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Schwartz spaces and local zeta integrals for powers of Hecke characters

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SCHWARTZ SPACES AND LOCAL ZETA INTEGRALS FOR POWERS OF HECKE CHARACTERS

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by

Natalya Jackson

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Abstract

Schwartz Spaces and Local Zeta Integrals for Powers of Hecke Characters

by

Natalya Jackson

We provide alternative constructions for the Local Langlands Correspondence for certain reductive groups through elementary integrals. Following the work of Matchett, Tate, Iwasawa, and Godement-Jacquet, we generate L-functions through integral representations which facilitate alternative proofs of their functional equations. We investigate, via difference operators, viable spaces as replacements for the usual Schwartz spaces, which produce L-functions which were previously not accessible through integrals. In particular, we realize the vision of Braverman-Kazhdan for the case where $G = GL_1(K)$ and ρ is the n^{th} power character.

Dedication

I dedicate this to the would-be mathematician who lacks the racial, gender, or socioeconomic privilege required to thrive in academia, whose potential contributions to mathematics are lost to us and to our descendants.

Acknowledgments

I am infinitely grateful to my doctoral advisor, Martin Weissman, for his suggestion of this problem when all I knew was that I was drawn to the Langlands Program. His support and guidance throughout this process has been incredibly valuable and I hope to pay it forward someday.

I would like to acknowledge my children, Legacy and Scott, for their support through my academic journey, both voluntary and involuntary.

I extend my deepest gratitude to my sister, Laura Thacker, for supporting me through this process in more ways than she will ever know.

I would like to thank a man whose name will go unrecognized, who vetoed my hire for a position I thought I wanted, without whose prejudice I would never have thought to pursue a doctorate in mathematics. I discovered my true passion by being rejected for the path which was not mine.

Chapter 1

Introduction and Background

1.1 Motivation and Overview

Much of the philosophy of the Langlands program rests on certain functoriality properties of L-functions for corresponding representations. To this end, many methods have been employed to construct L-functions and demonstrate their corresponding functional equations, some more straightforward than others. The constructions which seem the most intuitive are the integral constructions, following Matchett [9], Tate [11], and Godement-Jacquet [5]. Braverman-Kazhdan have envisioned a sweeping framework for generalizing these constructions [2]. Wen-Wei Li has done significant work toward resolving some of the technical difficulties which arise when generalizing these constructions, as well as developing a formal foundation for such generalizations [8]. In this work, we show that an integral construction can be used, with a generalized Schwartz space of test functions, to produce L-functions which are currently known by other methods, and that these integral constructions can be used to more elegantly prove the known functional equations for these L-functions. Our hope is that these methods will generalize to arbitrary reductive groups over local nonarchimedean fields using the relationship between a reductive group and its maximal tori.

We will first introduce some background, beginning with a brief history of integral constructions for L-functions and their functional equations. Specializing to nonarchimedean local fields, we will discuss some constructions for GL_1.

1.2 Matchett, Tate, and Iwasawa

For this section, K is a global field, $K_\mathfrak{p}$ the completion at some (finite or infinite) prime $\mathfrak{p}$, $\mathcal{O}_\mathfrak{p}$ the ring of integers of the completion $K_\mathfrak{p}$, $\varpi_\mathfrak{p}$ a uniformizer for $\mathcal{O}_\mathfrak{p}$.

The history of adèlic integral constructions for L-functions goes back to Margaret Matchett [9] and John Tate [11], both doctoral students of Emil Artin, as well as Kenkichi Iwasawa [7]. Artin's contributions to class field theory and his quest for a nonabelian analogue likely informed the suggestions he gave the students he advised [4]. Prior to the first publication of Matchett's thesis in 2015 [4, Chapter 3], the only widely known reference to it was Tate's comment in his own thesis [11, Section 1.2]:

> Artin suggested to me the possibility of generalizing the notion of ζ-function, and simplifying the proof of the analytic continuation and functional equation for it, by making fuller use of analysis in the spaces of valuation vectors and idèles themselves than Matchett had done. This thesis is the result of my work on his suggestion. I replace the classical notion of ζ-function, as the sum over integral ideals of a certain type of ideal character, by the corresponding notion for idèles, namely, the integral over the idèle group of a rather general weight function times an idèle character which is

> trivial on field elements. The role of Hecke's complicated theta-formulas for theta functions formed over a lattice in the n-dimensional space of classical number theory can be played by a simple Poisson Formula for general functions of valuation vectors, summed over the discrete subgroup of field elements.

For a reader encountering this quote, it may not be immediately apparent that Matchett had also translated the classical ζ-functions from the ideal perspective to the idèle perspective, following the suggestion of Artin to make use of the idèlic framework developed by Chevalley. Another historical note is that the "fuller use of analysis" alluded to by Tate in 1950 depended largely on theory not fully developed in 1946 when Matchett completed her thesis. Indeed, Tate had access to the 1948 work of Jean Braconnier detailing harmonic analysis in an abstract locally compact abelian group [1], and the 1947 work of Cartan and Godement regarding duality theory in such groups [3]. Matchett, not having the benefit of such fully developed theory in 1946, defined a topology on $K_{\mathfrak{p}}$ for $\mathfrak{p}$ finite [9, Section 2] and constructed a measure function explicitly for $K_{\mathfrak{p}}$ with $\mathfrak{p}$ finite [9, Section 2] and infinite [9, Section 3].

Matchett shows that a character on the idèles can be factored into a product of local characters [9, Section 4, Theorem 4] and she interpreted Hecke's Grössencharakters as characters of the idèles [9, Section 4]. She fixes an idèle character C, and associates to it a divisor $\mathfrak{f} = \prod_{i=1}^{k} \mathfrak{p}_i^{\nu_i}$ made up of the ramified finite primes. For an idèle $\mathfrak{a}$ with components $\mathfrak{a}_{\mathfrak{p}_{\infty,i}}$ at infinity, she defines [9, Section 6]

$$I(\mathfrak{a}, s, C) = C(\mathfrak{a}) \prod_{i=1}^{n} \mathfrak{a}_{\mathfrak{p}_{\infty,i}}^{a_i}.$$

She then defines F to be the region in the idèle space defined by $\mathfrak{a} \in F$ if and only if $\mathfrak{a}$ is integral and $\mathfrak{a} \equiv 1 \pmod{\mathfrak{f}}$. It is worth noting that the resulting integral $\int_F I(\mathfrak{a}, s, C)\, d\phi$ is precisely what Tate would later call a zeta integral, using the characteristic function of F as the specific element of the (global) Schwartz space. Matchett decomposes the space F with respect to non-associated $\mathfrak{b}^*$ meeting the same conditions, writing $F = \sum_{\mathfrak{b}^*}^{\circ} E_{\mathfrak{b}^*}$. This allows her to compute $\int_F I(\mathfrak{a}, s, C)\, d\phi$ by computing the associated $\int_{E_{\mathfrak{b}^*}} I(\mathfrak{a})\, d\phi$. She does this by breaking each one into the finite and infinite components:

$$\int_{E_{\mathfrak{b}^*}} I(\mathfrak{a})\, d\phi = \int_{E_{\mathfrak{b}^*,\infty}} I(\mathfrak{a}_\infty)\, d\phi_\infty \cdot \int_{E_{\mathfrak{b}^*,1}} I(\mathfrak{a}_1)\, d\phi_1 = J_\infty \cdot J_1.$$

Her explicit computations of the infinite components, in particular, reveal the connection between the known gamma function factors appearing in the functional equation for Hecke's L-functions, and the archimedean places of the idèlic factorization:

$$J_\infty = k \prod_{p=1}^{r+1} \Gamma\left(\frac{z_p + 1}{2}\right).$$

It should be noted that this conceptual link between the archimedean places and the presence of the gamma function in the functional equation is usually attributed to Tate, but clearly appears in Matchett's work. She in fact uses this explicit computation and its relationship to the known convergence of the L-function to prove convergence of her integral [9, Theorem 13]. Following this, she observes that the more natural integral decomposition

$$\int_F I(\mathfrak{a}, s, C)\, d\phi = \prod_{\mathfrak{p}} \int_{F_\mathfrak{p}} I_\mathfrak{p}(\mathfrak{a}, s, C)\, d\phi_\mathfrak{p},$$

where for finite $\mathfrak{p}$ prime to f we have:

$$\int_{F_\mathfrak{p}} I_\mathfrak{p}\, d\phi_\mathfrak{p} = \sum_\nu \frac{C_\mathfrak{p}(\pi)^\nu}{(N\mathfrak{p}^\nu)^{s+1}}.$$

shows the product decomposition of the zeta function into local factors.

Her goal had been to translate the classic tools used to study L-functions into the idèlic language, including the use of the theta function to prove the functional equation. It is generally agreed that there is a conceptual gap in her translation, however, when it comes to the expression of the partial L-function used by Hecke in his proof of the functional equation

$$\phi(s, \lambda, \mathcal{K}) = \zeta(x, \lambda, \mathcal{K}) \prod_{p=1}^{r+1} \Gamma\left(\frac{z_p}{2}\right)$$

in the idélic framework. Matchett fails to explicitly define the specific idèles corresponding to $\mathcal{K}$ [4, Section 3.2] and thus her idèlic interpretation of Hecke's proof of the functional equation is generally considered to be incomplete.

