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EMBEDDINGS AND COMPRESSIONS OF POLYHEDRA AND SMOOTH MANIFOLDS†

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§1. INTRODUCTION

IT IS OFTEN desirable to compress a subset X of a manifold V into a submanifold V' by an isotopy of V. For example, V might be a Euclidean space R^n and V' might be R^k, with k much smaller than n. If X has some special properties ensuring that such a compression is always possible, then we conclude that X embeds in R^k. In this article the problem of compressing X into the boundary of an s-cell is studied, where $s = \dim V$. As applications several theorems are proved about embedding polyhedra and smooth manifolds in Euclidean space.

The main theorems say that a compact polyhedron or smooth submanifold $X \subset V$ compresses into the boundary of an s-cell provided (1) there exists a "Dehn cone" on X, and (2) $V-X$ is highly connected. A *Dehn cone* on $X \subset V$ is an embedding $X \times I \subset V$ with $X \times 0 = X$ together with a null homotopy of $X \times 1$ in $V - X \times 0$. If $V-X$ is sufficiently connected, a regular neighborhood theorem of Hudson and Zeeman, together with an engulfing theorem due to Zeeman and the author, provides an s-cell $E \subset V$ with $X \subset \partial E$. If $V = R^s$, this implies the compressibility of X into R^{s-1} or S^{s-1}.

First the piecewise linear ($=PL$) theory is developed, then the smooth case is reduced to the PL case.

In the applications we take $X \subset R^{q+1}$ and try to compress X into R^q. There are three steps:

(1) extend X to $X \times I \subset R^{q+1}$;
(2) choose the extension so that $X \times 1 \simeq 0$ in $R^{q+1} - X$;
(3) prove $R^{q+1} - X$ is sufficiently connected for the Engulfing Theorem to apply.

Step (1) is difficult in the PL case, so in the applications it is assumed as part of the hypothesis. In the smooth case, however, (1) is equivalent to the existence of a normal vector field on the smooth submanifold X.

Step (2) is accomplished through algebraic topology, sometimes by luck—as when X is a smooth homology sphere—more usually by just assuming that the obstructions to a

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null homotopy vanish. These lie in groups $H^i(X, \pi_i(R^{q+1} - X))$. Since $H_i(R^{q+1} - X) = H^{q-i}(X)$, the hypothesis takes the form: $H^j(X) = 0$ for $j \geqq k$.

Step (3) is handled similarly. The relevant property of X is its "collapsibility dimension"; call X *d-collapsible* if X collapses to Y of dimension d. The engulfing theorem requires $\pi_i(R^{q+1} - X) = 0$ for $i \leqq 2d - q + 2$, or equivalently, $H^j(X) = 0$ for $j \geqq 2q - 2d - 2$. It also requires $q \geqq d + 3$. Observe that a connected bounded polyhedral m-manifold is necessarily $m - 1$ collapsible.

Some examples of embedding theorems proved by these methods:

THEOREM A. *Let P be a d-collapsible compact polyhedron such that $P \times I^r$ embeds in R^{q+r}. Suppose $q \geqq d + 3$, and $H^i(P) = 0$ for*

$$i \geqq \min\left\{\left[\frac{q+1}{2}\right], 2q - 2d - 2\right\}$$

Then P embeds in R^q. In fact if $f: P \times I^r \to R^{q+r}$ is an embedding, then $f(P \times 0)$ compresses into R^q.

Proof. Induct on r, writing $P \times I^r = (P \times I^{r-1}) \times I$, and use Theorem 3 below.

THEOREM B. *Let M be a compact PL m-manifold with trivial cohomology. Then M embeds in R^{m+2}.*

Proof. According to Milnor [14], M embeds in R^{m+k} with a normal microbundle ν, for some k. Since M has trivial cohomology, the classifying map $M \to BPL(k)$ of ν is null homotopic; hence ν is trivial. In other words $M \times I^k$ embeds in R^{m+k}; now apply Theorem A.

ADDENDUM. *The hypothesis of Theorem B can be weakened to the following:*

(i) $H^i(M) = 0$ *for* $i \geqq \min\left\{\left[\frac{m+3}{2}\right], 4\right\}$,

(ii) $W_1(M) = W_2(M) = 0$.

To see this, observe that $\pi_i(BPL) = \pi_i(BO)$ for $i \leqq 3$ (in fact for $i \leqq 7$), and that (ii) therefore ensures that ν is trivial over the 3-skeleton of M, while (i) makes ν trivial over the rest of M and allows Theorem A to be applied.

