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Bounded Domains on Kobayashi Hyperbolic Manifolds Covering Compact Complex Manifolds

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ABSTRACT OF THE DISSERTATION

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Since there is no Uniformization Theorem in several complex variables, there is a desire to classify all of the simply connected domains. We use a result of Zimmer and a localization technique of Lin and Wong to extend a result of Cheung et al. In particular, we show that if a domain with $C^{1,1}$ boundary on a Kobayashi hyperbolic complex manifold contains a totally real boundary point and covers a compact manifold, then its universal cover must be the Euclidean ball.

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Chapter 1

Introduction

The Riemann Mapping Theorem states that every proper, simply connected, open subset of $\mathbb{C}$ is biholomorphic to the disk. However, there is no analogue for this result in higher dimensions. In fact, most domains in $\mathbb{C}^n$ that are "close to" the ball are not biholomorphic to the ball; the set of equivalence classes of such domains is uncountable. See [1] & [15], [16].

The first result that suggested the failure of a higher dimensional analogue of the Riemann Mapping Theorem was proved by Poincaré: the ball in $\mathbb{C}^n$ and the polydisk in $\mathbb{C}^n$ are not biholomorphic. His technique was to show that the automorphism groups of these domains are not isomorphic.

The previous discussion is all to say that there is no canonical topologically trivial domain in several complex variables like there is in $\mathbb{C}$, namely the disk. This means that the study of holomorphic functions of several complex variables must depend on the domains, themselves.

In $\mathbb{C}$, the Uniformization Theorem allows us to reduce analytic questions about planar domains to analytic questions on the disk. But, again, there is no Uniformization Theorem in higher dimensions. This shortcoming gives rise to a big problem in several complex variables: Classifying the simply connected domains. This problem is the inspiration for the current work.

Our main result is

Theorem 1.0.1. *Let M^n be a (Kobayashi) hyperbolic complex manifold, and let $\Omega \subset M$ be a subdomain with nonempty boundary, and assume $\partial\Omega$ is $C^{1,1}$. Suppose there exists a totally real boundary point $p \in \partial\Omega$. Suppose further that Ω covers a compact complex manifold. Then Ω is biholomorphic to the Euclidean ball in $\mathbb{C}^n$.*

Theorem 1.0.1 weakens slightly a condition of a result by Cheung et al [5], and makes use of a result of Zimmer [35]. Theorem 1.0.1 also extends a result of Wong [30].

Proposition 1.0.2 (Proposition 3.1 in [5]). *Let Ω be a relatively compact subdomain of an n-dimensional hyperbolic complex manifold M. If Ω admits a compact quotient, and $\partial\Omega$ is smooth and strictly pseudoconvex near a point $p \in \partial\Omega$, then Ω is biholomorphic to the ball.*

Theorem 1.0.3 (Theorem 1.1 in [35]). *Suppose $\Omega \subset \mathbb{C}^n$ is a bounded domain which covers a compact manifold. If $\partial\Omega$ is $C^{1,1}$, then Ω is biholomorphic to the unit ball in $\mathbb{C}^n$.*

Theorem 1.0.4 (Wong [30]). *Suppose $\Omega \subset \mathbb{C}^n$ is a bounded domain which covers a compact manifold. If $\partial\Omega$ is C^2, then Ω is biholomorphic to the unit ball.*

Theorem 1.0.4 was proved by Wong for strongly pseudoconvex domains. Rosay later extended the result to any bounded domain with C^2 boundary.

The proof of Proposition 1.0.2 – as well as other rigidity results similar to it and Theorem 1.0.4 – relies on the idea that the interior complex geometry of the domain near a strongly pseudoconvex boundary point is close to that of the ball. Moreover, every bounded domain in $\mathbb{C}^n$ with C^2 boundary has at least one strongly pseudoconvex point on the boundary. Then since the domain covers a compact complex manifold, the interior geometry must be *everywhere* close to that of the ball. Then use a limiting argument.

Zimmer's contribution for the $C^{1,1}$ case is to notice that localizing around a strongly pseudoconvex point is no longer possible. Zimmer's proof of Theorem 1.0.3 uses a rescaling technique of Frankel to show that the domain $\Omega \subset \mathbb{C}^n$ is biholomorphic to a domain $D \subset \mathbb{C}^n$ containing a one-parameter subgroup. It then follows by a theorem of Frankel and Nadel [12], [26] that Ω is a bounded symmetric domain. Zimmer then uses the geometry of the rescaled domain to show that Ω is the unit ball.

Our proof follows much of the same logic as Zimmer's. The difference in the arguments is in the construction of the rescaling maps. For this construction, we use a localization technique of Lin and Wong [24] to force the domain $\Omega \subset M$ to be considered as a domain in $\mathbb{C}^n$. This construction alters slightly the statements of Zimmer's results, and, for this reason, we will present our argument in full noting that it is, in some places, identical to Zimmer's. We will, of course, point out each instance of this.

The rest of the document is laid out as follows. In Chapter 2, we give a brief overview of the necessary background material. In Chapter 3, we will discuss some geometric properties of domains in $\mathbb{C}^n$ that will allow us to study such sets in better detail. In Chapter 4, we describe the aforementioned localization technique that will be instrumental in the construction of our rescaling maps, and give some other applications of it from the original paper. In Chapter 5, we present some basic information on bounded symmetric domains. The proof of Theorem 1.0.1 is presented in Chapter 6. In the final chapter, we recall the inspiration for our main result and present some ideas for future work.

Chapter 2

Background

2.1 Several Complex Variables

We begin with some basic background information on several complex variables. We will define holomorphic functions in several variables, discuss briefly domains in $\mathbb{C}^n$, and finish with an introduction to complex manifolds.

2.1.1 Holomorphic Functions of Several Complex Variables

As with single variable complex analysis, there are many equivalent ways to define holomorphic functions of several variables, i.e., Cauchy-Riemann equations, power series, integral formula, etc. We will use the following.

Definition 2.1.1. Let $\Omega \subset \mathbb{C}^n$ be a domain (open, connected). A function $f : \Omega \to \mathbb{C}$ is said to be *holomorphic* if for each $j = 1, \ldots, n$ and each fixed $z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n$, the function

$$\zeta \mapsto f(z_1, \ldots, z_{j-1}, \zeta, z_{j+1}, \ldots, z_n)$$

is holomorphic in the classical one-variable sense on the set

$$\{\zeta \in \mathbb{C} \mid (z_1, \ldots, z_{j-1}, \zeta, z_{j+1}, \ldots, z_n) \in \Omega\}.$$

In other words, we require that f be holomorphic in each variable, separately.

Similarly, a function $g : \Omega \to \mathbb{C}^m$ is holomorphic if $\pi_k \circ g$ is holomorphic for each $0 \leq k \leq m$. Here, π_k is the projection onto the k^{th} coordinate.

As stated above, there are many equivalent definitions of holomorphic function. Anyone familiar with single variable complex analysis will recognize the obvious extension to the several variable case.

Theorem 2.1.2. *Let $D_r^n(w) = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid |z_j - w_j| < r, 1 \leq j \leq n\}$, $\Omega \subset \mathbb{C}^n$ be a domain, and $f : \Omega \to \mathbb{C}$ be continuous in each variable, separately. Then the following are equivalent.*

1. *f is holomorphic.*

2. *f satisfies the Cauchy-Riemann equations in each variable separately.*

3. *For each $w \in \Omega$, there is $r = r(w) > 0$ such that $\overline{D_r^n(w)} \subset \Omega$ and f can be written as an absolutely and uniformly convergent power series*

$$f(z) = \sum_{\alpha} a_\alpha (z - w)^\alpha$$

for all $z \in D_r^n(w)$.

4. *For each $w \in \Omega$ there is an $r = r(w) > 0$ such that $\overline{D_r^n(w)} \subset \Omega$ and*

$$f(z) = \frac{1}{(2\pi i)^n} \int_{|\zeta_n - w_n| = r} \cdots \int_{|\zeta_1 - w_1| = r} \frac{f(\zeta_1, \ldots, \zeta_n)}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} d\zeta_1 \ldots d\zeta_n$$

for all $z \in D_r^n(w)$.

A detailed discussion and proof of the above theorem can be found in [23]. At a glance, it appears that the definition is nothing more than the use of indicies. However, several complex variables is a rich and surprising field of study. Many results, such as the Riemann Mapping Theorem, are no longer true.

Before proceeding, we introduce the notion of equivalence in several complex variables.

Definition 2.1.3. Let $\Omega_1, \Omega_2 \subset \mathbb{C}^n$ be domains, and $f : \Omega_1 \to \Omega_2$ be a holomorphic function. If f is a bijection, we call it a *biholomorphism*, and, subsequently, call Ω_1 and Ω_2 *biholomorphic*. If $\Omega_1 = \Omega_2$, we call f an *automorphism*.

Automorphisms of domains will become very useful for us later, so we take this opportunity to denote the set of automorphisms of $\Omega \subset \mathbb{C}^n$ by $\mathrm{Aut}(\Omega)$.

2.1.2 Domains in $\mathbb{C}^n$

As we have already stated, we define a domain in $\mathbb{C}^n$ to be an open and connected set. Consider a holomorphic function f defined on a domain $\Omega \subset \mathbb{C}^n$. If we were able to extend f to a larger domain $\Omega' \supset \Omega$, then Ω would be of little interest to us. This is similar to the real-variable case. For instance, we are not interested in the behavior of the function $\frac{1}{1-x}$ on the interval $[11, 38]$; we want to study it on its maximal domain of definition.

So, if we have a domain $\Omega \subset \mathbb{C}^n$ on which every holomorphic function $f : \Omega \to \mathbb{C}$ extends to a larger domain, Ω is decidedly uninteresting. However, a domain that is a maximal domain of definition for some holomorphic function is interesting. These domains are called domains of holomorphy. More formally, we have:

Definition 2.1.4. A domain $\Omega \subset \mathbb{C}^n$ is called a *domain of holomorphy* if there do not exist nonempty open sets U_1, U_2, with U_2 connected, $U_2 \not\subset \Omega$, $U_1 \subset U_2 \cap \Omega$, such that for every holomorphic function f on Ω there is a holomorphic function g on U_2 such that $f = g$ on U_1.

Domains of holomorphy are typically not studied in a single complex variable text because every domain is a domain of holomorphy in this case. This does not hold in several complex variables, which motivates the study of such objects. The following is a classical example of Hartogs [18] describing this phenomenon.

Example 2.1.5. Consider the domain

$$\Omega = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1| < 3, |z_2| < 3\} \setminus \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1| \leq 1, |z_2| \leq 1\}.$$

We will show that every holomorphic function $f : \Omega \to \mathbb{C}$ extends to the domain

$$\Omega' = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1| < 3, |z_2| < 3\}.$$

For a fixed z_1 with $|z_1| < 3$, we write

$$f_{z_1}(z_2) = f(z_1, z_2) = \sum_{j=-\infty}^{\infty} a_j(z_1) z_2^j,$$

where the coefficeints of the Laurent expansion are given by

$$a_j(z_1) = \frac{1}{2\pi i} \int_{|\zeta|=2} \frac{f(z_1, \zeta)}{\zeta^{j+1}} d\zeta.$$

In particular, $a_j(z_1)$ depends holomorphically on z_1 by Morera's Theorem. But $a_j(z_1) = 0$ for all $j < 0$ and $1 < |z_1| < 3$. Therefore, by analytic continuation, $a_j \equiv 0$ for all $j < 0$. But then the series expansion becomes

$$\sum_{j=0}^{\infty} a_j(z_1) z_2^j,$$

and this series defines a holomorphic function f' on Ω' such that $f'|_{\Omega} = f$. Since f was chosen arbitrarily, we see that any holomorphic function on Ω can be extended to Ω', and thus Ω is not a domain of holomorphy.

One of the first problems in several complex variables is the Levi problem: to characterize the domains of holomorphy in terms of some geometric properties of the boundary. It can be shown that any geometrically convex domain is a domain of holomorphy.

Definition 2.1.6. A subset $X \subset \mathbb{R}^n$ is *(geometrically) convex* if for all $x, y \in X$,

$$\{xt + (1-t)y \mid t \in [0,1]\} \subset X.$$

It would certainly be nice if convexity were a necessary and sufficient condition, for it is purely geometric and makes no mention of holomorphicity. Unfortunately, convexity is not preserved under holomorphic mappings. However, convexity is almost enough, giving rise to the aforementioned Levi problem.

Since we are interested in extending holomorphic functions to larger domains, it is useful to describe domains in terms of functions.

Definition 2.1.7. Let $\Omega \subset M$ be a domain of a complex manifold. A function $f : \Omega \to \mathbb{R}$ is said to be *α-Hölder continuous* if there exist constants $C \geq 0$ and $\alpha > 0$ such that

$$|f(x) - f(y)| \leq C\|x - y\|^{\alpha}$$

for all $x, y \in \Omega$.

The function f is said to be $C^{k,\alpha}$ if it is both a C^k function in the usual sense and all of its k^{th} partial derivatives are α-Hölder continuous, where $0 < \alpha \leq 1$.

Definition 2.1.8. Let $\Omega \subset \mathbb{C}^n$ be a domain. A function $\rho : \mathbb{C}^n \to \mathbb{R}$ is said to be a *defining function for* Ω if

1. $\rho(z) < 0$ for all $z \in \Omega$
2. $\rho(z) > 0$ for all $z \notin \Omega$
3. $\nabla \rho(x) \neq 0$ for all $z \in \partial\Omega$

Definition 2.1.9. We say a domain $\Omega \subset M$ has $C^{k,\alpha}$ *boundary* if there is a $C^{k,\alpha}$ defining function ρ for Ω.

Example 2.1.10. Below are some classical domains and their defining functions. It is worth noting that all of these domains are convex.

1. The unit disk in $\mathbb{C}$ is given by $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$.
2. The half-plane in $\mathbb{C}$ is given by $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$.
3. The unit polydisk in $\mathbb{C}^n$ is given by

$$\mathbb{D}^n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid |z_1| < 1, \ldots, |z_n| < 1\}.$$

4. The ball in $\mathbb{C}^n$ centered at $a = (a_1, \ldots, a_n)$ of radius r is given by

$$B_r^n(a) = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid |z_1 - a_1|^2 + \cdots + |z_n - a_n|^2 < r\}.$$

2.2 Complex Manifolds

Here we recall some basic definitions of complex manifolds. Roughly speaking, a complex manifold (much like smooth manifolds in $\mathbb{R}^n$) is a topological space that, locally, looks like $\mathbb{C}^n$.

