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*Patterns of Drainage Areas with Random Topology**

After several years of rather successful investigation of the geometry and topology of channel networks by theoretical analysis and simulation models, one is tempted to pose the question of whether other morphological patterns of the physical landscape can be approached with a similar methodology. The problem with which this paper is concerned is a direct transfer of Shreve's investigation of channel networks [7] to the subject of drainage divide patterns. Thus, the problem can be formulated as follows: Assume that, in the absence of environmental control, all topologically different patterns formed by the drainage areas (drainage polygons) of the links of a channel network of given magnitude n are equally likely to occur. What will be the expected distribution of drainage polygons when grouped according to the number of their sides? Do empirical data support the hypothetical assumption above through close correspondence to the theoretical distribution deduced from this assumption?

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THE TOPOLOGY OF DRAINAGE PATTERNS: BASIC DEFINITIONS AND DESCRIPTIVE STATEMENTS

The drainage area of a channel network can be subdivided into a set of smaller areas, each of which is drained by one network link. But rather than defining the subdivision on the basis of the total drainage, we will consider drainage by surface runoff only. Thus, this definition is equivalent with an alternative definition based on the concept of channel links and the notion of steepest slope; namely, that all points of the area under consideration belong to the same subarea if and only if the lines of steepest decline passing through these points enter the same channel link. Obviously this latter definition is altogether independent from the notion of drainage and provides a strictly topographical subdivision.

The pattern of individual drainage areas (drainage polygons) as defined above produces an exhaustive and exclusive subdivision of the total area under consideration. The boundaries are formed by a pattern of interlinked drainage divides consisting of *divide links* and *divide nodes*. Each divide node joins at least three divide links, and, in general, no more than three drainage divide links merge in one divide node. Since, in any real case, the location of a drainage divide can never be identified with absolute (mathematical) precision, this observation will be adopted as an axiom, i.e., we will assume, that every node in a drainage divide pattern joins exactly three drainage divide links. It follows, then,

(a) the pattern of drainage divides subdiving the drainage area of a given channel network forms a cubic graph;

(b) if the magnitude[1] of the channel network is n, then the number of individual drainage areas is $2n - 1$, i.e., the number of channel links;

(c) For a channel network of magnitude n the number of drainage divide nodes is $4(n-1)$ and the number of drainage divide links is $6(n-1)$. This follows from Euler's equation $v - e + c = 1$ and the relationship $e = 3v/2$ in cubic graphs, where e refers to the number of divide links, v to the number of divide nodes, and c to the number of individual drainage areas or drainage polygons. Each of the $n - 1$ channel network nodes is, at the same time, a divide node, so that only ¾ of the divide nodes are located on slopes[2] or ridges.

(d) In calculating the average number of divide links per drainage polygon (topologically speaking, the average "*size*" of a polygon), we have to take into account that the inner divide links are always

[1] The magnitude of a channel network is defined as the number of first order streams (or sources) it possesses.

[2] It might be appropriate here to emphasize that ridges are only a subset of drainage divide lines. A slope which approximates an inclined plane can very well be drained by two or more channel links. Since each channel link would drain a certain subarea of the plane, the boundaries between those subareas are, by definition, drainage divide lines. Obviously, these drainage divides are not ridges, and the drainage divide nodes interconnecting them can also be located on the plane.

shared by two polygons, whereas each of the links of the outer divide enclosing the whole basin of the channel network delimits only one drainage polygon. Let $I(n)$ be the number of inner divide links and $O(n)$ the number of outer divide links as defined above, and let X be the average number of links per drainage polygon. Then it is

$$I(n) + O(n) = 6(n-1)$$

and

$$(2n-1) \cdot X = 2 \cdot I(n) + O(n)$$

Hence,

$$X = \frac{12(n-1) - O(n)}{2n-1} = 6 - \frac{6 + O(n)}{2n-1}$$

The theoretically possible minimum value of $O(n)$ is 1, and the maximum value is $3(n-1)$. The following table shows maximum and minimum values of X for the range $2 \leq n \leq \infty$.