COROLLARY C. *A PL m-manifold M with the cohomology of S^m embeds in R^{m+3}.*

Proof. By Theorem B, the complement M_0 of the interior of an m-cell of M embeds in R^{m+2}. Adjoin the cone on ∂M_0 from a point of $R^{m+3} - R^{m+2}$.

The smooth analogue of Theorem B (but not of Corollary C) is true:

THEOREM D. *A smooth compact m-manifold M with trivial homology embeds in R^{m+2}. In fact any embedding in R^n compresses into R^{m+2} if $n > m + 2$.*

Proof. Apply Theorem 6 with $t = 0$, and $q = m + 2$.

THEOREM E. *Let M be the boundary of a smooth compact parallelizable manifold. Suppose $H_i(M) = 0$ for $0 < i \leqq k$. Then M embeds in R^q provided*

$$q \geqq \max\{2m - 2k - 1, \tfrac{1}{2}(3m - k + 2), m + 3\}.$$

Proof. This is a special case Theorem 12.

A new proof is given for Haefliger's theorem that a smooth homology m-sphere embeds in R^q where $q \geqq \frac{3}{4}(m + 1)$. Other applications, on which Theorems A through E are based, will be found in §3. The two main theorems are stated and proved in §2.

§2. THE MAIN THEOREMS

We start with some terminology which will apply to both the *piecewise linear category* of polyhedra and *PL* maps and to the *smooth category* of smooth manifolds and smooth maps. Observe that Euclidean n-space R^n has natural smooth and *PL* structures; thus R^n denotes an object in both categories. Similarly for a closed half-n-space E^n and the closed interval $I = [0, 1] \subset R^1$.

A *manifold* means a paracompact object locally isomorphic to E^n. An object X is a *subobject* of an object Y, written $X \subset Y$, if X is a subset of Y and there is a retraction (in the category!) of an open set of Y onto X.

A *homeomorphism* is an isomorphism in the category. An *embedding* of X in Y is an isomorphism of X onto a subobject of Y. An *isotopy* of Y is a homotopy $F : Y \times I \to Y$ in the category such that, putting $F(x, t) = F_t(x)$, we have $F_0 = 1_Y$ and the map $(x, t) \to (F_t(x), t)$ is an isomorphism $Y \times I \to Y \times I$. If $A \subset Y$ and $B \subset Y$, then A is *isotopic* to B if there is an isotopy F of Y with $F_1(A) = B$, and A *compresses into* B if there is an isotopy F of Y such that $F_1(A) \subset B$.

An n-cell in the *PL* category is a homeomorph of I^n; in the smooth category an n-cell means a homeomorph of the closed unit ball $D^n \subset R^n$. A *sphere* is the boundary of a cell.

A basic fact, true in both categories is that *two m-cells in the interior of an m-manifold are isotopic*; see for example [20, 18]. Of course a set meeting the boundary of a manifold M cannot be isotopic to a set in the interior.

If M is an m-manifold and $X \subset M$, then X is *compressible* if $X \subset \partial B$ where $B \subset M$ is an m-cell. If $M = R^m$ and X is not a sphere, then X is compressible if and only if X compresses into R^{m-1}. "Compressible" is a convenient generalization of "compressible into R^{m-1}"; the main theorems concern the compressibility of a subobject of a manifold.

A *Dehn cone* on $X \subset V$ is a null homotopy g of X in V with no double points for $t \leqq \frac{1}{2}$, and such that g embeds $X \times [0, \frac{1}{2}]$. That is $g : X \times I \to Y$ is a map in the category such that

(i) $g(x, 0) = x$
(ii) $g(X \times 1)$ is a point
(iii) $g|X \times [0, \frac{1}{2}]$ is an embedding
(iv) $g^{-1}g(X \times [0, \frac{1}{2}]) = X \times [0, \frac{1}{2}]$.

A polyhedron is *d-collapsible* if it collapses in the sense of Whitehead [20] to a d-dimensional polyhedron. A smooth manifold is d-collapsible if it can be smoothly triangulated by a d-collapsible polyhedron (which is necessarily a *PL* manifold).

We can now state the two main theorems.

THEOREM 1. *Let V be a PL-manifold of dimension $q + 1$, and $P \subset V$ a compact polyhedron contained either in ∂V or in $V - \partial V = \text{int } V$. Suppose:*

(a) *P is d-collapsible;*

(b) *$V - P$ is $2d - q + 2$ connected;*

(c) $q \geqq d + 3$;

(d) *there is a Dehn cone on P.*

Then P is compressible.

