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Abstract

We reconsider the question of the representation of an isotropic symmetric second-order tensor-valued function of two symmetric second-order tensors. This question goes back roughly 70 years and represents the attempt to duplicate the stunning result possible when the function only depends on one tensor input. Over the years there have been numerous attempts to derive a general representation theorem for such functions and the literature is replete with conflicting statements as to the appropriate form. By focusing on the Cayley-Hamilton theorem for three-by-three matrices we are able to arrive at a general result applicable to isotropic functions using nine tensor-valued basis functions. With the addition of an argument exploiting the Cayley-Hamilton theorem for two-by-two matrices we are able to show that the complete function basis reduces to only involves eight tensor-valued basis functions. Our result clarifies which historically advocated representations are complete and non-redundant without need for complex qualifying cases based upon the eigen-structure of the input tensors. The arguments are straightforward and only involve basic algebraic considerations with an intentional focus on Cayley-Hamilton-based reductions.

Keywords: isotropic tensor functions, representation theorem

Dedicated to Rohan Abeyaratne on the occasion of his 70th birthday.

1 Revisiting isotropic representations

In this contribution, we reconsider the representation formulas for isotropic symmetric second-order tensor-valued functions of two symmetric second-order tensors. The topic has been studied for over 70 years, beginning with works by Rivlin and Ericksen [1955], Smith [1960], followed by Wang [1970], further refinements by Smith [1971], and the works of Zheng [1993], among quite a few others. In the case of symmetric tensor-valued functions of *one* symmetric tensor, the derivation of isotropic representations can be "cleanly" achieved through usage of the Cayley-Hamilton theorem, which guarantees a tensor $\mathbf{A}$ satisfies its own characteristic polynomial; see e.g. Bowen and Wang [2009, Theorem 26.1] or Gurtin et al. [2010, §2.16]. Thus, powers of $\mathbf{A}$ higher than 3 can be represented as a combination of powers of $\mathbf{A}$ of order 2 or less. This results in the astonishing representation of *all* isotropic symmetric tensor-valued functions of a single symmetric tensor argument being expressible as

$$\mathbf{B} = \hat{\mathbf{B}}(\mathbf{A}) = \varphi_0(\mathcal{I}_{\mathbf{A}})\mathbf{1} + \varphi_1(\mathcal{I}_{\mathbf{A}})\mathbf{A} + \varphi_2(\mathcal{I}_{\mathbf{A}})\mathbf{A}^2\,, \tag{1}$$

where $\varphi_0, \varphi_1, \varphi_2$ are scalar-valued functions of the principal invariants, $\mathcal{I}_{\mathbf{A}} = \{I_1(\mathbf{A}), I_2(\mathbf{A}), I_3(\mathbf{A})\} \equiv \big\{\mathrm{tr}(\mathbf{A}),\ \frac{1}{2}((\mathrm{tr}(\mathbf{A}))^2 - \mathrm{tr}(\mathbf{A}^2)),\ \det(\mathbf{A})\big\}$, of $\mathbf{A}$ [see e.g., Gurtin, 1981, §37] – astonishing, since there is no requirement that $\mathbf{B}$ be polynomial in $\mathbf{A}$ (see Appendix A).

For functions of two or more tensors, however, the representations obtained since the 1950's have not relied (solely) on implications of the Cayley-Hamilton theorem. Rather, they have been surmised using algebraic considerations that are sometimes daunting to follow and have resulted in proposed forms that have required amendment over time. While it is certainly true that the necessity of a term within a representation can be shown by example [see Pennisi and Trovato, 1987], it is rare to see discussions of span; that is to say, whether the proposed representation can be proven to generate all isotropic functions or at least some broad subset.

In this brief note, we take up this foundational issue and first look at how general of a result one can arrive at on the basis of applications of the three-dimensional Cayley-Hamilton theorem alone. We will see that the best one can do is the reduction to the form:

$$\begin{aligned}\mathbf{C} = \hat{\mathbf{C}}(\mathbf{A},\mathbf{B}) = c_{00}\mathbf{1} + c_{10}\mathbf{A} + c_{11}\mathbf{B} + c_{20}\mathbf{A}^2 + c_{21}\mathbf{B}^2 + c_{22}(\mathbf{AB}+\mathbf{BA}) \\ + c_{32}(\mathbf{A}^2\mathbf{B}+\mathbf{BA}^2) + c_{33}(\mathbf{B}^2\mathbf{A}+\mathbf{AB}^2) + c_{42}(\mathbf{A}^2\mathbf{B}^2+\mathbf{B}^2\mathbf{A}^2),\end{aligned} \tag{2}$$

where $\hat{\mathbf{C}}$ is an isotropic symmetric second-order tensor-valued function of symmetric second-order tensors $\mathbf{A}$ and $\mathbf{B}$, and the coefficients c_{ij} are any functions of the joint invariants of $\mathbf{A}$ and $\mathbf{B}$. Thus, the tensorial function basis has nine elements, no more and no less, when only utilizing the three-dimensional Cayley-Hamilton theorem. We will, however, show through an additional argument based on the two-dimensional Cayley-Hamilton theorem and a lemma from Rivlin and Ericksen [1955], that the fourth-order term, the last term in (2), is unneeded, leaving us with the final representation

$$\begin{aligned}\mathbf{C} = \hat{\mathbf{C}}(\mathbf{A},\mathbf{B}) = c_{00}\mathbf{1} + c_{10}\mathbf{A} + c_{11}\mathbf{B} + c_{20}\mathbf{A}^2 + c_{21}\mathbf{B}^2 + c_{22}(\mathbf{AB}+\mathbf{BA}) \\ + c_{32}(\mathbf{A}^2\mathbf{B}+\mathbf{BA}^2) + c_{33}(\mathbf{B}^2\mathbf{A}+\mathbf{AB}^2)\,.\end{aligned} \tag{3}$$

There can be no fewer than the number of terms in Eq. (3) as can be shown by direct example [Pennisi and Trovato, 1987], thus no more reduction is possible and the correct dimension of the tensor function basis is eight. This result only relies on the assumed existence of a tensorial expansion of $\hat{\mathbf{C}}$ (see Eq (4)) in terms of its arguments, irrespective of the eigen-structure of $\mathbf{A}$ and $\mathbf{B}$, with coefficients that can be arbitrary functions of the scalar invariants. Our reasoning is detailed in what follows and differs somewhat from what is available in the current literature. Beyond adding clarity to the form of the representation, we also aim herein to deliver a clear step-wise derivation showing how it is obtained.

2 Brief historical recap

The literature contains a number of conflicting and difficult to follow developments by some of the seminal figures in continuum mechanics. Without in any way attempting to be comprehensive in citation of all works, we provide a brief road map to some of the most cited developments that took place with respect to the special representation question addressed in this note, viz. representations for isotropic symmetric second-order-tensor-valued functions of two symmetric second-order tensors. The developments are shown in Table 1 in chronological order. Some useful definitions and facts to understanding the papers are as follows:

- $\hat{\mathbf{C}}$ is a polynomial isotropic tensor function if it is a sum of joint powers of $\mathbf{A}$ and $\mathbf{B}$ with scalar coefficients. We will denote this case simply as polynomial. This does not imply that as a series expansion in terms of $\mathbf{A}$ and $\mathbf{B}$ that the scalar coefficients are constants – a restriction that would be too severe for practical work [Truesdell and Noll, 1965, §7].
- The ten basic invariants are $\mathcal{I}_{\mathbf{A},\mathbf{B}} = \big(\mathrm{tr}\mathbf{A}, \mathrm{tr}\mathbf{A}^2, \mathrm{tr}\mathbf{A}^3, \mathrm{tr}\mathbf{B}, \mathrm{tr}\mathbf{B}^2, \mathrm{tr}\mathbf{B}^3, \mathrm{tr}\mathbf{AB}, \mathrm{tr}\mathbf{AB}^2, \mathrm{tr}\mathbf{BA}^2, \mathrm{tr}\mathbf{A}^2\mathbf{B}^2\big)$. The coefficients of the final representation are often determined or simply stated to be either rational, polynomial, or general functions in the ten basic invariants (or at times of simply the components of $\mathbf{A}$ and $\mathbf{B}$).
- The methods of proof usually differ in their approach. If a paper utilizes an approach invoking Cayley-Hamilton arguments, we denote it by CH. If a paper utilizes solvability conditions based on non-zero determinants of linear equations, we denote it as S. If a paper utilizes a co-set invariance argument, we denote it as Co. If the method of analysis is unclear to us, we denote it as unc.