AVERAGE NUMBER OF LINKS
X PER DRAINAGE POLYGON

	X_{min}	X_{max}
n=2	3	3⅔
n=∞	4½	6

It is important to note that X_{min} and X_{max} are monotone functions within the domain $2 \leq n \leq \infty$, so that the tabulated figures represent the extreme values for the whole domain. Also, these are the theoretically possible values; in reality, n will never approach infinity, and the drainage patterns will never come close to the extreme cases established above. Hence, the empirical range of X will be even smaller. In particular, a strictly hexagonal pattern will never be found in reality; on the other hand, real world patterns cannot deviate very much from this situation, at least insofar as the average size of polygons is concerned.

Let us now consider the possible topological arrangements of drainage polygons in a channel network basin of lower magnitude. Since empirical data have shown that first order streams tend to be twice as long as the interior links of the network,[3] we will assume that the area along the outer drainage divide delimiting the network basin is drained by all and only the outer links of the network (i.e., all first order streams plus the outlet of the network). Although this will introduce at least a small error term in the following calculations, it eases the mathematical manageability substantially. Combined, these assumptions provide a very convenient geometrical construct composed of the links of the channel network and the corresponding pattern of drainage polygons. An example is shown in Figure 1.

The following definitions and descriptive statements will prove useful

[3] See. for example, James and Krumbein [3, p. 545].