In the smooth category a Dehn cone $g : X \times I \to V$ can be identified with a *nonlinking normal vector field* ϕ on X. This is a normal vector field ϕ such that $\phi(X) \simeq 0$ in $V - \text{int } T$, where $T \subset V$ is a closed tubular neighborhood of X identified with the normal disk bundle of X, and ϕ is identified with an embedding $\phi : X \to \partial T$. If $X \subset \partial V$ then ϕ is required to point into V.

If $B \subset V$ is a smooth cell with $X \subset \partial B$, and ϕ points into B, then X is called *compressible along* ϕ. Observe that if $X \subset \partial V$ and ϕ is any nonlinking vector field, then if X is compressible, X is necessarily compressible along ϕ.

THEOREM 2. *Let V be a smooth $q + 1$ manifold and $X \subset V$ a smooth compact submanifold with normal bundle ν. Suppose either $X \subset \partial V$ or $X \subset \text{int } V$. Let ϕ be a nonlinking normal vector field on X, and let ν' be the orthogonal complement of ϕ in ν. Suppose:*

(a) *X is d-collapsible;*

(b) *$V - X$ is $2d - q + 2$ connected;*

(c) $q \geqq d + 3$.

Then X is compressible along ϕ. In fact there is a smooth $q + 1$ cell $B \subset V$ with $X \subset \partial B$ such that the normal bundle of X in ∂B is ν'.

The rest of this section is devoted to the proofs of Theorems 1 and 2.

Proof of Theorem 1

The main tool, a criterion for the existence of collapsible polyhedra, is the following theorem, which is joint work with E. C. Zeeman. A proof of a more general result will appear elsewhere.

ENGULFING THEOREM (with E. C. Zeeman). *Suppose $P \subset \partial V$. Assume:*

(a) *$P \simeq 0$ in V*

(b) *P is d-collapsible*

(c) *V is $2d - s + 2$ connected*

(d) $s \geqq d + 4$.

Then there exists a collapsible C such that $P \subset C \subset V$.

Recall that a subset $A \subset B$ is called *collared* if there is an embedding $f: A \times I \to B$ such that $f(x, 0) = x$ for all $x \in A$, and $f(A \times I)$ is a neighborhood of A in B.

LEMMA (Hudson–Zeeman [9]). *Let X be a manifold. Suppose $A \subset B \subset X$, with A collared in B. There exists a regular neighborhood N of B in X such that $A \subset B \cap \partial N$.*

Moreover, if $B \cap \partial X \subset A$, *then* N *can be chosen so that* $A = B \cap \partial N$.

Next, observe that (i) $P \subset V$ *is compressible if there is a collapsible* C *such that* $P \subset C \subset V$, *and* P *is collared in* C. The reason is that any regular neighborhood N of C is a cell [20]. By Hudson–Zeeman, N can be chosen so that $P \subset \partial N$.

To prove Theorem 1, let $f: P \times I \to V$ be a Dehn cone on P. Let $f(P \times t) = P_t$. If $P \subset \text{int } V$, choose f so that $f(P \times I) \subset \text{int } V$. If $P \subset \partial V$, choose f so that $f(P \times I) \cap \partial V = P$. Under these conditions it is easy to find a regular neighborhood N of $f(P \times [0, \frac{1}{2}])$ such that:

(ii) $P \cup P_{\frac{1}{2}} \subset \partial N$

(iii) $P_{\frac{1}{2}} \simeq 0$ in $V - \text{int } N$, and therefore $P_{\frac{1}{2}} \simeq 0$ in $W = V - (P \cup \text{int } N)$.

To achieve (ii), use Hudson–Zeeman. For (iii) use the conditions on f. These conditions also imply that the inclusion $W \to V - P$ is a homotopy equivalence. The Engulfing Theorem, with $s = q + 1$, gives a collapsible $C' \subset W$ with $P_{\frac{1}{2}} \subset C'$. Let

$$C = C' \cup f(P \times [0, \tfrac{1}{2}]).$$

Since $C' \cap f(P \times [0, \frac{1}{2}]) = P_{\frac{1}{2}}$, it follows that C collapses to C'. Therefore C is collapsible. Moreover, P is collared in C, making P compressible by (i).

