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“Rigidity Phenomena in random point sets”

by

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Abstract

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Let Π be a translation invariant point process on the Euclidean space E and let $\mathcal{D} \subset E$ be a bounded open set whose boundary has zero Lebesgue measure. We ask what does the point configuration Π_{out} obtained by taking the points of Π outside $\mathcal{D}$ tell us about the point configuration Π_{in} of Π inside $\mathcal{D}$? We focus mainly on translation invariant point processes on the plane. We show that for the Ginibre ensemble, Π_{out} determines the number of points in Π_{in}. We refer to this kind of behaviour as "rigidity". For the translation-invariant zero process of a planar Gaussian Analytic Function, we show that Π_{out} determines the number as well as the centre of mass of the points in Π_{in}. Further, in both models we prove that the outside says "nothing more" about the inside, in the sense that the conditional distribution of the inside points, given the outside, is mutually absolutely continuous with respect to the Lebesgue measure on its supporting submanifold. We further show that the conditional density (of the inside points given the outside) is, roughly speaking, comparable to a squared Vandermonde density. In particular, this shows that even under spatial conditioning, the points exhibit repulsion which is quadratic in their mutual separation. We apply these results to the study of continuum percolation on these point processes, and establish the existence of a non-trivial critical radius and the uniqueness of infinite cluster in the supercritical regime. En route, we obtain new estimates on hole probabilities for zeroes of the planar Gaussian Analytic Function. Finally, we apply these ideas to prove completeness properties of random exponential functions originating from "rigid" determinantal point processes. We conclude by establishing miscellaneous other results on determinantal point processes. These include answers to two questions of Lyons and Steif on certain models of stationary determinantal processes on $\mathbb{Z}$, one involving insertion and deletion tolerance, and the other regarding the recovery of the driving function from the process.

To my mother
Rina Ghosh

Contents

Acknowledgments

A great philosopher had once said, “If I have seen further it is by standing on the shoulders of giants”. Indeed, I am deeply indebted to several individuals, whose assistance, compassion and thoughtful guidance have helped my journey at difficult junctures.

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Chapter 1

The Basic Models

A point process Π on $\mathbb{C}$ is a random locally finite point configuration on the two dimensional Euclidean plane. The probability distribution of the point process Π is a probability measure $\mathbb{P}[\Pi]$ on the Polish space of locally finite point sets on the plane. Point processes on the plane have been studied extensively. For a lucid exposition on general point processes one can look at [DV97]. The group of translations of $\mathbb{C}$ acts in a natural way on the space of locally finite point configurations on $\mathbb{C}$. Namely, the translation by z, denoted by T_z, takes the point configuration Λ to the configuration $T_z(\Lambda) := \{x+z \big| x \in \Lambda\}$. A point process Π is said to be translation invariant if Π and $T_z(\Pi)$ have the same distribution for all $z \in \mathbb{C}$. In this work we will focus primarily on translation invariant point processes on $\mathbb{C}$ whose intensities are absolutely continuous with respect to the Lebesgue measure. We will consider simple point processes, namely, those in which no two points are at the same location. A simple point process can also be looked upon as a random discrete measure $[\Pi] = \sum_{z\in\Pi} \delta_z$.

In this thesis, we will chiefly concern ourselves with translation invariant point processes in the complex plane. We will focus on the two main natural examples of translation invariant point processes on the plane (other than the Poisson process) - namely, the Ginibre ensemblle and the zeroes of the standard planar Gaussian analytic function. In the final chapter, there will be some references to other classes of point processes, including those on the real line and on $\mathbb{Z}^d$.

In this introductory section, we will discuss our main models under study, which will come up repetedly in the subsequent chapters of this thesis. Other models and processes will be introduced as and when necessary.

1.1 The Ginibre Ensemble

Let us consider an $n \times n$ matrix $X_n, n \geq 1$ whose entries are i.i.d. standard complex Gaussians. The vector of its eigenvalues, in uniform random order, has the joint density (with respect to the Lebesgue measure on $\mathbb{C}^n$) given by

$$p(z_1, \cdots, z_n) = \frac{1}{\pi^n \prod_{k=1}^n k!} e^{-\sum_{k=1}^n |z_k|^2} \prod_{i<j} |z_i - z_j|^2.$$

Recall that a determinantal point process on the Euclidean space $\mathbb{R}^d$ with kernel K and background measure μ is a point process on $\mathbb{R}^d$ whose k-point intensity functions with respect to the measure $\mu^{\otimes k}$ are given by

$$\rho_k(x_1, \cdots, x_k) = \det\Big[(K(x_i, x_j))_{i,j=1}^k \Big].$$

Typically, K has to be such that the integral operator defined by K is a non-negative trace class contraction mapping $L^2(\mu)$ to itself. For a detailed study of determinantal point processes, we refer the reader to [HKPV10] or [Sos00]. A simple calculation involving Vandermonde determinants shows that the eigenvalues of X_n (considered as a random point configuration) form a determinantal point process on $\mathbb{C}$. Its kernel is given by $K_n(z,w) = \sum_{k=0}^{n-1} \frac{(z\bar{w})^k}{k!}$ with respect to the background measure $d\gamma(z) = \frac{1}{\pi} e^{-|z|^2} d\mathcal{L}(z)$ where $\mathcal{L}$ denotes the Lebesgue measure on $\mathbb{C}$. This point process is the Ginibre ensemble (of dimension n), which we will denote by $\mathcal{G}_n$. As $n \to \infty$, these point processes converge, in distribution, to a determinantal point process given by the kernel $K(z,w) = e^{z\bar{w}} = \sum_{k=0}^{\infty} \frac{(z\bar{w})^k}{k!}$ with respect to the same background measure γ. This limiting point process is the infinite Ginibre ensemble $\mathcal{G}$. It is known that the distribution of $\mathcal{G}$ is invariant and ergodic under the natural action of the translations of the plane.

1.2 The GAF zero process

Let $\{\xi_k\}_{k=0}^{\infty}$ be a sequence of i.i.d. standard complex Gaussians. Define

$$f_n(z) = \sum_{k=0}^{n} \xi_k \frac{z^k}{\sqrt{k!}} \,(\text{ for } n \geq 0)\,, \qquad f(z) = \sum_{k=0}^{\infty} \xi_k \frac{z^k}{\sqrt{k!}}\,.$$

These are Gaussian processes on $\mathbb{C}$ with covariance kernels given by $K_n(z,w) = \sum_{k=0}^{n} \frac{(z\bar{w})^k}{k!}$ and $K(z,w) = \sum_{k=0}^{\infty} \frac{(z\bar{w})^k}{k!} = e^{z\bar{w}}$ respectively. A.s. f_n and f are entire functions and the functions f_n converge to f (in the sense of the uniform convergence of holomorphic functions on compact sets). It is not hard to see (e.g., via Rouche's theorem) that this implies that the corresponding point processes of zeroes, denoted by $\mathcal{F}_n$, converge a.s. to the zero process $\mathcal{F}$ of the GAF (in the sense of locally finite point configurations converging on compact sets). It is known that the distribution of $\mathcal{F}$ is invariant and ergodic under the natural action of the translations of the plane.

Chapter 2

Rigidity and Tolerance in point processes

2.1 Introduction

In this chapter, we study the following question: if we know the configuration Π_{out}, what can we conclude about Π_{in}? We consider, with regard to this question, the main natural examples of translation invariant point processes on the plane, and provide a complete answer in each case.

The (homogeneous) Poisson point process is the most canonical example of a translation-invariant point process on the plane. Its crucial property is that the point configurations in two disjoint measurable sets are independent of each other. Therefore, our question is a triviality for the Poisson process: the points outside $\mathcal{D}$ do not provide any information about the points inside $\mathcal{D}$.

The two natural examples of translation invariant point processes in the plane that have non-trivial spatial correlations are the Ginibre ensemble and the zeroes of the planar Gaussian Analytic Function. We refer the reader to [HKPV10] for a detailed study of these ensembles. The Ginibre ensemble was introduced in the physics literature by Ginibre [Gin65] as a model based on non-Hermitian random matrices. Like the Poisson process, it is translation-invariant and ergodic under rigid motions of the plane. In fact, it is a determinantal point process with the determinantal kernel $K(z,w) = \sum_{j=0}^{\infty} \frac{(z\overline{w})^j}{j!}$ and the background measure $e^{-|z|^2} d\mathcal{L}(z)$, where $\mathcal{L}$ denotes the Lebesgue measure on $\mathbb{C}$. Krishnapur [Kr06] has shown that the Ginibre ensemble is essentially the unique translation invariant process among a certain class of determinantal point processes on $\mathbb{C}$ that have a sesqui-holomorphic kernel (i.e., the determinantal kernel is holomorphic in the first variable and anti-holomorphic in the second), a radially symmetric background measure and is normalized to have unit intensity.

The standard planar Gaussian Analytic Function (abbreviated henceforth as GAF) is the random entire function defined by the series development $f(z) = \sum_{k=0}^{\infty} \frac{\xi_k}{\sqrt{k!}} z^k$ where ξ_k-s are i.i.d. standard complex Gaussians. We are interested in the point configuration on $\mathbb{C}$ given by the zeroes of this GAF. The GAF zero process is translation invariant and ergodic, and exhibits local repulsion. It has been studied intensively by several authors including Nazarov, Sodin, Tsirelson, and others (see, e.g., [FH99], [STs1-04],[STs2-06],[STs3-05],[NSV07],[NS10]). Sodin and Tsirelson [STs1-04] have shown that in the class of Gaussian power series, the standard planar GAF is the only one to have a translation invariant zero-set (up to scaling and multiplication by a deterministic entire function with no zeroes).

For further details on these models, we refer the reader to Chapter 1.

For a pair of random variables (X, Y) which has a joint distribution on a product of Polish spaces $\mathcal{S}_1 \times \mathcal{S}_2$, we can define the *regular conditional distribution* γ of Y given X by the family of probability measures measures $\gamma(s_1, \cdot)$ parametrized by the elements $s_1 \in \mathcal{S}_1$ such that for any Borel sets $A \subset \mathcal{S}_1$ and $B \subset \mathcal{S}_2$ we have

$$\mathbb{P}(X \in A, Y \in B) = \int_A \gamma(s_1, B) \, d\,\mathbb{P}[X](s_1)$$

where $\mathbb{P}[X]$ denotes the marginal distribution of X. For details on regular conditional distributions, see, e.g., [Pa00] or [Bil95].

Recall that $\mathcal{S}_{\text{in}}$ and $\mathcal{S}_{\text{out}}$ are Polish spaces. Hence, by abstract nonsense, there exists a *regular conditional distribution* ϱ of Π_{in} given Π_{out}. Clearly, ϱ can be seen as the distribution of a point process on $\mathcal{D}$ which depends on Υ_{out}.

Let $\underline{\zeta}$ be the vector (of variable length) whose co-ordinates are the points of Π_{in} taken in uniform random order. We will denote the conditional distribution of $\underline{\zeta}$ given Π_{out} by ρ. Formally, it is a family of probability measures $\rho(\Upsilon_{\text{out}}, \cdot)$ on $\bigcup_{m=0}^{\infty} \mathcal{D}^m$ parametrized by $\Upsilon_{\text{out}} \in \mathcal{S}_{\text{out}}$.

There is a simple relationship between $\rho(\Pi_{\text{out}}, \cdot)$ and $\varrho(\Pi_{\text{out}}, \cdot)$. Consider the natural map ϕ from $\bigcup_{m=0}^{\infty} \mathcal{D}^m$ to $\mathcal{S}_{\text{in}}$ which makes a point configuration Υ_{in} of size m from a vector $\underline{\zeta}$ in $\mathcal{D}^m$ by forgetting the order of the co-ordinates of $\underline{\zeta}$. It is easy to see that $\mathbb{P}[\Pi_{\text{out}}]$-a.s. $\phi_* \rho(\Pi_{\text{out}}, \cdot) = \varrho(\Pi_{\text{out}}, \cdot)$.

For a vector $\underline{\alpha}$, we denote by $\Delta(\underline{\alpha})$ the Vandermonde determinant generated by the co-ordinates of α. Note that $|\Delta(\underline{\alpha})|$ is invariant under permutations of the co-ordinates of $\underline{\alpha}$.

For two measures μ_1 and μ_2 defined on the same measure space Ω with $\mu_1 \ll \mu_2$ (meaning μ_1 is absolutely continuous with respect to μ_2), we will denote by $\frac{d\mu_1}{d\mu_2}(\omega)$ the Radon Nikodym derivative of μ_1 with respect to μ_2 evaluated at $\omega \in \Omega$. Similarly, by $\mu_1 \equiv \mu_2$ we mean that the measures μ_1 and μ_2 are mutually absolutely continuous, which implies that both $\mu_1 \ll \mu_2$ and $\mu_2 \ll \mu_1$ are simultaneously true.

In the case of the Ginibre ensemble, we prove that almost surely (henceforth abbreviated as "a.s."), the points outside $\mathcal{D}$ determine the number of points inside $\mathcal{D}$, and "nothing more".

In Theorems 2.1.1-2.1.4 we denote the Ginibre ensemble by $\mathcal{G}$ and the GAF zero ensemble by $\mathcal{F}$. As before, $\mathcal{D}$ is a bounded open set in $\mathbb{C}$ whose boundary has zero Lebesgue measure.

Theorem 2.1.1. *For the Ginibre ensemble, there is a measurable function $N : \mathcal{S}_{\text{out}} \to \mathbb{N}\cup\{0\}$ such that a.s.*

$$\textit{Number of points in } \mathcal{G}_{\text{in}} = N(\mathcal{G}_{\text{out}}) \,.$$

Since a.s. the length of $\underline{\zeta}$ equals $N(\mathcal{G}_{\text{out}})$, we can as well assume that each measure $\rho(\Upsilon_{\text{out}}, \cdot)$ is supported on $\mathcal{D}^{N(\Upsilon_{\text{out}})}$.

Theorem 2.1.2. *For the Ginibre ensemble, $\mathbb{P}[\mathcal{G}_{\text{out}}]$-a.s. the measure $\rho(\mathcal{G}_{\text{out}}, \cdot)$ and the Lebesgue measure $\mathcal{L}$ on $\mathcal{D}^{N(\mathcal{G}_{\text{out}})}$ are mutually absolutely continuous.*

In the case of the GAF zero process, we prove that the points outside $\mathcal{D}$ determine the number as well as the centre of mass (or equivalently, the sum) of the points inside $\mathcal{D}$, and “nothing more”.

Theorem 2.1.3. *For the GAF zero ensemble,*
(i) There is a measurable function $N : \mathcal{S}_{\text{out}} \to \mathbb{N} \cup \{0\}$ such that a.s.

$$\textit{Number of points in } \mathcal{F}_{\text{in}} = N(\mathcal{F}_{\text{out}}).$$

(ii) There is a measurable function $S : \mathcal{S}_{\text{out}} \to \mathbb{C}$ such that a.s.

$$\textit{Sum of the points in } \mathcal{F}_{\text{in}} = S(\mathcal{F}_{\text{out}}).$$

Define the set

$$\Sigma_{S(\mathcal{F}_{\text{out}})} := \{\underline{\zeta} \in \mathcal{D}^{N(\mathcal{F}_{\text{out}})} : \sum_{j=1}^{N(\mathcal{F}_{\text{out}})} \zeta_j = S(\mathcal{F}_{\text{out}})\}$$

where $\underline{\zeta} = (\zeta_1, \cdots, \zeta_{N(\mathcal{F}_{\text{out}})})$.

Since a.s. the length of $\underline{\zeta}$ equals $N(\mathcal{F}_{\text{out}})$, we can as well assume that each measure $\rho(\Upsilon_{\text{out}}, \cdot)$ gives us the distribution of a random vector in $\mathcal{D}^{N(\Upsilon_{\text{out}})}$, supported on $\Sigma_{S(\Upsilon_{\text{out}})}$.

Theorem 2.1.4. *For the GAF zero ensemble, $\mathbb{P}[\mathcal{F}_{\text{out}}]$-a.s. the measure $\rho(\mathcal{F}_{\text{out}}, \cdot)$ and the Lebesgue measure $\mathcal{L}_\Sigma$ on $\Sigma_{S(\mathcal{F}_{\text{out}})}$ are mutually absolutely continuous.*

The central question of this chapter is partially motivated by the concepts of insertion and deletion tolerance of a point process. Consider a point process Π. Fix a bounded open set $\mathcal{D}$, and add to Π a random point uniformly distributed in $\mathcal{D}$. Let us call this perturbed point process $\Pi_{\mathcal{D}}$. The point process Π is said to be **insertion tolerant** if $\mathbb{P}[\Pi_{\mathcal{D}}] \ll \mathbb{P}[\Pi]$ for all bounded open sets $\mathcal{D}$. **Deletion tolerance** is similarly defined by deleting a point inside $\mathcal{D}$ (if one exists) uniformly at random. Insertion and deletion tolerance have been investigated by Burton and Keane ([BK89]) in the context of percolation, by Holroyd and Peres ([HP05]) for studying invariant allocation on the plane, and by Heicklen and Lyons

([HLy03]) in the setting of random spanning forests. They have also been studied as topics of their own importance by Holroyd and Soo ([HS10]).

Notice that the question whether $\mathbb{P}[\Pi_{\mathcal{D}}] \ll \mathbb{P}[\Pi]$ for a given $\mathcal{D}$ can be phrased in terms of the conditional distribution ϱ, and the same holds for the main questions addressed in this chapter. Therefore, in more general terms, we are interested in the regularity properties of ϱ.

For a finite point process of fixed size n, the phenomenon of outside points determining the number of inside points is a triviality. However, the behaviour of infinite point processes can be quite different, even when they arise as distributional limits of finite point processes. For example, let us take the disk of area n centred at the origin and consider the point process Π_n given by n uniform points inside it. Let $\mathcal{D}$ be the unit disk centred at the origin. The process Π_n clearly has the property that the point configuration outside $\mathcal{D}$ determines the number of points inside $\mathcal{D}$. However, the distributional limit of Π_n-s, as $n \to \infty$, is the Poisson point process (of intensity 1), which has no such property. Hence, the reason behind this phenomenon to occur in the case of the Ginibre ensemble or the Gaussian zeroes is fundamentally different, and is connected with the spatial correlation properties of the corresponding ensembles.

In addition to answering our central question mentioned in the beginning, Theorems 2.1.1-2.1.4 also provide information on the relative strength of spatial correlations in the Ginibre and the GAF zero ensembles. While a simple visual inspection suffices to (heuristically) distinguish a sample of the Poisson process from that of either the Ginibre or the GAF zero process (of the same intensity), the latter two are hard to set apart between themselves. It is therefore an interesting question to devise mathematical statistics that distinguish them. The qualitative idea is that the spatial correlation is much stronger in the GAF zero process than in the Ginibre ensemble. There can be several possible approaches to quantify this heuristic observation. One important feature to look at, for instance, is the rate of decay of the hole probabilities. However, it turns out that both the Ginibre ensemble and the Gaussian zero process behave similarly in this respect. For more details, one can refer to citeHKPV. Our results clearly demonstrate that the GAF zeroes have much greater spatial dependence, in the sense that the point configuration in the exterior of an open set dictates much more about the one in its interior.

In [STs1-04], Sodin and Tsirelson compared the GAF zero process (CAZP in their terminology) with various models of perturbed lattices. They noticed that the lattice process

$$\{\sqrt{3\pi}(k+li) + ce^{2\pi i m/3}\eta_{k,l} : k,l \in \mathbb{Z}, m = 0,1,2\},$$

(where $\eta_{k,l}$ are i.i.d. standard complex Gaussians, $c \in (0,\infty)$ is a parameter and i denotes the imaginary unit) achieves "asymptotic similarity" with the GAF zero process (in the sense that the variances of scaled linear statistics have similar asymptotic behaviour to those for the GAF zeroes). Sodin and Tsirelson further observed that the above perturbed lattice model satisfied two conservation laws: one pertaining to the "mass" and another pertaining to the "centre of mass". They predicted similar conservation laws for the GAF zero process, although the sense in which such laws would hold was left open to interpretation. Theorem

2.1.3 establishes two conservation laws, one of which preserves the "mass" (i.e., the number of points), and the other one preserves the "centre of mass". Moreover, Theorem 2.1.4 says that these are the only conservation laws for the GAF zero process. We further note that among the perturbed lattice models in [STs1-04], the one that achieves "asymptotic similarity" with the Ginibre ensemble is the process

$$\{\sqrt{\pi}(k+li)+c\eta_{k,l} : k,l \in \mathbb{Z}\}$$

where $\eta_{k,l}$ and c are as before. In this model, we have one conserved quantity (namely, the "mass"). In Theorems 2.1.1 and 2.1.2, we obtain a conservation law for the "mass" (i.e., the number of points) in the Ginibre ensemble, and further, show that there are no other conserved quantities.

En route proving the main theorems mentioned above, we obtain results that are interesting in their own right. For example we prove that the harmonic sum $\left(\sum_{z\in\Pi}\frac{1}{z}\right)$, for the Ginibre ensemble as well as for the GAF zero process, is a.s. finite (in a precise technical sense specified in Propositions 2.9.3 and 2.13.3 and the remarks thereafter). In fact, we show that this sum has a finite first moment for both processes. It is not hard to see that the corresponding sum for the Poisson process does not converge in any reasonable sense. Even for the Ginibre or the GAF zero ensembles, the corresponding sum does not converge absolutely. The underlying reason for the conditional convergence is the mutual cancellation arising from the higher degree of symmetry (compared to the Poisson process) exhibited by a typical point configuration in the Ginibre or the GAF zero process. This is yet another manifestation of the fact that the Gaussian zeros or the Ginibre eigenvalues exhibit a much more regular arrangement (which indicates greater rigidity) than, say, the Poisson process.

In a more precise sense, we can define the finite sums $\alpha_k(n)=\left(\sum_{z\in\Pi}1/z^k\right)$ when $\Pi=\mathcal{G}_n$ or $\mathcal{F}_n$. Here $\mathcal{G}_n$ is the approximation to G by eigenvalues of $n\times n$ random matrices, and $\mathcal{F}_n$ is the approximation to $\mathcal{F}$ by zeroes of random polynomials of degree n (for details refer to Chapter 1). Our results, as in Proposition 2.9.6 and Proposition 2.13.6, establish that both for the Ginibre and the GAF zeroes, these sums converge in probability as $n\to\infty$. The limit α_k can be justifiably taken to be an analogue of the sum $\left(\sum_{z\in\Pi}1/z^k\right)$ for the respective limiting process $\mathcal{G}$ or $\mathcal{F}$.

We also prove a reconstruction theorem for the planar GAF from its zeroes, which essentially says that the zeroes of the GAF determine a.s. the GAF itself, up to a factor of modulus 1. In what follows, α_k will denote the random variable introduced above for GAF zeroes, P_k will be the k-th Newton polynomial (for details, see Section 2.16). Define $a_k=P_k(\alpha_1,\cdots,\alpha_k)$ and $\chi=\lim_{k\to\infty}k^{1/2}\left(\sum_{j=0}^{k-1}|P_j(\alpha_1,\cdots,\alpha_j)|^2\right)^{-1/2}$ (the existence of the limit will be proved in the course of proving Theorem 2.1.5). We state the reconstruction theorem as:

Theorem 2.1.5. *Consider the random analytic function $g(z)=\sum_{k=0}^{\infty}\chi a_k z^k$, which is measur-*

able with respect to the GAF zeroes. There is a random variable ζ with uniform distribution on $\mathbb{S}^1$ and independent of the GAF zeroes, such that a.s. we have $f(z) = \zeta g(z)$.

It is interesting to compare Theorem 2.1.5 with the Weierstrass Factorization Theorem (see, e.g., [Rud87] Chapter 15) for reconstructing analytic functions from their zeroes. In the case of the Weierstrass factorization, the main problem is that there is a (random) analytic function (with no zeroes) that occurs as a factor in front of the canonical product formed from the Gaussian zeroes, and a priori no concrete information is available about this function. E.g., it can, in principle, depend on the GAF zeroes. However, in Theorem 2.1.5 we are able to give a concrete description of the factor ζ and also the precise dependence of g on GAF zeroes.

The description in Theorem 2.1.5 is optimal in the sense that the factor ζ cannot be done away with. This can be seen from the fact that if θ is a random variable that is uniform in $\mathbb{S}^1$ and independent of the ξ_i-s, then the random analytic functions θf and f are both distributed as planar GAF-s but have the same zeroes; hence from the zeroes of f we can hope to recover the coefficients of f only up to such a factor θ.

Theorem 2.1.5 can be compared with Theorem 6 in [PV05], where a similar result is established for the zeroes of the Gaussian analytic function on the hyperbolic plane. However, such a result for the planar case is not known, and our approach here is distinct from [PV05], relying crucially on the estimates we obtain in Section 2.13.

We view the conservation laws as the **"rigidity"** properties of the respective point processes. The absolute continuity (with respect to the Lebesgue measure on the conserved submanifold) of the conditional distribution of the vector of inside points can be viewed as **"tolerance"**. The heuristic is that due to such mutual absolute continuity, the inside points can form (almost) any configuration on this conserved submanifold.

2.2 Plan of the chapter

In Section 2.3 we provide an abstract framework in which other models having similar characteristics can be investigated. Further, we show in Section 2.4 that for proving the main Theorems 2.1.1 - 2.1.4, it suffices to establish them in the case where $\mathcal{D}$ is a disk.