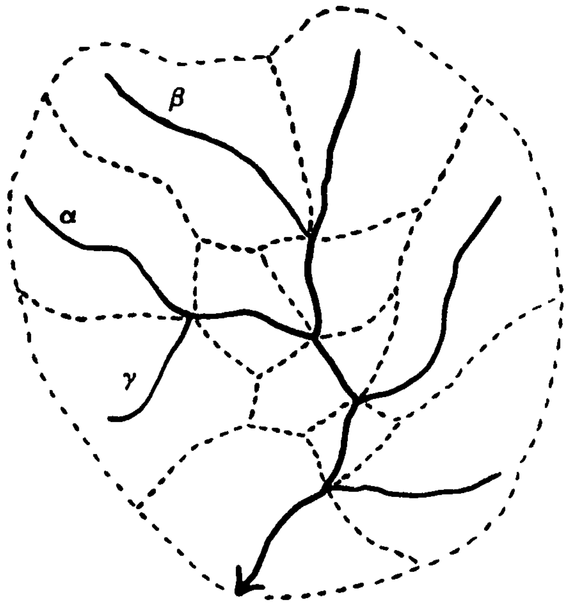


Figure 1

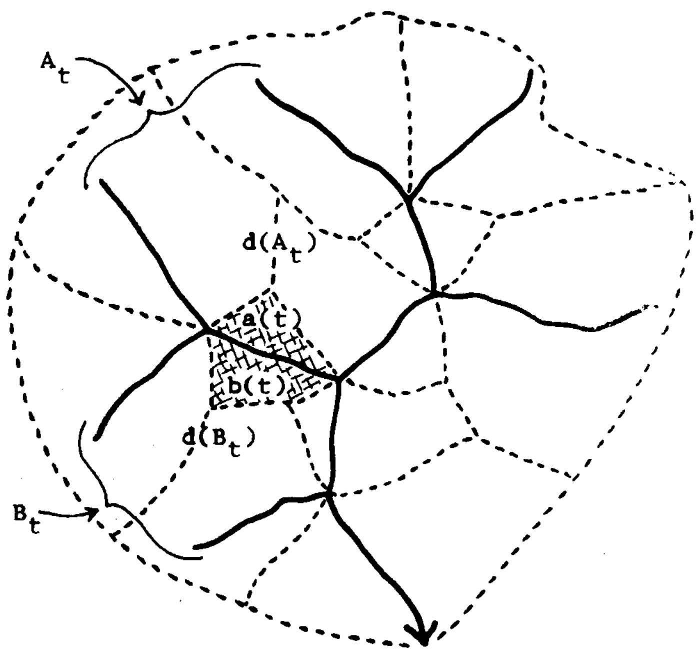


Figure 2

during the later course of investigation. We will call two outer network links ***neighboring*** if one can be reached from the other by a path consisting only of left (or right) turns. In other words: if a path of the network starts in an outer link α and continues on the left of the two links of the first node it reaches and further on on the left of the two links of the second node, and so on, then the path will finally reach an outer link β and β is called the ***left neighbor*** of α. Similarly, a right neighbor γ of α can be defined, and it is clear, that α is at the same time the right neighbor of β (see Fig. 1). Each outer link has exactly two neighbors, and if we order all outer links using this definition of neighborhood, then the first and the last outer link will be neighbors. A path connecting two neighbors will be called a ***bowl*** (e.g., A_t or B_t in Fig. 2), and the ***size*** of the bowl will be defined as the number of links it consists of. Two bowls are called ***adjacent*** when they have an outer link of the network in common, and they are called ***opposite*** to each other when they have an interior link in common (e.g., A_t and B_t in Fig. 2). In either case we will say that the two bowls are joined by the link. Clearly, every drainage network of magnitude n is composed of $n+1$ bowls, and the sequence of the numbers representing the sizes of the bowls, where the latter are ordered according to adjacency, describes completely the topology of the network under consideration. Since each network link belongs to two and only two different bowls, the sum of the bowl sizes is twice the number of network links, i.e., the average size of a bowl is

$$\frac{2(2n-1)}{n+1} = 4 - \frac{6}{n+1}. \tag{1}$$

Always three bowls meet in each channel network node, and each of them contains one of the three drainage divide lines radiating from that node. For any bowl of size t there are $t-1$ such divide lines separating the catchment areas (drainage polygons) of adjacent links from each other. Since we assumed that the area adjacent to the basin boundary is drained by all and only the outer links of the network there will be exactly one drainage divide line located between the two outer links of the bowl; it connects the pattern of drainage divide lines of the area "enclosed" by the bowl with the basin boundary (which itself is of course a drainage divide). The t drainage divide lines are interlinked by others so as to form a dendritic tree possessing t outer links. This is of course a most welcome observation because it will permit us to utilize the existing body of theory regarding dendritic stream channel networks. We will call the tree of drainage divides "inside" a given bowl, separating the various catchment areas of the bowl links, the ***drainage divide network*** corresponding to this bowl.

We can now describe the pattern of drainage divide lines of the total network basin as a system of interlinked divide networks, each of which is located in a channel network bowl and thereby separated from the others, the only connections being the channel network nodes, and, of course, the basin boundary, which forms the main divide. Going back for the moment to a single channel network bowl and the corresponding divide network it "encloses," it becomes apparent that the divide network in turn can be studied in terms of the bowls from which it is made. If the size of the channel network bowl is t, then the magnitude of the corresponding divide network is $t - 1$, and it consists of t divide bowls, each of which lies in juxtaposition to a link of the channel network bowl, such that adjacent divide bowls correspond to adjacent links of the channel network bowl and vice versa. In subject matter terms, each divide bowl delimits the surface run-off catchment area of one side of the corresponding channel network bowl link, which in turn drains all and only the area defined by the divide bowl under consideration.

To reiterate this situation, using Figure 2 as an illustration the following statement can be made:

(a) Let t be a link of a channel network;

(b) Let A_t and B_t be the two bowls of the channel network, which have t in common;

(c) Let $d(A_t)$ and $d(B_t)$ denote the two divide networks which correspond to the network bowls A_t and B_t in the predefined sense;

(d) Let $a(t)$ and $b(t)$ be the bowls of the divide networks $d(A_t)$ and $d(B_t)$, which correspond to the two sides of t, i.e., which delimit the drainage area of t on both sides of t;

(e) Then the drainage polygon corresponding to t is the closed polygon defined by the union of two open polygons, namely the bowls $a(t)$ and $b(t)$.

n SOURCES PRODUCING i NETWORKS[4]

Before we can proceed with the theoretical analysis of drainage area patterns, we need the following:

Theorem: The number $Z(i,n)$ of topologically different ways in which n first order streams can merge to form i networks is

$$Z(i,n) = \frac{\binom{2n-i}{n} i}{2n-i} (i \leq n) \tag{2}$$

The following observations about this relationship are noteworthy and will in part be used for subsequent analysis.