Proof of Theorem 2

We reduce the case $X \subset \text{int } V$ to the case $X \subset \partial V$. If $X \subset \text{int } V$, let $f: X \times I \to \text{int } V$ be a (smooth) Dehn cone. Since X is isotopic to $f(X \times \frac{1}{2})$, it suffices to prove that $f(X \times \frac{1}{2})$ is compressible. We may choose f so that $f(X \times \frac{1}{2})$ lies on the boundary of a tubular neighborhood T of X in int V. Replacing X by $f(X \times \frac{1}{2})$, and V by $V - \text{int } T$, we may thus assume $X \subset \partial V$.

If $X \subset \partial V$, give V a smooth triangulation so that ∂V and X are subcomplexes [16]. Consider V as a *PL*-manifold and $X \subset \partial V$ as a polyhedron. By Theorem 1 there is a *PL* $q + 1$ cell $B' \subset V$ with $X \subset \partial B'$. We may assume that $B' \cap \partial V$ is a neighborhood of X in ∂V, for if this is not already the case, we replace B' by a suitable regular neighborhood of itself.

By [7] there is a smooth submanifold $B \subset V$ which can be smoothly triangulated by the *PL*-cell B, such that $B \cap \partial V$ is a neighborhood of X in $B' \cap \partial V$. (See especially the remark on p. 106 of [6].) By Munkres [15] B must be a smooth $q + 1$ cell. Clearly $X \subset \partial B$, and ϕ points into B. The proof is complete.

§3. APPLICATIONS OF THE MAIN THEOREMS

It is easy to see that if $X \subset R^{q+1}$ is d-collapsible, then $R^{q+1} - X$ is $q - d - 1$ connected. Indeed, if X collapses to a d-dimensional $Y \subset X$, then $R^{q+1} - X$ has the homotopy type of $R^{q+1} - Y$ (they are in fact homeomorphic), and $R^{q+1} - Y$ is $q - d - 1$ connected by a simple general position argument. In particular if $q \geqq d + 2$, then $R^{q+1} - X$ is 1-connected.

Assume further that $t \geqq 0$ is an integer such that the reduced integer cohomology of X satisfies $\tilde{H}^i(X) = 0$ for $i \geqq t$. Then if $R^{q+1} - X$ is 1-connected, Alexander duality and Hurewicz isomorphism show that $R^{q+1} - X$ is $q - t$ connected. These remarks are used to prove the next theorem.

THEOREM 3. *Let $X \subset R^{q+1}$ be a compact d-collapsible polyhedron. Assume:*

(a) *there is an embedding $g : X \times I \to R^{q+1}$ with $g(x, 0) = x$;*

(b) *$\bar{H}^i(X) = 0$ for $i \geqq t$, where $t \geqq 0$.*

(c) $q \geqq \max\{2t - 1, \frac{1}{2}(t + 2d + 2), d + 3\}$

Then X is compressible.

Proof. Theorem 1 will apply if

(1) there is a Dehn cone on X,

(2) $R^{q+1} - X$ is $2d - q + 2$ connected, and

(3) $q \geqq d + 3$ (which is assumed in (c)).

To prove (1) it suffices to show that $g(X \times 1) \simeq 0$ in $R^{q+1} - X$. The obstructions to such a homotopy lie in $\bar{H}^i(X; \pi_i(R^{q+1} - X))$. The coefficient group vanishes for $i \leqq q - t$; the first inequality in (c) gives $q - t + 1 \geqq t$, so that the cohomology group vanishes if $i > q - t$. Hence X has a Dehn cone. To prove (2), observe that the second inequality of (c) is equivalent to $q - t \geqq 2d - q + 2$.

THEOREM 4. *Let M be a compact smooth manifold. Suppose:*

(a) *M is d-collapsible*

(b) *M embeds in R^{q+r} with a field of normal $(r + s)$-frames $(\phi_1, \dots, \phi_{r+s})$.*

(c) *$\bar{H}^i(X) = 0$ for $i \geqq t$, where $t \geqq 0$.*

(d) $q \geqq \max\{2t - 1, \frac{1}{2}(t + 2d + 2), d + 3\}$.

Then M embeds in R^q with a field of normal s-frames. In fact any embedding as in (b) compresses into R^q by a diffeotopy of R^{q+r} which carries $\phi_1, \dots, \phi_s$ into R^q.

Proof. It suffices to prove that ϕ_{r+s} is nonlinking and that the connectivity hypothesis (b) of Theorem 2 is satisfied, for then M compresses along ϕ_{r+s} into R^{q+r-1}, and iteration proves the theorem. The arguments are similar to those in the proof of Theorem 3, and are left to the reader.