Definition 2.2.1. Let M be a connected, Hausdorff, and second countable topological space. Then M is called a *complex manifold* of (complex) dimension n if there is an open cover $\{U_\alpha\}$ of M such that for each α there exists a homeomorphism f_α from U_α to an open set in $\mathbb{C}^n$ such that for any pair α, β with $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \emptyset$, the map $f_\alpha \circ f_\beta^{-1}$ is a biholomorphism between $f_\beta(U_{\alpha\beta})$ and $f_\alpha(U_{\alpha\beta})$. We call U_α a *coordinate neighborhood.*

We now define the notion of holomorphic maps between complex manifolds.

Definition 2.2.2. A map $\varphi : M_1^n \to M_2^m$ between two complex manifolds is called *holomorphic at* $p \in M_1$ if there exist holomorphic coordinate neighborhoods (U_α, f_α) of p in M_1 and (V_β, g_β) of $\varphi(p)$ in M_2 such that the map $g_\beta \circ \varphi \circ f_\alpha^{-1}$ is holomorphic at $f_\alpha(p)$. The map φ is said to be *holomorphic* if it is holomorphic at all points in M_1.

Definition 2.2.3. A holomorphic map from a complex manifold M to $\mathbb{C}$ is called a *holomorphic function.* We denote the set of holomorphic functions of M by $\mathcal{O}(M)$. Naturally, it forms a ring.

Remark 2.2.4. When M is compact, $\mathcal{O}(M) = \mathbb{C}$. Indeed, if M is compact, then any holomorphic function on M must be constant via the Maximum Principle.

2.2.1 Examples of Complex Manifolds

Here we list some examples of complex manifolds. We will not provide all of the details, but the interested reader can look at [34].

Example 2.2.5. (Riemann Surfaces). A complex manifold of dimension one is called a *Riemann surface.* The Uniformization Theorem for Riemann Surfaces states that there are only three simply connected Riemann surfaces: the complex plane, the complex projective plane $\mathbb{CP}^1$, and the unit disk $\mathbb{D}$.

Example 2.2.6. (Domains in $\mathbb{C}^n$). The complex Euclidean space $\mathbb{C}^n$ is a complex manifold of dimension n. A domain (open, connected subset) in $\mathbb{C}^n$ is also a complex n-manifold. This provides a large class of noncompact examples. The unit ball in $\mathbb{C}^n$ is one such example.

Example 2.2.7. (Complex Projective Space). The complex projective space $\mathbb{CP}^n$ is the set of all complex lines passing through the origin in $\mathbb{C}^{n+1}$. That is, $\mathbb{CP}^n$ is the set of all 1-dimensional complex linear subspaces of $\mathbb{C}^{n+1}$. Complex projective space can also be realized as the quotient of $\mathbb{C}^{n+1} \setminus \{0\}$ by the equivalence relation

$$(z_1, \ldots, z_{n+1}) \sim (\lambda z_1, \ldots, \lambda z_n), \quad \lambda \in \mathbb{C}^*.$$

Example 2.2.8. (Stein Manifolds). A *Stein manifold* is a closed complex submanifold in $\mathbb{C}^n$. Equivalently, any complex manifold M satisfying the following is a Stein manifold.

1. M is *holomorphically convex*, i.e., for any compact $K \subset M$, the set

$$\widehat{K} = \left\{ x \in M \mid |f(x)| \leq \sup_K |f|, \text{ for all } f \in \mathcal{O}(M) \right\}$$

   is also compact.

2. M is *holomorphically separable*, i.e, for any two distinct points $x, y \in M$, there is $f \in \mathcal{O}(M)$ such that $f(x) \neq f(y)$.

3. Given any $x \in M$, there exist $f_1, \ldots, f_n \in \mathcal{O}(M)$ such that $(f_1, \ldots, f_n)$ gives a holomorphic coordinate in a neighborhood of x.

By the Maximum Principle, a compact complex manifold does not admit any bounded, nonconstant holomorphic functions. So a Stein manifold is always noncompact. In this way, Stein manifolds can be viewed as generalizations of the noncompact Riemann surfaces in higher dimensions. Stein manifolds are also generalizations of domains of holomorphy, and, as such, are the subject of extensive study in the field of several complex variables.

2.2.2 The Almost Complex Structure

The following is adapted from [34], and many of the omitted details can be found there.

Let M be an n-dimensional complex manifold. Since $\mathbb{C}^n \cong \mathbb{R}^{2n}$, and biholomorphisms are diffeomorphisms, we have that M is also a (real) $2n$-dimensional differentiable manifold, denoted by $M_{\mathbb{R}}$. We call $M_{\mathbb{R}}$ the *underlying differentiable manifold* of the complex manifold M.

Definition 2.2.9. The complex manifold M is called a *complex structure* on the underlying differentiable manifold $M_{\mathbb{R}}$.

The complex structure on $M_{\mathbb{R}}$ induces a splitting of the complexifictation of the tangent bundle $TM_{\mathbb{R}}^{\mathbb{C}}$ into the sum of complex subbundles of equal rank. Denote this splitting by

$$TM_{\mathbb{R}}^{\mathbb{C}} = TM^{(1,0)} \oplus TM^{(0,1)}$$

and note that each summand is the complex conjugate of the other. We will describe this splitting using local coordinates.

Let $(z_1, \ldots, z_n)$ be a local holomorphic coordinate in a neighborhood U of $p \in M$, and write $z_j = x_j + iy_j$. Here $i = \sqrt{-1}$ is the imaginary constant. Then

$$(x_1, \ldots, x_n, y_1, \ldots, y_n)$$

is a smooth coordinate in U, and

$$\left\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}\right\}$$

gives a local frame of the tangent bundle $TM_{\mathbb{R}}$.

For $1 \leq j \leq n$, let

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right).$$

Then $TM^{(1,0)}$ is the complex subbundle of $TM_{\mathbb{R}}^{\mathbb{C}}$ spanned by

$$\left\{\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}\right\},$$

and $TM^{(0,1)}$ is spanned by

$$\left\{\frac{\partial}{\partial \overline{z}_1}, \ldots, \frac{\partial}{\partial \overline{z}_n}\right\}.$$

As usual, these two subbundles are independent of choice of coordinates.

Now consider the bundle isomorphism on $TM_{\mathbb{R}}$ defined by

$$\begin{aligned} J : TM_{\mathbb{R}} &\to TM_{\mathbb{R}} \\ \frac{\partial}{\partial x_j} &\mapsto \frac{\partial}{\partial y_j} \\ \frac{\partial}{\partial y_j} &\mapsto -\frac{\partial}{\partial x_j} \end{aligned}$$

for each $1 \leq j \leq n$. We can extend J linearly over $\mathbb{C}$ to an isomorphism of $TM_{\mathbb{R}}^{\mathbb{C}}$. By an abuse of notation, call this new isomorphism J. We then have

$$J\frac{\partial}{\partial z_j} = i\frac{\partial}{\partial z_j} \quad \text{and} \quad J\frac{\partial}{\partial \overline{z}_j} = -i\frac{\partial}{\partial \overline{z}_j}$$

for each $1 \leq j \leq n$. Again, this definition of J is independent of choice of coordinates.

For a real tangent vector X on $M_{\mathbb{R}}$, the vector $X - iJX$ is in $TM^{(1,0)}$, and any vector in $TM^{(1,0)}$ is of this form. Thus the map that sends X to $X - iJX$ defines an isomorphism between $TM_{\mathbb{R}}$ and $TM^{(1,0)}$.

Definition 2.2.10. Sections of $TM_{\mathbb{R}}^{\mathbb{C}}$ ($TM^{(1,0)}$, or $TM^{(0,1)}$) are called *complex vector fields* (*of type* $(1,0)$ or $(0,1)$) on M.

Remark 2.2.11. From the definition and preceding discussion, it is clear that a complex vector field is of type $(1,0)$ if and only if it is in the form $X - iJX$ for some real vector field X.

Definition 2.2.12. On a complex manifold M, the bundle of type $(1,0)$ complex vector fields $TM^{(1,0)}$ is a holomorphic vector bundle called the *holomorphic tangent bundle of* M, and will be denoted T_M. Similarly, the complex dual bundle of T_M is called the *holomorphic cotangent bundle of* M, and will be denoted T_M^*.

Definition 2.2.13. An endomorphism J of the tangent bundle of a differentiable manifold N satisfying $J^2 = -I$, where I denotes the identity map of TN, is called an *almost complex structure* on N.

The map J constructed above satisfies $J^2 = -I$ on $TM_{\mathbb{R}}$, and is called the almost complex structure on $TM_{\mathbb{R}}$ induced by the complex structure of M.

If a differentiable manifold N admits an almost complex structure, its (real) dimension is necessarily even, and it is orientable. This means that the tangent bundle TN has a reduction from the $GL(2n, \mathbb{R})$-structure to the $GL(n, \mathbb{C})$-structure. Hence, the existence problem for an almost complex structure resides purely in the field of algebraic topology. This problem is understood fairly well; for example, it can be shown that the only spheres that admit almost complex structures are S^2 and S^6.

2.2.3 Kähler Manifolds

In this section we define a special class of complex manifolds called Kähler manifolds. These manifolds have significantly more structure than other complex manifolds in that they possess three mutually compatible structures: a complex structure, a Riemannian structure, and a symplectic structure. For our purposes, we need only consider Kähler manifolds from a complex analytic viewpoint.

We begin with the definition of Hermitian manifolds, which are the complex analogue of Riemannian manifolds.

Definition 2.2.14. A *Hermitian metric* on a complex manifold, M^n is a Hermitian metric on the holomorphic tangent bundle, T_M. That is, a covariant 2-tensor

$$h = \sum_{i,j=1}^{n} h_{i\bar{j}} dz_i \otimes d\overline{z_j}$$

where $(z_1, \ldots, z_n)$ is a local holomorphic coordinate, and $(h_{i\bar{j}})$ is an $n \times n$ matrix of smooth functions which is Hermitian symmetric and positive definite.

A *Hermitian manifold* is a complex manifold with a Hermitian metric on its holomorphic tangent space.

Remark 2.2.15. Every complex manifold admits a Hermitian metric. The construction uses partition of unity, much like the Riemannian case.

Definition 2.2.16. Let h be a Hermitian metric. Its associated *Kähler form* is denoted by

$$\omega_h = -\frac{1}{2}\mathrm{Im}(h) = \frac{\sqrt{-1}}{2}\sum_{i,j=1}^{n} h_{i\bar{j}} dz_i \wedge d\overline{z_j}$$

Definition 2.2.17. A Hermitian metric h on M^n is a called a *Kähler metric* if $d\omega_h = 0$; equivalently, if the Hermitian connection has vanishing torsion tensor.

A *Kähler manifold* is a complex manifold equipped with a Kähler metric.

Theorem 2.2.18 (Criteria for Kähler Manifolds)**.** *Let (M^n, h) be a Hermitian manifold. Then the following are equivalent.*

1. *h is Kähler*
2. $\dfrac{\partial h_{i\bar{j}}}{\partial z_k} = \dfrac{\partial h_{k\bar{j}}}{\partial z_i}$, *$1 \leq i, j, k \leq n$ under any local holomorphic coordinate system*
3. *the Kähler form ω_h is closed, i.e. $d\omega_h = 0$*
4. *For any $p \in M$, there exists a local holomorphic coordinate $(z_1, \ldots, z_n)$ in a neighborhood of p such that $h_{i\bar{j}}(p) = \delta_{ij}$, $dh_{i\bar{j}}(p) = 0$. Such a coordinate is said to be normal at p.*

The following example illustrates the aforementioned additional structure possessed by Kähler manifolds. See [17] for more details.

Example 2.2.19. Consider the real and complex Laplacians: The real Laplacian Δ of a real valued function f is

$$\Delta f = \frac{1}{\sqrt{|g|}}\sum_{i,j}\frac{\partial f}{\partial x^j}\left(\sqrt{|g|}g^{ij}\frac{\partial f}{\partial x^i}\right)$$

where $g = \det(g_{ij})$ and (g^{ij}) is the inverse matrix of (g_{ij}). The complex Laplacian $\Box$ is

$$\Box f = 2\sum_{i,j} G^{ij} \frac{\partial^2 f}{\partial z^i \partial \overline{z}^j}.$$

If the metric G is Kähler, then $\Delta f = 2\Box f$. This implies that the real and imaginary parts of a holomorphic function on a Kähler manifold are harmonic.

Definition 2.2.20. The *Levi form* of a real valued function f on a complex manifold is

$$Lf = 4\sum_{i,j} \frac{\partial^2 f}{\partial z^i \partial \overline{z}^j} dz^i d\overline{z}^j.$$

Remark 2.2.21. Lf is (up to a constant factor) just the Hermitian tensor associated to the $(1,1)$-form $\partial\overline{\partial} f$. A more fundamental definition for the complex Laplacian is

$$\Box f = \frac{1}{2}\operatorname{Tr}(Lf).$$

Therefore, when the manifold is Kähler, $\Delta f = \operatorname{Tr}(Lf)$.

A remarkable property of Kähler manifolds is illustrated in a result by Wu, which says that the universal cover of any complete Kähler manifold with nonpositive sectional curvature is a Stein manifold. Proofs of this result can be found in [17] and [34].

Theorem 2.2.22 (Wu's Thm, [17])**.** *A simply connected, complete Kähler manifold with nonpositive sectional curvature is a Stein manifold.*

Given that Stein manifolds are generalizations of domains of holomorphy, Wu's Theorem makes clear the desire to study Kähler manifolds satisfying the above conditions.

2.3 Chern Classes

This section is adapted from [34] and [25]. As such, many of the missing details can be found in those sources.

Chern classes, named for Shiing-Shen Chern [4], are characteristic classes that act as topological invariants associated with vector bundles on a smooth manifold. They can be constructed in various ways. We will describe their construction via the Euler class here, but will point out their relation to Ricci curvature in Section 3.2. Chern classes, in particular the first Chern class, will be of use in the proof of Theorem 1.0.1.

We begin with the Euler class for orientable vector bundles. Let X be a smooth manifold, and E an oriented vector bundle over X of real rank m. Let $\overline{E}$ be the S^m bundle over X obtained by compactifying the fibers. Since E is oriented, we have a consistent choice of a preferred generator $\sigma_x \in H^m(\overline{E_x}, \mathbb{Z})$ for each $x \in X$. By "consistent" we mean that for any $x \in X$, there exists a neighborhood $U \subset X$ of x and a frame $\{s_1, \ldots, s_m\}$ of E over U which gives a positive basis E_y at any $y \in U$. The Thom Isomorphism Theorem (see [25]) states that there is a unique $\sigma \in H^m(\overline{E}, \mathbb{Z})$ such that $\sigma|_{\overline{E_x}} = \sigma_x$ for each x.