(a) If we ask for the number of topologically different ways in which n first order streams can form a single network we get

$$Z(1,n) = \frac{\binom{2n-1}{n}}{2n-1} = N(n) \qquad (3)$$

which shows that this result, previously reported by Shreve [7, p. 29] and Werner [9, p. 158], is only a special case of the above theorem.

(b) Obviously, $Z(n,n) = 1$ for all n.

(c) It is interesting to note, that

$$Z(1,n) = Z(2,n) \qquad (4)$$

i.e., the number of ways to form one network from n first order streams is equal to the number of ways to form two networks from them:

$$Z(2,n) = \frac{(2n-2)\cdot 2}{n!(n-2)!(2n-2)} = \frac{(2n-1)!\cdot 2}{(2n-1)\cdot n!} = Z(1,n) \qquad (5)$$

Actually, this result is trivial and can be shown without using the theorem. Whenever n first order streams merge to form two networks, the merger of the latter will produce a single network; and whenever n first order streams merge into a single network, the elimination of the outlet together with the last downstream node will produce exactly two networks. Since the described transformation and its inverse are one-valued functions, i.e., establish a one-to-one relationship, we get $Z(1,n) = Z(2,n)$.

(d) Let us assume, that the n first order streams form an ordered set, and that the first j of them form k networks $(k \leq j)$, whereas the remaining $n - j$ first order streams form $i - k$ networks $(i - k \leq n - j)$. If one considers all possible mergers for given n, i, j and k, i.e., $Z(k,j) \cdot Z(i - k, n - j)$, then the sum of the number of topologically different arrangements taken over all j, $l \leq j \leq n$, will be equal to the total number of different patterns possible:

$$Z(i,n) = \sum_{j=1}^{n} Z(k,j)\,Z(i-k,n-j) \qquad \text{For any } k,\ 1 \leq k < i \qquad (6)$$

For equation (6) as well as later applications of the theorem we need the following definitions:

$$Z(m,n) = 0 \text{ for } m < 0$$

$$Z(0,0) = 1$$

$$Z(m,n) = 0 \text{ for } m > n \qquad (7)$$

(e) We will show the following relationship

$$Z(i,n) = Z(i-1,n) - Z(i-2,n-1) \qquad (8)$$

by mathematical induction. Equation (8) is apparently correct for $i = 2$ and any $n \geq 2$ (see equation (4) and subsequent discussion). Assume that equation (8) is correct for all positive integers $u < i$ and $v < n$, i.e.,

$$Z(u,v) = Z(u-1,v) - Z(u-2, v-1) \tag{9}$$

Multiplying equation (9) by a factor $Z(i-u,n-v)$ and summing over v, keeping u fixed:

$$\sum_{v=1}^{n-1} Z(u,v) Z(i-u, n-v) = \sum_{v=1}^{n-1} Z(u-1,v) Z(i-u,n-v)$$

$$- \sum_{v=1}^{n-1} Z(u-2,v-1) Z(i-u,n-v) \tag{10}$$

Since we assumed $u < i$, $Z(i-u,n-v)$ will always be zero for $v = n$, and

$$Z(u,n) Z(i-u,o) = Z(u-1,n) Z(i-u,o)$$
$$- Z(u-2,n-1)\ Z(i-u,o) \tag{11}$$

is a trivial statement. Adding equation (11) to equation (10) will increase the index set of v to $\{1, \ldots, n\}$. According to equation (6), the three sums will then represent the individual terms in equation (8), q.e.d.

(f) Still another relationship, relating $Z(i,n)$ to the number $N(n) = Z(1,n)$, i.e., relating the number of ways in which n first order streams can merge to produce i channel networks to the number of ways that n first order streams can produce a single channel network, is

$$Z(i,n) = \sum \prod_{j=1}^{i} N(t_j) \tag{12}$$

where the sequence t_j, $j = 1, \ldots i$, assumes all possible partitions of n into i positive integers.