In order to study rigidity phenomena, we devise a unified approach in Section 2.5, where Theorem 2.5.1 gives general criteria for a function (of the inside points) to be rigid with respect to a point process. We complete the proofs of Theorems 2.1.1 and 2.1.3 by establishing the relevant criteria for the Ginibre and the GAF zero processes.

In Section 2.6, we study tolerance properties in the general setup introduced in Section 2.3. Theorem 2.6.2 lays down conditions under which certain tolerance behaviour of a point process can be established. Proving Theorems 2.1.2 and 2.1.4, therefore, amounts to showing that the relevant conditions hold for our models. However, unlike the rigidity phenomena, this requires substantially more work, and is carried out in two stages for each process. First, we obtain some estimates for the point processes $\mathcal{G}_n$ and $\mathcal{F}_n$, which are finite approximations to $\mathcal{G}$ and $\mathcal{F}$ respectively (see Sections 1.1 and 1.2 for definitions). For the Ginibre ensemble, this is done in Section 2.7, and for the GAF zeroes this is done in Section 2.11. Finally, we apply these estimates to deduce that the relevant conditions for tolerance behaviour hold for our models; this is carried out for the Ginibre ensemble in Section 2.10 and for the GAF zeroes in Section 2.14.

2.3 The General Setup

Fix a Euclidean space $\mathcal{E}$ equipped with a non-negative regular Borel measure μ. Let $\mathcal{S}$ denote the Polish space of countable locally finite point configurations on $\mathcal{E}$. Endow $\mathcal{S}$ with its canonical topology, namely the topology of convergence on compact sets (which gives $\mathcal{S}$ a canonical Borel σ-algebra). Fix a bounded open set $\mathcal{D} \subset \mathcal{E}$ with $\mu(\partial\mathcal{D}) = 0$. Corresponding to the decomposition $\mathcal{E} = \mathcal{D} \cup \mathcal{D}^c$, we have $\mathcal{S} = \mathcal{S}_{\text{in}} \times \mathcal{S}_{\text{out}}$, where $\mathcal{S}_{\text{in}}$ and $\mathcal{S}_{\text{out}}$ denote the spaces of finite point configurations on $\mathcal{D}$ and locally finite point configurations on $\mathcal{D}^c$ respectively.

Let Ξ be a measure space equipped with a probability measure $\mathbb{P}$. For a random variable $Z : \Xi \to \mathcal{X}$ (where $\mathcal{X}$ is a Polish space), we define the push forward $Z_*\mathbb{P}$ of the measure $\mathbb{P}$ by $Z_*\mathbb{P}(A) = \mathbb{P}(Z^{-1}(A))$ where A is a Borel set in $\mathcal{X}$. Also, for a point process $Z' : \Xi \to \mathcal{S}$, we can define point processes $Z'_{\text{in}} : \Xi \to \mathcal{S}_{\text{in}}$ and $Z'_{\text{out}} : \Xi \to \mathcal{S}_{\text{out}}$ by restricting the random configuration Z' to $\mathcal{D}$ and $\mathcal{D}^c$ respectively.

Let $X, X^n : \Xi \to \mathcal{S}$ be random variables such that $\mathbb{P}$-a.s., we have $X^n \to X$ (in the topology of $\mathcal{S}$). We demand that the point processes X, X^n have their first intensity measures absolutely continuous with respect to μ. We can identify X_{in} (by taking the points in uniform random order) with the random vector $\underline{\zeta}$ which lives in $\bigcup_{m=0}^{\infty} \mathcal{D}^m$. The analogous quantity for X^n will be denoted by $\underline{\zeta}^n$.

For our models we can take $\mathcal{E}$ to be $\mathbb{C}$, μ to be the Lebesgue measure, and $\mathcal{D}$ to be a bounded open set whose boundary has zero Lebesgue measure.

In the case of the Ginibre ensemble, we can define the processes $\mathcal{G}_n$ and $\mathcal{G}$ on the same underlying probability space so that a.s. we have $\mathcal{G}_n \subset \mathcal{G}_{n+1} \subset \mathcal{G}$ for all $n \geq 1$. For reference, see [Go10]. We take $(\Xi, \mathbb{P})$ to be this underlying probability space, $X^n = \mathcal{G}_n$ and $X = \mathcal{G}$.

In the case of the Gaussian zero process, we take $(\Xi, \mathbb{P})$ to be a measure space on which we have countably many standard complex Gaussian random variables denoted by $\{\xi_k\}_{k=0}^{\infty}$. Then X^n is the zero set of the polynomial $f_n(z) = \sum_{k=0}^{n} \xi_k \frac{z^k}{\sqrt{k!}}$, and X is the zero set of the entire function $f(z) = \sum_{k=0}^{\infty} \xi_k \frac{z^k}{\sqrt{k!}}$. The fact that $X^n \to X$ $\mathbb{P}$-a.s. follows from Rouche's theorem.

2.4 Reduction from a general $\mathcal{D}$ to a disk

In this Section we intend to prove that to obtain Theorems 2.1.1-2.1.4, it suffices to consider the case where $\mathcal{D}$ is an open disk centred at the origin. We will demonstrate the proof for Theorems 2.1.3 and 2.1.4, the arguments for Theorems 2.1.1 and 2.1.2 are on similar lines.

Let $\mathcal{D}$ be a bounded open set in $\mathbb{C}$ whose boundary has zero Lebesgue measure. By translation invariance of the Ginibre ensemble, we take the origin to be in the interior of $\mathcal{D}$. Let $\mathcal{D}_0$ be a disk (centred at the origin) which contains $\overline{\mathcal{D}}$ in its interior (where $\overline{\mathcal{D}}$ is the closure of $\mathcal{D}$).

Suppose we know the point configuration $\mathcal{F}_{\text{out}}$ to be equal to Υ_{out}. Further, suppose that we can show that the point configuration $\Upsilon_{\text{out}}^{\mathcal{D}_0}$ outside $\mathcal{D}_0$ determines the number N_0 and the sum S_0 of the points inside $\mathcal{D}_0$ a.s. Since we also know the number and the sum of the points inside $\mathcal{D}_0 \setminus \mathcal{D}$, we can determine the number N as well as the sum S of the points in $\mathcal{D}$. This proves the rigidity theorem for the GAF zero ensemble (Theorem 2.1.3) for a general $\mathcal{D}$.

Now suppose we have the tolerance Theorem 2.1.4 for a disk. To obtain Theorem 2.1.4 for $\mathcal{D}$, we appeal to the tolerance Theorem 2.1.4 for the disk $\mathcal{D}_0$. Define

$$\Sigma := \left\{ (\lambda_1, \cdots, \lambda_N) : \sum_{j=1}^{N} \lambda_j = S, \lambda_j \in \mathcal{D} \right\}$$

and

$$\Sigma_0 := \left\{ (\lambda_1, \cdots, \lambda_{N_0}) : \sum_{j=1}^{N_0} \lambda_j = S_0, \lambda_j \in \mathcal{D}_0 \right\}.$$

The conditional distribution of the vector of points inside $\mathcal{D}_0$, given $\Upsilon_{\text{out}}^{\mathcal{D}_0}$, lives on Σ_0, in fact it has a density f_0 which is positive a.e. with respect to Lebesgue measure on Σ_0. Let there be k points in $\mathcal{D}_0 \setminus \mathcal{D}$ and let their sum be s, clearly we have $N = N_0 - k$ and $S = S_0 - s$. We parametrize Σ by the last $N-1$ co-ordinates. Note that the set $U := \{(\lambda_2, \cdots, \lambda_N) : (S - \sum_{j=2}^{N} \lambda_j, \lambda_2, \cdots, \lambda_N) \in \Sigma\}$ is an open subset of $\mathcal{D}^{N-1}$. Further, we define the set $V := \{(\lambda_1, \cdots, \lambda_k) : \lambda_i \in \mathcal{D}_0 \setminus \mathcal{D}, \sum_{i=1}^{k} \lambda_i = s\}$.

Let the points in $\mathcal{D}_0 \setminus \mathcal{D}$, taken in uniform random order, form the vector $\underline{\mathbf{z}} = (z_1, \cdots, z_k)$. Then we can condition the vector of points in $\mathcal{D}_0$ to have its last k co-ordinates equal to $\underline{\mathbf{z}}$, to obtain the following formula for the conditional density of the vector of points in $\mathcal{D}$ at $(\zeta_1, \cdots, \zeta_N) \in \Sigma$ (with respect to the Lebesgue measure on Σ):

$$f(\zeta_1, \zeta_2, \cdots, \zeta_N) = \frac{f_0(\zeta_1, \zeta_2, \cdots, \zeta_N, z_1, \cdots, z_k)}{\int_U f_0(s - (\sum_{j=2}^{N} w_j), w_2, \cdots, w_N, z_1, \cdots, z_k) dw_2 \cdots dw_N}. \tag{2.1}$$

It is clear that for a.e. $\underline{\mathbf{z}} \in V$, we have f is strictly positive a.e. with respect to Lebesgue measure on Σ, because the same is true of f_0 on Σ_0.

2.5 Rigidity Phenomena

We begin by giving a precise definition of rigidity. Recall the general setup in Section 2.3.

Definition 1. *A measurable function* $f_{\text{in}} : \mathcal{S}_{\text{in}} \to \mathbb{C}$ *is said to be* ***rigid*** *with respect to the point process* X *on* $\mathcal{S}$ *if there is a measurable function* $f_{\text{out}} : \mathcal{S}_{\text{out}} \to \mathbb{C}$ *such that a.s. we have* $f_{\text{in}}(X_{\text{in}}) = f_{\text{out}}(X_{\text{out}})$.

In this section, we prove that the number of points in $\mathcal{D}$ in the case of the Ginibre ensemble and the number as well as the sum of the points in $\mathcal{D}$ for the GAF zero process are rigid. In fact, we will state some general conditions that ensure such rigid behaviour, and then show that the Ginibre and the GAF satisfy the relevant conditions.

We will use linear statistics of point processes as the main tool that will enable us to obtain the rigidity results.

Definition 2. *Let* φ *be a compactly supported continuous function on* $\mathbb{C}$. *The* ***linear statistic*** *corresponding to* φ *is the random variable* $\int \varphi \, d[\pi] = \sum_{z\in\pi} \varphi(z)$.

By a C_c^k function on a Euclidean space $\mathcal{E}$ we denote the space of compactly supported C^k functions on $\mathcal{E}$.

We can now state:

Theorem 2.5.1. *Let* π *be a point process on* $\mathbb{C}$ *whose first intensity is absolutely continuous with respect to the Lebesgue measure, and let* $\mathcal{D}$ *be a bounded open set whose boundary has zero Lebesgue measure. Let* φ *be a continuous function on* $\mathbb{C}$. *Suppose for any* $1 > \varepsilon > 0$, *we have a* C_c^2 *function* Φ^ε *such that* $\Phi^\varepsilon = \varphi$ *on* $\mathcal{D}$, *and* $\text{Var}\left(\int_{\mathbb{C}} \Phi^\varepsilon d[\pi]\right) < \varepsilon$. *Then* $\int_{\mathcal{D}} \varphi d[\pi]$ *is rigid with respect to* π.

Proof. Consider the sequence of C_c^2 functions $\Phi^{2^{-n}}, n \geq 1$. Note that $\mathbb{E}\left[\int_{\mathbb{C}} \Phi^{2^{-n}} d[\pi]\right] = \int_{\mathbb{C}} \Phi^{2^{-n}} \rho_1 d\mathcal{L}$ where $\rho_1(z)$ is the one point intensity function of π. By Chebyshev's inequality, it is clear that

$$\mathbb{P}\left(\left|\int_{\mathbb{C}} \Phi^{2^{-n}} d[\pi] - \mathbb{E}\left[\int_{\mathbb{C}} \Phi^{2^{-n}} d[\pi]\right]\right| > 2^{-n/4}\right) \leq 2^{-n/2}.$$

The Borel Cantelli lemma implies that with probability 1, as $n \to \infty$ we have

$$\left|\int_{\mathbb{C}} \Phi^{2^{-n}} d[\pi] - \mathbb{E}\left[\int_{\mathbb{C}} \Phi^{2^{-n}} d[\pi]\right]\right| \to 0.$$

But

$$\int_{\mathbb{C}} \Phi^{2^{-n}} d[\pi] = \int_{\mathcal{D}} \Phi^{2^{-n}} d[\pi] + \int_{\mathcal{D}^c} \Phi^{2^{-n}} d[\pi].$$

Thus we have, as $n \to \infty$

$$\left|\int_{\mathcal{D}} \Phi^{2^{-n}} d[\pi] + \int_{\mathcal{D}^c} \Phi^{2^{-n}} d[\pi] - \int_{\mathbb{C}} \Phi^{2^{-n}} \rho_1 d\mathcal{L}\right| \to 0. \tag{2.2}$$

If we know π_{out}, we can compute $\int_{\mathcal{D}^c} \Phi^{2^{-n}} d[\pi]$ exactly, also ρ_1 is known explicitly; in case of a translation invariant point process π it is, in fact, a constant $c(\pi)$. Hence, from the limit in (2.2), a.s. we can obtain $\int_{\mathcal{D}} \Phi^{2^{-n}} d[\pi] = \int_{\mathcal{D}} \varphi d[\pi]$ as the limit

$$\lim_{n\to\infty} \int_{\mathbb{C}} \left(\Phi^{2^{-n}} \rho_1 d\mathcal{L} - \int_{\mathcal{D}^c} \Phi^{2^{-n}} d[\pi] \right).$$

■

We now use Theorem 2.5.1 to establish Theorems 2.1.1 and 2.1.3.

Proof of Theorems 2.1.1 and 2.1.3. We have already seen in Section 2.4 that it suffices to take $\mathcal{D}$ to be a disk. By translation invariance of $\mathcal{F}$ and $\mathcal{G}$, we can assume that $\mathcal{D}$ is centred at the origin. We intend to construct functions Φ^ε as in Theorem 2.5.1.

Let $r_0 =$ Radius $(\mathcal{D})$. Fix $\varepsilon > 0$.

We begin with a continuous function $\widetilde{\Psi}$ on $\mathbb{R}_+ \cup \{0\}$ such that $\widetilde{\Psi}(r) = 1$ for $0 \le r \le r_0$, $\widetilde{\Psi}'(r) = -\varepsilon/r$ and $\widetilde{\Psi}''(r) = \varepsilon/r^2$ for $r_0 \le r \le r_0 \exp(1/\varepsilon)$, and $\widetilde{\Psi}(r) = 0$ for $r \ge r_0 \exp(1/\varepsilon)$. This can be obtained, e.g., by solving the relevant boundary value problem between r_0 and $r_0 \exp(1/\varepsilon)$, and extending to $\mathbb{R}_+ \cup \{0\}$ in the obvious manner. We then smooth the function $\widetilde{\Psi}$ at r_0 and $r_0 \exp(1/\varepsilon)$ such that the resulting function Ψ_1 is C^2 on the positive reals, and satisfies $|\Psi_1'(r)| \le \varepsilon/r$ and $|\Psi''(r)| \le \varepsilon/r^2$ for all $r > 0$. Finally, we define the radial C_c^2 function Ψ on $\mathbb{C}$ as $\Psi(z) = \Psi_1(|z|)$.

For $\mathcal{G}$, we know (see [RV07] Theorem 11) that there exists a constant $C_1 > 0$ such that for every radial C_c^2 function Ψ we have

$$\text{Var}\left(\int_{\mathbb{C}} \Psi \, d[\mathcal{G}] \right) \le C_1 \int_{\mathbb{C}} \|\nabla \Psi(z)\|_2^2 \, d\mathcal{L}(z).$$

But from the definition of Ψ it is clear that $\int_{\mathbb{C}} \|\nabla \Psi(z)\|_2^2 \, d\mathcal{L}(z) \le C_2\varepsilon$.We apply Theorem 2.5.1 with $\varphi \equiv 1$, and choose $\Phi^{C_1 C_2 \varepsilon} = \Psi$ as defined above; recall that $\Psi \equiv 1$ on $\mathcal{D}$.

For $\mathcal{F}$, we know (see [NS11] Theorem 1.1) that there exists a constant $C_3 > 0$ such that every C_c^2 function ϑ satisfies

$$\text{Var}\left(\int_{\mathbb{C}} \vartheta \, d[\mathcal{F}] \right) \le C_3 \int_{\mathbb{C}} \|\Delta \vartheta(z)\|_2^2 \, d\mathcal{L}(z).$$

For the rigidity of the number of points of $\mathcal{F}$ in $\mathcal{D}$ we make exactly the same choice as we did for $\mathcal{G}$, and note that $\int_{\mathbb{C}} \|\Delta \Psi(z)\|_2^2 \, d\mathcal{L}(z) \le C_4 \varepsilon^2$. For the rigidity of the sum of points of $\mathcal{F}$ in $\mathcal{D}$ we intend to apply Theorem 2.5.1 with $\varphi(z) = z$. We consider the function $\theta(z) = z\Psi(z)$, Ψ as before. Observe that $\Delta\theta(z) = 4\frac{\partial \Psi}{\partial \bar{z}}(z) + z\Delta\Psi(z)$. Using this, for $\varepsilon < 1$, we get $\int_{\mathbb{C}} |\Delta(z\Psi(z))|^2 \, d\mathcal{L}(z) \le C_5 \varepsilon$. It remains to note that for $z \in \mathcal{D}$ we have $\Psi(z) = 1$ and $\theta(z) = z$. ■

2.6 Tolerance: Limits of Conditional Measures

The tolerance properties are established for both models by obtaining explicit bounds on conditional probability measures for finite approximations (finite matrices in case of Ginibre and polynomials for GAF) and then passing to the limit. In this Section, we state and prove some general conditions (in the context of the abstract setup considered in Section 2.3), which will enable us to make the transition from the finite ensembles to the infinite one.

We start with the following general proposition:

Proposition 2.6.1. *Let Γ be second countable topological space. Let Σ be a countable basis of open sets in Γ and let $\mathcal{A} := \{\cup_{i=1}^{k}\sigma_i : \sigma_i \in \Sigma, k \geq 1\}$. Let $c > 0$. To verify that two non-negative regular Borel measures μ_1 and μ_2 on Γ satisfy $\mu_1(B) \leq c\mu_2(B)$ for all Borel sets B in Γ, it suffices to verify the inequality for all sets in $\mathcal{A}$.*

Proof. Any open set $U \subset \Gamma$ is a countable union $\bigcup_{i=1}^{\infty} \sigma_i, \sigma_i \in \Sigma$ because the sets in Σ form a basis for the topology on Γ. If we have $\mu_1(\bigcup_{i=1}^{n} \sigma_i) \leq c\mu_2(\bigcup_{i=1}^{n} \sigma_i)$, then we can let $n \to \infty$ to obtain $\mu_1(U) \leq c\mu_2(U)$. Once we have the inequality for all open sets U, we can extend it to all Borel sets, because for a regular Borel measure μ and for any Borel set $B \subset \Gamma$, we have $\mu(B) = \inf_{B\subset U}\mu(U)$ where the infimum is taken over all open sets U containing B. ∎

In this Section, we will work in the setup of Section 2.3, specifying $\mathcal{D}$ to be an open ball, and requiring that the first intensity of our point process X is absolutely continuous with respect to the Lebesgue measure on $\mathcal{E}$. We further assume that X exhibits rigidity of the number of points. In other words, there is a measurable function $N : \mathcal{S}_{\text{out}} \to \mathbb{N} \cup \{0\}$ such that a.s. we have

$$\text{Number of points in } X_{\text{in}} = N(X_{\text{out}}).$$

In such a situation, we can identify X_{in} (by taking the points in uniform random order) with a random vector $\underline{\zeta}$ taking values in $\mathcal{D}^{N(X_{\text{out}})}$. Studying the conditional distribution $\varrho(X_{\text{out}}, \cdot)$ of X_{in} given X_{out} is then the same as studying the conditional distribution of this random vector given X_{out}. We will denote the latter distribution by $\rho(X_{\text{out}}, \cdot)$. Note that it is supported on $\mathcal{D}^{N(X_{\text{out}})}$ (see Section 2.1 for details).

For $m > 0$, let $\mathfrak{W}_{\text{in}}^{m}$ denote the countable basis for the topology on $\mathcal{D}^m$ formed by open balls contained in $\mathcal{D}^m$ and having rational centres and rational radii. We define the collection of sets $\mathfrak{A}^m := \{\bigcup_{i=1}^{k} A_i : A_i \in \mathfrak{W}_{\text{in}}^{m}, k \geq 1\}$.

Fix an integer $n \geq 0$, a closed annulus $B \subset \mathcal{D}^c$ whose centre is at the origin and which has a rational inradius and a rational outradius, and a collection of n disjoint open balls B_i with rational radii and centres having rational co-ordinates such that $\{B_i \cap \mathcal{D}^c\}_{i=1}^{n} \subset B$. Let $\Phi(n, B, B_1, \cdots, B_n)$ be the Borel subset of $\mathcal{S}_{\text{out}}$ defined as follows:

$$\Phi(n, B, B_1, \cdots, B_n) = \{\Upsilon \in \mathcal{S}_{\text{out}} : |\Upsilon \cap B| = n, |\Upsilon \cap \mathcal{B}_i| = 1\}.$$

Then the countable collection $\Sigma_{\text{out}} = \{\Phi(n, B, B_1, \cdots, B_n) : n, B, B_i \text{ as above }\}$ is a basis for the topology of $\mathcal{S}_{\text{out}}$. Define the collection of sets $\mathcal{B} := \{\bigcup_{i=1}^{k} \Phi_i : \Phi_i \in \Sigma_{\text{out}}, k \geq 1\}$.

We will denote by Ω^m the event that $|X_{\text{in}}| = m$, and by Ω^m_n we will denote the event $|X^n_{\text{in}}| = m$.

Definition 3. *Let p and q be indices (which take values in potentially infinite abstract sets), and $\alpha(p,q)$ and $\beta(p,q)$ be non-negative functions of these indices. We write $\alpha(p,q) \asymp_q \beta(p,q)$ if there exist positive numbers $k_1(q), k_2(q)$ such that*

$$k_1(q)\alpha(p,q) \le \beta(p,q) \le k_2(q)\alpha(p,q) \text{ for all } p, q.$$

The main point is that k_1, k_2 in the above inequalities are uniform in p, that is, all the indices in question other than q.

We will also use the notation introduced in Section 2.3. We will define an "exhausting" sequence of events as:

Definition 4. *A sequence of events $\{\Omega(j)\}_{j\ge1}$ is said to exhaust another event Ω if $\Omega(j) \subset \Omega(j+1) \subset \Omega$ for all j and $\mathbb{P}(\Omega \setminus \Omega(j)) \to 0$ as $j \to \infty$.*

Let $\mathcal{M}(\mathcal{D}^m)$ denote the space of all probability measures on $\mathcal{D}^m$. For two random variables U and V defined on the same probability space, we say that U is measurable with respect to V if U is measurable with respect to the sigma algebra generated by V. Finally, recall the definition of $\mathfrak{A}^m$ and $\mathcal{B}$ from the beginning of this section.

Now we are ready to state the following important technical reduction:

Theorem 2.6.2. *Let $m \ge 0$ be such that $\mathbb{P}(\Omega^m) > 0$. Suppose that:*

(a) There is a map $\nu : \mathcal{S}_{\text{out}} \to \mathcal{M}(\mathcal{D}^m)$ such that for each Borel set $A \subset \mathcal{D}^m$, the $\nu(\cdot, A)$ is a measurable real valued function.

(b) For each fixed j we have a sequence $\{n_k\}_{k\ge1}$ (which might depend on j) and corresponding events $\Omega_{n_k}(j)$ such that:

(i) $\Omega_{n_k}(j) \subset \Omega^m_{n_k}$.

(ii) $\Omega(j) := \underline{\lim}_{k\to\infty} \Omega_{n_k}(j)$ exhaust Ω^m as $j \uparrow \infty$.

(iii) For all $A \in \mathfrak{A}^m$ and $B \in \mathcal{B}$ we have

$$\mathbb{P}[(X^{n_k}_{in} \in A) \cap (X^{n_k}_{\text{out}} \in B) \cap \Omega_{n_k}(j)] \asymp_j \int_{(X^{n_k}_{\text{out}})^{-1}(B)\cap\Omega_{n_k}(j)} \nu(X_{\text{out}}(\xi), A) d\mathbb{P}(\xi) + \vartheta(k; j, A, B). \tag{2.3}$$

where $\lim_{k\to\infty} \vartheta(k; j, A, B) = 0$ for each fixed j, A and B .

Then a.s. on the event Ω^m we have

$$\rho(X_{\text{out}}, \cdot) \equiv \nu(X_{\text{out}}, \cdot) \tag{2.4}$$

We defer the proof of Theorem 2.6.2 to Section 2.18.

We conclude this section with the following simple observation:

Remark 2.6.1. *If we have Theorem 2.6.2 for all $m \ge 0$ then we can conclude that (2.4) holds a.e. $\xi \in \Xi$.*

2.7 Tolerance of the Ginibre Ensemble for a disk

In this section we obtain several estimates necessary to prove Theorem 2.1.2 in the case where $\mathcal{D}$ is a disk.

