p]$ with $z_p \in \mathfrak{g}_0$ and $y_p \in \mathfrak{g}_s$ for $1 \leq p \leq q$. This means that,

$$x \otimes t^m fg = \sum [z_p \otimes f, y_p \otimes t^m g], \quad f, g \in \mathbb{C}[t].$$

If $g \in S_m$ then $y_p \otimes t^m g \in \iota$ by definition of S_m and so the right hand side of the preceeding equation is an element of ι. Hence $x \otimes t^m fg \in \iota$ for all $f \in \mathbb{C}[t^{ka_j}]$ and $g \in S_m$ proving that S_k is an ideal for all $0 \leq m \leq ka_j - 1$. A similar argument using $[g_\ell, \mathfrak{g}_{m-\ell}] = \mathfrak{g}_m$ proves the inclusions in (3.2).

For $0 \leq m \leq ka_j - 1$ let $f_m \in \mathbb{C}[t^{ka_j}]$ be a non-zero generator for the ideal S_m. By (2.1.2) there exists $g_0, \ldots, g_{ka_j-1} \in \mathbb{C}[t^{ka_j}]$ such that

$$f_r = g_r f_{r+1} 0 \leq r \leq ka_j - 2, \quad t^{ka_j} f_{ka_j-1} = g_{ka_j-1} f_0.$$

This implies

$$g_{ka_j-1}f_0 = g_0 \cdots g_{ka_j-1}g_{ka_j-1} = t^{ka_j} f_{ka_j-1}.$$

Hence there exists a unique $\ell \in \{0, \ldots, ka_j - 1\}$ such that $g_\ell = t^{ka_j}$ and $g_p = 1$ if $p \neq \ell$. Taking $f = f_{\ell+1}$, where we understand $f_{ka_j} = f_0$, we see that

$$I_{m,f} \subset \iota, m = \ell + 1 - k\alpha_j \delta_{\ell, ka_j-1}$$

$\square$

2.1.3 Equivariant Map Algebras

We now show that $\mathfrak{g}[t]^{\tau\sigma}$ is never a current algebra or more generally an equivariant map algebra with free action. For this we recall from [14] the definition of an equivariant map algebra. Let $\mathfrak{a}$ be any complex Lie algebra and A a finitely generated commutative associative algebra. Assume also that Γ is a finite abelian group acting on $\mathfrak{a}$ by Lie algebra automorphisms and on A by algebra automorphisms. Then we have an induced action on the Lie algebra $(\mathfrak{a} \otimes A)$ such that $\gamma(x \otimes f) = \gamma x \otimes \gamma f$, where $\gamma \in \Gamma$. An equivariant map algebra is defined to be the fixed point subalgebra:

$$(\mathfrak{a} \otimes A)^\Gamma := \{z \in (\mathfrak{a} \otimes A) | \gamma(z) = z \, \forall \, \gamma \in \Gamma\}.$$

The finite-dimensional irreducible representations of such algebras(and hence for $(\mathfrak{g}[t]^{\tau\sigma})$ were given in [14] and generalized earlier work on affine Lie algebras. An independent proof can be found in [CKO] when Γ acts freely on maxSpec A. In the case when Γ acts without fixed points on A, many aspects of the representation theory of the equivariant map algebra are the same as the representation theory of $\mathfrak{a} \otimes A$ ([9]). The importance of the following proposition is now clear.

Proposition 2.1.2. *The Lie algebra $\mathfrak{g}[t]^{\tau\sigma}$ is not isomorphic to an equivariant map algebra $(\mathfrak{a} \otimes A)^\Gamma$ with $\mathfrak{a}$ semisimple and Γ acting without fixed points on A.*

Proof. Recall our assumption that $a_j > 1$ and assume for a contradiction that

$$\mathfrak{g}[t]^{\tau\sigma} \cong (\mathfrak{a} \otimes A)^\Gamma$$

where $\mathfrak{a}$ is semi−simple. Write $\mathfrak{a} = \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_m$ where each $\mathfrak{a}_s$ is a direct sum of copies of a simple Lie algebra $\mathfrak{g}_s$ and $\mathfrak{g}_s \ncong \mathfrak{g}_r$ if $r \neq s$. Clearly Γ preserves $\mathfrak{a}_s$ for all $1 \leq s \leq m$ and hence

$$\mathfrak{g}[t]^{\tau\sigma} \cong (\mathfrak{a} \otimes A)^\Gamma \cong \bigoplus_{s=1}^{m} (\mathfrak{a}_s \otimes A)^\Gamma.$$

Since $\mathfrak{g}[t]^{\tau\sigma}$ is infinite−dimensional at least one of the summands $(\mathfrak{a}_s \otimes A)^\Gamma$ is infinite−dimensional, say $s = 1$ without loss of generality. But this means $\bigoplus_{s=2}^{m} (\mathfrak{a}_s \otimes A)^\Gamma$ is an ideal which is not of finite codimension which contradicts the previous proposition. Hence we must have $m = 1$. It was proven in [14, Proposition 5.2] that if Γ acts freely on A then any finite−dimensional simple quotient of $(\mathfrak{a} \otimes A)^\Gamma$ is a quotient of $\mathfrak{a}$; in particular in our situation it follows that all finite−dimensional simple quotients of $(\mathfrak{a} \otimes A)^\Gamma$ are isomorphic. On the other hand Equation 1.1.1 shows that $\mathfrak{g}[t]^{\tau\sigma}$ has both $\mathfrak{g}_0^{\tau\sigma}$ and $\mathfrak{g}^\sigma$ as quotients. Since $\mathfrak{g}_0^{\tau\sigma}$ is not isomorphic to $\mathfrak{g}^\sigma$ we have the desired contradiction. $\square$

2.1.4 Twisted affine Lie algebras

The twisted affine Lie Algebra $\widehat{\mathfrak{g}}(\sigma)$ is defined as follows: the vector space structure is given by

$$\widehat{\mathfrak{g}}(\sigma) = (\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}])^\sigma \oplus \mathbb{C}c \oplus \mathbb{C}d$$

with commutator defined so c is central and

$$[d, x \otimes f] = x \otimes t(\partial f/\partial t) \quad [x \otimes f, y \otimes g] = [x, y] \otimes fg + Res(\partial f/\partial tg)\kappa(x, y)c.$$

In this definition κ is the Killing form and $Res : \mathbb{C}[t, t^{-1}] \to \mathbb{C}$ pulls the coefficient of t^{-1}. The Cartan subalgebra is

$$\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

where $\delta \in \widehat{\mathfrak{h}^\sigma}^*$ is defined so $\delta(d) = 1$ and $\delta(\mathfrak{h}^\sigma \oplus \mathbb{C}c) = 0$. We extend $\alpha \in \mathfrak{h}^*$ to an element of $\widehat{\mathfrak{h}^\sigma}^*$ by defining $\alpha(c) = \alpha(d) = 0$. The set of roots, respectively simple roots, associated to $\widehat{\mathfrak{h}}^*$ is given by

$$\widehat{R^\sigma} = \{\alpha + r\delta : \alpha \in R_0, r \in \mathbb{Z}\} \cup \{s\delta : s \in \mathbb{Z}, s \neq 0\}, \quad \widehat{\Delta} = \{\alpha_i : i \in I\} \cup \{\delta - \theta\}.$$

The Borel subalgebra defined by $\widehat{\Delta}$ is

$$\widehat{\mathfrak{b}} = ((\mathfrak{h} \otimes \mathfrak{n}^+) \otimes \mathbb{C}[t])^\sigma \oplus (\mathfrak{n}^- \otimes t\mathbb{C}[t])^\sigma \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

Similar to simple finite dimensional Lie algebras, we define a parabolic subalgebra of $\widehat{\mathfrak{g}}$ as a subalgebra that contains $\widehat{\mathfrak{b}}$. The biggest difference is we lose the property that any two Borel subalgebras are conjugate. The Borel subalgebra that we defined is called the the standard Borel subalgebra and this is a natural restriction. For $\alpha \in \widehat{R}$ let $\widehat{\mathfrak{g}}_\alpha$ be the corresponding root space and given $\Delta' \subset \widehat{\Delta}$ we define $\widehat{R}(\Delta') \subset \widehat{R}$ to be the subset consisting of the $\mathbb{Z}$-span of Δ'.

Lemma 2.1.1. *Suppose that $\widehat{\mathfrak{p}}$ is a proper parabolic subalgebra of $\widehat{\mathfrak{g}}$ and assume that $\widehat{\mathfrak{b}} \neq \widehat{\mathfrak{p}}$. Then there exists a proper subset Δ' of Δ such that*

$$\widehat{\mathfrak{p}} = \widehat{\mathfrak{b}} + \sum_{\alpha \in \widehat{R}(\Delta')} \widehat{\mathfrak{g}}_\alpha.$$

Moreover $\widehat{\mathfrak{p}}$ is maximal iff $|\Delta'| = |I|$.

Proof. It is a simple observation that $\widehat{\mathfrak{g}}$ is generated by $\widehat{\mathfrak{b}}$ and $h \otimes t^{-s-1}$ for any $h \in \mathfrak{h}$ and $s \in \mathbb{Z}_+$. Since $\widehat{\mathfrak{p}}$ is a proper sublagebra, $h \otimes t^{-s-1} \notin \widehat{\mathfrak{p}}$ for any $h \in \mathfrak{h}_0$ and $s \in \mathbb{Z}_+$. This also means $x_\alpha^- \otimes t^{-r}, x_\alpha^+ \otimes t^{-r-1} \notin \widehat{\mathfrak{p}}$, since otherwise $[x_\alpha^+ \otimes t^{-r-1}, x_\alpha^- \otimes t^{-r}] = [x_\alpha^+ \otimes t^{-r}, x_\alpha^- \otimes t^{r}] = h_\alpha \otimes t^{-1} \in \widehat{\mathfrak{p}}$. Now set

$$\Delta' = \begin{cases} \{\alpha_i : i \in I, x_i^- \in \widehat{\mathfrak{p}}\}, & \text{if } x_{\theta_0^s}^+ \otimes t^{-1} \notin \widehat{\mathfrak{p}} \\ \{\delta - \theta_0^s, \alpha_i, : i \in I, x_i^- \in \widehat{\mathfrak{p}}\} & \text{if } x_{\theta_0^s}^+ \otimes t^{-1} \in \widehat{\mathfrak{p}} \end{cases}$$

then $\Delta' \neq \emptyset$. Clearly

$$\widehat{\mathfrak{p}} \supseteq \widehat{\mathfrak{b}} + \sum_{\alpha \in \widehat{R}(\Delta')} \widehat{\mathfrak{g}}_\alpha$$

By using the commutator with $x_i^+ \otimes 1$ we can see that if $x_\alpha^+ \otimes t^{-1} \in \widehat{\mathfrak{p}}$, then $x_\theta^+ \otimes t^{-1} \in \widehat{\mathfrak{p}}$. Similarly if $x_\alpha^- \otimes 1 \in \widehat{\mathfrak{p}}$, then $x_i^- \otimes 1 \in \widehat{\mathfrak{p}}$ for $i \in I$ such that $\alpha_i \preceq \alpha$. For the reverse inclusion it suffices to show that $x_\alpha^+ \otimes t^{-1} \in \widehat{\mathfrak{p}}$ (resp. $x_\alpha^- \otimes 1 \in \widehat{\mathfrak{p}}$) only if $-\alpha + \delta \in \Delta' (\alpha \in \Delta')$.