In 1950, Tate extended Matchett's work by proving an adèlic Poisson summation (a number-theoretic Riemann Roch theorem) which allowed him to use Fourier analysis to give an alternative proof of the (global) functional equation [1] for the Riemann Zeta function using the more fully developed theoretical tools available as mentioned before [11].

Let $f \in L^1(K_\mathfrak{p})$ be such that its Fourier transform $\hat{f}$ is in $L^1(K_\mathfrak{p})$ as well. Tate made the additional constraint that $f(x)|x|^s$ and $\hat{f}(x)|x|^s$ are in $L^1(K_\mathfrak{p}^\times)$ as well, for $\Re(s) > 0$. Note the modern treatment of Tate's work usually refers to Schwartz-Bruhat functions, which are rapidly decaying smooth functions for $K_\mathfrak{p}$ archimedean, and locally constant functions with compact support when $K_\mathfrak{p}$ is nonarchimedean. [10,

[1]The functional equation is often expressed in terms of the completed Riemann Zeta function, $\xi(s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \xi(1-s)$. Here the factorization includes the infinite primes, thus the notation $\zeta(s)$ is here the completed version.

Section 7.1] It should also be noted that Tate referred to quasi-characters and characters, rather than characters and unitary characters, to specify the codomain as $\mathbb{C}^\times$ or S^1, respectively. Where Tate used the notation $c(\alpha) = \tilde{c}(\alpha)|\alpha|^s$ for a quasicharacter of exponent $\sigma = \Re(s)$, we commonly use χ for $\tilde{c}$ and separate the nonunitary part.

Tate used ξ to denote a typical element of the additive group $K_\mathfrak{p}$ and α a member of the multiplicative group $K_\mathfrak{p}^\times$. Thus he distinguished between the additive and multiplicative Haar measure by $d\xi$ and $d\alpha$, respectively. In what follows we will use the subscripted $\mathfrak{p}$ notation to indicate local components, and its absence to indicate global theory. He defined [11, Definition 2.4.1] the local zeta integral, for $f_\mathfrak{p}, c_\mathfrak{p}, \alpha_\mathfrak{p}$:

$$\zeta_\mathfrak{p}(f_\mathfrak{p}, c_\mathfrak{p}) = \int f_\mathfrak{p}(\alpha_\mathfrak{p}) c_\mathfrak{p}(\alpha_\mathfrak{p}) \, d\alpha_\mathfrak{p},$$

and showed [11, Lemma 2.4.1] that this converged absolutely for characters $c_\mathfrak{p}$ with exponent greater than zero. He proved the local functional equation in [11, Theorem 2.4.1]:

> A ζ-function has an analytic continuation to the domain of all quasi-characters given by a functional equation of the type
>
> $$\zeta(f, c) = \rho(c)\zeta(\hat{f}, \hat{c}).$$
>
> The factor $\rho(c)$, which is independent of the function f, is a meromorphic function of quasi-characters defined in the domain $0 <$ exponent $c < 1$ by the functional equation itself, and for all quasi-characters by analytic continuation.

Note Tate uses the same notation for both local and global theory, where we have added subscripted $\mathfrak{p}$ to indicate local theory in our discussion of his work. He showed in [11, Section 4.5], using [11, Theorem 3.3.1] (allowing the global integral to be expressed as a product of local integrals) that the product of the local zeta integrals gave the globally defined zeta integral

$$\zeta(f,c) = \prod_{\mathfrak{p}} \zeta_{\mathfrak{p}}(f_{\mathfrak{p}}, c_{\mathfrak{p}}).$$

He remarked further that the global zeta integral in fact gave the classically known zeta functions for Hecke characters [11, Section 4.5]. He proved the global functional equation[11, Theorem 4.4.1]

$$\zeta(\hat{f}, \hat{c}) = \zeta(f, c).$$

Here

$$\zeta(f,c) = \prod_{\mathfrak{p}} \zeta_{\mathfrak{p}}(f_{\mathfrak{p}}, c_{\mathfrak{p}}),$$

and

$$\zeta_{\mathfrak{p}}(f_{\mathfrak{p}}, c_{\mathfrak{p}}) = \rho_{\mathfrak{p}}(c_{\mathfrak{p}})\zeta_{\mathfrak{p}}(\hat{f}_{\mathfrak{p}}, \hat{c}_{\mathfrak{p}}),$$

noting that $\rho_{\mathfrak{p}}$ is not dependent on f_p. In our notation $Z(\chi, s)$, in which $c = \chi|\cdot|^s$, this yields the functional equation

$$Z(\chi^{-1}, 1-s) = \prod_{\mathfrak{p}\in S} \rho_{\mathfrak{p}}(\chi_{\mathfrak{p}}|\cdot|_{\mathfrak{p}}^{s+it_{\mathfrak{p}}}) \cdot \prod_{\mathfrak{p}\notin S} \mathcal{N}\mathfrak{d}_{\mathfrak{p}}^{s-\frac{1}{2}}\chi^{-1}(\mathfrak{d}_{\mathfrak{p}}) \cdot Z(\chi, s).$$

Note the factor $\rho_{\mathfrak{p}}$ also does not depend on f and is in fact a rational function of q^{-s} where q is the size of the residue field.

Matchett's interpretation of the gamma factors which appear in the functional equation as coming from the infinite places of $\mathbb{Q}$, combined with Tate's more extensive use of newly developed analytical tools in the measure space, led to a fuller understanding of the overall theory. It should be remarked that Iwasawa proved the results in [11] independently at or around the same time, presenting his results at the International Congress of Mathematics in 1950 [7].

1.3 Godement-Jacquet Constructions

Integral constructions for L-functions in the GL_n case were detailed by Roger Godement and Hervé Jacquet [5] in 1972, extending the results of [11] to n dimensions. One major difference in approach is that while Tate explicitly defined specific functions for which zeta integrals produced the desired L-functions, Godement-Jacquet showed that the set of all possible zeta integral outputs for a suitable space was in fact a finitely-generated submodule of a PID. The canonical generator was then defined to be the L-function, without specifying the input function required to produce it. It is worth noting that one can in fact specify the "test vector" of the Schwartz space which produces the L-function for GL_n in the nonarchimedean case. This was done for spherical π in [5, Lemma 6.10], but the case for ramified π was open until a recent work by Peter Humphries [6].

Let $\mathcal{S}(M_n(K_\mathfrak{p}))$ be the space of compactly-supported complex-valued functions on $M_n(K_\mathfrak{p})$. For a smooth admissible representation (π, V), denote by $(\tilde{\pi}, \tilde{V})$ its contragredient. Let $v \in V$, $\lambda \in \tilde{V}$, $\phi \in \mathcal{S}(M_n(K_\mathfrak{p}))$, $s \in \mathbb{C}$. Denote by dg the normalized Haar measure on $GL_n(K_\mathfrak{p})$. They define the zeta integral [5, Section 3]:

$$Z(v, \lambda, f, s) = \int_{GL_n(K_\mathfrak{p})} m_{v,\lambda}(g) f(g) |\det(g)|^s \, dg.$$

Regarding notation here, Godement-Jacquet use a single variable f to express the data of the matrix coefficient $m_{v,\lambda}$ and they use ϕ instead of f for the member of the Schwartz space. They prove [5, Theorem 3.3] that these zeta integrals converge, for $\Re(s) \gg 0$, to a rational function of q^{-s}. They define

$$J(\pi) = \{Z(v, \lambda, f, s) : v \in V, \lambda \in \tilde{v}, f \in \mathcal{S}(M_n(K_\mathfrak{p}))\} \subset \mathbb{C}(q^{-s}),$$

and prove that it forms a finitely-generated $\mathbb{C}[q^{\pm s}]$-submodule of $\mathbb{C}(q^s)$. Godement and Jacquet define $L(\pi, s)$ to be the canonical generator for $J(\pi)$. Expressing the data (v, λ) as $\varphi \in C(\pi)$ a choice of matrix coefficient for π, there is a functional equation for the zeta integrals:

$$Z(\hat{f}, \check{\varphi}, 1 - s) = \gamma(\pi, \psi, s) Z(f, \varphi, s),$$

where γ is a rational function in $\mathbb{C}(q^{-s})$ which does not depend on f or on the choice of matrix coefficient $\varphi \in C(\pi)$, but does depend on the choice of ψ used to define the Fourier transform. Passing to the L-function defined by the canonical generator of $J(\pi)$, we have the functional equation

$$\epsilon(\pi, \psi, s) = \gamma(\pi, \psi, s) \cdot \frac{L(\pi, s)}{L(\check{\pi}, 1 - s)},$$

where γ is as before and ϵ is a unit in $\mathbb{C}[q^{\pm s}]$, thus a monomial in q^{-s}.