It can be seen in the table that the $\mathbf{A}^2\mathbf{B}^2 + \mathbf{B}^2\mathbf{A}^2$ term in the representation appears and disappears over the last seven decades, even over common sets of assumptions. It is interesting to note that this 4th-order term comes and goes in papers with common authors without any comment, and similarly when the term disappeared in the 1970s, it did so also without any explicit comments on why the prior analyses were in error. One of our goals in this paper is to show that this term is indeed not necessary.

Table 1: Brief historical summary of the fourth-order term and the chronological developments.

Publication	Assumptions on $\hat{\mathbf{C}}$	$\mathbf{A}^2\mathbf{B}^2 + \mathbf{B}^2\mathbf{A}^2$ included?	Coefficients	Method
Rivlin and Ericksen [1955, §27,40]	no	no	rational	S
Rivlin [1955, eqn (1.7)]	polynomial	yes	polynomial	CH
Spencer and Rivlin [1959, eqn (4.4) & Thm 5]	polynomial	yes	polynomial	CH
Wang [1969, eqn (1.14)]	no	yes	general	Co
Smith [1970, eqn (2.7)]	no	yes	general	unc[1]
Wang [1970, p. 215]	no	yes	general	Co
Smith [1971, eqns (4.4) & (4.5)]	no	no	general	Co[2]
Pennisi and Trovato [1987, eqn (2.3)]	no	no	general	unc[3]
Zheng [1993, eqn (3.23)]	no	no	general	S+Co

3 The two tensor argument case [4]

We take a moment to attempt a derivation which appears different from what we see in the existing literature, focused on the case of symmetric tensor-valued functions of two symmetric tensors. The representation derived below relies on reductions from Cayley-Hamilton considerations and is (i) guaranteed to span a particular isotropic function space, and (ii) never encounters 'corner cases' where certain choices of $\mathbf{A}$ and $\mathbf{B}$ make one or more coefficients in the representation indeterminate or infinite.[5] Letting $\mathbf{A}$ and $\mathbf{B}$ be two symmetric tensors, observe that any general polynomial series in $\mathbf{A}$ and $\mathbf{B}$ with symmetric output:

$$\begin{aligned}\mathbf{C} = \hat{\mathbf{C}}(\mathbf{A},\mathbf{B}) = c_{00}\mathbf{1} + c_{10}\mathbf{A} + c_{11}\mathbf{B} + c_{20}\mathbf{A}^2 + c_{21}\mathbf{B}^2 + c_{22}(\mathbf{AB}+\mathbf{BA}) + c_{30}\mathbf{A}^3 + c_{31}\mathbf{B}^3 \\ + c_{32}(\mathbf{A}^2\mathbf{B}+\mathbf{BA}^2) + c_{33}(\mathbf{B}^2\mathbf{A}+\mathbf{AB}^2) + c_{34}\mathbf{ABA} + c_{35}\mathbf{BAB} + c_{40}\mathbf{A}^4 \\ + c_{41}\mathbf{B}^4 + c_{42}(\mathbf{A}^2\mathbf{B}^2+\mathbf{B}^2\mathbf{A}^2) + c_{43}(\mathbf{BABA}+\mathbf{ABAB}) + c_{44}\mathbf{AB}^2\mathbf{A} + \dots \end{aligned} \tag{4}$$

is an isotropic symmetric-tensor-valued function.[6] The scalars $c_{ij} = \hat{c}_{ij}(\mathcal{I}_{\mathbf{A},\mathbf{B}})$ are arbitrary scalar-valued functions of $\mathcal{I}_{\mathbf{A},\mathbf{B}}$, the joint invariants of $\mathbf{A}$ and $\mathbf{B}$. Note the convention that the first index i represents the order of the term c_{ij} multiplies, and j counts through all the terms of the same order.

- *Our goal is to collapse this space of tensor-valued functions into a representation with the fewest terms allowable using repeated application of Cayley-Hamilton-based reduction.*

The intentionally-broad form of Eq (4) is chosen to be able to represent general isotropic tensor functions, noting it contains infinitely more terms than any of the previously suggested representations. However, we cannot rule out at this stage the possibility that there may exist an isotropic function that is inexpressible via Eq (4). But we further recognize that Eq (4) is sufficiently general for most applications. As such we restrict our attention to relations of this form.

3.1 Cayley-Hamilton reducers

First, we determine which joint powers of $\mathbf{A}$ and $\mathbf{B}$ can be expressed in terms of lower order powers using the Cayley-Hamilton theorem. The Cayley-Hamilton theorem ensures that any (not necessarily symmetric)

[1] The main point of the paper is to provide counter examples to Wang [1969] for the case of more than 2 arguments; proofs of the basic results are not given.

[2] Smith [1971] states that he follows the method of Wang [1970], while sharpening it, but does not provide any proofs or details of where Wang [1970] can be sharpened; he only points out where he feels Wang [1970] is incomplete.

[3] Pennisi and Trovato [1987] adopt the results of Smith [1971] without proof but declare them to be correct. Pennisi and Trovato [1987] further go on to show that the Smith [1971] results are irreducible via use of the Rouché-Capelli theorem.

[4] Throughout we will assume that the qualifiers of symmetric, second-order, and tensor-valued apply without, in general, making all qualifications explicit.

[5] This is a problem with some of the existing representation results found in the literature; see e.g. Rivlin and Ericksen [1955, §26] or Zheng [1993, Case 6] as just two such instances.

[6] $\mathbf{Q}\hat{\mathbf{C}}(\mathbf{A},\mathbf{B})\mathbf{Q}^T = \hat{\mathbf{C}}(\mathbf{QAQ}^T,\mathbf{QBQ}^T)$ for all orthogonal $\mathbf{Q}$.

tensor $\mathbf{M}$ satisfies its own characteristic polynomial. In three dimensions, this implies

$$\mathbf{M}^3 = \mathrm{tr}(\mathbf{M})\,\mathbf{M}^2 - I_2(\mathbf{M})\,\mathbf{M} + \det(\mathbf{M})\,\mathbf{1}. \tag{5}$$

Let us apply this rule to progressively higher-order polynomials in $\mathbf{A}$ and $\mathbf{B}$. Starting with an order one polynomial, we have, for arbitrary scalars s_0 and s_1,

$$\begin{aligned}(\mathbf{1}+s_0\mathbf{A}+s_1\mathbf{B})^3 = {} & \mathrm{tr}(\mathbf{1}+s_0\mathbf{A}+s_1\mathbf{B})\,(\mathbf{1}+s_0\mathbf{A}+s_1\mathbf{B})^2 - I_2(\mathbf{1}+s_0\mathbf{A}+s_1\mathbf{B})\,(\mathbf{1}+s_0\mathbf{A}+s_1\mathbf{B}) \\ & + \det(\mathbf{1}+s_0\mathbf{A}+s_1\mathbf{B})\,\mathbf{1}\,. \end{aligned} \tag{6}$$

We introduce the shorthand $o(\mathbf{i}, j)$ to denote a well-defined ith-order tensor polynomial in $\mathbf{A}$ and $\mathbf{B}$ whose scalar coefficients are ($\mathcal{I}_{\mathbf{A},\mathbf{B}}$-dependent) jth-order or lower polynomials in the s variables. For example, the right-hand side of Eq (6) is $o(\mathbf{2}, 3)$ because no tensorial monomial exceeds order 2, but the scalar coefficients that multiply the tensorial monomials can depend on s_0 and s_1 up to order 3. We shall use $o(\mathbf{i}) \equiv o(\mathbf{i}, 0)$ for tensor polynomials with no dependence on s variables. Expanding the left-hand side of Eq (6) gives

$$(\mathbf{1}+s_0\mathbf{A}+s_1\mathbf{B})^3 = s_0^3\,\mathbf{A}^3 + s_0^2 s_1(\mathbf{A}^2\mathbf{B}+\mathbf{ABA}+\mathbf{BA}^2) + s_0 s_1^2(\mathbf{B}^2\mathbf{A}+\mathbf{BAB}+\mathbf{AB}^2) + s_1^3\,\mathbf{B}^3. \tag{7}$$