We prove the former by downward induction on ht α. Induction starts since if $\alpha = \theta_0^s$ then $\delta - \theta_0^s \in \Delta'$ by construction. Let $\alpha_i \in \Delta_0$ and $\alpha + \alpha_i \in \mathrm{wt}(\mathfrak{g}_k^\sigma)$ for $k \neq 0$. Since $x_{\alpha+\alpha_i}^+ = [x_i^+ \otimes 1, x_\alpha^+ \otimes t^{-1}] \in \widehat{\mathfrak{p}}$, by the inductive hypothesis $-(\alpha + \alpha_i) + \delta \in \mathbb{Z}_+\Delta'$. By assumption of the inductive step $x_\alpha^- \otimes t \in \widehat{\mathfrak{p}}$. Thus, $x_i^- \otimes 1 = [x_{\alpha+\alpha_i}^- \otimes t, x_\alpha^+ \otimes t] \in \widehat{\mathfrak{p}}$. This means $-\alpha_i \in \Delta'$ and $-\alpha + \delta \in \mathbb{Z}_+\Delta'$. The last step, $x_\alpha^- \otimes 1 \in \widehat{\mathfrak{p}}$ then $-\alpha \in \mathbb{Z}_+\Delta'$, is the same as in [4].

To prove the result when $x_\alpha^- \otimes 1 \in \hat{\mathfrak{p}}$ we proceed by induction on ht α. If $\alpha \in \Delta$ then by definition $\alpha \in \Delta'$ and so induction begins. For the inductive step, choose $\alpha_i \in \Delta$ such that $\beta = \alpha - \alpha_i \in R^+$. Since $x_\beta^- \otimes 1$ is a non-zero scalar multiple of $[x_i^+, x_\alpha^-] \otimes 1$ we see that $x_\beta^- \otimes 1 \in \hat{\mathfrak{p}}$. By the inductive hypothesis we have β is in the $\mathbb{Z}_+$-span of Δ'. On the other hand $[x_\beta^+ \otimes 1, x_\alpha^- \otimes 1]$ is a non-zero scalar multiple of $x_i^- \otimes 1$ and hence $\alpha_i \in \Delta'$ and the inductive step is proved. The second statement of the lemma is now obvious. $\square$

2.1.5 Maximal Parabolic subalgebras and $\mathfrak{g}[t]^{\tau\sigma}$

We can now make the connection between maximal parabolic subalgebras of $\widehat{\mathfrak{g}}$ and $\mathfrak{g}[t]^{\tau\sigma}$. We set

$$\Delta' = \{\alpha_i : i \in I(j)\} \cup \{\delta - \theta_0^s\},$$

and denote the corresponding maximal parabolic by $\widehat{\mathfrak{p}}$. It is easily shown that

$$\mathbb{Z}_+\Delta' \cap \widehat{R} = \{\delta - \alpha : \alpha \in R^+, \mathbf{a}_j(\alpha) = \alpha_j\} \cup \{\alpha \in R^+ : \mathbf{a}_j(\alpha) = 0\}.$$

This means $\widehat{\mathfrak{p}}$ is spanned by $\widehat{\mathfrak{b}}$ and

$$\{x_\alpha^+ \otimes t^{-1} : \alpha \in R^+, \mathbf{a}_j(\alpha) = a_j\} \cup \{x_\alpha^- \otimes 1 : \alpha \in R^+, \mathbf{a}_j(\alpha) = 0\}.$$

Let $\widetilde{\mathfrak{p}}$ be the quotient of the derived subalgebra of $\widehat{\mathfrak{p}}$ by the subspace $\mathbb{C}c$. It is clear that $\widetilde{\mathfrak{p}}$ is isomorphic to subalgebra of $\mathfrak{g}[t,t^{-1}]^\sigma$. We define a new grading on $\mathfrak{g}[t,t^{-1}]^\sigma$ given by

$$gr(x_\alpha^\pm \otimes t^r) = ra_j \pm \mathbf{a}_j(\alpha), \alpha \in R^+, r \in \mathbb{Z}.$$

We can observe $\widetilde{\mathfrak{p}}$ is a graded subalgebra of $\mathfrak{g}[t,t^{-1}]^\sigma$ under this new grading. The following is now easily checked.

Proposition 2.1.3. *The map $\phi : (\mathfrak{g} \otimes \mathbb{C}[t,t^{-1}])^\sigma \to (\mathfrak{g} \otimes \mathbb{C}[t,t^{-1}])^\sigma$ of Lie algebras given on graded elements by $\phi(x \otimes t^r) = x \otimes t^{gr(x\otimes t^r)}$ is a graded isomorphism $\widetilde{\mathfrak{p}} \cong \mathfrak{g}[t]^\tau$*

Chapter 3

Category $\tilde{\mathcal{I}}$

3.1 Motivation

In this section we develop the representation theory of $\mathfrak{g}[t]^{\tau\sigma}$. Following [1, 3], we define the notion of global Weyl modules, the associated commutative algebra, $\mathbf{A}_\lambda$, and the local Weyl modules associated to maximal ideals in this algebra. In the case of $\mathfrak{g}[t]$ it was shown in [7] that the commutative algebra associated with a global Weyl module is a polynomial ring in finitely many variables. This is no longer true for $\mathfrak{g}[t]^{\tau\sigma}$; however we shall see that modulo the Jaobson radical, $\mathbf{A}_\lambda$ is a quotient of a finitely generated polynomial ring by a squarefree monomial ideal. As a consequence we see that under suitable conditions a global Weyl module can be finite-dimensional and irreducible. More precise statements can be found in Section 5.

3.1.1 Fundamental Weights

Fix a set of fundamental weights $\{\lambda_i : i \in I(j) \cup \{0\}\}$ for $\mathfrak{g}_0^{\tau\sigma}$ with respect to $\Delta_0^{\tau\sigma}$ and denote by $(P_0)^{\tau\sigma}, (P_0^{\tau\sigma})^+$ the set of integral, dominant integral weights. Note that the subset

$$(P^\sigma)^+ = \{\lambda \in (P_0^{\tau\sigma})^+ : \lambda(h_j) \in \mathbb{Z}_+\}$$

is precisely the set of dominant integral weights for $\mathfrak{g}^\sigma$ with respect to Δ. Also note that $(P^\sigma)^+$ is properly contained in $(P_0^{\tau\sigma})^+$. For example, in the $A_{2n-1}^{(2)}$ case $\lambda_{n-1} \in (P_0^{\tau\sigma})^+$ and $\lambda_{n-1}(h_n) = -1$. It is the existence of these types of weights that causes the representation theory of $\mathfrak{g}[t]^{\tau\sigma}$ to be different from that of $\mathfrak{g}[t]^\sigma$.

For $\lambda \in P_0^+$, we denote by $V_{\mathfrak{g}_0^{\tau\sigma}}(\lambda)$ the finite dimensional representation of $\mathfrak{g}_0^{\tau\sigma}$ with highest weight λ. We will denote by v_λ the highest weight vector, and similarly define $V_{\mathfrak{g}^\sigma}(\lambda)$ for $\lambda \in (P^\sigma)^+$.

3.1.2 The category $\widetilde{\mathcal{I}}$

Let $\widetilde{I}$ be the category whose objects are $\mathfrak{g}[t]^{\tau\sigma}$-modules that are $\mathfrak{g}_0^{\tau\sigma}$-integrable with the morphisms being $\mathfrak{g}[t]^{\tau\sigma}$-module maps. Since $\mathfrak{g}_0^{\tau\sigma}$ is a semisimple Lie algebra, if V is an object in $\widetilde{I}$ it decomposes into a direct sum of finite-dimensional $\mathfrak{g}_0^{\tau\sigma}$-modules. This means V admits a weight space decomposition:

$$V = \bigoplus_{\mu \in P_0^{\tau\sigma}} V_\mu, \quad V_\mu = \{v \in V : hv = \mu(h)v, h \in \mathfrak{h}_0^{\tau\sigma}\},$$

and we set $\mathrm{wt}\, V = \{\mu \in P_0^{\tau\sigma} : V_\mu \neq 0\}$. Since the weights of finite-dimensional $\mathfrak{g}_0^{\tau\sigma}$-modules are closed under the action of the Weyl group of $\mathfrak{g}_0^{\tau\sigma}$, $W_0^{\tau\sigma}$, we see that

$$w(\text{wt V}) \subset \text{wt } V,\ w \in W_0^{\tau\sigma}.$$

For $\lambda \in P_0^+$ we denote by $\widetilde{I}^\lambda$ the full subcategory of $\widetilde{I}$ whose objects have the property that wt$V \subset \lambda - (Q^\sigma)^+$; note this is a weaker condition than requiring the set of weights be contained in $\lambda - (Q_0^{\tau\sigma})^+$

Lemma 3.1.1. *Suppose that V is an object of $\tilde{\mathcal{I}}^\lambda$ and $\mu \in$ wt V and $\alpha \in R^+$. Then $\mu - s\alpha \in$ wt V for only finitely many $s \in \mathbb{Z}$.*

Proof. Since $\mu \in$ w V we can write $\lambda - \mu = \sum_{i \in I^\sigma} s_i\alpha_i$ for some $s_i \in \mathbb{Z}_+, i \in I$. If $s < 0$ and $p \in I$ is such that $\mathbf{a}_p(\alpha) > 0$ then $-s\mathbf{a}_p(\alpha) - s_p < 0$ or equivalently $-s_p < s\mathbf{a}_p(\alpha) < 0$ for only finitely many values of s. If $\mu - s\alpha \in$ wt V then $\lambda - (\mu - s\alpha) \in (Q^{\tau\sigma})^+$ and $\mu - s\alpha \prec \lambda$. It follows that the set of negative integers such that $\mu - s\alpha \in$ wt V is finite.

Suppose that $s > 0$. Since $\alpha \in P_0^{\sigma\tau}$ we can choose $w \in W_0^{\tau\sigma}$ such that $w\alpha$ is in the antidominant chamber for the action of $W_0^{\tau\sigma}$ on $\mathfrak{h}^\sigma$. This implies that $w\alpha = -r_0\alpha_0 - \sum_{i \in I(j)} r_i\alpha_i$ where the r_i are non-negative rational numbers. Since $W_0^{\tau\sigma}$ is a subgroup of W^σ it follows that $-w\alpha \in (R^\sigma)^+$. Since $w\mu = -(-s)(-w\alpha) = w\mu - sw\alpha \in$ wt V, it follows by applying the argument in the case $s < 0$ to the elements $w\mu \in$ wt V and $-w\alpha \in (R^\sigma)^+$ that $-s$ is bounded below and hence s is bounded above. This completes the proof of the lemma. $\square$

3.1.3 Triangular Decomposition

Let

$$(\mathfrak{n}^\sigma)^+ \oplus \mathfrak{h}^\sigma \oplus (\mathfrak{n}^\sigma)^- = \mathfrak{g}^\sigma, \text{ where } \mathfrak{n}^\pm = \bigoplus_{\alpha \in R_0^\sigma} \mathfrak{g}_\alpha^\sigma$$

Since τ preserves $(\mathfrak{n}^\sigma)^\pm$ and $\mathfrak{h}^\sigma$, we have

$$\mathfrak{n}^+[t]^{\tau\sigma} \oplus \mathfrak{h}[t]^{\tau\sigma} \oplus \mathfrak{n}^-[t]^{\tau\sigma} = \mathfrak{g}[t]^{\tau\sigma}.$$

Additionally, $\mathfrak{h}[t]^{\tau\sigma}$ is a commutative subalgebra of $\mathfrak{g}[t]^{\tau\sigma}$. For $\lambda \in (P_0^{\tau\sigma})^+$ the global Weyl module $W(\lambda)$ is the cyclic $\mathfrak{g}[t]^{\tau\sigma}$-module generated by w_λ such that for $h \in \mathfrak{h}^{\tau\sigma}$ and $i \in I(j) \cup \{0\}$:

$$\mathfrak{n}^+[t]^{\tau\sigma}.w_\lambda = 0, \quad h_i.w_\lambda = \lambda(h_i)w_\lambda, \quad (x_i^- \otimes 1)^{\lambda(h_i+1)}.w_\lambda = 0. \tag{3.1}$$

We can easily observe that $W(\lambda) \in \widetilde{\mathcal{I}}$ since $x_i^\pm \otimes 1$ acts nilpotently by definition. Further if we declare the grade of w_λ to be zero $W(\lambda)$ inherits a $\mathbb{Z}_+$-grading from $\mathfrak{g}[t]^{\tau\sigma}$.

