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Testing for Unit Roots with Stationary Covariates

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Abstract.

We derive the family of tests for a unit root with maximal power against a point alternative when an arbitrary number of stationary covariates are modeled with the potentially integrated series. We show that very large power gains are available when such covariates are available. We then derive tests which are simple to construct (involving the running of vector autoregressions) and achieve at a point the power envelopes derived under very general conditions. These tests have excellent properties in small samples. We also show that these are obvious and internally consistent tests to run when identifying structural VAR's using long run restrictions.

Keywords: Unit Roots, Power Envelopes, Structural vector autoregressions.

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1. Introduction.

Due to the effects of the assumption of a unit root in a variable on both the econometric method used and the economic interpretation of the model examined, it is quite common to pre-test the data for unit roots. This is typically done by either (or both) testing variables one by one for unit roots or by examining cointegrating rank using Johansen (1988) tests or their asymptotic equivalent.

In testing variables one by one, commonly the t-test method of Dickey and Fuller (1979) is employed. This hypothesis test is asymptotically optimal when the data is stationary and is a natural statistic to consider. However in the unit root case there are many other tests available that have greater power. Elliott et. al (1996) (denoted ERS in the remainder of the paper) showed that there is no uniformly most powerful test for this problem and derived tests that were approximately most powerful in the sense that they have asymptotic power close to the envelope of most powerful tests for this problem.

This paper considers a model where there is one series that potentially has a unit root, and that this series potentially covaries with some available stationary variables. In a model similar to the one examined here, Hansen (1995) demonstrated in a model with no deterministic terms that no uniformly most powerful test for a unit root in the presence of stationary covariates exists and that power gains are to be had from using these covariates. He suggested covariate augmented Dickey Fuller (CADF) tests and showed that these tests had greater power than tests that ignored these covariates[1].

This paper extends the results in Hansen (1995) in a number of ways. First, we show that the point optimal tests implicit in the power envelope derived in Hansen (1995) and computed when all nuisance parameters are known are feasible when these parameters are not known. We also extend the results by deriving the power envelope in the more empirically relevant cases of where constants and/or time trends are also included in the regression. We propose tests that are feasible to construct with data and attain the power envelope at a point. These tests have good power at other points as well. We then show that these are natural tests to report in justifying the unit root assumption in the popular method of identifying structural vector autoregressions (VAR's) from long run restrictions (as suggested by Blanchard and Quah (1989)).

The paper is set up as follows. In the next section the model is introduced, and the power bounds for the problem are established. In the third section, tests which feasibly attain these power bounds at a point are derived and discussed. Section four examines the tests empirically using Monte Carlo methods. A fifth section discusses the tests as they relate to identifying structural VAR's from long run restrictions. The

[1] There is also a discussion of this work in Caporale and Pittis (1999).

final section concludes. All proofs are contained in a separate appendix, available from the authors upon request.

2. Model and Power Envelopes.

Consider the model

$$z_t = b_0 + b_1 t + u_t \qquad t = 1,...,T \tag{1}$$

and

$$A(L)\begin{pmatrix}(1-rL)u_{y,t}\\ u_{x,t}\end{pmatrix} = e_t \tag{2}$$

where z_t=[y_t,x_t']', x_t is an mx1 vector, y_t is 1x1, β_0 =[β_{y0} ,β'_{x0}]', β_1 =[β_{y1} ,β'_{x1}]' , u_t =[$u_{y,t}$,$u'_{x,}$]' and A(L) is a matrix polynomial of finite order k in the lag operator L. For the constructed test statistics we will assume that

A1. |A(z)|=0 has roots outside the unit circle.

A.2 $E_{t-1}(e_t)$=0. $E_{t-1}(e_t e_t')$=Σ and $\sup_t \|e_t\|^{2+d} < \infty$ (a.s.) for some δ>0, where Σ is positive definite and E_{t-1}(.) denotes conditional expectation with respect to $\{e_{t-1}, e_{t-2},...\}$.

A3. u_0,u_{-1},…,u_{-k} are $0_p(1)$.

Define $u_t(r) = [(1-rL)u_{y,t} \quad u_{xt}']'$ with spectral density at frequency zero (scaled by 2π) Ω, so we have $\Omega = A(1)^{-1}\Sigma A(1)^{-1}{}'$ where we can partition this after the first column and row so that

$$\Omega = \begin{bmatrix} w_{yy} & w_{yx} \\ w_{yx}' & \Omega_{xx}\end{bmatrix}$$

(we partition Σ similarly). We will further define $R^2 = w_{yy}^{-1} w_{yx} \Omega_{xx}^{-1} w_{yx}'$, the frequency zero correlation between the shocks to x_t and the quasi differences of y_t. The R^2 value represents the contribution of the stationary variables - it is equal to zero when there is no long run correlation and one if there is perfect correlation. We impose that R^2<1, hereby ruling out the case under the null where the partial sums of x_t cointegrate with y_t. If there is such a cointegrating relation, this should be modeled in the system taking the model outside this framework (unless the coefficients of the cointegrating vector is known, in which case the model can be rotated back into this framework, see Elliott et. al. (2002)).

We consider five cases indexed by superscript i (i=1,2,3,4,5) for the deterministic part of the model (where parameters are free unless otherwise stated)

Case 1: $b_{y0} = b_{y1} = 0$ and $b_{x0} = b_{x1} = 0$.

Case 2: $b_{y1} = 0$ and $b_{x0} = b_{x1} = 0$.

Case 3. $b_{y1} = 0$ and $b_{x1} = 0$.

Case 4: $b_{x1} = 0$.

Case 5: No restrictions.

Each of these cases can be characterized by the restriction $\left(I_{2(m+1)} - S_i\right)b = 0$ where β = [β_0' β_1']', S_i is a 2(m+1)x2(m+1) matrix where S_1=0, $S_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $S_3 = \begin{pmatrix} I_{m+1} & 0 \\ 0 & 0 \end{pmatrix}$, $S_4 = \begin{pmatrix} I_{m+2} & 0 \\ 0 & 0 \end{pmatrix}$ and S_5 is the identity matrix.

This represents a fairly general set of models in which we have a VAR in the model of x and the quasi difference of y. We wish to test that the parameter ρ is equal to one (y_t has a unit root) against alternatives that this root is less than one. Following the general methods of King (1980, 1988) we will examine Neyman-Pearson tests for this hypothesis. Following the application of these methods to testing for unit roots in ERS and Elliott (1999) we will examine Neyman-Pearson tests for this hypothesis under simplifying assumptions, and then in the following section we will derive general tests that are asymptotically equivalent to these optimal tests.

With the assumption that A(L)=I (so that Ω=Σ) and assuming the e_t are normally distributed and u_{y0}=0 we will examine tests against the local alternative that c = $\bar{c}$ <0 where $r = 1 + c/T$ and $\bar{r} = 1 + \bar{c}/T$ with c, $\bar{c}$ fixed (we will suppress the dependence of ρ on T in the notation).

The system likelihood ratio test statistic for the hypothesis is given by

$$\Lambda^i(1, \bar{r}) = \sum_{t=1}^{T} \hat{u}_t^i(\bar{r})'\Sigma^{-1}\hat{u}_t^i(\bar{r}) - \sum_{t=1}^{T} \hat{u}_t^i(1)'\Sigma^{-1}\hat{u}_t^i(1)$$

where we have for r = $\bar{r}$,1 that

$$\hat{u}_t^i(r) = z_t(r) - d_t(r)'\hat{b}^i(r)$$

where $z_t(r) = [(1-rL)y_t, x_t']'$ for t>1 and $z_1(r) = [y_1, x_1']'$,

$d_t(r)' = \begin{bmatrix} 1-r & 0 & (1-rL)t & 0 \\ 0 & I_m & 0 & I_m t \end{bmatrix}$ for t>1, $d_1(r)' = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & I_m & 0 & I_m \end{bmatrix}$, and

$\hat{b}^i(r) = \left[S_i \left(\sum_{t=1}^{T} d_t(r) \Sigma^{-1} d_t(r)' \right) S_i \right]^{-} \left[S_i \sum_{t=1}^{T} d_t(r) \Sigma^{-1} z_t(r) \right]$ where D^- is the Moore Penrose inverse of D. The test has rejection regions of the form $\{y_t, x_t : \Lambda^i(1, \bar{r}) - \bar{c} < b\}$ where b is a critical value.

Theorem 1[2].