Since right-hand side of Eq (6) is $o(\mathbf{2}, 3)$, and hence has no tensorially third-order terms, when we gather terms with like powers of the s variables in Eq (6), after applying the expansion in Eq (7), we obtain

$$\begin{aligned} & s_0^3\,[\mathbf{A}^3 + o(\mathbf{2})] + s_0^2 s_1[\mathbf{A}^2\mathbf{B}+\mathbf{ABA}+\mathbf{BA}^2+o(\mathbf{2})] \\ & \qquad + s_0 s_1^2[\mathbf{B}^2\mathbf{A}+\mathbf{BAB}+\mathbf{AB}^2+o(\mathbf{2})] + s_1^3\,[\mathbf{B}^3+o(\mathbf{2})] + o(\mathbf{2},2) = \mathbf{0}. \end{aligned}$$

The above is a polynomial in s_0 and s_1 equal to the zero function. Thus, each term in brackets must be $\mathbf{0}$, yielding the following three third-order Cayley-Hamilton reductions:

$$\mathbf{A}^3 = o(\mathbf{2}), \quad \mathbf{A}^2\mathbf{B}+\mathbf{ABA}+\mathbf{BA}^2 = o(\mathbf{2}), \quad \mathbf{B}^2\mathbf{A}+\mathbf{BAB}+\mathbf{AB}^2 = o(\mathbf{2}), \quad \mathbf{B}^3 = o(\mathbf{2}). \tag{8}$$

The first and last reductions in the above are well-known from the one-input case; e.g., that $\mathbf{A}^3$ can be expressed in terms of $\mathbf{A}^2$, $\mathbf{A}$, and $\mathbf{1}$, the usual Cayley-Hamilton result for a tensor. But the middle two arise only when considering a second input tensor. Heretofore, we use the term "n-reducer" to refer to a polynomial comprised of only nth degree powers in $\mathbf{A}$ and $\mathbf{B}$, which can be rewritten in terms of lower-order powers; i.e. Eq (8) shows four 3-reducers.

The same technique can be applied to determine reductions at higher orders. For example, if we use a second-order tensor polynomial on the left-hand side of Eq (5), we get

$$\begin{aligned} & (\mathbf{1}+s_0\mathbf{A}+s_1\mathbf{B}+s_2\mathbf{A}^2+s_3\mathbf{B}^2+s_4\mathbf{AB}+s_5\mathbf{BA})^3 \\ & = \mathrm{tr}(\mathbf{1}+s_0\mathbf{A}+s_1\mathbf{B}+s_2\mathbf{A}^2+s_3\mathbf{B}^2+s_4\mathbf{AB}+s_5\mathbf{BA})\,(\mathbf{1}+s_0\mathbf{A}+s_1\mathbf{B}+s_2\mathbf{A}^2+s_3\mathbf{B}^2+s_4\mathbf{AB}+s_5\mathbf{BA})^2 \\ & \quad - I_2(\mathbf{1}+s_0\mathbf{A}+s_1\mathbf{B}+s_2\mathbf{A}^2+s_3\mathbf{B}^2+s_4\mathbf{AB}+s_5\mathbf{BA})\,(\mathbf{1}+s_0\mathbf{A}+s_1\mathbf{B}+s_2\mathbf{A}^2+s_3\mathbf{B}^2+s_4\mathbf{AB}+s_5\mathbf{BA}) \\ & \quad + \det(\mathbf{1}+s_0\mathbf{A}+s_1\mathbf{B}+s_2\mathbf{A}^2+s_3\mathbf{B}^2+s_4\mathbf{AB}+s_5\mathbf{BA})\,\mathbf{1}. \end{aligned} \tag{9}$$

Expanding both sides of the above gives very long expressions. However, all terms of order five or higher in $\mathbf{A}$ and $\mathbf{B}$ on the left side have no terms on the right to cancel them. With these fifth- and sixth-order terms, as before, we can separate and match terms by their corresponding powers of the s variables. These groupings, in turn, give rise to a set of twenty 5-reducers all equal to $o(\mathbf{4})$,

$$\begin{aligned} & 2\mathbf{A}^3\mathbf{B}^2+\mathbf{A}^2\mathbf{B}^2\mathbf{A}+\mathbf{AB}^2\mathbf{A}^2+2\mathbf{B}^2\mathbf{A}^3 = o(\mathbf{4}), && \mathbf{ABAB}^2+\mathbf{AB}^2\mathbf{AB}+\mathbf{BABAB} = o(\mathbf{4}), \\ & 2\mathbf{A}^3\mathbf{BA}+\mathbf{A}^2\mathbf{BA}^2+\mathbf{ABA}^3+2\mathbf{BA}^4 = o(\mathbf{4}), && \mathbf{AB}^4+\mathbf{B}^2\mathbf{AB}^2+\mathbf{B}^4\mathbf{A} = o(\mathbf{4}), \\ & \mathbf{A}^2\mathbf{BAB}+\mathbf{ABA}^2\mathbf{B}+\mathbf{ABABA} = o(\mathbf{4}), && \mathbf{BABAB}+\mathbf{BAB}^2\mathbf{A}+\mathbf{B}^2\mathbf{ABA} = o(\mathbf{4}) \\ & \mathbf{ABABA}+\mathbf{BA}^2\mathbf{BA}+\mathbf{BABA}^2 = o(\mathbf{4}), && \mathbf{A}^4\mathbf{B}+\mathbf{A}^2\mathbf{BA}^2+\mathbf{BA}^4 = o(\mathbf{4}), \\ & 2\mathbf{A}^2\mathbf{B}^3+\mathbf{BA}^2\mathbf{B}^2+\mathbf{B}^2\mathbf{A}^2\mathbf{B}+2\mathbf{B}^3\mathbf{A}^2 = o(\mathbf{4}), && 2\mathbf{A}^4\mathbf{B}+\mathbf{A}^3\mathbf{BA}+\mathbf{A}^2\mathbf{BA}^2+2\mathbf{ABA}^3 = o(\mathbf{4}), \\ & 2\mathbf{AB}^4+\mathbf{BAB}^3+\mathbf{B}^2\mathbf{AB}^2+2\mathbf{B}^3\mathbf{AB} = o(\mathbf{4}), && 2\mathbf{BAB}^3+\mathbf{B}^2\mathbf{AB}^2+\mathbf{B}^3\mathbf{AB}+2\mathbf{B}^4\mathbf{A} = o(\mathbf{4}), \end{aligned}$$

$$\mathbf{A}^2\mathbf{BAB} + \mathbf{A}^2\mathbf{B}^2\mathbf{A} + \mathbf{BA}^3\mathbf{B} + \mathbf{BA}^2\mathbf{BA} + \mathbf{BABA}^2 + \mathbf{B}^2\mathbf{A}^3 = o(\mathbf{4}),$$
$$\mathbf{AB}^2\mathbf{AB} + \mathbf{AB}^3\mathbf{A} + \mathbf{BA}^2\mathbf{B}^2 + \mathbf{BABAB} + \mathbf{BAB}^2\mathbf{A} + \mathbf{B}^2\mathbf{A}^2\mathbf{B} = o(\mathbf{4}),$$
$$\mathbf{A}^2\mathbf{B}^3 + \mathbf{ABAB}^2 + \mathbf{AB}^2\mathbf{AB} + \mathbf{AB}^3\mathbf{A} + \mathbf{B}^2\mathbf{A}^2\mathbf{B} + \mathbf{B}^2\mathbf{ABA} = o(\mathbf{4}),$$
$$\mathbf{ABAB}^2 + \mathbf{AB}^3\mathbf{A} + \mathbf{BA}^2\mathbf{B}^2 + \mathbf{BAB}^2\mathbf{A} + \mathbf{B}^2\mathbf{ABA} + \mathbf{B}^3\mathbf{A}^2 = o(\mathbf{4}),$$
$$\mathbf{A}^2\mathbf{B}^2\mathbf{A} + \mathbf{ABA}^2\mathbf{B} + \mathbf{ABABA} + \mathbf{AB}^2\mathbf{A}^2 + \mathbf{BA}^3\mathbf{B} + \mathbf{BA}^2\mathbf{BA} = o(\mathbf{4}),$$
$$\mathbf{A}^3\mathbf{B}^2 + \mathbf{A}^2\mathbf{BAB} + \mathbf{ABA}^2\mathbf{B} + \mathbf{AB}^2\mathbf{A}^2 + \mathbf{BA}^3\mathbf{B} + \mathbf{BABA}^2 = o(\mathbf{4}),$$
$$\mathbf{B}^5 = o(\mathbf{4}), \quad \mathbf{A}^5 = o(\mathbf{4}),$$

and a set of twenty 6-reducers also equal to $o(\mathbf{4})$:

$$\mathbf{A}^4\mathbf{B}^2 + \mathbf{A}^2\mathbf{B}^2\mathbf{A}^2 + \mathbf{B}^2\mathbf{A}^4 = o(\mathbf{4}),$$
$$\mathbf{A}^4\mathbf{BA} + \mathbf{A}^2\mathbf{BA}^3 + \mathbf{BA}^5 = o(\mathbf{4}),$$
$$\mathbf{A}^3\mathbf{BAB} + \mathbf{ABA}^3\mathbf{B} + \mathbf{ABABA}^2 = o(\mathbf{4}),$$
$$\mathbf{BABAB}^2 + \mathbf{BAB}^3\mathbf{A} + \mathbf{B}^3\mathbf{ABA} = o(\mathbf{4}),$$
$$\mathbf{ABAB}^2\mathbf{A} + \mathbf{AB}^2\mathbf{A}^2\mathbf{B} + \mathbf{BA}^2\mathbf{BAB} = o(\mathbf{4}),$$
$$\mathbf{BABABA} = o(\mathbf{4})$$
$$\mathbf{BAB}^4 + \mathbf{B}^3\mathbf{AB}^2 + \mathbf{B}^5\mathbf{A} = o(\mathbf{4}),$$
$$\mathbf{A}^5\mathbf{B} + \mathbf{A}^3\mathbf{BA}^2 + \mathbf{ABA}^4 = o(\mathbf{4}),$$
$$\mathbf{A}^2\mathbf{B}^4 + \mathbf{B}^2\mathbf{A}^2\mathbf{B}^2 + \mathbf{B}^4\mathbf{A}^2 = o(\mathbf{4}),$$
$$\mathbf{A}^2\mathbf{BABA} + \mathbf{BA}^3\mathbf{BA} + \mathbf{BABA}^3 = o(\mathbf{4}),$$
$$\mathbf{ABABAB} = o(\mathbf{4}),$$
$$\mathbf{AB}^2\mathbf{ABA} + \mathbf{BA}^2\mathbf{B}^2\mathbf{A} + \mathbf{BABA}^2\mathbf{B} = o(\mathbf{4}),$$
$$\mathbf{AB}^5 + \mathbf{B}^2\mathbf{AB}^3 + \mathbf{B}^4\mathbf{AB} = o(\mathbf{4}),$$
$$\mathbf{ABAB}^3 + \mathbf{AB}^3\mathbf{AB} + \mathbf{B}^2\mathbf{ABAB} = o(\mathbf{4}),$$

$$\mathbf{A}^3\mathbf{B}^2\mathbf{A} + \mathbf{A}^2\mathbf{BA}^2\mathbf{B} + \mathbf{ABA}^2\mathbf{BA} + \mathbf{AB}^2\mathbf{A}^3 + \mathbf{BA}^4\mathbf{B} + \mathbf{BA}^2\mathbf{BA}^2 = o(\mathbf{4}),$$
$$\mathbf{AB}^2\mathbf{AB}^2 + \mathbf{AB}^4\mathbf{A} + \mathbf{BA}^2\mathbf{B}^3 + \mathbf{BAB}^2\mathbf{AB} + \mathbf{B}^2\mathbf{AB}^2\mathbf{A} + \mathbf{B}^3\mathbf{A}^2\mathbf{B} = o(\mathbf{4}),$$
$$\mathbf{A}^3\mathbf{B}^3 + \mathbf{A}^2\mathbf{B}^2\mathbf{AB} + \mathbf{ABA}^2\mathbf{B}^2 + \mathbf{AB}^3\mathbf{A}^2 + \mathbf{B}^2\mathbf{A}^3\mathbf{B} + \mathbf{B}^2\mathbf{ABA}^2 = o(\mathbf{4}),$$
$$\mathbf{A}^2\mathbf{BAB}^2 + \mathbf{A}^2\mathbf{B}^3\mathbf{A} + \mathbf{BA}^3\mathbf{B}^2 + \mathbf{BAB}^2\mathbf{A}^2 + \mathbf{B}^2\mathbf{A}^2\mathbf{BA} + \mathbf{B}^3\mathbf{A}^3 = o(\mathbf{4}),$$
$$\mathbf{B}^6 = o(\mathbf{4}), \quad \mathbf{A}^6 = o(\mathbf{4}).$$

It is clear that some of these reductions could have been inferred from the 3-reducers in Eq (8). For example, $\mathbf{A}^6$ clearly reduces down since we known $\mathbf{A}^3$ does. However, many of the above reductions are independent of the ones derived previously at third order, such as the 6-reducer $\mathbf{ABABAB}$. Our goal at the moment is to be exhaustive rather than concise, to make sure we include *every* reducible grouping that follows from three-dimensional Cayley-Hamilton, even if there is some duplication.

With a careful eye, it can be seen that Eq (9) also discloses a set of 4-reducers. The left and right sides of Eq (9) each produce tensorially fourth order terms. While many of these cancel each other, some do not. For example, fourth-order terms multiplying $s_0^2 s_5$ will exist on the left side but not the right. The surviving fourth-order terms generate a set of 4-reducers, listed below:

$$\mathbf{A}^3\mathbf{B} + \mathbf{A}^2\mathbf{BA} + \mathbf{ABA}^2 = o(\mathbf{3})$$
$$\mathbf{A}^2\mathbf{B}^2 + \mathbf{AB}^2\mathbf{A} + \mathbf{B}^2\mathbf{A}^2 = o(\mathbf{3})$$
$$\mathbf{A}^2\mathbf{B}^2 + 2\mathbf{ABAB} + \mathbf{AB}^2\mathbf{A} + \mathbf{BA}^2\mathbf{B} + \mathbf{BABA} = o(\mathbf{3})$$
$$\mathbf{ABAB} + \mathbf{AB}^2\mathbf{A} + \mathbf{BA}^2\mathbf{B} + 2\mathbf{BABA} + \mathbf{B}^2\mathbf{A}^2 = o(\mathbf{3})$$
$$\mathbf{AB}^3 + \mathbf{BAB}^2 + \mathbf{B}^2\mathbf{AB} = o(\mathbf{3})$$
$$3\mathbf{B}^4 = o(\mathbf{3})$$
$$\mathbf{A}^2\mathbf{BA} + \mathbf{ABA}^2 + \mathbf{BA}^3 = o(\mathbf{3})$$
$$2\mathbf{A}^3\mathbf{B} + \mathbf{A}^2\mathbf{BA} + \mathbf{ABA}^2 + 2\mathbf{BA}^3 = o(\mathbf{3})$$
$$2\mathbf{AB}^3 + \mathbf{BAB}^2 + \mathbf{B}^2\mathbf{AB} + 2\mathbf{B}^3\mathbf{A} = o(\mathbf{3})$$
$$\mathbf{A}^2\mathbf{B}^2 + \mathbf{BA}^2\mathbf{B} + \mathbf{B}^2\mathbf{A}^2 = o(\mathbf{3})$$
$$\mathbf{BAB}^2 + \mathbf{B}^2\mathbf{AB} + \mathbf{B}^3\mathbf{A} = o(\mathbf{3})$$
$$3\mathbf{A}^4 = o(\mathbf{3}).$$

Aside from the *primary* reductions listed so far, reductions of a *secondary* nature must also be considered. That is to say, any reduction shown on one of the above lists can be multiplied on the left or right by $\mathbf{A}$ or $\mathbf{B}$ to produce a reduction at the next order higher. For example, the 3-reducer $\mathbf{A}^2\mathbf{B} + \mathbf{ABA} + \mathbf{BA}^2$ implies $\mathbf{A}^2\mathbf{B}^2 + \mathbf{ABAB} + \mathbf{BA}^2\mathbf{B}$ is a 4-reducer. By repeatedly multiplying all the Cayley-Hamilton reductions on the left/right by $\mathbf{A}$ or $\mathbf{B}$, we can construct all secondary reductions up to some desired order — for reasons that will become apparent, one need only do this up to order six in order to obtain the final result. We neglect writing these all down for brevity and obviousness! The accumulated set of all primary and secondary n-reducers is denoted $R(n)$.

