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Targeted Mathematical Equivalence Training Lessens the Effects of Early Misconceptions on Equation Encoding and Solving

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Abstract

Many students fail to develop adequate understanding of mathematical equivalence in early grades, with detrimental consequences for later algebra learning. The change resistance account (McNeil, 2014) proposes that students struggle with equivalence because traditional arithmetic practice overexposes students to mathematical expressions where all the operators are on the left of the equal sign. Students erroneously believe the equal sign means to "do something" or "give the answer" – and fail to see equations as relations between two expressions. These operations-based misconceptions affect how they perceive, conceptualize, and approach math problems and interfere with developing correct understandings of equivalence. The current paper explores 1) are these misconceptions evident as encoding errors in second graders? 2) do item properties make specific error types more or less likely? 3) do misconceptions in encoding impact solving performance? and 4) can targeted training mitigate the effects of prior misconceptions on both equation encoding and solving? We identify a category of misconception-based encoding errors that negatively impacts equation solving and replicate findings that a conceptually rich research-based intervention program is maximally effective in training students to overcome problematic misconceptions.

Keywords: Mathematical representations; relational reasoning; mathematics education; randomized control trial

Introduction

How do early conceptions about equivalence impact children's ability to correctly encode, and later solve, arithmetic equations? Research suggests that understanding mathematical equivalence is a critical component of algebraic reasoning (Carpenter, Franke, & Levi, 2003; Charles, 2005; Knuth, Stephens, McNeil, & Alibali, 2006). However, the majority of US students fail to reason with and apply concepts of equivalence (McNeil & Alibali, 2005), making encoding errors when reconstructing mathematical equations (e.g., McNeil & Alibali, 2004), and interpreting the equal sign to mean "calculate the total" rather than "two amounts are the same" (e.g., Behr, Erlwanger, & Nichols, 1980).

McNeil and Alibali (2005; McNeil 2014) proposed a change-resistance account of children's difficulty with mathematical equivalence. Traditional arithmetic instruction, which focuses on procedures (i.e., solving problems such as 7 + 2 = _), reinforces a *misconception* of the equal sign as a request for an answer, which, in turn, interferes with the development of relational concepts. Most arithmetic problems in early elementary math curricula show operations (e.g., addition and subtraction) on the left of the equal sign and the "answer" on the right (Seo & Ginsburg, 2003; McNeil, 2008). Children detect and extract patterns from these examples and ultimately construct long-term memory representations. McNeil and Alibali characterize these representations as "operational patterns" as they reflect an understanding of arithmetic that focuses on the operators (e.g., +, -, ×, ÷) rather than the relational nature of mathematical expressions. Although default representations typically speed computation in the problem-solving contexts that children encounter most frequently, these representations may lead to difficulties when operational patterns are mistakenly transferred to similar, but non-applicable, problem types (e.g., Bruner, 1957).

Alibali and colleagues (Crooks & Alibali, 2013; McNeil & Alibali, 2004, 2005) have identified three different sub-patterns, described below, that reflect a distorted view of arithmetic and hinder conceptual understanding of equivalence and underlying mathematics. Once entrenched, children rely on these potentially misleading patterns when encoding, interpreting, and solving novel mathematics problems. In the current study, we group these three types of errors as "misconception errors" (see Table 2) to differentiate them from errors believed to stem from working memory constraints or performance demands.

Perceptual pattern errors. Through over-exposure to traditional arithmetic problems, children learn to expect math problems to have **all operations on the left side of the equal sign**, with the equal sign immediately before the answer blank on the right, an "operations = answer" problem format (McNeil & Alibali, 2004, Carpenter et al., 2003). Students who expect all problems to have operations on the left fail to correctly encode the problem before them. For instance, after briefly viewing the problem "7 + 4 + 5 = 7 + __" children who rely on their representations of the "operations = answer" problem format erroneously remember the problem as "7 + 4 + 5 + 7 = __" (McNeil & Alibali, 2004).

Conceptual pattern errors. Children learn to interpret the equal sign operationally as a symbol to do something (Baroody & Ginsburg, 1983; Behr et al., 1980). When asked to define the equal sign—even in the context of a mathematical equivalence problem—many children treat it like an arithmetic operator (like + or -) **that means they should calculate the total of everything on the left side of the equal sign** (McNeil & Alibali, 2005).