For the model in (1) and (2) with A(L)=I, e_t independent N(0,S) random variables and A3 holding then with $r = 1 + c/T$ and $\bar{r} = 1 + \bar{c}/T$ with c, $\bar{c}$ fixed as T ® ¥ the most powerful test of H_0: c=0 vs. H_a: c= $\bar{c}$ <0 has asymptotic power functions

$$P(c, \bar{c}, R^2) = \Pr[y^i(c, \bar{c}, R^2) < b(\bar{c}, R^2)]$$

where $y^i(c, \bar{c}, R^2) = g^i(c, \bar{c}) + (\bar{c}^2 - 2\bar{c}c) Q \int (W_{1c}^i)^2 + 2\bar{c} Q^{1/2} \int W_{1c}^i dW_2 + h^i(c, \bar{c}, R^2)$,

$b(\bar{c}, R^2)$ *is a constant,* $Q = R^2/(1-R^2)$, $W_{1c}^1 = W_{1c}$, $W_{1c}^i = W_{1c} - \int W_{1c}$ *for i=2,3,4 and*

$W_{1c}^5 = W_{1c} - (4-6s)\int W_{1c} - (12s-6)\int sW_{1c}$, $g^i(c, \bar{c}) = \bar{c}^2 \int W_{1c}^2 - \bar{c} W_{1c}(1)^2$ *for i=1,2,3 and*

$g^i(c, \bar{c}) = \bar{c}^2 \int W_{1c}^2 + (1-\bar{c}) W_{1c}(1)^2 - k^{-1}\left[(1-\bar{c})W_{1c}(1) + \bar{c}^2 \int sW_{1c}\right]^2$ *for i=4,5,* $h^i(c, \bar{c}, R^2)$ *is*

zero except for $h^4(c, \bar{c}, R^2) = k^{-1}\left[(1-\bar{c})W_{1c}(1) + \bar{c}^2 \int sW_{1c}\right]^2 -$

$(k + Q\frac{\bar{c}^2}{12})^{-1}\left\{(1-\bar{c})W_{1c}(1) + \bar{c}^2 \int sW_{1c} + Q\left[\frac{\bar{c}}{2}(c-\bar{c})\int W_{1c} - \bar{c}(c-\bar{c})\int sW_{1c}\right] + Q^{1/2}\left[\bar{c}\int s dW_2 - \frac{\bar{c}}{2}\int dW_2\right]\right\}^2$,

and $k = 1 - \bar{c} + \bar{c}^2/3$. *All integrals are 0 to 1 over s with s suppressed, so e.g.* $\int W_{1c} = \int_0^1 W_{1c}(s) ds$ *and*

$W_{1c}(s) = c\int_0^s e^{c(s-l)} W_1(l) dl + W_1(s)$, *$W_1$ and W_2 are independent univariate standard Brownian motions.*

In case 1 this is apart from a scale factor the same as that reported in Hansen (1995)[3]. A number of features are noteworthy. Firstly, the dependence of the test on $\bar{c}$ indicates that no uniformly most powerful test is

[2] Proofs are available in a UCSD discussion paper version of this paper.

available for this problem, power depends on the choice of the alternative. Second, the distribution of the test is nonstandard. Third, the optimal test statistic depends on Σ and its distribution depends on the parameter R^2. When $R^2=0$ then $y^i(c,\bar{c},R^2)=g^i(c,\bar{c})$ which is equivalent to the asymptotic limits of the tests derived in ERS, thus the most powerful tests coincide asymptotically with tests with the relevant invariance properties with respect to deterministic terms which do not use the information in the covariates (Cases 1-3 equate to the constant included case, Cases 4-5 are the time trend included result). When R^2 is nonzero the optimal univariate and system tests are different, indicating that information is lost when information in the covariates is ignored.

The results in the Theorem give the local power for any choice of $\bar{c}$ at any local alternative c. When we set $\bar{c}$ =c, we obtain by construction the test that has the highest attainable power. By evaluating the powers setting c=$\bar{c}$ we obtain the envelope of greatest asymptotic power, which we call the power envelope. Figure 1 examines the power envelopes for various R^2. The power envelope when $R^2=0$ has the lowest power - this is the relevant envelope if no covariate information is employed. When R^2 is greater than zero, the power attainable increases considerably above this lower bound. Hence, use of covariates has the potential to greatly increase the power of tests for a unit root, as indicated by Hansen (1995). The larger is R^2, the more powerful the optimal test[4]. These results are true for each of the various assumptions on the deterministic terms[5]. Comparing the first two panels in Figure 1 we see the effect of estimating the constant terms. This effect is small, e.g. when $R^2=0.5$ and c=-5 the power envelope in the constants known case is 70% whilst when the constants are unknown this power is 62%. Both of these powers are substantially above that of the case where no covariates are employed, where the envelope attains a power of 32%.

As in the case where there are no covariates, the effect on the power envelopes for the case where the trend terms (coefficients on time trends) are not known is quite large. In the case mentioned above, where $R^2=0.5$ and c= -5 the maximal power in case 5 is 33%, far below the 62% when only coefficients on the constants are known. Notice though that the maximal power in this case even when constants and coefficients on the time trend are estimated is (just) above that for the case where stationary covariates are ignored and the coefficient on the time trend is known. In general the power losses from not knowing the coefficient on the trends in the x_t regressions is small (differences between cases 4 and 5, not pictured in the figures), between zero (when R^2 is small) and 6% or so (when R^2 is large). There is clearly the potential for much to be gained

[3] We also have a notational difference in that our R^2 is defined in Hansen (1995) as $1-R^2$. We changed the notation to accord with the usual use of R^2.
[4] The asymptotic results are not appropriate at $R^2=1$, which is readily seen from the limit expression which would not be finite at this point.
[5] Case 2 and case 1 are asymptotically identical, so we omit case 2. Case 4 has functions similar to Case 5 and is omitted.

in terms of power from exploiting stationary covariates in constructing tests for a unit root. The construction of tests that achieve these gains is addressed in the next section.

3. Feasible Tests.

In this section we derive families of tests that asymptotically attain the power bounds derived above at pre-specified points, relaxing the normality and known nuisance parameter assumptions. The method for constructing the test is set out in 4 steps

(a) Estimate nuisance parameters for detrending and R^2.

Run a VAR $A(L)z_t(1)$ = deterministics + e_t including no deterministic terms for case 1, constants for cases 2 and 3, constants and time trends for cases 4 and 5. Using the residuals from the VAR[6] construct $\hat{\Sigma} = T^{-1}\sum_{t=k+1}^{T}\hat{e}_t\hat{e}_t'$, $\hat{\Omega} = \hat{A}(1)^{-1}\hat{\Sigma}\hat{A}(1)^{-1\prime}$ and $\hat{R}^2 = \hat{w}_{yx}\hat{\Omega}_{xx}^{-1}\hat{w}_{yx}'/\hat{w}_{yy}$ where $\hat{A}(1) = I + \sum_{i=1}^{k}\hat{A}_i$ and A_i is the (i+1)th matrix element of A(L).

(b) Construct detrended data under the null and alternative hypotheses, i.e. construct for r=(1, ρ)

$$\tilde{u}_t^i(r) = z_t(r) - d_t(r)'\tilde{b}^i(r)$$

where

$$\tilde{b}^i(r) = \left[S_i\left(\sum_{t=1}^{T} d_t(r)\hat{\Omega}^{-1}d_t(r)'\right)S_i\right]^{-}\left[S_i\sum_{t=1}^{T} d_t(r)\hat{\Omega}^{-1}z_t(r)\right] \tag{3}$$

(c) Run VAR's (for r= 1, $\bar{r}$), i.e. run $\tilde{A}(L)\tilde{u}_t^i(r) = \tilde{e}_t(r)$ and construct the estimated variance covariance matrices

$$\tilde{\Sigma}(r) = T^{-1}\sum_{t=k+1}^{T}\tilde{e}_t(r)\tilde{e}_t(r)'$$

(d) Construct the test statistic

$$\tilde{\Lambda}^i(1,\bar{r}) = T\left(tr\left[\tilde{\Sigma}(1)^{-1}\tilde{\Sigma}(\bar{r})\right] - (m+\bar{r})\right)$$

[6] In practice one can choose the lag length of the VAR through theory or a consistent lag length estimator such as the BIC information criterion.

This test will have asymptotic power that achieves the power bound at $\bar{c}$ under the assumptions.

Theorem 2.

For the model in (1) and (2) with assumptions A1, A2 and A3 holding and deterministic terms correctly specified for each case then as $T \to \infty$

$$\tilde{\Lambda}^i(1,\bar{r}) \Rightarrow y^i(c,\bar{c},R^2)$$

where $\Rightarrow$ *denotes weak convergence.*

Thus the critical values for the test depend on the alternative chosen ($\bar{c}$) and R^2. The feasible test asymptotically achieves the highest power possible at $\bar{c}$. We have chosen here to let $\bar{c}$ = -7 for cases 1-3 and $\bar{c}$ =-13.5 for cases 4 and 5 (which follows the choice of ERS). In principle and practice we could choose different values for $\bar{c}$ depending on R^2, however as R^2 rises above zero lack of power is becoming less problematic so it seems reasonable to us to choose $\bar{c}$ for the worst case scenario.

Asymptotic critical values for the test for selected values of R^2 are given in Table 1. The relevant critical value is determined for the estimated $\hat{R}^2$. For values of R^2 between the ones given in Table 1, interpolation can be used to approximate the critical value.