With $R(n)$ in hand up to $n = 6$, we can begin to remove terms from Eq (4). Since our ultimate goal is to represent symmetric-tensor-valued outputs, we first symmetrize by adding each element of $R(n)$ to its transpose to produce the set of symmetric-valued n-reducers, $S(n)$. We can then compute the dimension, $D_S(n)$, of the polynomial space spanned by $S(n)$. For example, a brute-force computer calculation[7] of the dimension of the polynomial space spanned by $S(3)$ can be done by writing each element of $S(3)$ as a vector in terms of monomial basis elements, and computing the rank of the set, which gives $D_S(3) = 4$. We also define $D_{\text{Tot}}(n)$, which is the dimension of the space of all symmetric-valued combinations of nth-degree powers of $\mathbf{A}$ and $\mathbf{B}$; this is also equal to the total number of nth degree terms showing up in Eq (4). A simple counting problem reveals that $D_{\text{Tot}}(n) = (2^n - 2^{\lceil n/2 \rceil})/2 + 2^{\lceil n/2 \rceil}$, where $\lceil x \rceil$ is the ceiling function of x, the smallest integer greater than x. For example, this formula gives $D_{\text{Tot}}(3) = 6$ and indeed, from Eq (4), it can be seen that there are 6 independent third-order terms, i.e. $\mathbf{A}^3, \mathbf{B}^3, \mathbf{B}^2\mathbf{A} + \mathbf{A}\mathbf{B}^2, \mathbf{A}^2\mathbf{B} + \mathbf{B}\mathbf{A}^2, \mathbf{ABA}, \text{and } \mathbf{BAB}$. Altogether, we obtain $D_{\text{Tot}}(3) - D_S(3) = 2$, so exactly two 'irreducible' third order terms from Eq (4) will remain in our final representation.

The same method for determining the number of irreducible terms can be used at progressively higher orders without affecting results from previous orders. We find the number of irreducible terms at fourth-, fifth-, and sixth-order are, respectively, $D_{\text{Tot}}(4) - D_S(4) = 1$, $D_{\text{Tot}}(5) - D_S(5) = 0$, and $D_{\text{Tot}}(6) - D_S(6) = 0$. Moreover, another calculation reveals that when n reaches 6, the space of $R(6)$ — the space generated by all not-necessarily-symmetric n-reducers — has dimension $D_R(6) = 2^6$. This is precisely the dimension of the entire space generated by sixth-order monomials. Likewise, all sixth-order polynomial terms are reducible. Since any seventh-order monomial can be constructed by multiplying a sixth-order monomial on the left or right by $\mathbf{A}$ or $\mathbf{B}$, it follows that secondary reductions from $R(6)$ are sufficient to span the entire space of seventh-order polynomial terms. By continuing the same construction order-by-order, we conclude that all polynomial terms of order $n \geq 7$ must be reducible and can be eliminated from the representation. Altogether, we find **any function of the form of Eq** (4) **can be truncated after fourth-order.** Note that generating higher-order Cayley-Hamilton reductions beyond those from Eqs (6) and (9) is unnecessary since these additional results would only affect orders > 6, which have already been shown to be fully reducible.

3.2 Greatest reduction via strictly 3D Cayley-Hamilton-based arguments

A particular representation can be found upon identifying two third-order terms and one fourth-order term, which, when included within the sets $S(3)$ and $S(4)$, respectively increase the dimension of the spaces spanned by those sets to $D_{\text{Tot}}(3)$ and $D_{\text{Tot}}(4)$. Multiple options exist. Below is one such solution:

$$\begin{aligned}\mathbf{C} = \hat{\mathbf{C}}(\mathbf{A}, \mathbf{B}) = c_{00}\mathbf{1} + c_{10}\mathbf{A} + c_{11}\mathbf{B} + c_{20}\mathbf{A}^2 + c_{21}\mathbf{B}^2 + c_{22}(\mathbf{AB} + \mathbf{BA}) \\ + c_{32}(\mathbf{A}^2\mathbf{B} + \mathbf{B}\mathbf{A}^2) + c_{33}(\mathbf{B}^2\mathbf{A} + \mathbf{A}\mathbf{B}^2) + c_{42}(\mathbf{A}^2\mathbf{B}^2 + \mathbf{B}^2\mathbf{A}^2).\end{aligned} \tag{10}$$

This final result, which includes the fourth-order term, is identical to that of (e.g.) Rivlin [1955], Spencer and Rivlin [1959], Wang [1969], Smith [1970], Wang [1970] but differs from (e.g.) Rivlin and Ericksen [1955], Smith [1971], Pennisi and Trovato [1987], Zheng [1993] whose representations are the same except without the fourth-order term. Any representation without the fourth-order term as we have derived, would require reductive algebra beyond 3D Cayley-Hamilton considerations.

4 Showing the $\mathbf{A}^2\mathbf{B}^2 + \mathbf{B}^2\mathbf{A}^2$ term is unnecessary

The previous argument concluded that the representation for $\hat{\mathbf{C}}$ needs no more than the nine terms in equation (10). We shall now show through a different kind of argument that the $\mathbf{A}^2\mathbf{B}^2 + \mathbf{B}^2\mathbf{A}^2$ term can always be expressed as a linear combination of the other 8 terms regardless of the choice of $\mathbf{A}$ and $\mathbf{B}$. Thus, it can always be removed from the representation.

[7]The Matlab code for this and all ensuing computer-assisted linear algebra is in the Supplemental Materials.

First, Rivlin and Ericksen [1955] (in §27) showed that the set of tensors

$$K \equiv \{\mathbf{1}, \mathbf{A}, \mathbf{B}, \mathbf{A}^2, \mathbf{B}^2, \mathbf{AB} + \mathbf{BA}, \mathbf{A}^2\mathbf{B} + \mathbf{BA}^2, \mathbf{B}^2\mathbf{A} + \mathbf{AB}^2\}$$

is a basis for the space of all 3×3 symmetric tensors in the special case that (i) **A** and **B** do not both have repeated eigenvalues, and (ii) **A** and **B** share no common eigenvectors.[8] These conditions exemplify the special cases one sees frequently in the existing literature on this topic. However, the result can be shown in a rather straightforward fashion by writing each element of K as a six-dimensional vector using Voigt notation in the eigenbasis of **A**, and showing that the matrix of these vectors has non-zero determinant when the aforementioned conditions are met. Consequently, for any **A** and **B** fulfilling these conditions, $\mathbf{A}^2\mathbf{B}^2 + \mathbf{B}^2\mathbf{A}^2$ can be expressed as a linear combination of the members of elements of K and thus the function $\hat{\mathbf{C}}$ can be written without the $\mathbf{A}^2\mathbf{B}^2 + \mathbf{B}^2\mathbf{A}^2$ term.

To show that the $\mathbf{A}^2\mathbf{B}^2 + \mathbf{B}^2\mathbf{A}^2$ term is in fact never needed, we must show that $\mathbf{A}^2\mathbf{B}^2 + \mathbf{B}^2\mathbf{A}^2$ is a linear combination of elements of K even when condition (i) or (ii) is not met.

Condition (i) not met — Repeated eigenvalues. We will prove the stronger result that $\mathbf{A}^2\mathbf{B}^2 + \mathbf{B}^2\mathbf{A}^2$ can be expressed as a linear combination of the members of K if **A** or **B** has a repeated eigenvalue. Suppose, without loss of generality, that **A** has a repeated eigenvalue. Let its (unordered) eigenvalues be $\{a_1, a_1, a_3\}$. In this case, it always follows that

$$\mathbf{A}^2\mathbf{B}^2 + \mathbf{B}^2\mathbf{A}^2 = (a_1 + a_3)(\mathbf{AB}^2 + \mathbf{B}^2\mathbf{A}) - 2a_1a_3\mathbf{B}^2 \tag{11}$$

and thus $\mathbf{A}^2\mathbf{B}^2 + \mathbf{B}^2\mathbf{A}^2$ is a linear combination of elements of K. One way to obtain this result is to utilize the principal basis of **A**, in which we can write

$$[\mathbf{A}] = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_1 & 0 \\ 0 & 0 & a_3 \end{bmatrix} .$$

The repeated eigenvalue means that **A** satisfies the analogous 2D Cayley-Hamilton theorem, i.e.

$$\mathbf{A}^2 = (a_1 + a_3)\mathbf{A} - a_1a_3\mathbf{1}$$

where $a_1 + a_3$ is the analogous trace and a_1a_3 the analogous determinant. Equation (11) thus follows directly upon multiplying this result on either side by $\mathbf{B}^2$ and adding the two results together.