1.4 Braverman-Kazhdan

In 2000, Alexander Braverman and David Kazhdan published a conjectural framework for generalizing the Godement-Jacquet process to produce additional L-functions [2]. Consider G a split reductive p-adic group (of which $GL_n(K)$, considered in [5], is an example) and π a smooth irreducible representation of G. Let ρ be an algebraic representation of $G^\vee$, the dual group. The Langlands conjectures predict the existence of an L-function associated to this data, $L(\pi, \rho, s)$ satisfying certain desirable properties. Braverman-Kazhdan denote this as $L(l_\rho(\pi), s)$ [2, Section 1.4], in reference to the conjectured functorial lifting. They conjecture the existence of a space $\mathcal{S}_\rho$, associated to G and ρ, such that the Godement-Jacquet process produces $L(l_\rho(\pi), s)$ as the normalized generator of a finitely generated submodule consisting of zeta integral outputs for $\mathcal{S}_\rho$. These $\mathcal{S}_\rho$ are assumed to contain the usual Schwartz space of smooth compactly supported functions on G.

In this work, we explicitly describe such a Schwartz space $\mathcal{S}_\rho$ for the case in which $G = GL_1(K)$, where ρ is the n^{th} power character. We prove that this generalized Schwartz space gives the known Hecke L-functions for this choice of ρ, and can in fact be used to directly prove the local functional equation in this case.

Chapter 2

Constructions for GL_1

Let K be a local nonarchimedean field, $\mathcal{O}$ the ring of integers, and ϖ a uniformizer. Let f be the residue degree, and q the order of $\mathcal{O}/\varpi\mathcal{O}$. We normalize the valuation so $\mathrm{val}(\varpi) = 1$ and normalize the absolute value so that $|\varpi| = q^{-1}$. Denote by $\mathrm{Vol}(\cdot)$ the measure of a set with respect to the additive Haar measure dx, and $\mathrm{Vol}^\times(\cdot)$ the measure of a set with respect to the multiplicative Haar measure $d^\times x$. We are assuming that dx is normalized so that $\mathrm{Vol}(\mathcal{O}) = 1$, and that $d^\times x$ is normalized so that $\mathrm{Vol}^\times(\mathcal{O}^\times) = 1$.

2.1 L-functions for GL_1

Associated to a character of $K^\times$ is its L-function, defined as follows:

Definition 1. *Let $\chi : K^\times \to \mathbb{C}^\times$ be a continuous character. We define the local L-function $L(\chi, s) = (1 - \chi(\varpi)q^{-s})^{-1}$ when χ is unramified, and $L(\chi, s) = 1$ when χ is* *ramified.*

We also define the n^{th} power L-functions:

Definition 2. *Let* $L(\chi, n, s) = L(\chi^n, s)$*.*

Note that within the framework of the Langlands program, we have L-functions of the form $L(\pi, \rho, s)$, where π is an irreducible representation of G and ρ is a finite dimensional representation of $G^\vee$, the Langlands dual group. In the case of $L(\chi, s)$ we have $G = GL_1(K)$, $G^\vee = GL_1(\mathbb{C})$, and $\rho : G^\vee \to \mathbb{C}^\times$ the identity representation. For the case of $L(\chi, n, s)$, the groups G and $G^\vee$ remain the same but we replace ρ with the n^{th} power representation.

In this work we will define a substitution for the usual Schwartz space found in [11], [5]. We hope to then use this new space to generate $L(\chi, s, n)$ via integrals, not just $L(\chi, s)$ as has been done previously. Thus we achieve the vision of [2] for the case $G = GL_1(K)$ where ρ is the n^{th} power character.

2.2 Zeta Integrals and Schwartz spaces

Definition 3. *Following [11] we define the* local zeta integral

$$Z(f, \chi, s) = \int_{K^\times} f(x)\chi(x)|x|^s \, d^\times x,$$

for all continuous $f : K^\times \to \mathbb{C}$*, all continuous characters* $\chi : K^\times \to \mathbb{C}^\times$*, and all* $s \in \mathbb{C}$ *such that the integral converges.*

Lemma 4. *For $f \in C_c^\infty(K^\times)$, and χ a continuous character, the local zeta integral $Z(f, \chi, s)$ lies in $\mathbb{C}[q^{\pm s}]$.*

Proof. For $f \in C_c^\infty(K^\times)$ we may view f as a linear combination of characteristic functions of compact sets Ω, where we note that Ω can be taken sufficiently small that f, χ, and $|\cdot|$ are constant on Ω. Let $f(x) = a, \chi(x) = b, |x| = q^{-n}$ for $x \in \Omega$. It suffices to consider zeta integrals for the restriction to such Ω. Then we have:

$$\begin{aligned} Z(f\big|_\Omega, \chi, s) &= \int_\Omega f(x)\chi(x)|x|^s \, d^\times x, \\ &= \int_\Omega a \cdot b \cdot q^{-ns} \, d^\times x, \\ &= \mathrm{Vol}^\times(\Omega) a \cdot b \cdot (q^{-s})^n. \end{aligned}$$

This is visibly in $\mathbb{C}[q^{\pm s}]$, as required. The result follows by the linearity of the integral.

□

Definition 5. *Following the notation of [5] we define, for S a subspace of $C(K^\times)$ and χ a continuous character on $K^\times$, $J(S, \chi) = \{Z(f, \chi, s) : f \in S\}$.*

Example 6. *For $S = C_c^\infty(K^\times)$ we have $J(C_c^\infty(K^\times), \chi) = \mathbb{C}[q^{\pm s}]$ which can be viewed as the trivial fractional ideal within itself.*

Proposition 7. *$J(C_c^\infty(K), \chi)$ is the fractional ideal of $\mathbb{C}[q^{\pm s}]$ generated by $L(\chi, s)$.*

Proof. Godement and Jacquet attribute this result to Tate in [5, Section 3], but Tate did not explicitly consider the fractional ideal framework in [11]. The statement is true by [5, Theorem 3.3] and thus it seems better attributed to Godement-Jacquet. □

Combining Proposition 7 with Example 6, we see that

$$J(C_c^\infty(K), \chi) = L(\chi, s) J(C_c^\infty(K^\times), \chi).$$

We want to explore replacements for $C_c^\infty(K)$. We begin by defining what constitutes a *viable* space of functions.

Definition 8. *A* viable *space of functions on $K^\times$ is a complex vector space S which has the following properties:*

1. *$C_c^\infty(K^\times) \subset S \subset C^\infty(K^\times)$,*

2. *For every $f \in S$ and for every $t \in K^\times$, g defined by $g(x) = f(t^{-1}x)$ is also in S,*

3. *For every $f \in S$ and for every continuous character $\chi : K^\times \to \mathbb{C}^\times$, $Z(f, \chi, s)$ converges "loosely" to an element of $\mathbb{C}(q^s)$.*

We need to define "loosely convergent," and to this end we note that for any $f \in C^\infty(K^\times)$ we may write $f = f^+ + f^-$ where $f^+(x) = 0$ for $\mathrm{val}(x) \gg 0$ and $f^-(x) = 0$ for $\mathrm{val}(x) \ll 0$. (Note this is not a unique decomposition.)

Definition 9. *We say $Z(f, \chi, s)$* converges loosely *to an element of $\mathbb{C}(q^s)$ if the following two statements hold:*

- *$Z(f^+, \chi, s)$ converges to an element of $\mathbb{C}(q^s)$ for $\Re(s) \ll 0$,*
- *$Z(f^-, \chi, s)$ converges to an element of $\mathbb{C}(q^s)$ for $\Re(s) \gg 0$.*

Definition 10. *For $f \in C^\infty(K^\times)$, if $Z(f, \chi, s)$ is loosely convergent, then we define $Z(f, \chi, s) = Z(f^+, \chi, s) + Z(f^-, \chi, s)$.*

It remains to show that this is well-defined.

Proposition 11. *For $f \in C^\infty(K^\times)$, we have:*

1. *The condition of $Z(f, \chi, s)$ being loosely convergent does not depend on the choice of decomposition $f = f^+ + f^-$;*

2. *If $Z(f, \chi, s)$ is in fact loosely convergent, the element of $\mathbb{C}(q^s)$ defined by $Z(f, \chi, s)$ does not depend on the choice of f^+ and f^- for decomposition of f.*

Proof. Note that if $f = f_1^+ + f_1^- = f_2^+ + f_2^-$, then $f_1^+ - f_2^+ = f_2^- - f_1^-$. Thus since

$$[f_1^+ - f_2^+](x) = 0 \text{ for } \operatorname{val}(x) \gg 0 \quad \text{ and } \quad [f_2^- - f_1^-](x) = 0 \text{ for } \operatorname{val}(x) \ll 0,$$

we see that both sides are actually in $C_c^\infty(K^\times)$. So for the first statement, we use the linearity of the zeta integral to see that $Z(f_1^+, \chi, s) - Z(f_2^+, \chi, s) \in \mathbb{C}(q^s)$, so either both $Z(f_1^+, \chi, s)$ and $Z(f_2^+, \chi, s)$ converge to an element of $\mathbb{C}(q^s)$, or both do not. The same holds for f_1^- and f_2^-, so the condition of being loosely convergent does not depend on the choice of f^+ and f^-. For the second statement, we have

$$Z(f_1^+, \chi, s) - Z(f_2^+, \chi, s) = Z(f_2^-, \chi, s) - Z(f_1^-, \chi, s),$$

$$\Rightarrow Z(f_1^+, \chi, s) + Z(f_1^-, \chi, s) = Z(f_2^+, \chi, s) + Z(f_2^-, \chi, s),$$

which shows that $Z(f, \chi, s)$ is well-defined for loosely convergent $f \in C^\infty(K^\times)$. $\square$

2.3 Operators and Identities

Here we collect a series of identities, presented as lemmas, related to operators on function spaces on K and $K^\times$. We begin by describing the relationships between these function spaces.