4. Evaluation of the Tests.

4.1. Large Sample Evaluation.

Figure 2 examines the power of the feasible test for Cases 1, 3 and 5 in each panel respectively. The figures give the results for R^2 = 0.3, 0.5 and 0.7. Accompanying the power curves are the power envelopes for comparison. The feasible point optimal test has power that is close to the power envelope, suggesting that there is little asymptotic power loss at points away from where the test is optimal, especially for lower values for R^2. This is similar to results of ERS, where for R^2=0 this was found to be true. When R^2 = 0.5 the difference between the power envelope and the asymptotic power of the feasible test is small for alternatives a moderate distance and further from the null, but a little larger for alternatives close to the null. This becomes more apparent for larger R^2. To the extent that very large values for R^2 are probably not too relevant empirically, this may not be too much of a problem. The suggestion from these graphs appears to be that the most useful choice of $\bar{c}$ in practice may depend on R^2. We also examined the power curves for the case where $\bar{c}$ =-7 to perhaps improve the closeness of the power curves to the envelopes for these near alternatives. When this alternative is chosen this indeed happens, however the tradeoff is that the power curves for R^2 small are not as close to the envelope for more distant alternatives. Thus we recommend choosing $\bar{c}$ =-13.5 as power is more of a concern when R^2 is small.

The power gains are clearly substantial for each of the cases for the deterministic terms (results for case 4 are similar to those in case 5). Consider the gains from using covariates when R^2=0.5. At the local alternative $\bar{c}$ = -5, in case 3 power rises by 30% and in case 5 power rises by 35%. Such gains in power substantially improve the odds of correctly distinguishing a process with a unit root from a slowly mean reverting process.

4.2. Small Sample Evaluation.

We will examine various special case models in samples of 100 observations. Along with the above tests, we report results for the commonly applied test of Dickey and Fuller (1979) and also the P_T test of ERS as well as the Hansen (1995) CADF test.

Table 2 reports results of simulations of the model in (1) and (2) for each of the cases 1, 3 and 5 respectively where A(L)=I (and this is known), e_t is normally distributed with variances equal to 1 and R^2 as reported in the Table. Size is given in the row corresponding to ρ =1 and (empirical) power against the indicated alternatives in the following rows. When there are no deterministic terms in the model the DF and P_T single equation tests do similarly well (see ERS for a discussion of this similarity). In the test proposed here, when R^2=0 power and size are comparable to the univariate tests indicating that even in small samples little may be lost by including extraneous information and doing the system test. As R^2 increases, size remains well controlled whilst power rises considerably. Consider the case of the true ρ being equal to 0.96, the P_T test has power around 23% whilst if R^2 =0.25 the system test has power equal to 34%, roughly a 50% gain.

When a constant is included, the P_T statistic gains in power over the Dickey and Fuller (1979) t test are very large. Again, when R^2=0 the test proposed here has similar size and power to the P_T statistic indicating that little is lost adding extraneous stationary covariates. In general, size is less well controlled, especially for R^2 close to one (where the asymptotic theory would no longer be relevant, however it would not be expected that such models would be appropriate for real world data) . There is some evidence of power losses from not knowing the constant term. At a value of ρ = 0.96 the power when the constant is known (or zero) power is 49% compared to the unknown constant power of 45% when R^2=0.49. Even so, power for the test with the constant unknown is quite high in many cases, and is far beyond that achievable when covariates are not employed.

Similar results are found for the detrended (case 5) model. In both of these cases power when using covariates is substantially greater than when relevant covariates are ignored (for example, in case 3 when ρ =0.9, power of the test proposed here when R^2=0.25 is 20% for the Dickey and Fuller test and is 49% for the test with covariates employed. There are as usual power losses in including a time trend. In the case of ρ = 0.96 and R^2=0.25 the power drops from 36% in case 3 to 13% in case 5.

The effect of estimating R^2 in the computation of the test is examined in Table 3 (for cases 3 and 5 in each of the panels respectively). Here the results when R^2 is estimated are repeated from Table 1 on the right hand side panels, whilst the same results using the critical value chosen using the true R^2 are given in the left hand panels. There is very little difference, even in a sample of 100 observations. Most of the differences in size and power are at the third decimal place. It is only for case 5 when R^2 is a little larger that there is much of an effect, but the effect is minor (in these cases there is a small power loss from estimating R^2).

Table 4 compares the CADF test of Hansen (1995) with the feasible test derived here (again for the leading cases 3 and 5 respectively). The CADF test augments the usual Dickey and Fuller (1979) test with lags, leads and the contemporaneous values of x_t. In this table, with no serial correlation, this amounts to including x_t as a regressor in the ADF regression and then constructing the t-test of the unit root hypothesis as normal. As shown in Hansen (1995) this test also depends on R^2. In the comparison we use the same value of R^2 to compute critical values for each of the tests. In the first column of the CADF results, where R^2=0, we have essentially the same results as the Dickey and Fuller (1979) test in Tables 3 and 5 that ignores the covariates. This should be the case, the included x_t variable in the ADF regression has a population coefficient of zero in this case. Likewise, the first column of the $\hat{\Lambda}(1,\bar{r})$ test matches with the P_T test for the reasons we have described. This gives an insight into the difference in the two approaches, the difference between the CADF and $\hat{\Lambda}(1,\bar{r})$ is similar to the difference between the Dickey and Fuller (1979) approach and the ERS approach. When R^2>0, we see that the $\hat{\Lambda}(1,\bar{r})$ test outperforms the CADF test in terms of power, although is slightly worse in size performance. The increases in power can be quite large. In the case 3 when R^2= 0.09 the power of the $\hat{\Lambda}(1,\bar{r})$ test is two to three times that of the CADF test. For case 5 the effects are not as dramatic, but still power gains of 50% or so are available from using the covariates test proposed here over the CADF test.

5. Unit Root Tests and Long Run Structural VAR Estimation.

Blanchard and Quah (1989) derive a method for identifying structural VAR's from restrictions placed on the spectral density of the data at frequency zero when there are known unit roots in the system. Consider the bivariate version of the model considered in this paper when we impose that the root ρ is equal to unity,

$$A(L)\begin{bmatrix}\Delta y_t \\ x_t\end{bmatrix} = e_t .$$

Inverting the lag polynomial gives us

$$\begin{bmatrix}\Delta y_t \\ x_t\end{bmatrix} = C(L)e_t = C(L)KK^{-1}e_t = D(L)h_t$$

where $C(L)=A(L)^{-1}$ and $E[\eta_t\eta_t']=I$. This model is not identified in the usual sense as for any of the infinite possible invertible matrices K we obtain a different structural model. In this bivariate system we require a single restriction so that the rotation K is unique for the model to be identified (this would be the order condition).

In such systems, y_t is permanently affected by shock(s) since it is an integrated process. On economic grounds, it may be interesting to identify the model such that only one of the structural shocks has a permanent effect on y_t. In Blanchard and Quah (1989) this argument meant that demand shocks could not have a permanent effect. In King et. al (1991) cointegration was used to imply a smaller number of permanent shocks than total shocks. In such cases it is possible to identify the model as the cumulated sum of the structural impulse responses, D(1), will be triangular as only one of the shocks has a long run effect on y_t.

For the model above, the identification scheme would set the (1,2) component of D(1) equal to zero. Since the spectral density of the data at frequency zero (scaled by 2π) is $\Omega = D(1)D(1)'$ this amounts to taking the choleski decomposition of the estimated matrix $\hat{\Omega}$. Such a restriction is only interesting and useful in identification when the off diagonals for Ω are indeed nonzero, i.e. when $R^2>0$.

The crux of this approach to identification clearly is that y_t indeed does have a unit root. If instead there were no permanent effects then we would interpret D(1) differently and would have no reason to make this matrix triangular. So in practice a useful hypothesis test to report in undertaking this method would be a test for a unit root in y_t. Further, when the imposed restriction is indeed informative, then $R^2>0$ and hence we are exactly in the cases where the tests of this paper yield power gains over univariate testing. Typically, such tests for a unit root to provide evidence of the validity of this restriction are undertaken using Dickey Fuller

(1979) tests (see Gali (1999) for example), which neither use the full information in the model nor are they the most powerful univariate tests. The tests derived in this paper provide a natural test of the basic identification assumption of the Blanchard and Quah identification scheme.

We apply the tests derived here and other common tests to the Blanchard-Quah dataset. The data is quarterly data on income and unemployment for the US from 1950:2 to 1987:4, where unemployment is the stationary variable x_t and income is the y_t variable. We include constants and time trends in both unemployment and income[7] (so the tests are from case 5) and follow Blanchard and Quah in choosing eight lags. The DF statistic is -1.78 and the DF-GLS statistic of ERS is -1.37. Neither is close to rejecting for a 5% or 10% test. The $\tilde{\Lambda}^5(1, \bar{r})$ test is 17.93. For the estimated R^2 of 0.76 the critical value is 16.56, so we have a p-value of 0.07 and fail (but only just) to reject at 5% and so find some support for the Blanchard and Quah assumption[8].