Remark 2.7.1. *By the translation invariance of the Ginibre ensemble, we can take $\mathcal{D}$ to be centred at the origin.*

Notation 1. *For a set $\Lambda \subset \mathbb{C}$ and $R > 0$, we denote $R \cdot \Lambda = \{Rz : z \in \Lambda\}$.*

2.8 Matrix Approximations

Let $\delta \in (0,1)$. We consider the event $\Omega_n^{m,\delta}$ which entails that $\mathcal{G}_n$ has exactly m points inside $\mathcal{D}$, and there is at least a δ separation between the sets $\partial\mathcal{D}$ and $(\mathcal{G}_n)_{\text{out}}$. The analogous event for $\mathcal{G}$ will be denoted by $\Omega^{m,\delta}$. Notice that $\Omega^{m,\delta}$ has positive probability (which is bounded away from 0 uniformly in δ for δ small enough), so by the convergence of $\mathcal{G}_n$-s to $\mathcal{G}$, we obtain that $\Omega_n^{m,\delta}$-s have positive probability that is uniformly bounded away from 0 for large enough n. We denote the points of $\mathcal{G}_n$ inside $\mathcal{D}$ (in uniform random order) by $\underline{\zeta} = (\zeta_1, \cdots, \zeta_m)$ and those outside $\mathcal{D}$ (in uniform random order) by $\underline{\omega} = (\omega_1, \omega_2, \cdots, \omega_{n-m})$. Following the notation introduced in Chapter 1, for a vector $(\underline{\zeta}, \underline{\omega})$ as above, we denote $\Upsilon_{\text{in}} = \{\zeta_i\}_{i=1}^m$, $\Upsilon_{\text{out}} = \{\omega_j\}_{j=1}^{n-m}$, and $\Upsilon = \Upsilon_{\text{in}} \cup \Upsilon_{\text{out}}$. For a vector $\underline{\gamma} = (\gamma_1, \cdots, \gamma_N)$ in $\mathbb{C}^N$, we denote by $\Delta(\underline{\gamma})$ the Vandermonde determinant $\prod_{i<j}(\gamma_i - \gamma_j)$. For two vectors $\underline{\gamma}_1, \underline{\gamma}_2$ we set $\Delta(\underline{\gamma}_1, \underline{\gamma}_2) = \Delta(\underline{\gamma})$ where $\underline{\gamma} = (\underline{\gamma}_1, \underline{\gamma}_2)$ denotes the concatenated vector.

Then the conditional distribution $\rho(\Upsilon_{out}, \underline{\zeta})$ of $\underline{\zeta}$ given Υ_{out} has the density

$$\rho^n_{\underline{\omega}}(\underline{\zeta}) = C(\underline{\omega}) \left|\Delta(\underline{\zeta}, \underline{\omega})\right|^2 \exp\left(-\sum_{k=1}^m |\zeta_k|^2\right) \tag{2.5}$$

with respect to the Lebesgue measure on $\mathcal{D}^m$, where $C(\underline{\omega})$ is the normalizing constant (which depends on $\underline{\omega}$). Let $(\underline{\zeta}, \underline{\omega})$ and $(\underline{\zeta}', \underline{\omega})$ correspond to two configurations such that the event $\Omega_n^{m,\delta}$ occurs in both cases. Then the ratio of the conditional densities at these two points is given by

$$\frac{\rho^n_{\underline{\omega}}(\underline{\zeta}')}{\rho^n_{\underline{\omega}}(\underline{\zeta})} = \frac{|\Delta(\underline{\zeta}', \underline{\omega})|^2}{|\Delta(\underline{\zeta}, \underline{\omega})|^2} \frac{\exp(-\sum_{k=1}^m |\zeta_k'|^2)}{\exp(-\sum_{k=1}^m |\zeta_k|^2)}. \tag{2.6}$$

Clearly, $\exp\left(-\sum_{k=1}^m |\zeta_k'|^2\right) \Big/ \exp\left(-\sum_{k=1}^m |\zeta_k|^2\right)$ is bounded above and below by constants which are functions of m and $\mathcal{D}$.

To study the ratio of the Vandermonde determinants, we define

$$\Gamma(\underline{\zeta}, \underline{\omega}) = \prod_{1 \le i \le m, 1 \le j \le n-m} (\zeta_i - \omega_j).$$

Then we have

$$\frac{|\Delta(\underline{\zeta}',\underline{\omega})|^2}{|\Delta(\underline{\zeta},\underline{\omega})|^2} = \frac{|\Delta(\underline{\zeta}')|^2}{|\Delta(\underline{\zeta})|^2}\frac{|\Gamma(\underline{\zeta}',\underline{\omega})|^2}{|\Gamma(\underline{\zeta},\underline{\omega})|^2}. \tag{2.7}$$

In order to bound $\dfrac{|\Gamma(\underline{\zeta}',\underline{\omega})|^2}{|\Gamma(\underline{\zeta},\underline{\omega})|^2}$ from above and below uniformly in $\underline{\zeta},\underline{\zeta}' \in \mathcal{D}^m$, it suffices to bound $\dfrac{|\Gamma(\underline{\zeta},\underline{\omega})|}{|\Gamma(\underline{0},\underline{\omega})|}$ from above and below uniformly in $\underline{\zeta} \in \mathcal{D}^m$. Here $\underline{0}$ is the vector in $\mathcal{D}^m$ all whose co-ordinates are 0-s. We observe that

$$\frac{|\Gamma(\underline{\zeta},\underline{\omega})|}{|\Gamma(\underline{0},\underline{\omega})|} = \prod_{i=1}^{m}\left(\prod_{j=1}^{n-m}\left|\frac{\zeta_i-\omega_j}{\omega_j}\right|\right).$$

Since m is fixed, it suffices to bound $\displaystyle\prod_{j=1}^{n-m}\left|\frac{\zeta_0-\omega_j}{\omega_j}\right|$ from above and below uniformly in $\zeta_0 \in \mathcal{D}$. To this end we prove

Proposition 2.8.1. *Let $\zeta_0 \in \mathcal{D}$ and $\omega_1,\cdots,\omega_{n-m}$ be such that $|\omega_j| > r+\delta$ for all j where r is the radius of $\mathcal{D}$. Then we have*

$$\left|\log\left(\prod_{j=1}^{n-m}\left|\frac{\zeta_0-\omega_j}{\omega_j}\right|\right)\right| \le K_1(\mathcal{D})\left|\sum_{j=1}^{n-m}\frac{1}{\omega_j}\right| + K_2(\mathcal{D})\left|\sum_{j=1}^{n-m}\frac{1}{\omega_j^2}\right| + K_3(\mathcal{D},\delta)\left(\sum_{j=1}^{n-m}\frac{1}{|\omega_j|^3}\right).$$

Here $K_1(\mathcal{D})$, $K_2(\mathcal{D})$ and $K_3(\mathcal{D},\delta)$ are constants depending on $\mathcal{D}$ and δ.

Proof. We begin with $\log\left|\frac{\zeta_0-\omega_j}{\omega_j}\right| = \log\left|1-\frac{\zeta_0}{\omega_j}\right|$. Due to the δ-separation between $\partial\mathcal{D}$ and ω, the ratio $\theta_j := \frac{\zeta_0}{\omega_j}$ satisfies $|\theta_j| \le \frac{r}{r+\delta} < 1$. Let log be the branch of the complex logarithm given by the power series development $\log(1-z) = -\left(\sum_{k=1}^{\infty}\frac{z^k}{k}\right)$ for $|z|<1$. Then we have $\log|1-\theta_j| = \Re\log(1-\theta_j) = -\Re(\theta_j) - \frac{1}{2}\Re(\theta_j^2) + h(\theta_j)$ where $|h(\theta_j)| \le K_3'(\mathcal{D},\delta)|\theta_j|^3$, where K_3' is a constant depending on $\mathcal{D}$ and δ. Hence,

$$\left|\log\left(\prod_{j=1}^{n-m}\left|\frac{\zeta_0-\omega_j}{\omega_j}\right|\right)\right| = \left|\sum_{j=1}^{n-m}\log\left(\left|\frac{\zeta_0-\omega_j}{\omega_j}\right|\right)\right| = \left|-\Re\left(\sum_{j=1}^{n-m}\theta_j\right) - \frac{1}{2}\Re\left(\sum_{j=1}^{n-m}\theta_j^2\right) + \sum_{j=1}^{n-m}h(\theta_j)\right|.$$

Recall that $\theta_j = \frac{\zeta_0}{\omega_j}$ and $|\zeta_0| \le r$ and $|h(\theta_j)| \le K_3'(\mathcal{D},\delta)|\theta_j|^3$. The triangle inequality applied to the above gives us the statement of the Proposition with $K_1(\mathcal{D}) = r$, $K_2(\mathcal{D}) = \frac{1}{2}r^2$ and $K_3(\mathcal{D},\delta) = K_3'(\mathcal{D},\delta)r^3$. ■

As a result, we have

Proposition 2.8.2. *On $\Omega_n^{m,\delta}$, we have a constant $K(\mathcal{D},\delta) > 0$ such that*

$$\exp\Big(- 4mK(\mathcal{D},\delta)\boldsymbol{X}_n\Big)\frac{|\Delta(\underline{\zeta}')|^2}{|\Delta(\underline{\zeta})|^2} \le \frac{\rho^n_{\underline{\omega}}(\underline{\zeta}')}{\rho^n_{\underline{\omega}}(\underline{\zeta})} \le \exp\Big(4mK(\mathcal{D},\delta)\boldsymbol{X}_n\Big)\frac{|\Delta(\underline{\zeta}')|^2}{|\Delta(\underline{\zeta})|^2},$$

where $\boldsymbol{X}_n = \left|\sum_{\omega\in\mathcal{G}_n\cap\mathcal{D}^c}\frac{1}{\omega}\right| + \left|\sum_{\omega\in\mathcal{G}_n\cap\mathcal{D}^c}\frac{1}{\omega^2}\right| + \left(\sum_{\omega\in\mathcal{G}_n\cap\mathcal{D}^c}\frac{1}{|\omega|^3}\right)$ *and* $\mathbb{E}[\boldsymbol{X}_n] \le c_1(\mathcal{D},m) < \infty$.

Proof. Clearly, it suffices to bound $|\Delta(\underline{\zeta}',\underline{\omega})|^2/|\Delta(\underline{\zeta},\underline{\omega})|^2$ from above and below.
From Proposition 2.8.1, on $\Omega_n^{m,\delta}$ we have $\left|\log\left(\prod_{j=1}^{n-m}\left|\frac{\zeta_0-\omega_j}{\omega_j}\right|\right)\right| \le K(\mathcal{D},\delta)\boldsymbol{X}_n$ for any $\zeta_0\in\mathcal{D}$, where $K(\mathcal{D},\delta) = \max\{K_1,K_2,K_3\}$. Considering this for each $\zeta_i, i = 1,\cdots,m$, exponentiating and taking product over $i = 1,\cdots,m$, we get

$$\exp\Big(- mK(\mathcal{D},\delta)\boldsymbol{X}_n\Big) \le \frac{|\Gamma(\underline{\zeta},\underline{\omega})|}{|\Gamma(\underline{0},\underline{\omega})|} \le \exp\Big(mK(\mathcal{D},\delta)\boldsymbol{X}_n\Big).$$

The same estimate holds for ζ'. Since $\frac{|\Gamma(\underline{\zeta}',\underline{\omega})|}{|\Gamma(\underline{\zeta},\underline{\omega})|} = \frac{|\Gamma(\underline{\zeta}',\underline{\omega})|}{|\Gamma(\underline{0},\underline{\omega})|}\Big/\frac{|\Gamma(\underline{\zeta},\underline{\omega})|}{|\Gamma(\underline{0},\omega)|}$, we have

$$\exp\Big(- 2mK(\mathcal{D},\delta)\boldsymbol{X}_n\Big) \le \frac{|\Gamma(\underline{\zeta}',\underline{\omega})|}{|\Gamma(\underline{\zeta},\underline{\omega})|} \le \exp\Big(2mK(\mathcal{D},\delta)\boldsymbol{X}_n\Big).$$

In view of (2.6) and (2.7), this leads to the desired bound $\frac{\rho^n_{\underline{\omega}}(\zeta')}{\rho^n_{\underline{\omega}}(\underline{\zeta})}$. In Section 2.9, we will see that each of the three random sums defining $\boldsymbol{X}_n$ has finite expectation. Moreover, those expectations are uniformly bounded by quantities depending only on $\mathcal{D}$ and m. This yields the statement $\mathbb{E}[\boldsymbol{X}_n] \le c_1(\mathcal{D},m) < \infty$. ■

Corollary 2.8.3. *Given $M > 0$, we can replace $\boldsymbol{X}_n$ in Proposition 2.8.2 by a uniform bound M except on an event of probability less than $c_1(m,\mathcal{D})/M$.*

2.9 Estimates for Inverse Powers

Our aim in this section is to estimate the sums of inverse powers of the points in $\mathcal{G}$ and $\mathcal{G}_n$ outside a disk containing the origin. To this end, we first discuss certain estimates on the variance of linear statistics, which are uniform in n. Let $B(0;r)$ denote the disk of radius r centred at the origin.

Proposition 2.9.1. *Let φ be a compactly supported Lipschitz function, supported inside the disk $B(0;r)$ with Lipschitz constant $\kappa(\varphi)$. Let $\varphi_R(z) := \varphi(z/R)$. Then* $\mathrm{Var}\left(\int\varphi_R(z)\,d[\mathcal{G}_n](z)\right) \le$ *$C(\varphi)$, where $C(\varphi)$ is a constant that is independent of n. The same conclusion holds for $\mathcal{G}$ in place of $\mathcal{G}_n$.*

To prove Proposition 2.9.1, we will make use of a general fact about determinantal point processes:

Lemma 2.9.2. *Let* Π *be a determinantal point process with Hermitian kernel* K*. Let* K *be a reproducing kernel with respect to its background measure* γ*, which means* $K(x,y) = \int K(x,z)K(z,y)\,d\gamma(z)$ *for all* x,y*. Let* φ, ψ *be compactly supported continuous functions.* $\text{Cov}\left[\int \varphi\, d[\Pi], \int \psi\, d[\Pi]\right] = \frac{1}{2}\iint (\varphi(z)-\varphi(w))\overline{(\psi(z)-\psi(w))}|K(z,w)|^2\,d\gamma(z)\,d\gamma(w)$.

Proof. Expanding the two point correlation in its determinantal formula gives the covariance as

$$\int \varphi(z)\overline{\psi(z)}K(z,z)\,d\gamma(z) - \int\int \varphi(z)\overline{\psi(w)}|K(z,w)|^2\,d\gamma(z)\,d\gamma(w).$$

Using $K(z,z) = \int K(z,w)K(w,z)\,d\gamma(w)$ and $K(z,w) = \overline{K(w,z)}$, elementary calculations give us the final result. ∎

Proof of Proposition 2.9.1. We give the proof when $r = 1$, from here the general case is obtained by scaling. In what follows we deal with $\mathcal{G}_n$, the result for $\mathcal{G}$ follows, for instance, from taking limits as $n \to \infty$ for the result for $\mathcal{G}_n$.

Using lemma 2.9.2, we have

$$\text{Var}\left(\int \varphi_R(z)\,d[\mathcal{G}_n](z)\right) = \frac{1}{2}\int\int |\varphi_R(z)-\varphi_R(w)|^2|K_n(z,w)|^2\,d\gamma(z)\,d\gamma(w)$$

where γ is the standard complex Gaussian measure. Now,

$$|\varphi_R(z)-\varphi_R(w)|^2 = |\varphi(z/R)-\varphi(w/R)|^2 \le \frac{1}{R^2}\kappa(\varphi)^2|z-w|^2.$$

Therefore, it suffices to bound the integral $\int_{A(R)}|z-w|^2|K_n(z,w)|^2\,d\gamma(z)\,d\gamma(w)$ on the set

$$A(R) := \{(z,w) : |z| \wedge |w| \le R\}$$

because outside $A(R)$, we have $\varphi_R(z) = \varphi_R(w) = 0$. We begin with

$$\int_{A(R)} |z-w|^2|K_n(z,w)|^2\,d\gamma(z)\,d\gamma(w) \le \int_{A_1(R)} |z-w|^2|K_n(z,w)|^2\,d\gamma(z)\,d\gamma(w) + \int_{A_2(R)} |z-w|^2|K_n(z,w)|^2\,d\gamma(z)\,d\gamma(w)$$

where

$$A_1(R) = \{|z| \le 2R, |w| \le 2R\} \text{ and } A_2(R) = \{|z| \le R, |w| \ge 2R\} \cup \{|w| \le R, |z| \ge 2R\}.$$

We first address the case of $A_2(R)$. By symmetry, it suffices to bound the integral over the region $\{|z| \le R, |w| \ge 2R\}$. In this region, $||z|-|w|| \ge R$, and $|K_n(z,w)|^2 e^{-|z|^2-|w|^2} \le e^{-||z|-|w||^2}$. The integral is bounded from above by

$$\frac{1}{\pi}\int_{|z|\le R}\left(\int_{|w|\ge 2R} (|z|+|w|)^2 e^{-(|z|-|w|)^2}\,d\mathcal{L}(w)\right)\,d\mathcal{L}(z).$$

It is not hard to see that the inner integral is $O(e^{-\frac{1}{2}R^2})$, where the constant in O is universal. Integrating over $|z| \leq R$ gives another factor of R^2, so the total contribution is $o(1)$ as $R \to \infty$.

For the integral over $A_1(R)$, we proceed as

$$\begin{aligned}&\int |z-w|^2 |K_n(z,w)|^2 \, d\gamma(z) \\ &= \int \left(|z|^2 - z\bar{w} - \bar{z}w + |w|^2\right) \left(\sum_{k=0}^{n} (z\bar{w})^k/k!\right) \left(\sum_{k=0}^{n} (\bar{z}w)^k/k!\right) e^{-|z|^2-|w|^2} \, d\mathcal{L}(z) \, d\mathcal{L}(w).\end{aligned}$$

Now, we integrate the $|z-w|^2$ part term by term. Due to radial symmetry, only some specific terms from $|K_n(z,w)|^2$ contribute. For example, when we integrate the $|z|^2$ term in $|z-w|^2$, only the $\frac{(z\bar{w})^k}{k!}\frac{(\bar{z}w)^k}{k!}$,$0 \leq k \leq n$ terms in the expanded expression for $|K_n(z,w)|^2$ contribute. When we integrate $z\bar{w}$, only the $\frac{(z\bar{w})^k}{k!}\frac{(\bar{z}w)^{k+1}}{(k+1)!}$,$0 \leq k \leq n-1$ terms provide non-zero contributions.

Due to symmetry between z and w, it is enough to bound the contribution from $(|z|^2 - z\overline{w})$ by $O(R^2)$ in order to obtain Proposition 2.9.1.

• $|z|^2$ term:

Let us denote $|z|^2$ by x and $|w|^2$ by y. Then the contribution coming from

$$\frac{(z\bar{w})^j}{j!}\frac{(\bar{z}w)^j}{j!}$$

as discussed above can be written as (a constant times)

$$\int_0^{4R^2}\int_0^{4R^2} \frac{x^{j+1}}{j!}e^{-x}\frac{y^j}{j!}e^{-y}dxdy.$$

So, the total contribution due to all such terms, ranging from $k = 0, \cdots, n$ is

$$\sum_{j=0}^{n}\left(\int_0^{4R^2}\frac{x^{j+1}}{j!}e^{-x}dx\right)\left(\int_0^{4R^2}\frac{y^j}{j!}e^{-y}dy\right).$$

• $z\bar{w}$ term:

As above, the contribution coming from the

$$\frac{(z\bar{w})^k}{k!}\frac{(\bar{z}w)^{k+1}}{(k+1)!}$$

is given by

$$\int_0^{4R^2}\int_0^{4R^2} \frac{x^{j+1}}{j!}e^{-x}\frac{y^{j+1}}{(j+1)!}e^{-y}dxdy.$$

Therefore the total contribution from $0 \le k \le n-1$ is

$$\sum_{j=0}^{n-1} \left(\int_0^{4R^2} \frac{x^{j+1}}{j!} e^{-x} dx \right) \left(\int_0^{4R^2} \frac{y^{j+1}}{(j+1)!} e^{-y} dy \right).$$

We interpret $\frac{x^j}{j!} e^{-x} dx$ as a gamma density, the corresponding random variable being denoted by Γ_{j+1}.

The contribution due to the $|z|^2$ term is $\sum_{j=0}^{n} \mathbb{E}[\Gamma_{j+1} 1_{(\Gamma_{j+1} \le 4R^2)}] \mathbb{P}[\Gamma_{j+1} \le 4R^2]$ and that due to the $z\bar{w}$ term is $\sum_{j=0}^{n-1} \mathbb{E}[\Gamma_{j+1} 1_{(\Gamma_{j+1} \le 4R^2)}] \mathbb{P}[\Gamma_{j+2} \le 4R^2]$.

The difference between the above two terms can be written as:

$$\mathbb{E}[\Gamma_{n+1} 1_{(\Gamma_{n+1} \le 4R^2)}] \mathbb{P}[\Gamma_{n+1} \le 4R^2] + \sum_{j=1}^{n} \mathbb{E}[\Gamma_j 1(\Gamma_j \le 4R^2)] \Big(\mathbb{P}[\Gamma_j \le 4R^2] - \mathbb{P}[\Gamma_{j+1} \le 4R^2] \Big). \tag{2.8}$$

All the expectations in the above are $\le 4R^2$, and $\mathbb{P}[\Gamma_j \le 4R^2] \ge \mathbb{P}[\Gamma_{j+1} \le 4R^2]$ because Γ_{j+1} stochastically dominates Γ_j. Therefore the absolute value of (2.8), by triangle inequality and telescpoing sums, is $\le 4R^2 \mathbb{P}[\Gamma_1 \le 4R^2] \le 4R^2$.

Combining all of these, we see that $\text{Var}\left(\int \varphi_R(z)\, d[\mathcal{G}_n](z)\right)$ is bounded by some constant $C(\varphi)$. ∎

Let $r_0 = \text{radius}\ (\mathcal{D})$. Let φ be a non-negative radial C_c^∞ function supported on $[r_0, 3r_0]$ such that $\varphi = 1$ on $[\frac{3}{2}r_0, 2r_0]$ and $\varphi(r_0 + r) = 1 - \varphi(2r_0 + 2r)$, for $0 \le r \le \frac{1}{2}r_0$. In other words, φ is a test function supported on the annulus between r_0 and $3r_0$ and its "ascent" to 1 is twice as fast as its "descent". Let $\widetilde{\varphi}$ be another radial function with the same support as φ, satisfying $\widetilde{\varphi}(r_0 + xr_0) = 1$ for $0 \le x \le \frac{1}{2}$ and $\widetilde{\varphi} = \varphi$ otherwise. Recall that for a function ψ and $L > 0$ we denote by ψ_L the scaled function $\psi_L(z) = \psi(z/L)$.

Proposition 2.9.3. *Let r_0 be the radius of $\mathcal{D}$. Let φ and $\widetilde{\varphi}$ be defined as above.*

(i) The random variables

$$S_l(n) := \int \frac{\widetilde{\varphi}(z)}{z^l}\, d[\mathcal{G}_n](z) + \sum_{j=1}^{\infty} \int \frac{\varphi_{2^j}(z)}{z^l}\, d[\mathcal{G}_n](z) = \sum_{\omega \in \mathcal{G}_n \cap \mathcal{D}^c} \frac{1}{\omega^l} \quad (\textit{ for } l \ge 1)$$

and

$$\widetilde{S}_l(n) := \int \frac{\widetilde{\varphi}(z)}{|z|^l}\, d[\mathcal{G}_n](z) + \sum_{j=1}^{\infty} \int \frac{\varphi_{2^j}(z)}{|z|^l}\, d[\mathcal{G}_n](z) = \sum_{\omega \in \mathcal{G}_n \cap \mathcal{D}^c} \frac{1}{|\omega|^l} \quad (\textit{ for } l \ge 3)$$

have finite first moments which, for every fixed l, are bounded above uniformly in n.

(ii) There exists $k_0 = k_0(\varphi) \geq 1$, uniform in n and l, such that for $k \geq k_0$ the "tails" of $S_l(n)$ and $\widetilde{S}_l(n)$ beyond the disk $2^k \cdot \mathcal{D}$, given by

$$\tau_l^n(2^k) := \sum_{j=k}^{\infty} \int \frac{\varphi_{2^j}(z)}{z^l} \, d[\mathcal{G}_n](z) \ (\text{ for } l \geq 1)$$

$$\text{and} \quad \widetilde{\tau}_l^n(2^k) := \sum_{j=k}^{\infty} \int \frac{\varphi_{2^j}(z)}{|z|^l} \, d[\mathcal{G}_n](z) \ (\text{ for } l \geq 3)$$

satisfy the estimates

$$\mathbb{E}\left[\left|\tau_l^n(2^k)\right|\right] \leq C_1(\varphi, l)/2^{kl} \quad \text{and} \quad \mathbb{E}\left[\left|\widetilde{\tau}_l^n(2^k)\right|\right] \leq C_2(\varphi, l)/2^{k(l-2)}.$$

All of the above remain true when $\mathcal{G}_n$ is replaced by $\mathcal{G}$, for which we use the notations S_l and $\widetilde{S}_l$ to denote the quantities corresponding to $S_l(n)$ and $\widetilde{S}_l(n)$.