We may view $C^\infty(K)$ as a subspace of $C^\infty(K^\times)$, via the restriction map, because every locally constant function on K restricts to a locally constant function on the open subset $K^\times$. We may then view $C_c^\infty(K^\times)$ as a subspace of $C_c^\infty(K)$, via extension by zero. Explicitly, if $f \in C_c^\infty(K^\times)$, it can be viewed as a function on K by setting $f(0) = 0$. The resulting function is then also compactly supported in K.

It is sometimes convenient to work within a larger space, of distributions. The space of distributions on K, denoted by $C^{-\infty}(K)$, is the linear dual of $C_c^\infty(K)$. Similarly, $C^{-\infty}(K^\times)$ is the space of distributions on $K^\times$, the linear dual of $C_c^\infty(K^\times)$. Having fixed a normalized Haar measure on K and $K^\times$, we see that the space of functions $C^\infty(K)$ can be embedded naturally in the space of distributions $C^{-\infty}(K)$, and similarly for $C^\infty(K^\times)$ in $C^{-\infty}(K^\times)$. Thus every linear operator on $C_c^\infty(K)$ canonically defines a linear operator on $C^{-\infty}(K)$, and the same is true for linear operators on $C_c^\infty(K^\times)$ and $C^{-\infty}(K^\times)$.

Every distribution on K restricts to a distribution on $K^\times$, and in fact this restriction from $C^{-\infty}(K)$ to $C^{-\infty}(K^\times)$ is the dual of the extension-by-zero map which embeds $C_c^\infty(K^\times)$ in $C_c^\infty(K)$. Note there are some functions $f \in C^\infty(K^\times)$, namely those which are constant near zero, which arise as restrictions of functions in $C^\infty(K)$.

Some do not, yet these still arise as elements of $C^{-\infty}(K^\times)$ which are restrictions of distributions in $C^{-\infty}(K)$.

Definition 12. *For $a \in K$ and $f \in C_c^\infty(K)$, let T_a be defined by $[T_a f](x) = f(x-a)$.*

Note $T_a \in \operatorname{End} C_c^\infty(K)$, and as an operator on $C_c^\infty(K)$ extends canonically to an operator on $C^{-\infty}(K)$.

Definition 13. *For $b \in K^\times$, and $f \in C^\infty(K)$, let M_b be defined by $[M_b f](x) = f(b^{-1}x)$.*

Lemma 14. *$Z(M_b f, \chi, s) = \chi(b)|b|^s Z(f, \chi, s)$.*

Proof. We have

$$
\begin{aligned}
Z(M_b f, \chi, s) &= \int_{K^\times} f(b^{-1}x)\chi(x)|x|^s \, d^\times x, \\
&= \int_{K^\times} f(u)\chi(bu)|bu|^s \, d^\times u, \\
&= \chi(b)|b|^s Z(f, \chi, s).
\end{aligned}
$$

□

Corollary 15. *Let S be a viable space of functions and χ a continuous character on $K^\times$. Then $J(S, \chi)$ is a $\mathbb{C}[q^{\pm s}]$-submodule of $\mathbb{C}(q^{\pm s})$.*

Proof. The linearity of the zeta integral allows addition and subtraction, and scaling by multiples of $q^{\pm s}$ follows from the above lemma. □

Definition 16. *Let $\mathcal{F} : C_c^\infty(K) \to C^{-\infty}(K)$ be defined by $[\mathcal{F} f](y) = \hat{f}(y) = \int_K f(x)\overline{\psi(xy)}\, dx$ where ψ is Tate's additive character on K.*

Note that although initially defined on $C_c^\infty(K)$, the map $\mathcal{F}$ canonically extends to an operator from $C^{-\infty}(K)$ to itself.

Definition 17. *For an additive character ψ and $a \in K^\times$, let ψ_a denote the multiplicative shift of ψ by a, i.e. $\psi_a(x) = \psi(ax)$.*

Lemma 18. *$[\mathcal{F} \circ T_a](f) = [\overline{\psi_a} \cdot \mathcal{F}](f)$ for all $f \in C^\infty(K)$.*

Proof.

$$\begin{aligned}
\big[\,(\mathcal{F} \circ T_a)\, f\big](y) &= \big[\mathcal{F}\,(T_a f)\,\big](y), \\
&= \int_K f(x-a)\overline{\psi(xy)}\, dx, \\
&= \int_K f(u)\overline{\psi\left((u+a)y\right)}\, du, \\
&= \int_K f(u)\overline{\psi(uy)\psi(ay)}\, du, \\
&= \overline{\psi(ay)}\hat{f}(y), \\
&= \overline{\psi_a}[\mathcal{F}f](y).
\end{aligned}$$

□

Note this is also true for all distributions in $C^{-\infty}(K)$.

Lemma 19. *For all $f \in C_c^\infty(K)$, we have $\mathcal{F}M_b(f) = |b|M_{b^{-1}}\mathcal{F}\ (f)$.*

Proof. Here we use the substitution $u = b^{-1}x$ which gives $dx = |b|\, du$:

$$[(\mathcal{F}M_b)f](y) = \widehat{[M_b f]}(y) = \int_K f(b^{-1}x)\overline{\psi(xy)}\, dx,$$
$$= |b| \int_K f(u)\psi(u \cdot by)\, du,$$
$$= [(|b| M_{b^{-1}}\mathcal{F})f](y).$$

□

2.4 Difference Operators and Viable Spaces

Definition 20. *For d a positive integer, let ∂_ϖ^d be defined by $[\partial_\varpi^d f](x) = f(x) - f(\varpi^{-d}x)$ for any $f \in C^\infty(K^\times)$ and for any choice of uniformizer ϖ. Note that $\partial_\varpi^d = 1 - M_{\varpi^d}$ in $\mathrm{End}(C^\infty(K^\times))$.*

Lemma 21. *For $f \in C^\infty(K) \cap C^{-\infty}(K)$, we have $\mathcal{F}\partial_\varpi^d(f) = \mathcal{F} - q^{-d}M_{\varpi^{-d}}\mathcal{F}(f)$.*

Proof. This follows from applying Lemma 19 to Definition 20. □

Corollary 22. $\mathcal{F}\partial_\varpi^1 = \mathcal{F} - q^{-1}M_\varpi \mathcal{F}$.

Proof. This is simply the case where $d = 1$. □

It is clear that ∂_ϖ^d defines a linear operator on $C^\infty(K^\times)$, and indeed on any viable space S.

Definition 23. *Let $S_\varpi^d = \{f \in C^\infty(K^\times) : \partial_\varpi^d f \in C_c^\infty(K^\times)\}$.*

Definition 24. *Let* $S^d = \bigcap_{\varpi} S^d_{\varpi} = \{f \in C^{\infty}(K^{\times}) : \partial^d_{\varpi} f \in C^{\infty}_c(K^{\times})$ *for all uniformizers* $\varpi\}$.

The motivation for the use of ∂^d_{ϖ} becomes more apparent in the following claim:

Proposition 25. $Z(\partial^d_{\varpi} f, \chi, s) = (1 - \chi(\varpi)^d q^{-ds}) Z(f, \chi, s)$ *for all* $f \in C^{\infty}(K^{\times})$, *for any uniformizer* ϖ, *and for all* $s \in \mathbb{C}$ *such that the integrals are loosely convergent.*

Proof. This is seen by applying the observation in Definition 20 and considering Lemma 14 with $b = \varpi^d$:

$$\begin{aligned} Z(\partial^d_{\varpi} f, \chi, s) &= Z((1 - M_{\varpi^d}) f, \chi, s), \\ &= Z(f, \chi, s) - Z(M_{\varpi^d} f, \chi, s), \\ &= (1 - \chi(\varpi)^d q^{-ds}) Z(f, \chi, s). \end{aligned}$$

∎

Considering Definition 2 we immediately obtain the following corollary:

Corollary 26. *If* χ^d *is unramified, then*

$$Z(f, \chi, s) = L\left(\chi^d, ds\right) Z(\partial^d_{\varpi} f, \chi, s),$$

for all $f \in C^{\infty}(K^{\times})$, *for any uniformizer* ϖ, *and for all* $s \in \mathbb{C}$ *such that the integrals are loosely convergent. In particular, under these conditions we have*

$$Z(f, \chi, s) = L(\chi, s) Z(\partial^1_{\varpi} f, \chi, s).$$

In order to address the case when χ is ramified and develop some more useful results, we will first explore the structure of S^d_{ϖ}. We begin by defining the following:

Definition 27. *Let Ω be a compact-open subset of $K^\times$. We define*

$$\varphi_\Omega^{d,+} = \sum_{k\geq 0} \mathrm{Char}(\varpi^{kd}\Omega),$$

and

$$\varphi_\Omega^{d,-} = \sum_{k\leq 0} \mathrm{Char}(\varpi^{kd}\Omega),$$

where $\mathrm{Char}(\cdot)$ *denotes the characteristic function.*

It is worth noting that the $\varphi_\Omega^{d,\pm}$ extend to distributions on K. Specifically, we note that $\varphi_\Omega^{d,\pm}$ need not be constant near zero, thus as a function in $C^\infty(K^\times)$ may not arise as a restriction of a function in $C^\infty(K)$. But it does still arise as an element of $C^{-\infty}(K^\times)$ which is a restriction of a distribution in $C^{-\infty}(K)$. Thus for any $g \in C_c^\infty(K)$, we have convergence for additive integrals of the form $\int_K g(x)\varphi_\Omega^{d,\pm}\, dx$. In particular, the identities we proved in the various lemmas can also be applied to $\varphi_\Omega^{d,\pm}$.