6. Conclusion.

Typically in economics correlation between the variables is the rule rather than the exception. Often these are implied by theory. Either way, this information can be extremely valuable in testing assumptions that are ancillary to the modeling process. This appears to be especially true in the case of testing for a unit root. Hansen (1995) showed this with tests he developed based around the statistic of Dickey and Fuller (1979). In a related paper Horvath and Watson (1995) showed that power gains are available when there are known cointegrating relationships (which are then stationary variables). We have shown here that even greater gains are possible. The statistics are simple to implement and yield extremely large gains in power when the covariates are relevant.

The statistics we generate, useful in many areas, are directly applicable to testing the unit root assumption in the identification of structural VAR's from long run restrictions. These restrictions do not make sense unless there is a process with a unit root in the model, yet typically very low power tests are used to examine this assumption. The tests derived here will have much better power at detecting the mistaken use of this procedure.

[7] Blanchard and Quah included a time trend in unemployment on the grounds that it was increasing over the sample. They had the equivalent of a time trend with a break for the oil shocks in income. We do not include a 'known' break such as this, however not including the break if it were truly there (tests which search for such a break typically fail to reject the hypothesis of no break) biases us away from rejecting the unit root.

[8] We do reject for 7 lags, but not for shorter lags than this.

References.

Blanchard, O. and D. Quah (1989), The Dynamic effect of Aggregate Demand and Aggregate Supply Disturbances, American Economic Review, 79, 655-73.

Caporale, G.M. and N. Pittis (1999), "Unit Root Testing Using Covariates: Some Theory and Evidence", Oxford Bulletin of Economic Studies, 61, 583-95.

Dickey, D.A. and W.A.Fuller (1979), Distribution of Estimators for Autoregressive Time Series with a Unit Root, Journal of the American Statistical Association, 74, 427-431.

Elliott, G. (1999), Efficient Tests for a Unit Root when the Initial Observation is Drawn From its Unconditional Distribution, International Economic Review, 40, 767-783.

Elliott, G., M. Jansson and E. Pesavento (2002), Optimal Power for Testing Potential Cointegrating Vectors with Known Parameters for Nonstationarity, manuscript, UCSD.

Elliott G., T.J. Rothenberg, and J.H. Stock (1996), Efficient Tests for an Autoregressive Unit Root, Econometrica, 64, 813-836.

Gali, J. (1999), Technology, Employment and the Business Cycle: Do Technology Shocks Explain Aggregate Fluctuations?, American Economic Review, 89, 249-271.

Hansen, B. E. (1995), Rethinking the Univariate Approach to Unit Root Testing: Using Covariates to Increase Power, Econometric Theory, 11, 1148-1172.

Horvath, M.T.K and M.W. Watson (1995), Testing for Cointegration when Some of the Cointegrating Vectors are Known, Econometric Theory, 5, 952-983.

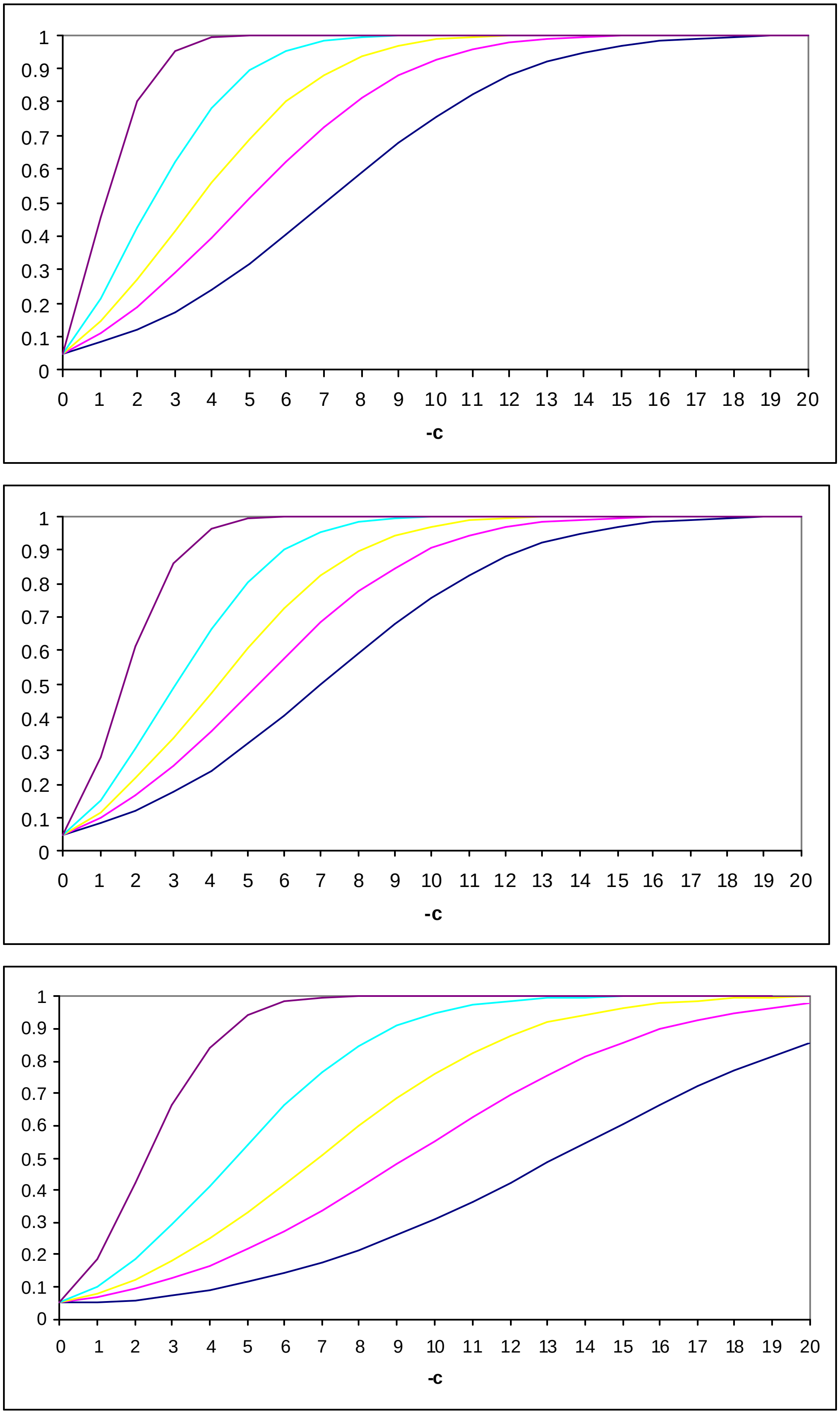
Johansen, S. (1988), Statistical Analysis of Cointegrating Vectors, Journal of Economic Dynamics and Control, 12, 231-54.

King, M.L. (1980), Robust Tests for Spherical Symmetry and their Application to Least Squares Regression, Annals of Statistics, 8, 1265-1271.

King, M.L. (1988), Towards a Theory of Point Optimal Testing, Econometric Reviews, 6, 169-218.