Definition 2.3.1. If $\iota : E \hookrightarrow \overline{E}$ is the inclusion map, then

$$\iota^*\sigma \in H^m(E, \mathbb{Z}) \cong H^m(X, \mathbb{Z})$$

is called the *Euler class* of E. We will denote it by $e(E)$.

Now let E be a rank r complex vector bundle over a smooth manifold X. Since $\mathbb{C}^r$ can be identified with an oriented $\mathbb{R}^{2r}$, E is also an oriented real vector bundle of rank $2r$. Let $X' \subset E$ be the complement of the zero section of E, and let $\pi : X' \to X$ be the projection map. Lastly, let E' be the rank $(r-1)$ complex vector bundle over X' whose fiber at $v \in E_x$ is the quotient space $E_x/\mathbb{C}v$, where $\mathbb{C}v$ is the 1-dimensional vector space spanned by the vector $v \neq 0$.

Definition 2.3.2. For any integer k, the k^{th} *Chern class* $c_k(E) \in H^{2k}(X, \mathbb{Z})$ of E is defined by induction on the rank of E:

$$c_k(E) = \begin{cases} 0, & \text{if } k < 0 \text{ or } k > r; \\ 1, & \text{if } k = 0; \\ e(E), & \text{if } k = r; \\ \pi^* c_k(E'), & \text{if } 0 < k < r. \end{cases}$$

We will adopt the convention that for any complex manifold M, we write $c_k(M)$ to mean $c_k(T_M)$, and call it the k^{th} *Chern class* of M.

For the purposes of proving Theorem 1.0.1, we will be interested in the first Chern class, $c_1(M)$. The first Chern class turns out to be a complete invariant with which to classify complex line bundles.

Chapter 3

The Geometry of Domains on Complex Manifolds

In this chapter, we introduce some geometric tools that will allow us to better understand domains on complex manifolds, and work with them in a more concrete way. We begin with some motivation for the first section via the Schwarz Lemma from single variable complex analysis.

Lemma 3.0.1. *Let $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic function such that $f(0) = 0$ and $|f(z)| \leq 1$ for all $z \in \mathbb{D}$. Then $|f'(0)| \leq 1$ and $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$. Moreover, if $|f'(0)| = 1$, or if $|f(z)| = |z|$ for some nonzero z, then there is a constant $c, |c| = 1$, such that $f(w) = cw$ for all $w \in \mathbb{D}$.*

A simple corollary of the Schwarz Lemma is the following inequality (see [6]).

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}$$

This inequality bears a striking resemblance to the Poincaré metric.

Definition 3.0.2. The Poincaré metric on the unit disk $\mathbb{D}$ is defined by

$$P_{\mathbb{D}}(z, v) = \frac{|v|}{1 - |z|^2}.$$

This metric yields a pseudodistance

$$d^P_{\mathbb{D}}(z, w) = \tanh^{-1} \left| \frac{z - w}{1 - z\overline{w}} \right| = \frac{1}{2} \log \left(\frac{|1 - z\overline{w}| + |z - w|}{|1 - z\overline{w}| - |z - w|} \right).$$

Definition 3.0.3. A *pseudodistance* or *pseudometric* is a distance that doesn't distinguish points. That is, $d(x, y) = 0$ does not always imply that $x = y$.

Looking at both the previously stated inequality and the definition of the Poincaré metric, the resemblance is clear. In fact, it is the case that on the unit disk, the Poincaré metric has a distance decreasing property with respect holomorphic functions. That is, the distance between the holomorphic images of points is at most the distance between the original points. It follows that automorphisms of the disk are, necessarily, isometries. This property is the motivation for the famous Kobayashi metric.

3.1 The Kobayashi Pseudometric

The Kobayashi pseudometric is a generalization of the Poincaré metric to arbitrary domains in $\mathbb{C}^n$, and will be the the most used metric in this work.

Definition 3.1.1. For a domain $\Omega \subset \mathbb{C}^n$, the *Kobayashi-Royden pseudometric* is defined by

$$k_\Omega(x, v) = \inf \left\{ \frac{1}{\lambda} \mid f \in \mathrm{Hol}(\mathbb{D}, \Omega), f(0) = x, f'(0) = \lambda v, \lambda > 0 \right\}.$$

We let $d_\Omega^K(x,y)$ denote the induced Kobayashi pseudodistance of Ω. It has the explicit form

$$d_\Omega^K(z,w) = \inf_\gamma \int_0^1 k_\Omega(\gamma(t), \gamma'(t))dt.$$

Here, $z, w \in \Omega$ and $\gamma : [0,1] \to \Omega$ is a curve starting at z and ending at w.

Definition 3.1.2. A complex manifold M is called *(Kobayashi) hyperbolic* if d_M^K is indeed a distance. M is called *completely hyperbolic* if d_M^K is a complete distance.

The following example shows that not every domain is hyperbolic.

Example 3.1.3. Consider the complex plane $\mathbb{C}$ equipped with the Kobayashi pseudodistance $d_\mathbb{C}^K$. For any two distinct complex numbers z and w, there exists a holomorphic function $f : \mathbb{D} \to \mathbb{C}$ and a real number r such that $f(0) = z$ and $f(r) = w$. This can be easily accomplished by a rescaling and a rotation. However, we could make r arbitrarily small, which yields $d_\mathbb{C}^K(z,w) = 0$. Therefore, the complex plane is not hyperbolic.

A detailed look at hyperbolic manifolds can be found in [21].

Remark 3.1.4. A domain containing no complex lines is hyperbolic, and so bounded domains are hyperbolic. We, therefore, restrict ourselves to studying such domains.

As mentioned earlier, the Kobayashi pseudometric has a distance decreasing property which will be useful in the proof of our main result.

Proposition 3.1.5. *Suppose $\Omega_1 \subset \mathbb{C}^n$ and $\Omega_2 \subset \mathbb{C}^m$ are domains. If $f : \Omega_1 \to \Omega_2$ is holomorphic, then*

$$d_{\Omega_2}^K(f(p), f(q)) \leq d_{\Omega_1}^K(p,q)$$

for all $p, q \in \Omega_1$.

Proof. Suppose that $g : \mathbb{D} \to \Omega_1$ is such that $g(0) = x$ and $g'(0) = \lambda v$. Then $(g \circ f)(0) = f(x)$ and $(g' \circ f')(0) = f(\lambda v)$. We then have the following set inclusion

$$\left\{\frac{1}{\lambda} \mid h \in \mathrm{Hol}(\mathbb{D}, \Omega_1), h(0) = x, h'(0) = \lambda v, \lambda > 0\right\} \subset \left\{\frac{1}{\lambda} \mid h \in \mathrm{Hol}(\mathbb{D}, \Omega_2),\right.$$

$$\left. h(0) = x, h'(0) = \lambda v, \lambda > 0\right\}$$

Thus $k_{\Omega_2}(f(x), f'(v)) \leq k_{\Omega_1}(x, v)$.

It then follows that

$$\begin{aligned} d^K_{\Omega_2}(f(p), f(q)) &= \inf_\gamma \int_0^1 k_{\Omega_2}((f \circ \gamma)(t), (f' \circ \gamma')(t))dt \\ &\leq \inf_\gamma \int_0^1 k_{\Omega_1}(\gamma(t), \gamma'(t))dt \\ &= d^K_{\Omega_1}(p, q). \end{aligned}$$

□

As a simple corollary, the Kobayashi pseudometric is a biholomorphic invariant. That is, all inequalities in Proposition 3.1.5 can be replaced by equalities when f is a biholomorphism.

As it is pertinent to our main result, we mention one last result concerning the Kobayashi metric on bounded domains. Recall that a *proper metric space* is one in which bounded sets are relatively compact.

Proposition 3.1.6. *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain such that* $\mathrm{Aut}(\Omega)$ *acts cocompactly on Ω. Then (Ω, d^K_Ω) is a proper metric space.*

3.2 The Curvature of a Hermitian or Kähler Metric

The material presented here is adapted from [34]. All of the differnetial geometric terminology used in this section (and not defined in this document) can be found in [2] and [34].

Suppose (E, h) is a Hermitian vector bundle over a complex manifold M. Let $\{e_1, \ldots, e_r\}$ be a frame for E, and suppose that $u = \sum u_j e_j$ and $v = \sum v_j e_j$ are sections in E. Define the $(1,1)$-form $\Theta_{u\overline{v}}$ by

$$\Theta_{u\overline{v}} = \sum_{j,k,\ell=1}^{r} \Theta_{j\ell} h_{\ell\overline{k}} u_j \overline{v_k}.$$

Certainly, $\Theta_{u\overline{v}}$ is independent of choice of frame.

Definition 3.2.1. The Hermitian bundle (E, h) is said to be *positively curved* if for any nowhere zero section u of E, the $(1,1)$-form $i\Theta_{u\overline{u}} > 0$ in the domain where u is defined. Similarly, (E, h) is called *nonnegatively* (*negatively*, or *nonpositively*) *curved* if $i\Theta_{u\overline{u}}$ is so.

For sections u, v of E and tangent vectors X, Y of type $(1,0)$ in M, we will write

$$R_{X\overline{Y}u\overline{v}} = \Theta_{u\overline{v}}(X, \overline{Y}).$$

Since $\overline{\Theta_{u\overline{v}}} = -\Theta_{v\overline{u}}$, we have

$$\overline{R_{X\overline{Y}u\overline{v}}} = R_{Y\overline{X}v\overline{u}}.$$

Definition 3.2.2. We call $R_{X\overline{X}u\overline{u}}$ the *curvature* of (E, h) in the direction of u and X.

Remark 3.2.3. From the definition, we see that (E, h) is positively curved if and only if $R_{X\overline{X}u\overline{u}} > 0$. For ease of reading, we will write this as $\Theta > 0$ or $R > 0$.

We now restrict our attention to the tangent bundle. Again, let h be a Hermitian metric on a complex manifold M. Then the curvature becomes a covariant 4-tensor

$$R_{X\overline{Y}Z\overline{W}} = \Theta_{Z\overline{W}}(X, \overline{Y})$$

where X, Y, Z, W are tangent vectors of type $(1, 0)$.

Definition 3.2.4.

1. The normalized curvature in the direction of X and Z,

$$B(X, Z) = \frac{R_{X\overline{X}Z\overline{Z}}}{|X|^2|Z|^2},$$

   is called the *bisectional curvature* of h in the directions of X and Z.

2. The curvature

$$H(X) = B(X, X) = \frac{R_{X\overline{X}X\overline{X}}}{|X|^4}$$

   is called the *holomorphic sectional curvature* of h in the direction of X.

Here, $|X|^2$ means $\langle X, X\rangle$, where $\langle , \rangle$ is the underlying Riemannian metric of M.

The covariant 4-tensor R satisfies the symmetry property

$$\overline{R_{X\overline{Y}Z\overline{W}}} = R_{Y\overline{X}W\overline{Z}},$$

but in general we cannot swap the first two positions with the last two.

Remark 3.2.5. This R is not the Riemannian curvature tensor of the Riemannian metric $\mathrm{Re}(h)$ unless h is Kähler.

3.2.1 The Kähler Case

Let M be a Kähler manifold with h its Kähler metric, and J its almost complex structure. Denote by $\langle , \rangle$ the underlying Riemannian metric, by ∇ the Riemannian connection, and by R the Riemannian curvature tensor. Clearly, the curvature tensor R of h is just the linear extension over $\mathbb{C}$ of the Riemannian curvature of the underlying Riemannian metric. Since h is a Kähler metric, we have that $\langle Ju, Jv \rangle = \langle u, v \rangle$ and $\nabla_u(Jv) = J\nabla_u v$ for any two real vector fields u and v.

With the additional structure J, the Riemannian curvature satisfies an additional symmetry

$$R(u, v, Jz, Jw) = R(u, v, z, w)$$

for any four real vector fields.

In practice, it is often more convenient to write R in terms of its complex components. The map

$$u \to \tilde{u} = \frac{1}{\sqrt{2}}(u - iJu)$$

is a linear isomorphism over $\mathbb{R}$ between $TM_{\mathbb{R}}$ and T_M (the tangent bundle of the underlying differentiable manifold of M and the holomorphic tangent bundle of M). If we extend $\langle , \rangle$ linearly over $\mathbb{C}$ to $T_M \oplus \overline{T_M}$, then $\langle , \rangle$ becomes a complex bilinear form, and

$$h(X, Y) = 2\langle X, \overline{Y} \rangle, \quad \langle X, Y \rangle = \langle \overline{X}, \overline{Y} \rangle = 0$$

for any $X, Y \in T_M$.

Next, let us extend the Riemannian curvature tensor R linearly over $\mathbb{C}$ to a complex quadrilinear map on $T_M \oplus \overline{T_M}$. Then for any $X, Y \in T_M$, we have

$$R(X, Y, \cdot, \cdot) = R(\overline{X}, \overline{Y}, \cdot, \cdot) = R(\cdot, \cdot, X, Y) = R(\cdot, \cdot, \overline{X}, \overline{Y}) = 0.$$

Since R is skew symmetric with respect to its first two, or last two, positions, the only nontrivial components are $R(X, \overline{Y}, Z, \overline{W})$, where $X, Y, Z, W \in T_M$. It then follows from the first Bianchi identity that

$$R(X, \overline{Y}, Z, \overline{W}) = R(Z, \overline{Y}, X, \overline{W}) = R(X, \overline{W}, Z, \overline{Y})$$

for any $X, Y, Z, W \in TM^{(1,0)}$.

For real tangent vectors u and v, let $X = \tilde{u}$ and $Y = \tilde{v}$. Then

$$\begin{aligned} R(X, \overline{X}, Y, \overline{Y}) &= -R(u, Ju, v, Jv) \\ &= R(v, u, Ju, Jv) + R(Ju, v, u, Jv) \\ &= R(v, u, u, v) + R(Ju, v, v, Ju). \end{aligned}$$

Therefore, when X and Y are nonzero, we have

$$B(X, Y) = \frac{|Ju \wedge v|^2}{|u|^2|v|^2} K(Ju \wedge v) + \frac{|u \wedge v|^2}{|u|^2|v|^2} K(u \wedge v),$$

where K is the sectional curvature. Thus

$$H(X) = B(X, X) = K(u \wedge Ju).$$

So, for a Kähler manifold, the bisectional curvature B is dominated by the sectional curvature K in the sense that B will be positive (negative, nonpositive, or nonnegative) if K is so. Additionally, the holomorphic sectional curvature is just the sectional curvature in a 2-plane.