We make use of this definition in the following observation regarding the structure of S_ϖ^d:

Lemma 28. *The function space S_ϖ^d is spanned by $C_c^\infty(K^\times)$ and $\{\varphi_\Omega^{d,\pm} : \Omega \subset K^\times$ compact-open$\}$.*

Proof. Suppose $f \in S_\varpi^d$. Since $\partial_\varpi^d f \in C_c^\infty(K^\times)$, there exists a positive integer N such that $|\,\mathrm{val}(x)| > N$ implies $f(x) - f(\varpi^d x) = 0$, so $f(x) = f(\varpi^d x)$ when $\mathrm{val}(x) > N$ or $\mathrm{val}(x) < -N$. Define

$$f^+(x) = \begin{cases} \lim_{n\to\infty} f(\varpi^{nd}x), & \text{for } \mathrm{val}(x) \geq 0 \\ 0, & \text{for } \mathrm{val}(x) < 0 \end{cases},$$

and

$$f^{-}(x) = \begin{cases} \lim_{n\to-\infty} f(\varpi^{nd}x), & \text{for } \mathrm{val}(x) \le 0 \\ \\ 0, & \text{for } \mathrm{val}(x) > 0 \end{cases},$$

where the nonzero portions are limits of sequences which are eventually constant, thus well-defined. Note we may choose N such that $f(x) = f^{+}(x)$ for $\mathrm{val}(x) > N$ and $f(x) = f^{-}(x)$ for $\mathrm{val}(x) < -N$. We may thus write $f = f^{+} + f^{-} + g$, where $g \in C_c^{\infty}(K^{\times})$. We claim f^{+} is a linear combination of $\varphi_{\Omega}^{d,+}$ for a finite number of Ω, and similarly f^{-} for $\varphi_{\Omega}^{d,-}$. Consider the set $D^{+} = \{x \in K^{\times} : 0 \le \mathrm{val}(x) < d\}$, which can be viewed as a fundamental domain for the action of the semigroup $\{\varpi^{nd} : n \ge 0\}$ on $\mathcal{O} \setminus \{0\}$, which is a disjoint union of multiplicative translates of D^{+} by ϖ^{d}. Note we may cover this by a finite number of compact open sets $\Omega_1, \ldots, \Omega_m$, with f^{+} constant on each Ω_i, say $f^{+}\big|_{\Omega_i}(x) = a_i$. Then $f^{+} = \sum_{i=1}^{m} a_i \varphi_{\Omega_i}^{d,+}$ as required. The case for f^{-} and $\varphi_{\Omega}^{d,-}$ is similar, using $D^{-} = \{x \in K^{\times} : -d < \mathrm{val}(x) \le 0\}$. $\square$

We observe that Lemma 28 implies that any $f \in S_{\varpi}^{d}$ extends to a distribution on K, as we previously observed for $\varphi_{\Omega}^{\pm}$. This allows us to apply the identities we proved within the space S_{ϖ}^{d}. We'd like to consider zeta integrals in S_{ϖ}^{d}, and we claim:

Proposition 29. *S_{ϖ}^{d} is a viable space in the sense of Definition 8. In particular, if $f \in S_{\varpi}^{d}$, then $Z(f, \chi, s)$ is loosely convergent.*

Proof. It is clear that $C_c^{\infty}(K^{\times}) \subset S_{\varpi}^{d} \subset C^{\infty}(K^{\times})$ by Lemma 28, thus the first condition holds. For the second condition, we note that S_{ϖ}^{d} is stable under multiplicative translation. For the third condition, loose convergence for zeta integrals in S_{ϖ}^{d}, we combine

Lemmas 4 and 28. Then it suffices to show $Z(\varphi_\Omega^{d,\pm}, \chi, s) \in \mathbb{C}(q^{-s})$. We consider the cases $\varphi_\Omega^{d,+}$ and $\varphi_\Omega^{d,-}$ separately, recalling that we can take Ω sufficiently small that χ and $|\cdot|$ are constant on Ω. Note that

$$\begin{aligned}
Z(\varphi_\Omega^{d,+}, \chi, s) &= \int_{K^\times} \varphi_\Omega^{d,+}(x)\chi(x)|x|^s \, d^\times x, \\
&= \int_{K^\times} \big[\sum_{k\geq 0} \mathrm{Char}(\varpi^{dk}\Omega)\big](x)\chi(x)|x|^s \, d^\times x, \\
&= \sum_{k\geq 0} \int_{\varpi^{dk}\Omega} \chi(x)|x|^s \, d^\times x, \\
&= \sum_{k\geq 0} \int_{\Omega} \chi(\varpi^{dk}x)|\varpi^{dk}x|^s \, d^\times x, \\
&= \sum_{k\geq 0} \chi(\varpi)^{dk} q^{-dks} \int_{\Omega} \chi(x)|x|^s \, d^\times x, \\
&= (1-\chi(\varpi)^d q^{-ds})^{-1} \cdot \mathrm{Vol}^\times(\Omega) \cdot a \cdot q^n \in \mathbb{C}(q^s).
\end{aligned}$$

Above we have absolute convergence for $\Re(s) > 0$ by properties of geometric series, where a is the constant value of χ on Ω, and q^n is the constant value of $|\cdot|$ on Ω. Note we use the countable additivity of the measure to move the sum outside the integral, and the invariance of the measure to perform the substitution. For the $\varphi_\Omega^{d,-}$ case we simply replace k with $-k$ and follow the same procedure, noting convergence of the geometric series for $\Re(s) < 0$. $\square$

Having shown that S_ϖ^d is a viable space, we arrive at the main result of this section, which applies when χ is either ramified or unramified:

Theorem 30. *$J(S^d, \chi) = L(\chi^d, ds)\mathbb{C}[q^{\pm s}]$, and in particular, $J(S^1, \chi) = J(C_c^\infty(K), \chi)$.*

Proof. Because we've shown, in particular, that the zeta integrals for S^d_ϖ are loosely convergent, we can apply Corollary 26 to the case where χ^d is unramified to see that $J(S^d_\varpi, \chi) \subset L(\chi^d, ds)\mathbb{C}[q^{\pm s}]$ for every choice of uniformizer ϖ. Because $C_c^\infty(K^\times) \subset S^d_\varpi$ it is clear that $\mathbb{C}[q^{\pm s}] \subset J(S^d_\varpi, \chi)$, so we have

$$\mathbb{C}[q^{\pm s}] \subset J(S^d_\varpi, \chi) \subset L(\chi^d, ds)\mathbb{C}[q^{\pm s}].$$

For the case that χ^d is unramified, it remains to show that there exists $f \in S^d_\varpi$ such that $Z(f, \chi, s) = L(\chi^d, ds)$. Let $\Omega = \chi^{-1}(1) \cap \mathcal{O}^\times$, and recall the calculation showing the convergence of $Z(\varphi_\Omega^{d,+}, \chi, s)$ for Ω a compact-open subset of $K^\times$, from the proof of Proposition 29. Clearly $|x| = 1$ for all $x \in \Omega$. Note that χ^d unramified does not necessarily imply χ unramified. For χ also unramified, we see that $\Omega = \mathcal{O}^\times$ and we have $\mathrm{Vol}^\times(\Omega) = 1$. Thus

$$Z(\varphi_\Omega^{d,+}, \chi, s) = (1 - \chi(\varpi)^d q^{-ds})^{-1},$$

as required. If χ^d is unramified but χ is ramified, $\chi(\varpi)^d = \chi^d(\varpi)$ is still well-defined but Ω may not be all of $\mathcal{O}^\times$. In this case we may choose $f = \left(\mathrm{Vol}^\times(\Omega)\right)^{-1} \varphi_\Omega^{d,+}$ and scale using the linearity of the zeta integral arrive at the desired result. Because $J(S^d_\varpi, \chi) = L(\chi^d, ds)\mathbb{C}[q^{\pm s}]$ for all uniformizers ϖ, we have $J(S^d, \chi) = L(\chi^d, ds)\mathbb{C}[q^{\pm s}]$ as claimed.