Condition (i) not met — Common eigenvector. Suppose **A** and **B** share a common eigenvector. In the principal basis of **A**, the matrices of **A** and **B** can be expressed as

$$[\mathbf{A}] = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} , \quad [\mathbf{B}] = \begin{bmatrix} b_1 & 0 & 0 \\ 0 & b_{22} & b_{23} \\ 0 & b_{23} & b_{33} \end{bmatrix} .$$

Here, we obtain the result

$$\mathbf{A}^2\mathbf{B}^2 + \mathbf{B}^2\mathbf{A}^2 = (a_2 + a_3)(\mathbf{AB}^2 + \mathbf{B}^2\mathbf{A}) - 2a_2a_3\mathbf{B}^2 + 2b_1^2\left(\mathbf{A}^2 - (a_2 + a_3)\mathbf{A} + a_2a_3\mathbf{1}\right) \tag{12}$$

which shows, again, $\mathbf{A}^2\mathbf{B}^2 + \mathbf{B}^2\mathbf{A}^2$ is a linear combination of elements of K. This result can be verified through direct calculation, and can be most easily deduced by appealing to the 2D Cayley-Hamilton theorem.

Since we have now shown that $\mathbf{A}^2\mathbf{B}^2 + \mathbf{B}^2\mathbf{A}^2$ is always representable as a linear combination of elements of K for all choices of **A** and **B**, we have thus proven that

$$\begin{aligned}\mathbf{C} = \hat{\mathbf{C}}(\mathbf{A}, \mathbf{B}) = c_{00}\mathbf{1} + c_{10}\mathbf{A} + c_{11}\mathbf{B} + c_{20}\mathbf{A}^2 + c_{21}\mathbf{B}^2 + c_{22}(\mathbf{AB} + \mathbf{BA}) \\ + c_{32}(\mathbf{A}^2\mathbf{B} + \mathbf{BA}^2) + c_{33}(\mathbf{B}^2\mathbf{A} + \mathbf{AB}^2) .\end{aligned} \tag{13}$$

Since, as shown by counterexample in Pennisi and Trovato [1987], the representation for $\hat{\mathbf{C}}$ cannot be reduced further than the terms included in Eq (13), we conclude Eq (13) is the minimal representation needed to represent any series-expandable (as in Eq (4)) isotropic function $\hat{\mathbf{C}}$.

[8]Rivlin and Ericksen [1955] proved only the forward statement written here, but one can show the converse is also true.

5 Summary

In this paper we have revisited the topic of isotropic representations of tensor-valued functions of multiple tensors, a topic with a long and somewhat daunting literature of development over the last 70 years or so. Our emphasis here was twofold. Our first goal was to help resolve the question of how many terms the representation needs in the case of two symmetric inputs and one symmetric output. This case carries prime relevance in mechanics where one might want, for example, a function for stress in terms of a symmetric deformation variable (e.g., strain or strain-rate) and a symmetric tensorial variable representing the structural state of the material. The second goal of this paper was to carry out the derivation in a clear fashion relying primarily on consequences of the Cayley-Hamilton theorem in three dimensions and then in two dimensions.

o The derivation described herein takes the following route to obtain the final result. First, we consider the broad class of functions in Eq (4). We then apply the three-dimensional Cayley-Hamilton theorem to first- and second-order combinations of monomials of $\mathbf{A}$ and $\mathbf{B}$ to obtain a set of Cayley-Hamilton-based reductions — i.e. polynomial expressions in $\mathbf{A}$ and $\mathbf{B}$ that are identical to lower-order polynomials. Second, we use computer-assisted linear algebra[9] to determine the dimensionality of the order-by-order space of tensorial monomials that can be built from the Cayley-Hamilton reducers (including secondary reductions) and hence removed from inclusion in the representation. This wipes away all terms in the function expansion above fourth order, and many lower-order terms as well, leaving a nine-term expression that includes the disputed fourth-order term $\mathbf{A}^2\mathbf{B}^2 + \mathbf{B}^2\mathbf{A}^2$. We then appeal to a theorem in Rivlin and Ericksen [1955] whose proof is straightforward from algebra, which we can assert to see that there are only two conditions in which the fourth-order term might not be expressible in terms of the other eight. In each of those two cases, using the two-dimensional Cayley-Hamilton theorem, we find explicit formulas for the fourth-order term in terms of the other eight thereby finalizing the proof. The only explicit limitation of the proof offered here is the initial assumption that the tensor-valued function can be expanded into a non-constant coefficient tensorial power series, which is an assumption that need not be made in the classical and far-simpler one-input case shown in Eq (1); see Appendix A for such a proof.

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[9]The code for performing these computations is available in the Supplementary Material.

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A Aside: Proving the one-input case without assuming a series

The proof shown in the prior section presumes at the outset that the isotropic function at hand is expressible as an expansion per Eq (4). For most physically relevant situations, this is hardly a restriction, since one mostly deals with analytic functions. Notwithstanding, a more general proof would remove this assumption and such proofs are well known in the literature. For completeness and it may be instructive to consider the following proof of the one-input tensor case, which is able to arrive at Eq (1) without assuming at the outset that $\hat{\mathbf{B}}(\mathbf{A})$ is expressible as a tensorial power series.

First a lemma:

- Lemma: If $\mathbf{B} = \hat{\mathbf{B}}(\mathbf{A})$ is an isotropic function, the eigenvectors of $\mathbf{A}$ must be the same as the eigenvectors of $\mathbf{B}$.

This lemma can be proven by considering a rotation tensor $\mathbf{Q_e}$ which rotates by π radians about $\mathbf{e}$ where $\mathbf{e}$ is one of the eigenvectors of $\mathbf{A}$. Observe for this particular rotation, we have that $\mathbf{Q_e A Q_e}^T = \mathbf{A}$. Thus, by isotropy, we can write

$$\mathbf{Q_e B Q_e}^T = \mathbf{Q_e}\hat{\mathbf{B}}(\mathbf{A})\mathbf{Q_e}^T = \hat{\mathbf{B}}(\mathbf{Q_e A Q_e}^T) = \hat{\mathbf{B}}(\mathbf{A}) = \mathbf{B}$$

But since $\mathbf{Q_e B Q_e}^T = \mathbf{B}$, it follows that $\mathbf{e}$ is also an eigenvector of $\mathbf{B}$. Therefore, every eigenvector of $\mathbf{A}$ is also an eigenvector of $\mathbf{B}$, and thus we have that the two tensors are necessarily coaxial. □

Now, for each $\mathbf{A}$, let $\mathbf{Q}$ be a rotation tensor that diagonalizes $\mathbf{A}$ and hence, by the lemma, also diagonalizes $\mathbf{B}$. Then we have diagonal tensors defined by $\mathbf{Diag}(A_1, A_2, A_3) = \mathbf{QAQ}^T$ and $\mathbf{Diag}(B_1, B_2, B_3) = \mathbf{QBQ}^T$, where (A_1, A_2, A_3) and (B_1, B_2, B_3) are the (possibly unordered) eigenvalues of $\mathbf{A}$ and $\mathbf{B}$, respectively. With this choice of $\mathbf{Q}$, isotropy gives that

$$\mathbf{Diag}(B_1, B_2, B_3) = \mathbf{QBQ}^T = \hat{\mathbf{B}}(\mathbf{QAQ}^T) = \hat{\mathbf{B}}(\mathbf{Diag}(A_1, A_2, A_3))\,.$$

This implies the following formula for B_1,

$$B_1 = \hat{B}_{11}(\mathbf{Diag}(A_1, A_2, A_3)) \equiv \hat{f}(A_1, A_2, A_3)\,.$$

But notice that there are six ways to choose $\mathbf{Q}$ that diagonalize the $\mathbf{A}$ and $\mathbf{B}$ tensors, corresponding to the six permutations of the eigenvalues. Going through each such $\mathbf{Q}$ and writing the resulting $_{11}$-component of the $\hat{\mathbf{B}}$ function gives the following set of formulas

$$B_i = \hat{f}(A_i, A_j, A_k) = \hat{f}(A_i, A_k, A_j) \qquad \text{for distinct } i, j, k\,. \tag{14}$$

Likewise, $\hat{f}$ is the only function one needs to determine each eigenvalue of $\mathbf{B}$ from the eigenvalues of $\mathbf{A}$.

Our task is now to determine a general representation for this function $\hat{f}$, which could be any function that commutes in its last two inputs, i.e. $\hat{f}(A_i, A_j, A_k) = \hat{f}(A_i, A_k, A_j)$.