Procedural pattern errors. Through early practice with traditional problems (e.g., all operations on the left of '='), children learn to **perform all of the listed operations on all given numbers in a math problem** (e.g., add up all the numbers in an addition problem, McNeil & Alibali, 2004, 2005). This incorrect representation of equations misleads students to solve the problem "7 + 4 + 5 = 7 + __" by performing all given operations on all given numbers and put 23 (instead of 9) in the blank (McNeil, 2007; Rittle-Johnson, 2006, Falkner et al., 1999).

A history of findings supports the hypothesis that children's difficulties with mathematical equivalence are partially due to inappropriate knowledge of the perceptual structure, conceptual meaning, and procedural routine associated with encoding and solving equations. The change-resistance account further suggests that these faulty representations are derived from overly narrow experience with traditional arithmetic. Recent studies have documented the effects of incorrect representations of equivalence in fourth-graders (McNeil & Alibali, 2004) and have induced similar error patterns in adults (Crooks and Alibali, 2013). We build on the work of McNeil, Fyfe, and Dunwiddie (2015), who examined the impact of an early intervention on second-graders multi-faceted understanding of equivalence, replicate and extending these findings to more closely examine the nature of early equivalence encoding and its relationship to equation solving in a large representative sample of students.

In the current study, we sought to more deeply examine the nature of second-grade students' encoding responses, looking for evidence of the misconception-based (i.e., perceptual, conceptual, and procedural) error patterns that have been theorized in past work from McNeil, Alibali, and others (McNeil et al., 2019, McNeil & Alibali, 2005), and induced in adults by Crooks and Alibali (2013).

We further explore the relationship between encoding and solving of equivalence problems, asking whether the specific misconceptions identified through encoding errors are predictive of equation solving performance. We then examine the impacts of research-based equivalence training activities on encoding and solving accuracy. Specifically, we randomly assigned classrooms to training using an intensive treatment intervention or an active control condition consisting solely of non-traditional mathematical practice and measured the training impact on students' ability to encode equations and solve equivalence problems post-training. We organize our findings to explore four related questions:

Do second-grade students make encoding errors consistent with overgeneralizing patterns from early arithmetic?

Do encoding errors systematically vary across items with different structure and length? How does the frequency of different types of encoding errors change with targeted training?

How do misconception-based errors in students' equation encoding predict equation solving?

Does targeted, conceptually rich equivalence training impact encoding and equation solving?

Measuring Equation Encoding and Solving. We assessed second-grade students' ability to correctly encode and solve non-traditional equivalence problems before and after the intervention training using the same measures of equation encoding, equation solving sign used in previous work by McNeil and colleagues (Johannes et al., 2017; McNeil et al., 2012; McNeil & Alibali, 2005b).

Equation encoding. The equation encoding measure consisted of recalling four math expressions (e.g., 2 + 6 =2 + _) presented one at a time. Each expression was visible for five seconds and students were instructed to remember and write down exactly what they saw. Responses were coded as correct if the student wrote the equation exactly as shown (i.e., the correct numbers and symbols in the correct order). We discuss the coding of relevant erroneous response types in the results.

Equation solving. The equation solving measure consisted of eight equations with operations on both sides of the equal sign (e.g., 3 + 5 + 6 = 3 + __). For a response to be coded as correct, a student needed to write the value that would make the equivalence relation hold.

Our sample of encoding and solving items is listed in Table 1. All items included one addend and a blank on the *right* side of the equal sign. The items varied on two dimensions: the number of addends (two or three) on the *left* side of the equal sign, and the position of the blank (at the end of the equation or directly after the equal sign).

Table 1. Equation encoding and solving items administered pre- and post-intervention

Addends	**Position of blank**	**Encoding items**	**Solving items**
Two	End of equation	4+5=3+_	3+7=3+_ 2+7=6+_
	After '='	7+1=_+6	5+3=_+3 8+2=_+6
Three	End of equation	2+3+6=2+_	3+5+6=3+_ 6+2+8=4+_
	After '='	3+5+4=_+4	7+2+4=_+4 7+4+6=_+3

ICUE: Improving Children's Understanding of Equivalence Intervention

As current math practice seems to promote the development of faulty representations, the change resistance account of "operational patterns" offers design principles for instruction to improve students' understanding of equivalence. Initially, researchers hypothesized that greater exposure to "non-traditional arithmetic" problems (e.g., presenting operations on the right side of the equation, "_ = 2 + 4" and using relational phrases such as "is equal to" instead of the equal sign in practice problems) may prevent students from developing operational patterns (McNeil et al., 2011). Though practice with non-traditional arithmetic led to improved outcomes over traditional instruction, a number of students failed to reach proficiency (McNeil, Fyfe, & Dunwiddle, 2015).