Remark 2.9.1. *For $\mathcal{G}$, by the sum $\left(\sum_{\omega \in \mathcal{G} \cap \mathcal{D}^c} \frac{1}{\omega^l} \right)$ we denote the quantity*

$$S_l = \int \frac{\widetilde{\varphi}(z)}{z^l} \, d[\mathcal{G}](z) + \sum_{j=1}^{\infty} \int \frac{\varphi_{2^j}(z)}{z^l} \, d[\mathcal{G}](z)$$

due to the obvious analogy with $\mathcal{G}_n$, where the corresponding sum $S_l(n)$ is indeed equal to $\left(\sum_{\omega \in \mathcal{G}_n \cap \mathcal{D}^c} \frac{1}{\omega^l} \right)$ with its usual meaning.

Proof. Observe that the functions $\widetilde{\varphi}$ and φ_{2^j} for $j \geq 1$ form a partition of unity on $\mathcal{D}^c$, hence we have the identities appearing in part (i).

Fix $n, l \geq 1$. Set $\psi_k = \int \frac{\varphi_{2^k}(z)}{z^l} \, d[\mathcal{G}_n](z)$ for $k \geq 1$, and $\psi_0 = \int \frac{\widetilde{\varphi}(z)}{z^l} \, d[\mathcal{G}_n](z)$. When $l \geq 3$ we also define $\gamma_k = \int \frac{\varphi_{2^k}(z)}{|z|^l} \, d[\mathcal{G}_n](z)$ for $k \geq 1$, and $\gamma_0 = \int \frac{\widetilde{\varphi}(z)}{|z|^l} \, d[\mathcal{G}_n](z)$. Let Ψ_k and Γ_k denote the analogous quantities defined with respect to $\mathcal{G}$ instead of $\mathcal{G}_n$.

We begin with the observation that for $k \geq 1$ we have $\mathbb{E}[\psi_k] = 0$. This implies that

$$\mathbb{E}[|\psi_k|] \leq \left(\mathbb{E}[|\psi_k|^2]\right)^{1/2} = \sqrt{\mathrm{Var}[\psi_k]}$$

We then apply Proposition 2.9.1 to the function $\varphi(z)/z^l$ and $R = 2^k$ for $k \geq 1$ to obtain $\mathbb{E}[|\psi_k|] \leq C(\varphi, l)/(2^k)^l$. We also note that

$$\mathbb{E}[|\psi_0|] \leq \int_{2 \cdot \mathcal{D} \setminus \mathcal{D}} \frac{K_n(z,z) e^{-|z|^2}}{|z|^l} \, d\mathcal{L}(z) \leq \int_{2 \cdot \mathcal{D} \setminus \mathcal{D}} \frac{1}{|z|^l} d\mathcal{L}(z) = c(l).$$

This implies that for $l \geq 1$

$$\mathbb{E}[|S_l(n)|] \leq \sum_{k=0}^{\infty} \mathbb{E}[|\psi_k|] < \infty.$$

The desired bound for $\mathbb{E}[\widetilde{S}_l(n)]$ follows from a direct computation of the expectation using the first intensity, and noting that the first intensity of $\mathcal{G}_n$ (with respect to Lebesgue measure) is $K_n(z,z)e^{-|z|^2} \leq 1$ for all z.

The estimates for τ and $\widetilde{\tau}$ follow by using the above argument for the sums $\sum_{j=k}^{\infty} \psi_j$ and $\sum_{j=k}^{\infty} \gamma_j$.

■

Corollary 2.9.4. *For $R = 2^k$ for $k \geq k_0$ (as in Proposition 2.9.3), we have $\mathbb{P}[|\tau_l^n(R)| > R^{-l/2}] \leq c_1(\varphi, l)R^{-l/2}$ and $\mathbb{P}[|\widetilde{\tau}_l^n(R)| > R^{-(l-2)/2}] \leq c_2(\varphi, l)R^{-(l-2)/2}$, and these estimates remain true when $\mathcal{G}_n$ is replaced with $\mathcal{G}$.*

Proof. We use the estimates on the expectation of $|\tau_l^n(R)|$ and $|\widetilde{\tau}_l^n(R)|$ from Proposition 2.9.3 and apply Markov's inequality. ■

With notations as above, we have

Proposition 2.9.5. *For each $l \geq 1$ we have $S_l(n) \to S_l$ in probability, and for each $l \geq 3$ we have $\widetilde{S}_l(n) \to \widetilde{S}_l$ in probability, and hence we have such convergence a.s. along some subsequence, simultaneously for all l.*

Proof. Fix $\delta > 0$. Given $\varepsilon > 0$, we choose $R = 2^k$ large enough such that $c_1 R^{-l/2} < \varepsilon/4$ (for $l \geq 1$) and $c_2 R^{-(l-2)/2} < \varepsilon/4$ for $l \geq 3$ (as in Corollary 2.9.4), as well as $R^{-l/2} < \delta/4$ for $l = 1, 2$ and $R^{-(l-2)/2} < \delta/4$ for $l \geq 3$. By definition, we have $S_l(n) = \sum_{j=0}^{k} \psi_{2^j,l} + \tau_l(2^k)$. On the disk of radius R, we have $\mathcal{G}_n \to \mathcal{G}$ a.s. Now choose n large enough so that we have $|\sum_{j=0}^{k} \psi_{2^j,l}^n - \sum_{j=0}^{k} \psi_{2^j,l}| < \delta/2$ except on an event of probability $\varepsilon/2$. By choice of R, we have $|\tau_l^n(2^k)| < \delta/4$ and $|\tau_l(2^k)| < \delta/4$ except on an event of probability $< \varepsilon/2$. Combining all these, we have $\mathbb{P}(|S_l(n) - S_l| > \delta) \leq \varepsilon$, proving that $S_l(n) \to S_l$ in probability.

For each l, given any sequence we can find a subsequence along which this convergence is a.s. A diagonal argument now gives us a subsequence for which a.s. convergence holds simultaneously for all l.

The argument for $\widetilde{S}_l$ is similar. ■

Define $S_k(\mathcal{D}, n) = \sum_{z \in \mathcal{G}_n \cap \mathcal{D}} 1/z^k$ and $S_k(\mathcal{D}) = \sum_{z \in \mathcal{G} \cap \mathcal{D}} 1/z^k$. Set $\alpha_k(n) = S_k(\mathcal{D}, n) + S_k(n)$ and $\alpha_k = S_k(\mathcal{D}) + S_k$. Observe that $\alpha_k(n) = \sum_{z \in \mathcal{G}_n} 1/z^k$. Then we have:

Proposition 2.9.6. *For each k, $\alpha_k(n) \to \alpha_k$ in probability as $n \to \infty$. Hence, there is a subsequence such that $\alpha_k(n) \to \alpha_k$ a.s. when $n \to \infty$ along this subsequence, simultaenously for all k.*

Proof. Since a.s. the finite point configurations given by $\mathcal{G}_n|_{\mathcal{D}} \to \mathcal{G}|_{\mathcal{D}}$ and there is no point at the origin, therefore $S_k(\mathcal{D}, n) \to S_k(\mathcal{D})$ a.s. This, combined with Proposition 2.9.5, gives us the desired result. ■

2.10 Limiting Procedure for the Ginibre Ensemble

The aim of this section is to use the estimates derived in Section 2.7 to verify the conditions laid out in Theorem 2.6.2, so that the limiting procedure outlined in Section 2.6 can be executed. This will lead us to a proof of Theorem 2.1.2 for a disk $\mathcal{D}$. We have already seen in Section 2.4 that this implies Theorem 2.1.2 for general $\mathcal{D}$.

Proof of Theorem 2.1.2 for a disk. We will appeal to Theorem 2.6.2 with $\mathcal{D}$ an open disk. We already know from Theorem 2.1.1 that the number of points in $\mathcal{D}$ is rigid. In terms of the notation used in Section 2.6, we set $X = \mathcal{G}$ and $X^n = \mathcal{G}_n$.

Fix an integer $m \geq 0$. Consider the event that there are m points of $\mathcal{G}$ inside $\mathcal{D}$. The result in Theorem 2.1.2 is trivial for $m = 0$. Hence, we focus on the case $m > 0$. For $\omega \in \mathcal{S}_{\text{out}}$, our candidate for $\nu(\omega, \cdot)$ (refer to Theorem 2.6.2) is the probability measure $Z^{-1}|\Delta(\underline{\zeta})|^2 d\mathcal{L}(\underline{\zeta})$ on $\mathcal{D}^m$, where Z is the normalizing constant and $\mathcal{L}$ is the Lebesgue measure. Notice that ν is a constant function when considered as a function mapping $\mathcal{S}_{\text{out}}$ to $\mathcal{M}(\mathcal{D}^m)$. Since a.s. $\nu(X_{\text{out}}, \cdot)$ is mutually absolutely continuous with respect to the Lebesgue measure on $\mathcal{D}^m$, Theorem 2.6.2 would imply that the same holds true for the conditional distribution $\rho(X_{\text{out}}, \cdot)$ of the points inside $\mathcal{D}$ (treated as a vector in the uniform random order).

Now we construct, for each j, the sequences $\{n_k(j)\}_{k\geq 1}$ and the events $\Omega_{n_k}(j)$. We proceed as follows. From Proposition 2.9.5, we get a subsequence n_k such that a.s. $S_1(n_k) \to S_1$, $S_2(n_k) \to S_2$ and $\widetilde{S}_3(n_k) \to \widetilde{S}_3$. This is going to be our subsequence $n_k(j)$ for all j.

Let $M_j \uparrow \infty$ be a sequence of positive numbers such that none of them is an atom of the distributions of $|S_1|, |S_2|$ and $|\widetilde{S}_3|$, and $\frac{1}{M_j} <$ radius($\mathcal{D}$) for each j.

We will first **define the events** $\Omega_{n_k}(j)$ by the following conditions:

- (i) There are exactly m points of $\mathcal{G}_{n_k}$ inside $\mathcal{D}$.
- (ii) There is at least a $1/M_j$ separation between $\partial\mathcal{D}$ and the points of $\mathcal{G}_{n_k}$ outside $\mathcal{D}$, that is, there is no point of $\mathcal{G}_{n_k}$ in the annulus between $\mathcal{D}$ and the $1/M_j$-thickening of $\mathcal{D}$.
- (iii) $|S_1^{n_k}| < M_j, |S_2^{n_k}| < M_j, |\widetilde{S}_3^{n_k}| < M_j$.

Clearly, each $\Omega_{n_k}(j)$ is measurable with respect to $\mathcal{G}_{\text{out}}^{n_k}$. On the event Ω_n^m (refer to the statement of Theorem 2.6.2 and the notations introduced just before that), the points in $(\mathcal{G}_n)_{\text{out}}$, considered in uniform random order, yield a vector $\underline{\omega}$ in $(\mathcal{D}^c)^{n-m}$. We have $\Omega_{n_k}(j) \subset \Omega_{n_k}^{m,\delta}$ with $\delta = \frac{1}{M_j}$. Denoting the conditional distribution of $\underline{\zeta}$ given $\underline{\omega}$ to be $\rho_{\underline{\omega}}^n(\underline{\zeta})$ we recall from Proposition 2.8.2 that

$$\exp\Big(-4mK(\mathcal{D},\delta)\boldsymbol{X}_n\Big)\frac{|\Delta(\underline{\zeta}')|^2}{|\Delta(\underline{\zeta})|^2} \leq \frac{\rho_{\underline{\omega}}^n(\underline{\zeta}')}{\rho_{\underline{\omega}}^n(\underline{\zeta})} \leq \exp\Big(4mK(\mathcal{D},\delta)\boldsymbol{X}_n\Big)\frac{|\Delta(\underline{\zeta}')|^2}{|\Delta(\underline{\zeta})|^2}.$$

Therefore, the bounds on $S_1(n_k), S_2(n_k)$ and $\widetilde{S}_3(n_k)$ as in condition (iii) above imply that on $\Omega_{n_k}(j)$ we have, with $\delta = 1/M_j$,

$$\exp\Big(- 12mK(\mathcal{D},\delta)M_j\Big)\frac{|\Delta(\underline{\zeta}')|^2}{|\Delta(\underline{\zeta})|^2} \leq \frac{\rho^n_{\underline{\omega}}(\underline{\zeta}')}{\rho^n_{\underline{\omega}}(\underline{\zeta})} \leq \exp\Big(12mK(\mathcal{D},\delta)M_j\Big)\frac{|\Delta(\underline{\zeta}')|^2}{|\Delta(\underline{\zeta})|^2}\ .$$

Let us consider the inequality on the right hand side, namely,

$$\frac{\rho^n_{\underline{\omega}}(\underline{\zeta}')}{\rho^n_{\underline{\omega}}(\underline{\zeta})} \leq \exp\Big(12mK(\mathcal{D},\delta)M_j\Big)\frac{|\Delta(\underline{\zeta}')|^2}{|\Delta(\underline{\zeta})|^2}.$$

Cross multiplying, we get

$$\rho^n_{\underline{\omega}}(\underline{\zeta}')|\Delta(\underline{\zeta})|^2 \leq \rho^n_{\underline{\omega}}(\underline{\zeta})\exp\Big(12mK(\mathcal{D},\delta)M_j\Big)|\Delta(\underline{\zeta}')|^2.$$

We now integrate the above inequalities, first with respect to the Lebesgue measure in the variable $\underline{\zeta}' \in A$, then with respect to the Lebesgue measure in the variable $\underline{\zeta} \in \mathcal{D}^m$ and finally with respect to the distribution of $\underline{\omega}$ on the set $B \cap \Omega_{n_k}(j)$ (recall that $\Omega_{n_k}(j)$ is measurable with respect to $\mathcal{G}^{n_k}_{\text{out}}$). We can carry out the same procedure with the inequality on the left hand side. It can be seen that together they give us condition (2.3) (in fact, the additive error term in 2.3 is actually 0).

All that remains now is to show that events $\Omega(j) := \underline{\lim}_{k\to\infty}\Omega_{n_k}(j)$ exhausts Ω^m, and that $\mathbb{P}\left(\Omega(j)\Delta\Omega_{n_k}(j)\right) \to 0$ as $k \to \infty$ for each fixed j.

Recall that Ω^m is the event that there are m points of $\mathcal{G}$ inside $\mathcal{D}$. On each $\Omega_{n_k}(j)$ there are m points of $\mathcal{G}_{n_k}$ in $\mathcal{D}$. And finally, $\mathcal{G}_{n_k} \to \mathcal{G}$ a.s. on $\mathcal{D}$. There three facts together imply that $\Omega(j) \subset \Omega^m$ for each j. Since the M_j-s are increasing, we automatically have $\Omega(j) \subset \Omega(j+1)$. It only remains to check that $\mathbb{P}(\Omega^m \setminus \Omega(j)) \to 0$ as $j \to \infty$. This is the goal of the next proposition, which will complete the proof of Theorem 2.1.1 for a disk. ■

Proposition 2.10.1. *Let $\Omega(j)$ be as defined above. Then $\mathbb{P}(\Omega^m \setminus \Omega(j)) \to 0$ as $j \to \infty$. Morever, there exists $\Omega^{corr}(j)$, measurable with respect to $\mathcal{G}_{\text{out}}$, such that $\mathbb{P}(\Omega(j)\Delta\Omega^{corr}(j)) = 0$.*.

Proof. We will first construct the event $\Omega^{corr}(j)$. Similar ideas will then be used to prove that $\mathbb{P}(\Omega^m \setminus \Omega(j)) \to 0$ as $j \to \infty$.

Let $\varepsilon > 0$. We will demonstrate an event $A_\varepsilon(j)$ such that $A_\varepsilon(j)$ is measurable with respect to X_{out} and there is a bad set Ω^ε_{bad} of probability $< \varepsilon$ such that $\Omega(j) \setminus \Omega^\varepsilon_{bad} = A_\varepsilon(j) \setminus \Omega^\varepsilon_{bad}$. As a result, $\mathbb{P}(\Omega(j)\Delta A_\varepsilon(j)) < \varepsilon$. Then $\Omega^{corr}(j) = \underline{\lim}_{n\to\infty} A_{2^{-n}}(j)$ will give us the desired event. It is easy to check that $\mathbb{P}(\Omega(j)\Delta\Omega^{corr}(j)) = 0$.

Let $\delta = \delta(\varepsilon) < 1$ be a small number, depending on ε, to be chosen later.

We **define the event** $A_\varepsilon(j)$ by the following conditions:

- (i) $N(\mathcal{G}_{\text{out}}) = m$, where N is as in Theorem 2.1.1.

- (ii) There is at least a $\frac{1}{M_j}$ separation between $\partial\mathcal{D}$ and the points ω of $\mathcal{G}$ outside $\mathcal{D}$.
- (iii) $|S_1| < M_j - \delta, |S_2| < M_j - \delta, |\widetilde{S}_3| < M_j - \delta$.

It is clear from the definition of $A_\varepsilon(j)$ that it is measurable with respect to X_{out}.

By Proposition 2.9.5, $S_i(n_k) \to S_i, i = 1,2$ and $\widetilde{S}_3(n_k) \to \widetilde{S}_3$ a.s. along our chosen subsequence. By Egorov's Theorem, there is a bad event Ω^1 of probability $< \varepsilon/4$ such that outside Ω^1, this convergence is uniform. Therefore, on $(\Omega^1)^c$ there is a large k_0 such that $|S_i(n_k) - S_i| < \delta, i = 1,2$ and $|\widetilde{S}_i(n_k) - \widetilde{S}_i| < \delta$ for all $k \geq k_0$. Recall that we have a.s. convergence of $\mathcal{G}_{n_k}$-s to $\mathcal{G}$ on the compact set $2 \cdot \overline{\mathcal{D}}$ (which contains the $\frac{1}{M_j}$-thickening of $\mathcal{D}$ for each j, by our assumption that $\frac{1}{M_j} <$ radius($\mathcal{D}$) for all j). Thus, we have an event Ω^2 of probability $< \varepsilon/4$ outside which conditions (i) and (ii) in the definition of $\Omega_{n_k}(j)$ are true or false simultaneously for all $\mathcal{G}_{n_k}, k \geq k_1$, for some integer k_1. By the coupling of $\mathcal{G}_{n_k}$-s and $\mathcal{G}$ (which entails that $\mathcal{G}_n \subset \mathcal{G}_{n+1}\ subset \mathcal{G}$ for all n), this would imply that the conditions (i) and (ii) in the definition of $A_\varepsilon(j)$ are also true or false respectively.

Since M_j is not an atom of the distribution of S_1, S_2 or $\widetilde{S}_3$, there is an event Ω^3 (of probability $< \varepsilon/2$) outside which each S_1, S_2 or $\widetilde{S}_3$ is either $> M_j + 2\delta$ or $< M_j - 2\delta$ ($\delta < 1$ is to be chosen based on ε so that this condition is satisfied).

Define $\Omega_{bad} = \Omega^1 \cup \Omega^2 \cup \Omega^3$; clearly $\mathbb{P}(\Omega_{bad}) < \varepsilon$. We note that on Ω_{bad}^c, the conditions (i) and (ii) in the definition of Ω_{n_k} and the conditions (i) and (ii) in the definition of $A_\varepsilon(j)$ are simultaneously true or false (for all k large enough). This is because of the a.s. convergence $\mathcal{G}_n \to \mathcal{G}$ on the compact set $2\cdot\overline{\mathcal{D}}$. On $\Omega(j) \setminus \Omega_{bad}$, we have $|S_i(n_k)| < M_j$ for all large enough k, hence $|S_i(n_k) - S_i| < \delta$ implies $|S_i| < M_j + \delta$, where $i = 1,2$. But we are on $(\Omega^3)^c$, so $|S_i| \in (M_j - 2\delta, M_j + 2\delta)^c$, hence $|S_i| < M_j + \delta$ implies $|S_i| < M_j - 2\delta$, hence we are inside $A_\varepsilon(j)$. Conversely, on $A_\varepsilon(j) \setminus \Omega_{bad}$, we have $|S_i| < M_j - \delta$. But $|S_i(n_k) - S_i| < \delta$ for all large enough k. This implies $|S_i(n_k)| < M_j$ for all large enough k which means we are on $\Omega(j)$. Hence $\Omega(j) \setminus \Omega_{bad} = A_\varepsilon(j) \setminus \Omega_{bad}$. As a result, we have $\Omega(j)\Delta A_\varepsilon(j) \subset \Omega_{bad}$, and $\mathbb{P}(\Omega(j)\Delta A_\varepsilon(j)) < \varepsilon$.

To show that $\mathbb{P}(\Omega^m \setminus \Omega(j)) \to 0$ as $j \to \infty$, we define events $A'(j) \subset A_\varepsilon(j)$ above by replacing δ by 1 in the condition (iii) in the definition of $A_\varepsilon(j)$. Clearly, $A'(j) \subset A_\varepsilon(j)$ for each $0 < \varepsilon < 1$, and therefore $A'(j) \subset \underline{\lim}_{n\to\infty} A_{2^{-n}}(j) = \Omega^{corr}(j)$. It is also clear that $\mathbb{P}(\Omega^m \setminus A'(j)) \to 0$ as $j \to \infty$, because S_1, S_2 and $\widetilde{S}_3$ are well defined random variables (with no mass at ∞). These two facts imply that $\mathbb{P}(\Omega^m \setminus \Omega(j)) \to 0$ as $j \to \infty$.

■

This completes the proof of the translation tolerance of the Ginibre ensemble in the case of $\mathcal{D}$ being a disk.

2.11 Tolerance of the GAF zeros for a disk

In this section we obtain the estimates necessary to prove Theorem 2.1.2 in the case where $\mathcal{D}$ is a disk. By translation invariance of $\mathcal{F}$, we can take $\mathcal{D}$ to be centred at the origin.

2.12 Polynomial Approximations

We focus on the event $\Omega_n^{m,\delta}$ which entails that f_n has exactly m zeroes inside $\mathcal{D}$, and there is at least a δ separation between $\partial\mathcal{D}$ and the outside zeroes. The corresponding event for the GAF zero process has positive probability, so by the distributional convergence $\mathcal{F}_n \to \mathcal{F}$, we have that $\Omega_n^{m,\delta}$ has positive probability (which is bounded away from 0 as $n \to \infty$).

Let us denote the zeroes of f_n (in uniform random order) inside $\mathcal{D}$ by $\underline{\zeta} = (\zeta_1, \cdots, \zeta_m)$ and those outside $\mathcal{D}$ by $\underline{\omega} = (\omega_1, \omega_2, \cdots, \omega_{n-m})$. Let $s = \sum_{j=1}^m \zeta_j$ and

$$\Sigma_S := \{(\zeta_1, \ldots, \zeta_m) \in \mathcal{D}^m : \sum_{j=1}^m \zeta_j = s\}.$$

For a vector $\underline{v} = (v_1, \cdots, v_N)$ we define

$$\sigma_k(\underline{v}) = \sum_{1 \le i_1 < \cdots < i_k \le N} v_{i_1} \cdots v_{i_k}$$

and for two vectors $\underline{u}$ and $\underline{v}$, $\sigma_k(\underline{u}, \underline{v})$ is defined to be $\sigma_k(\underline{w})$ where the vector $\underline{w}$ is obtained by concatenating the vectors $\underline{u}$ and $\underline{v}$. Then the conditional density $\rho^n_{\underline{\omega},s}(\underline{\zeta})$ of $\underline{\zeta}$ given $\underline{\omega}, s$ is of the form (see, e.g., [FH99])

$$\rho^n_{\underline{\omega},s}(\underline{\zeta}) = C(\underline{\omega}, s) \frac{|\Delta(\zeta_1, \ldots, \zeta_m, \omega_1, \ldots, \omega_{n-m})|^2}{\left(\sum_{k=0}^n \left| \frac{\sigma_k(\underline{\zeta}, \underline{\omega})}{\sqrt{\binom{n}{k} k!}} \right|^2\right)^{n+1}} \tag{2.9}$$

where $C(\underline{\omega}, s)$ is the normalizing factor.

Throughout this Subsection 2.12, the zeroes will be those of f_n with n fixed.

Let $(\underline{\zeta}, \underline{\omega})$ and $(\underline{\zeta}', \underline{\omega})$ be two vectors of points (under $\mathcal{F}_n$). Then the ratio of the conditional densities at these two vectors is given by

$$\frac{\rho^n_{\underline{\omega},s}(\underline{\zeta}')}{\rho^n_{\underline{\omega},s}(\underline{\zeta})} = \frac{|\Delta(\underline{\zeta}', \underline{\omega})|^2}{|\Delta(\underline{\zeta}, \underline{\omega})|^2} \left(\sum_{k=0}^n \left| \frac{\sigma_k(\underline{\zeta}, \underline{\omega})}{\sqrt{\binom{n}{k} k!}} \right|^2\right)^{n+1} \Bigg/ \left(\sum_{k=0}^n \left| \frac{\sigma_k(\underline{\zeta}', \underline{\omega})}{\sqrt{\binom{n}{k} k!}} \right|^2\right)^{n+1}. \tag{2.10}$$

The expression (2.10) has two distinct components: the ratio of two squared Vandermonde determinants and the ratio of certain expressions involving the elementary symmetric functions of the zeroes. We will consider these two components in two separate sections.

Ratio of Vandermondes

Here we consider the quantity $|\Delta(\underline{\zeta}', \underline{\omega})|^2/|\Delta(\underline{\zeta}, \underline{\omega})|^2$. We proceed exactly as in the case of the Ginibre ensemble. We refer the reader to section 2.8. The estimates here are valid for all pairs $(\underline{\zeta}, \underline{\zeta}') \in \mathcal{D}^m \times \mathcal{D}^m$.