Lastly, we discuss the Ricci curvature.

Definition 3.2.6. The *Ricci curvature tensor* of a Kähler manifold M is defined to be the trace of the Riemannian curvature tensor R:

$$r(X, \overline{Y}) = \sum_{j=1}^{n} R(X, \overline{Y}, e_j, \overline{e_j})$$

for any unitary frame $\{e_j\}$.

This definition is clearly independent of choice of frame. The Ricci curvature is also Hermitian symmetric:

$$\overline{r(X, \overline{Y})} = r(Y, \overline{X}).$$

In fact, it is just the Ricci tensor of the Riemannian metric.

Definition 3.2.7. Define the $(1,1)$-form on M

$$\eta_h = \frac{i}{2\pi} \sum_{j,k=1}^{n} r(e_j, \overline{e_k}) \varphi_j \wedge \overline{\varphi_k}$$

where $\{e_j\}$ is any tangent frame, and $\{\varphi_j\}$ is the dual coframe. It is called the *Ricci form of h*.

Again, this definition is independent of choice of frame. Under a local holomorphic coordinate $(z_1, \ldots, z_n)$, we have

$$\eta_h = \frac{i}{2\pi} \sum r_{j\overline{k}} dz_j \wedge d\overline{z_k} = -\frac{i}{2\pi} \partial\overline{\partial} \log \det h.$$

More generally, for a Hermitian bundle (E, h) over M, the trace

$$\mathrm{Tr}(\Theta) = \mathrm{Tr}\, \overline{\partial}(\partial h h^{-1}) = \overline{\partial}\partial \log \det h$$

is independent of choice of local frames. Hence, it is a globally defined closed $(1,1)$-form on M. The $\frac{i}{2\pi}$ multiple of it is called the *Ricci form of* E, and is denoted η_h. Thus η represents the first Chern class $c_1(E)$. This fact will become important in the proof of Proposition 6.3.4.

3.3 A Rescaling Method

To prove the main result, we will need to rescale a domain and look at its limit with respect to the local Hausdorff topology. This method of rescaling is used by Zimmer [35], and we use it much the same way. A detailed treatment of this technique can be found in [11].

Definition 3.3.1. Here we define the *local Hausdorff topology* on the set of all convex domains in $\mathbb{C}^n$. First, define the *Hausdorff distance* between two compact sets $X, Y \subset \mathbb{C}^n$ by

$$d_H(X,Y) = \max\left\{\max_{x\in X}\min_{y\in Y}\|x-y\|, \max_{y\in Y}\min_{x\in X}\|y-x\|\right\}.$$

To obtain a topology on the set of all convex domains in $\mathbb{C}^n$, we consider the *local Hausdorff pseudodistances* defined by

$$d_H^{(R)}(X,Y) = d_H\left(X \cap \overline{B_R(0)}, Y \cap \overline{B_R(0)}\right),\ \ R > 0.$$

Then a sequence of convex domains Ω_j converges to a convex domain Ω if there exists some $R_0 \geq 0$ such that

$$\lim_{j\to\infty} d_H^{(R)}\left(\overline{\Omega_j}, \overline{\Omega}\right) = 0$$

for all $R \geq R_0$.

Remark 3.3.2. The Kobayashi metric is continuous with respect to the local Hausdorff topology.

Theorem 3.3.3. *Suppose $\Omega_j \subset \mathbb{C}^n$ is a sequence of convex domains and $\Omega = \lim_{j\to\infty} \Omega_j$ in the local Hausdorff topology. Assume the Kobayashi metric is nondegenerate on Ω and each Ω_j. Then*

$$d_\Omega^K(p,q) = \lim_{j\to\infty} d_{\Omega_j}^K(p,q)$$

for all $p, q \in \Omega$. Moreover, the convergence is uniform on compact subsets of $\Omega \times \Omega$.

Example 3.3.4. Consider the domain $\Omega = \{z \in \mathbb{C} \mid |z-i|^2 < 1\}$. This is simply the unit disk translated up so that 0 sits on the boundary. We will rescale this domain using the sequence of affine transformations $\Lambda_j(z) = jz$. Denote $\lim_{j\to\infty} \Lambda_j(\Omega)$ by D. For any $\zeta \in \mathbb{C}$ with $\mathrm{Im}(\zeta) > 0$, it follows that for sufficiently large j, $\zeta \in \Lambda_j(\Omega)$. Thus the upper half plane $\mathbb{H} \subset D$.

Now consider $\zeta' \in \mathbb{C}$ with $\mathrm{Im}(\zeta') < 0$. Since every point in Ω has positive real part, and we are rescaling by positive integers, we must have that $\zeta' \notin D$. Therefore, $D = \mathbb{H}$ in the local Hausdorff topology.

Remark 3.3.5. While it is true that the unit disk is biholomorphic to the upper half plane, the previous example does not show it.

The rescaling method is most useful when the rescaled domain is actually biholomorphic to the original one. A theorem of Frankel [11] gives a condition for this to be true when the domain is convex.

Theorem 3.3.6. *Suppose $\Omega \subset \mathbb{C}^n$ is a convex domain which does not contain a complex line in its boundary. Let $K \subset \Omega$ be compact and $\varphi_j \in \mathrm{Aut}(\Omega)$. If there exists a sequence $p_j \in K$ and complex affine maps Λ_j such that*

1. $\lim_{j \to \infty} \Lambda_j(\Omega) = D$

2. $\varphi_j(p_j) \to p \in D$

where D does not contain a complex line in its boundary, then Ω is biholomorphic to D.

Example 3.3.7. Consider the following sequence of automorphisms of the unit disk:

$$\varphi_j(z) = \frac{z + \frac{j-1}{j}}{1 + \frac{j-1}{j} z}$$

Note that $\lim_{j \to \infty} \varphi_j(0) = 1$. Define the *Frankel rescaling sequence* to be

$$f_j(z) = [d\varphi_j|_0]^{-1}(\varphi_j(z) - \varphi_j(0)).$$

We can then explicitly compute the Frankel rescaling map.

$$\begin{aligned}
f_j(z) &= [d\varphi_j|_0]^{-1}(\varphi_j(z) - \varphi_j(0)) \\
&= \left(\frac{1}{1 - \left| \frac{j-1}{j} \right|^2} \right) \left(\frac{z + \frac{j-1}{j}}{1 + \frac{j-1}{j} z} - \frac{j-1}{j} \right) \\
&= \left(\frac{1}{1 - \left| \frac{j-1}{j} \right|^2} \right) \left(\frac{z + \frac{j-1}{j} - \frac{j-1}{j} - \left(\frac{j-1}{j} \right)^2 z}{1 + \frac{j-1}{j} z} \right) \\
&= \frac{z \left(1 - \left(\frac{j-1}{j} \right)^2 \right)}{\left(1 - \left| \frac{j-1}{j} \right|^2 \right) \left(1 + \frac{j-1}{j} z \right)} \\
&= \frac{z}{1 + \frac{j-1}{j} z}
\end{aligned}$$

We then have that

$$f(z) = \lim_{j \to \infty} f_j(z) = \frac{z}{1 + z.}$$

Moreover, $f(0) = 0$. This gives an explicit biholomorphism from the unit disk to the upper half plane. We refer the reader to Frankel's paper [11] for a detailed use of this rescaling sequence.

We conclude this section with an example that will appear in the proof on Theorem 1.0.1.

For $\alpha > 0$, define

$$\mathcal{P}_\alpha = \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n \mid \mathrm{Re}(z_1) > \alpha \sum_{j=2}^{n} |z_j|^2 \right\}$$

Note that $\mathcal{P}_\alpha$ is biholomorphic to a ball.

Example 3.3.8. Fix $r > 0$, a sequence $r_j > 0$ converging to 0, and the sequence of linear maps

$$\Lambda_j(z_1, \ldots, z_n) = \left(\frac{1}{r_j} z_1, \frac{1}{\sqrt{r_j}} z_2, \ldots, \frac{1}{\sqrt{z_j}} z_n \right).$$

Then

$$\mathcal{P}_{1/(2r)} = \lim_{j \to \infty} \Lambda_j(B_r(re_1))$$

in the local Hausdorff topology.

3.4 The Bergman Kernel and Some Properties

Definition 3.4.1. Let μ denote the standard Lebesgue measure on $\mathbb{C}^n$. For a domain $\Omega \subset \mathbb{C}^n$, let

$$H^2(\Omega) = \left\{ f \in \mathrm{Hol}(\Omega, \mathbb{C}) \;\middle|\; \int_\Omega |f|^2 d\mu < \infty \right\}.$$

Then $H^2(\Omega)$ is a Hilbert space. If $\{\phi_j\}$ is an orthonormal basis of $H^2(\Omega)$, then the function

$$\kappa_\Omega : \Omega \times \Omega \to \mathbb{C}$$

$$\kappa_\Omega(z, w) = \sum_j \phi_j(z)\overline{\phi_j(w)}$$

is called the *Bergman kernel* of Ω.

The convergence of the series is absolute and uniform on compact subsets of $\Omega \times \Omega$, and for any $z \in \Omega$, the diagonal $\kappa_\Omega(z, z)$ is strictly positive.

Next we recall two properties of the Bergman kernel that we will use later in the proof of Theorem 1.0.1.

Proposition 3.4.2. *If* $\Omega_1 \subset \Omega_2 \subset \mathbb{C}^n$ *are domains, then*

$$\kappa_{\Omega_2}(z, z) \leq \kappa_{\Omega_1}(z, z)$$

for all $z \in \Omega_1$.

Proposition 3.4.3. *If* $\Omega_1, \Omega_2 \subset \mathbb{C}^n$ *are domains, and* $F : \Omega_1 \to \Omega_2$ *is a biholomorphism, then*

$$\kappa_{\Omega_1}(z, w) = \kappa_{\Omega_2}(F(z), F(w)) \det(F'(z))\overline{\det(F'(w))}$$

for all $z, w \in \Omega_1$.

Propositions 3.4.2 and 3.4.3 are typically referred to as the monotonicity property and change of variable formula, respectively. Proofs can be found in any standard several complex variables text. See, for instance, [23].

While we will not explicitly use the Bergman metric in our work, it would be unwise not to mention it since we have already defined the Bergman kernel.

Definition 3.4.4. We obtain the *Bergman metric* B from the Bergman kernel κ by way of

$$B = \partial\overline{\partial} \log \kappa(z, z).$$

That is, the $(ij)^{\text{th}}$ component of the Bergman metric is given by

$$B_{i\bar{j}} = (\log \kappa)_{i\bar{j}} = \frac{\partial^2}{\partial z_i \partial \overline{z}_j} \log \kappa(z, z).$$

Remark 3.4.5.

1. The Bergman metric is a Kähler metric which is invariant under biholomorphisms.

2. Every bounded domain admits the Bergman metric; the metric is nondegenerate.

Next is an example of the Bergman kernel and Bergman metric on the unit ball in $\mathbb{C}^n$. Full details of the following results can be found in [23].

Proposition 3.4.6. *The Bergman kernel of the unit ball $\mathbb{B}^n \subset \mathbb{C}^n$ is given by*

$$\kappa_{\mathbb{B}^n}(z,w) = \frac{n!}{\pi^n} \cdot \frac{1}{(1-z\overline{w})^{n+1}}.$$

Proposition 3.4.7. *The Bergman metric on the unit ball $\mathbb{B}^n \subset \mathbb{C}^n$ is given by*

$$B_{i\bar{j}}(z) = \frac{n+1}{(1-|z|^2)^2}\left[(1-|z|^2)\delta_{ij} + \overline{z}_i z_j\right].$$

Proof. We perform a routine calculation using the Bergman kernel for the unit ball from Proposition 3.4.6. The diagonal Bergman kernel on the ball is

$$\kappa_{\mathbb{B}^n}(z,z) = \frac{n!}{\pi^n} \cdot \frac{1}{(1-z\overline{z})^{n+1}}.$$

So

$$\begin{aligned}\log \kappa(z,z) &= \log\left(\frac{n!}{\pi^n}\right) - (n+1)\log(1-z\overline{z}) \\ &= \log\left(\frac{n!}{\pi^n}\right) - (n+1)\log\left(1-\sum_{k=1}^{n} z_k\overline{z}_k\right).\end{aligned}$$

Taking derivatives yields

$$\begin{aligned}(\log \kappa(z,z))_i &= (n+1)\frac{\overline{z}_i}{1-\sum_{k=1}^n z_k\overline{z}_k} \\ (\log \kappa(z,z))_{i\bar{j}} &= \frac{n+1}{(1-|z|^2)^2}\left[(1-|z|^2)\delta_{ij} + \overline{z}_i z_j\right].\end{aligned}$$

□

We conclude this section with a calculation that will be useful in the proof of Theorem 1.0.1.

Proposition 3.4.8. *Recall that for $\alpha > 0$, we define*

$$\mathcal{P}_\alpha = \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n \mid \mathrm{Re}(z_1) > \alpha \sum_{j=2}^{n} |z_j|^2 \right\}.$$

There is a constant $C_\alpha > 0$ such that

$$\kappa_{\mathcal{P}_\alpha} \left((\lambda, 0, \ldots, 0), (\lambda, 0, \ldots, 0) \right) = C_\alpha \mathrm{Re}(\lambda)^{-(n+1)}$$

for all $(\lambda, 0, \ldots, 0) \in \mathcal{P}_\alpha$.

Proof. We exhibit explicitly the constant. Let

$$C_\alpha = \kappa_{\mathcal{P}_\alpha} \left((1, 0, \ldots, 0), (1, 0, \ldots, 0) \right)$$

and consider the automorphisms $a_t, u_t \in \mathrm{Aut}(\mathcal{P}_\alpha)$ defined by

$$a_t(z_1, \ldots, z_n) = (e^t z_1, e^{\frac{t}{2}} z_2, \ldots, e^{\frac{t}{2}} z_n)$$

$$u_t(z_1, \ldots, z_n) = (z_1 + it, z_2, \ldots, z_n)$$

Then

$$(\lambda, 0, \ldots, 0) = \left(u_{\mathrm{Im}(\lambda)} \circ a_{\log(\mathrm{Re}(\lambda))} \right) (1, 0, \ldots, 0).$$

Thus, by Proposition 3.4.3,

$$\kappa_{\mathcal{P}_\alpha} \left((\lambda, 0, \ldots, 0), (\lambda, 0, \ldots, 0) \right) = C_\alpha \mathrm{Re}(\lambda)^{-(n+1)}.$$

□

Chapter 4

A Localization Procedure

In this chapter, we present a localization technique of Lin and Wong [24] which we will use in tandem with Frankel's rescaling sequence from Chapter 3 in Step 1 of the proof of Theorem 1.0.1.