3.1.4 $\mathbf{A}_\lambda$

Similar to [1], $W(\lambda)$ has a right action of $\mathfrak{h}[t]^{\tau\sigma}$ that is compatible the the left module structure in the following way

$$(uw_\lambda)h = uhw_\lambda \text{ for } u \in \mathfrak{g}[t]^{\tau\sigma}, h \in \mathfrak{h}[t]^{\tau\sigma}.$$

Further, if we define

$$Ann_{\mathfrak{h}[t]^{\tau\sigma}}(w_\lambda) = \{h \otimes t^r \in \mathfrak{h}[t]^{\tau\sigma} : h \otimes t^r.w_\lambda = 0\} \quad \mathbf{A}_\lambda = \mathbf{U}(\mathfrak{h}[t]^{\tau\sigma})/Ann_{\mathfrak{h}[t]^{\tau\sigma}}(w_\lambda)$$

we can see that $Ann_{\mathfrak{h}[t]^{\tau\sigma}}(w_\lambda)$ is an ideal of $\mathbf{U}(\mathfrak{h}[t]^{\tau\sigma})$ and $W(\lambda)$ is a $(\mathfrak{g}[t]^{\tau\sigma}, \mathbf{A}_\lambda)$-bimodule. Since $\mathbf{A}_\lambda$ is a graded ideal of $\mathbf{U}(\mathfrak{h}[t]^{\tau\sigma})$, $\mathbf{A}_\lambda$ is a graded algebra with a unique graded maximal ideal I_0. Since $\mathbf{A}_\lambda/I_0 \cong \mathbb{C}$ we can note that $I_0 = m + (\mathbf{A}_\lambda)_1 + (\mathbf{A}_\lambda)_2 + \cdots$, where $(\mathbf{A}_\lambda)_r$ is the r^{th} graded component of $\mathbf{A}_\lambda$ and m is a maximal ideal of $(\mathbf{A}_\lambda)_0$. We observe that since $\lambda \in P_0^+$, we have no choice for m. It is then easy to check that

$$W(\lambda)_\lambda \cong \mathbf{A}_\lambda \tag{3.2}$$

as right $\mathbf{A}_\lambda$-modules.

3.1.5 Another description of $\mathbf{A}_\lambda$

Before we study the structure of $W(\lambda)$ as an $\mathbf{A}_\lambda$-module we need to know more about the structure of $\mathbf{A}_\lambda$. Following [2] and [4], for $\alpha \in (R^\sigma)^+$, if α is a short root of $\mathfrak{g}_0^{\tau\sigma}$, and $r \in \mathbb{Z}_+$ we define $P_{\alpha,r} \in U(\mathfrak{h}[t]^{\tau\sigma})$ recursively by

$$P_{\alpha,0} = 1, \quad P_{\alpha,r} = -\frac{1}{r}\sum_{p=1}^{r}(h_\alpha \otimes t^{a_j p})P_{\alpha,r-p}, \quad r \geq 1$$

where $h_\alpha \in \mathfrak{g}_1^\sigma$. If β is a long root of $\mathfrak{g}_0^{\tau\sigma}$ we define for $r \in \mathbb{Z}_+$, $P_{\beta,r} \in \{\mathbf{U}\mathfrak{h}[t]^{\tau\sigma}\}$ recursively by

$$P_{\beta,0} = 1, \quad P_{\beta,r} = -\frac{1}{r}\sum_{p=1}^{r}(h_\beta \otimes t^{a_j p})P_{\beta,r-p}, \quad r \geq 1$$

where h_β is the coroot in $\mathfrak{g}[t]^{\tau\sigma}$.

We note that since α is a short root there is $x_\alpha^\pm \in \mathfrak{g}_0^\sigma$ and $x_\alpha^\pm \in \mathfrak{g}_m^\sigma$ for $0 \leq m < ka_j$. The grade, exponent of t, allows one to know whether $h_\alpha \in \mathfrak{g}_0$ or $h_\alpha \in \mathfrak{g}_m$ for some value of m. The same holds for $x_\alpha^\pm$. This removes the ambiguity and will be used heavily in Section 6.

Equivalently $P_{\alpha,r}$ is the coefficient of u^r in

$$P_\alpha(u) = \exp\left(-\sum_{r\geq 1} \frac{h_\alpha \otimes t^{a_j r}}{r} u^r\right).$$

Writing $h_\alpha = \sum_{i=1}^{n} \check{\mathbf{a}}_i(\alpha) h_i$, we see that

$$P_\alpha(u) = \prod_{i=1}^{n} P_{\alpha_i}(u)^{\check{\mathbf{a}}_i(\alpha)}, \quad \alpha \in (R^\sigma)^+$$

We set $P_{\alpha_i,r} = P_{i,r}$ for $i \in I \cup \{0\}$. The following is now clear from the Poincare-Birkhoff-Witt Theorem and the recursive definition of $P_{i,r}$

Lemma 3.1.2. *The algebra $\mathbf{U}(\mathfrak{h}[t]^{\tau\sigma})$ is the polynomial algebra in the variables*

$$\{P_{i,r} : i \in I(j) \cup \{0\}, r \in \mathbb{N}\}$$

and the variables

$$\{P_{i,r} : i \in I, r \in \mathbb{N}\}$$

The comultiplication satisfies

$$\widetilde{\Delta}(P_\alpha(u)) = P_\alpha(u) \otimes P_\alpha(u), \quad \alpha \in (R^\sigma)^+. \tag{3.3}$$

For $x \in \mathfrak{g}[t]^{\tau\sigma}, r \in \mathbb{Z}_+$ we also set $x^{(r)} = \dfrac{x^r}{r!}$.

3.1.6 Useful Lemma

The following can be found in [7, Lemma 1.3] and is a reformulation of [11, Lemma 7.1].

Lemma 3.1.3. *Let $x^\pm, h$ be the standard basis for $\mathfrak{sl}_2$ and V be a representation of the Lie subalgebra generated by $(x^+ \otimes 1)$ and $(x^- \otimes t)$. Assume $0 \neq v \in V$ is such that $(x^+ \otimes t^r)v = 0$ for $r \in \mathbb{Z}_+$. Then for all $r \in \mathbb{Z}_+$ we have*

$$(x^+ \otimes 1)^{(r)}(x^- \otimes t)^{(r)}v = (x^+ \otimes t)^{(r)}(x^- \otimes 1)^{(r)}v = (-1)^r P_r v, \tag{3.4}$$

where P_r is defined similarly to $P_{i,r}$. Furthermore,

$$(x^+ \otimes 1)^{(r)}(x^- \otimes t)^{(r+1)}v = (-1)^r \sum_{s=0}^{r} (x^- \otimes t^{s+1}) P_{r-s} v. \tag{3.5}$$

3.1.7 Some structure of $W(\lambda)$

Proposition 3.1.1. *For all $\lambda \in (P_0^{\tau\sigma})^+$ the algebra $\mathbf{A}_\lambda$ is finitely generated and $W(\lambda)$ is a finitely generated $\mathbf{A}_\lambda$-module.*

Proof. The proof of the proposition is very similar to the one given in [1, Theorem 2] but we sketch the proof below for the reader's convenience and also to set up some further necessary notation. Given $\alpha \in (R^{\tau\sigma})_0^+$, it is easily that the element $(x_\alpha^+ \otimes t^s)$ and $(x_\alpha^- \otimes t^{s'})$ generate a subalgebra of $\mathfrak{g}[t]^{\tau\sigma}$ which is isomorphic to the subalgebra of $\mathfrak{sl}_2[t]$ generated by $(x^+ \otimes 1)$ and $(x^- \otimes t)$. It should be noted that the value of s, s' determines whether $\alpha \in \mathrm{wt}\mathfrak{g}_0^\sigma$ or $\mathfrak{g}_m^\sigma$ for some $m \neq 0$. Using the defining relations of $W(\lambda)$ and 3.4 we get that

$$P_{\alpha,r} w_\lambda = 0, \quad r \geq \lambda(h_\alpha) + 1, \quad \alpha \in (R_0^{\tau\sigma})^+. \tag{3.6}$$

It also follows from 3.1.1 that $p_{j,r} w_\lambda = 0$ for all $r >> 0$. Using 3.1.1 we see that $\mathbf{A}_\lambda$ is finitely generated by the images of the elements

$$\{P_{i,r} : i \in I(j) \cup \{0\},\, r \leq \lambda(h_i)\}.$$

Fix an enumeration $\beta_1, \ldots, \beta_M$ of $(R^\sigma)^+$. Using the PBW theorem it is clear that $W(\lambda)$ is spanned by elements of the form $X_1 X_2 \cdots X_M \mathbf{U}(\mathfrak{h}[t]^{\tau\sigma}) w_\lambda$ where each X_p is either a constant or a monomial in the elements $\{x_{\beta_p}^- \otimes t^s\} \subset \mathfrak{g}[t]^{\tau\sigma}$. The length of each X_r is bounded by 3.1.1 and equation 3.5 proves that for any $\gamma \in (R^\sigma)^+$ and $r \in \mathbb{Z}_+$, the element $(x_\gamma^- \otimes t^{s_\gamma}) \mathbf{U}(\mathfrak{h}[t]^{\tau\sigma}) w_\lambda$ is in the span of elements $\{(x_\gamma^- \otimes t^{ska_j + s_\gamma}) \mathbf{U}(\mathfrak{h}[t]^{\tau\sigma}) w_\lambda : 0 \leq s \leq N\}$ for some N sufficiently large. An obvious induction on the length of the product of monomials shows that the values of s are bounded for each β and the proof is complete. $\square$

Remark: Notice that the preceding arguments proves that the set wt $W(\lambda)$ is finite. This is not obvious since wt $W(\lambda)$ is not a subset of $\lambda - Q_0^+$.

3.1.8 Local Weyl modules

For $\lambda \in P_0^+$ and I_λ, a maximal ideal of $\mathbf{A}_\lambda$, we define

$$W(\lambda, I_\lambda) = W(\lambda) \otimes_{\mathbf{A}_\lambda} \mathbf{A}_\lambda / I_\lambda.$$

By 3.1.1 since $W(\lambda, I_\lambda)$ is a quotient of $W(\lambda)$, $W(\lambda, I_\lambda)$ is finite dimensional. Using the standard arguments $W(\lambda, I_\lambda)$ contains a unique irreducible quotient $V(\lambda, I_\lambda)$. In particular, $W(\lambda, I_0)$ is a $\mathbb{Z}_+$-graded $\mathfrak{g}[t]^{\tau\sigma}$-module and

$$V(\lambda, I_0) \cong ev_0^* V_{\mathfrak{g}_0}(\lambda) \tag{3.7}$$

where $ev_0 V_{\mathfrak{g}_0}(\lambda)$ is the representation of $\mathfrak{g}[t]^{\tau\sigma}$ obtained by pulling back the irreducible representation $V(\lambda)$ of $\mathfrak{g}_0$.