The proof of the theorem follows directly from equation (8) by mathematical induction. It has already been verified for $i = 1$ and any n [see equation (3) with comments]. Assuming the theorem to be correct for $Z(i-1,n)$ and $Z(i-2,n-1)$, equation (8) becomes

$$Z(i,n) = \frac{\binom{2n-(i-1)}{n}(i-1)}{2n-(i-1)} - \frac{\binom{2(n-1)-(i-2)}{n-1}(i-2)}{2(n-1)-(i-2)} \tag{13}$$

$$= \frac{(2n-i)!}{n!\,(n-i)!\,(2n-i)}\left(\frac{(i-1)(2n-i)-n(i-2)}{n-i+1}\right) = \frac{\binom{2n-i}{n} i}{2n-i} \quad \text{q.e.d.}$$

Combinatorial Analysis of the Frequency Distribution of Drainage Polygons

Let us summarize the explication of the problem stated at the beginning, as it has developed so far.

Any channel network can be described as a sequence of bowls, which are paths of the network connecting "neighboring" outer links of the network. The drainage basin of a channel network is subdivided into individual drainage areas (drainage polygons), each draining in one and only one link of the channel network. In topological terms, these drainage areas form a pattern of drainage polygons. Their boundaries are the drainage divides separating the catchment areas of the individual channel links. Leaving, for the moment, the boundary of the basin out of consideration, then the drainage divide lines which are separated from the rest of the divide pattern by a bowl of the channel network form a dendritic network on their own. Under the previous assumption, that the basin boundary is drained by all and only the outer links of the channel network, this divide network has as many outer links as the corresponding bowl has links, i.e., if the size of the bowl is t, then the number of links of the corresponding divide network is $2t - 3$. If we define the drainage divide link connecting the divide network with the basin's boundary as its root, then its magnitude is $t - 1$. Each divide network of magnitude $t - 1$ consists of t bowls, each of which delimits the drainage area of one side of one of the links of the corresponding channel network bowl. The union of the bowls of the various divide networks constitutes the drainage polygons we want to analyze.

The calculation of the expected number $E(w,n)$ of drainage polygons of size w (i.e., having w sides) for a channel network of magnitude n can now be broken down into the following problems:

(a) What is the probability $p(u,n)$, that a bowl drawn randomly from a channel network of magnitude n is of size u?

(b) What is the probability $p(u,v,n)$, that a link drawn randomly from a channel network of magnitude n joins two bowls of sizes u and v?

(c) Given a channel network bowl of size u and a randomly drawn link t of this bowl, what is the probability $p(i,u)$ that the bowl of the corresponding divide network delimiting the drainage area of t is of size i?

As pointed out before, the drainage polygon of any channel network link consists of the union of two open polygons, which are the divide network bowls corresponding to that link. Thus, the size of the drainage polygon is the same for any combination of open polygons (divide network bowls), whose respective sizes add up to a fixed value. Furthermore, we have to allow for the dependency of the size of a divide network bowl on the magnitude of the corresponding divide network which in turn is dependent on the size of the channel network bowl to which it corresponds. Disregarding, for the moment, those sides of drainage polygons which are part of the basin boundary we get

$$E\,(w,n) = (2n-1) \sum_{\substack{u \\ u \leq v}} \sum_{v} \sum_{\substack{i \\ i+j=w}} \sum_{j} p(u,v,n)\, p(i,u)\, p(j,v) \qquad (14)$$

Solution to Problem A: Any bowl of size u in a network of magnitude n possesses $u-1$ nodes. In each of these nodes the bowl is linked to a sub-tree of the network. If one considers each node as the outlet of the respective sub-tree, then the sum of the magnitudes of the sub-trees is $n-1$. Hence, the number of ways in which a network of magnitude n can have a bowl of size u is equal to the number of ways in which $n-1$ sources can generate $u-1$ networks. This number, however, is given by the Theorem. Dividing by the number $N(n)$ of all topologically distinct configurations a network of magnitude n can have we get

$$p(u,n) = \frac{Z(u-1, n-1)}{N(n)} = \frac{Z(u-1, n-1)}{Z(1,n)} \qquad (15)$$

Solution to Problem B: If in a network of given magnitude n two bowls of sizes u and v have a network link in common, this link can be either an inner link (case I) or an outer link (case II).