To see why the case where χ^d is ramified holds as well, recall the definition of S^d as the intersection of S^d_ϖ for all possible uniformizers ϖ. Note that when χ^d is ramified, χ^d will take two different values on two different uniformizers, i.e. there exist two uniformizers ϖ_1, ϖ_2 such that $\chi^d(\varpi_1) \neq \chi^d(\varpi_2)$. Thus for $f \in S^d$ we have

$$Z(f, \chi, s) = (1 - \chi(\varpi_i)^d q^{-ds})^{-1} Z(\partial^d_{\varpi_i} f, \chi, s),$$

for $i = 1, 2$. So

$$J(S^d_{\varpi_i}, \chi) = (1 - \chi(\varpi_i)^d q^{-ds})^{-1}\mathbb{C}[q^{\pm s}],$$

for each i, using the calculation in the unramified case to show equality for each choice of ϖ. Because

$$J(S^d_{\varpi_1}, \chi) \cap J(S^d_{\varpi_2}, \chi) = J(S^d_{\varpi_1} \cap S^d_{\varpi_2}, \chi),$$

and these two fractional ideals are coprime, this intersection is simply $\mathbb{C}[q^{\pm s}]$, i.e. the fractional ideal generated by $L\left(\chi^d, ds\right) = 1$ since χ^d is ramified. This proves Theorem 30 in general, showing that it can not only replace $C_c^\infty(K)$ in Tate's thesis for the case $d = 1$, but that for general d we can also use S^d to generate the d^{th}-power L-functions. In this way, S^d is seen to play the role of the conjectural $\mathcal{S}_\rho$ alluded to in [2, Section 1.4], where ρ is the d^{th} power map. $\square$

2.5 Functional Equation

Our goal here is to prove a functional equation in line with [11] and [5] for the space S^d_ϖ, of the form

$$Z(f, \chi, s)Z(\mathcal{F}g, \chi^{-1}, 1 - s) = Z(\mathcal{F}f, \chi^{-1}, 1 - s)Z(g, \chi, s),$$

for all unitary characters χ and all complex numbers s within some given vertical strip of $\mathbb{C}$. (Note here that we use χ^{-1} and $1 - s$ to express the information of $\check{\chi}$ in the classical notation of [11].)

In what follows, we explore the behavior of $\mathcal{F}$ on S^d_ϖ. With the insight which Lemma 28 gave us into the structure of S^d_ϖ, it suffices to consider functions in $C_c^\infty(K^\times)$

and functions of the form $\varphi_\Omega^{d,\pm}$. We will heavily use Lemma 19, which provided a rule for commuting the Fourier transform with the multiplicative translation operator, to investigate the behavior of $\mathcal{F}$ on functions of these types.

We begin by computing the Fourier transform of characteristic functions of compact-open subsets of K.

Lemma 31. *Let $\Omega = a + \varpi^k\mathcal{O}$, for some $a \in K$ and some integer k, be a compact-open subset of K. Then we have $\mathcal{F}[\mathrm{Char}(\Omega)] = (\overline{\psi_a})q^{-k}M_{\varpi^{-k}}[\mathrm{Char}(\mathcal{O})]$.*

Proof. Noting that $\mathrm{Char}(a + \varpi^k\mathcal{O}) = T_a \circ M_{\varpi^k}[\mathrm{Char}(\mathcal{O})]$, and that $\mathcal{F}[\mathrm{Char}(\mathcal{O})] = \mathrm{Char}(\mathcal{O})$, then applying Lemmas 18 and 19, we compute:

$$\begin{aligned}\mathcal{F}[\mathrm{Char}(\Omega)] &= \mathcal{F}[\mathrm{Char}(a + \varpi^k\mathcal{O})],\\ &= (\mathcal{F} \circ T_a \circ M_{\varpi^k})\,[\mathrm{Char}(\mathcal{O})],\\ &= (\overline{\psi_a})q^{-k}M_{\varpi^{-k}}[\mathrm{Char}(\mathcal{O})],\end{aligned}$$

as claimed. □

For what comes later it will be useful to define the following:

Definition 32. *We say $\omega \subset K^\times$ is a* small open subset *if it is compact, open, and has constant absolute value (i.e. $|x| = |y|$ for all $x, y \in \omega$).*

Continuing with our exploration of the behavior of $\mathcal{F}$ on S_ϖ^d using the decomposition in Lemma 28, we claim the following:

Lemma 33. *If Ω is a compact open subset of $K^\times$, then*

$$\mathcal{F}\varphi_\Omega^{d,+} \in \mathit{Span}\left(\mathrm{Char}(\mathcal{O}), \{|\cdot|^{-1}\varphi_\omega^{d,-} : \omega \in K^\times \textit{ small open }\}\right).$$

Proof. For $\varphi_\Omega^{d,+}$, noting that $x \in \varpi\Omega \Leftrightarrow \varpi^{-1}x \in \Omega$, we have

$$\begin{aligned}
\mathcal{F}\varphi_\Omega^{d,+} &= \mathcal{F}\left(\sum_{k\geq 0} \mathrm{Char}(\varpi^{dk}\Omega)\right), \\
&= \mathcal{F}\left(\sum_{k\geq 0} M_{\varpi^{dk}}[\mathrm{Char}(\Omega)]\right), \\
&= \left[\mathcal{F}\circ\left(\sum_{k\geq 0} M_{\varpi^{dk}}\right)\right][\mathrm{Char}(\Omega)], \\
&= \left[\sum_{k\geq 0} (\mathcal{F}\circ M_{\varpi^{dk}})\right][\mathrm{Char}(\Omega)], \\
&= \left[\sum_{k\geq 0} \left(|\varpi^{dk}| M_{\varpi^{-dk}}\right)\circ\mathcal{F}\right][\mathrm{Char}(\Omega)], \\
&= \left[\sum_{k\geq 0} q^{-dk} M_{\varpi^{-dk}}\right]\left[\mathcal{F}\left(\mathrm{Char}(\Omega)\right)\right],
\end{aligned}$$

and we note that $\mathrm{Char}(\Omega) \in C_c^\infty(K^\times)$ for $\Omega \subset K^\times$ compact-open. So

$$\mathcal{F}\,\mathrm{Char}(\Omega) = [\mathcal{F}\,\mathrm{Char}(\Omega)](0)\cdot\mathrm{Char}(\mathcal{O}) + h,$$

for some $h \in C_c^\infty(K^\times)$. Let $a = [\mathcal{F}\,\mathrm{Char}(\Omega)](0)$. Substituting this into the result above, we are left to consider:

$$\left[\sum_{k\geq 0} q^{-dk} M_{\varpi^{-dk}}\right]\left[\mathcal{F}\left(\mathrm{Char}(\Omega)\right)\right] = \left[\sum_{k\geq 0} q^{-dk} M_{\varpi^{-dk}}\right]\left(a\cdot \mathrm{Char}(\mathcal{O}) + h\right),$$

$$= \left[\sum_{k\geq 0} q^{-dk} M_{\varpi^{-dk}}\right] a\cdot \mathrm{Char}(\mathcal{O}) + \left[\sum_{k\geq 0} q^{-dk} M_{\varpi^{-dk}}\right] h,$$

and we can consider these two terms separately.

We note trivially that

$$C_c^\infty(K^\times) = Span\{\mathrm{Char}(\omega) : \omega \text{ is a small open subset of } K^\times\}.$$

Using this characterization we see that $\mathcal{F}(\varphi_\Omega^{d,+})$ is a linear combination of functions taking the form $\left[\sum_{k\geq 0} q^{-dk} M_{\varpi^{-dk}}\right] \mathrm{Char}(\mathcal{O})$ and $\left[\sum_{k\geq 0} q^{-dk} M_{\varpi^{-dk}}\right] \mathrm{Char}(\omega)$. We further compute, by considering the restriction of this function to $\mathcal{O}$ (i.e. $\mathrm{val}(x)\geq 0$) and to $K\setminus\mathcal{O}$ (i.e. $\mathrm{val}(x)<0$):

$$\left[\sum_{k\geq 0} q^{-dk} M_{\varpi^{-dk}}\right] \mathrm{Char}(\mathcal{O}) = \sum_{k\geq 0} q^{-dk}\, \mathrm{Char}(\varpi^{-dk}\mathcal{O}),$$

$$= (1-q^{-d})^{-1}\, \mathrm{Char}(\mathcal{O}) + \sum_{k\geq 1} q^{-dk}\, \mathrm{Char}(\varpi^{-dk}\mathcal{O}^\times),$$