3.1.9 Evaluation Modules

We now explicitly construct a family of modules for $\mathfrak{g}[t]^{\tau\sigma}$ which will be critical in the study of $\mathbf{A}_\lambda$. For $0 \neq z_1, \ldots, z_m \in \mathbb{C}$ such that $z_r^{ka_j} \neq z_s^{ka_j}$ $\forall 1 \leq r \neq s \leq m$ we have a surjective Lie algebra homomorphism $ev_0 \bigoplus_{s=1}^{m} ev_{z_s}\colon \mathfrak{g}[t]^{\tau\sigma} \to \mathfrak{g}_0 \oplus (\mathfrak{g}^\sigma)^{\oplus m}$ For a $\mathfrak{g}^\sigma$-representation, V, and $z \neq 0$ we denote by $ev_z^* V$ the pullback representation. It is quickly verifiable from the recursive definition of $P_{\alpha,r}$ that for $ev_z^* V_{\mathfrak{g}_0}(\lambda)$, $\lambda \in P^+$, and $ev_0^* V_{\mathfrak{g}_0}(\mu)$, $\mu \in P_0^+$:

$$\mathfrak{n}^+[t]^{\tau\sigma} v_\lambda = 0 \quad P_{i,r} v_\lambda = \binom{\lambda(h_i)}{r}(-1)^r z^{a_j r} v_\lambda \quad i \in I, r \in \mathbb{N}$$

$$\mathfrak{n}^+[t]^{\tau\sigma} v_\mu = 0, \quad P^{\tau\sigma}_{i,r} v_\mu = 0 \quad i \in I, r \in \mathbb{N}.$$

By the comultiplication of $P_{\alpha,r}$ and the discussion of the action of $P_{\alpha,r}$ the next proposition follows from [14, Proposition 4.9]

Proposition 3.1.2. *Suppose that $\lambda_1, \ldots, \lambda_m \in P^+$ and $\mu \in P_0^+$. Let $z_1, \ldots, z_m$ be non-zero complex numbers such that $z_r^{a_j} \neq z_s^{a_j}$ for all $1 \leq r \neq s \leq m$. Then*

$$ev_0^* V_{\mathfrak{g}_0}(\mu) \otimes ev_{z_1}^* V_{\mathfrak{g}}(\lambda_1) \otimes \cdots \otimes ev_{z_m}^* V_{\mathfrak{g}}(\lambda_m)$$

is an irreducible $\mathfrak{g}[t]^\tau$-module. Moreover,

$$\mathfrak{n}^+[t]^{\tau\sigma}(v_\mu \otimes v_{\lambda_1} \otimes \cdots \otimes v_{\lambda_m}) = 0,$$

$$(P_{i,r} - \pi_{i,r})(v_\mu \otimes v_{\lambda_1} \otimes \ldots \otimes v_{\lambda_m}) = 0, \quad i \in I, \quad r \in \mathbb{Z}_+,$$

where

$$\sum_{r \in \mathbb{Z}_+} \pi_{i,r} u^r = \prod_{s=1}^m (1 - z_s^{a_j} u)^{\lambda_s(h_i)}, \quad i \in I.$$

Remark The module we constructed in the Proposition is of the form $V(\lambda, I)$ where $I = \mu + \lambda_1 + \ldots + \lambda_m$.

Chapter 4

4.1 $\mathbf{A}_\lambda$ as a Stanley-Reisner ring

For this section we denote by $Jac(\mathbf{A}_\lambda)$ the Jacobson radical of $\mathbf{A}_\lambda$. Since $\mathbf{A}_\lambda$ is a commutative algebra we will utilize the fact that $Jac(\mathbf{A}_\lambda)$ coincides with the nilradical of $\mathbf{A}_\lambda$.

4.1.1 Structure of $\mathbf{A}_\lambda/Jac(\mathbf{A}_\lambda)$

The main result of this section is:

Theorem 4.1.1. *The algebra $\mathbf{A}_\lambda/Jac(\mathbf{A}_\lambda)$ is isomrophic to the algebra $\tilde{\mathbf{A}}_\lambda$ which is the quotient of $\boldsymbol{U}(\mathfrak{h}[t]^\tau)$ by the ideal generated by the elements*

$$P_{i,s} \quad i \in I(j), \quad s \geq \lambda(h_i) + 1, \tag{4.1}$$

and

$$P_{1,r_1} \cdots P_{n,r_n}, \quad \sum_{i=1}^{n} \mathbf{a}_i(\alpha_0) r_i > \lambda(h_0) \tag{4.2}$$

Moreover, $Jac(\mathbf{a}_\lambda)$ is generated by 4.2.

4.1.2 Properties of $\mathbf{A}_\lambda/Jac(\mathbf{A}_\lambda)$

In this section we note some more interesting consequences that reveal more information about how the structure of $\mathbf{A}_\lambda$ is tied to the structure of $W(\lambda)$.

Proposition 4.1.1. $\mathbf{A}_\lambda/Jac(\mathbf{A}_\lambda)$ *is either infinite dimensional or isomorphic to* $\mathbb{C}$. *Moreover the latter is true iff the following two conditions hold:*

(i) for $i \in I(j)$, *we have* $\lambda(h_i) > 0$ *only if* $\mathbf{a}_i(\alpha_0) > 0$,

(ii) $\lambda(h_0) < \mathbf{a}_i(\alpha_0)$ *if* $i = j$ *or if* $i \in I(j)$ *and* $\lambda(h_i) > 0$.

.

Proof. Suppose that λ satisfies the conditions in (i) and (ii). To prove that $\dim \mathbf{A}_\lambda/Jac(\mathbf{A}_\lambda) = 1$ it suffices to prove that the elements $P_{i,s} \in Jac(\mathbf{A}_\lambda)$ for all $i \in I$ and $s \geq 1$. Assume first that $i \neq j$. If $\lambda(h_i) = 0$ then equation 3.6 gives $P_{i,s}w_\lambda$ for all $s \geq 1$. If $\lambda(h_i) > 0$ then the conditions imply that $\lambda(h_0) < \mathbf{a}(\alpha_0)$, by condition (ii), and hence equation 4.2 shows that $P_{i,s} \in Jac(\mathbf{A}_\lambda)$ for all $s \geq 1$. If $i = j$ then again the result follows from equation 4.2 and condition (ii).

We now prove the converse direction. Suppose that (i) does not hold. Then, there exists $i \neq j$ with $ba_i(\alpha_0) = 0$ and $\lambda(h_i) > 0$. Equation 4.2 implies that the preimage of $Jac(\mathbf{A}_\lambda)$ is contained in the ideal of $\mathbf{U}(\mathfrak{h}[t]^{\tau\sigma})$ generated by the elements $\{P_{i,s} : i \in I, \mathbf{a}_i(\alpha_0) > 0\}$. Hence, using 3.1.2 we see that the image of the elements $\{P_{i,1}^r : r \in \mathbb{N}\}$ in $\mathbf{A}_\lambda/Jac(\mathbf{A}_\lambda)$ must remain linearly independent showing that the algebra is infinite-dimensional.

Suppose that (ii) does not hold. Then either $\lambda(h_0) \geq \mathbf{a}_j(\alpha_0)$ for some $i \in I(j)$ with $\lambda(h_i) > 0$. In either case 4.2 and 3.1.2 show that the image of the set $\{P_{i,1}^r : r \in \mathbb{N}\}$

in $\mathbf{A}_\lambda / Jac(\mathbf{A}_\lambda)$ must remain linearly independent showing that the algebra is infinite-dimensional.

□

Corollary 4.1.1. *The following are equivalent:*

(i) $\mathbf{A}_\lambda$ *is finite dimensional*

(ii) $W(\lambda)$ *is finite dimensional*

(iii) $\mathbf{A}_\lambda$ *is a local ring*

Proof. If $\mathbf{A}_\lambda$ is finite-dimensional then so is $\mathbf{A}_\lambda / Jac(\mathbf{A}_\lambda)$ and the corollary is immediate from the proposition. Conversely suppose that $\mathbf{A}_\lambda$ is a local ring. By the proposition and equation 3.6, we have

$$P_{i,s} w_\lambda = 0, \quad \text{if } \mathbf{a}_i(\alpha_0) = 0, \quad s \in \mathbb{N}.$$

If $\mathbf{a}_i(\alpha_0) \neq 0$ we still have from 3.6 that $P_{i,s} w_\lambda = 0$ if s is sufficiently large. Otherwise Equation 4.2 shows that there exists $N \in \mathbb{Z}_+$ such that

$$P_{i,s}^N w_\lambda = 0, \quad \text{for all } i \in I, s \in \mathbb{N}.$$

This proves that $\mathbf{A}_\lambda$ is generated by finitely many nilpotent elements and since it is a commutative algebra it is finite-dimensional. The second statement of the corollary is now immediate from 3.1.1.

□

4.1.3 Generators of $Jac(\mathbf{A}_\lambda)$

We now return to the proof of Theorem 1. It is clear that terms in 4.1 map to zero. So it suffices to show that 4.2 generates $Jac(\mathbf{A}_\lambda)$.

Let α and β be roots such that $\ell\alpha + \beta$ is also a root. Define $c(\ell, \alpha, \beta) \in \mathbb{C}^\times$ by

$$ad^{\ell}_{x_\alpha}(x_\beta) = c(\ell, \alpha, \beta)x_{\ell\alpha+\beta}$$

Lemma 4.1.1. *Let β be a root in $\mathfrak{g}_1^{\tau\sigma}$. There exists $\gamma \in \Delta_0^{\tau\sigma}$ such that $(\beta, \gamma) < 0$. Then*

$$\begin{aligned}&(x^+_{\gamma,0} \otimes t^p)^{(s+k)}(x^+_{\beta-\gamma,0} \otimes t^m)^{(s)}(x^-_{\beta,1} \otimes t^n)^{(k+s)}\\ &= A(x^-_{\beta-\gamma,1} \otimes t^{n+p})^{(k)}(x^+_{\gamma,0} \otimes t^p)^{(s)}(x^-_{\gamma,1} \otimes t^{m+n})^{(s)} \quad \mathrm{mod}\ \mathbf{U}(\mathfrak{g}[t]^{\tau\sigma})\mathbf{U}(n^+[t]^{\tau\sigma})_+\end{aligned}$$

where

$$A = C(1, \gamma, -\beta)^k c(1, \beta-\gamma, -\beta)^s$$

Proof. First induct on s, and then on k.

We first show

$$(x^+_{\beta-\gamma,0} \otimes t^m)^{(s)}(x^-_{\beta,1} \otimes t^n)^{(k+s)} = c(1, \beta-\gamma, -\beta)^s(x^-_{\beta,1} \otimes t^n)^{(k)}(x^-_{\gamma,1} \otimes t^{m+n})^{(s)} \quad \mathrm{mod}\ \mathbf{U}(\mathfrak{g}[t]^{\tau\sigma})\mathbf{U}(n^+[t]^{\tau\sigma}) \tag{4.3}$$

For the $s = 1$ case, we have

$$\begin{aligned}&(x^+_{\beta-\gamma,0} \otimes t^m)(x^-_{\beta,1} \otimes t^n)^{(k+1)}\\ &= (x^-_{\beta,1} \otimes t^n)^{(k+1)}(x^+_{\beta-\gamma,0} \otimes t^m) + c(1, \beta-\gamma, -\beta)(x^-_{\beta,1} \otimes t^n)^{(k)}(x^-_{\gamma,1} \otimes t^{n+m})\end{aligned} \tag{4.4}$$

since any term with weight $\beta - 2\gamma$ is of the form $x^+_{\beta-2\gamma}$, and 2γ is not a root. For the inductive step, let $s > 1$ and consider

$$(x^+_{\beta-\gamma,0}\otimes t^m)^{(s)}(x^-_{\beta,1}\otimes t^n)^{(k+s)} = \frac{1}{s}(x^+_{\beta-\gamma,0}\otimes t^m)(x^+_{\beta-\gamma,0}\otimes t^m)^{(s-1)}(x^-_{\beta,1}\otimes t^n)^{(s-1+k+1)}. \quad (4.5)$$

By inductive hypothesis, we have

$$\frac{c(1,\beta-\gamma,-\beta)^{s-1}}{s}(x^+_{\beta-\gamma,0}\otimes t^m)(x^-_{\beta,1}\otimes t^n)^{(k+1)}(x^-_{\gamma,1}\otimes t^{m+n})^{(s-1)} \mod \mathbf{U}(\mathfrak{g}[t]^{\tau\sigma})\mathbf{U}(\mathfrak{n}^+[t]^{\tau\sigma})_+. \tag{4.6}$$

Finally since $\beta - 2\gamma$ is not a weight in $\mathfrak{g}_1$ we have

$$c(1,\beta-\gamma,-\beta)^s(x^-_{\beta,1}\otimes t^n)^{(k)}(x^-_{\gamma,1}\otimes t^{m+n})^{(s)} \mod \mathbf{U}(\mathfrak{g}[t]^{\tau\sigma})\mathbf{U}(\mathfrak{n}[t]^{\tau\sigma})_+, \tag{4.7}$$

proving the induction on s.