(a) Case I: The number of inner links of the two bowls are $u-2$ and $v-2$ respectively. Keeping one bowl fixed in one of the possible $n+1$ positions in the network, the number of possible ways in which the two bowls can be joined along an inner link is $(u-2)\,(v-2)$, leaving together $[(u-1)+(v-1)-4]$ nodes in which the remaining parts of the network will be attached as sub-trees whose joint magnitude is $n-3$. Consequently, the total number of ways in which a network of magnitude n can have two bowls of sizes u,v joining along an inner link of the network is $\frac{n+1}{\delta_u^v + 1}(u-2)\,(v-2)\,Z(u+v-6, n-3)$. δ_u^v is the Kronecker delta, and is defined as

$$\delta_u^v = \begin{cases} 1 \text{ for } u = v \\ 0 \text{ for } u \neq v \end{cases}$$

It assures that the combination of two bowls A,B joined by a network link is counted only once rather than twice (A vs. B and B vs. A) when the sizes of A and B are the same.

(b) Case II: There are only two ways in which two bowls can have an outer link in common, leaving $[(u-1+(v-1)-2]$ nodes in which the remaining network is attached as a set of sub-trees. The joint magnitude of those $u+v-4$ sub-trees is $(n+1)-3$. Hence, the total number of ways in which a network of magnitude n can have two bowls of sizes u,v joining along an outer link of the network is $\frac{n+1}{\delta_u^v + 1} \cdot 2Z(u+v-4, n-2)$. Combining the solution of cases I and II we obtain

$$p(u,v,n) = \frac{(n+1)\left[(u-2)(v-2)Z(u+v-6,n-3) + 2Z(u+v-4,n-2)\right]}{(\delta_u^v+1)\quad(2n-1)Z(i,n)} \qquad (16)$$

The two expressions on the right side of the equation are the probabilities $p_{in}(u,v,n)$ and $P_{out}(u,v,n)$ referring to inner and outer links respectively.

Solution to Problem C: The problem is mathematically analogous to the problem A. Let u be the size of a channel network bowl and t be a randomly drawn link of it. Then the magnitude of the divide network corresponding to this bowl is $u-1$, and the probability $p(i,u)$ that the divide network bowl corresponding to link t is of size i is

$$p(i,u) = \frac{Z(i-1,u-2)}{Z(1,u-1)} \qquad (17)$$

We are now prepared to construct the probability distribution $p(w,n)$ of drainage polygons by size w for a channel network of magnitude n explicitly. Let us first determine the probability distribution $p_{in}(w,n)$ for all inner polygons, i.e., drainage polygons of inner links of the channel network. The probability of an inner link joining bowls of sizes u and v is $p_{in}(u,v,n)$, and the probability that an inner link joining bowls of sizes u and v has a drainage polygon of size w is

$$\sum_{\substack{i \; j \\ i+j=w}} \sum \mathrm{p}(i,u)\,p(j,v) \qquad (18)$$

Hence,

$$p_{in}(w,n) = \sum_{\substack{u \; v \\ u \le v}} \sum \sum_{\substack{i \; j \\ i+j=w}} \sum p_{in}(u,v,n)\,p(i,u)\,p(j,v) \qquad (19)$$

Drainage polygons corresponding to outer channel network links have an additional link located on the boundary of the network basin. Hence, their probability distribution by size is

$$p_{out}(w,n) = \sum_{\substack{u \; v \\ u \le v}} \sum \sum_{\substack{i \; j \\ i+j=w-1}} \sum p_{out}(u,v,n)\,p(i,u)\,p(j,v) \qquad (20)$$

Combining the two distributions and substituting the individual expressions yields:

$$p(w,n) = p_{\text{in}}(w,n) + p_{\text{out}}(w,n)$$

$$= \sum_{\substack{u\ v \\ u \le v}} \sum_{\substack{i\ j \\ i+j=w}} \frac{(n+1)}{\delta_u{}^v + 1} \cdot \frac{(u-2)(v-2)Z(u+v-6,n-3)}{(2n-1)Z(1,n)} \cdot \frac{Z(i-1,u-2)Z(j-1,v-2)}{Z(1,u-1)Z(1,v-1)}$$

$$+ \sum_{\substack{u\ v \\ u \le v}} \sum_{\substack{i\ j \\ i+j=w-1}} \frac{(n+1)}{\delta_u{}^v + 1} \cdot \frac{2Z(u+v-4,n-2)}{(2n-1)Z(1,n)} \cdot \frac{Z(i-1,u-2)Z(j-1,v-2)}{Z(1,u-1)Z(1,v-1)} \quad (21)$$