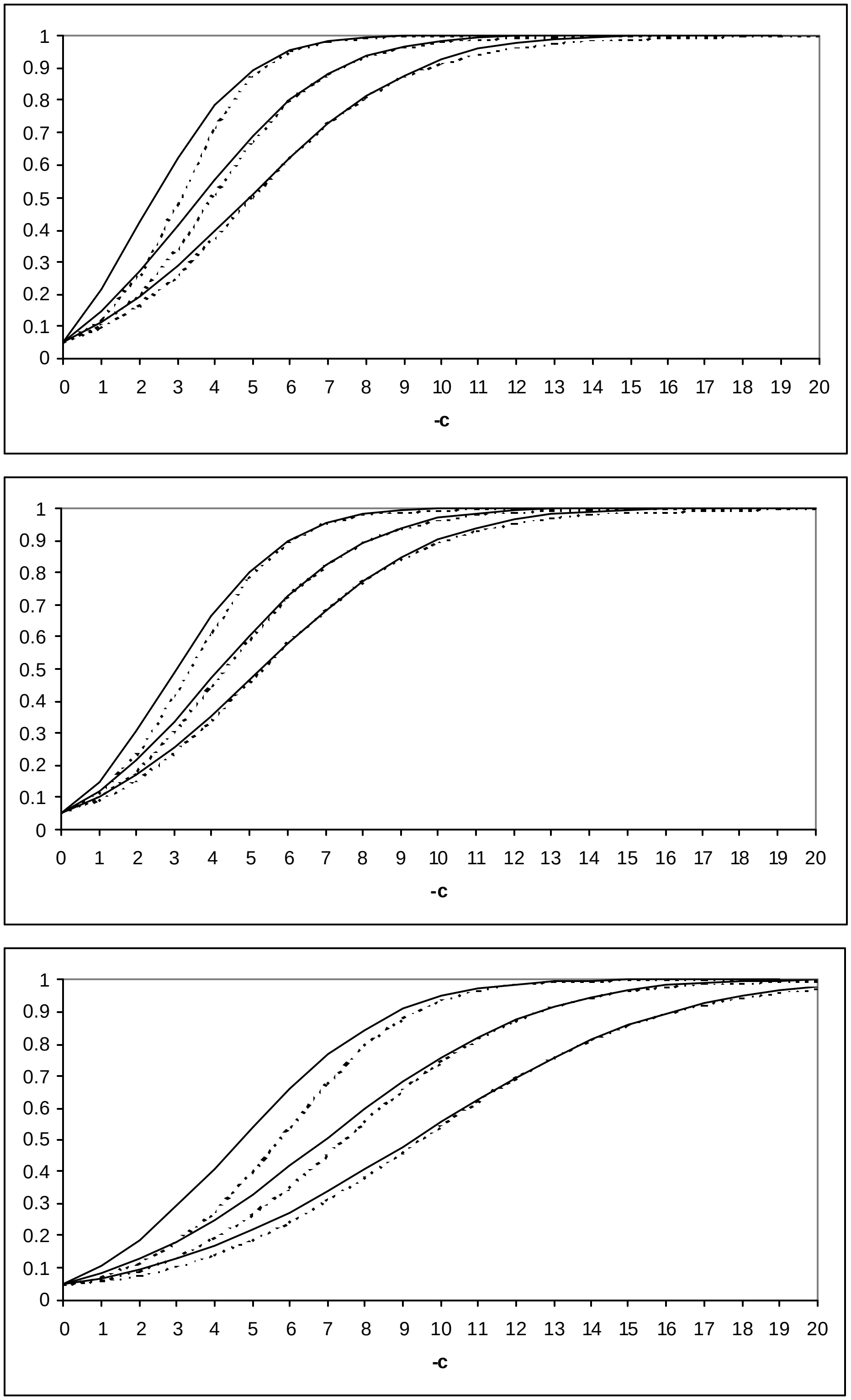
King, R.G, Plosser, C.I., Stock, J.H. and Watson, M.W. (1991), Stochastic Trends and Economic Fluctuations, American Economic Review, 81, 819-40.

Figure 1: Power Envelopes for Cases 1,3 and 5 respectively.



Note: Envelopes for R^2=0,0.3,0.5,0.7 and 0.9 where power is increasing in R^2.

Figure 2: Power Envelopes and Power curves for Cases 1,3 and 5



Note: Unbroken lines are Envelopes for R^2=0.3,0.5 and 0.7 and broken lines are power of Point Optimal tests for each R^2 where power is increasing in R^2.

Table 1: Asymptotic Critical Values (Distribution in Theorem 3)

R^2	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Cases 1,2	3.34	3.41	3.54	3.76	4.15	4.79	5.88	7.84	12.12	25.69
Case 3	3.34	3.41	3.54	3.70	3.96	4.41	5.12	6.37	9.17	17.99
Case 4	5.70	5.79	5.98	6.38	6.99	7.97	9.63	12.6	19.03	39.62
Case 5	5.70	5.77	6.00	6.40	7.07	8.15	10.00	13.36	20.35	41.87

Notes: Critical values were computed using 1500 steps as approximations to the Brownian Motion terms in the limit theorem representations and 60000 replications. The critical values reported are for tests of size 5% with $\bar{c}$ =-7 for cases 1, 2 and 3 and $\bar{c}$ = -13.5 for cases 4 and 5.

Table 2: Small Sample results for $\tilde{\Lambda}^i(1, \bar{r})$

	DF	PT	$\tilde{\Lambda}^i(1, \bar{r})$				
R^2 =	0	0	0	0.09	0.25	0.49	0.81
r							
Case 1: No Determinstic Terms							
1	0.05	0.048	0.051	0.049	0.05	0.05	0.044
0.98	0.117	0.113	0.119	0.132	0.153	0.195	0.306
0.96	0.237	0.229	0.239	0.276	0.342	0.493	0.848
0.94	0.407	0.396	0.407	0.463	0.576	0.782	0.992
0.92	0.594	0.581	0.59	0.655	0.774	0.926	0.999
0.9	0.758	0.744	0.748	0.807	0.896	0.977	1
0.88	0.878	0.865	0.867	0.905	0.954	0.993	1
0.86	0.947	0.939	0.936	0.957	0.981	0.998	1
Case 3: Constants in each Regression							
r							
1	0.054	0.059	0.064	0.061	0.06	0.054	0.039
0.98	0.075	0.138	0.145	0.154	0.167	0.192	0.254
0.96	0.105	0.273	0.285	0.308	0.355	0.445	0.716
0.94	0.159	0.453	0.466	0.499	0.572	0.709	0.946
0.92	0.235	0.64	0.648	0.685	0.759	0.875	0.991
0.9	0.332	0.795	0.797	0.825	0.879	0.951	0.998
0.88	0.448	0.899	0.897	0.914	0.943	0.981	1
0.86	0.573	0.956	0.951	0.959	0.974	0.992	1
Case 5: Constants and Time Trends in each Regression							
r							
1	0.057	0.039	0.053	0.053	0.051	0.044	0.021
0.98	0.062	0.049	0.065	0.069	0.076	0.085	0.08
0.96	0.078	0.076	0.099	0.111	0.131	0.172	0.262
0.94	0.106	0.119	0.152	0.173	0.223	0.32	0.599
0.92	0.147	0.184	0.226	0.267	0.345	0.511	0.871
0.9	0.204	0.27	0.325	0.379	0.488	0.699	0.971
0.88	0.277	0.377	0.441	0.507	0.634	0.834	0.993
0.86	0.365	0.503	0.564	0.635	0.758	0.919	0.998

Notes: Based on 20000 replications of the model with T=100, normal errors as discussed in the text. The system test is implemented with R^2 estimated.

Table 3: Effect of estimating R^2

	R^2 known					Estimated R^2				
R^2 =	0	0.09	0.25	0.49	0.81	0	0.09	0.25	0.49	0.81
Case 3: Constants in each equation										
r										
1	0.063	0.06	0.061	0.056	0.053	0.064	0.061	0.06	0.054	0.039
0.98	0.144	0.152	0.167	0.193	0.29	0.145	0.154	0.167	0.192	0.254
0.96	0.283	0.305	0.356	0.45	0.758	0.285	0.308	0.355	0.445	0.716
0.94	0.465	0.497	0.573	0.716	0.967	0.466	0.499	0.572	0.709	0.946
0.92	0.647	0.684	0.761	0.882	0.997	0.648	0.685	0.759	0.875	0.991
0.9	0.796	0.824	0.881	0.956	1	0.797	0.825	0.879	0.951	0.998
0.86	0.951	0.958	0.975	0.994	1	0.951	0.959	0.974	0.992	1
Case 5: Constants and Time Trends in each equation										
r										
1	0.053	0.052	0.052	0.048	0.05	0.053	0.053	0.051	0.044	0.021
0.98	0.065	0.068	0.076	0.087	0.131	0.065	0.069	0.076	0.085	0.08
0.96	0.099	0.109	0.131	0.176	0.342	0.099	0.111	0.131	0.172	0.262
0.94	0.152	0.172	0.221	0.327	0.686	0.152	0.173	0.223	0.32	0.599
0.92	0.225	0.265	0.345	0.522	0.923	0.226	0.267	0.345	0.511	0.871
0.9	0.323	0.377	0.489	0.714	0.989	0.325	0.379	0.488	0.699	0.971
0.86	0.562	0.633	0.764	0.93	1	0.564	0.635	0.758	0.919	0.998

Notes: As per Table 2.

Table 4: CADF and $\tilde{\Lambda}^i(1, \bar{r})$

	CADF					$\tilde{\Lambda}^i(1, \bar{r})$				
R^2 =	0	0.09	0.25	0.49	0.81	0	0.09	0.25	0.49	0.81
Case 3: Constants in each equation										
r										
1	0.053	0.055	0.056	0.054	0.051	0.064	0.061	0.06	0.054	0.039
0.98	0.075	0.082	0.098	0.135	0.321	0.145	0.154	0.167	0.192	0.254
0.96	0.107	0.123	0.162	0.272	0.675	0.285	0.308	0.355	0.445	0.716
0.94	0.16	0.188	0.262	0.456	0.885	0.466	0.499	0.572	0.709	0.946
0.92	0.234	0.285	0.396	0.639	0.965	0.648	0.685	0.759	0.875	0.991
0.9	0.332	0.4	0.542	0.79	0.991	0.797	0.825	0.879	0.951	0.998
0.86	0.566	0.654	0.798	0.947	0.999	0.951	0.959	0.974	0.992	1
Case 5: Constants and Time Trends in each equation										
r										
1	0.057	0.058	0.057	0.053	0.046	0.053	0.053	0.051	0.044	0.021
0.98	0.061	0.067	0.079	0.106	0.219	0.065	0.069	0.076	0.085	0.08
0.96	0.079	0.093	0.121	0.197	0.525	0.099	0.111	0.131	0.172	0.262
0.94	0.105	0.131	0.182	0.327	0.78	0.152	0.173	0.223	0.32	0.599
0.92	0.147	0.186	0.268	0.479	0.916	0.226	0.267	0.345	0.511	0.871
0.9	0.203	0.257	0.375	0.635	0.973	0.325	0.379	0.488	0.699	0.971
0.86	0.363	0.451	0.613	0.861	0.998	0.564	0.635	0.758	0.919	0.998

Notes: As per table 3. The CADF refers to the test procedure in Hansen (1995). In each case the same R^2 estimate is used to determine the critical value.