4.1 Definitions

Definition 4.1.1. Let Ω be a domain on a complex manifold M, and let $p \in \partial\Omega$ be a fixed boundary point. A *boundary neighborhood of* p is an open set $\widehat{\Omega} = \Omega \cap U$, where U is an open set in M containing p.

We note here that, by construction, a boundary neighborhood is biholomorphic to a bounded domain in $\mathbb{C}^n$.

Definition 4.1.2. We say a point $p \in \partial\Omega$ is *totally real* if there exists no complex analytic variety containing p of positive dimension lying on $\partial\Omega$.

Definition 4.1.3. A domain Ω on a complex manifold M is said to *cover a compact manifold* if there exists a discrete group $\Gamma \leq \mathrm{Aut}(\Omega)$ such that Γ acts freely, properly discontinuously, and cocompactly on Ω. Recall that $\mathrm{Aut}(\Omega)$ is the group of biholomorphisms of Ω.

Definition 4.1.4. We say a domain Ω on a complex manifold M *admits a compact quotient* if $\Omega/\mathrm{Aut}(\Omega)$ is compact.

Remark 4.1.5. A domain Ω admits a compact quotient if Ω covers a compact manifold.

4.2 The Technique

Lemma 4.2.1. *Let Ω be a domain on a hyperbolic complex manifold admitting a compact quotient. Then there exists a compact set $K \subset \Omega$ such that for every $y \in \Omega$ there is $t \in K$ and $g \in \mathrm{Aut}(\Omega)$ such that $g(t) = y$ (i.e., $\mathrm{Aut}(\Omega) \cdot K = \Omega$).*

Proof. We can exhaust Ω by a sequence of relatively compact open sets $\{\Omega_j\}_{j=1}^{\infty}$ such that $\Omega_j \subset\subset \Omega_{j+1}$ and $\bigcup\limits_{j=1}^{\infty} \Omega_j = \Omega$. Let $\pi : \Omega \to \Omega/\mathrm{Aut}(\Omega)$ be the canonical projection.

Since π is an open map and $\Omega/\mathrm{Aut}(\Omega)$ is compact, there is a positive integer m such that

$$\pi(\Omega_m) = \Omega/\mathrm{Aut}(\Omega).$$

We can then take K to be the closure of Ω_m. $\square$

Definition 4.2.2. The compact set K in Lemma 4.2.1 is called the *fundamental set* for $\mathrm{Aut}(\Omega)$.

Lemma 4.2.3. *Let Ω be a domain on a hyperbolic complex manifold admitting a compact quotient. Then Ω is complete hyperbolic.*

Proof. Let $\{p_j\} \subset \Omega$ be a Cauchy sequence with respect to d_Ω^K. Let K be the fundamental set of $\mathrm{Aut}(\Omega)$. Then there is a sequence $\{g_j\} \subset \mathrm{Aut}(\Omega)$ such that $g_j(p_j) \in K$. Passing to a subsequence if necessary, we may assume that $g_j(p_j) \to q \in K$. Let ε be small enough so that

$$\{z \in \Omega \mid d_\Omega^K(z,q) \leq \varepsilon\} \subset\subset \Omega.$$

Since

$$\begin{aligned} d_\Omega^K(g_j(p_k), q) &\leq d_\Omega^K(g_j(p_k), g_j(p_j)) + d_\Omega^K(g_j(p_j), q) \\ &= d_\Omega^K(p_k, p_j) + d_\Omega^K(g_j(p_j), q) \end{aligned}$$

there exists a positive integer N such that $d_\Omega^K(g_N(p_k), q) \leq \varepsilon$ for all $k \geq N$. Therefore, there exists a subsequence $g_N(p_{k_\ell})$ of $g_N(p_k)$ and a point $q' \in \Omega$ such that

$$d_\Omega^K(p_{k_\ell}, g_N^{-1}(q')) = d_\Omega^K(g_N(p_{k_\ell}), q') \to 0 \text{ as } \ell \to \infty.$$

Hence $p_k \to g_N^{-1}(q')$. Thus Ω is complete hyperbolic. □

Lemma 4.2.4 (Montel)**.** *Let Ω be a relatively compact subset of a hyperbolic complex manifold M. Let N be a complex manifold. Then for any sequence $\{f_j\} \subset \mathrm{Hol}(N, \Omega)$, there exists a subsequence $\{f_{j_k}\}$ that converges local uniformly to a holomorphic map $f : N \to \overline{\Omega}$.*

We refer to [32] for the details of this lemma, which is a generalization of the classical Montel theorem.

Lemma 4.2.5 (H. Cartan)**.** *Let Ω be a relatively compact domain on a hyperbolic complex manifold M. Suppose that a sequence $\{m_j\} \subset \mathrm{Aut}(\Omega)$ converges local uniformly on Ω to $m : \Omega \to \overline{\Omega}$. Then either $m \in \mathrm{Aut}(\Omega)$, or $m(\Omega) \subset \partial\Omega$.*

Lemma 4.2.6. *Let Ω be a domain on a hyperbolic complex manifold M admitting a compact quotient. Let $\{x_j\} \subset \Omega$ be a sequence converging to a boundary point $p \in \partial\Omega$. Then there exists $\{m_j\} \subset \mathrm{Aut}(\Omega)$ such that $\{\overline{x}_j = m_j^{-1}(x_j)\}$ converges, through a subsequence if necessary, to a point $x \in \Omega$. Furthermore, the sequence $\{m_j(x)\}$ will also converge to p.*

Proof. Let K be the fundamental set for $\mathrm{Aut}(\Omega)$, and let $m_j \in \mathrm{Aut}(\Omega)$ be such that $\overline{x}_j = m_j^{-1}(x_j) \in K$. Through a subsequence if necessary, $\{\overline{x}_j\}$ will converge to a point $x \in K \subset \Omega$. Now consider the Kobayashi metric d^K_Ω. We have

$$d^K_\Omega(m_j(x), x_j) = d^K_\Omega(x, m_j^{-1}(x_j)) = d^K_\Omega(x, \overline{x}_j).$$

The following inequality is clear by Proposition 3.1.5:

$$d^K_\Omega \geq d^K_M \text{ on } \Omega.$$

Observe that $d^K_\Omega(m_j(x), x_j) \to 0$ as $j \to \infty$ because $d^K_\Omega(x, \overline{x}_j) \to 0$ as $\overline{x}_j \to x$. Hence $d^K_M(m_j(x), x_j) \to 0$ as $j \to \infty$. Since d^K_M is finite around an open set of $p \in \partial\Omega$, we have that $d^K_M(x_j, p) \to 0$ as $x_j \to p$. By the Triangle Inequality for d^K_M, we see that $d^K_M(m_j(x), p) \to 0$ as $j \to \infty$. Thus $m_j(x) \to p$. □

Lemma 4.2.7. *Let Ω be a domain on a hyperbolic complex manifold M, and suppose there is a totally real boundary point $p \in \partial\Omega$. Let $\{m_j\} \subset \mathrm{Aut}(\Omega)$ be a sequence such that $m_j(x) \to p$ for some $x \in \Omega$. Then for any compact set $K \subset \Omega$ and any boundary neighborhood $\widehat{\Omega}$ of p, $m_j(K) \subset \widehat{\Omega}$ for sufficiently large j. In particular, $m_j(y) \to p$ for any $y \in \Omega$.*

Proof. By a standard normal family argument, through a subsequence, $\{m_j\}$ will converge on compact sets to a holomorphic map $m : \Omega \to M$ such that $m(\Omega) \subset \partial\Omega$ and $m(x) = p$.

However, since p is totally real, there is no complex analytic variety of positive dimension lying on the boundary of Ω through p. This implies that m must be constant with $m(\Omega) = p$. The statement follows from this fact. $\square$

The importance of the localization technique is in the power of Lemma 4.2.7. Using this technique, we can biholomorphically fit our domain Ω inside a boundary neighborhood $\widehat{\Omega}$ which, itself, is biholomorphic to a bounded domain in $\mathbb{C}^n$.

4.3 Applications of the Technique

Here we present three applications of the technique from [24], as well as their significance to the field of several complex variables.

Definition 4.3.1. Let Ω_1 and Ω_2 be two domains on two complex manifolds, respectively. Ω_1 is said to be *locally biholomorphic* to Ω_2 at two boundary points $p_1 \in \partial\Omega_1$ and $p_2 \in \partial\Omega_2$ if:

1. There exist boundary neighborhoods $\widehat{\Omega}_1$ of p_1 and $\widehat{\Omega}_2$ of p_2 with a biholomorphism $f : \widehat{\Omega}_1 \to \widehat{\Omega}_2$ between them.
2. There is a sequence $\{x_j\} \subset \widehat{\Omega}_1$ converging to p_1 such that the sequence $\{f(x_j)\} \subset \widehat{\Omega}_2$ converges to p_2.

Theorem 4.3.2 (Theorem 1 from [24])**.** *Let Ω_1 and Ω_2 be two domains on two hyperbolic manifolds X_1 and X_2, respectively. Suppose both Ω_1 and Ω_2 admit compact quotients, and that Ω_1 is locally biholmorphic to Ω_2 at two totally real boundary points $p_1 \in \partial\Omega_1$ and $p_2 \in \partial\Omega_2$. Then Ω_1 is biholomorphic to Ω_2.*

Definition 4.3.3. For a domain $\Omega \subset \mathbb{C}^n$, the *Eisenman differential measure* is defined as

$$k_\Omega^\ell(x) = \inf \left\{ \frac{1}{R^{2\ell}} \;\mid\; \exists f \in \mathrm{Hol}(B_R^\ell(0), \Omega), f(0) = x, \det(df_0) = 1 \right\}$$

for $1 \leq \ell \leq n$.

When $\ell = n$, it associates with a volume form, k_Ω^n. When $\ell = 1$, it corresponds to the Kobayashi-Royden pseudometric, k_Ω, defined in Chapter 3.

Definition 4.3.4. For a domain $\Omega \subset \mathbb{C}^n$, the *Carathéodory differential measure* is defined as

$$c_\Omega^\ell(x) = \sup \left\{ \frac{1}{R^{2\ell}} \;\mid\; \exists f \in \mathrm{Hol}(\Omega, B_R^\ell(0)), f(x) = 0, \det(df_x) = 1 \right\}$$

for $1 \leq \ell \leq n$.

When $\ell = n$, it associates with a volume form, c_Ω^n. When $\ell = 1$, it corresponds to the *Carathéodory-Reiffen differential metric*, denoted c_Ω.

The *Carathéodory distance function* is defined as

$$d_\Omega^C(x, y) = \sup \left\{ P_{\mathbb{D}}(f(x), f(y)) \mid f \in \mathrm{Hol}(\Omega, \mathbb{D}) \right\}.$$

Remark 4.3.5.

1. The Eisenman measure measures how well the ball fits inside a domain, while the Carathéodory measure measures how well a domain fits inside the ball.

2. The Eisenman measure and the Carathéodory measure enjoy the same decreasing and biholomorphic invariance properties as the Kobayashi metric.

Theorem 4.3.6 (Theorem 2 from [24])**.** *Let Ω be a domain admitting a compact quotient on a hyperbolic manifold M. Suppose there is a boundary neighborhood $\widehat{\Omega}$ of a totally real boundary point $p \in \partial\Omega$ satisfying one of the following local conditions:*

1. $\dfrac{c_{\widehat{\Omega}}}{k_{\widehat{\Omega}}} \geq a^2 > 0$ *on* $\widehat{\Omega}$ *for some constant* $a^2 > 0$.

2. $\dfrac{c^n_{\widehat{\Omega}}}{k^n_{\widehat{\Omega}}} \geq b^2 > 0$ *on* $\widehat{\Omega}$ *for some constant* $b^2 > 0$.

Then Ω *admits a lot of bounded holomorphic functions which give local coordinate functions at each point of* Ω.

Theorem 4.3.7 (Theorem 3 from [24])**.** *Let* Ω *be a domain admitting a compact quotient on a hyperbolic manifold* M. *Suppose there exists a boundary neighborhood of a totally real boundary point* $p \in \partial\Omega$ *with the condition*

$$\frac{d^C_{\widehat{\Omega}}(x, y)}{d^K_{\widehat{\Omega}}(x, y)} \geq c^2 > 0$$

for all distinct points $x, y \in \Omega$, *for some constant* $c^2 > 0$. *Then* Ω *is holomorphically convex with respect to the set of all bounded holomorphic functions.*

Remark 4.3.8. Theorems 4.3.6 and 4.3.7 together imply that the domain Ω is a *bounded* Stein manifold.

Theorems 4.3.6 and 4.3.7 provide some insight into a major open problem in several complex variables and Kähler geometry. Namely, does the universal cover of a compact Kähler manifold with negative sectional curvature admit a nontrivial, bounded holomorphic function? In view of the discussion of Kähler manifolds in Chapter 2 and Remark 4.3.8, these results give some hope of obtaining a positive answer to this question.

Chapter 5

Bounded Symmetric Domains

In this chapter, we present some basic results on bounded symmetric domains that will be useful in Step 4 of the proof of Theorem 1.0.1.

5.1 Definitions

Definition 5.1.1. A bounded domain $\Omega \subset \mathbb{C}^n$ is called *symmetric* if $\mathrm{Aut}(\Omega)$ is a semisimple Lie group which acts transitively on Ω.

Definition 5.1.2. Suppose Ω is a bounded symmetric domain. The *real rank of* Ω is the largest integer r with the property that there exists a holomorphic isometric embedding $f : (\mathbb{D}^r, d^K_{\mathbb{D}^r}) \to (\Omega, d^K_\Omega)$.

E. Cartan [3] characterized the bounded symmetric domains, showing that each one has real rank at least 1. Moreover, the only bounded symmetric domain of rank $r = 1$ is the ball.

Definition 5.1.3. Let $\Omega \subset \mathbb{C}^n$ be a bounded symmetric domain.