- Lemma: If $\hat{f}(A_i, A_j, A_k) = \hat{f}(A_i, A_k, A_j)$ for all (A_i, A_k, A_j), then there exist three functions ϕ_1, ϕ_2, ϕ_3 such that
$$\hat{f}(A_i, A_j, A_k) = \phi_1(\mathcal{I}_\mathcal{A}) + \phi_2(\mathcal{I}_\mathcal{A})A_i + \phi_2(\mathcal{I}_\mathcal{A})A_i^2 \tag{15}$$
where $\mathcal{I}_\mathcal{A} = \{A_1+A_2+A_3,\ A_1A_2+A_2A_3+A_1A_3,\ A_1A_2A_3\}$ is a complete set of permutation invariants of (A_1, A_2, A_3).

In the most general case where the three eigenvalues of $\mathbf{A}$ are distinct, the functions ϕ_i can be uniquely determined by solving
$$\begin{bmatrix} \hat{f}(A_1, A_2, A_3) \\ \hat{f}(A_2, A_1, A_3) \\ \hat{f}(A_3, A_2, A_1) \end{bmatrix} = \begin{bmatrix} 1 & A_1 & A_1^2 \\ 1 & A_2 & A_2^2 \\ 1 & A_3 & A_3^2 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}$$
which has a unique solution because the *Vandermonde matrix* above is known to be invertible when A_1, A_2, and A_3 are distinct. Upon solving the linear system for the ϕ_i, one can observe the solutions are invariant to permutations of $\{A_1, A_2, A_3\}$ and thus the ϕ_i must be expressible in terms of the permutation invariants. One can also see this fact without directly computing the solution. Simply notice that for any i and j, permuting A_i and A_j in the equation system above is equivalent to permuting the ith and jth equations of the system (recalling the last two inputs to $\hat{f}$ necessarily commute). As a result, we can infer ϕ_1, ϕ_2 and ϕ_3 are unchanged by permutations of A_1, A_2, and A_3 and thus they must be functions only of the permutation invariants. Now, if any two of the A_i are equal, the above system does not have a unique solution; rather it has infinitely many. This can be seen by observing that two equations will become the same and thus the system has more unknowns than equations. The same logic holds if all of the A_i are equal. Thus, in all cases, there exist functions of the form $\phi_1(\mathcal{I}_\mathcal{A})$, $\phi_2(\mathcal{I}_\mathcal{A})$, and $\phi_3(\mathcal{I}_\mathcal{A})$ that solve Eq (15). □

Finally, using Eqs (14) and (15), we can write
$$\begin{aligned} \mathbf{B} &= \mathbf{Q}^T \, \mathbf{Diag}(B_1, B_2, B_3) \, \mathbf{Q} \\ &= \mathbf{Q}^T \, \mathbf{Diag}\left(\hat{f}(A_1, A_2, A_3), \hat{f}(A_2, A_1, A_3), \hat{f}(A_3, A_2, A_1)\right) \mathbf{Q} \\ &= \mathbf{Q}^T \left(\phi_1(\mathcal{I}_\mathcal{A}) \, \mathbf{1} + \phi_2(\mathcal{I}_\mathcal{A}) \, \mathbf{Diag}(A_1, A_2, A_3) + \phi_3(\mathcal{I}_\mathcal{A}) \, \mathbf{Diag}(A_1^2, A_2^2, A_3^2)\right) \mathbf{Q} \\ &= \phi_1(\mathcal{I}_\mathcal{A}) \, \mathbf{1} + \phi_2(\mathcal{I}_\mathcal{A}) \, \mathbf{A} + \phi_3(\mathcal{I}_\mathcal{A}) \, \mathbf{A}^2 \, . \end{aligned}$$

This concludes the proof of Eq (1).