Appendix.

Lemma 1. Distribution results.

Under the Assumptions of the model in (1) and (2) with A1, A2 and A3 we have that $T^{-1/2}\sum_{s=1}^{[T\bullet]} e_t \Rightarrow \Sigma^{1/2}\left[W_1(\cdot)\ \ V(\cdot)'\right]'$, *where $W_1(.)$ is a univariate standard Brownian Motion on C[0,1], V(.) is and mx1 standard Brownian Motion and so*

a) $T^{-1/2}u_{y[T\cdot]} \Rightarrow w_{yy}^{1/2}W_{1c}(\cdot)$

b) $\dfrac{1}{Tw_{yy}^{1/2}}\sum_{t=2}^{T} u_{yt-1}\left(\Sigma^{-1/2}e_t(r)\right)' \Rightarrow \int W_{1c}(l)d\left[W_1(l)\ \ V(l)'\right]$

where $\bar{d}'V(l) = \sqrt{\dfrac{R^2}{1-R^2}}W_2(l)$, $\bar{d}' = w_{yy}^{-1/2}w_{yx}\Omega_{x.y}^{-1/2}$, $\Omega_{x.y} = \Omega_{xx} - w_{yx}'w_{yx}w_{yy}^{-1}$,

$W(l) = \begin{bmatrix} W_1(l) \\ W_2(l) \end{bmatrix}$ *are univariate independent standard Brownian Motions on C[0,1] and*

$W_{1c}(l) = c\int_0^l e^{c(l-s)}W_1(s)ds + W_1(l)$.

Proof: (a) follows as $u_{y,t} = ru_{y,t-1} + v_t$ where $v_t = s_1 A(L)^{-1}e_t(r)$. The partial sum $T^{-1/2}\sum_1^{[T.]} v_s \Rightarrow s_1\Omega^{1/2}\begin{pmatrix} W_1(.) \\ V(.) \end{pmatrix} = w_{yy}^{1/2}W_1(.)$ where s_1 = [1 0] is an 1xm+1 vector with partition after the first column. The result then follows setting ρ =1+c/T from Phillips (1987). Part (b) follows from Chan and Wei (1988), Park and Phillips (1988). The relationship between V(λ) and W_2(λ) follows from the relation $\bar{d}'\bar{d} = \dfrac{R^2}{1-R^2}$.

Proof of Theorem 1.

Throughout we use r for results general for ρ , $\bar{r}$ and 1.

First, define $\hat{u}_t^i(r) = z_t(r) - d_t(r)\hat{b}^i(r) = e_t(r) - d_t(r)'\left(\hat{b}^i(r) - b\right)$, and $e_t(r) = A(L)u_t(r)$.

From the algebra of GLS

$$\sum_{t=1}^{T}\hat{u}_t^i(r)'\Sigma^{-1}\hat{u}_t^i(r) = \sum_{t=1}^{T} e_t(r)'\Sigma^{-1}e_t(r) - \left(S_i N_T(r)\right)'\left(S_i D_T(r) S_i\right)^{-}\left(S_i N_T(r)\right)$$

where

$$N_T(r) = \Psi_T^{-1}\left(\sum_{t=1}^{T} d_t(r)\Sigma^{-1}e_t(r)\right)$$
$$D_T(r) = \Psi_T^{-1}\left(\sum_{t=1}^{T} d_t(r)\Sigma^{-1}d_t(r)'\right)\Psi_T^{-1}{}'$$

and

$$\Psi_T = \begin{pmatrix} w_{yy}^{-1/2} & 0 & 0 & 0 \\ 0 & T^{1/2}\Omega_{x.y}^{-1/2}{}' & 0 & 0 \\ 0 & 0 & T^{1/2}w_{yy}^{-1/2} & 0 \\ 0 & 0 & 0 & T^{3/2}\Omega_{x.y}^{-1/2}{}' \end{pmatrix}$$

Thus,

$$\begin{aligned}\Lambda^i(1,\bar{r}) &= \sum_{t=1}^{T} e_t(\bar{r})'\Sigma^{-1}e_t(\bar{r}) - \sum_{t=1}^{T} e_t(1)'\Sigma^{-1}e_t(1) \\ &+ \left(S_iN_T(1)\right)'\left(S_iD_T(1)S_i\right)^{-}\left(S_iN_T(1)\right) - \left(S_iN_T(\bar{r})\right)'\left(S_iD_T(\bar{r})S_i\right)^{-}\left(S_iN_T(\bar{r})\right)\end{aligned} \quad \text{(A1)}$$

Notice that for t>1

$$\Sigma^{-1/2}e_t(r) = \varepsilon_t + (\bar{r} - r)\Sigma^{-1/2}s_1'u_{y,t-1} \quad \text{(A2)}$$

(and is ε_1 for t=1) where $\varepsilon_t = \Sigma^{1/2}e_t$. Using the results $s_1\Sigma^{-1}s_1' = \left(1+\bar{d}'\bar{d}\right)w_{yy}^{-1}$ and $s_1\Sigma^{-1/2}{}' = w_{yy}^{-1/2}[1 \quad -\bar{d}']'$ then in case 1 where S_i=0 we have

$$\begin{aligned}\Lambda^1(1,\bar{r}) &= \sum_{t=1}^{T} e_t(\bar{r})'\Sigma^{-1}e_t(\bar{r}) - \sum_{t=1}^{T} e_t(1)'\Sigma^{-1}e_t(1) \\ &= (\bar{c}^2 - 2c\bar{c})(1+\bar{d}'\bar{d})w_{yy}^{-1}\frac{1}{T^2}\sum_{t=1}^{T} u_{y,t-1}^2 \\ &- 2\bar{c}\frac{1}{T}\sum_{t=1}^{T}\left[u_{y,t-1}w_{yy}^{-1/2}[1 \quad -\bar{d}']\varepsilon_t\right]\end{aligned}$$

From the limit results in lemma 1

$$\Lambda^1(1,\overline{r}) \Rightarrow (\overline{c}^2 - 2c\overline{c})\left(\frac{1}{1-R^2}\right)\int W_{1c}(l)^2\,dl - 2\overline{c}\left[\int W_{1c}(l)dW_1(l) - \frac{R}{\sqrt{1-R^2}}\int W_{1c}(l)dW_2(l)\right]$$

where the right hand side of this expression equals $y^{\,1}(c,\overline{c},R^2)+\overline{c}$. For the other cases, extra terms arise from the final two terms in equation (A1). Defining c_r=T(r-1) we have

$$\lim_{T\to\infty}\left\| \left(\Psi_T^{-1} d_1(r)\Sigma^{-1/2}{}'\right) - \begin{pmatrix} 1 & -\overline{d}\,' \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \right\| = 0$$

and

$$\lim_{T\to\infty}\left(\sup_{2/T\le s\le 1}\left\| \left(T^{1/2}\Psi_T^{-1} d_{[Ts]}(r)\Sigma^{-1/2}{}'\right) - \begin{pmatrix} 0 & 0 \\ 0 & I_m \\ 1-c_r s & -(1-c_r s)\overline{d} \\ 0 & sI_m \end{pmatrix} \right\| \right) = 0$$

Using these two results and the continuous mapping theorem $\left(S_i D_T(r) S_i\right)^- \to \left(S_i D(c_r,\overline{d}) S_i\right)^-$ where

$$D(c_r,\overline{d}) = \begin{pmatrix} 1+\overline{d}\,'\overline{d} & 0 & 0 & 0 \\ 0 & I_m & -\left(1-\frac{c_r}{2}\right)\overline{d} & \frac{1}{2}I_m \\ 0 & -\left(1-\frac{c_r}{2}\right)\overline{d}\,' & (1+\frac{c_r}{3}-c_r)(1+\overline{d}\,'\overline{d}) & -\left(\frac{1}{2}-\frac{c_r}{3}\right)\overline{d}\,' \\ 0 & \frac{1}{2}I_m & -\left(\frac{1}{2}-\frac{c_r}{3}\right)\overline{d} & \frac{1}{3}I_m \end{pmatrix}$$

Using the continuous mapping theorem, equation (A2) and results from lemma 1 we have $N_T(r) \Rightarrow N(c,c_r,\overline{d})$ where

$$N(c,c_r,\overline{d}) = \begin{pmatrix} e_{y,1} - \overline{d}\,' e_{x,1} \\ V(1) - (c-c_r)\overline{d}\int W_{1c}(s)ds \\ \int (1-c_r s)d\left[W_1(s) - \overline{d}\,'V(s)\right] + (c-c_r)(1+\overline{d}\,'\overline{d})\int (1-c_r s)W_{1c}(s)ds \\ \int sdV(s) - (c-c_r)\overline{d}\int W_{1c}(s)ds \end{pmatrix}$$

(all integrals are zero to one). Applying these results to (A1) yields

$$\begin{aligned}\Lambda^i(1,\overline{r}) \Rightarrow\, & y^{\,1}(c,\overline{c},R^2) + \left(S_i N(c,0,\overline{d})\right)'\left(S_i D(0,\overline{d})S_i\right)^-\left(S_i N(c,0,\overline{d})\right) \\ & - \left(S_i N(c,\overline{c},\overline{d})\right)'\left(S_i D(\overline{c},\overline{d})S_i\right)^-\left(S_i N(c,\overline{c},\overline{d})\right) + \overline{c}\end{aligned}$$

The individual results follow by using the relevant S_i and rearranging.