(1) There is an embedding $F : \Omega \hookrightarrow \mathbb{C}^n$ whose image is convex and bounded, called the *Harish-Chandra embedding*. We denote the image of the embedding by $F(\Omega) = \Omega_{HC}$.

(2) We say Ω is in *standard form* if it coincides with the image of its Harish-Chandra embedding, i.e., $\Omega = \Omega_{HC}$.

Next we recall the well-known description of the Bergman kernel on a bounded symmetric domain. See [9] for a detailed proof.

Theorem 5.1.4. *Suppose $\Omega \subset \mathbb{C}^n$ is a bounded symmetric domain in standard form with real rank r. Let $\Phi : (\mathbb{D}^r, d^K_{\mathbb{D}^r}) \to (\Omega, d^K_\Omega)$ be a holomorphic isometric embedding with $\Phi(0) = 0$. Then there exist constants $q, C > 0$ such that*

$$\kappa_\Omega(\Phi(z), \Phi(z)) = C \left(\prod_{j=1}^{n} \left(1 - |z_j|^2\right) \right)^{-q}$$

for all $z \in \mathbb{D}^r$. Furthermore, $q \geq \dfrac{n+r}{r}$.

5.2 Complex Geodesics in Polydisks in Bounded Symmetric Domains

Step 4 of the proof of Theorem 1.0.1 will require two technical lemmas concerning complex geodesics in polydisks.

Definition 5.2.1. Suppose $\Omega \subset \mathbb{C}^n$ is a domain. A holomorphic map $\gamma : \mathbb{D} \to \Omega$ is called a *complex geodesic* if

$$d^K_\Omega(\gamma(z), \gamma(w)) = d^K_{\mathbb{D}}(z, w)$$

for all $z, w \in \mathbb{D}$.

Lemma 5.2.2. *Suppose $z \in \mathbb{D}^r$ and $|z_j| \neq |z_k|$ for some $1 \leq j, k \leq r$. Then there are two complex geodesics $\gamma_1, \gamma_2 : \mathbb{D} \to \mathbb{D}^r$ whose images contain z and 0, but $\gamma_1(\mathbb{D}) \neq \gamma_2(\mathbb{D})$.*

Proof. We will construct the two geodesics via transformations of the disk. Without loss of generality, we may assume that

$$0 \leq |z_1| \leq |z_2| \leq \cdots \leq |z_r|.$$

Since $|z_1| \leq |z_r|$ there exist holomorphic functions $f_1, f_2 : \mathbb{D} \to \mathbb{D}$ such that

(i) $f_1(0) = f_2(0) = 0$,

(ii) $f_1(z_r) = f_2(z_r) = z_1$, and

(iii) $f_1 \neq f_2$.

For $2 \leq j \leq r-1$, choose $w_j \in \overline{\mathbb{D}}$ such that $w_j z_r = z_j$. Then for $j = 1, 2$, define the maps

$$\gamma_j : \mathbb{D} \to \mathbb{D}^r$$

$$\gamma_j(\lambda) = (f_j(\lambda), \lambda w_2, \ldots, \lambda w_{r-1}, \lambda)$$

Since each γ_j is holomorphic, we have

$$d^K_{\mathbb{D}^r}(\gamma_j(\lambda_1), \gamma_j(\lambda_2)) \leq d^K_{\mathbb{D}}(\lambda_1, \lambda_2)$$

for all $\lambda_1, \lambda_2 \in \mathbb{D}$. Additionally, by projecting onto the last coordinate, we obtain

$$d^K_{\mathbb{D}^r}(\gamma_j(\lambda_1), \gamma_j(\lambda_2)) \geq d^K_{\mathbb{D}}(\lambda_1, \lambda_2)$$

for all $\lambda_1, \lambda_2 \in \mathbb{D}$. Hence,

$$d^K_{\mathbb{D}^r}(\gamma_j(\lambda_1), \gamma_j(\lambda_2)) = d^K_{\mathbb{D}}(\lambda_1, \lambda_2)$$

for all $\lambda_1, \lambda_2 \in \mathbb{D}$, and so γ_1 and γ_2 are both complex geodesics. Lastly, since $f_1 \neq f_2$, we have that $\gamma_1(\mathbb{D}) \neq \gamma_2(\mathbb{D})$. □

Lemma 5.2.3. *Suppose $z \in \mathbb{D}^r$ with*

$$0 < |z_1| = |z_2| = \cdots = |z_r|.$$

If $\gamma : \mathbb{D} \to \mathbb{D}^r$ is a complex geodesic with $\gamma(0) = 0$ and $\gamma(\lambda_0) = z$, then $|\lambda_0| = |z_1|$, and

$$\gamma(\lambda) = \left(\frac{z_1}{\lambda_0}\lambda, \ldots, \frac{z_r}{\lambda_0}\lambda \right)$$

for all $\lambda \in \mathbb{D}$.

Proof. Since

$$d^K_{\mathbb{D}}(0, \lambda_0) = d^K_{\mathbb{D}^r}(0, z) = \max_{1 \leq j \leq r} d^K_{\mathbb{D}}(0, z_j) = d^K_{\mathbb{D}}(0, z_1),$$

we have that $|\lambda_0| = |z_1|$. Next, applying the Schwarz Lemma to each of the component functions of γ shows that

$$\gamma(\lambda) = \left(\frac{z_1}{\lambda_0}\lambda, \ldots, \frac{z_r}{\lambda_0}\lambda \right)$$

for all $\lambda \in \mathbb{D}$. □

The next result says that there are many ismetric embeddings of polydisks.

Theorem 5.2.4 (Polydisk Theorem, [29])**.** *Suppose $\Omega \subset \mathbb{C}^n$ is a bounded symmetric domain of real rank r. For any two points $z_1, z_2 \in \Omega$, there exists a holomorphic isometric embedding $f : (\mathbb{D}^r, d^K_{\mathbb{D}^r}) \to (\Omega, d^K_\Omega)$ whose image contains z_1, z_2.*

5.3 Modern Results on Bounded Symmetric Domains

In an attempt at classifying those complex manifolds which are biholomorphic to the unit ball, a common first step is to show that the manifold is biholomorphic to a bounded symmetric domain. As such, many rigidity results exist. We include two of them here.

We begin with an observation of Bun Wong [31] which follows easily from theorems of Hano [19] and Koszul [22].

Theorem 5.3.1. *Let $D = G/H$ be a homogeneous complex manifold, where G is a connected Lie group acting on D effectively, and H is the isotopy subgroup of G. If there exists a discrete subgroup $\Gamma < G$ such that $M = D/\Gamma$ is a compact complex manifold with negative definite first Chern class, then D is a bounded symmetric domain in $\mathbb{C}^n$.*

Remark 5.3.2. Recall that a manifold is called *homogeneous* if the isometry group acts transitively on it.

The proof of Theorem 5.3.1 requires showing that the Ricci form of the Eisenman-Kobayashi measure is negative definite, and then employing Theorems 5.3.3 and 5.3.4. Theorem 5.3.1 appears in a paper whose main result (Theorem 7.0.1), is the initial inspiration for Theorem 1.0.1.

Theorem 5.3.3 (Hano [19])**.** *If the Ricci curvature of a Kähler homogeneous space of a connected unimodular Lie group is nondegenerate, and the group acts effectively on the space, then the group is semisimple.*

Theorem 5.3.4 (Koszul [22]). *Let G/H be a homogeneous complex manifold with G a connected semisimple Lie group. Suppose there exists a G-invariant volume form V such that its associated Ricci form is negative definite. Then G/H is a bounded symmetric domain in $\mathbb{C}^n$.*

The next result is an extension of Theorem 5.3.1.

Theorem 5.3.5 (Frankel, Nadel, [12], [26]). *Let X be a compact complex manifold with $c_1(X) < 0$, and let $\widetilde{X}$ be the universal cover. If $\mathrm{Aut}\left(\widetilde{X}\right)$ is nondiscrete, then $\widetilde{X}$ is biholomorphic to either*

(a) a bounded symmetric domain, or

(b) a nontrivial product $D_1 \times D_2$, where D_1 is a bounded symmetric domain and $\mathrm{Aut}(D_2)$ is discrete.

Nadel proved Theorem 5.3.5 in dimension 2, and Frankel extended it to all dimensions. Nadel's contribution to the result was in removing the bounded domain hypothesis from [11] needed to show the semisimplicity of $\mathrm{Aut}\left(\widetilde{X}\right)$. In fact, Nadel proved

Theorem 5.3.6 (Theorem 0.1 in [26]). *Let X be a connected compact complex manifold with ample canonical bundle. Then the identity component of the group of automorphisms of $\widetilde{X}$, denoted by $\mathrm{Aut}_0\left(\widetilde{X}\right)$, is a real semisimple Lie group without compact factors.*

Nadel's proof of Theorem 5.3.6 requires the study of finite dimensional linear spaces of global holomorphic vector fields on $\widetilde{X}$, and locating zeros of the Ricci curvature of any complex volume manifold admitting a suitable abelian Lie algebra of infinitesimal automorphisms. To prove the 2-dimensional case of Theorem 5.3.5, Nadel showed that if

$\mathrm{Aut}_0\left(\widetilde{X}\right)$ has real dimension at least six, then $\widetilde{X}$ is symmetric. He used a topological argument to rule out the case where $\mathrm{Aut}_0\left(\widetilde{X}\right)$ has three real dimensions. The main theme throughout his paper is that the identity component should be nontrivial only in the presence of symmetric factors.

Frankel's proof of the n-dimensional case is purely differential geometric. It relies on the theory of geodesic flow for the construction of a center of mass, as well as on the theory of harmonic maps and previously developed rigidity theory of negatively curved complex manifolds. See, for instance, [8].

Chapter 6

Main Result

Here we present the proof of Theorem 1.0.1. We note that the method is very similar, and in some places identical, to the one used by Zimmer in the proof of Theorem 1.0.3, but we include it in full for completeness, as well as to verify that the calculations still work out. The proof is split into four parts:

1. Show that $\Omega \subset M^n$ is biholomorphic to a domain $D \subset \mathbb{C}^n$ such that
$$\mathcal{P}_\alpha \subset D \subset \mathcal{P}_\beta.$$
2. Show that $\mathrm{Aut}(D)$ contains the one-parameter subgroup
$$u_t(z_1, \ldots, z_n) = (z_1 + it, z_2, \ldots, z_n).$$
3. Show that Ω is a bounded symmetric domain.
4. Show that Ω is biholomorphic to the unit ball.

In Step 1, we use the aforementioned rescaling argument of Frankel described in Chapter 3. We note that it is similar to Zimmer's use of the argument. However, in the construction of the rescaling maps, we use the localization technique from Chapter 4. Herein lies our contribution.

Step 2 is simply a verification of Zimmer's calculations using the new rescaling maps. Most of the argument is technical, and, while necessary, is ultimately uninteresting.

In Step 3, we make use of the theorem of Frankel and Nadel (Theorem 5.3.5) to show that our domain is a bounded symmetric domain. The argument exploits the geometry of the rescaled domain from Step 1 to show that the latter potentiality of that theorem is impossible. In order to use Theorem 5.3.5, we need a result on Chern classes.

Lastly, in Step 4, we use the theory of bounded symmetric domains to conclude that Ω is, in fact, the ball. We do this in much the same way as Zimmer by introducing a holomorphic function which measures the volume contraction (or expansion) of the biholomorphism from Step 1 along a linear slice of the rescaled domain from Step 1. We then use a contradiction argument on the rank of Ω as a bounded symmetric domain to show that Ω must be the ball. An important part of the argument is showing that we can parameterize the diagonal of a maximal polydisk in the Harish-Chandra embedding of Ω via the Jacobian matrix of the biholomorphism.

6.1 $\Omega \subset M^n$ is Biholomorphic to $D \subset \mathbb{C}^n$

Lemma 6.1.1. *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with $C^{1,1}$ boundary. After applying an affine transformation, we can assume that*

$$B_r(re_1) \subset \Omega \subset B_1(e_1)$$

for some $r \in (0, 1)$.

Proof. By translating, we can assume that $e_1 \in \Omega$. Choose a boundary point $p_0 \in \partial\Omega$ such that

$$|p_0 - e_1| = \max\{|p - e_1| \mid p \in \partial\Omega\}.$$

By rotating and scaling Ω about e_1, we may assume that $p_0 = 0$. Then $\Omega \subset B_1(e_1)$.

Next, for $p \in \partial\Omega$, let $n_\Omega(p)$ be the inward pointing normal unit vector at p. Since the boundary of Ω is $C^{1,1}$, there exists $r > 0$ such that

$$B_r(p + rn_\Omega(p)) \subset \Omega$$

for every $p \in \partial\Omega$. So, since $\Omega \subset B_1(e_1)$, we have that $n_\Omega(0) = e_1$. Thus $B_r(re_1) \subset \Omega$. $\square$

The following theorem is due to Deng et al., [7]. It is a higher dimensional version of Hurwitz's theorem.

Theorem 6.1.2. *Suppose that $\Omega \subset \mathbb{C}^n$ is a bounded domain and let $x \in \Omega$. Let $f_j : \Omega \to \mathbb{C}^n$ be a sequence of injective holomorphic maps such that $f_j(x) = 0$ for all j, and f_j converges local uniformly to a map $f : \Omega \to \mathbb{C}^n$. If there exists $\varepsilon > 0$ such that $B_\varepsilon(0) \subset f_j(\Omega)$ for all j, then f is injective.*

Recall that the set $\mathcal{P}_\alpha$ for $\alpha > 0$ is defined to be

$$\mathcal{P}_\alpha = \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n \mid \mathrm{Re}(z_1) > \alpha \sum_{j=2}^{n} |z_j|^2 \right\}.$$

The main result of this section is the following:

Proposition 6.1.3. *Let Ω be as in Theorem 1.0.1. Then Ω is biholomorphic to a domain $D \subset \mathbb{C}^n$ such that*

$$\mathcal{P}_\alpha \subset D \subset \mathcal{P}_\beta.$$

Proof. Suppose that M^n is a hyperbolic complex manifold. Let $\Omega \subset M$ be a bounded domain with $C^{1,1}$ boundary admitting a compact quotient. Let $K \subset \Omega$ be the fundamental set of $\mathrm{Aut}(\Omega)$, i.e., K is compact and $\mathrm{Aut}(\Omega) \cdot K = \Omega$. Let $p \in \partial\Omega$ be totally real and let $\widehat{\Omega}$ be a boundary neighborhood of p. Note that $\widehat{\Omega}$ is biholomorphic to a bounded domain $\widetilde{\Omega} \subset \mathbb{C}^n$. Let ϕ be the biholomorphism.