Proposition 2.12.1. *On $\Omega_n^{m,\delta}$ there are quantities $K(\mathcal{D}, \delta) > 0$ and $\boldsymbol{X}_n(\underline{\omega}) > 0$ such that for any $(\underline{\zeta}, \underline{\zeta}') \in \mathcal{D}^m \times \mathcal{D}^m$ we have*

$$\exp\Big(- 2mK(\mathcal{D}, \delta)\boldsymbol{X}_n(\underline{\omega})\Big)\frac{|\Delta(\underline{\zeta}')|^2}{|\Delta(\underline{\zeta})|^2} \leq \frac{|\Delta(\underline{\zeta}', \underline{\omega})|^2}{|\Delta(\underline{\zeta}, \underline{\omega})|^2} \leq \exp\Big(2mK(\mathcal{D}, \delta)\boldsymbol{X}_n(\underline{\omega})\Big)\frac{|\Delta(\underline{\zeta}')|^2}{|\Delta(\underline{\zeta})|^2}$$

where $\boldsymbol{X}_n(\underline{\omega}) = \left|\sum_{\omega_j \in \mathcal{G}_n \cap \mathcal{D}^c} \frac{1}{\omega_j}\right| + \left|\sum_{\omega_j \in \mathcal{G}_n \cap \mathcal{D}^c} \frac{1}{\omega_j^2}\right| + \left(\sum_{\omega_j \in \mathcal{G}_n \cap \mathcal{D}^c} \frac{1}{|\omega_j|^3}\right)$ *and* $\mathbb{E}[\boldsymbol{X}_n(\underline{\omega})] \leq c_1(\mathcal{D}, m) < \infty$.

Remark 2.12.1. *The estimates on* $\left|\sum \frac{1}{\omega_j}\right|$, $\left|\sum \frac{1}{\omega_j^2}\right|$ *and* $\left(\sum \frac{1}{|\omega_j|^3}\right)$ *which are necessary for Proposition 2.12.1 are proved in section 2.13.*

Corollary 2.12.2. *Given $M > 0$, we can replace the $\boldsymbol{X}_n(\underline{\omega})$ in Proposition 2.12.1 by a uniform bound M except on an event of probability less than $c(m, \mathcal{D})/M$.*

Ratio of Symmetric Functions

In this section we will restrict $\underline{\zeta}$ and $\underline{\zeta}'$ to lie in the same constant-sum hyperplane. For $s \in \mathbb{C}$, define

$$\Sigma_s := \{\underline{\zeta} \in \mathcal{D}^m : \sum_{i=1}^m \zeta_i = s\}.$$

Let $D(\underline{\zeta}, \underline{\omega}) = \left(\sum_{k=0}^n \left|\frac{\sigma_k(\underline{\zeta}, \underline{\omega})}{\sqrt{\binom{n}{k}k!}}\right|^2\right)$.

We want to bound the ratio $\big(D(\underline{\zeta}', \underline{\omega})/D(\underline{\zeta}, \underline{\omega})\big)^{n+1}$ from above and below. Our main goal is:

Proposition 2.12.3. *Given $M > 0$ large enough, $\exists n_0$ such that for all $n \geq n_0$ the following is true: with probability $\geq 1 - C/M$ we have, on $\Omega_n^{m,\delta}$,*

$$e^{-2K(m,\mathcal{D})M \log M} \leq \Big(D(\underline{\zeta}', \underline{\omega})\Big/D(\underline{\zeta}, \underline{\omega})\Big)^{n+1} \leq e^{2K(m,\mathcal{D})M \log M}$$

for all $\underline{\zeta}' \in \Sigma_s$, where $s = \sum_{i=1}^m \zeta_i$ and $(\underline{\zeta}, \underline{\omega})$ is randomly generated from $\mathcal{F}_n$.

We will first prove several auxiliary propositions.

We begin with the observation

$$\sigma_k(\underline{\zeta},\underline{\omega}) = \sum_{i=0}^{m} \sigma_i(\underline{\zeta})\sigma_{k-i}(\underline{\omega}). \tag{2.11}$$

Note that $\sigma_0(\underline{\zeta}) = 1$, $\sigma_1(\underline{\zeta}) = s = \sigma_1(\underline{\zeta}')$ and $|\sigma_i(\underline{\zeta})| < \binom{m}{i} r_0^i$ for all $i \leq m$, where r_0 is the radius of $\mathcal{D}$. Since both $\underline{\zeta}$ and $\underline{\zeta}' \in \Sigma_s$, we have

$$\sigma_k(\underline{\zeta}',\underline{\omega}) = \sigma_k(\underline{\zeta},\underline{\omega}) + \sum_{i=2}^{m} [\sigma_i(\underline{\zeta}') - \sigma_i(\underline{\zeta})]\sigma_{k-i}(\underline{\omega}). \tag{2.12}$$

Taking modulus squared on both sides, we have $|\sigma_k(\underline{\zeta}',\underline{\omega})|^2 =$

$$|\sigma_k(\underline{\zeta},\underline{\omega})|^2 + \sum_{i=2}^{m} |\sigma_i(\underline{\zeta}') - \sigma_i(\underline{\zeta})|^2 |\sigma_{k-i}(\underline{\omega})|^2 + \sum_{i=2}^{m} 2\Re\left((\sigma_i(\underline{\zeta}') - \sigma_i(\underline{\zeta}))\overline{\sigma_k(\underline{\zeta},\underline{\omega})}\sigma_{k-i}(\underline{\omega}) \right)$$
$$+ \sum_{i,j=2}^{m} \sum_{i\neq j} 2\Re\left(\overline{(\sigma_i(\underline{\zeta}') - \sigma_i(\underline{\zeta}))}(\sigma_j(\underline{\zeta}') - \sigma_j(\underline{\zeta}))\overline{\sigma_{k-i}(\underline{\omega})}\sigma_{k-j}(\underline{\omega}) \right).$$

Summing the above over $k = 0, \cdots, n$, we obtain

$$\sum_{k=0}^{n} |\sigma_k(\underline{\zeta}',\underline{\omega})|^2 = \sum_{k=0}^{n} |\sigma_k(\underline{\zeta},\underline{\omega})|^2 + \sum_{i=2}^{m} |\sigma_i(\underline{\zeta}') - \sigma_i(\underline{\zeta})|^2 \left(\sum_{k=0}^{n} |\sigma_{k-i}(\underline{\omega})|^2 \right)$$
$$+ \sum_{i=2}^{m} 2\Re\left((\sigma_i(\underline{\zeta}') - \sigma_i(\underline{\zeta})) \left(\sum_{k=0}^{n} \overline{\sigma_k(\underline{\zeta},\underline{\omega})}\sigma_{k-i}(\underline{\omega}) \right) \right)$$
$$+ \sum_{i,j=2}^{m} \sum_{i\neq j} 2\Re\left(\overline{(\sigma_i(\underline{\zeta}') - \sigma_i(\underline{\zeta}))}(\sigma_j(\underline{\zeta}') - \sigma_j(\underline{\zeta})) \left(\sum_{k=0}^{n} \overline{\sigma_{k-i}(\underline{\omega})}\sigma_{k-j}(\underline{\omega}) \right) \right).$$

Dividing throughout by $\left(\sqrt{\binom{n}{k}k!}\right)^2$ and applying triangle inequality we get

$$D(\underline{\zeta}',\underline{\omega}) - A(\underline{\zeta},\underline{\zeta}',\underline{\omega}) \leq D(\underline{\zeta}',\underline{\omega}) \leq D(\underline{\zeta},\underline{\omega}) + A(\underline{\zeta},\underline{\zeta}',\underline{\omega}) \tag{2.13}$$

where we recall the fact that $D(\underline{\zeta},\underline{\omega}) = \left(\sum_{k=0}^{n} \left| \frac{\sigma_k(\underline{\zeta},\underline{\omega})}{\sqrt{\binom{n}{k}k!}} \right|^2 \right)$ and $A(\underline{\zeta},\underline{\zeta}',\underline{\omega}) =$

$$\sum_{i=2}^{m} |\sigma_i(\underline{\zeta}') - \sigma_i(\underline{\zeta})|^2 \left(\sum_{k=0}^{n} \left| \frac{\sigma_{k-i}(\underline{\omega})}{\sqrt{\binom{n}{k}k!}} \right|^2 \right) + \sum_{i=2}^{m} 2|\sigma_i(\underline{\zeta}') - \sigma_i(\underline{\zeta})| \left| \sum_{k=0}^{n} \frac{\overline{\sigma_k(\underline{\zeta},\underline{\omega})}}{\sqrt{\binom{n}{k}k!}} \frac{\sigma_{k-i}(\underline{\omega})}{\sqrt{\binom{n}{k}k!}} \right|$$
$$+ \sum_{i\neq j=2}^{m} 2|\sigma_i(\underline{\zeta}') - \sigma_i(\underline{\zeta})||\sigma_j(\underline{\zeta}') - \sigma_j(\underline{\zeta})| \left| \sum_{k=0}^{n} \frac{\overline{\sigma_{k-i}(\underline{\omega})}}{\sqrt{\binom{n}{k}k!}} \frac{\sigma_{k-j}(\underline{\omega})}{\sqrt{\binom{n}{k}k!}} \right|.$$

Divide throughout by $D(\underline{\zeta}, \underline{\omega})$ in the above inequalities, and observe that
(a) i and j vary between 2 and m (which is fixed and finite),
(b) $|\sigma_i(\underline{\zeta}') - \sigma_i(\underline{\zeta})|$ is bounded for each i as discussed immediately before (2.12).

In view of these facts, we conclude that in order to bound $\big(D(\underline{\zeta}', \underline{\omega})/D(\underline{\zeta}, \underline{\omega})\big)^n$ between two constants with high probability, it suffices to show that the quantities

$$\left|\sum_{k=0}^{n} \frac{\overline{\sigma_k(\underline{\zeta}, \underline{\omega})}}{\sqrt{\binom{n}{k}k!}} \frac{\sigma_{k-i}(\underline{\omega})}{\sqrt{\binom{n}{k}k!}}\right| \Big/ \left(\sum_{k=0}^{n} \left|\frac{\sigma_k(\underline{\zeta}, \underline{\omega})}{\sqrt{\binom{n}{k}k!}}\right|^2\right) \tag{2.14}$$

and

$$\left|\sum_{k=0}^{n} \frac{\overline{\sigma_{k-i}(\underline{\omega})}}{\sqrt{\binom{n}{k}k!}} \frac{\sigma_{k-j}(\underline{\omega})}{\sqrt{\binom{n}{k}k!}}\right| \Big/ \left(\sum_{k=0}^{n} \left|\frac{\sigma_k(\underline{\zeta}, \underline{\omega})}{\sqrt{\binom{n}{k}k!}}\right|^2\right) \tag{2.15}$$

for $m \geq i, j \geq 2$ are bounded above by M/n with high probability (depending on M) where $M > 0$ is a constant.

Observe that $\sigma_k(\underline{\zeta}, \underline{\omega})/\sqrt{\binom{n}{k}k!} = \xi_{n-k}/\xi_n$, where the ξ_j-s are standard complex Gaussians. On the other hand, we do not have good control over $\sigma_k(\underline{\omega})$. To obtain such control, the idea is to obtain a convenient expansion of $\sigma_k(\underline{\omega})$ in terms of $\sigma_i(\underline{\zeta}, \underline{\omega})$. To this end, we formulate:

Proposition 2.12.4. *On the event $\Omega_n^{m,\delta}$ we have, for $0 \leq k \leq n - m$,*

$$\sigma_k(\underline{\omega}) = \sigma_k(\underline{\zeta}, \underline{\omega}) + \sum_{r=1}^{k} g_r \sigma_{k-r}(\underline{\zeta}, \underline{\omega})$$

where a.s. the random variables g_r are $O(K(\mathcal{D}, m)^r)$ as $r \to \infty$, for a deterministic quantity $K(\mathcal{D}, m)$ and the constant in O being deterministic and uniform in n and δ.

Proof. We begin with the observation that

$$\sigma_k(\underline{\zeta}, \underline{\omega}) = \sum_{r=0}^{m} \sigma_r(\underline{\zeta}) \sigma_{k-r}(\underline{\omega}). \tag{2.16}$$

From this, we have

$$\sigma_k(\underline{\omega}) = \sigma_k(\underline{\zeta}, \underline{\omega}) - \sum_{r=1}^{m} \sigma_r(\underline{\zeta}) \sigma_{k-r}(\underline{\omega}). \tag{2.17}$$

Proceeding inductively in (2.17) we can similarly expand each of the lower order $\sigma_{k-r}(\underline{\omega})$ in terms of $\sigma_j(\underline{\zeta}, \underline{\omega})$ and obtain an expansion of $\sigma_k(\underline{\omega})$ in terms of $\sigma_j(\underline{\zeta}, \underline{\omega}), j = 1, \cdots, k$:

$$\sigma_k(\underline{\omega}) = \sigma_k(\underline{\zeta}, \underline{\omega}) + \sum_{r=1}^{k} g_r \sigma_{k-r}(\underline{\zeta}, \underline{\omega}). \tag{2.18}$$

Notice that the coefficients g_r do not depend on n.

The coefficient of $\sigma_k(\underline{\zeta},\underline{\omega})$ is 1 and the rest of the coefficients are polynomials in $\sigma_j(\underline{\zeta}), j = 1, \cdots, m$. Due to the inductive structure, it is easy to see that g_r-s satisfy a recurrence relation:

$$\begin{pmatrix} g_i \\ g_{i-1} \\ \cdots \\ g_{i-m+1} \end{pmatrix} = \begin{pmatrix} -\sigma_1(\underline{\zeta}) & -\sigma_2(\underline{\zeta}) & \dots & -\sigma_m(\underline{\zeta}) \\ 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} g_{i-1} \\ g_{i-2} \\ \cdots \\ g_{i-m} \end{pmatrix} \tag{2.19}$$

with the boundary conditions $g_i = -\sigma_i(\underline{\zeta})$ for $i = 0, \cdots, m-1, g_0 = 1$ and $g_i = 0$ for $i < 0$.

Denoting the matrix in (2.19) above by A, we observe that its eigenvalues are precisely $-\zeta_1, \cdots, -\zeta_m$, where $\{\zeta_i\}_{i=1}^m$ are the zeroes inside $\mathcal{D}$. Due to the recursive structure in (2.19), we note that g_k is an element of A^k applied to the vector $(-\sigma_{m-1}(\underline{\zeta}), \cdots . -\sigma_0(\underline{\zeta}))$. This implies that g_k is a linear combination of the entries of A^k (with the coefficients in the linear combination being independent of k, but depending on $\mathcal{D}$ and m).

From Proposition 2.12.5, we see that if the eigenvalues of a matrix A have absolute value $< \rho'$, then the entries of the matrix A^k are $o(\rho'^k)$. Now, the eigenvalues of A are ζ_i-s and for each i we have $|\zeta_i| \leq \text{radius}(\mathcal{D}) = K(\mathcal{D})$. Combining the last two paragraphs, we have $g_k = O((K(\mathcal{D})+1)^k)$ Crucially, the constant in the O above is uniform in n and δ (although it can depend on m). ■

We complete this discussion with the following result on the growth of matrix entries:

Proposition 2.12.5. *Let A be a $d \times d$ matrix. Let $\rho' > \rho(A) := max\{|\lambda| : \lambda$ an eigenvalue of $A\}$. Then $(A^r)_{ij} = o(\rho'^r)$ as $r \to \infty$, for all i, j.*

Proof. This follows from the well known result (Gelfand's spectral radius theorem): $\rho(A) = \lim_{r\to\infty} \|A^r\|^{1/r}$ where $\|\cdot\|$ denotes the L^2 operator norm of the matrix, and the fact that $\sup_{1\leq i,j\leq d} |B_{ij}| \leq c(d)\|B\|$ for any $d \times d$ matrix B. ■

The aim of the next proposition is to show that in the expansion of $\sigma_k(\underline{\omega})$ in terms of $\sigma_r(\underline{\zeta},\underline{\omega})$ with $r \leq k$, it is essentially the leading term $\sigma_k(\underline{\zeta},\underline{\omega})$ that matters. For clarity, we will switch to a different indexing of the σ-s and involve the Gaussian coefficients of the polynomial more directly. Recall that $\dfrac{\sigma_{n-r}(\underline{\zeta},\underline{\omega})}{\sqrt{\binom{n}{n-r}(n-r)!}} = \dfrac{\xi_r}{\xi_n}$. Divide both sides of the expansion proved in Proposition 2.12.4 by $\sqrt{\binom{n}{k}k!}$ and note that

$$\sqrt{\binom{n}{k}k!} = \sqrt{\binom{n}{k-r}(k-r)!}\sqrt{(n-k+1)\cdots(n-k+r)} \quad .$$

Notation: We denote the falling factorial (of order k) of an integer x by

$$(x)_k := x(x-1)\cdots(x-k+1).$$

For $k=0$ we set $(x)_0 = 1$.

Changing variables to $l = n-k$, we can rewrite the expansion in Proposition 2.12.4 as

$$\frac{\sigma_{n-l}(\underline{\zeta}, \underline{\omega})}{\sqrt{\binom{n}{n-l}(n-l)!}} = \frac{\xi_l}{\xi_n} + \sum_{r=1}^{n-l} g_r \frac{\xi_{l+r}}{\xi_n} \frac{1}{\sqrt{(l+r)_r}}$$

for $l \geq m$. Define for any integer $l \geq 1$

$$\eta_l^{(n)} = \sum_{r=1}^{n-l} g_r \xi_{l+r} \frac{1}{\sqrt{(l+r)_r}}. \tag{2.20}$$

As a result, for $l \geq m$ we have

$$\frac{\sigma_{n-l}(\underline{\omega})}{\sqrt{\binom{n}{n-l}(n-l)!}} = \frac{1}{\xi_n}\left[\xi_l + \eta_l^{(n)}\right]. \tag{2.21}$$

Proposition 2.12.6. *Let $\eta_l^{(n)}$ be as in (2.20) and $\gamma = \frac{1}{8}$. Then $\exists$ positive random variables η_l (not depending on n) such that a.s. $\left|\eta_l^{(n)}\right| \leq \eta_l$, and for fixed $l_0 \in \mathbb{N}$ and large enough $M > 0$ we have*

$$(i)\mathbb{P}\left[\eta_l > \frac{M}{l^\gamma} \text{ for some } l \geq 1\right] \leq e^{-c_1 M^2}$$

$$(ii)\mathbb{P}\left[\eta_l > \frac{M}{l^\gamma} \text{ for some } l \geq l_0\right] \leq e^{-c_2 M^2 l_0^{\frac{1}{4}}}$$

where c_1, c_2 are constants that depend only on the domain $\mathcal{D}$ and on m.

Proof. We begin by recalling from Proposition 2.12.4 that a.s. $|g_r| \leq K^r$ for some $K = K(\mathcal{D}, m)$. Moreover, $\sqrt{l+p} \geq \sqrt{2} l^{\frac{1}{4}} p^{\frac{1}{4}}$ Hence,

$$|\eta_l^{(n)}| \leq \sum_{r=1}^{n-l} \frac{K^r}{2^{r/2} l^{r/4} (r!)^{1/4}} |\xi_{l+r}|.$$

Let B be such that $\sup_{r\geq 1} \frac{K^r}{2^{r/2}(r!)^{1/8}} \leq B$. Then $|\eta_l^{(n)}| \leq \sum_{r=1}^{n-l} \frac{B|\xi_{l+r}|}{l^{r/4}(r!)^{1/8}}$.

Define $\eta_l = \sum_{r=1}^{\infty} \frac{B|\xi_{l+r}|}{l^{r/4}(r!)^{1/8}}$. Clearly, $|\eta_l^{(n)}| \leq \eta_l$. Now, $|\xi_{l+r}|^2$ is an exponential random variable with mean 1, we have

$$\mathbb{P}\left(|\xi_{l+r}| \geq \frac{M l^{r/8} (r!)^{1/16}}{B}\right) \leq \exp\left(-\frac{M^2 l^{r/4} (r!)^{1/8}}{B^2}\right). \tag{2.22}$$

Moreover,

$$\sum_{l=1}^{\infty}\sum_{r=1}^{\infty}\exp\left(-\frac{M^2 l^{r/4}(r!)^{1/8}}{B^2}\right) \le C_1 \exp\left(-M^2/B^2\right).$$

So we can have $|\xi_{l+r}| \le \frac{Ml^{r/8}(r!)^{1/16}}{B}$ for all $l \ge 1, r \ge 1$, except on an event of probability $\le C_1\exp\left(-M^2/B^2\right)$. The last expression can be made arbitrarily small by fixing M to be large enough. Note that for large enough M, we have $C\exp\left(-M^2/B^2\right) \le \exp\left(-cM^2\right)$. On the complement of this small event, we have for all $l \ge 1$

$$\eta_l \le \sum_{r=1}^{\infty}\frac{M}{l^{r/8}(r!)^{1/16}} \le \sum_{r=1}^{\infty}\frac{M}{l^{r/8}(r!)^{1/16}} \le C\frac{M}{l^{1/8}}.$$

We then absorb this constant C into M by changing the constant c_1 appearing in the exponent of the tail estimate (i) we want to prove.

For proving (ii) we proceed exactly as in the case of (i) above and sum (2.22) over $r \ge 1$ and $l \ge l_0$. ■

We are now ready to prove bounds on the quantities in (2.14) and (2.15). To this end, we first express them in terms of ξ-s and η-s, using $\sigma_{n-l}(\underline{\zeta},\underline{\omega})\Big/\sqrt{\binom{n}{n-l}(n-l)!} = \xi_l/\xi_n$ and (2.21). For $2 \le i,j \le m$ these expressions, (after clearing out $|\xi_n|^2$ from numerator and denominator) reduce to :

$$\left|\sum_{l=0}^{n-i}\frac{\overline{\xi_l}\left(\xi_{l+i}+\eta_{l+i}^{(n)}\right)}{\sqrt{(l+i)_i}}\right|\Bigg/\left(\sum_{l=0}^{n}|\xi_l|^2\right) \tag{2.23}$$

and

$$\left|\sum_{l=0}^{n-i\wedge n-j}\frac{\overline{\left(\xi_{l+i}+\eta_{l+i}^{(n)}\right)}\left(\xi_{l+j}+\eta_{l+j}^{(n)}\right)}{\sqrt{(l+i)_i}\sqrt{(l+j)_j}}\right|\Bigg/\left(\sum_{l=0}^{n}|\xi_l|^2\right). \tag{2.24}$$

Define

$$\boldsymbol{E}_n = \sum_{l=0}^{n}|\xi_l|^2, \qquad \boldsymbol{L}_{ij}^{(n)} = \sum_{l=0}^{n-i\wedge n-j}\frac{\overline{\xi_{l+i}}\xi_{l+j}}{\sqrt{(l+i)_i(l+j)_j}},$$

$$\boldsymbol{M}_{ij}^{(n)} = \sum_{l=0}^{n-i\wedge n-j}\frac{|\xi_{l+i}|\eta_{l+j}}{\sqrt{(l+i)_i(l+j)_j}}, \qquad \boldsymbol{N}_{ij}^{(n)} = \sum_{l=0}^{n-i\wedge n-j}\frac{\eta_{l+i}\eta_{l+j}}{\sqrt{(l+i)_i(l+j)_j}}.$$

For $i = 0$ the product $(l+i)_i$ in the denominator is replaced by 1. Expanding out the products and applying the triangle inequality in (2.23) and(2.24), we observe that to upper bound the expressions in (2.23) and (2.24) it suffices to upper bound the following quantities for $2 \leq i, j \leq m$ (remember that $|\eta_l^{(n)}| \leq \eta_l$ for all l):

$$\left(\frac{\left|\boldsymbol{L}_{0j}^{(n)}\right|}{\boldsymbol{E}_n} + \frac{\boldsymbol{M}_{0j}^{(n)}}{\boldsymbol{E}_n} \right), \qquad \left(\frac{\left|\boldsymbol{L}_{ij}^{(n)}\right|}{\boldsymbol{E}_n} + \frac{\boldsymbol{M}_{ij}^{(n)}}{\boldsymbol{E}_n} + \frac{\boldsymbol{M}_{ji}^{(n)}}{\boldsymbol{E}_n} + \frac{\boldsymbol{N}_{ij}^{(n)}}{\boldsymbol{E}_n} \right) \ . \tag{2.25}$$

Let

$$\boldsymbol{Y}_n = \sum_{i=2}^{m} \left|\boldsymbol{L}_{0i}^{(n)}\right| + \sum_{i=2}^{m} \boldsymbol{M}_{0i}^{(n)} + \sum_{i,j\geq 2}^{m} \left|\boldsymbol{L}_{ij}^{(n)}\right| + \sum_{i,j\geq 2}^{m} \boldsymbol{M}_{ij}^{(n)} + \sum_{i,j\geq 2}^{m} \boldsymbol{M}_{ji}^{(n)} + \sum_{i,j\geq 2}^{m} \boldsymbol{N}_{ij}^{(n)} \ .$$

Recall (2.13) and recall that for fixed m and $\mathcal{D}$, we have that $\sigma_i(\underline{\zeta})$ are uniformly bounded. Putting all these together, we deduce that: for some constant $K(m, \mathcal{D})$ we have

$$1 - K(m, \mathcal{D}) \frac{\boldsymbol{Y}_n}{\boldsymbol{E}_n} \leq \frac{D(\underline{\zeta}', \underline{\omega})}{D(\underline{\zeta}, \underline{\omega})} \leq 1 + K(m, \mathcal{D}) \frac{\boldsymbol{Y}_n}{\boldsymbol{E}_n} \ . \tag{2.26}$$

Regarding $\boldsymbol{Y}_n$ and $\boldsymbol{E}_n$, we have the following estimates:

Proposition 2.12.7. *Given $M > 0$ we have:*
(i) $\mathbb{P}[\boldsymbol{Y}_n \geq M \log M] \leq c(m, \mathcal{D})/M$,
(ii) Given $M > 0$ there exists n_0 (depending only on M) such that for $n \geq n_0$ we have

$$\mathbb{P}[\frac{n}{2} \leq |\mathbb{E}_n| \leq 2n] \geq 1 - \frac{1}{M}.$$

Proof. Part (i):
We will show that whenever $X = \boldsymbol{L}_{ij}^{(n)}$,$X = \boldsymbol{M}_{ij}^{(n)}$ or $X = \boldsymbol{N}_{ij}^{(n)}$, we have that X satisfies $\mathbb{P}[|X| > M \log M] < c'(m, \mathcal{D})/M$. This would imply $\mathbb{P}[\boldsymbol{Y}_n > M \log M] < c_1(m, \mathcal{D})/M$ albeit for a different constant $c(m, \mathcal{D})$. The very last statement is easily seen as follows. The number of summands in the definition of $\boldsymbol{Y}_n$ is a polynomial in m, let us call it $p(m)$. Now, for each summand X of $\boldsymbol{Y}_n$, we have

$$\mathbb{P}\left[|X| > \frac{M \log M}{p(m)}\right] \leq \mathbb{P}\left[|X| > \frac{M}{p(m)} \log \frac{M}{p(m)}\right] \leq c'(m, \mathcal{D}) p(m)/M.$$

If $\boldsymbol{Y}_n > M \log M$ and there are $p(m)$ summands, at least one of the summands must be $> M \log M/p(m)$. By a union bound over the summands, this implies

$$\mathbb{P}[\boldsymbol{Y}_n > M \log M] \leq c'(m, \mathcal{D}) p(m)^2/M.$$

Setting $c(m, \mathcal{D}) = c'(m, \mathcal{D}) p(m)^2$ as the new constant, this would give us part (i) of the proposition.