In case 2, we have

$$\left(S_i N(c,c_r,\bar{d})\right)\left(S_i D(c_r,\bar{d})S_i\right)^{-}\left(S_i N(c,c_r,\bar{d})\right) = (1+\bar{d}'\bar{d})^{-1}(e_{y,1}-\bar{d}'e_{x,1})^2$$

thus the terms offset giving the result in the Theorem.

In case 3, we have

$$\left(S_3 D(c_r,\bar{d})S_3\right)^{-} = \begin{pmatrix} (1+\bar{d}'\bar{d})^{-1} & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and so

$$\left(S_i N(c,c_r,\bar{d})\right)\left(S_i D(c_r,\bar{d})S_i\right)^{-}\left(S_i N(c,c_r,\bar{d})\right) = (1+\bar{d}'\bar{d})^{-1}(e_{y,1}-\bar{d}'e_{x,1})^2 + V(1)'V(1) + (c-c_r)^2\bar{d}'\bar{d}\left(\int W_{1c}\right)^2 - 2(c-c_r)\bar{d}'V(1)\int W_{1c}$$

Plugging in 0 and $\bar{c}$ for c_r and taking the difference yields the result.

Case 4.

Here $\left(S_4 D(c_r,\bar{d})S_4\right)^{-} = \left(S_3 D(c_r,\bar{d})S_3\right)^{-} + \frac{1}{h(r)}\begin{pmatrix} 0 \\ (1-\frac{c_r}{2})\bar{d} \\ 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ (1-\frac{c_r}{2})\bar{d} \\ 1 \\ 0 \end{pmatrix}'$

Where $h(r) = 1+\frac{c_r^2}{3}-c_r+\frac{c_r^2}{12}\frac{R^2}{1-R^2}$. The result follows after some rearrangement.

Case 5.

Here $\left(S_5 D(c_r,\bar{d})S_5\right)^{-} = \begin{pmatrix} (1+\bar{d}'\bar{d})^{-1} & 0 & 0 & 0 \\ 0 & 4I_m & 0 & -6I_m \\ 0 & 0 & 0 & 0 \\ 0 & -6I_m & 0 & 12I_m \end{pmatrix} + \frac{1}{a(r)}\begin{pmatrix} 0 \\ \bar{d} \\ 1 \\ -c_r\bar{d} \end{pmatrix}\begin{pmatrix} 0 \\ \bar{d} \\ 1 \\ -c_r\bar{d} \end{pmatrix}'$

Where $a(r) = 1+\frac{c_r^2}{3}-c_r$.

We have

$$N(c,c_r,\bar{d})'\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 4I_m & 0 & -6I_m\\ 0 & 0 & 0 & 0\\ 0 & -6I_m & 0 & 12I_m\end{pmatrix}N(c,c_r,\bar{d}) = \begin{pmatrix}\int dV(s)\\ \int sdV(s)\end{pmatrix}'\begin{pmatrix}4I_m & -6I_m\\ -6I_m & 12I_m\end{pmatrix}\begin{pmatrix}\int dV(s)\\ \int sdV(s)\end{pmatrix}$$

$$+(c-c_r)^2\bar{d}'\bar{d}\begin{pmatrix}\int W_{1c}\\ \int sW_{1c})\end{pmatrix}'\begin{pmatrix}4 & -6\\ -6 & 12\end{pmatrix}\begin{pmatrix}\int W_{1c}\\ \int sW_{1c})\end{pmatrix}$$

$$-2(c-c_r)\begin{pmatrix}\int W_{1c}\\ \int sW_{1c})\end{pmatrix}'\begin{pmatrix}4 & -6\\ -6 & 12\end{pmatrix}\begin{pmatrix}\int d\bar{d}'V(s)\\ \int sd\bar{d}'V(s)\end{pmatrix}$$

and also

$$\begin{pmatrix}0\\ \bar{d}\\ 1\\ -c_r\bar{d}\end{pmatrix}'N(c,c_r,\bar{d}) = (1-c_r)W_{1c}(1) + c_r^2\int sW_{1c}$$

The result follows from straightforward algebra.

Proof of Theorem 2.

First, note that

$$\tilde{\Lambda}^i(1,\bar{r}) = T\left(tr\left[\tilde{\Sigma}(1)^{-1}\left(\tilde{\Sigma}(\bar{r})-\tilde{\Sigma}(1)\right)\right]\right)-\bar{c}$$

so we need to show that $T\left(tr\left[\tilde{\Sigma}(1)^{-1}\left(\tilde{\Sigma}(\bar{r})-\tilde{\Sigma}(1)\right)\right]\right) \Rightarrow y^i(c,\bar{c},R^2)+\bar{c}$. To show this we will show

(a) $\sum_{t=k+1}^{T}\tilde{e}_t^i(\bar{r})\tilde{e}_t^i(\bar{r})' - \sum_{t=k+1}^{T}\tilde{e}_t^i(1)\tilde{e}_t^i(1)' = \sum_{t=k+1}^{T}\hat{e}_t^i(\bar{r})\hat{e}_t^i(\bar{r})' - \sum_{t=k+1}^{T}\hat{e}_t^i(1)\hat{e}_t^i(1)' + o_p(1)$

where $\hat{e}_t^i(r) = A(L)\tilde{u}_t^i(r)$.

(b) $\sum_{t=k+1}^{T}\hat{e}_t^i(r)'\Sigma^{-1}\hat{e}_t^i(r) - \sum_{t=k+1}^{T}e_t(r)'\Sigma^{-1}e_t(r) \Rightarrow -\left(S_iN(c,c_r,\bar{d})\right)'\left(S_iD(c_r,\bar{d})S_i\right)^-\left(S_iN(c,c_r,\bar{d})\right)$

(c) $\sum_{t=k+1}^{T}e_t(\bar{r})'\Sigma^{-1}e_t(\bar{r}) - \sum_{t=k+1}^{T}e_t(1)'\Sigma^{-1}e_t(1) \Rightarrow y^1(c,c_r,R^2)$

We take part (b) first.

We have

$$\hat{e}_t^i(r) = A(L)\left[z_t(r) - d_t(r)'\tilde{b}^i(r)\right]$$
$$= e_t(r) - A(L)d_t(r)'S_i\left(S_i\sum d_t(r)\tilde{\Omega}^{-1}d_t(r)'S_i\right)^{-}\left(S_i\sum d_t(r)\tilde{\Omega}^{-1}u_t(r)\right)$$

so

$$\sum_{t=k+1}^{T}\hat{e}_t^i(r)'\Sigma^{-1}\hat{e}_t^i(r) = \sum_{t=k+1}^{T}e_t(r)'\Sigma^{-1}e_t(r)$$
$$+\left(S_iN_T(r)\right)'\left(S_iD_T(r)S_i\right)^{-}\left(S_i\Psi_T^{-1}\sum\left[A(L)d_t(r)'\right]'\Sigma^{-1}\left[A(L)d_t(r)'\right]\Psi_T^{-1}{}'S_i\right)\left(S_iD_T(r)S_i\right)^{-}\left(S_iN_T(r)\right)$$
$$-2\left(S_iN_T(r)\right)'\left(S_iD_T(r)S_i\right)^{-}\left(S_i\Psi_T^{-1}\sum\left[A(L)d_t(r)'\right]'\Sigma^{-1}e_t(r)\right)+o_p(1)$$

where $N_T(r)$ is defined as before replacing $e_t(r)$ by $u_t(r)$ and Σ by Ω and similarly for $D_T(r)$ (these are the generalizations to $A(L)\neq I$) and the $o_p(1)$ term arises from replacing the estimated Ω with its true value.