By Lemma 6.1.1, we can assume that

$$B_r(re_1) \subset \widetilde{\Omega} \subset B_1(e_1)$$

for some $r \in (0, 1)$.

By Lemma 4.2.7 there exists a sequence $\{m_j\} \subset \mathrm{Aut}(\Omega)$ such that for any compact set $L \subset \Omega$, $m_j(L) \subset \widehat{\Omega}$ for sufficiently large j. Fix a sequence $p_j \in \widehat{\Omega}$ converging to p and a sequence $k_j \in K$ such that $m_j(k_j) = p_j$.

Fix a sequence $r_j \in (0, r)$ converging to 0. Then pick $g_j \in \mathrm{Aut}(\widetilde{\Omega})$ such that $g_j(\phi(p_j)) = r_j e_1$. Consider the $\widetilde{\Omega}$-dilations

$$\Lambda_j(z_1, \ldots, z_n) = \left(\frac{1}{r_j} z_1, \frac{1}{\sqrt{r_j}} z_2, \ldots, \frac{1}{\sqrt{r_j}} z_n \right).$$

Let $\widetilde{\Omega}_j := \Lambda_j(\widetilde{\Omega})$ and $F_j := \Lambda_j \circ g_j \circ \phi \circ m_j : \Omega \to \widetilde{\Omega}_j$. Then

$$\Lambda_j \left(B_r(re_1) \right) \subset \widetilde{\Omega}_j \subset \Lambda_j \left(B_1(e_1) \right).$$

Moreover, by Example 3.3.8,

$$\mathcal{P}_\alpha = \lim_{j \to \infty} \Lambda_j \left(B_r(re_1) \right), \text{ where } \alpha = \frac{1}{2r}$$

and

$$\mathcal{P}_\beta = \lim_{j \to \infty} \Lambda_j \left(B_1(e_1) \right), \text{ where } \beta = \frac{1}{2}$$

in the local Hausdorff topology.

We claim now that after passing to a subsequence, F_j converges to a holomorphic embedding $F : \Omega \to \mathbb{C}^n$. Furthermore, if $D = F(\Omega)$, then

$$\mathcal{P}_\alpha \subset D \subset \mathcal{P}_\beta.$$

By construction, $F_j(k_j) = e_1$ and the decreasing property of the Kobayashi pseudodistance implies that

$$d^K_{\widetilde{\Omega}_j}(z,w) \leq d^K_{\Lambda_j(B_1(e_1))}(z,w)$$

for all $z, w \in \widetilde{\Omega}_j$. Theorem 3.3.3 implies that $d^K_{\Lambda_j(B_1(e_1))}$ converges to $d^K_{\mathcal{P}_\beta}$ locally uniformly. So by Arzelá-Ascoli, we can pass to a subsequence where F_j converges locally uniformly to a holomorphic map $F : \Omega \to \mathbb{C}^n$.

Let $D = F(\Omega)$. Since

$$\Lambda_j\left(B_r(re_1)\right) \subset \widetilde{\Omega}_j \subset \Lambda_j\left(B_1(e_1)\right)$$

for each j, it follows that

$$\overline{\mathcal{P}_\alpha} \subset D \subset \overline{\mathcal{P}_\beta}.$$

It remains to show that F is injective. We will do this using Theorem 6.1.2. Since

$$\mathcal{P}_\alpha = \lim_{j\to\infty} \Lambda_j\left(B_r(re_1)\right)$$

in the local Hausdorff topology and $\Lambda_j\left(B_r(re_1)\right) \subset \widetilde{\Omega}_j$ for each j, there exists $\varepsilon > 0$ such that

$$B_\varepsilon(e_1) \subset F_j(\Omega)$$

for each j. Passing to a subsequence if necessary, we can suppose $k_j \to k \in K$. Then consider the maps

$$G_j(q) = F_j(q) - F_j(k).$$

Since

$$\lim_{j\to\infty} F_j(k) = \lim_{j\to\infty} F_j(k_j) = F(k) = e_1,$$

G_j converges locally uniformly to $F - e_1$. Moreover, by passing to another subsequence, we can assume that $\|e_1 - F_j(k)\| < \frac{\varepsilon}{2}$ for each j. Then for every j, the map G_j is injective, $G_j(k) = 0$, and

$$B_{(\varepsilon/2)}(0) \subset G_j(\Omega).$$

So by Theorem 6.1.2, F is injective. Thus F is an embedding.

Since F is an embedding, D is an open set and we can write

$$\mathcal{P}_\alpha = \operatorname{int}(\overline{\mathcal{P}_\alpha}) \subset D \subset \operatorname{int}(\overline{\mathcal{P}_\beta}) = \mathcal{P}_\beta.$$

□

6.2 Aut(D) Contains a One-Parameter Subgroup

Lemma 6.2.1. *Let $\widetilde{\Omega}_j$ be as in the proof of Proposition 6.1.3. Suppose z_j is a sequence such that $z_j \in \widetilde{\Omega}_j$ for each j, $z_j \to z$, and*

$$\liminf_{j\to\infty} d^K_{\widetilde{\Omega}_j}(e_1, z_j) < \infty,$$

then $z \in D$.

Proof. Fix $z_0 \in \Omega \subset M$ and let

$$P = \max_{k \in K} d_\Omega^K(z_0, k).$$

Pick $j_s \to \infty$ such that

$$Q = \lim_{s\to\infty} d^K_{\widetilde{\Omega}_{j_s}}(e_1, z_{j_s}) < \infty.$$

Let $F_j : \Omega \to \widetilde{\Omega}_j$ and $k_j \in K$ be as in the proof of Proposition 6.1.3. Since $F_j(k_j) = e_1$, for each s, there exists

$$w_s \in \{y \in \Omega \mid d_\Omega^K(z_0, y) \leq P + Q\}$$

such that $F_{j_s}(w_s) = z_{j_s}$. By Proposition 3.1.6, d_Ω^K is proper. So we can pass to a subsequence where $w_s \to w \in \Omega$. Since $F_j \to F$ local uniformly, we have

$$F(w) = \lim_{s\to\infty} F_{j_s}(w_s) = \lim_{s\to\infty} z_s = z.$$

Thus $z \in F(\Omega) = D$. $\square$

The next lemma is highly technical, and serves only as a tangible representation of an obvious fact. For this reason, we omit the proof. It can be found in [35].

Lemma 6.2.2. *Let $\widetilde{\Omega}$ be as in the proof of Proposition 6.1.3. For every $m > 0$, there exists $\delta_m > 0$ such that if $z_0 \in \widetilde{\Omega} \cap B_{\delta_m}(0)$, $T > 0$, and*

$$\{z_0 + xe_1 \mid -T < x < T\} \subset \widetilde{\Omega},$$

then

$$\left\{ z_0 + (x+iy)e_1 \;\middle|\; -\frac{T}{2} \leq x \leq \frac{T}{2}, -mT \leq y \leq mT \right\} \subset \widetilde{\Omega}.$$

Proposition 6.2.3. *Let D be as in Proposition 6.1.3. Then* $\mathrm{Aut}(D)$ *contains the one-parameter subgroup*

$$u_t(z_1, \ldots, z_n) = (z_1 + it, z_2, \ldots, z_n).$$

In particular, $\mathrm{Aut}(\Omega) \cong \mathrm{Aut}(D)$ *is nondiscrete.*

Proof. It suffices to fix $w_0 \in D$ and $t \in \mathbb{R}$, and show that $w_0 + ite_1 \in D$. Since F_j converges local uniformly to F, there exits $\varepsilon > 0$ and $J \in \mathbb{N}$ such that

$$B_\varepsilon(w_0) \subset F_j(\Omega) = \Lambda_j(\widetilde{\Omega})$$

for all $j \geq J$.

Define the sequence $\{w_j = \Lambda_j^{-1}(w_0)\}$. Then

$$\{w_j + xe_1 \mid -r_j\varepsilon < x < r_j\varepsilon\} \subset \Lambda_j^{-1}(B_\varepsilon(w_0)) \subset \widetilde{\Omega} \tag{6.1}$$

whenever $j \geq J$. Here, the sequence r_j is as in the proof of Proposition 6.1.3.

Fix $m \in \mathbb{N}$ such that $|t| < m\varepsilon$. Let δ_m be the associated constant from Lemma 6.2.2. Since $r_j \to 0$, we have that $w_j \to 0$. So, by increasing J if necessary, we may assume that

$$w_j \in B_{\delta_m}(0)$$

whenever $j \geq J$. Then by (6.1) and Lemma 6.2.2,

$$\left\{ w_j + (x+iy)e_1 \;\middle|\; -\frac{r_j\varepsilon}{2} \leq x \leq \frac{r_j\varepsilon}{2}, -mr_j\varepsilon \leq y \leq mr_j\varepsilon \right\} \subset \widetilde{\Omega}$$

for all $j \geq J$. Thus

$$W := \left\{ w_0 + (x+iy)e_1 \;\middle|\; -\frac{\varepsilon}{2} \leq x \leq \frac{\varepsilon}{2}, -m\varepsilon \leq y \leq m\varepsilon \right\} \subset \Lambda_j(\widetilde{\Omega}).$$

Then for all $j \geq J$,

$$d^K_{\Lambda_j(\widetilde{\Omega})}(w_0, w_0 + ite_1) \leq d^K_W(w_0, w_0 + ite_1).$$

Thus

$$\sup_{j \geq J} d^K_{\Lambda_j(\widetilde{\Omega})}(w_0, w_0 + ite_1) < \infty.$$

By Lemma 6.2.1, $w_0 + ite_1 \in D$. $\square$

6.3 Ω is Symmetric

We begin with some preliminary material. For the rest of this section, let D be the rescaled domain obtained in Step 1.

Define

$$\mathcal{H}_D = D \cap \mathbb{C} \cdot e_1 = \{(z, 0, \ldots, 0) \in \mathbb{C}^n \mid \mathrm{Re}(z) > 0\},$$

and, for a domain $\Omega \subset \mathbb{C}^n$, let $\mathrm{Aut}_0(\Omega)$ denote the connected component in $\mathrm{Aut}(\Omega)$ containing the identity.

We will require the following technical (and standard) result.

Proposition 6.3.1. *Suppose $\{z_j\}, \{w_j\} \subset D$ are sequences such that*

$$z_j \to \eta \in \{(ix, 0, \ldots, 0) \in \mathbb{C}^n \mid x \in \mathbb{R}\} = \overline{\mathcal{H}}_D \cap \partial D$$

and

$$\limsup_{j \to \infty} d^K_D(z_j, w_j) < \infty.$$

Then $w_j \to \eta$.

We will also need a result from [30], which is attributed in that paper to R. E. Greene.

Lemma 6.3.2. *If $\Omega \subset \mathbb{C}^n$ is a bounded domain,* $\mathrm{Aut}(\Omega)$ *acts cocompactly on Ω, and $\partial\Omega$ is C^1, then for every $m \geq 1$, the m^{th} homotopy group $\pi_m(\Omega)$ is trivial. In particular, Ω is simply connected.*

The last ingredient is the following short lemma which will allow us to use Theorem 5.3.5.

Lemma 6.3.3. *Let $\Gamma \leq \mathrm{Aut}(\Omega)$ be the discrete subgroup guaranteed by the fact that Ω covers a compact manifold. Let $M = \Omega/\Gamma$. Then the first Chern class $c_1(M) < 0$.*

Proof. Since Ω is biholomorphic to a bounded domain in $\mathbb{C}^n$ (see 6.1.3), the Bergman kernel function for Ω is nontrivial and strictly positive along the diagonal. Note that M is a compact Kähler manifold. The Bergman kernel function can be considered as a volume form on M which gives a metric on the anticanonical bundle such that the Ricci curvature is negative definite. It is well known that on a Kähler manifold, the Ricci curvature determines the first Chern class (see [25], [33]). In particular, since the Ricci curvature is negative definite, we have that $c_1(M) < 0$. □

The main result for this section is the following.

Proposition 6.3.4. *Let Ω be a subdomain of a hyperbolic complex manifold. Suppose $\partial\Omega$ is $C^{1,1}$, and that there exists a totally real boundary point $p \in \partial\Omega$. If Ω covers a compact manifold, then Ω is biholomorphic to a bounded symmetric domain in $\mathbb{C}^n$.*

Proof. By Proposition 6.2.3, we have that $\mathrm{Aut}(\Omega)$ is nondiscrete. By Proposition 6.3.2, Ω is simply connected. Let $M = \Omega/\Gamma$. Then by Lemma 6.3.3, $c_1(M) < 0$. So by Theorem 5.3.5, Ω is biholomorphic to either

(1) a bounded symmetric domain, or

(2) a product $D_1 \times D_2$ where D_1 is a bounded symmetric domain and $\mathrm{Aut}(D_2)$ is infinite and discrete.

By way of contradiction, suppose the latter. By Propositions 6.1.3 and 6.2.3, there is a biholomorphism $F : \Omega \cong D_1 \times D_2 \to D$, where $D \subset \mathbb{C}^n$ is the domain from Step 1 with the properties

$$\mathcal{P}_\alpha \subset D \subset \mathcal{P}_\beta$$

for some $\alpha > \beta > 0$, and $\mathrm{Aut}(D)$ contains the one-parameter subgroup

$$u_t(z_1, \ldots, z_n) = (z_1 + it, z_2, \ldots, z_n).$$

We define the infinite, discrete group

$$G = F \circ (\{\mathrm{Id}_1\} \times \mathrm{Aut}(D_2)) \circ F^{-1} \leq \mathrm{Aut}(D).$$

We note that G commutes with $\mathrm{Aut}_0(D)$. We will obtain a contradiction by showing that G is finite.