It remains to show that each of the summands in $\boldsymbol{Y}_n$ satisfies $\mathbb{P}[|X| > M \log M] < c(m, \mathcal{D})/M$ with the quantity $c(m, \mathcal{D})$ being uniform in n. The understanding is that c will depend on the particular summand, and we will take a maximum over the $p(m)$ summands in order to obtain $c'(m,\mathcal{D})$ as in the previous paragraph. We will take up the cases of $\boldsymbol{L},\boldsymbol{M}$ and $\boldsymbol{N}$ separately.

In what follows, $i\vee j$ denotes $\max(i,j)$ and $i\wedge j$ denotes $\min(i,j)$.

Estimating $\boldsymbol{L}_{ij}^{(n)}, i\vee j\geq 2$:

We have

$$\mathbb{E}\left[|\boldsymbol{L}_{ij}^{(n)}|^2\right] = \sum_{l=0}^{n-i\wedge n-j} \frac{1}{(l+i)_i(l+j)_j} \leq \sum_{l=0}^{\infty} \frac{1}{(l+i)_i(l+j)_j} = c(i,j) \leq c < \infty$$

(since either i or $j\geq 2$). An application of Chebyshev's inequality proves the desired tail bound on $\boldsymbol{L}_{ij}^{(n)}$.

Estimating $\boldsymbol{M}_{ij}^{(n)}, j\geq 2$:

By Proposition 2.12.6 we can assume (except on an event A of probability $\leq e^{-c(\log M)^2}$) that $\max_{l\geq 1}\eta_l \leq \log M/l^{1/8}$, and $\left|\eta_l^{(n)}\right| \leq \eta_l$ for each l. Applying triangle inequality to the definition of $\boldsymbol{M}_{ij}^{(n)}$ and using the upper bound on η_{l+j} we have

$$\mathbb{E}\left[|\boldsymbol{M}_{ij}^{(n)}|1_{A^c}\right] \leq \log M \mathbb{E}\left[\sum_{l=0}^{n-i\wedge n-j} \frac{|\xi_{l+i}|}{(l+j)^{1/8}\sqrt{(l+i)_i(l+j)_j}}\right] \leq m(i,j)\log M \leq c\log M < \infty$$

where $m(i,j) = \sum_{l=0}^{\infty} \frac{\mathbb{E}[|\xi|]}{(l+j)^{1/8}\sqrt{(l+i)_i(l+j)_j}} \leq c < \infty$, ξ is a standard complex Gaussian and c is uniform in i and j. In the last step, we have used the fact that $j\geq 2$ in order to upper bound $m(i,j)$ uniformly in i and j . Applying Markov inequality gives us the desired tail estimate $\mathbb{P}(|M_{ij}^{(n)}|1_{A^c} > M\log M) \leq c/M$. This, along with $\mathbb{P}(A) < e^{-c(\log M)^2}$ completes the proof.

Estimating $\boldsymbol{N}_{ij}^{(n)}, i\wedge j\geq 2$:

We consider the event A as in the case of $\boldsymbol{M}_{ij}^{(n)}$. On A^c, we use $\eta_{l+i}\eta_{l+j} \leq (\log M)^2 < M\log M$ (for large M), and the rest is bounded above by

$$\sum_{l=0}^{n-i\wedge n-j} \frac{1}{(l+i)^{1/8}(l+j)^{1/8}\sqrt{(l+i)_i(l+j)_j}} \leq \sum_{l=0}^{\infty} \frac{1}{l^{1/4}\sqrt{(l+i)_i(l+j)_j}} = c(i,j) \leq c < \infty.$$

This, along with $\mathbb{P}[A] < e^{-c(\log M)^2}$ establishes the desired tail bound on $N_{ij}^{(n)}$.

All these arguments complete the proof of part (i) of Proposition 2.12.7.

Part(ii):

We simply observe that $|\xi|^2$-s are i.i.d. exponentials and by the strong law of large numbers, $\boldsymbol{E}_n/n \to 1$ a.s. From this, the statement of part (ii) follows.

■

Define

$$\boldsymbol{L}_{ij} = \sum_{l=0}^{\infty} \frac{\overline{\xi_{l+i}}\xi_{l+j}}{\sqrt{(l+i)_i(l+j)_j}} \ , \qquad \tau(L_{ij}^{(n)}) = \boldsymbol{L}_{ij} - \boldsymbol{L}_{ij}^{(n)}, (i \vee j \geq 2)$$

$$\boldsymbol{M}_{ij} = \sum_{l=0}^{\infty} \frac{|\xi_{l+i}|\eta_{l+j}}{\sqrt{(l+i)_i(l+j)_j}} \ , \qquad \tau(M_{ij}^{(n)}) = \boldsymbol{M}_{ij} - \boldsymbol{M}_{ij}^{(n)}, (j \geq 2)$$

$$\boldsymbol{N}_{ij} = \sum_{l=0}^{\infty} \frac{\eta_{l+i}\eta_{l+j}}{\sqrt{(l+i)_i(l+j)_j}} \ , \qquad \tau(N_{ij}^{(n)}) = \boldsymbol{N}_{ij} - \boldsymbol{N}_{ij}^{(n)}, (i \wedge j \geq 2).$$

For $i = 0$ the product $(l+i)_i$ in the denominator is, as usual, replaced by 1. The above random variables have finite first moments, as can be seen by arguing on similar lines to the estimates in Proposition 2.12.7. Similar arguments with first moments of $\tau(L_{ij}^{(n)}), \tau(M_{ij}^{(n)}), \tau(N_{ij}^{(n)})$ also show that:

Proposition 2.12.8. *As $n \to \infty$, we have each of the random variables $\tau(L_{ij}^{(n)}), \tau(M_{ij}^{(n)})$, $\tau(N_{ij}^{(n)})$, as defined above, converge to 0 in L^1, and hence, in probability.*

We omit the proof to avoid repetitiveness.

Now we are ready to prove Proposition 2.12.3.

Proof. **Proof of Proposition 2.12.3**
We refer back to equation (2.26). By virtue of Proposition 2.12.7 we have, dropping an event of probability $\leq c/M$, we have $|\boldsymbol{Y}_n/\boldsymbol{E}_n| \leq 2M \log M/n$ (for $n \geq n_0$, where n_0 is large enough such that Proposition 2.12.7 part(ii) holds). Raising this to the $(n+1)$th power we obtain the desired result. All constants are absorbed in $K(m, \mathcal{D})$. ■

For the following corollary we recall the definition of Σ_s from the beginning of Section 2.12.

Corollary 2.12.9. *Given $M > 0$ large enough, $\exists n_0$ such that for all $n \geq n_0$, we have, on $\Omega_n^{m,\delta}$ (except on an event of probability $\leq C/M$) the following inequalities:*

$$e^{-4K(m,\mathcal{D})MlogM} \leq \left(D(\underline{\zeta}'', \underline{\omega}) \Big/ D(\underline{\zeta}', \underline{\omega}) \right)^{n+1} \leq e^{4K(m,\mathcal{D})MlogM}$$

for all $(\underline{\zeta}'', \underline{\zeta}') \in \Sigma_s \times \Sigma_s$ where $(\underline{\zeta}, \underline{\omega})$ is picked randomly from the distribution $\mathbb{P}[\mathcal{F}_n]$ and $s = \sum_{i=1}^{m} \zeta_i$.

Proof. Let $(\underline{\zeta}, \underline{\omega})$ be picked randomly from the distribution $\mathbb{P}[\mathcal{F}_n]$ and $s = \sum_{i=1}^{m} \zeta_i$. We take the ratio

$$\frac{D(\underline{\zeta}'', \underline{\omega})}{D(\underline{\zeta}', \underline{\omega})} = \frac{D(\underline{\zeta}'', \underline{\omega})}{D(\underline{\zeta}, \underline{\omega})} \Bigg/ \frac{D(\underline{\zeta}', \underline{\omega})}{D(\underline{\zeta}, \underline{\omega})}$$

for $(\underline{\zeta}', \underline{\zeta}'') \in \Sigma_s \times \Sigma_s$ where $s = \sum_{i=1}^{m} \zeta_i$, and apply the bounds in Proposition 2.12.3.

■

Estimate for Ratio of Conditional Densities

Proposition 2.12.10. *There exist constants $K(m, \mathcal{D}, \delta)$ such that given $M > 0$ large enough, we have for $n \geq n_0(m, M, \mathcal{D})$ the following inequalities hold on $\Omega_n^{m,\delta}$, except for an event of probability $\leq c(m, \mathcal{D})/M$:*

$$\exp\left(-K(m,\mathcal{D},\delta)MlogM\right)\frac{|\Delta(\underline{\zeta}'')|^2}{|\Delta(\underline{\zeta}')|^2} \leq \frac{\rho^n_{\underline{\omega},s}(\underline{\zeta}'')}{\rho^n_{\underline{\omega},s}(\underline{\zeta}')} \leq \exp\left(K(m,\mathcal{D},\delta)MlogM\right)\frac{|\Delta(\underline{\zeta}'')|^2}{|\Delta(\underline{\zeta}')|^2}$$

uniformly for all $(\underline{\zeta}', \underline{\zeta}'') \in \Sigma_s \times \Sigma_s$, where $(s, \underline{\omega})$ corresponds to a point configuration picked randomly from $\mathbb{P}[\mathcal{F}_n]$.

Proof. We simply put together the estimates for the ratios of the Vandermondes as well as the symmetric functions and subsume all relevant constants in $c(m, \mathcal{D})$ and $K(m, \mathcal{D}, \delta)$. ■

2.13 Estimates for Inverse Powers of Zeroes

In this section we prove estimates on the (smoothed) sum of inverse powers of GAF zeroes.

Proposition 2.13.1. *Let Φ be a C_c^∞ radial function supported on the annulus between r_0 and $3r_0$. Let $\Phi_R = \Phi(z/R)$. We have*

$$(i)\mathbb{E}\left[\left|\int \frac{\Phi_R(z)}{z^l}\, d[\mathcal{F}_n](z)\right|\right] \leq C(r_0, \Phi, l)/R^l$$

$$(ii)\mathbb{E}\left[\int \frac{\Phi_R(z)}{|z|^l}\, d[\mathcal{F}_n](z)\right] \leq C_1(r_0, \Phi, l)/R^{l-2}\ .$$

The same is true with for $\mathcal{F}$ in place of $\mathcal{F}_n$. The constants $C(r_0, \Phi, l)$ and $C_1(r_0, \Phi, l)$ are uniform in n.

Proof. • Part (i): We begin with

$$\int \frac{1}{z^l}\Phi_R(z)\, d[\mathcal{F}_n](z) = \int \frac{1}{z^l}\Phi_R(z)\Delta\log(|f_n(z)|)\, d\mathcal{L}(z).$$

Now, $\log(\sqrt{\mathbb{E}[|f_n(z)|^2]})$ is a radial function, and Laplacian of a radial function is also radial. Hence,

$$\int \frac{1}{z^l}\Phi_R(z)\Delta\log(\sqrt{\mathbb{E}[|f_n(z)|^2]})\, d\mathcal{L}(z) = 0$$

because for $l \geq 1$, $1/z^l$ when integrated against a radial function is always 0. Let $\hat{f}_n(z) = f_n(z)/\sqrt{\mathbb{E}[|f_n(z)|^2]}$. Then the above argument implies

$$\int \frac{1}{z^l}\Phi_R(z)\, d[\mathcal{F}_n](z) = C\int \frac{1}{z^l}\Phi_R(z)\Delta\log(|\hat{f}_n(z)|)\, d\mathcal{L}(z). \tag{2.27}$$

Integrating by parts on the right hand side of (2.27) we have

$$\mathbb{E}\left[\left|\int \frac{1}{z^l}\Phi_R(z)\, d[\mathcal{F}_n](z)\right|\right] \leq C\int \left|\Delta\left(\frac{1}{z^l}\Phi_R(z)\right)\right| \mathbb{E}\left[\left|\log(|\hat{f}_n(z)|)\right|\right]\, d\mathcal{L}(z). \tag{2.28}$$

Now, the integrand is non-zero only for $Rr_0 \leq |z| \leq 3Rr_0$. Hence it is easy to see that $\left|\Delta\left(\frac{1}{z^l}\Phi_R(z)\right)\right| \leq C(r_0,\Phi,l)/R^{l+2}$. Further, $\mathbb{E}\left[\left|\log(|\hat{f}_n(z)|)\right|\right]$ is a constant because $\hat{f}_n$ is $N_{\mathbb{C}}(0,1)$.

Finally, $\int_{3Rr_0.D\setminus Rr_0.D} d\mathcal{L}(z) = 8\pi r_0^2 R^2$, where D is the unit disk.

Putting all these together, we have

$$\int \left|\Delta\left(\frac{1}{z^l}\Phi_R(z)\right)\right| \mathbb{E}\left[\left|\log(|\hat{f}_n(z)|)\right|\right]\, d\mathcal{L}(z) \leq C(r_0,\Phi,l)/R^l$$

as desired (all constants are subsumed in $C(r_0,\Phi,l)$).

• Part (ii): Since Φ is a radial function on $\mathbb{C}$, therefore there exists a function $\widetilde{\Phi}$ on non-negative reals such that $\Phi(z) = \widetilde{\Phi}(|z|)$.

We have, with $r = |z|$,

$$\mathbb{E}\left[\int \frac{1}{|z|^l}\Phi_R(z)\, d[\mathcal{F}_n](z)\right] = c\int \frac{1}{r^l}\widetilde{\Phi}_R(r)\Delta\log(\sqrt{K_n(z,z)})rdr$$

where the integral on the right hand side is over the non-negative reals. Integrating by parts and using triangle inequality,

$$\left|\int \frac{1}{r^l}\widetilde{\Phi}_R(r)\Delta\log(\sqrt{K_n(z,z)})rdr\right| \leq \int \left|\Delta\left(\frac{1}{r^l}\widetilde{\Phi}_R(r)\right)\right| \log(\sqrt{K_n(z,z)})rdr.$$

But $\log(\sqrt{K_n(z,z)}) \leq \log(\sqrt{K(z,z)}) = \frac{1}{2}r^2 \leq \frac{9}{2}r_0^2R^2$ and $\left|\Delta\left(\frac{1}{z^l}\widetilde{\Phi}_R(z)\right)\right| \leq C(r_0,\Phi,l)/R^{l+2}$, hence

$$\int \left|\Delta\left(\frac{1}{r^l}\widetilde{\Phi}_R(r)\right)\right| \log(\sqrt{K_n(z,z)})rdr \leq C_1(r_0,\Phi,l)/R^{l-2}$$

for another constant C_1 (which is clearly uniform in n). ■

Remark 2.13.1. *Tracing the constants in the above computation, it can be seen that the constant $C(r_0,\Phi,l)$ above is of the form $p(l)\left(\frac{1}{r_0}\right)^l C(\Phi)$ where p is a fixed polynomial (of degree 2), and $C(\Phi)$ is a constant that depends only on Φ. Both p and $C(\Phi)$ are uniform in n. Similarly, $C_1(r_0,\Phi,l)$ is of the form $p_1(l)\left(\frac{1}{r_0}\right)^{l-2} C_1(\Phi)$; where p_1 is another degree 2 polynomial and $C_1(\Phi)$ is uniform in n.*

Let $r_0 =$ radius $(\mathcal{D})$. Let φ be a non-negative radial C_c^∞ function supported on $[r_0, 3r_0]$ such that $\varphi = 1$ on $[r_0 + \frac{r_0}{2}, 2r_0]$ and $\varphi(r_0 + r) = 1 - \varphi(2r_0 + 2r)$, for $0 \leq r \leq \frac{1}{2}r_0$. In other words, φ is a test function supported on the annulus between r_0 and $3r_0$ and its "ascent" to 1 is twice as fast as its "descent".

Let $\widetilde{\varphi}$ be another radial test function with the same support as φ, and $\widetilde{\varphi}(r_0 + xr_0) = 1$ for $0 \le x \le \frac{1}{2}$ and $\widetilde{\varphi} = \varphi$ otherwise.

Proposition 2.13.2.

$$\mathbb{E}\left[\left|\int \frac{\widetilde{\varphi}(z)}{z^l}\, d[\mathcal{F}_n](z)\right|\right] \le \mathbb{E}\left[\left|\int \frac{\widetilde{\varphi}(z)}{|z|^l}\, d[\mathcal{F}_n](z)\right|\right] \le c(r_0, \widetilde{\varphi}, l)$$

where $c(r_0, \widetilde{\varphi}, l)$ is uniform in n, and the same result remains true when $\mathcal{F}_n$ is replaced by $\mathcal{F}$.

Proof. For fixed l, let $a(n) = \displaystyle\int \frac{\widetilde{\varphi}(z)}{z^l}\, d[\mathcal{F}_n](z)$ and $b(n) = \displaystyle\int \frac{\widetilde{\varphi}(z)}{|z|^l}\, d[\mathcal{F}_n](z)$.

We have, $\mathbb{E}\left[|a(n)|\right] \le \mathbb{E}\left[b(n)\right] = C \int \frac{\widetilde{\varphi}(z)}{|z|^l} \Delta \log\left(K_n(z,z)\right) d\mathcal{L}(z)$ for some constant C. Using the uniform convergence of the continuous functions $\Delta \log\left(K_n(z,z)\right) \to \Delta \log\left(K(z,z)\right) < \infty$ on the (compact) support of $\widetilde{\varphi}$, we deduce that $\mathbb{E}\left[b(n)\right]$-s are uniformly bounded by constants that depend on $r, \widetilde{\varphi}$ and l. It is obvious from the above argument that the same holds true for $\mathcal{F}$ instead of $\mathcal{F}_n$. ∎

Proposition 2.13.3. *Let r_0 be the radius of $\mathcal{D}$. Let φ and $\widetilde{\varphi}$ be defined as above.*

(i) The random variables

$$S_l(n) := \int \frac{\widetilde{\varphi}(z)}{z^l}\, d[\mathcal{F}_n](z) + \sum_{j=1}^{\infty} \int \frac{\varphi_{2^j}(z)}{z^l}\, d[\mathcal{F}_n](z) = \sum_{\omega \in \mathcal{F}_n \cap \mathcal{D}^c} \frac{1}{\omega^l} \quad (\text{ for } l \ge 1)$$

and

$$\widetilde{S}_l(n) := \int \frac{\widetilde{\varphi}(z)}{|z|^l}\, d[\mathcal{F}_n](z) + \sum_{j=1}^{\infty} \int \frac{\varphi_{2^j}(z)}{|z|^l}\, d[\mathcal{F}_n](z) = \sum_{\omega \in \mathcal{F}_n \cap \mathcal{D}^c} \frac{1}{|\omega|^l} \quad (\text{ for } l \ge 3)$$

have finite first moments which, for every fixed l, are bounded above uniformly in n.

(ii) There exists $k_0 = k_0(\varphi) \ge 1$, uniform in n and l, such that for $k \ge k_0$ the "tails" of $S_l(n)$ and $\widetilde{S}_l(n)$ beyond the disk $2^k \cdot \mathcal{D}$, given by

$$\tau_l^n(2^k) := \sum_{j=k}^{\infty} \int \frac{\varphi_{2^j}(z)}{z^l}\, d[\mathcal{F}_n](z) \ (\text{ for } l \ge 1)$$

$$\text{and} \quad \widetilde{\tau}_l^n(2^k) := \sum_{j=k}^{\infty} \int \frac{\varphi_{2^j}(z)}{|z|^l}\, d[\mathcal{F}_n](z) \ (\text{ for } l \ge 3)$$

satisfy the estimates

$$\mathbb{E}\left[\left|\tau_l^n(2^k)\right|\right] \le C_1(\varphi, l)/2^{kl/2} \quad \text{and} \quad \mathbb{E}\left[\left|\widetilde{\tau}_l^n(2^k)\right|\right] \le C_2(\varphi, l)/2^{k(l-2)/2}.$$

All of the above remain true when $\mathcal{F}_n$ is replaced by $\mathcal{F}$, for which we use the notations S_l and $\widetilde{S}_l$ to denote the analogous quantities corresponding to $S_l(n)$ and $\widetilde{S}_l(n)$.

Remark 2.13.2. *For $\mathcal{F}$, by the sum* $\left(\sum_{\omega\in\mathcal{F}\cap\mathcal{D}^c}\frac{1}{\omega^l}\right)$ *we denote the quantity*

$$S_l = \int \frac{\widetilde{\varphi}(z)}{z^l}\, d[\mathcal{F}](z) + \sum_{j=1}^{\infty}\int \frac{\varphi_{2^j}(z)}{z^l}\, d[\mathcal{F}](z)$$

due to the obvious analogy with $\mathcal{F}_n$, where the corresponding sum $S_l(n)$ is indeed equal to $\left(\sum_{\omega\in\mathcal{F}_n\cap\mathcal{D}^c}\frac{1}{\omega^l}\right)$ *with its usual meaning.*

Proof. The functions $\widetilde{\varphi}$ and $\varphi_{2^j}, j \geq 1$ form a partition of unity on $\mathcal{D}^c$, hence the identities appearing in part (i). That the sum of integrals in the statement of part (i) has finite expectation can be seen from Propositions 2.13.1 and 2.13.2; it is uniformly bounded in n because so are $C(r_0, \varphi, l)$ and $c(r_0, \widetilde{\varphi}, l)$ in Propositions 2.13.1 and 2.13.2.

Fix $n, l \geq 1$. Set $\psi_k = \int \frac{\varphi_{2^k}(z)}{z^l}\, d[\mathcal{F}_n](z)$ for $k \geq 1$, and $\psi_0 = \int \frac{\widetilde{\varphi}(z)}{z^l}\, d[\mathcal{F}_n](z)$. When $l \geq 3$ we also define $\gamma_k = \int \frac{\varphi_{2^k}(z)}{|z|^l}\, d[\mathcal{F}_n](z)$ for $k \geq 1$, and $\gamma_0 = \int \frac{\widetilde{\varphi}(z)}{|z|^l}\, d[\mathcal{F}_n](z)$. Let Ψ_k and Γ_k denote the analogous quantities defined with respect to $\mathcal{F}$ instead of $\mathcal{F}_n$.

We begin with the observation that for $k \geq 1$ we have $\mathbb{E}[\psi_k] = 0$. This implies that

$$\mathbb{E}[|\psi_k|] \leq \left(\mathbb{E}[|\psi_k|^2]\right)^{1/2} = \sqrt{\mathrm{Var}[\psi_k]}$$

We then apply Proposition 2.13.1 part (i) to the function $\Phi = \varphi$ and $R = 2^k$ to obtain

$$\mathbb{E}[|\psi_k|] \leq C(r_0, \varphi, l)/(2^k)^l. \tag{2.29}$$

We also observe that $\mathbb{E}[|\psi_0|] \leq \mathbb{E}[|\gamma_0|]$ which, for fixed l, is uniformly bounded above in n, using Proposition 2.13.2.

The above arguments imply that for $l \geq 1$

$$\mathbb{E}[|S_l(n)|] \leq \sum_{k=0}^{\infty}\mathbb{E}[|\psi_k|] < \infty.$$

Moreover, the infinite sum on the right hand side of the above display are uniformly bounded aboce in n, using (2.29) and the observation on γ_0 that follows (2.29). The results for $\widetilde{S}_l(n)$ are similar, utilizing part (ii) of Proposition 2.13.1.