The proof of the lemma now follows from the equality

$$\begin{aligned}&(x^+_{\gamma,0}\otimes t^p)^{(s+k)}(x^-_{\beta,1}\otimes t^n)^{(k)}(x^-_{\gamma,1}\otimes t^{m+n})^{(s)} = \\ &c(1,\gamma,-\beta)^k(x^-_{\beta-\gamma,1}\otimes t^{n+p})^{(k)}(x^+_{\gamma,0}\otimes t^p)^{(s)}(x^-_{\gamma,1}\otimes t^{m+n})^{(s)} \mod \mathbf{U}(\mathfrak{g}[t]^{\tau\sigma})\mathbf{U}(\mathfrak{n}^+[t]^{\tau\sigma})_+\end{aligned} \tag{4.8}$$

which we prove by induction on k. For the case $k = 1$ consider

$$(x^+_{\gamma,0}\otimes t^p)^{(s+1)}(x^-_{\beta,1}\otimes t^n)(x^-_{\gamma,1}\otimes t^{m+n})^{(s)} \tag{4.9}$$

Recall that for any two elements $X, Y \in \mathfrak{g}[t]^{\tau\sigma}$ and $\ell \in \mathbb{N}$ we have the following identity in $\mathbf{U}(\mathfrak{g}[t]^{\tau\sigma})$:

$$[Y^\ell, X] = \sum_{q=1}^{\ell}\binom{\ell}{q} ad^q_Y(X)Y^{\ell-q}. \tag{4.10}$$

Applying the above equation to the equation above it gives

$$\frac{1}{(s+1)!}(s+1)c(1,\gamma,-\beta)(x^-_{\beta-\gamma,1}\otimes t^{n+p})(x^+_{\gamma,0}\otimes t^p)^s(x_{\gamma,1}\otimes t^{m+n})^{(s)} \mod \mathbf{U}(\mathfrak{g}[t]^{\tau\sigma})\mathbf{U}(\mathfrak{n}^+[t]^{\tau\sigma}) \tag{4.11}$$

which equals

$$c(1,\gamma-\beta)(x^-_{\gamma-\beta,1}\otimes t^{n+p})(x^+_{\gamma,0}\otimes t^p)^{(s)}(x^-_{\gamma,1}\otimes t^{m+n})^{(s)} \mod \mathbf{U}(\mathfrak{g}[t]^{\tau\sigma})\mathbf{U}(\mathfrak{n}^+[t]^{\tau\sigma})_+, \tag{4.12}$$

proving the base case. For the inductive step, let $k>1$ and consider

$$(x^+_{\gamma,0}\otimes t^p)^{(s+k+1)}(x^-_{\beta,1}\otimes t^n)^{(k+1)}(x^-_{\gamma,1}\otimes t^{m+n})^{(s)} \tag{4.13}$$

equals

$$\frac{1}{(s+k+1)!(k+1)!}(x^+_{\gamma,0}\otimes t^p)^{(s+k+1)}(x^-_{\beta,1}\otimes t^n)(x^-_{\beta,1}\otimes t^n)^k(x^-_{\gamma,1}\otimes t^{m+n})^{(s)}. \tag{4.14}$$

By the adjoint equation, the above equation equals

$$\frac{c(1,\gamma,-\beta)}{(s+k+1)!(k+1)!}(s+k+1)(x^-_{\beta-\gamma,1}\otimes t^{n+p})(x^+_{\gamma,0}\otimes t^p)^{s+k}(x^-_{\beta,1}\otimes t^n)^k$$
$$(x^-_{\gamma,1}\otimes t^{m+n})^{(s)} \mod \mathbf{U}(\mathfrak{g}[t]^{\tau\sigma})\mathbf{U}(\mathfrak{n}^+[t]^{\tau\sigma})_+ \tag{4.15}$$

which equals

$$\frac{c(1,\gamma,-\beta)}{k+1}(x^-_{\beta-\gamma,1}\otimes t^{n+p})(x^+_{\gamma,0}\otimes t^p)^{(s+k)}(x^-_{\beta,1}\otimes t^n)^{(k)}(x^-_{\gamma,1}\otimes t^{m+n})^{(s)}$$
$$\mod \mathbf{U}(\mathfrak{g}[t]^{\tau\sigma})\mathbf{U}(\mathfrak{n}^+[t]^{\tau\sigma})_+. \tag{4.16}$$

Applying the induction hypothesis to the equation above yields

$$\frac{c(1,\gamma,-\beta)^{k+1}}{k+1}(x^-_{\beta-\gamma,1}\otimes t^{n+p})(x^-_{\beta-\gamma,1}\otimes t^{n+p})^{(k)}(x^+_{\gamma,0}\otimes t^p)^{(s)}(x^-_{\gamma,1}\otimes t^{m+n})^{(s)}$$
$$\mod \mathbf{U}(\mathfrak{g}[t]^{\tau\sigma})\mathbf{U}(\mathfrak{n}^+[t]^{\tau\sigma})_+, \tag{4.17}$$

and we have finished the proof of the lemma. □

It is immediate from that under they hypothesis of the lemma we have for all $P \in \mathbf{U}(\mathfrak{h}[t]^{\tau\sigma})$ that

$$(x_\gamma^+ \otimes t^p)^{(s+d_\gamma q)}(x_{\beta-\gamma}^+ \otimes t^m)^{(s)}(x_\beta^- \otimes t^n)^{(q+s)}Pw_\lambda$$
$$= C(x_{s_\gamma(\beta)} \otimes t^{n+d_\gamma p})^{(q)}(x_\gamma^+ \otimes t^p)^{(s)}(x_\gamma^- \otimes t^{m+n})^{(s)}Pw_\lambda \quad (4.18)$$

4.1.4 Setup Lemma

Recall that given any root $\beta \in (R_1^\sigma)^+$ we can choose $\alpha \in \Delta_0^\sigma$ with $(\beta, \alpha) > 0$. This means $\beta + \alpha \notin R_1^+$. Setting α_{i_0}, $\beta_0 = \alpha_0$, we set $\beta_1 = s_{i_0}\beta_0$ and note that $\beta_1 \in R_1^+$. If $\beta_1 \notin \Delta^\sigma$ then we choose $\alpha_{i_1} \in \Delta$ with $(\beta, \alpha_{i_1}) > 0$ and set $\beta_2 = s_{i_1}\beta_1$. Repeating this if necessary we reach a stage when $k \geq 1$ and $\beta_k \in \Delta^\sigma$. In this case we set $\alpha_{i_k} = \beta_{i_k}$. We claim that

$$|\{0 \leq r \leq k : i_r = i\}| = \mathbf{a}_i(\alpha_0), \quad 1 \leq i \leq n. \quad (4.19)$$

This equation follows from understanding how σ induces a map $\psi : \Delta \to \Delta^\sigma$.

4.1.5 Setup Fact

We retain the notation from the previous section. We now prove that

$$P_{i_k,s_k} \cdots P_{i_0,s_0} w_\lambda = 0, \quad \text{if } (s_0 + \cdots + s_k) \geq \lambda(h_0) + 1. \quad (4.20)$$

We begin with the equality

$$w = (x_0^- \otimes 1)^{(s_0+\cdots+s_k)}w_\lambda = 0, \quad (s_0 + \cdots + s_k) \geq \lambda(h_0) + 1,$$

which is a defining relation for $W(\lambda)$. Recalling that $j = i_0$ and setting

$$X_1 = (x_j^+ \otimes t)^{s_0 + d_{a_j}(s_1+\cdots+s_k)}(x_{\alpha_0-\alpha_j}^+ \otimes t^{ka_j-1})^{(s_0)}$$

we get by applying 4.18

$$0 = X_1 w = (x^-_{\beta_1} \otimes t^{d_{a_j}})^{(s_1+\cdots+s_k)} P_{i_0,s_0} w_\lambda.$$

More generally, if we set

$$X_{r+1} = (x^+_{\alpha_{i_r}} \otimes t^{\delta_{i_r,j}})^{s_r + d_{a_{i_r}}(s_r+\cdots+r_k)} (x^+_{\beta_r-\alpha_{i_r}} \otimes t^{m_r})^{(s_R)},$$

where $m_r = a_j - \delta_{i_r,j} - d_{a_j}|\{0 \leq q < r | i_q = j\}|$ we find after repeatedly applying 4.18 that

$$0 = (x^+_{\beta_k} \otimes t^{\delta_{i_k},j})^{(s_k)} X_k \cdots X_1 w = P_{i_k,s_k} \cdots P_{i_0.s_0} w_\lambda.$$

This proves the assertion.

4.1.6 Proof of generators of $Jac(\mathbf{A}_\lambda)$

We can now prove that

$$P_{1,r_1} \cdots P_{n,r_n} \in Jac(\mathbf{A}_\lambda) \text{ if } \sum_{i=1}^{n} \mathbf{a}_i(\alpha_0) r_i > \lambda(h_0)$$

By selecting $s_p = r_m$ whenever $i_p = m$ in equation (4.16, will change when I go to finish the formatting) and using equation 4.20 we see that

$$P^{\mathbf{a}_1(\alpha_0)}_{1,r_1} \cdots P^{\mathbf{a}_n(\alpha_0)}_{n,r_n} w_\lambda = 0 \text{ if } \sum_{i=1}^{n} \mathbf{a}_i(\alpha_0) r_i > \lambda(h_0). \tag{4.21}$$

Multiplying by the appropriate P'_{i,r_i} for $1 \leq i \leq n$ we find that for some $s \geq 0$ we have

$$P^s_{1,r_1} \cdots P^s_{n,r_n} w_\lambda = 0 \text{ if } \sum_{i=1}^{n} \mathbf{a}_i(\alpha_0) r_i > \lambda(h_0).$$

Hence $P^s_{1,r_1} \cdots P^s_{n,r_n} = 0$ in $\mathbf{A}_\lambda$ proving that $P_{1,r_1} \cdots P_{n,r_n} \in Jac(\mathbf{A}_\lambda)$. This argument proves that there exists a well-defined morphism of algebras

$$\varphi : \tilde{\mathbf{A}}_\lambda \twoheadrightarrow \mathbf{A}_\lambda / Jac(\mathbf{A}_\lambda). \tag{4.22}$$

Lemma 4.1.2. *If $\mathbf{a}_j(\alpha_0) = 1$ then φ factors through $\mathbf{A}_\lambda$; i.e.the following diagram commutes.*

$$\begin{tikzcd} F_t(x) \arrow[r, two heads] \arrow[dr, dashed, two heads] & F(x) \arrow[d, two heads] \\ & A \end{tikzcd}$$

Proof. From 4.21 it suffices to show that if $\mathbf{a}_j(\alpha_0) = 1$ then $\mathbf{a}_i(\alpha_0) = 1$ for all $i \in I$. However this is obvious from the table in Section 1.1.5. $\square$