Using the Beveridge Nelson decomposition $A(L)=A(1)+A^*(L)(1-L)$ we have

$$A(L)d_t(r)'\Psi_T^{-1} = A(1)d_t(r)'\Psi_T^{-1} + A^*(L)\Delta d_t(r)\Psi_T^{-1}$$
$$= A(1)d_t(r)'\Psi_T^{-1} + 0(T^{-3/2})$$

so

$$S_i\Psi_T^{-1}\sum\left[A(L)d_t(r)'\right]'\Sigma^{-1}\left[A(L)d_t(r)'\right]\Psi_T^{-1}{}'S_i = S_iD_T(r)S_i + o(1)$$

and also

$$\Psi_T^{-1}\sum d_t(r)A(1)'\Sigma^{-1}e_t(r) = \Psi_T^{-1}\sum d_t(r)A(1)'\Sigma^{-1}A(L)u_t(r)$$
$$= \Psi_T^{-1}\sum d_t(r)\Omega^{-1}u_t(r) + \Psi_T^{-1}\sum d_t(r)A(1)'\Sigma^{-1}A^*(L)\Delta u_t(r)$$
$$= \Psi_T^{-1}\sum d_t(r)\Omega^{-1}u_t(r) + o_p(1)$$

This gives the result

$$\sum_{t=k+1}^{T}\hat{e}_t^i(r)'\Sigma^{-1}\hat{e}_t^i(r) = \sum_{t=k+1}^{T}e_t(r)'\Sigma^{-1}e_t(r) - \left(S_iN_T(r)\right)'\left(S_iD_T(r)S_i\right)^{-}\left(S_iN_T(r)\right) + o_p(1)$$

Finally, following steps analogous to those in the proof of Theorem 1 we have that

$$\left(S_iN_T(r)\right)'\left(S_iD_T(r)S_i\right)^{-}\left(S_iN_T(r)\right) \Rightarrow \left(S_iN(c,c_r,\bar{d})\right)'\left(S_iD(c_r,\bar{d})S_i\right)^{-}\left(S_iN(c,c_r,\bar{d})\right).$$

Part (c) follows from noting that

$$\Sigma^{-1/2}e_t(r) = e_t + (r - r)\Sigma^{-1/2}A(L)s_1'u_{y,t-1}$$

so using the Beveridge Nelson decomposition and results above

$$\sum e_t(r)'\Sigma^{-1}e_t(r) = \sum e_t'e_t + (r-r)^2 s_1\Omega^{-1}s_1'\sum u_{y,t-1}^2 + 2(r-r)\sum u_{y,t-1}s_1\Omega^{-1/2}e_t$$

Thus

$$\sum_{t=1}^{T} e_t(\bar{r})'\Sigma^{-1}e_t(\bar{r}) - \sum_{t=1}^{T} e_t(1)'\Sigma^{-1}e_t(1) = (\bar{c}^2 - 2c\bar{c})(1+\bar{d}'\bar{d})w_{yy}^{-1}\frac{1}{T^2}\sum_{t=1}^{T}u_{y,t-1}^2$$
$$- 2\bar{c}\frac{1}{T}\sum_{t=1}^{T}\left[u_{y,t-1}w_{yy}^{-1/2}[1 \ -\bar{d}']e_t(r)\right]$$

Applying the convergence results in lemma 1 completes the result.

Finally, it remains only to show part (a), that estimating the VAR coefficients assuming the largest root for y_t is r does not matter asymptotically.

We have that

$$\begin{aligned}\tilde{e}_t^{\,i}(r) &= \tilde{A}(L,r)\tilde{u}_t^{\,i}(r) \\ &= \hat{e}_t^{i}(r) - \left(\sum U_{t-1}(r)\hat{e}_{t-1}(r)'\right)\left(\sum U_{t-1}(r)U_{t-1}(r)'\right)^{-1}U_{t-1}(r)\end{aligned}$$

where $U_{t-1}(r) = \left[\tilde{u}_{t-1}^{\,i}(r)' \ \ \tilde{u}_{t-1}^{\,i}(r)' \ \cdots \ \cdots \ \tilde{u}_{t-k}^{\,i}(r)'\right]'$

(i.e. the regressors in the VAR to be run). Note that

$$U_{t-1}(r) = \begin{bmatrix}\tilde{u}_{t-1}^{\,i}(r) \\ \vdots \\ \vdots \\ \vdots \\ \tilde{u}_{t-k}^{\,i}(r)\end{bmatrix} = \begin{bmatrix}\tilde{u}_{t-1}^{\,i}(r) \\ \vdots \\ \vdots \\ \vdots \\ \tilde{u}_{t-1}^{\,i}(r)\end{bmatrix} + \begin{bmatrix}\begin{pmatrix}(r-r)\tilde{y}_{t-2}^{\,i} \\ 0\end{pmatrix} \\ \vdots \\ \vdots \\ \vdots \\ \begin{pmatrix}(r-r)\tilde{y}_{t-k-1}^{\,i} \\ 0\end{pmatrix}\end{bmatrix} = U_{t-1}(r) + (r-r)V_y$$

where $\tilde{y}_t^{\,i} = y_t - s_1 d_t'\tilde{b}(r)$ (i.e. y_t detrended under the hypothesis that ρ =r).

Now,

$$\begin{aligned}\sum_{t=k+1}^{T}\tilde{e}_t^{\,i}(r)\tilde{e}_t^{\,i}(r)' &= \sum_{t=k+1}^{T}\hat{e}_t^{i}(r)\hat{e}_t^{i}(r)' \\ &\quad -\left(T^{-1/2}\sum U_{t-1}(r)\hat{e}_t^{i}(r)\right)'\left(T^{-1}\sum U_{t-1}(r)U_{t-1}(r)'\right)^{-1}\left(T^{-1/2}\sum U_{t-1}(r)\hat{e}_t^{i}(r)\right)\end{aligned}$$

and

$$\begin{aligned}\left(T^{-1}\sum U_{t-1}(r)U_{t-1}(r)'\right) &= \left(T^{-1}\sum U_{t-1}(r)U_{t-1}(r)'\right) + T^2(r-r)^2T^{-3}\sum V_yV_y' \\ &\quad + 2T(r-r)T^{-2}\sum V_yU_{t-1}(r)\end{aligned}$$

The second of these terms is $o_p(1)$ as typical terms involve $T^{-3}\sum \tilde{y}_{t-i}^{2}$. These converge to zero as $T^{-1/2}\tilde{y}_t^i$ is $0_p(1)$. This follows as

$$\begin{aligned} T^{-1/2}\tilde{y}_t^i &= T^{-1/2}u_{y,t-1} - s_1 T^{-1/2} d_t\left(\tilde{b}^i - b^i\right) \\ &= T^{-1/2}u_{y,t-1} - s_1 T^{-1/2} d_t \Psi_T^{-1}\left(S_i D_T(r) S_i\right)^{-}\left(S_i N_T(r)\right) \\ &= T^{-1/2}u_{y,t-1} - w_{yy}^{-1/2}(T^{-1}t) s_3\left(S_i D_T(r) S_i\right)^{-}\left(S_i N_T(r)\right) + o_p(1) \end{aligned}$$

where s_3 is (2m+2)x1 with the (m+2) element one and is zero everywhere else. Similar results follow for the cross product terms. So we have

$$\begin{aligned} &\sum_{t=k+1}^{T}\tilde{e}_t^i(\bar{r})\tilde{e}_t^i(\bar{r})' - \sum_{t=k+1}^{T}\tilde{e}_t^i(1)\tilde{e}_t^i(1)' = \sum_{t=k+1}^{T}\hat{e}_t^i(\bar{r})\hat{e}_t^i(\bar{r})' - \sum_{t=k+1}^{T}\hat{e}_t^i(1)\hat{e}_t^i(1)' \\ &- \left(T^{-1/2}\sum U_{t-1}(r)\hat{e}_t^i(r)\right)'\left(T^{-1}\sum U_{t-1}(r)U_{t-1}(r)'\right)^{-1}\left(T^{-1/2}\sum U_{t-1}(r)\hat{e}_t^i(r)\right) \\ &+ \left(T^{-1/2}\sum U_{t-1}(r)\hat{e}_t^i(r)\right)'\left(T^{-1}\sum U_{t-1}(r)U_{t-1}(r)'\right)^{-1}\left(T^{-1/2}\sum U_{t-1}(r)\hat{e}_t^i(r)\right) + o_p(1) \end{aligned}$$

and the third and fourth terms cancel obtaining the result in (a).

References for Appendix.

Chan, N.H. and C.Z.Wei (1988), "Limiting Distributions of Least Squares Estimates of Unstable Autoregressive Processes", *Annals of Statistics*, 16, 367-401.

Park, J.Y and P.C.B. Phillips (1988), "Statistical Inference in Regressions with Integrated Processes: Part 1", *Econometric Theory*, 4, 468-97.

Phillips, P.C.B. (1987), "Towards a Unified Asymptotic Theory for Autoregression", *Biometrika*, 74, 535-547.