To further promote mastery of equivalence, McNeil and colleagues added additional design features beyond non-traditional arithmetic practice. The current version of the materials, dubbed Improving Children's Understanding of Equivalence (ICUE), consists of second grade student activities that reduce reliance on operational patterns and promote deep understanding of mathematical equivalence through four key components, outlined below, that have independently been shown to be effective. Multiple pilot studies have since found that the ICUE treatment intervention is successful in improving student understanding of mathematical equivalence (Byrd et al., 2015; Johannes et al. 2017).

1. Nontraditional arithmetic practice (McNeil, Fyfe, & Dunwiddle, 2015, Chesney et al., 2012),
2. Lessons that first introduce the equal sign outside of arithmetic contexts (e.g., "28 = 28") before introducing arithmetic expressions (e.g., Baroody & Ginsburg, 1983).
3. Concreteness fading exercises in which concrete, real-world, relational contexts (e.g., sharing stickers, balancing a scale) are gradually faded into the corresponding abstract mathematical symbols (e.g., Fyfe, McNeil, Son, & Goldstone, 2014), and
4. Activities that require students to compare and explain different problem formats and problem-solving strategies (e.g., Carpenter, Franke, & Levi, L. 2003).

Methods

Design

We used a cluster-randomized control trial design to examine the impacts of the ICUE intervention training relative to an active control program. Teachers were randomly assigned to use the either the ICUE Treatment intervention or Active Control materials. The Active Control consisted of workbook activities to control for time on task and contained non-traditional arithmetic practice but not the additional components present in the Treatment ICUE condition, described above.

Participants. 44 second-grade teachers (24 treatment, 20 control) used the activities in their classrooms in California. Class sizes ranged from 18 to 25, and we analyzed data from 482 students who completed the Treatment activities and 406 students who completed the Control activities and measures.

Procedure and Materials

The procedure for ICUE Treatment and Active Control conditions were identical, differing only in the content of the materials used by teachers and students. Each teacher received training on the study purpose, features of the activities, and strategies for integrating the activities into their typical mathematics curriculum.

Prior to starting the study, participating teachers completed online surveys assessing their mathematics teaching experience and classroom structure and dynamics.

After administering a pre-test, teachers used the study materials for approximately 15 minutes twice each week for 16 weeks. In both conditions, teachers were asked to use the study materials to supplement, rather than replace current math instruction, and to limit the duration of the activities to 20 minutes per session.

After completing the 32 sessions, teachers administered the same pre-intervention measure of mathematical equivalence understanding, which included the equation encoding and solving items reported here, along with an item prompting children to name and define the "=" symbol, not reported. Teachers administered additional post-intervention measures of transfer and computation fluency, we do not report these here.

Active Control. Teachers in the Active Control condition received a set of student workbooks and a teacher guide.

ICUE Treatment. Teachers in the ICUE Treatment condition received a set of student workbooks, a teacher guide, a set of classroom manipulatives including balance scales and flashcards.

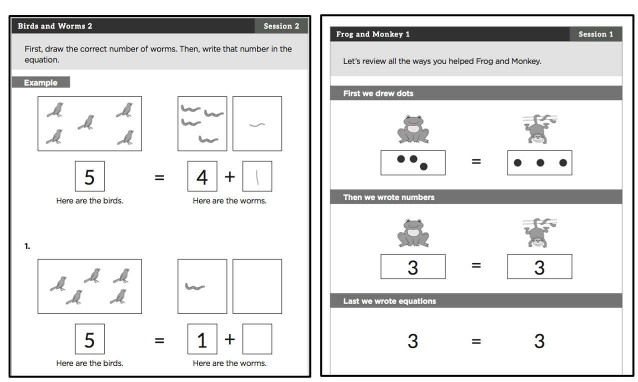

Figure 1. Sample workbook page from the Active Control (left) and ICUE Treatment (right) condition materials.

Results

Do second graders make misconception-based encoding errors?

Crooks and Alibali (2013) identified three categories of errors in encoding and solving, reviewed above, which stem from different types of knowledge and misconceptions. We asked whether, after only one year of formal mathematics instruction, second graders produced erroneous encoding responses that align with these any of these three related facets of misunderstanding, grouping them together as "misconception errors". We examined the frequency with which theoretically-relevant types of encoding errors, which can be induced in adults (Crooks & Alibali, 2013) naturally occur in young students. We differentiated these misconception-based errors from other types of errors, including performance errors, which we hypothesize stem from memory-based constraints in this population.