4.1.7 Proof of Theorem 4.1.1

We can now see that 4.1.1 is complete if we show that $\varphi : \tilde{A_\lambda} \twoheadrightarrow \mathbf{A}_\lambda / Jac(\mathbf{A}_\lambda)$ is injective. Since $\tilde{A_\lambda}$ is the quotient of $\mathbf{U}(\mathfrak{h}[t]^{\tau\sigma})$ by a square free ideal, $Jac(\tilde{A_\lambda}) = 0$. This means $\forall 0 \neq f \in \tilde{A}$ there is a maximal ideal I_f such that $f \notin I_f$, $Jac(\mathbf{A}_\lambda) = \bigcap_{I \in MaxSpec(\mathbf{A}_\lambda)} I$. Following Section 4.9 we will construct an evaluation representation that is a quotient of $W(\lambda)$ such that the action of f on the quotient is non-zero. By 3.1.2 there is a choice of tuple $(\pi_{i,r}), i \in I, r \in \mathbb{N}$ satisfying the defining relations of $\tilde{A_\lambda}$ such that under the evaluation fo $P_{i,r}$ to $\pi_{i,r}$, f is mapped to a non-zero scalar. Now we can choose $z_1, \ldots, z_m$ and $\lambda_1, \ldots, \lambda_m \in P^+$ such that

$$\pi_i(u) = 1 + \sum_{r \in \mathbb{N}} \pi_{i,r} u^r = \prod_{s=1}^{m} (1 - z_s^{r a_j} u)_s^\lambda (h_i), \quad i \in I$$

and set $\mu = \lambda - (\lambda_a + \cdots + \lambda_k) \in P_0$. Since the $(\pi_{i,r})$ satisfy the first defining relation of $\tilde{A_\lambda}$, we have $\mu(h_i) \in P_0^+$. We can see that $\mu(h_0) \in \mathbb{Z}_+$ by utilizing the fact that $(\pi_{i,r})$ satisfy the second defining relation of $\tilde{A_\lambda}$. We start by noting that the coefficient of u^r in

$\prod_{i\in I}\pi_i(u)^{\mathbf{a}(\alpha_0)}$ is given by

$$\sum_{(r_{i_k})}\prod_{i\in I}\prod_{s=1}^{\mathbf{a}(\alpha_0)}\pi_{r_{i_s}} \tag{4.23}$$

where the sum runs over all tuples (r_{i_s}) such that $\sum_{i\in I}\sum_{s=1}^{\mathbf{a}(\alpha_0)} r_{i_s} = r$. Setting $r_i = \max\{r_{i_s}, 1 \leq s \leq \mathbf{a}(\alpha_0)\}, i \in I$ and observing that if $r > \lambda(h_0)$ then

$$\sum_{i\in I}\mathbf{a}(\alpha_0)r_i \geq r > \lambda(h_0)$$

and hence the previous equation vanishes. It is immediate from observation that

$$\mu(h_0) = \lambda(h_0) - \left(\prod_{i\in I}\pi_i(u)^{\mathbf{a}(\alpha_0)}\right) \in \mathbb{Z}_+.$$

By Proposition 4.9 we have constructed the desired quotient of $W(\lambda)$ such that f acts by a non-zero scalar on the highest weight vector. Hence, $f^N \notin Ann_{\mathfrak{h}[t]^{\tau\sigma}}(w_\lambda)$ for $N \geq 1$. This proves $\varphi : \tilde{A}_\lambda \twoheadrightarrow \mathbf{A}_\lambda/Jac(\mathbf{A}_\lambda)$ is injective and completes the proof of 4.1.1.

Chapter 5

Structure of global Weyl modules

5.1 Irreducible global Weyl modules

In this section we give necessary and sufficient conditions for a global Weyl module to be irreducible.

5.1.1 Irreducibility conditions

Recall from 3.7 that $ev_0^* V_{\mathfrak{g}_0}(\lambda) \cong V(\lambda, I_0)$ is a quotient of $W(\lambda)$ for all $\lambda \in (P_0^{\tau\sigma})^+$.

Proposition 5.1.1. *Let $\lambda \in (P_0^{\tau\sigma})^+$ and $\iota : V_{\mathfrak{g}_0^{\tau\sigma}}(\lambda) \hookrightarrow W(\lambda)$ be the inclusion of $\mathfrak{g}_0^{\tau\sigma}$-modules with $\iota(v_\lambda) = w_\lambda$. Define $\Phi : (\mathfrak{g}_1^{\tau\sigma}) \otimes V_{\mathfrak{g}_0^{\tau\sigma}}(\lambda) \to W(\lambda)$ by*

$$\Phi(x \otimes v) = (x \otimes t)\iota(v), \quad x \in \mathfrak{g}_1^{\sigma\tau}, \quad v \in V_{\mathfrak{g}_0^{\tau\sigma}}(\lambda).$$

The following are equivalent.

(a) The module $W(\lambda)$ is irreducible.

(b) The canonical map of $\mathfrak{g}[t]^{\tau\sigma}$-modules $W(\lambda) \to V(\lambda, I_0) \to 0$ is an isomorphism.

(c) $(\mathfrak{g}_1^{\tau\sigma} \otimes t)w_\lambda = 0$

(d) $\Phi = 0$.

(e) For all $\mu \in (P_0^{\tau\sigma})^+$ *with* $\lambda - \mu \in Q^+$ *we have* $Hom_{\mathfrak{g}_0^{\tau\sigma}}(\mathfrak{g}_1^{\tau\sigma} \otimes V_{\mathfrak{g}_0^{\tau\sigma}}(\lambda), V_{\mathfrak{g}_0^{\tau\sigma}}(\mu)) = 0$

Proof. It is clear that $(a) \Rightarrow (b)$ and since $V(\lambda, I_0) \cong ev_0^* V_{\mathfrak{g}_0^{\tau\sigma}}(\lambda)$, $(b) \Rightarrow (c)$ is also clear. To show $(c) \Rightarrow (a)$ we utilize the fact that $\mathfrak{g}_m^{\tau\sigma} \otimes t^m = [\mathfrak{g}_1^{\tau\sigma} \otimes t, \mathfrak{g}_{m-1}^{\tau\sigma} \otimes t^{m-1}]$ and apply induction on m to see that $(\mathfrak{g}_m^{\tau\sigma} \otimes t^m)w_\lambda = 0$ for $1 \le m < ka_j$. Again since $\mathfrak{g}_0 \otimes t^{ka_j} = [\mathfrak{g}_1^{\tau\sigma} \otimes t, \mathfrak{g}_{ka_j-1}^{\tau\sigma} \otimes t^{ka_j-1}]$, we have $\mathfrak{g}_0^{\tau\sigma} \otimes t^{ka_j} w_\lambda = 0$. It then follows that $\mathfrak{g}_m^{\tau\sigma} \otimes t^m \mathbb{C}[t^{ka_j}]w_\lambda = 0$. By an immediate application of the PBW theorem $W(\lambda) = \mathbf{U}(\mathfrak{g}_0^{\tau\sigma})w_\lambda$, is an irreducible $\mathfrak{g}_0$-module and hence an irreducible $\mathfrak{g}[t]^{\tau\sigma}$. $(d) \Rightarrow (c)$ is immediate from construction. It is easily verifiable that Φ is a $\mathfrak{g}_0^{\tau\sigma}$-module homomorphism and since $\mathfrak{g}_1^{\tau\sigma} = [\mathfrak{g}_1^{\tau\sigma}, \mathfrak{g}_0^{\tau\sigma}]$ by the PBW Theorem, $(d) \Rightarrow (c)$. We now show that (e) is equivalent to (a). Suppose $W(\lambda)$ is reducible. By the equivalence of (a) and (d), $\Phi \neq 0$. Since $\mathfrak{h}^\sigma \cap \mathfrak{g}_1^{\tau\sigma} = \{0\}$, Im $\Phi \cap W(\lambda)_\lambda = \{0\}$. Hence, $\exists \mu \in P_0^+$ with $\lambda - \mu \in Q^+ \backslash \{0\}$ such that $V_{\mathfrak{g}_0^{\tau\sigma}}(\mu)$ is a quotient of ImΦ. For the converse, assume that $\Psi : \mathfrak{g}_1 \otimes V_{\mathfrak{g}_0^{\tau\sigma}}(\lambda) \to V_{\mathfrak{g}_0^{\tau\sigma}}(\mu)$ is a non-zero map of $\mathfrak{g}_0$-modules. Set $V = V_{\mathfrak{g}_0^{\tau\sigma}}(\lambda) \oplus V_{\mathfrak{g}_0^{\tau\sigma}}(\mu)$. We extend the $\mathfrak{g}_0$ structure to a $\mathfrak{g}[t]^{\tau\sigma}$ structure by

$$(x \otimes 1)(v_1, v_2) = (xv_1, xv_2)$$

$$(y \otimes t)(v_1, v_2) = (0, \Psi(y \otimes v_1))$$

$$\mathfrak{g}[t]^{\tau\sigma}[s](v_1, v_2) = 0, \quad s \ge 2,$$

for $x \in \mathfrak{g}_0^{\tau\sigma}, y \in \mathfrak{g}_1^{\tau\sigma}$, and $(v_1, v_2) \in V$. Since $\lambda - \mu \in (Q^\sigma)^+$ is is easy to see that V is a quotient of $W(\lambda)$. $\square$

Lemma 5.1.1. *Suppose that $\lambda \in (P_0^{\tau\sigma})^+$ is such that $W(\lambda)$ is reducible. Then $W(\lambda+\nu)$ is reducible for all $\nu \in (P_0^{\tau\sigma})^+$*

Proof. A simple checking of defining relations shows that we have a map of $\mathfrak{g}[t]^{\tau\sigma}$-modules $W(\lambda+\nu) \to W(\lambda) \otimes W(\nu)$ which sends $w_{\lambda+\nu} \to w_\lambda \otimes w_\nu$. If $W(\lambda+\nu)$ is irreducible then by part (c) of 5.1.1 we would have

$$(x_\alpha^- \otimes t)(w_{\lambda+\nu}) = 0, \quad \alpha \in R_1^+.$$

Since this implies that $(x_\alpha^- \otimes t)(w_\lambda \otimes w_\nu) = 0$ we would get $(\mathfrak{g}_1 \otimes t)(w_\lambda \otimes w_\nu) = 0$. Then 5.1.1 implies that $W(\lambda)$ is irreducible which is a contradiction.

□

Proposition 5.1.2. *The global Weyl module is infinite dimensional if and only if* $\dim \mathbf{A}_\lambda = \infty$ *and in this case $W(\lambda)$ is reducible*

Proof. By 3.1.1 and 4.2 we know that $\dim W(\lambda) = \infty$ if and only if $\dim \mathbf{A}_\lambda = \infty$. By 4.1.1, $\mathbf{A}_\lambda$ is not a local ring. Thus, there is a second maximal ideal $I_1 \neq I_0$ and $W(\lambda)$ has two non-isomorphic quotients, $V(\lambda, I_0)$ and $V(\lambda, I_1)$. This means there are two submodules, one of which is non-zero and proper, corresponding the kernel of each projection map completing the proof. □

Corollary 5.1.1. *Suppose that $\lambda(h_i) > 0$ for some $i \in I(j)$. Then $W(\lambda)$ is a reducible $\mathfrak{g}[t]^{\tau\sigma}$-module if $\lambda(h_0) \geq \mathbf{a}_i(\alpha_0)$.*

Proof. By 5.1.2 it suffices to prove that $\mathbf{A}_\lambda/Jac(\mathbf{A}_\lambda)$ is infinite dimensional. If $\mathbf{a}_i(\alpha_0) = 0$ then condition (i) of 4.1.1 is not satisfied so $\mathbf{A}_\lambda/Jac(\mathbf{A}_\lambda)$ is infinite dimensional. If

$\mathbf{a}_i(\alpha_0) > 0$ then condition (ii) is violated and we again see that $\mathbf{A}_\lambda/Jac(\mathbf{A}_\lambda)$ is infinite dimensional. $\square$

5.1.2 Useful Remark

The following remarks will be useful in what follows. Suppose that $\beta \in R_0$ is such that $\mathbf{a}_j(\beta) = a_j$. If $\beta \neq \alpha_0$ then $\mathbf{a}_i(\beta) > 0$ for some $i \in I(j)$. Recall the elements $\theta_m^{\tau\sigma} \in (R_m^{\tau\sigma})^+$ defined in Section 2.5. These can be characterized as follows: if $\alpha \in R_m^{\tau\sigma}$ and $\alpha \neq \theta_m^{\tau\sigma}$ then there exists $i \in I(j)$ such that $\mathbf{a}_i(\theta_m^{\tau\sigma}) > \mathbf{a}_i(\alpha)$. Further the elements $-\theta_m^{\tau\sigma} \in R_{ka_j-m}$ and if $\alpha \in R_{ka_j-m}$ with $\alpha \neq -\theta_m$ then there exists $i \in I(j)$ with $\mathbf{a}_i(\alpha) > \mathbf{a}_i(-\theta_m^{\tau\sigma})$. In particular $-\theta_{ka_j-m}^{\tau\sigma}$ is the lowest weight of $\mathfrak{g}_m^{\tau\sigma}$ regarded as a $\mathfrak{g}_0^{\tau\sigma}$-module.