where for integers $n<0$ and $i=0,\ldots,d-1$, for $val(x)=i+nd$ the second term has the value $q^{nd}(1-q^{-d})^{-1}$, and in particular takes the value $|x|^{-1}(1-q^{-d})^{-1}$ when $i=0$. More importantly, if we consider $\omega_i = \varpi^{-i}\mathcal{O}^\times$ then we may write the second term as $\sum_{i=0}^{d-1} q^{d-i}|\cdot|^{-1}\varphi_{\omega_i}^{d,-}$. Returning to functions of the form $\left[\sum_{k\geq 0} q^{-dk} M_{\varpi^{-dk}}\right] \mathrm{Char}(\omega)$

for ω a small open subset of $K^\times$, suppose $|x| = q^m$ for all $x \in \omega$. Note that if $x \in \omega$ then $x \notin \varpi^{-dk}\omega$ for $k \neq 0$ because $val(x) = -m$ is constant. Let $y \in K^\times$ be such that $\mathrm{val}(y) = -dl - m$, with $\left[\sum_{k\geq 0} q^{-dk} M_{\varpi^{-dk}}\right] \mathrm{Char}(\omega)(y) \neq 0$, so $y \in \varpi^{-dl}\omega$. Note $|y| = q^{dl+m}$ and $|y|^{-1} = q^{-dl-m}$. We evaluate

$$\begin{aligned}\left[\sum_{k\geq 0} q^{-dk} M_{\varpi^{-dk}}\right] \mathrm{Char}(\omega)(y) &= \left[\sum_{k\leq 0} q^{dk} M_{\varpi^{dk}}\right] \mathrm{Char}(\omega)(y), \\ &= q^{-dl}\varphi_\omega^{d,-}(y), \\ &= q^m |y|^{-1} \varphi_\omega^{d,-}(y),\end{aligned}$$

showing that $\left[\sum_{k\geq 0} q^{-dk} M_{\varpi^{-dk}}\right] \mathrm{Char}(\omega)$ is simply $c|\cdot|\varphi_\omega^{d,-}$ where $c = |x|$ for all $x \in \omega$, proving the claim. $\square$

When we look at $\mathcal{F}\varphi_\Omega^{d,-}$, however, we run into trouble. While we might suspect, based on Lemma 33, that $\mathcal{F}\varphi_\Omega^{d,-} \in Span\left(\mathrm{Char}(\mathcal{O}), \{|\cdot|^{-1}\varphi_\omega^{d,+} : \omega \in K^\times \text{ small open }\}\right)$, it turns out we fail to have convergence:

$$\begin{aligned}
\mathcal{F}\varphi_{\Omega}^{d,-} &= \mathcal{F}\left(\sum_{k\leq 0}\operatorname{Char}(\varpi^{dk}\Omega)\right), \\
&= \mathcal{F}\left(\sum_{k\leq 0} M_{\varpi^{dk}}[\operatorname{Char}(\Omega)]\right), \\
&= \left[\mathcal{F}\circ\left(\sum_{k\leq 0} M_{\varpi^{dk}}\right)\right][\operatorname{Char}(\Omega)], \\
&= \left[\sum_{k\leq 0}(\mathcal{F}\circ M_{\varpi^{dk}})\right][\operatorname{Char}(\Omega)], \\
&= \left[\sum_{k\leq 0}\left(|\varpi^{dk}|M_{\varpi^{-dk}}\right)\circ\mathcal{F}\right][\operatorname{Char}(\Omega)], \\
&= \left[\sum_{k\leq 0} q^{-dk}M_{\varpi^{-dk}}\right]\left[\mathcal{F}\left(\operatorname{Char}(\Omega)\right)\right].
\end{aligned}$$

If we break this down, similar to the process for $\mathcal{F}\varphi_{\Omega}^{d,+}$ to consider functions of the form $\sum_{k\geq 0} q^{dk}\operatorname{Char}(\varpi^{dk}\mathcal{O})$ and $\sum_{k\geq 0} q^{dk}\operatorname{Char}(\varpi^{dk}\mathcal{O})$, we find that the second term is simply $c| \cdot |\varphi_{\omega}^{d,+}$ where ω is a small open such that $|x| = c$ for all $x \in \omega$, and the first term doesn't converge. We remedy this by making the following restriction:

Definition 34. *Let $S^{d,+} = \{f \in S^d : f$ vanishes at infinity$\}$.*

We immediately see the following:

Lemma 35. *The space $S^{d,+}$ is spanned by $C_c^{\infty}(K^{\times})$ and $\{\varphi_{\omega}^{d,+} : \omega \subset K^{\times}$ small open$\}$.*

We further claim:

Proposition 36. *$S^{d,+}$ is a viable space.*

Proof. We immediately see that

$$C_c^\infty(K^\times) \subset S^{d,+} \subset S^d \subset C^\infty(K^\times).$$

We note that $\varphi_\Omega^{d,+}(t^{-1}x) = \varphi_{t\Omega}^{d,+}(x)$, showing the second condition. Loose convergence of zeta integrals in $S^{d,+}$ follows from loose convergence in S^d by inclusion. □

In Theorem 30 we showed that $J(S^d, \chi) = L(\chi^d, ds)\mathbb{C}[q^{\pm s}]$. We claim the same is true of $S^{d,+}$:

Theorem 37.

$$J(S^{d,+}, \chi) = J(S^d, \chi).$$

Proof. It is clear that $J(S^{d,+}, \chi) \subset J(S^d, \chi)$. To see the other inclusion, we recall the calculation from the proof of Theorem 30, showing that the L-function is actually generated by the zeta integral for $\varphi_\Omega^{d,+}$ for a well-chosen Ω. Because $\varphi_\Omega^{d,+}$ came from the $S^{d,+}$ portion of S^d in the decomposition found in Lemma 28, the result follows. □

Thus we see we can also take $S^{d,+}$ to be the conjectural $\mathcal{S}_\rho$ alluded to in [2, Section 1.4], and in fact this is the defintion which will allow us to prove the local functional equation for the d^{th} power L-functions.

The proof given in [11] for the local functional equation relies only on the absolute convergence of $Z(f, \chi, s)$ and $Z(\mathcal{F}g, \chi^{-1}, 1-s)$ in an appropriate vertical strip, allowing Tate to perform a substitution which yields a visible symmetry. Toward that end, we prove the following:

Proposition 38. *For any $f, g \in S^{d,+}$ the zeta integrals $Z(f, \chi, s)$ and $Z(\mathcal{F}g, \chi^{-1}, 1-s)$ converge absolutely to elements of $\mathbb{C}(q^s)$ in the vertical strip $0 < \Re(s) < 1$.*

Proof. We use Lemma 35 to break the proof into cases. The case where $f, g \in C_c^\infty(K^\times)$ is covered in Lemma 4. Within the proof of Proposition 29 we showed that $Z(\varphi_\Omega^{d,+}, \chi, s)$ converges absolutely for $\Re(s) > 0$. Using the result of Lemma 33 to consider the convergence of $Z(\mathcal{F}\varphi_\Omega^{d,+}, \chi^{-1}, 1-s)$, we consider the integrals $Z(\mathrm{Char}(\mathcal{O}), \chi^{-1}, 1-s)$ and $Z(|\cdot|^{-1}\varphi_\omega^{d,-}, \chi^{-1}, 1-s)$. Note that the former converges for $\Re(s) < 1$ by results in [11]. For the latter, we observe:

$$\begin{aligned} Z(|\cdot|^{-1}\varphi_\omega^{d,-}, \chi^{-1}, 1-s) &= \int_{K^\times} |x|^{-1}\varphi_\omega^{d,-}(x)\chi(x)^{-1}|x|^{1-s}\, d^\times x, \\ &= \int_{K^\times} \varphi_\omega^{d,-}(x)\chi(x)^{-1}|x|^{-s}\, d^\times x, \end{aligned}$$

which, after noting the $-s$ and applying the computations in the proof of Proposition 29, shows absolute convergence for $\Re(s) > 0$. The result follows. □

It is important to note that although there are functions $f \in S^{d,+}$ such that $\mathcal{F}f \notin S^{d,+}$, the zeta integrals associated to these $\mathcal{F}f$ still converge absolutely to elements of $\mathbb{C}(q^s)$ and are useful. Having shown convergence for the appropriate zeta integrals in the vertical strip $0 < \Re(s) < 1$, we have the following:

Theorem 39. *For $f, g \in S^{d,+}$, we have the local functional equation*

$$Z(f, \chi, s)Z(\mathcal{F}g, \chi^{-1}, 1-s) = Z(\mathcal{F}f, \chi^{-1}, 1-s)Z(g, \chi, s),$$

in the vertical strip $0 < \Re(s) < 1$.