We assessed students' accuracy in encoding four different equivalence problems (see Table 1 for items) and examined the frequency with which they made different types of errors. Student in both the Control and Treatment groups produced a range of responses for each encoding item and made multiple types of errors, including misconception-based errors, with different frequency. Examples and overall frequency of response types are listed in Table 2.

Students produced the misconception-based errors of interest in approximately 20% of their responses overall. The majority of misconception-based errors produced in both conditions aligned with the perceptual error type identified by Crooks and Alibali (2013); conceptual and procedural errors types were produced relatively infrequently.

Table 2. Response types and examples for encoding item 2+3+6=2+_, with overall frequency of response pre- and post-training.

Response type	*Examples*	*Control Pre/Post*	*Treatment Pre / Post*
Correct	2+3+6=2+_	0.25/0.47	0.35/0.56
Misconception errors	**2+3+6+2=_ 2+3+6=11+2 2+3+6=2+13**	**0.23/0.21**	**0.24/0.17**
Memory error	2+3+6	0.06 / 0.08	0.06/ 0.15
Other errors	2+3+7=6	0.39 / 0.21	0.28/ 0.09
No response	no response	0.07 / 0.02	0.05/ 0.03

How does the equation structure influence encoding errors?

We chose to focus on misconception-based and memory-based encoding error patterns and explored variation in error rates across the four encoding items that varied in A. the number of addends, and B. the position of the blank in the equation (see Table 1 for items). The larger number of addends was predicted to increase the working memory demands of the problem. The position of the blank at the end of the equation (e.g., 4+5=3+_) was predicted to increase the likelihood of perceptual pattern errors as these items are most perceptually similar to traditional arithmetic problems (e.g., 4+5+3=__), and may trigger operational, instead of relational, interpretations of the equal sign (e.g., as a symbol that means give the answer') that give rise to erroneous arithmetic procedures (e.g., add up all numbers and write the sum in the blank; see Crooks & Alibali, 2013; McNeil et al., 2011).

Students' pre-intervention encoding error frequency is displayed by item in Figure 2. The frequency of misconception- and memory-based errors varied based on both the position of blank (at the end of the equation – first and third items - or directly after the '=' sign – second and fourth items in Table 1) and the number of addends on the left side of the equation (two – first two items - or three – last two items in Table 1).

In line with our predictions, regression models confirmed that students in both conditions produced a reliably greater number of perception-based errors for items with the blank at the end of the equation (β=0.852, SE =0.12, p<.01), and this interacted with the number addends, such that students produced the greatest number of misconception errors for the three-addend item with the blank at the end: "2+3+6=2+_" (β=0.534, SE =0.09, p<.05).

Finally, students made a reliable number of memory-based errors, but only for items with three addends (β=0.472, SE =0.11, p<.05).

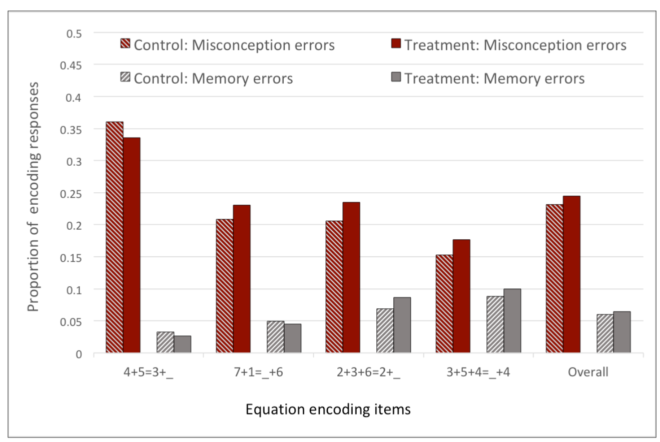

Figure 2. Pre-intervention patterns of misconception- and memory-based error responses for Treatment and Control students. Misconception errors were greatest for items with a blank space at the end of the equation (first and third items); memory errors were greatest for items with three addends (second and fourth items).

How does conceptually rich equivalence training change students' encoding responses?