5.1.3 Further reducibility conditions for $W(\lambda_i)$

Lemma 5.1.2. *Let $i \in I(j) \cup \{0\}$. Then $W(\lambda_i)$ is reducible if*

(i) $i \in I(j)$ *and* $\mathbf{a}_i(\alpha_0) = 0$,

(ii) $i = 0$

Proof. Part (i) is immediate from 5.1.1. Recall the element $w_\circ$ defined in Section 2.4. To prove that $W(\lambda_0)$ is reducible, it suffices from 5.1.1 to show that

$$w_\circ\theta_1 \in (R_1^{\tau\sigma})^+, \quad \mu_0 = \lambda_0 - w_\circ\theta_1 \in P_0^+, \quad \mathrm{Hom}_{\mathfrak{g}_0^{\tau\sigma}}(\mathfrak{g}_1^{\tau\sigma} \otimes V_{\mathfrak{g}_0^{\tau\sigma}}(\lambda_0), V_{\mathfrak{g}_0^{\tau\sigma}}(\mu_0)) \neq 0.$$

The first assertion is clear since $w_\circ\alpha_j \in (R^\sigma)^+$. If $i \in I(j)$ then $-w_\circ^{-1}\alpha_i \in (R^\sigma)^+$ and hence $-w_\circ^{-1}(h_i)$ is in the $\mathbb{Z}_+$-span of $h_i, i \in I(j)$. It also implies that $\alpha_0 + w_\circ\theta_1 \notin R^\sigma$ and $0 \leq w_\circ\theta_1(h_0) \leq 1$. It follows that

$$\mu_0(h_i) = -w_\circ\theta_1(h_i) = \theta_1(-w_\circ^{-1}(h_i)) \geq 0, i \in I(j).$$

It follows again that $\mu_0(h_0) \geq 0$ and the second assertion is proved. The last assertion follows from the PRV Theorem [13, Theorem 2.10].

$\square$

5.1.4 irreducibility condition

Theorem 5.1.1. *Let $\lambda \in P_0^+$. Then $W(\lambda)$ is an irreducible $\mathfrak{g}[t]^{\tau\sigma}$-module iff the following holds:*

$$\{i \in I(j) \cup \{0\} : \lambda(h_i) > 0\} \subset \{i \in I(j) : \mathbf{a}_i(\alpha_0) = \mathbf{a}_i(\theta_{ka_j-1})\} \tag{5.1}$$

Proof. Suppose that λ satisfies the conditions in the above equation. By 5.1.1 it suffices to prove that $(\mathfrak{g}_1 \otimes t)w_\lambda = 0$. Using the irreducibility of $\mathfrak{g}_1$ it suffices to prove that

$$(x^-_{\theta_{ka_j-1}} \otimes t)w_\lambda = 0. \tag{5.2}$$

By 1.5 we can write

$$\theta_{ka_j-1} - \alpha_0 + \alpha_j = \sum_{i \in I(j)} p_i\alpha_i, \quad p_i \in \mathbb{Z}_+, \quad i \in I(j).$$

Since $\lambda(h_0) = 0$ and $\lambda(h_i) = 0$ for all $i \in I(j)$ with $p_i > 0$ we have the defining relations

$$(x_0^- \otimes 1)w_\lambda = 0, \quad (x_i^- \otimes 1)w_\lambda = 0, \quad i \in I(j), \quad p_i > 0.$$

Hence

$$(x_j^+ \otimes t)(x_0^- \otimes 1)w_\lambda = A(x^-_{\alpha_0-\alpha_j} \otimes t)w_\lambda = 0,$$

for some $0 \neq A \in \mathbb{C}$. Equation 5.2 follows by noting that there exists $0 \neq B \in \mathbb{C}$ such that $x_{\theta_{a_j}-1} = B[x_{i_1}^-, \cdots [x_{i_{s-1}}^-, [x_{i_s}^-, x_{\alpha_0-\alpha_j}^-]], \cdots]$ where $i_1, \ldots, i_s$ are elements of the set $\{i \in I(j) : p_i > 0\}$.

For the converse suppose that $\lambda(h_0) \neq 0$ and let $\mu = \lambda - \lambda_0$. Since $W(\lambda_0)$ is reducible by 5.1.1 we can use 5.1.1 to conclude that $W(\lambda)$ is reducible. The proof if $\lambda(h_i) > 0$ for some $i \in I(j)$ with $\mathbf{a}_i(\theta_{ka_j-1}) \neq \mathbf{a}_i(\alpha_0)$ is identical. $\square$

Chapter 6

Local Weyl Modules

6.1 Consolidation of facts

Recall from Section 3 that the equivariant map algebra $\mathfrak{g}[t]^{\sigma\tau}$ is not isomorphic to an equivariant map algebra where the group Γ acts freely on the set of maximal ideals of A. When Γ acts freely, the finite dimensional representation theory of the equivariant map algebra is closely related to that of the map algebra $\mathfrak{g} \otimes A$. We have already seen a major difference between the finite dimensional representation theory of $\mathfrak{g}[t]^{\sigma\tau}$ and that of $\mathfrak{g}[t]^{\sigma}$. Specifically we showed the global Weyl module for $\mathfrak{g}[t]^{\sigma\tau}$ can be finite-dimensional and irreducible for nontrivial dominant integral weights. In this section we discuss the structure of local Weyl modules for the base case of $(A_{2n-1}^{(2)}, D_n)$ where λ is a fundamental weight, in which case $\mathbf{A}_\lambda$ is a polynomial algebra.

6.1.1 Local Weyl modules definition

Remark: We note that since we are looking at $A_{2n-1}^{(2)}$ with $j = n$ we have $\alpha_0 = \alpha_{n-1} + \alpha_n$ This means $x_\alpha^- \otimes t$ with $\mathbf{a}_n(\alpha) = 0$ means $x_\alpha^- \in (\mathfrak{g}^\sigma)_1$. If $x_\alpha^- \otimes t$ with $\mathbf{a}_n(\alpha) = 1$ means $x_\alpha^- \in (\mathfrak{g}^\sigma)_0$

Recall that we have a well established theory of local Weyl modules for the current algebra $\mathfrak{g}[t]^\sigma$. $W_{loc}(\lambda_i)$ is the $\mathfrak{g}[t]^\sigma$-module generated by an element w_{λ_i}, which we will denote by w_i, and defining relations

$$\mathfrak{n}^+[t]^{\sigma\tau} w_1 = 0 \quad (h \otimes t^{mk+r}) w_1 = \delta_{r,0}\lambda(h) w_1 = 0 \quad (x_i^- \otimes 1)^2 w_1 = 0. \tag{6.1}$$

We remind the reader that $\{\omega_i : 1 \leq i \leq n\}$ is a set of fundamental weights for $\mathfrak{g}_0^\sigma$ with respect to Δ_0.

We can clearly regard $W_{loc}^g(\omega_i)$ as a graded $\mathfrak{g}[t]^{\sigma\tau}$-module by restriction, however it is not the case that this restriction gives a local Weyl module for $\mathfrak{g}[t]^{\sigma\tau}$. The relationship between local Weyl modules for $\mathfrak{g}[t]^{\sigma\tau}$ and the restriction of local Weyl modules for $\mathfrak{g}[t]^{\sigma\tau}$ is more complicated as we now explain.

6.1.2 Associated Graded Space

Given $z \in \mathbb{C}^\times$ we have an isomorphism of Lie algebras $\eta_z : \mathfrak{g}[t]^\sigma \to \mathfrak{g}[t]^\sigma$ given by $(x \otimes t^{r+mk}) \to (x \otimes (t+z)^{r+mk})^\sigma$ and let $\eta_z^* V$ be the pull-back through this homomorphism of a representation V of $\mathfrak{g}[t]^\sigma$. Suppose that V is such that there exists $N \in \mathbb{Z}_+$ with $(\mathfrak{g} \otimes t^m)^\sigma V = 0$ for all $m \geq N$. Then $(\mathfrak{g} \otimes (t-z)^m)^\sigma \eta_z^* V = 0$ for all $m \geq N$. In particular we can regard the module $\eta_z^* V$ as a module for the finite-dimensional Lie algebra $(\mathfrak{g} \otimes \mathbb{C}[t]/(t-z)^N)^\sigma$. Following [9, Proposition 2.2], since $z \in \mathbb{C}^\times$ we have

$$\mathfrak{g}[t]^\sigma/(\mathfrak{g}\otimes(t-z)^N)^\sigma\mathbb{C}[t]\cong\mathfrak{g}[t]/(\mathfrak{g}\otimes(t-z)^N\mathbb{C}[t])\cong\mathfrak{g}[t]^{\sigma\tau}/(\mathfrak{g}\otimes(t-z)^N\mathbb{C}[t])^{\sigma\tau},$$

so if V is a cyclic module for $\mathfrak{g}[t]^\sigma$ then η_z^*V is a cyclic module for $\mathfrak{g}[t]^{\sigma\tau}$.

We now need a general construction. Given any finite-dimensional cyclic $\mathfrak{g}[t]^{\sigma\tau}$-module V with cyclic vector v define an increasing filtration of $\mathfrak{g}_0^{\sigma\tau}$-modules

$$0=V_0=\mathbf{U}(\mathfrak{g}[t]^{\sigma\tau})[0]v\subset\cdots\subset V_r=\sum_{s=0}^{r}\mathbf{U}(\mathfrak{g}[t]^{\sigma\tau})[s]v\subset\cdots\subset V.$$

The associated graded space gr V is naturally a graded module for $\mathfrak{g}[t]^{\sigma\tau}$ via the action

$$(x\otimes t^s)\overline{w}=\overline{(x\otimes t^s)w},\ \overline{w}\in V_r/V_{r-1}.$$

Suppose that v satisfies the relations

$$\mathfrak{n}^+[t]^{\sigma\tau}v=0,\quad (h\otimes t^{ka_jr})v=d_r(h)v,\quad d_r(h)\in\mathbb{C},\quad r\in\mathbb{Z}_+,\quad h\in\mathfrak{h}.$$

Then since $\dim V<\infty$ it follows that $d_0(h)\in\mathbb{Z}_+$; in particular there exists $\lambda\in P_0^+$ such that $d_0(h)=\lambda(h)$ and a simple checking shows that gr V is a quotient of $W_{loc}(\lambda):=W(\lambda,I_0)$. The following is now immediate.

Lemma 6.1.1. *Let $\lambda\in P^+$ and $z\in\mathbb{C}^\times$. The $\mathfrak{g}[t]^{\sigma\tau}$-module $gr(\eta_z^*W_{loc}^{\mathfrak{g}}(\lambda))$ is a quotient of $W_{loc}(\lambda)$ and hence*

$$\dim W_{loc}(\lambda)\geq\dim W_{loc}^{\mathfrak{g}}(\lambda).$$

6.1.3 Fundamental Local Weyl Modules

For the rest of this section we consider the case of $A_{2n-1}^{(2)}$, and study local Weyl modules corresponding to weights $r\lambda_i\in P_0^+$, where $r\in\mathbb{Z}_+$, and $0\leq i\leq n-2$ (the $i=n-1$ case is discussed in Section 6, where these local Weyl modules are shown to be finite-dimensional and irreducible). We remind the reader that $\lambda_0=\omega_n,\lambda_i=\omega_i,1\leq i\leq n-2$

and $\lambda_{n-1} = \omega_{n-1} - \omega_n$. We will now further investigate the dimension of these local Weyl modules

6.1.4 Setup Propositions

The next proposition summarizes some results on local Weyl modules which are needed for our study. Part (i) was proved in [7, Lemma 6.4, Proposition 6.1]. Parts (ii), (iii) can be found in [5, Theorem 1], where we remind the reader that the fundamental Kirillov–Reshitikin modules are the same as the local Weyl modules associated to a fundamental weight.