Proof. Here we follow [11] and observe:

$$\begin{aligned}
Z(f,\chi,s)Z(\mathcal{F}g,\chi^{-1},1-s) &= \int_{K^\times} f(x)\chi(x)|x|^s\,d^\times x \cdot \int_{K^\times} [\mathcal{F}g](y)\chi^{-1}(y)|y|^{1-s}\,d^\times y, \\
&= \iint_{K^\times \times K^\times} f(x)[\mathcal{F}g](y)\chi(xy^{-1})|x|^s|y|^{1-s}\,d^\times x\,d^\times y,
\end{aligned}$$

where the double integral is absolutely convergent, representing the product of two absolutely convergent integrals. We use the substitution $(x,y) \to (x,xy)$, which is invariant under $d^\times x\,d^\times y$ to continue:

$$\begin{aligned}
&= \iint_{K^\times \times K^\times} f(x)[\mathcal{F}g](xy)\chi\left(x(x^{-1}y^{-1})\right)|x|^s|xy|^{1-s}\,d^\times x\,d^\times y, \\
&= \iint_{K^\times \times K^\times} f(x)[\mathcal{F}g](xy)\chi(y^{-1})|x||y|^{1-s}\,d^\times x\,d^\times y.
\end{aligned}$$

By Fubini, this can be written as an iterated integral:

$$= \int_{K^\times} \left(\int_{K^\times} f(x)[\mathcal{F}g](xy)|x|\,d^\times x\right)\chi(y^{-1})|y|^{1-s}\,d^\times y.$$

Since only the inner integral contains expressions involving f or g it suffices to show that this inner integral is symmetric in f and g to prove the theorem. We express the inner integral as an additive integral:

$$\int_{K^\times} f(x)[\mathcal{F}g](xy)|x|\,d^\times x = c\int_K f(x)[\mathcal{F}g](xy)\,dx,$$

for some value of c. We then expand the Fourier transform, and use Fubini to express the result as an iterated integral:

$$\int_K f(x) \left(\int_K g(z)\overline{\psi(xyz)}\, dz \right) dx = \iint_{K\times K} f(x)g(z)\overline{\psi(xyz)}\, dx\, dz,$$

which is clearly symmetric in f and g, as required. $\square$

We further claim:

Theorem 40. *For every unitary character $\chi : K^\times \to \mathbb{C}^\times$ and every $f \in S^{d,+}$, there exists a function $\gamma(\chi, x) \in \mathbb{C}(q^s)$ such that*

$$Z(\mathcal{F}f, \chi^{-1}, 1-s) = \gamma(\chi, s)Z(f, \chi, s).$$

Proof. By Theorem 39 we see that the quotient

$$\frac{Z(\mathcal{F}f, \chi^{-1}, 1-s)}{Z(f, \chi, s)},$$

is independent of the choice of $f \in S^{d,+}$, so $\gamma(\chi, s)$ is at least well-defined. To see that it is in $\mathbb{C}(q^s)$, we note that the proof of [11, Theorem 2.4.1] includes, for the p-adic case, an explicit f for which the denominator is nonzero. We note that this $f \in C_c^\infty(K)$ (using the modern reference to Schwartz spaces as described in [10, Chapter 7]) and $C_c^\infty(K) \subset S^{d,+}$. Thus we may infer that $\gamma(\chi, s)$ is in fact a rational function of q^s for all $f \in S^{d,+}$, as claimed, being a quotient of two elements of $\mathbb{C}[q^{\pm s}]$. $\square$

Here we use γ and χ, where Tate refers to ρ and c. The γ notation is more in line with modern use, such as that in [2].

Definition 41. *We define the local ϵ-factor by*

$$\epsilon(\chi, d, s) = \gamma(\chi, s) \cdot \frac{L(\chi^d, ds)}{L(\chi^{-d}, 1 - ds)},$$

where $\gamma(\chi, s)$ is as described in Theorem 40.

We see that ϵ has the following property:

Proposition 42. *For every unitary character χ, $\epsilon(\chi, d, s)$ is a unit in $\mathbb{C}[\pm s]$.*

Proof. Combining Definition 41 with Theorem 40, we obtain

$$\epsilon(\chi, d, s) = \frac{Z(\mathcal{F}f, \chi^{-1}, 1 - s)}{Z(f, \chi, s)} \cdot \frac{L(\chi^d, ds)}{L(\chi^{-d}, 1 - ds)}.$$

We see immediately that

$$\frac{Z(\mathcal{F}f, \chi^{-1}, 1 - s)}{L(\chi^{-d}, 1 - ds)} = \epsilon(\chi, d, s) \cdot \frac{Z(f, \chi, s)}{L(\chi^d, ds)}.$$

We recall Theorem 30, that $J(S^d, \chi) = L(\chi^d, ds)\mathbb{C}[q^{\pm s}]$. (Note we reference Theorem 30, rather than Theorem 37, because $\mathcal{F}f$ may not be in $S^{d,+}$, but we recall the ideal generated by the zeta integrals over the two spaces is the same.) This implies that each of the quotients above lie in $\mathbb{C}[q^{\pm s}]$. In fact, we have shown the existence of f such that $Z(f, \chi, s) = L(\chi^d, ds)$, i.e. the quotient

$$\frac{Z(f, \chi, s)}{L(\chi^d, ds)} = 1,$$

for this f. Thus we see that $\epsilon(\chi, d, s) \in \mathbb{C}[\pm s]$. To see that it is a unit, note that we may apply Fourier inversion (noting that ϵ is independent of f, allowing us to choose an f for which $\mathcal{F}f$ lies in $S^{d,+}$) to obtain

$$\epsilon(\chi^{-1}, d, 1 - s) \cdot \frac{Z(\mathcal{F}f, \chi^{-1}, 1 - s)}{L(\chi^{-d}, 1 - ds)} = \frac{Z(f, \chi, s)}{L(\chi^d, ds)}.$$

We observe from this equation that $\epsilon(\chi, d, s) \cdot \epsilon(\chi^{-1}, d, 1-s) = 1$, showing the epsilon factor is in fact a unit in $\mathbb{C}[\pm s]$. $\square$

Thus we see that the space $S^{d,+}$ allows us to prove the known functional equation for the L-function associated to χ^d, using the methods of [5] and [11].

Chapter 3

Conclusions and Moving Forward

Braverman and Kazhdan suggest that new Schwartz spaces can be defined, to produce Langlands L-functions from zeta integrals using the method of Godement and Jacquet. In this work, we have defined such a space, for the case $G = GL_1(K)$ with ρ the n^{th} power map.

We began by exploring identities involving zeta integrals $Z(f, \chi, s)$ for unitary characters χ and functions $f \in C^\infty(K^\times)$, showing first that we had a loose convergence property for zeta integrals in $C^\infty(K^\times)$ which allowed us to explore these identities. We then defined a difference operator ∂_ϖ^d on $C^\infty(K^\times)$ and observed the relationship between $Z(f, \chi, s)$ and $Z(\partial_\varpi^d f, \chi, s)$. We found that the space S_ϖ^d, defined by the preimage of $C_c^\infty(K^\times)$ in $C^\infty(K^\times)$ under ∂_ϖ^d, was a viable space which could be used to generate the local L-factor as a zeta integral. Using the intersection of all S_ϖ^d for all choices of uniformizer ϖ, and thus defining the space S^d, allowed us to consider ramified χ as well as unramified. We proved that the fractional ideal in $\mathbb{C}[q^{\pm s}]$ of outputs of zeta integrals for S^d, denoted $J(S^d, \chi)$, was generated by $L(\chi^d, ds)$. For $d = 1$ this corresponds to $J(C_c^\infty(K), \chi)$. Thus the space S^d generalized $C_c^\infty(K)$ to produce the L-functions for $d > 1$, using the framework of Godement-Jacquet.

Studying the structure of S^d, we discovered that each S^d_ϖ was spanned by elements of $C_c^\infty(K^\times)$ and functions of the type $\varphi_\Omega^{d,\pm}$. This allowed us to investigate the behavior of the Fourier transform $\mathcal{F}$ on the space, by examining its behavior on typical elements of each S^d_ϖ. Lack of convergence for the Fourier transform of some functions in S^d_ϖ (such as $\varphi_\Omega^{d,-}$) led us to define a restricted space, $S^{d,+}$, using the additional constraint that functions vanish at infinity. We proved that $J(S^{d,+}, \chi) = J(S^d, \chi)$, showing the restricted space can be used without affecting its usefulness in generating L-functions.

Following the ideas of Tate, we used the space $S^{d,+}$ to prove the known local functional equation for the L-functions associated to powers of Hecke characters. This is a direct generalization of the methods used by Godement-Jacquet, and gives the known ϵ and γ factors in this case. Thus we showed that our space $S^{d,+}$ can be taken to be the $\mathcal{S}_\rho$ conjectured by Braverman-Kazhdan.

We have shown that Braverman-Kazhdan's vision of generalizing the Godement-Jacquet construction to produce additional L-functions is attainable, for the case $G = GL_1(K)$ with ρ the n^{th} power character. Having shown this, we ask whether the same is true for other G and ρ. The construction appears easily generalizable to split tori, using Fubini to consider zeta integrals over $K^\times \times \cdots \times K^\times$. Non-split tori might require additional work, but the relationship between an arbitrary reductive group and its maximal tori could provide for fruitful investigation. One might ask whether defining other operators on $C^{\infty}(K^{\times})$, similar to ∂^d_{ϖ}, and making use of the identities given in this work, could produce additional useful identities among zeta integrals in the associated spaces. Of particular interest is the case in which ρ is a symmetric power representation, for which no general integral construction of the L-function currently exists.

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