Test Results

Data have been sampled by Professor K. Wiek (University of California, Riverside) and students from the USGS 1:24,000 topographic maps covering an area of eastern Kentucky which is largely free from geologic control. In total, nine river networks (not channel networks!) of magnitude 7, and 17 river networks of magnitude 4 were subdivided into the areas drained by the individual network links. The frequency of these drainage polygons by size, i.e., by number of sides, for each set is shown in Tables I and II. The tables also include the expected frequencies according to equation (21). The chi-square values (X^2) are 2.73 for the data in Table I and 3.43 for the data in Table II, which correspond to percentiles of $\alpha = 55\%$ and 93% respectively.

TABLE I.
Drainage Polygons for 9 River Networks of Magnitude 7, by Number of Sides *w*

w	*Observed*	*Expected*
3&4	48	42.4
5	40	47.1
6	21	21.6
7	{7	{5.14
8	1}	0.59}

TABLE II.
Drainage Polygons for 17 River Networks of Magnitude 4, by Number of Sides *w*

w	*Observed*	*Expected*
3&4	78	68
5&6	41	51

Whereas the data of the first test support the theory, the data from the magnitude 4 river networks are inconclusive as to the adequacy of the theoretical approach developed in this paper. An inspection of the actual drainage divide patterns on the maps revealed that one of the assumptions of the paper was frequently violated; namely, that only outer links of the network drain the area along the boundary of the network basin. This assumption was therefore replaced by another one permitting every link of a channel network to drain an area stretching all the way to the basin boundary. Again we assume that no interior polygon is drained by an outer network link, and that all topologically different patterns will occur with equal likelihood. The probability that a randomly drawn link of a channel network bowl of size u has a drainage area delimited by a corresponding divide bowl of size i is

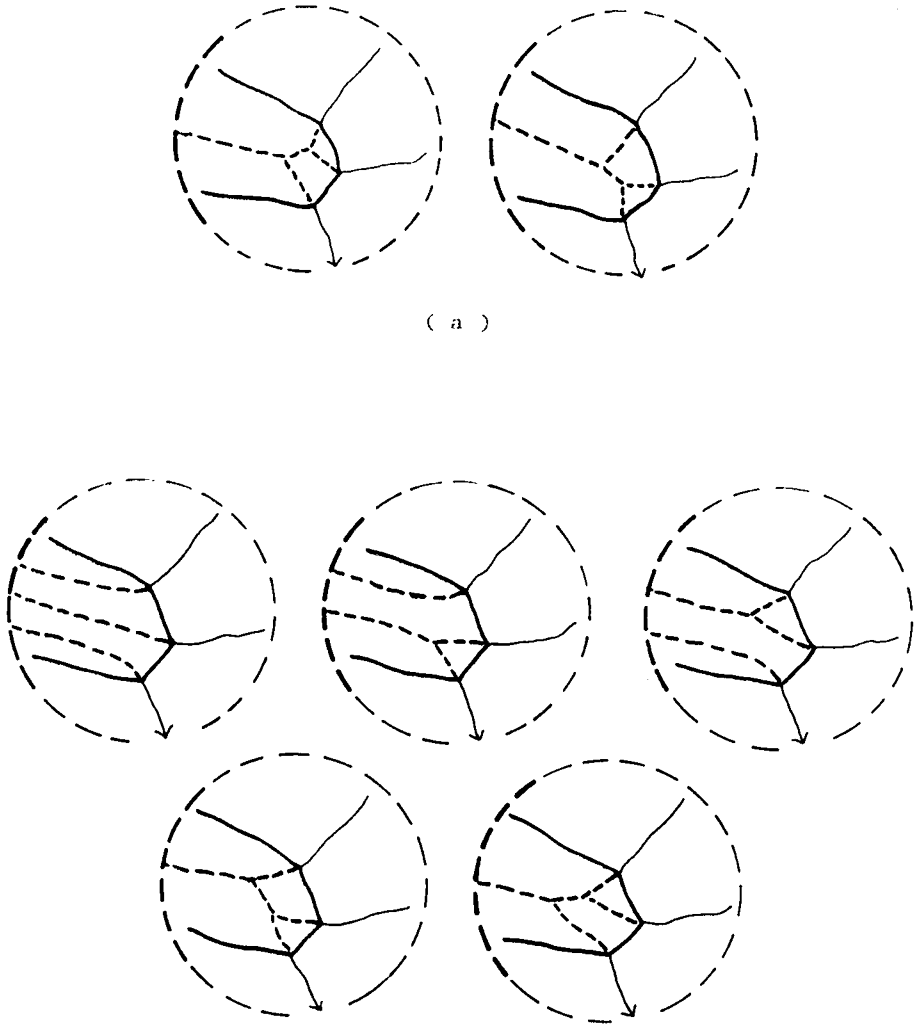
$$\bar{p}(i,u) = \frac{Z(i, -1,u-1)}{Z(1,u)} \tag{22}$$