We next examined the impact of the ICUE Treatment and Active Control training on misconception- and memory-based encoding errors. We asked whether exposure to non-traditional arithmetic, through the Active Control condition, was sufficient to maximally reduce these encoding errors, or whether more conceptually rich training, found in the ICUE Treatment intervention, would lead to greater error reduction. The change in students' encoding errors from pre- to post-intervention is displayed in Figure 3. Students in the Treatment condition showed a greater reduction, post-intervention, in misconception-based errors compared to students in the Control condition (β=0.921, SE =0.10, p<.01), and this reduction was greatest for items with a blank space at the end of the expression (β=0.633, SE =0.09, p<.05). Thus, for encoding, we find that conceptually rich training leads to greater reduction misconception-based errors. Training type did not significantly impact the frequency of memory errors for three-addend encoding items.

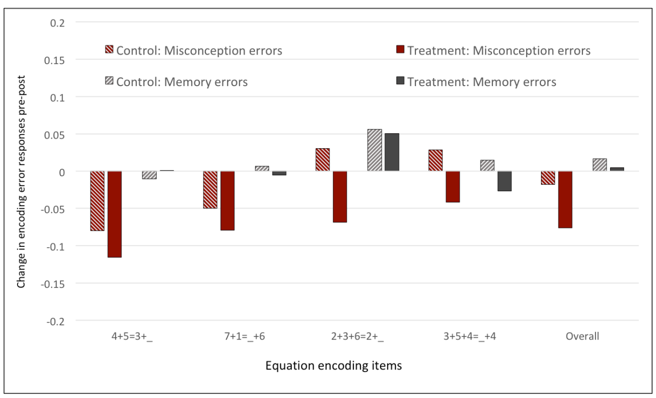

Figure 3. Pre- to post-intervention changes in error responses for Treatment and Control students. Students in the Treatment condition showed a greater reduction in misconception-based errors (solid red bars), which was greatest for items with a blank space at the end of the equation.

How do misconception-based errors in students' equation encoding impact equation solving?

Turning to equation solving, we tested whether perceptual errors in equation encoding reliably predicted students' equation solving performance before students had received any training through the Treatment or Control interventions. For each item type, students completed one encoding item and two solving items (see Table 1). Thus, for each type of item, a student could solve both solving items correct, one correct, or zero correct. We used ordinal regression models to capture this ordering and tested encoding performance (i.e., whether a student encoded that type of item correctly), error types (misconception- and memory-based), and item properties (number of addends and location of blank space) as predictors.

Pre-intervention solving performance was predicted by multiple aspects of encoding responses: students were more likely to solve an equivalence problem correctly if they had accurately encoded the same type of item correctly (β=0.778, SE=0.121, p<.05), and students were less likely to solve a problem correctly if they had produced a misconception-based error for that type of item on the encoding measure (β= -0.420, SE =0.097, p<.01).

Performance was also predicted by properties of the items: items with two addends were more likely to be solved correctly than those with three addends (β=0.360, SE =0.079, p<.01) and items with a blank space directly after "=" were more likely to be solved correctly than those with a blank at the end of the equation (β=0.226, SE =0.079, p<.05). However, pre-intervention solving performance was not reliably predicted by memory-based errors, or assigned condition (β=-0.061, SE =0.078, *ns*).

How does equivalence training impact equation solving?

We used a similar ordinal regression model to test the effect of training condition on students' post-intervention solving performance. Performance was best predicted by a combination of intervention condition and encoding responses; item properties (position of blank, number of addends) were not reliable predictors in the best-fitting model of solving performance. The strongest single predictor of post-intervention solving performance was training condition: students in the Treatment condition were more likely to correctly solve items post-intervention, compared to students in the Active Control condition (Figure 4; (β= 1.267, SE =0.074, p<.01)). As in the case of encoding, we found that conceptually richer training led to more accurate solving performance.

Students were also more likely to solve one or both equation solving problem correctly on the post-intervention measure if they had solved one correctly on the pre-intervention measure (β=0.636, SE =0.12, p<.01) and were increasingly likely to solve both items correctly if they had solved both items correctly pre-intervention (β=1.053, SE =0.15, p<.01).

Finally, as in the pre-intervention model, post-intervention solving performance was predicted by encoding responses: students were more likely to solve an equivalence problem correctly post-intervention if they had accurately encoded the same type of item correctly post-intervention (β=1.493, SE =0.08, p<.01), and students were less likely to solve a problem correctly if they had produced a misconception-based error for that type of item on the encoding measure (β= -0.749, SE =0.06, p<.05).