COROLLARY 5. *Let M be a bounded compact smooth m-manifold. Assume:*

(a) *$\bar{H}^i(M) = 0$ for $i \geqq t$, where $0 \leqq t \leqq m$;*

(b) $q \geqq \max\{2t - 1, \frac{1}{2}t + m, m + 2\}$.

Then M embeds in R^q with a field of normal s-frames if M immerses in R^q with a field of normal s-frames.

Proof. If M immerses in R^q with a field of normal s-frames, then M embeds in R^{q+r} with a field of normal $(r + s)$-frames for r sufficiently large. Apply Theorem 4 with $d = m - 1$.

In the next theorem the requirements for M are purely homological.

THEOREM 6. *Let M be a smooth compact m-manifold. Assume:*

(a) *$\bar{H}^i(M) = 0$ for $i \geqq t$, where $t \geqq 0$.*

(b) $\bar{W}_{q-m+1}(M) = 0$

(c) $q \geqq \max\{2t - 1, m + t - 2, m + \frac{1}{2}t, m + 2\}$.

Then M embeds in R^q.

Proof. If some component of M has no boundary, then (a) implies $t > m$ and (c) implies $q \geqq 2m + 1$ and the theorem is well known. Assume that each component of M has a boundary, so that M is $m - 1$ collapsible.

Embed M in R^{q+r} with $q + r \geqq 2m + 1$. The first obstruction to a field of normal r-frames is $\bar{W}_{q-m+1}$, assumed to vanish. There are no higher obstructions since $\bar{H}^i(M) = 0$ for $i \geqq t$, and the second inequality of (c) implies $q - m + 2 \geqq t$. The rest of (c) implies (d) of Theorem 4, taking $d = m - 1$. Now Theorem 6 follows from Theorem 4 (with $s = 0$).

THEOREM 7. *Let M be a smooth compact bounded m-manifold. Suppose*:
(a) $H^*(\partial M) = H^*(S^{m-1})$
(b) *M is parallelizable*
(c) *there is an integer k such that $H_i(M) = 0$ for $0 < i \leqq k$*
(d) $q \geqq \max\{2m - 2k - 1, \frac{1}{2}(3m - k), m + 2\}$
Then M embeds in R^q with a trivial normal bundle.

Proof. By Poincaré duality $H^i(M) = 0$ for $i \geqq m - k$. Apply Theorem 4 with $t = m - k$, $d = m - 1$, $s = q - m$.

In order to obtain embedding theorems for closed manifolds we use the following theorem of Haefliger.

THEOREM 8. (Haefliger) *Let M be a smooth closed m-manifold, $D \subset M$ a smooth m-cell, and $M_0 = M - \text{int } D$. Assume $q \geqq \frac{1}{2}(3m + 1)$. If $f: M_0 \to R^q$ is an embedding, the composition $M_0 \to R^q \subset R^{q+1}$ extends to an embedding of M.*

Proof. See [3] and [5, Theorem 3.2a].

THEOREM 9. *Let M be a closed smooth m-manifold. Suppose:*
(a) *$H_i(M) = 0$ for $0 \leqq i \leqq k$.*
(b) *M_0 immerses in R^q with a field of normal s-frames.*
(c) $q \geqq \max\{2m - 2k - 1, m + 2, \frac{1}{2}(3m + 1)\}$.
Then M embeds in R^{q+1} with a field of normal $s + 1$-frames on M_0.

Proof. Apply Corollary 5 with $t = m - k$, to embed M_0 in R^q with normal s-frames; use Theorem 8 to get the desired embedding of M in R^{q+1}.

This last theorem illustrates the influence of the tangent bundle on the embedding dimension. For example, it is known that a 5-connected 24-manifold V embeds in R^{43} since every k-connected m-manifold embeds in R^{2m-k} [5] (if $2k < m$). But if V is almost parallelizable in addition, then Theorem 9 embeds V in R^{38}.

By using the higher dimensional Poincaré conjecture as proved by Smale [19] and J. Levine's surgical techniques, further results can be proved, similar to those of Minkus [12] and De Sapio [2]. For these we need the following theorem of Levine [10].