Figure 3 shows a channel network bowl of size 4 and the topologically different drainage divide trees associated with it, (a) corresponding to distribution (17) and (b) corresponding to distribution (22). To reiterate: distribution (17) is based on the assumption that all and only outer links of the channel network drain the area along the basin boundary, and distribution (22) is based on the assumption that, in addition to all outer links, the inner links of the channel network may participate in the drainage of the area along the basin boundary.

The mathematical derivation of the probability distribution of (22) follows closely that of (17). For a given channel network bowl of size u there are $u - 1$ drainage divide links originating in the nodes of the bowl. If the two drainage divide links which are located on the basin boundary and establish a connection to the drainage divide trees of the adjacent network bowls are considered to be part of the divide tree associated with the channel network bowl under consideration, then the total number of outer links of this tree is $u + 1$ and its magnitude is u. Hence, the number of ways in which a divide bowl of size i can be constructed within the divide tree of magnitude u is, according to Theorem *I*, $Z(i - 1,u - 1)$, which leads directly to (22). It is important to note, that the way in which the distribution was established, the drainage divide links located on the basin boundary as specified above will be counted twice; thus the probability that an outer channel network link joining bowls of sizes u and v has a drainage polygon of size w is

$$\sum_{i}\sum_{j} p(i,u)\,p(j,v) \qquad i+j=w-1 \tag{23}$$

Adjusting equation (21) to the new probability distribution $\bar{p}(i,u)$ produces the following expected data for the 17 networks of magnitude 4:
Although the test results are quite encouraging, the theoretical approach as well as the data sampling procedure show several shortcomings and



(a)

(b)

Figure 3

need to be improved. On the theoretical side, the assumption of lower magnitude which was needed to assure that no outer network link drains an interior polygon, should be abolished. A preliminary study of actual drain-

[4] Results equivalent to those derived in this section have been independently established by Dacey [1].

TABLE III.
OBSERVED DRAINAGE POLYGON DATA FROM 17 RIVER NETWORKS OF MAGNITUDE 4; EXPECTED DATA ACCORDING TO (21) WITH MODIFIED PROBABILITY DISTRIBUTION (22)

w	*Observed*	*Expected*	
3	34	34.9	
4	44	45.9	$X^2 = .715$
5	30	26.3	$\alpha = 13.2\%$
6	{10}	{8.45}	
7	{1}	{3.45}	

age divide patterns seems to support the distribution (22) rather than (17). It might very well be that an adequate description of reality lies somewhere "in between," so that both (17) and (22) account only for part of the observed distribution. If this impression turns out to be correct, then it would either constitute a new example of topological non-randomness or a new definition of the concept of random network topology would be needed. As to the data: the selection of the river networks was based on availability rather than random sampling. The impact of the restriction of the sample networks to river networks which in effect results in a gross generalization of the pattern of channel networks, introduces another uncertainty with regard to the reliability and interpretation of the data. In view of the sampling procedures, the data do not really qualify for analysis by inferential statistics. In this context it might be interesting to note that similar data sampled in the coastal mountain ranges of Northern California, an area of pronounced geologic control, did not sufficiently deviate from the expected values so as to permit a rejection of the theory at the five percent level of significance.

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