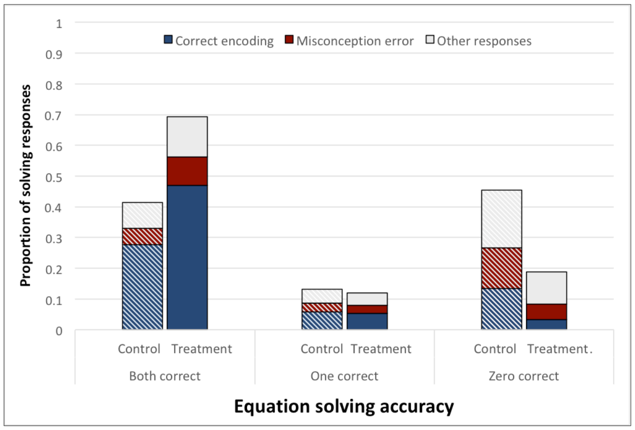

Figure 4. Treatment and Control students' post-intervention solving performance broken down by frequency of correct and misconception-based encoding responses. Conceptually rich training in the Treatment condition led to the greatest improvement in solving performance; students from both conditions were more likely to solve an item correctly if they had encoded the same type of item correctly, and less likely if they had made a misconception-based error in encoding.

Considering both our encoding and solving results, we found that, while manipulated item properties impacted students' ability to correctly encode and solve non-traditional equations pre-intervention, the magnitude of this impact was reduced, for encoding, and eliminated, for solving, by targeted conceptually rich training in mathematical equivalence. Specifically, students in the Treatment condition showed a greater reduction, post-intervention in misconception-based encoding errors compared to Control students, and this reduction was greatest for items with a blank space at the end of the equation (i.e., items that are perceptually most similar to traditional arithmetic problems). Treatment students were also more accurate on a post-intervention equation solving task (with no reliable condition differences pre-intervention) and, while encoding responses were predictive of solving performance, manipulated item properties (number of addends and position of blank space) were not.

Conclusions

Understanding equivalence is key for later mathematical understandings. The change-resistance account suggests that students fail to develop appropriate representations of equations and equivalence because instruction with traditional arithmetic problems encourages students to develop ineffective representations of problems.

In the current study, we explored the relationship between problematic representations and students' ability to accurately encode and later solve non-traditional equivalence problems. We examined encoding and solving abilities in second-grade students and found that a single year of formal instruction (i.e., first grade) with traditional arithmetic practice was sufficient to reliably lead to misconception-based errors at encoding, which predominantly consisted of perceptual pattern errors, in the framework of Crooks and Alibali (2013). Baseline performance on both tasks worsened when target problems perceptually resembled traditional arithmetic problems (i.e., when a blank was at the end of an equation), and when working memory load (number of addends) was increased. Misconception errors at encoding were predictive of solving performance, both at baseline (pre-intervention) and at post-intervention, suggesting that students who make these misconception-based errors at encoding may be activating similar faulty representations during the solving task. Finally, training improved both encoding and solving performance, demonstrating that erroneous response patterns can be overcome with intervention. However, students in the Treatment condition showed greatest improvements on both tasks, suggesting that deeper conceptual learning is required to resolve what, at first glance, might be thought of as a perceptual bias towards traditional arithmetic problem structure.

Interestingly, while manipulated properties of the encoding and solving items (see Table 1) predicted students' errors in the encoding task, these properties were only predictive of *pre-intervention* solving performance. Students' post-intervention solving performance was predicted by their post-intervention encoding responses, but not directly predicted by item properties, suggesting that any relationship between these manipulated item properties, such as the number of addends and position of blank space, and solving performance is potentially mediated by encoding. This is consistent with a mediation analysis performed by Crooks and Alibali (2013), in which the authors demonstrated that the impact of priming incorrect representations on adults' equation solving performance was mediated by problem reconstruction (or encoding). Future work will explore this relationship in more depth by using a greater number of items and possible combination of manipulated item properties.

Even after training, students in both conditions did not reach ceiling performance in either encoding or solving non-traditional equations. On the one hand, the equation encoding and solving assessment items were chosen to leave room for improvement and to avoid ceiling effects. However, in future work, we plan to explore how individual students resolve or persist in error patterns with training. We further plan to test whether different encoding errors give rise to specific solving responses, or whether any error simply creates noise in students' equation solving processes. Our preliminary findings suggest that misconception-based errors in encoding lead to greater error rates in solving, but a larger sample of items and responses may be required to support fine-grained conclusions about the nature of this relationship and, specifically, how perceptual, conceptual, and procedural misconceptions individually and collaboratively impact equation encoding and solving.

Acknowledgments

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