Since $\mathrm{Aut}_0(D_1 \times D_2) = \mathrm{Aut}_0(D_1) \times \{\mathrm{Id}_2\}$, we have that for any $(x_1, x_2) \in D_1 \times D_2$,

$$\mathrm{Aut}_0(D_1 \times D_2)_{(x_1,x_2)} = D_1 \times \{x_2\}.$$

Therefore, the orbit $\mathrm{Aut}_0(D_1 \times D_2)_{(x_1,x_2)}$ is a complex analytic variety in $D_1 \times D_2$. Hence, the orbit $\mathrm{Aut}_0(D)_w$ is a complex analytic variety in D for any $w \in D$. Moreover, since $\mathrm{Aut}(D)$

contains the one-parameter subgroup u_t, for any $w_0 \in \mathrm{Aut}_0(D)_w$, $ie_1 \in T_{w_0}(\mathrm{Aut}_0(D)_w)$, the tangent space of $\mathrm{Aut}_0(D)_w$ at w_0. Finally, since $\mathrm{Aut}_0(D)_w$ is a complex analytic variety,

$$\mathbb{C} \cdot e_1 \subset T_{w_0}(\mathrm{Aut}_0(D)_w).$$

Then $\mathcal{H}_D \subset \mathrm{Aut}_0(D)_{e_1}$, so for every $h \in \mathcal{H}_D$, there is $\phi_h \in \mathrm{Aut}_0(D)$ such that $\phi_h(e_1) = h$.

Let $g \in G$. Then since G commutes with $\mathrm{Aut}_0(D)$

$$\begin{aligned} d_D^K(h, g(h)) &= d_D^K(\phi_h(e_1), (g \circ \phi_h)(e_1)) \\ &= d_D^K(\phi_h(e_1), (\phi_h \circ g)(e_1)) \\ &= d_D^K(e_1, g(e_1)). \end{aligned}$$

Thus

$$\sup_{h \in \mathcal{H}_D} d_D^K(h, g(h)) = d_D^K(e_1, g(e_1)) < \infty.$$

Let $\mathcal{H} = \{\lambda \in \mathbb{C} \mid \mathrm{Re}(\lambda) > 0\}$, and define $\psi : \mathcal{H} \to D$ by $\psi(\lambda) = g(\lambda, 0, \ldots, 0)$, with $\psi_1, \ldots, \psi_n$ the coordinate functions. By Lemma 6.3.1,

$$\lim_{\lambda \to it} \psi(\lambda) = (it, 0, \ldots, 0)$$

when $t \in \mathbb{R}$. Then by applying the Schwarz Reflection Principle to each of the coordinate functions of ψ, we can extend it to a map $\tilde{\psi} : \mathbb{C} \to \mathbb{C}^n$ such that

$$\tilde{\psi}(it) = (it, 0, \ldots, 0).$$

But by the Identity Theorem for holomorphic functions, we have that

$$\tilde{\psi}(\lambda) = (\lambda, 0, \ldots, 0)$$

for all $\lambda \in \mathbb{C}$. In particular, $g(e_1) = e_1$.

Since g was arbitrary, we have that $G_{e_1} = \{e_1\}$. Since D is a bounded domain, $\mathrm{Aut}(D)$ acts properly on D, and so G must be compact. But G is also discrete. Therefore, G is finite, and we have reached our desired contradiction.

$\square$

6.4 Ω is Biholomorphic to the Ball

For the final part of the proof of Theorem 1.0.1, we again follow Zimmer's argument closely to verify that the calculations hold in our case. We will use the geometry of the rescaled domain D, the Harish-Chandra embedding of Ω, and the Bergman kernels of both of these spaces. Ultimately, we will show that the real rank of Ω as a bounded symmetric domain is 1. This will imply that Ω is the ball.

We begin with a Lemma concerning complex geodesics in the space $\mathcal{H}_D$ defined in the previous section. Recall

$$\mathcal{H}_D = D \cap \mathbb{C} \cdot e_1 = \{(z, 0, \ldots, 0) \in \mathbb{C}^n \mid \mathrm{Re}(z_1) > 0\}.$$

Lemma 6.4.1. *Suppose $a, b \in \mathcal{H}_D$ are distinct, and $\gamma : \mathbb{D} \to D$ is a complex geodesic with $a, b \in \gamma(\mathbb{D})$. Then there exists $g \in \mathrm{Aut}(\mathbb{D})$ such that*

$$(\gamma \circ g)(\lambda) = \left(\frac{1+\lambda}{1-\lambda}, 0, \ldots, 0\right)$$

for all $\lambda \in \mathbb{D}$. In particular, $\gamma(\mathbb{D}) = \mathcal{H}_D$.

In order to use Theorem 5.1.4, we require the following.

Lemma 6.4.2. *There exists a holomorphic isometric embedding* $\Phi : (\mathbb{D}^r, d^K_{\mathbb{D}^r}) \to (D, d^K_D)$ *such that*

$$\Phi(\lambda, \dots, \lambda) = \left(\frac{1+\lambda}{1-\lambda}, 0, \dots, 0 \right)$$

for all $\lambda \in \mathbb{D}$.

Proof. Let $\mathcal{H}_D$ be as before, and fix $w_0 \in \mathcal{H}_D \setminus \{e_1\}$. By Theorem 5.2.4, there exists a holomorphic isometric embedding $f : (\mathbb{D}^r, d^K_{\mathbb{D}^r}) \to (D, d^K_D)$ with $e_1, w_0 \in f(\mathbb{D}^r)$. By precomposing f with an element of $\mathrm{Aut}(\mathbb{D}^r) \geq \underbrace{\mathrm{Aut}(\mathbb{D}) \times \cdots \times \mathrm{Aut}(\mathbb{D})}_{r}$, we may assume that $f(0) = 0$ and $f(t_1, \dots, t_r) = w_0$ for some $t_1, \dots, t_r \in [0, 1)$.

By Lemma 6.4.1, every complex geodesic in D containing e_1 and w_0 has image $\mathcal{H}_D$. So by Lemma 5.2.2, we have $t_1 = \cdots = t_r$. Then by Lemma 5.2.3,

$$\mathcal{H}_D = \{ f(\lambda, \dots, \lambda) \mid \lambda \in \mathbb{D} \}.$$

By Lemma 6.4.1, there exists $g \in \mathrm{Aut}(\mathbb{D})$ such that

$$f(g(\lambda), \dots, g(\lambda)) = \left(\frac{1+\lambda}{1-\lambda}, 0, \dots, 0 \right)$$

for all $\lambda \in \mathbb{D}$. Then the map $\Phi = f \circ (g, \dots, g)$ has the desired properties. □

The last necessary result is the following technical lemma.

Lemma 6.4.3. *Let* κ_D *be the Bergman kernel of the rescaled domain* D. *Then there exist constants* $0 < C_\beta < C_\alpha$ *such that*

$$C_\beta \left(\frac{1-|\lambda|}{|1-\lambda|^2} \right)^{-(n+1)} \leq \kappa_D \left(\frac{1+\lambda}{1-\lambda}, 0, \dots, 0 \right) \leq C_\alpha \left(\frac{1-|\lambda|}{|1-\lambda|^2} \right)^{-(n+1)}$$

for all $\lambda \in \mathbb{D}$.

Proof. Define $\mathcal{H} = \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$. Since $D \subset \mathcal{P}_\beta$, Propositions 3.4.2 and 3.4.8 imply that there exists a constant $C_\beta > 0$ such that

$$C_\beta(\mathrm{Re}(z))^{-(n+1)} = d^K_{\mathcal{P}_\beta}(z, 0, \dots, 0) \leq d^K_D(z, 0, \dots, 0)$$

for all $z \in \mathcal{H}$. Moreover,

$$\frac{1 - |\lambda|}{|1 - \lambda|^2} \leq \mathrm{Re}\left(\frac{1 + \lambda}{1 - \lambda}\right) \leq 2\frac{1 - |\lambda|}{|1 - \lambda|^2}$$

for all $\lambda \in \mathbb{D}$. Combining these two statements give us the lower bound

$$C_\beta \left(\frac{1 - |\lambda|}{|1 - \lambda|^2}\right)^{-(n+1)} \leq \kappa_D\left(\frac{1 + \lambda}{1 - \lambda}, 0, \dots, 0\right).$$

Repeating the argument using $\mathcal{P}_\alpha \subset D$ will yield the upper bound. $\square$

We can now state and prove Step 4 of the proof of Theorem 1.0.1. We remark that the proof of the statement is identical to that of Theorem 8.1 in [35].

Proposition 6.4.4. *Suppose $\Omega \subset \mathbb{C}^n$ is a bounded symmetric domain. If $\partial\Omega$ is $C^{1,1}$, then Ω is biholomorphic to the unit ball.*

Proof. Suppose Ω is a bounded symmetric domain with real rank r, and with $C^{1,1}$ boundary. Let Ω_{HC} be the image of the Harish-Chandra embedding of Ω. By Proposition 6.1.3, there exists a biholomorphism $F : D \to \Omega_{HC}$. Recall that

$$\mathcal{P}_\alpha \subset D \subset \mathcal{P}_\beta$$

for some $\alpha > \beta > 0$. We may assume that $F(e_1) = 0$ by post composing F with an element of $\mathrm{Aut}(\Omega_{HC})$, if necessary.

Consider the function $J : \mathbb{D} \to \mathbb{C}$ defined by

$$J(\lambda) = \det\left(F'\left(\frac{1+\lambda}{1-\lambda}, 0, \ldots, 0\right)\right) = \det\left(F'(\Phi(\lambda, \ldots, \lambda))\right).$$

Here, $F'(z)$ is the complex Jacobian matrix of F, and $\Phi : \mathbb{D}^r \to D$ is the holomorphic isometric embedding from Lemma 6.4.2. Since F is a biholomorphism, J is nowhere zero. We will estimate J on $\partial\mathbb{D}$ to show that the real rank of Ω, r, must be equal to 1.

Define $\Phi_{HC} = F \circ \Phi : \mathbb{D}^r \to \Omega_{HC}$, and note that $\Phi_{HC}(0) = F(e_1) = 0$. Let κ_D and $\kappa_{\Omega_{HC}}$ be the Bergman kernels of D and Ω_{HC}, respectively. For ease of reading, we will use the convention

$$\kappa_D(z) = \kappa_D(z, z) \quad \text{and} \quad \kappa_{\Omega_{HC}}(w) = \kappa_{\Omega_{HC}}(w, w).$$

By Proposition 3.4.3,

$$|J(\lambda)|^2 = \frac{\kappa_D(\Phi(\lambda, \ldots, \lambda))}{\kappa_{\Omega_{HC}}(\Phi_{HC}(\lambda, \ldots, \lambda))} = \frac{\kappa_D\left(\dfrac{1+\lambda}{1-\lambda}, 0, \ldots, 0\right)}{\kappa_{\Omega_{HC}}(\Phi_{HC}(\lambda, \ldots, \lambda))} \tag{6.2}$$

for all $\lambda \in \mathbb{D}$.

Thus, by Equation (6.2), Theorem 5.1.4, and Lemma 6.4.3, there exist constants $C > 0$ and $q \geq \dfrac{n+r}{r}$ such that

$$\begin{aligned}
|J(\lambda)|^2 &\leq C\left(\frac{1-|\lambda|}{|1-\lambda|^2}\right)^{-(n+1)} (1-|\lambda|^2)^{rq} \\
&\leq C\left(\frac{1-|\lambda|}{|1-\lambda|^2}\right)^{-(n+1)} (1-|\lambda|^2)^{n+r} \\
&\leq C|1-\lambda|^{2(n+1)}(1+|\lambda|)^{n+r}(1-|\lambda|)^{r-1}
\end{aligned}$$

for all $\lambda \in \mathbb{D}$.

So J extends continuously to $\partial\mathbb{D}$, and if $r > 1$, $J|_{\partial\mathbb{D}} \equiv 0$. By the Maximum Principle, J is identically zero. This is a contradiction. Therefore, $r = 1$, and Ω is biholomorphic to the unit ball in $\mathbb{C}^n$. $\square$

6.5 Proof of Theorem 1.0.1

Proof. The result follows immediately from Propositions 6.1.3, 6.2.3, 6.3.4, and 6.4.4.

$\square$

Chapter 7

Conclusion

The initial inspiration for Theorem 1.0.1 is a result by Wong:

Theorem 7.0.1 (Theorem 1.5 in [31]). *Let M be a compact Kähler surface which is hyperbolic in the sense of Kobayashi. Suppose that*

1. *$M = \widetilde{M}/\Gamma$, where $\widetilde{M}$ is the universal cover of M and Γ is a discrete subgroup of the identity component of $\operatorname{Aut}\left(\widetilde{M}\right)$ acting freely on $\widetilde{M}$.*

2. *Γ is not isomorphic to the fundamental group of a compact real surface.*

Then $\widetilde{M}$ is biholomorphic to either the unit ball in $\mathbb{C}^2$ or the bidisk.

Our method of proof was inspired by the previously mentioned result of Cheung, et al. [5], as well as, of course, Zimmer's paper [35].

Theorem 7.0.2. *Let M be a hyperbolic complex surface. Let $\Omega \subset\subset M$ be a subdomain with smooth boundary (C^2 is enough). If Ω admits a compact quotient, then either*

1. *Ω is biholomorphic to a ball, or else*

2. The universal covering of Ω is biholomorphic to a bidisk.

The proof of Theorem 7.0.2 is split into two parts. The first part is concerned with when the boundary $\partial\Omega$ contains a strictly pseudoconvex point (totally real). In this case, we have Theorem 1.0.2, a stronger result which is true in any dimension. Our contribution was to replace the smooth boundary condition with the slightly weaker $C^{1,1}$ boundary condition.

The second part of the proof deals with the case when the boundary does not contain a strictly pseudoconvex point. In this case, the dimension 2 condition is necessary, and the argument is slightly more involved, requiring work from [13] and [14]. This is the case that yields the bidisk. The argument uses estimates on invariant volume forms on the bidisk.

We can replace the smooth boundary condition in Theorem 7.0.2 with the $C^{1,1}$ boundary condition to obtain the following result:

Proposition 7.0.3. *Let M be a hyperbolic complex surface, and let $\Omega \subset\subset M$ be a subdomain with $C^{1,1}$ boundary. If Ω admits a compact quotient, and $\partial\Omega$ does not contain a totally real boundary point, then the universal cover of Ω is biholomorphic to a bidisk.*

The proof of this result will likely be similar to the one presented in Section 4 of [5] with some minor modifications to the argument where the defining function for $\partial\Omega$ is concerned. In particular, we choose the defining function r to be a $C^{1,1}$ function rather than C^∞. We suspect some technical issues arising from this alteration, but the argument should still follow much of the same logic.

In n dimensions, the situation becomes much more complicated. We believe that in the absence of a totally real boundary point, we will obtain a universal cover biholomorphic to some higher rank symmetric space, or a compact manifold quotiented by a disk. Or we will be in the situation where, at each point, there will be a bidisk properly embedded in the universal cover. In any case, the argument will depend on the dimension of the disk sitting on the boundary of the domain Ω.

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