To obtain k_0 as in part (ii), we recall Remark 2.13.1 and find k_0 such that

$$p(l)\left(\frac{1}{r_0}\right)^l 2^{-kl/2}C(\varphi) \leq 1/2$$

for all $k \geq k_0$ and all $l \geq 1$.

Such a k_0 can be obtained as follows: first fix k and let $l \to \infty$ in $p(l)\left(\frac{1}{r_0}\right)^l 2^{-kl/2}C(\varphi)$; if k is large enough, this will $\to 0$. Fix such a k, then for large enough $l \geq l_0$ and this k, we have $p(l)\left(\frac{1}{r_0}\right)^l 2^{-kl/2}C(\varphi) \leq 1/2$. Note that if we increase k further, the inequality will still remain true for all $l \geq l_0$. To take care of the first l_0 terms, we simply need to pick a k much larger, so that $p(l)\left(\frac{1}{r_0}\right)^l 2^{-kl/2}C(\varphi) \leq 1/2$ holds for all $l \geq 1$. This new k is our k_0.

This k_0 will clearly be independent of n, because so is $C(\varphi)$, and by choice it is independent of l. To estimate $\mathbb{E}\left[\left|\tau_l^n(2^k)\right|\right]$, we now simply sum (2.29) for $k \geq k_0$ and use Remark 2.13.1.

The result for $\widetilde{\tau}_l^n(2^k)$ follows from a similar argument.

The same arguments yield the corresponding results when F_n is replaced by F. ■

Corollary 2.13.4. *For $R = 2^k, k \geq k_0$ as in Proposition 2.13.3. We have $\mathbb{P}[|\tau_l^n(R)| > R^{-l/4}] \leq R^{-l/4}$ and $\mathbb{P}[|\widetilde{\tau}_l^n(R)| > R^{-(l-2)/4}] \leq R^{-(l-2)/4}$, and these estimates remain true when f_n is replaced with f.*

Proof. We use the estimates on the expectation on $|\tau_l^{f_n}(R)|$ and $|\widetilde{\tau}_l^{f_n}(R)|$ from Proposition 2.13.3 and apply Markov's inequality. ■

With notations as above, we have

Proposition 2.13.5. *For each $l \geq 1$ we have $S_l(n) \to S_l$ in probability, and for each $l \geq 3$ we have $\widetilde{S}_l^n \to \widetilde{S}_l$ in probability, and hence we have such convergence a.s. along some subsequence, simultaneously for all l.*

Proof. We argue on similar lines to the proof of Proposition 2.9.5. ■

Define $S_k(\mathcal{D}, n) = \sum_{z \in \mathcal{F}_n \cap \mathcal{D}} 1/z^k$ and $S_k(\mathcal{D}) = \sum_{z \in \mathcal{F} \cap \mathcal{D}} 1/z^k$. Set $\alpha_k(n) = S_k(\mathcal{D}, n) + S_k(n)$ and $\alpha_k = S_k(\mathcal{D}) + S_k$. Observe that $\alpha_k(n) = \sum_{z \in \mathcal{F}_n} 1/z^k$. Then we have:

Proposition 2.13.6. *For each k, $\alpha_k(n) \to \alpha_k$ in probability as $n \to \infty$. Hence, there is a subsequence such that $\alpha_k(n) \to \alpha_k$ a.s. when $n \to \infty$ along this subsequence, simultaenously for all k.*

Proof. Since the finite point configurations given by $\mathcal{F}_n|_{\overline{\mathcal{D}}} \to \mathcal{F}|_{\overline{\mathcal{D}}}$ a.s. and a.s. there is no point (of $\mathcal{F}_n$ or of F) at the origin or on $\partial\mathcal{D}$, therefore $S_k(\mathcal{D}, n) \to S_k(\mathcal{D})$ a.s. This, combined with Proposition 2.13.5, gives us the desired result. ■

2.14 Limiting procedure for GAF zeroes

In this section, we use the estimates for $\mathcal{F}_n$ to prove Theorem 2.1.4 for a disk $\mathcal{D}$ centred at the origin. We know from Section 2.4 that this is sufficient in order to obtain Theorems 2.1.3 and 2.1.4 in the general case. We will work in the framework of Section 2.6. More specifically, we will show that the conditions for Theorem 2.6.2 are satisfied, which will give us the desired conclusion. We will introduce the definitions and check all the conditions here except the fact $\Omega(j)$ exhausts Ω^m. This last criterion will be verified in the subsequent Proposition 2.14.1.

In terms of the notation used in Section 2.6, we have $X^n = \mathcal{F}_n$ and $X = \mathcal{F}$.

Proof of Theorem 2.1.4 for a disk. We will invoke Theorem 2.6.2. We will define the relevant quantities (as in the statement of Theorem 2.6.2) and verify that they satisfy the required conditions for that theorem to apply.

Following the notation in Theorem 2.6.2, we begin with Ω^m, $m \geq 0$. The cases $m = 0$ and $m = 1$ are trivial, so we focus on the case $m \geq 2$.

Our candidate for $\nu(X_{\text{out}}(\xi), \cdot)$ (refer to Theorem 2.6.2) is the probability measure $Z^{-1}|\Delta(\underline{\zeta})|^2 d\mathcal{L}_\Sigma(\underline{\zeta})$ on $\Sigma_{S(X_{\text{out}}(\xi))}$, where $\Delta(\underline{\zeta})$ is the Vandermonde determinant formed by the coordinates of $\underline{\zeta}$, $\mathcal{L}_\Sigma$ is the Lebesgue measure on $\Sigma_{S(X_{\text{out}}(\xi))}$ and Z is the normalizing factor. Here we recall the definition of $S(X_{\text{out}}(\xi))$ from Theorem 2.1.3 and the definition of $\Sigma_{S(X_{\text{out}}(\xi))}$ from Section 2.12. Note that as soon as we define $\nu(X_{\text{out}}, \cdot)$ which maps ξ to $\mathcal{M}(\mathcal{D}^m)$, this automatically induces a map from $\mathcal{S}_{\text{out}}$ to $\mathcal{M}(\mathcal{D}^m)$ which satisfies the required measurability properties.

To find the sequence n_k (which will be the same for every j in our case), we proceed as follows. Let $N_g(K)$ denote the number of zeroes of the function g in a set K, Γ denote the closed annulus of thickness 1 around $\mathcal{D}$, and in the next statement let Z be any of the variables L, M or N as in Proposition 2.12.7 (and the immediately preceding discussion) with $p = 0$ or $2 \leq p \leq m$ and $2 \leq q \leq m$. We have the probabilities of each of the following events converging to 0 as $n \to \infty$: $\{|S_j(n) - S_j| > 1\}_{j=1,2}, \{|\widetilde{S}_3(n) - \widetilde{S}_3| > 1\}, \{|N_f(\overline{\mathcal{D}}) \neq N_{f_n}(\overline{\mathcal{D}})|\}, \{|N_f(\Gamma) \neq N_{f_n}(\Gamma)|\}, \{|\tau(Z^{(n)}_{pq})| > 1\}, \{\frac{1}{n}\left(\sum_{j=0}^{n} |\xi_j|^2\right) \leq 1/2\}$. Call the union of these events $\mathfrak{B}_n$. For a given $k \geq 1$, let n'_k be such that $\mathbb{P}(\mathfrak{B}_n) < 2^{-k}$ for all $n \geq n'_k$. From Proposition 2.13.5 we have a sequence such that $S_j(n) \to S_j$ ($j = 1, 2$) and $\widetilde{S}_3(n) \to \widetilde{S}_3$ a.s. as $n \to \infty$ along that sequence. For a given $k \geq 1$, we define n_k to be the least integer in that sequence which is $\geq n'_k$.

Fix a sequence of positive real numbers $M_j \uparrow \infty$.

On the event $\Omega^m_{n_k}$ (which entails that f_{n_k} has m zeroes inside $\mathcal{D}$), let $\underline{\zeta}(n_k)$ and $\Omega_{n(\delta)}$ respectively denote the vector of inside and outside zeroes of f_{n_k} (taken in uniform random order). Let $s(n_k)$ denote the sum of the inside zeroes: $s(n_k) := \sum_{j=1}^{m} \underline{\zeta}(n_k)_j$, where $\underline{\zeta}(n_k)_j$ are the co-ordinates of the vector $\underline{\zeta}(n_k)$. By $\Sigma_{s(n_k)}$ we will denote the (random) set $\{\underline{\zeta}' \in \mathcal{D}^m : \sum_{j=1}^{m} \underline{\zeta}'_j = s(n_k)\}$. Also recall the notation $\rho^{n_k}_{\underline{\omega},s}(\underline{\zeta})$ to be the conditional density (with respect to the Lebesgue measure on Σ_s) of the inside zeroes (at $\underline{\zeta} \in \mathcal{D}^m$) given the vector of outside zeroes to be $\underline{\omega}$ and the sum of the inside zeroes to be s.

We **define our event** $\Omega_{n_k}(j)$ by the following conditions on the zero set $(\underline{\zeta}(n_k), \Omega_{n(\delta)})$ of f_{n_k}:

1. There are exactly m zeroes of f_{n_k} in $\mathcal{D}$

2. There are no zeroes of f_{n_k} in the closed annulus of thickness $1/M_j$ around $\mathcal{D}$.

3. $|S_1(n_k)| \le M_j, |S_2(n_k)| \le M_j, |\widetilde{S}(n_k)| \le M_j$.

4. There exists $\underline{\zeta}' \in \Sigma_{s(n_k)}$ such that $(\underline{\zeta}', \Omega_{n(\delta)})$ satisfies ($\underline{\omega}$ here is an abbreviation for $\Omega_{n(\delta)}$):

   a) $$\left|\sum_{r=0}^{n_k} \frac{\overline{\sigma_r(\underline{\zeta}',\underline{\omega})}}{\sqrt{\binom{n}{r}r!}} \frac{\sigma_{r-i}(\underline{\omega})}{\sqrt{\binom{n}{r}r!}}\right| \Big/ \left(\sum_{r=0}^{n_k} \left|\frac{\sigma_r(\underline{\zeta}',\underline{\omega})}{\sqrt{\binom{n}{r}r!}}\right|^2\right) \le M_j/n_k \text{ for each } 2 \le i \le m.$$

   b) $$\left|\sum_{r=0}^{n_k} \frac{\overline{\sigma_{r-i}(\underline{\omega})}}{\sqrt{\binom{n}{r}r!}} \frac{\sigma_{r-j}(\underline{\omega})}{\sqrt{\binom{n}{r}r!}}\right| \Big/ \left(\sum_{r=0}^{n_k} \left|\frac{\sigma_r(\underline{\zeta}',\underline{\omega})}{\sqrt{\binom{n}{r}r!}}\right|^2\right) \le M_j/n_k \text{ for all } 2 \le i,j \le m.$$

Clearly, $\Omega_{n_k}(j)$ depends only on the quantities $s(n_k)$ and $\Omega_{n(\delta)}$. In particular, for a vector $\underline{\omega}$ of outside zeroes of f_{n_k}, if there exists $\underline{\zeta} \in \Sigma_s$ such that $(\underline{\zeta}, \underline{\omega})$ satisfies the conditions demanded in the definition of $\Omega_{n_k}(j)$, then all $(\underline{\zeta}', \underline{\omega})$ (with $\underline{\zeta}' \in \Sigma_s$) satisfies those conditions. From Proposition 2.12.1, (2.14), (2.15) and the discussion therein, it is clear that on the event $\Omega_{n_k}(j)$ we have

$$Z^{-1}m(j)|\Delta(\underline{\zeta})|^2 \le \rho^{n_k}_{\underline{\omega},s}(\underline{\zeta}) \le Z^{-1}M(j)|\Delta(\underline{\zeta})|^2 \tag{2.30}$$

for positive quantities $M(j)$ and $m(j)$ (that depend on M_j) and Z being a normalizing constant that makes $|\Delta(\underline{\zeta})|^2$ a probability density with respect to the Lebesgue measure on Σ_s.

To obtain (2.3), we introduce some further notations. On the event $\Omega^m_{n_k}$, let $\gamma_{n_k}(\underline{\omega}; s)$ denote the conditional probability measure on the sum s of inside zeroes given the vector of outside zeroes of f_{n_k} to be $\underline{\omega}$. Further, let $\mathcal{L}_s$ denote the Lebesgue measure on the set Σ_s.

For any $A \in \mathfrak{A}^m$ and $B \in \mathcal{B}$, we can write

$$\mathbb{P}[(X_{\text{in}} \in A) \cap (X^{n_k}_{\text{out}} \in B) \cap \Omega_{n_k}(j)] = \int_{B\cap\Omega_{n_k}(j)} \left(\int_{A\cap\Sigma_s} \rho^{n_k}_{\underline{\omega},s}(\underline{\zeta}) d\mathcal{L}_s(\underline{\zeta})\right) d\gamma_{n_k}(\underline{\omega}; s) d\mathbb{P}_{X^{n_k}_{\text{out}}}(\underline{\omega}) \tag{2.31}$$

Setting $h(s,A) := \left(\int_{A\cap\Sigma_s} |\Delta(\underline{\zeta})|^2 d\mathcal{L}_s(\underline{\zeta})\right) / \left(\int_{\Sigma_s} |\Delta(\underline{\zeta})|^2 d\mathcal{L}_s(\underline{\zeta})\right)$, we get from (2.30) that the right hand side of (2.31) is

$$\int_{B\cap\Omega_{n_k}(j)} \left(\int_{A\cap\Sigma_s} \rho^{n_k}_{\underline{\omega},s}(\underline{\zeta}) d\mathcal{L}_s(\underline{\zeta})\right) d\gamma_{n_k}(\underline{\omega}; s) d\mathbb{P}_{X^{n_k}_{\text{out}}}(\underline{\omega}) \asymp_j \int_{B\cap\Omega_{n_k}(j)} h(s,A) d\gamma_{n_k}(\underline{\omega}; s) d\mathbb{P}_{X^{n_k}_{\text{out}}}(\underline{\omega}). \tag{2.32}$$

The last expression can be written as $\mathbb{E}\Big[h(s(n_k),A)1[X^{n_k}_{\text{out}} \in B]1[\Omega_{n_k}(j)]\Big]$, where $1[E]$ is the indicator function of the event E. Note that for fixed A, the function $h(s,A)$ is continuous in s, because $A \in \mathcal{A}^m$, and $0 \le h(s, A) \le 1$. Since, as $k \to \infty$, we have $s(n_k) \to S(X_{\text{out}})$ a.s., therefore using the Dominated Convergence Theorem we get

$$\mathbb{E}\left[h(s(n_k), A)1[X_{\text{out}}^{n_k} \in B]1[\Omega_{n_k}(j)]\right] = \mathbb{E}\left[h(S(X_{\text{out}}), A)1[X_{\text{out}}^{n_k} \in B]1[\Omega_{n_k}(j)]\right] + o_k(1), \quad (2.33)$$

where $o_k(1)$ is a quantity which $\to 0$ as $k \to \infty$, for fixed A and B (actually, in this particular case, it can be easily seen that the convergence is uniform in $B \in \mathcal{B}$) . Also, as $k \to \infty$, we have $1[X_{\text{out}}^{n_k} \in B] \to 1[X_{\text{out}} \in B]$ since B is a compact set. Since $0 \le h(s, A) \le 1$ a.s., this implies that

$$\mathbb{E}\left[h(S(X_{\text{out}}), A)1[X_{\text{out}}^{n_k} \in B]1[\Omega_{n_k}(j)]\right] = \mathbb{E}\left[h(S(X_{\text{out}}), A)1[X_{\text{out}} \in B]1[\Omega_{n_k}(j)]\right] + o_k(1). \quad (2.34)$$

Observe that for $\xi \in \Xi$, we have $h(S(X_{\text{out}}(\xi)), A) = \nu(X_{\text{out}}(\xi), A)$. Therefore, putting (2.31)-(2.34) together, we obtain (2.3). The only condition in Theorem 2.6.2 that remains to be verified is the fact that $\Omega(j)$-s exhaust Ω^m, which will be taken up in Proposition 2.14.1.

By Theorem 2.6.2, this proves that a.s. we have $\rho(X_{\text{out}}, \cdot) \equiv \nu(X_{\text{out}}, \cdot)$. Since a.s. $\nu(X_{\text{out}}, \cdot) \equiv \mathcal{L}_\Sigma$, we have $\rho(X_{\text{out}}, \cdot) \equiv \mathcal{L}_\Sigma$, as desired. ∎

We end this section with a proof that $\Omega(j)$-s exhaust Ω^m:

Proposition 2.14.1. *With definitions as above,* $\Omega(j) := \underline{\lim}_{k\to\infty} \Omega_{n_k}(j)$ *exhausts* Ω^m *as* $j \to \infty$.

Proof. We begin by showing that $\Omega(j) \subset \Omega^m$ for each j. Due to the convergence of $X^{n_k} \to X$ on compact sets, we have $\underline{\lim}_{k\to\infty} \Omega_{n_k}^m = \Omega^m$. Since $\Omega_{n_k}(j) \subset \Omega_{n_k}^m$ for each k, therefore $\Omega(j) \subset \Omega^m$ for each j. Since $M_j < M_{j+1}$, it is also clear that $\Omega_{n_k}(j) \subset \Omega_{n_k}(j+1)$ for each k. Hence $\Omega(j) \subset \Omega(j+1)$.

To show that $\mathbb{P}(\Omega^m \setminus \Omega(j)) \to 0$ as $j \to \infty$, for each j we first **define the event** $\Omega_{n_k}^1(j)$ by demanding that the zeroes $(\underline{\zeta}, \underline{\omega})$ of f_{n_k} satisfy the following conditions :

1. There are exactly m zeroes of f_{n_k} in $\mathcal{D}$
2. There are no zeroes of f_{n_k} in the closed annulus of thickness $1/M_j$ around $\mathcal{D}$.
3. $|S_1(n_k)| \le M_j, |S_2(n_k)| \le M_j, |\widetilde{S}_3(n_k)| \le M_j$.
4. $\left|\sum_{k=0}^{n} \dfrac{\overline{\sigma_k(\underline{\zeta}, \underline{\omega})}}{\sqrt{\binom{n}{k}k!}} \dfrac{\sigma_{k-i}(\underline{\omega})}{\sqrt{\binom{n}{k}k!}}\right| \Big/ \left(\sum_{k=0}^{n} \left|\dfrac{\sigma_k(\underline{\zeta}, \underline{\omega})}{\sqrt{\binom{n}{k}k!}}\right|^2\right) \le M_j/n_k$ for each $2 \le i \le m$.
5. $\left|\sum_{k=0}^{n} \dfrac{\overline{\sigma_{k-i}(\underline{\omega})}}{\sqrt{\binom{n}{k}k!}} \dfrac{\sigma_{k-j}(\underline{\omega})}{\sqrt{\binom{n}{k}k!}}\right| \Big/ \left(\sum_{k=0}^{n} \left|\dfrac{\sigma_k(\underline{\zeta}, \underline{\omega})}{\sqrt{\binom{n}{k}k!}}\right|^2\right) \le M_j/n_k$ for all $2 \le i, j \le m$.

Clearly, $\Omega^1_{n_k}(j) \subset \Omega_{n_k}(j)$ and hence $\underline{\lim}_{k\to\infty} \Omega^1_{n_k}(j) = \Omega^1(j) \subset \Omega(j)$.

Next, we **define the event** $\Omega^2_{n_k}(j)$ by the following conditions (refer to Proposition 2.12.7 and the discussion immediately preceding it to recall the notations):

1. There are exactly m zeroes of f_{n_k} in $\mathcal{D}$
2. There are no zeroes of f_{n_k} in the closed annulus of thickness $1/M_j$ around $\mathcal{D}$.
3. $|S_1(n_k)| \le M_j, |S_2(n_k)| \le M_j, |\widetilde{S}_3(n_k)| \le M_j$.
4. Each of $|L^{(n_k)}_{0q}|, |M^{(n_k)}_{0q}|, |L^{(n_k)}_{pq}|, |M^{(n_k)}_{pq}|, |N^{(n_k)}_{pq}|$ is $\le M_j/8$ for all $2 \le p, q \le m$.
5. $E_{n_k}/n_k \ge 1/2$.

It is clear from (2.25) that $\Omega^2_{n_k}(j) \subset \Omega^1_{n_k}(j)$.

Finally, we define an event $\Omega^3(j)$ by the conditions:

1. There are exactly m zeroes of f in $\mathcal{D}$
2. There are no zeroes of f in the closed annulus of thickness $1/M_j$ around $\mathcal{D}$.
3. $|S_1| \le M_j, |S_2| \le M_j - 1, |\widetilde{S}_3| \le M_j - 1$.
4. Each of $|L_{0q}|, |M_{0q}|, |L_{pq}|, |M_{pq}|, |N_{pq}|$ is $\le \frac{M_j}{8} - 1$ for all $2 \le p, q \le m$.

Notice (refer to Proposition 2.12.8) that for each of the random variables $Z^{(n_k)}_{pq}$ in condition 4 defining $\Omega^2_{n_k}(j)$ and Z_{pq} in condition 4 defining $\Omega^3(j)$ (where $Z = L, M, N$) we have

$$|Z^{(n_k)}_{pq}| \le |Z_{pq}| + |\tau(Z^{(n_k)}_{pq})|. \tag{2.35}$$

Consider the event F_{n_k} where any one of the following conditions hold:

1. There are m zeroes of f in $\mathcal{D}$ but this is not true for f_{n_k}.
2. There are no zeroes of f in the closed annulus of thickness $1/M_j$ around $\mathcal{D}$, but this is not true for f_{n_k}.
3. $|S_1 - S_1(n_k)| > 1$ or $|S_2 - S_2(n_k)| > 1$ or $|\widetilde{S}_3 - \widetilde{S}_3(n_k)| > 1$.
4. $|\tau(Z^{(n_k)}_{pq})| > 1$ for any of the random variables appearing in condition 4 of $\Omega^2_{n_k}(j)$.
5. $E_{n_k}/n_k < 1/2$.

It is easy to see (using (2.35)) that $\Omega^3(j) \setminus F_{n_k} \subset \Omega^2_{n_k}(j) \subset \Omega_{n_k}(j)$. However, by our choice of the sequence n_k, we have $\mathbb{P}(F_{n_k}) < 2^{-k}$, hence by Borel Cantelli lemma,

$$\underline{\lim}_{k\to\infty} \left(\Omega^3(j) \setminus F_{n_k}\right) = \Omega^3(j).$$

Hence, $\Omega^3(j) \subset \Omega(j) = \underline{\lim}_{k\to\infty} \Omega_{n_k}(j)$.

However, each random variable used in defining $\Omega^3(j)$ does not put any mass at ∞, and the thickness of the annulus around $\mathcal{D}$ in condition 2 in its definition goes to 0 as $j \to \infty$. Hence, the probability that each of the conditions 2-4 deinining $\Omega^3(j)$ holds goes to 1 as $j \to \infty$. Finally, condition 1 is just the definition of Ω^m. Hence, as $j \to \infty$, we have $\mathbb{P}\left(\Omega^m \setminus \Omega^3(j)\right) \to 0$. But $\Omega^3(j) \subset \Omega(j)$, hence $\mathbb{P}\left(\Omega^m \setminus \Omega(j)\right) \to 0$ as $j \to \infty$, as desired.

■

2.15 Reconstruction of GAF from Zeroes and Vieta's formula

In this section we prove the reconstruction Theorem 2.1.5 for the planar GAF. En route, we establish an analogue of Vieta's formula for the planar GAF.

2.16 Vieta's formula for thr planar GAF

It is an elementary fact that for a polynomial

$$p(z) = \sum_{j=0}^{N} a_j z^j$$

whose roots are $\{z_j\}_{j=1}^N$, we have, for any $1 \le k \le N$,

$$a_{N-k}/a_N = \sum_{i_1<i_2<\cdots<i_k} z_{i_1} \cdots z_{i_k}. \tag{2.36}$$

When $a_0 \neq 0$, we equivalently have

$$a_k/a_0 = \sum_{i_1<i_2<\cdots<i_k} \frac{1}{z_{i_1} \cdots z_{i_k}}. \tag{2.37}$$

This kind of result is broadly referred to as Vieta's formula. For an entire function instead of a polynomial, such results do not hold in general; for example it may not be possible to provide any reasonable interpretation to the function of the zeroes appearing on the right hand side of (2.37).

The quantity on the right hand side of (2.36) is the elementary symmetric function of order k in the variables $z_1, \cdots, z_N$, denoted by $e_k(z_1, \cdots, z_N)$. If we introduce the power sum $\beta_k = \sum_{j=1}^N z_j^k$, then it is known that for each k we have

$$e_k(z_1, \cdots, z_N) = P_k(\beta_1, \cdots, \beta_k)$$

where P_k is a homogeneous symmetric polynomial of degree k in the variables $(\beta_1, \cdots, \beta_k)$. The polynomial P_k is called the Newton polynomial of degree k, and it is known that the coefficients of P_k depend only on k and do not depend on n (refer [Sta99], chapter 7).

At this point, we recall the definition of α_i and $\alpha_i(n)$ from the discussion immediately preceeding Proposition 2.13.6.

Proposition 2.16.1. *For the planar GAF zero process, we have, a.s.* $\frac{\xi_k}{\xi_0} = P_k(\alpha_1, \alpha_2, \cdots, \alpha_k)$ *for each* $k \geq 1$.