Proposition 6.1.1. *(i) Let x, y, h be the standard basis for $\mathfrak{sl}_2$ and set $y \otimes t^r = y_r$. For $\lambda \in P^+$ the local Weyl module $W_{loc}^{\mathfrak{sl}_2}(\lambda)$ has basis*

$$\{w_\lambda, y_{r_1} \cdots y_{r_k} w_\lambda, 0 \leq r_1 \leq \cdots \leq r_k \leq \lambda(h) - k\}$$

Moreover, $y_s w_\lambda = 0$ for all $s \geq \lambda(h)$.

(ii) Assume that $\mathfrak{g}$ is of type D_n and assume that $i \neq n$ (resp. $i \neq n-1, n$). Then

$$W_{loc}^{\mathfrak{g}}(\omega_i) \cong_{\mathfrak{g}} V_{\mathfrak{g}}(\omega_i) \oplus V_{\mathfrak{g}}(\omega_{i-2}) \oplus \cdots \oplus V_{\mathfrak{g}}(\omega_i),$$

where

$$V_{\mathfrak{g}}(\omega_i) = V_{\mathfrak{g}}(\omega_1), i \text{ odd}, V_{\mathfrak{g}}(\omega_i) = \mathbb{C}, \text{ even.}$$

(iii) Assume that $\mathfrak{g}$ is of type D_n and let $i = n$ (resp. $i \in \{n-1, n\}$). Then

$$W_{loc}^{\mathfrak{g}}(\omega_i) \cong_{\mathfrak{g}} V_{\mathfrak{g}}(\omega_i).$$

$\square$

We remind the reader of the following elementary facts on the dimension of the spin representations for D_n,

$$\dim V_{\mathfrak{g}}(\omega_i) = \binom{2n}{i}, \quad i \neq n-1, n.$$

Moreover, if $i \in \{n-1, n\}$, then

$$\dim V_{\mathfrak{g}}(\omega_i) = 2^{n-1}.$$

6.1.5 Spanning Sets

Our goal is to provide an upper bound on $\dim W_{loc}(\lambda_i)$. The proof needs several additional results, and we consider the cases $2 \leq i \leq n-2$ and $i = 0$ separately.

Recall that $\mathfrak{g}_0[t^2] \subset \mathfrak{g}[t]^{\sigma\tau}$, and so $W_{loc}(\lambda_i)$ can be regarded as a $\mathfrak{g}_0[t^2]$-module by pulling back along the inclusion map $\mathfrak{g}_0[t^2] \hookrightarrow \mathfrak{g}[t]^{\sigma\tau}$. For ease of notation we denote the element W_{λ_i} by w_i, $x^-_{p,\bar{q}} := x^-_{\alpha_p+\cdots+\alpha_i+2\alpha_p+\ldots+2\alpha_{n-1}+\alpha_n}$, and $x^-_{\bar{p}} := x^-_{2\alpha_p+\ldots+2\alpha_{n-1}+\alpha_n}$.

Lemma 6.1.2. *(i) For $2 \leq i \leq n-2$, $W_{loc}(\lambda_i)$ can be regarded as a $\mathfrak{g}_0[t^2]$-module by w_i and Yw_i where Y is a monomial in the elements*

$$(x^-_{p,\bar{i}} \otimes t^{2s+1}), \quad p \leq i, s \leq 1$$

(ii) $W_{loc}(\lambda_0)$ is generated as a $\mathfrak{g}_0[t^2]$-module by w_0 and Yw_0 where Y is a monomial in the elements

$$(x^-_{p,n-1} \otimes t^{2s+1}), \quad 1 \leq \quad p \leq n-1, \quad 0 \leq s \leq 1$$

Proof. First, for $2 \leq i \leq n-2$ the defining relation $x^-_0 w_r = 0$ implies that

$$(x_0^- \otimes t^{2s})w_i = (x_{n-1}^- \otimes t^{2s+1})w_i = 0, \quad s \geq 0$$

Since $(x_p^- \otimes 1)w_i = 0$ for $p \neq i$,

$$(x_{p,n}^- \otimes t^{2s+1})w_1 = (x_{p,n-1}^- \otimes t^{2s+1})w_1 = 0, \quad s \geq 0, p > i. \tag{6.2}$$

Since we also have

$$(x_i^- \otimes 1)^2 w_1 = 0 \Rightarrow (x_i^- \otimes t^{2s})w_1 = 0 \quad s \geq 1,$$

it follows that

$$\begin{aligned} 0 &= (x_n^- \otimes t)(x_{i,n-1}^- \otimes t)^2 w_1 \\ &= (x_{i,n}^- \otimes t^2)(x_{i,n-1}^- \otimes t)w_1 \\ &= (x_{2\alpha_i+\cdots+2\alpha_{n-1}+\alpha_n}^- \otimes t^3)w_1 \end{aligned}$$

and

$$(x_{i-1,\bar{i}}^- \otimes t^3)w_1 = 0.$$

From this we can conclude that $(x_{i-1,\bar{i}}^- \otimes t^{2s+1})w_1 = 0$ for $s \geq 1$, and (i) is clear with the additional use of the PBW Theorem.

For the case $i = 0$, we have

$$(x_{k,p}^- \otimes t^{2s})w_1 = 0, \quad 1 \leq k \leq p \leq n-1, s \geq 0.$$

Just like the previous case we have $(x_0^- \otimes 1)^s w_1 = 0, s \geq 2$ implies that

$$(x_0^- \otimes t^{2s})w_1 = 0, \quad s \geq 1.$$

Following the same approach as case (i) we find that

$$(x_{p-1,\overline{p}}^- \otimes t^{2s+1})w_1 = 0, \quad 2 \leq p \leq n, s \geq 1$$

$\square$

Before proceeding with the structure of $W_{loc}(w_1)$ we make a key observation:

Lemma 6.1.3. *For $2 \leq i \leq n-2$,*

$$(x^-_{i-1,\bar{i}} \otimes t)^2 w_1 = 0$$

Proof. From the previous lemma we have $(x^-_{i,n-1} \otimes t)^2 w_1 = 0$. Now we start by noting that

$$\begin{aligned}
0 &= (x^+_n \otimes t)(x^-_{i,n} \otimes 1)^2 w_1 \\
&= [(x^-_{i,n-1} \otimes t)(x^-_{i,n} \otimes 1) + (x^-_{i,n} \otimes 1)(x^-_{i,n-1} \otimes t)]w_1 \\
&= [(x^-_{2\alpha_i+\ldots+2\alpha_{n-1}+\alpha_n} \otimes t) + 2(x^-_{i,n} \otimes 1)(x^-_{i,n-1} \otimes t)]w_1
\end{aligned}$$

This means

$$\begin{aligned}
(x^-_{i-1,\bar{i}} \otimes t)w_1 &= (x^-_{i-1} \otimes 1)(x^-_{2\alpha_i+\ldots+2\alpha_{n-1}+\alpha_n} \otimes t)w_1 \\
&= -2(x^-_{i-1} \otimes 1)(x^-_{i,n} \otimes 1)(x^-_{i,n-1} \otimes t)w_1
\end{aligned}$$

and

$$\begin{aligned}
(x^-_{i-1,\bar{i}} \otimes t)^2 w_1 &= 4(x^-_{i-1} \otimes 1)(x^-_{i,n} \otimes 1)(x^-_{i,n-1} \otimes t)(x^-_{i-1} \otimes 1)(x^-_{i,n} \otimes 1)(x^-_{i,n-1} \otimes t)w_1 \\
&= 4(x^-_{i-1} \otimes 1)(x^-_{i,n} \otimes 1)[(x^-_{i-1} \otimes 1)(x^-_{i,n-1} \otimes t) + (x^-_{i-1,n-1} \otimes 1)](x^-_{i,n} \otimes 1)(x^-_{i,n-1} \otimes t)w_1 \\
&= 4(x^-_{i-1} \otimes 1)(x^-_{i,n} \otimes 1)(x^-_{i-1,n-1} \otimes t)(x^-_{i,n} \otimes 1)(x^-_{i,n-1} \otimes t)w_1 \\
&= 4(x^-_{i-1} \otimes 1)(x^-_{i,n} \otimes 1)(x^-_{i-1,\bar{i}} \otimes t)(x^-_{i,n-1} \otimes t)w_1 \\
&= -8(x^-_{i-1} \otimes 1)(x^-_{i,n} \otimes 1)(x^-_{i-1} \otimes 1)(x^-_{i,n} \otimes 1)(x^-_{i,n-1} \otimes t)^2 w_1 \\
&= 0
\end{aligned}$$

$\square$

6.1.6 Reduction of spanning set

Lemma 6.1.4. *(i) For $1 \le i \le n-2$, suppose that $Y = (x^-_{p_1,\bar{i}} \otimes t)\cdots(x^-_{p_k,\bar{i}} \otimes t)$ where $p_1 \le \cdots \le p_k \le i$. Then Yw_i is in the $\mathfrak{g}_0[t^2]$-module generated by elements Zw_i where Z is a monomial in the elements $\{(x^-_{i-1,\bar{i}} \otimes t)\}$.*

(ii) For $i = 0$, suppose that $Y = (x^-_{p_1,n-1} \otimes t^{2s_1+1})\cdots(x^-_{p_m,n-1} \otimes t^{2s_k+1})$ where $p_1 \le \cdots \le p_m \le n-1$. Then Yw_0 is in the $\mathfrak{g}_0[t^2]$-module generated by elements Zw_0 where Z is a monomial in the elements $(x^-_{n-1} \otimes t^{2s+1})$ with $s \in \mathbb{Z}_+$.

Proof. This follows immediately from 6.1.2 and the fact that

$$(x^-_{p,\bar{i}} \otimes t)(x^-_{i-1,\bar{i}} \otimes t)^\ell w_i = (x^-_{p,i-2} \otimes 1)(x^-_{i-1,\bar{i}} \otimes t)^{\ell+1} w_i.$$

The case $i = 0$ is identical. $\square$

6.1.7 Inequality about dimensionality

We now provide an upper bound for the dimension of fundamental local Weyl modules, first for $2 \le i \le n-2$. From 6.1.3 and 6.1.4 it is straightforward to see that we have an increasing filtration of $\mathfrak{g}_0[t^2]$-modules:

$$0 = U_0 \subset U_1 = \mathbf{U}(\mathfrak{g}_0[t^2])(x^-_{i-1,\bar{i}} \otimes t)w_i \subset U_2 = W_{loc}(\lambda_i).$$

Moreover U_2/U_1 (resp. U_1/U_0) is a quotient of the local Weyl module for $\mathfrak{g}_0[t^2]$ with highest weight ω_i (ω_{i-1}). Using Proposition 7.4(ii) we get

$$\dim U_2/U_1 \le \sum_{s=0}^{i} \binom{2n-1}{s}.$$

and

$$\dim U_1/U_0 \leq \sum_{s=0}^{i-1} \binom{2n-1}{s}.$$

Summing we get

$$\begin{aligned}\dim W_{loc}(\lambda_i) &\leq \binom{2n-1}{i} + 2\left(\binom{2n-1}{i-1} + \cdots + \binom{2n-1}{1}\right) \\ &= \left(\tbinom{2n}{i} + \tbinom{2n}{i-1} \cdots + \tbinom{2n}{1}\right).\end{aligned}$$

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