B Code for computing additional basis elements at order 3 and 4

```
import numpy as np
# Code to compute the order reducers in K. Kamrin and S. Govindjee
# "Clarifying the representation of isotropic symmetric tensor-valued
# functions of two symmetric tensors", Mathematics and Mechanics of Solids
# (2025), in submission.

# The code builds a row vector representation of each n-reducer in the
# general tensor basis from the use of the Cayley-Hamilton construction
# from the manuscript.  The list of all n-reducers is collected into a
# matrix mb[n][,]. These include secondary reducers generated from (n-1)-reducers
# and primary reducers that are deduced from the Cayley-Hamilton arguments
# in the manuscript.  The code then constructs a representation of all monomial
# permutations at a given order, symmetrizes them, and appends them as rows
# to the matrix of n-reducer representations one-by-one.  If doing so
# changes the rank, the term just added is outputted as a survivor in the
# representation.

# Helper functions
# Convert AB strings to binary
def abtb(s):
  s=s.replace('A','0')
  s=s.replace('B','1')
  return s

# Convert 01 strings to AB
```

```
def btab(s):
  s=s.replace('0','A')
  s=s.replace('1','B')
  return s

# Convert polynomial/monomial AB string to basis string for given degree
def convab(s,deg):
  basis_string = np.zeros(2**deg) # initialize basis_string
  monom = s.split('+')            # split individual monomials
  for i in range(len(monom)):
    fact = monom[i].split('*')    # pull out multipliers from monomials
    b    = abtb(fact[-1])         # convert monomial to binary
    c    = 1                      # default multiplier
    if len(fact) > 1:
      c = int(fact[0])            # set multiplier if present
    loc = int(b,2)                # compute decimal location
    basis_string[loc] = c         # set component value
  return basis_string

# Convert basis string to AB string
def convbs(bs):
  deg = int(np.log2(len(bs)))            # Extract degree of the basis_string
  out = ''
  for i in range(len(bs)):
    if bs[i] != 0:                       # Find non-zero entries
      b = format(i,'0{}b'.format(deg))   # determine monomial in binary form
      term = '+'+btab(b)                 # setup term
      if bs[i] != 1:
        term = '+{}*'.format(bs[i])+btab(b)  # mult by factor if not unity
      out = out + term
  if out[0]=='+':
    out = out[1:]                        # strip leading + if present
  return out


# Symmetrize a basis string
def symbasis(basis_string):
  deg = int(np.log2(len(basis_string))) # Extract degree of the basis_string
  sym_basis = np.zeros(2**deg)           # Initialize symmetrized string
  for i in range(len(basis_string)):
    if basis_string[i] != 0:             # Find non-zero entries
      b = format(i,'0{}b'.format(deg))   # determine monomial in binary form
      symb = b[::-1]                     # transpose the monomial
      symi = int(symb,2)                 # Compute integer location
      sym_basis[i] = basis_string[i]+basis_string[symi] # Set the symmetrized values
      sym_basis[symi] = sym_basis[i]                     # (without div by 2)
  return sym_basis

# Dimension of space of all symmetric-values combinations of nth degree monomials of A and B
def D_Tot(deg):
  return (2**deg - 2**np.ceil(deg/2))/2 + 2**np.ceil(deg/2)


# Initialize list to hold coefficients in the different monomial bases
# and the results for the irreducible monomials
mb                         = list([[],[],[],[],[],[],[]])
surviving_basis_dimension = list([0,0,0,0,0,0,0])

# Base case deg=3
# Set up 3-reducers in the monomial basis use Cayley-Hamilton results from the manuscript
deg = 3
mb[deg] = np.zeros((1,2**deg))                               # Initialize for stacking
mb[deg] = np.vstack((mb[deg],convab('AAA',deg)))
mb[deg] = np.vstack((mb[deg],convab('BBB',deg)))
mb[deg] = np.vstack((mb[deg],convab('AAB+ABA+BAA',deg)))
mb[deg] = np.vstack((mb[deg],convab('BAB+BBA+ABB',deg)))
mb[deg] = mb[deg][1:,:]                                      # Remove leading row of zeros

# Compute surviving basis dimension, the size of the irreducible set of monomials
surviving_basis_dimension[deg] = D_Tot(deg) - np.linalg.matrix_rank(mb[deg])

# Iterate of the other orders
for deg in [4,5,6]:
  # Take each row of mb[deg-1] and construct deg-reducers by pre- and post-multiplication by A and B
  mb[deg] = np.zeros((1,2**deg))                # Initialize for stacking
  for r in range(mb[deg-1].shape[0]):
    row = mb[deg-1][r,:]
    newrows = np.zeros((4,2**deg))              # Initialize 4 new rows
    for c in range(len(row)):
      if row[c]!=0:                              # Find non-zero columns and generate 4 new row entries
        m1 = '0'+format(c,'0{}b'.format(deg-1)) # pre-mult  by A
```

```
        m2 = format(c,'0{}b'.format(deg-1))+'0' # post-mult by A
        m3 = '1'+format(c,'0{}b'.format(deg-1)) # pre-mult  by B
        m4 = format(c,'0{}b'.format(deg-1))+'1' # post-mult by B
        newrows[0][int(m1,2)] += row[c]         # Assemble new terms into new rows
        newrows[1][int(m2,2)] += row[c]
        newrows[2][int(m3,2)] += row[c]
        newrows[3][int(m4,2)] += row[c]
    mb[deg]=np.vstack((mb[deg],newrows))        # Add in secondary reducers generated from deg-1
  mb[deg]=mb[deg][1:,:]                         # Remove leading row of zeros

  # Adding 4-reducers of o(3) from Cayley-Hamilton result in paper
  if deg == 4:
    mb[deg] = np.vstack((mb[deg],convab('AAAB+AABA+ABAA',deg)))
    mb[deg] = np.vstack((mb[deg],convab('AABA+ABAA+BAAA ',deg)))
    mb[deg] = np.vstack((mb[deg],convab('AABB+ABBA+BBAA',deg)))
    mb[deg] = np.vstack((mb[deg],convab('2*AAAB+AABA+ABAA+2*BAAA',deg)))
    mb[deg] = np.vstack((mb[deg],convab('AABB+2*ABAB+ABBA+BAAB+BABA',deg)))
    mb[deg] = np.vstack((mb[deg],convab('ABAB+ABBA+BAAB+2*BABA+BBAA',deg)))
    mb[deg] = np.vstack((mb[deg],convab('2*ABBB+BABB+BBAB+2*BBBA',deg)))
    mb[deg] = np.vstack((mb[deg],convab('AABB+BAAB+BBAA',deg)))
    mb[deg] = np.vstack((mb[deg],convab('ABBB+BABB+BBAB',deg)))
    mb[deg] = np.vstack((mb[deg],convab('BABB+BBAB+BBBA',deg)))
    mb[deg] = np.vstack((mb[deg],convab('3*BBBB',deg)))
    mb[deg] = np.vstack((mb[deg],convab('3*AAAA',deg)))

  # Adding 5-reducers of o(4) from Cayley-Hamilton result in paper
  if deg == 5:
    mb[deg] = np.vstack((mb[deg],convab('3*AAAAA',deg)))
    mb[deg] = np.vstack((mb[deg],convab('2*AAAAB+AAABA+AABAA+2*ABAAA',deg)))
    mb[deg] = np.vstack((mb[deg],convab('2*AAABA+AABAA+ABAAA+2*BAAAA',deg)))
    mb[deg] = np.vstack((mb[deg],convab('2*AAABB+AABBA+ABBAA+2*BBAAA',deg)))
    mb[deg] = np.vstack((mb[deg],convab('AABAB+ABAAB+ABABA',deg)))
    mb[deg] = np.vstack((mb[deg],convab('AABBA+ABAAB+ABABA+ABBAA+BAAAB+BAABA',deg)))
    mb[deg] = np.vstack((mb[deg],convab('AABBB+ABABB+ABBAB+ABBBA+BBAAB+BBABA',deg)))
    mb[deg] = np.vstack((mb[deg],convab('ABABA+BAABA+BABAA',deg)))
    mb[deg] = np.vstack((mb[deg],convab('ABABB+ABBBA+BAABB+BABBA+BBABA+BBBAA',deg)))
    mb[deg] = np.vstack((mb[deg],convab('ABBBB+BBABB+BBBBA',deg)))
    mb[deg] = np.vstack((mb[deg],convab('AAAAB+AABAA+BAAAA',deg)))
    mb[deg] = np.vstack((mb[deg],convab('AAABB+AABAB+ABAAB+ABBAA+BAAAB+BABAA',deg)))
    mb[deg] = np.vstack((mb[deg],convab('AABAB+AABBA+BAAAB+BAABA+BABAA+BBAAA',deg)))
    mb[deg] = np.vstack((mb[deg],convab('2*AABBB+BAABB+BBAAB+2*BBBAA',deg)))
    mb[deg] = np.vstack((mb[deg],convab('ABABB+ABBAB+BABAB',deg)))
    mb[deg] = np.vstack((mb[deg],convab('ABBAB+ABBBA+BAABB+BABAB+BABBA+BBAAB',deg)))
    mb[deg] = np.vstack((mb[deg],convab('2*ABBBB+BABBB+BBABB+2*BBBAB',deg)))
    mb[deg] = np.vstack((mb[deg],convab('BABAB+BABBA+BBABA',deg)))
    mb[deg] = np.vstack((mb[deg],convab('2*BABBB+BBABB+BBBAB+2*BBBBA',deg)))
    mb[deg] = np.vstack((mb[deg],convab('3*BBBBB',deg)))

  # Adding 6-reducers of o(5) from Cayley-Hamilton result in paper
  if deg == 6:
    mb[deg] = np.vstack((mb[deg],convab('AAAAAA',deg)))
    mb[deg] = np.vstack((mb[deg],convab('AAAAAB+AAABAA+ABAAAA',deg)))
    mb[deg] = np.vstack((mb[deg],convab('AAAABA+AABAAA+BAAAAA',deg)))
    mb[deg] = np.vstack((mb[deg],convab('AAAABB+AABBAA+BBAAAA',deg)))
    mb[deg] = np.vstack((mb[deg],convab('AAABAB+ABAAAB+ABABAA',deg)))
    mb[deg] = np.vstack((mb[deg],convab('AAABBA+AABAAB+ABAABA+ABBAAA+BAAAAB+BAABAA',deg)))
    mb[deg] = np.vstack((mb[deg],convab('AAABBB+AABBAB+ABAABB+ABBBAA+BBAAAB+BBABAA',deg)))
    mb[deg] = np.vstack((mb[deg],convab('AABABA+BAAABA+BABAAA',deg)))
    mb[deg] = np.vstack((mb[deg],convab('AABABB+AABBBA+BAAABB+BABBAA+BBAABA+BBBAAA',deg)))
    mb[deg] = np.vstack((mb[deg],convab('AABBBB+BBAABB+BBBBAA',deg)))
    mb[deg] = np.vstack((mb[deg],convab('ABABAB',deg)))
    mb[deg] = np.vstack((mb[deg],convab('ABABBA+ABBAAB+BAABAB',deg)))
    mb[deg] = np.vstack((mb[deg],convab('ABABBB+ABBBAB+BBABAB',deg)))
    mb[deg] = np.vstack((mb[deg],convab('ABBABA+BAABBA+BABAAB',deg)))
    mb[deg] = np.vstack((mb[deg],convab('ABBABB+ABBBBA+BAABBB+BABBAB+BBABBA+BBBAAB',deg)))
    mb[deg] = np.vstack((mb[deg],convab('ABBBBB+BBABBB+BBBBAB',deg)))
    mb[deg] = np.vstack((mb[deg],convab('BABABA',deg)))
    mb[deg] = np.vstack((mb[deg],convab('BABABB+BABBBA+BBBABA',deg)))
    mb[deg] = np.vstack((mb[deg],convab('BABBBB+BBBABB+BBBBBA',deg)))
    mb[deg] = np.vstack((mb[deg],convab('BBBBBB',deg)))

  # Symmetrize the reducers
  for r in range(mb[deg].shape[0]):
    mb[deg][r,:] = symbasis(mb[deg][r,:])

  # Compute size of surviving reducers
  surviving_basis_dimension[deg] = D_Tot(deg) - np.linalg.matrix_rank(mb[deg])

# Generate the irreducible basis elements
for deg in [3,4,5,6]:
  allm = np.eye(2**deg)  # representation of all monomials
```

```
# Symmetrize the monomials
for r in range(2**deg):
  allm[r,:] = symbasis(allm[r,:])

# Init counters and rank of the symmetrized reducers at deg
found_red = 0
cur_rank = np.linalg.matrix_rank(mb[deg])
row = 0

# Scan for rows in the symmetrized monomials that when added to the matrix of reducers changes the rank
# and hence identifies a non-reducible term, i.e. is a surviver, continue until surviving_basis_dimension[deg]
# elements are found
while found_red < surviving_basis_dimension[deg]:
  new_rank = np.linalg.matrix_rank(np.vstack((mb[deg],allm[row,:])))  # Add row from allm, get rank
  if new_rank > cur_rank:                                              # if rank goes up, keep
    print('Degree {} survivor {}'.format(deg,convbs(allm[row,:])))     # print found reducer
    cur_rank = new_rank
    found_red += 1
    mb[deg] = np.vstack((mb[deg],allm[row,:]))                         # append to reducers
  row += 1
```