THEOREM 10 (Levine). *A smooth $m - 1$ sphere $S \subset R^q$ bounds a smooth m-cell $B \subset R^q$ provided*:
(a) *$q \geqq m + 2$ and $m \geqq 5$*;

(b) *S bounds a smooth compact m-manifold $M \subset R^q$;*
(c) *M has a field Φ of normal $q - m$ frames;*
(d) *the Kervaire invariant of (M, Φ) vanishes if $m \equiv 2 \bmod 4$, and the signature of M vanishes if $m \equiv 0 \bmod 4$.*

Recall that the proof consists of surgering M in R^q until it is contractible; condition (c) ensures there is no difficulty near the middle dimension if m is even.

COROLLARY 11. *Let M be the complement of the interior of an m-cell D in a smooth closed manifold V. If $M \subset R^q$ and satisfies (a) through (d) of Theorem 10, then there is a smooth homotopy m-sphere T such that the embedding $M \to R^q$ extends to an embedding $V \circledast T \to R^{q+1}$.*

Proof. Let $B \subset R^q$ be the m-cell promised in Levine's theorem. Push B diffeomorphically onto an m-cell $B' \subset R^{q+1}$ with $B' \cap R^q = \partial M$. Then $M \cup B' \subset R^{q+1}$ has a smoothable corner along ∂M. Smoothing the corner gives an embedding of the connected sum $M \circledast T$ where T is obtained by gluing the m-cells D and B together along their common boundary ∂M.

Next, another embedding theorem.

THEOREM 12. *Let V be a smooth, closed stably parallelizable m-manifold. Assume:*
(a) *$H_i(V) = 0$ for $0 < i \leqq k$;*
(b) *if $m \equiv 6 \bmod 8$, V has a stable framing with vanishing Kervaire invariant.*
(c) *$p \geqq \max\{2m - 2k - 1, \frac{1}{2}(3m - k + 2), m + 3\}$ and $m \geqq 5$.*
Then there exists a homotopy m-sphere T such that the connected sum $V \circledast T$ embeds in R^p. If in addition V bounds a compact parallelizable manifold, or if the group $\Gamma_m = 0$, or if every homotopy m-sphere embeds in R^p, then V embeds in R^p. In all cases the embedding of $V \circledast T$ can be chosen to have a trivial normal bundle on the complement of a point.

Proof. Let $D \subset V$ be a smooth m-cell, put $M = V - \operatorname{int} D$. By Theorem 7 we may assume $M \subset R^{p-1}$ with a field Φ of normal $p - m - 1$ frames.

Observe that the embedding and Φ can be chosen to represent any given stable framing of M. Now the signature of M vanishes because V is stably parallelizable; if $m \equiv 2 \bmod 8$ the Kervaire invariant of (M, Φ) vanishes by Brown and Peterson [1]. Therefore (d) of Theorem 10 is satisfied, and Corollary 11 gives the embedding of $V \circledast T$. If V bounds a parallelizable manifold, so does T since (with the proper choice of framing) V is framed cobordant to T. If T bounds a parallelizable manifold then T, and hence $-T$, embeds in R^p. Therefore so does $(V \circledast T) \circledast (-T) \approx V$. The rest of the Theorem is obvious.

THEOREM 13 (Haefliger [4]). *Let M be a smooth m-manifold such that $H_*(M) = H_*(S^m)$. If $q \geqq \frac{3}{2}(m + 1)$, then M embeds in R^q.*

Proof. It suffices to prove that any embedding $M \subset S^{q+1}$ compresses into S^q. We may assume $q \geqq m + 3$, since otherwise $m \leqq 1$, a trivial case.

The normal sphere bundle of M in S^{q+1} is known to be fibre homotopically trivial [11]; that therefore M has a normal vector field $\phi : M \to S^{q+1} - M$.

Let $[\phi]$ denote the homotopy class of ϕ in $\pi_m(S^{q+1} - M)$. If $S^{q-m} \subset S^{q+1} - M$ is the fibre of the normal sphere bundle of M, the inclusion induces an isomorphism $i_* : \pi_m(S^{q-m}) \to \pi_m(S^{q+1} - M)$. Given $\alpha \in \pi_m(S^{q-m})$, there is a unique homotopy class of normal vector fields ψ such that α is the difference obstruction to a homotopy of sections from ϕ to ψ. It is easy to see that $[\psi] = [\phi] - i_*\alpha$. Therefore if $\alpha = i_*^{-1}[\phi]$, it follows that $\psi \simeq 0$ in $S^{q+1} - M$. In other words ψ is nonlinking. The assumption $q \geqq \frac{3}{2}(m+1)$ is equivalent to $q - m - 1 \geqq 2m - q + 2$. Since $S^{q+1} - M$ is $q - m - 1$ connected, Theorem 2 applies to compress M.

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