Proof. We begin by observing that under the natural coupling of the planar GAF f and its approximating polynomials f_n (obtained by using the same coefficients ξ_i), we have by the Vieta's formula for each f_n:

$$\frac{\xi_k}{\xi_0} = P_k(\alpha_1(n), \cdots, \alpha_k(n)) \tag{2.38}$$

where P_k is, as before, the Newton polynomial of degree k. Clearly, P_k is continuous in the input variables. We also know, from Proposition 2.13.6 that, as $n \to \infty$ (possibly along an appropriately chosen subsequence), $\alpha_k(n) \to \alpha_k$ a.s., simultaneously for all $k \geq 1$. Taking this limit in (2.38) and using the continuity of P_k, we get a.s.

$$\frac{\xi_k}{\xi_0} = P_k(\alpha_1, \cdots, \alpha_k). \tag{2.39}$$

■

2.17 Proof of Theorem 2.1.5

Since ξ_0 is a complex Gaussian, therefore a.s. $|\xi_0| \neq 0$. For $|\xi_0| \neq 0$, we can write

$$f(z) = \frac{\xi_0}{|\xi_0|}.|\xi_0| \left(1 + \frac{\xi_1}{\xi_0} z + \frac{\xi_2}{\xi_0} z^2 + \cdots + \frac{\xi_k}{\xi_0} z^k + \cdots \infty\right) \tag{2.40}$$

From Proposition 2.16.1 we have that for each k the random variable $\frac{\xi_k}{\xi_0}$ is measurable with respect to $\mathcal{F}$. From the strong law of large numbers, a.s. we have

$$\frac{|\xi_0|^2 + \ldots + |\xi_{k-1}|^2}{k} \to 1.$$

Therefore

$$|\xi_0| = \lim_{k \to \infty} k^{1/2} \left(\sum_{j=0}^{k-1} \frac{|\xi_j|^2}{|\xi_0|^2} \right)^{-1/2} = \chi.$$

Since $k^{1/2} \left(\sum_{j=0}^{k-1} \frac{|\xi_j|^2}{|\xi_0|^2} \right)^{-1/2}$ is measurable with respect to $\mathcal{F}$ for each k, therefore $\chi = |\xi_0|$ is also measurable with respect to $\mathcal{F}$.

Let us define the random variable $\zeta = \xi_0/|\xi_0|$ (it is set to be equal to 0 when $|\xi_0| = 0$) and the random function g as in the statement can clearly be written as

$$g(z) = |\xi_0| \left(1 + \frac{\xi_1}{\xi_0} z + \frac{\xi_2}{\xi_0} z^2 + \cdots + \frac{\xi_k}{\xi_0} z^k + \cdots \infty \right).$$

Then (2.40) can be re-written as

$$f(z) = \zeta g(z) \tag{2.41}$$

almost surely, and g is measurable with respect to $\mathcal{F}$ and ζ is distributed uniformly on $\mathbb{S}^1$.

Therefore, all that remains to complete the proof is to show that ζ and $\mathcal{F}$ are independent. For this, note that for any fixed $\theta \in \mathbb{S}^1$ the random function

$$\sum_{j \geq 1} \frac{\theta \xi_j}{\sqrt{j!}} z^j$$

has the same distribution (which does not depend on θ). This is because since the ξ_i-s are complex Gaussians with mean 0 and variance 1, the vectors $(\theta \xi_j)_{j \geq 1}$ and $(\xi_j)_{j \geq 1}$ have the same distribution for each fixed $\theta \in \mathbb{S}^1$. Also, observe that $|\xi_0|, \{\xi_j\}_{j \geq 1}$ and ζ are jointly independent random variables.

Now,

$$g(z) = |\xi_0| + \sum_{j \geq 1} \frac{\bar{\zeta} \xi_j}{\sqrt{j!}} z^j.$$

Therefore, the distribution of $g(z)$ given ζ does not depend on the value of ζ. Hence, the random function g and ζ are independent. But, g and f have the same zero set a.s. Hence, ζ and $\mathcal{F}$ are independent random variables. This completes the proof.

2.18 Proof of Theorem 2.6.2

The main proof. We claim that it suffices to show that for every j_0 , we have positive real numbers $M(j_0)$ and $m(j_0)$ such that for any $A \in \mathcal{A}^m$ and any Borel set B in $\mathcal{S}_{\text{out}}$

$$\mathbb{P}\left((X_{\text{in}} \in A) \cap (X_{\text{out}} \in B) \cap \Omega(j_0) \right) \leq M(j_0) \int_{X_{\text{out}}^{-1}(B)} \nu(\xi, A) d\mathbb{P}(\xi) \tag{2.42}$$

and

$$m(j_0) \int_{X_{\text{out}}^{-1}(B) \cap \Omega(j_0)} \nu(\xi, A) d\mathbb{P}(\xi) \leq \mathbb{P}\left(X_{\text{in}} \in A \cap (X_{\text{out}} \in B) \right). \tag{2.43}$$

Once we have (2.42), we can invoke Proposition 2.18.1. Setting $\mu_1(\xi, \cdot) = \rho(X_{\text{out}}(\xi), \cdot), \mu_2(\xi, \cdot) = \nu(X_{\text{out}}(\xi), \cdot), a(j_0) = 1$ and $b(j_0) = M(j_0)$ in (2.45) we get $\rho(X_{\text{out}}, \cdot) \ll \nu(X_{\text{out}}, \cdot)$ a.s.

If we have (2.43), we can again appeal to Proposition 2.18.1. Setting $\mu_1(\xi,\cdot)=\nu(X_{\text{out}}(\xi),\cdot), \mu_2(\xi,\cdot)=\rho(X_{\text{out}}(\xi),\cdot), a(j_0)=m(j_0)$ and $b(j_0)=1$ in (2.45) we get $\nu(X_{\text{out}},\cdot)\ll\rho(X_{\text{out}},\cdot)$ a.s.

The last two paragraphs together imply that $\rho(X_{\text{out}},\cdot)\equiv\nu(X_{\text{out}},\cdot)$ a.s., as desired.

To establish (2.42) and (2.43), we begin with a fixed j_0, a set $A\in\mathcal{A}^m$ and a Borel set B in $\mathcal{S}_{\text{out}}$. We will invoke Proposition 2.18.2 in the following manner. Let the multiplicative constants appearing in the $\asymp_j$ relation in (2.3) with $j=j_0$ be $m(j_0)$ and $M(j_0)$ respectively, with $m(j_0)\le M(j_0)$. In Proposition 2.18.2 we set $h_1(\xi)=1, h_2(\xi)=\nu(\xi,A), U_1=A, U_2=\mathcal{D}^m, V=B, a(j_0)=1, b(j_0)=M(j_0)$ to obtain (2.42). On the other hand, setting $h_1(\xi)=\nu(\xi,A), h_2(\xi)=1, U_1=\mathcal{D}^m, U_2=A, V=B, a(j_0)=m(j_0), b(j_0)=1$ in Proposition 2.18.2 we obtain (2.43).

This completes the proof of theorem 2.6.2. ■

We end this section with Propositions 2.18.1 and 2.18.2 used in the above proof.

Proposition 2.18.1. *Let μ_1 and μ_2 be two functions mapping $\Xi\to\mathcal{M}(\mathcal{D}^m)$ such that for each Borel set $A\subset\mathcal{D}^m$, the function $\xi\to\mu_j(\xi,A)$ is measurable, $j=1,2$. Suppose μ_1 and μ_2 satisfy, for each positive integer j_0, $A\in\mathcal{A}^m$ and Borel set $B\subset\mathcal{D}^m$*

$$a(j_0)\int_{X_{\text{out}}^{-1}(B)\cap\Omega(j_0)}\mu_1(\xi,A)d\mathbb{P}(\xi)\le b(j_0)\int_{X_{\text{out}}^{-1}(B)}\mu_2(\xi,A)d\mathbb{P}(\xi) \tag{2.44}$$

for some positive numbers $a(j_0)$ and $b(j_0)$. Then a.s. we have

$$\mathbb{E}\left[\mu_1(\xi,\cdot)|X_{\text{out}}(\xi)\right]\ll\mathbb{E}\left[\mu_2(\xi,\cdot)|X_{\text{out}}(\xi)\right]. \tag{2.45}$$

Proof. (2.44) implies that for each $A\in\mathcal{A}^m$ a.s. in X_{out} we have

$$a(j_0)\mathbb{E}\left[\mu_1(\xi,A)|X_{\text{out}}(\xi),\Omega(j_0)\text{ occurs }\right]\mathbb{P}\left(\Omega(j_0)\text{ occurs }|X_{\text{out}}(\xi)\right)\le b(j_0)\mathbb{E}[\mu_2(\xi,A)|X_{\text{out}}(\xi)]. \tag{2.46}$$

For almost every configuration $\omega\in S_{\text{out}}$ (with respect to the measure $\mathbb{P}_{X_{\text{out}}}$) we have (2.46) for all $A\in\mathcal{A}^m$, and therefore by the regularity of the Borel measures on the two sides, (2.46) extends to all Borel sets $A\subset\mathcal{D}^m$ (see Proposition 2.6.1). Now, for $\omega\in\mathcal{S}_{\text{out}}$, suppose that $A\subset\mathcal{D}^m$ is a Borel set such that $\mathbb{E}\left[\mu_2(\xi,A)|X_{\text{out}}(\xi)=\omega\right]=0$. Then (2.46) implies that

$$\mathbb{E}\left[\mu_1(\xi,A)|X_{\text{out}}(\xi)=\omega,\Omega(j_0)\text{ occurs }\right]\mathbb{P}\left(\Omega(j_0)\text{ occurs }|X_{\text{out}}(\xi)=\omega\right)=0, \tag{2.47}$$

for each j_0. But $\Omega(j_0)$ exhausts Ω^m, hence

$$\mathbb{E}\left[\mu_1(\xi,A)|X_{\text{out}}(\xi)=\omega,\Omega(j_0)\text{ occurs }\right]\mathbb{P}\left(\Omega(j_0)\text{ occurs }|X_{\text{out}}(\xi)\right)\to\mathbb{E}\left[\mu_1(\xi,A)|X_{\text{out}}(\xi)=\omega\right]$$

as $j_0\uparrow\infty$. By letting $j_0\uparrow\infty$ in (2.47), we obtain the fact that a.s. on Ω^m we have $\mathbb{E}\left[\mu_2(\xi,A)|X_{\text{out}}(\xi)\right]=0$ implies $\mathbb{E}\left[\mu_1(\xi,A)|X_{\text{out}}(\xi)\right]=0$. In other words, a.s. on Ω^m, we have (2.45). ■

Proposition 2.18.2. *Suppose we have measurable functions h_1 and h_2 mapping $\Xi \to [0,1]$, and measurable sets $U_1, U_2 \in \mathcal{A}^m$. Define $\mathbb{P}_{h_i}$ to be the finite non-negative measure on Ξ given by $d\mathbb{P}_{h_i}(\xi) = h_i(\xi)d\mathbb{P}(\xi), i = 1,2$. Suppose there are positive numbers $a(j_0)$ and $b(j_0)$ such that the following inequality holds for all $\widetilde{V} \in \mathcal{B}$:*

$$a(j_0)\mathbb{P}_{h_1}\left[(X_{\text{in}}^{n_k} \in U_1) \cap (X_{\text{out}}^{n_k} \in \widetilde{V}) \cap \Omega_{n_k}(j_0)\right] \leq b(j_0)\mathbb{P}_{h_2}\left[(X_{\text{in}}^{n_k} \in U_2) \cap (X_{\text{out}}^{n_k} \in \widetilde{V}) \cap \Omega_{n_k}(j_0)\right] + \vartheta(k) \tag{2.48}$$

where $\vartheta(k) = \vartheta(k; j_0, U_1, U_2, \widetilde{V}) \to 0$ as $k \to \infty$ for fixed $j_0, U_1, U_2, \widetilde{V}$. Then for all Borel sets V in $\mathcal{S}_{\text{out}}$, we have

$$a(j_0)\mathbb{P}_{h_1}\left[(X_{\text{in}} \in U_1) \cap (X_{\text{out}} \in V) \cap \Omega(j_0)\right] \leq b(j_0)\mathbb{P}_{h_2}\left[(X_{\text{in}} \in U_2) \cap (X_{\text{out}} \in V)\right]. \tag{2.49}$$

Proof. In what follows, we will denote by $\mathbb{P}_h$ the non-negative finite measure on Ξ obtained by setting $d\mathbb{P}_h(\xi) = h(\xi)d\mathbb{P}(\xi)$ where $h : \Xi \to [0,1]$ is a measurable function. We note that for any event E, we have $0 \leq \mathbb{P}_h(E) \leq \mathbb{P}(E)$. Fix a $U \in \mathcal{A}^m$.

For any Borel set V in $\mathcal{S}_{\text{out}}$, given $\varepsilon > 0$ we can find a $\widetilde{B} \in \mathcal{B}$ such that

$$\mathbb{P}\left(X_{\text{out}}^{-1}(V)\Delta X_{\text{out}}^{-1}(\widetilde{B})\right) < \varepsilon.$$

This can be seen by considering the push forward probability measure $(X_{\text{out}})_*\mathbb{P}$ on $\mathcal{S}_{\text{out}}$. The aim of this reduction is to exploit the fact that as $k \to \infty$, we have $1_{\widetilde{B}}(X_{\text{out}}^{n_k}) \to 1_{\widetilde{B}}(X_{\text{out}})$ a.s.

We start with $\mathbb{P}_h[(X_{\text{in}}^{n_k} \in U) \cap (X_{\text{out}}^{n_k} \in \widetilde{B}) \cap \Omega_{n_k}(j_0)]$. This is equal to

$$\mathbb{P}_h\left[(X_{\text{in}} \in U) \cap (X_{\text{out}} \in \widetilde{B}) \cap \Omega_{n_k}(j_0)\right] + o_k(1; \widetilde{B})$$

where $o_k(1; \widetilde{B})$ stands for a quantity that tends to 0 as $k \to \infty$ for fixed $\widetilde{B}$. This step uses the fact that $1\left[X_{\text{out}}^{n_k} \in \widetilde{B}\right] \to 1\left[X_{\text{out}} \in \widetilde{B}\right]$ and $1\left[X_{\text{in}}^{n_k} \in U\right] \to 1\left[X_{\text{in}} \in U\right]$ a.s. The expression in the last display above equals

$$\mathbb{P}_h\left[(X_{\text{in}} \in U) \cap (X_{\text{out}} \in V) \cap \Omega_{n_k}(j_0)\right] + o_\varepsilon(1) + o_k(1; \widetilde{B})$$

where $o_\varepsilon(1)$ denotes a quantity that tends to 0 uniformly in k as $\varepsilon \to 0$.

We upper bound the probability in the last display simply by $\mathbb{P}_h\left[(X_{\text{in}} \in U) \cap (X_{\text{out}} \in B)\right]$. Putting all these together, we have

$$\mathbb{P}_h[(X_{\text{in}}^{n_k} \in U) \cap (X_{\text{out}}^{n_k} \in \widetilde{B}) \cap \Omega_{n_k}(j_0)] \leq \mathbb{P}_h\left[(X_{\text{in}} \in U) \cap (X_{\text{out}} \in V)\right] + o_\varepsilon(1) + o_k(1; \widetilde{B}). \tag{2.50}$$

To obtain a comparable lower bound, we need to work more. We begin with

$$\mathbb{P}_h\Big[(X_{\text{in}} \in U) \cap (X_{\text{out}} \in V) \cap \Omega_{n_k}(j_0)\Big] \geq \mathbb{P}_h\Big[(X_{\text{in}} \in U) \cap (X_{\text{out}} \in V) \cap \Omega_{n_k}(j_0) \cap \Omega(j_0)\Big].$$

Observe that $\Omega(j_0) = \underline{\lim}_{k\to\infty} \Omega_{n_k}(j_0)$ and $\mathbb{P}\left(\underline{\lim}_{l\to\infty} \Omega_{n_l}(j_0) \Delta \left(\bigcap_{l\geq k} \Omega_{n_l}(j_0)\right)\right) = \mathfrak{o}_k(1)$ where $\mathfrak{o}_k(1)$ denotes a quantity $\to 0$ as $k \to \infty$, (for a fixed j_0) uniformly in all the other quantities.

Hence we have

$$\begin{aligned} &\mathbb{P}_h\Big[(X_{\text{in}} \in U) \cap (X_{\text{out}} \in V) \cap \Omega_{n_k}(j_0) \cap \Omega(j_0)\Big] \\ &= \mathbb{P}_h\Big[(X_{\text{in}} \in U) \cap (X_{\text{out}} \in V) \cap \Omega_{n_k}(j_0) \cap \Big(\bigcap_{l\geq k} \Omega_{n_l}(j_0)\Big)\Big] + \mathfrak{o}_k(1)\ . \end{aligned}$$

But $\Omega_{n_k}(j_0) \cap \left(\bigcap_{l\geq k} \Omega_{n_l}(j_0)\right) = \left(\bigcap_{l\geq k} \Omega_{n_l}(j_0)\right)$ and hence the probability in the last display equals

$$\mathbb{P}_h\Big[(X_{\text{in}} \in U) \cap (X_{\text{out}} \in V) \cap \Omega(j_0)\Big] + \mathfrak{o}_k(1).$$

The arguments above result in

$$\begin{aligned} &\mathbb{P}_h\Big[(X_{\text{in}}^{n_k} \in U) \cap (X_{\text{out}}^{n_k} \in \widetilde{B}) \cap \Omega_{n_k}(j_0)\Big] \\ &\geq \mathbb{P}_h\Big[(X_{\text{in}} \in U) \cap (X_{\text{out}} \in V) \cap \Omega(j_0)\Big] + o_k(1; \widetilde{B}) + o_\varepsilon(1) + \mathfrak{o}_k(1). \end{aligned} \tag{2.51}$$

Now, we wish to prove (2.49). We appeal to (2.50) with $h = h_2, U = U_2$ and to (2.51) with $h = h_1, U = U_1$ to obtain, using (2.48) (applied with $\widetilde{V} = \widetilde{B}$),

$$\begin{aligned} &a(j_0)\mathbb{P}_{h_1}\Big[(X_{\text{in}} \in U_1) \cap (X_{\text{out}} \in V) \cap \Omega(j_0)\Big] \\ &\leq b(j_0)\mathbb{P}_{h_2}\Big[(X_{\text{in}} \in U_2) \cap (X_{\text{out}} \in V)\Big] + o_\varepsilon(1) + o_k(1; \widetilde{B}) + \mathfrak{o}_k(1) + \vartheta(k). \end{aligned} \tag{2.52}$$

We first keep all the other quantities fixed and let $k \to \infty$, after that we let $\varepsilon \to 0$ to obtain (2.49), as desired. ■

Chapter 3

Rigidity and Tolerance in point processes: quantitative estimates

3.1 Introduction

In Chapter 2 we studied spatial conditioning in the two main natural examples of repelling point processes on the Euclidean plane, namely the Ginibre ensemble and the Gaussian zero process. They established certain "rigidity" phenomena, in the sense that for a bounded open set $\mathcal{D}$ (satisfying some minimal regularity conditions) the points outside $\mathcal{D}$ determine a.s. the number N of points inside $\mathcal{D}$ in the first case and both their number N and their sum S in the second case. Thus, the support of the conditional distribution of the inside points is contained in $\mathcal{D}^N$ in the first case, and in a (complex) co-dimension 1 subset of $\mathcal{D}^N$ in the second. It was further established in Chapter 2 that, on this restricted set, the conditional distribution is mutually absolutely continuous with respect to the Lebesgue measure. In this chapter, our aim is to refine the result in Chapter 2 and establish quantitative bounds on the conditional density of the inside points (considered as a vector in D^N by taking them in uniform random order).

In this chapter, we prove the following quantitative estimates on the density of the conditional distributions with respect to the Lebesgue measure on their support:

Theorem 3.1.1. *There exist positive quantities $m(\mathcal{G}_{\text{out}})$ and $M(\mathcal{G}_{\text{out}})$, measurable with respect to $\mathcal{G}_{\text{out}}$, such that a.s. we have*

$$m(\mathcal{G}_{\text{out}})|\Delta(\underline{\zeta})|^2 \leq \frac{d\rho(\mathcal{G}_{\text{out}},\cdot)}{d\mathcal{L}}(\underline{\zeta}) \leq M(\mathcal{G}_{\text{out}})|\Delta(\underline{\zeta})|^2$$

where $\Delta(\underline{\zeta})$ is the Vandermonde determinant formed by the co-ordinates of $\underline{\zeta}$ and $\mathcal{L}$ is the Lebesgue measure on $\mathcal{D}^N(\mathcal{G}_{\text{out}})$.

Theorem 3.1.2. *There exist positive quantities $m(\mathcal{F}_{\text{out}})$ and $M(\mathcal{F}_{\text{out}})$, measurable with respect to $\mathcal{F}_{\text{out}}$, such that a.s. we have*

$$m(\mathcal{F}_{\text{out}})|\Delta(\underline{\zeta})|^2 \leq \frac{d\rho(\mathcal{F}_{\text{out}},\cdot)}{d\mathcal{L}_\Sigma}(\underline{\zeta}) \leq M(\mathcal{F}_{\text{out}})|\Delta(\underline{\zeta})|^2$$

where $\Delta(\underline{\zeta})$ is the Vandermonde determinant formed by the co-ordinates of $\underline{\zeta}$ and $\mathcal{L}_\Sigma$ is the Lebesgue measure on $\Sigma_{S(\mathcal{F}_{\text{out}})}$.

Our results show, in particular, that even under spatial conditioning, the points of the Ginibre ensemble or the GAF zero process repel each other at close range, and the quantitative nature of the repulsion (quadratic in the mutual separation of the points) is similar to that of the unconditioned ensembles.

3.2 Limits of conditional measures: abstract setting

In this Section, we state and prove some general conditions (in the context of the abstract setup considered in Section 2.3 in Chapter 2), which will enable us to make the transition from the finite ensembles to the infinite one. The definitions and notations used in this section are from Section 2.3 in Chapter 2.

Now we are ready to state the following important technical reduction:

Theorem 3.2.1. *Let $m \geq 0$ be such that $\mathbb{P}(\Omega^m) > 0$. Suppose that:*

(a) There is a map $\nu : \Omega^m \to \mathcal{M}(\mathcal{D}^m)$ such that for each Borel set $A \subset \mathcal{D}^m$, the random variable $\nu(\cdot, A)$ is measurable with respect to X_{out}. Let $\widetilde{\nu} : S_{\text{out}} \to \mathcal{M}(\mathcal{D}^m)$ denote the map induced by ν.

(b) For each fixed j we have an event $\Omega(j)$, a sequence $\{n_k\}_{k\geq 1}$ (which might depend on j) and corresponding events $\Omega_{n_k}(j)$ such that:

(i) Each $\Omega(j)$ is measurable with respect to X_{out}, and the sequence of events $\Omega(j)$ exhausts Ω^m.

(ii) $\Omega_{n_k}(j)$ is measurable with respect to $X_{\text{out}}^{n_k}$ and $\Omega_{n_k}(j) \subset \Omega_{n_k}^m$.

(iii) $\Omega(j) \subset \underline{\lim}_{k\to\infty} \Omega_{n_k}(j)$.

(iv) For all $A \in \mathcal{A}$ and $B \in \mathcal{B}$ we have

$$\mathbb{P}[(\underline{\zeta}^{n_k} \in A) \cap (X_{\text{out}}^{n_k} \in B) \cap \Omega_{n_k}(j)] \asymp_j \int_{(X_{\text{out}}^{n_k})^{-1}(B)\cap\Omega_{n_k}(j)} \nu(\xi, A) d\mathbb{P}(\xi) + \vartheta(k; j). \tag{3.1}$$

where $\lim_{k\to\infty} \vartheta(k; j) \to 0$ for each fixed j.

Then, there are functions $\mathfrak{m}, \mathfrak{M} : \mathcal{S}_{\text{out}} \to \mathbb{R}_+$ such that a.s. on the event Ω^m we have

$$\mathfrak{m}(\omega)\widetilde{\nu}(\omega, A) \leq \rho(\omega, A) \leq \mathfrak{M}(\omega)\widetilde{\nu}(\omega, A) \tag{3.2}$$

for all Borel sets A in $\mathcal{D}^m$, where ω denotes $X_{\text{out}}(\xi), \xi \in \Xi$.

Remark 3.2.1. *The condition that $\Omega(j)$ is measurable with respect to X_{out} can be relaxed to the condition that $\widetilde{\Omega}(j)$ is measurable with respect to X_{out}, for some event $\widetilde{\Omega}(j)$ which satisfies $\mathbb{P}\left(\Omega(j)\Delta\widetilde{\Omega}(j)\right) = 0$.*

We defer the proof of Theorem 3.2.1 to Section 3.13.

We conclude this section with the following simple observation:

Remark 3.2.2. *If we have Theorem 3.2.1 for all $m \geq 0$ then we can conclude that (3.2) holds a.e. $\xi \in \Xi$.*

3.3 Reduction from a general $\mathcal{D}$ to a disk

We will provide a proof that Theorem 3.1.2 in the case where $\mathcal{D}$ is a disk is enough to deduce the case of a general $\mathcal{D}$. The corresponding reduction for Theorem 3.1.1 is on similar lines. The first part of this proof is very similar to the corresponding reduction in Chapter 2. Some additional arguments are introduced subsequently in order to take care of the comparison to the squared Vandermonde.

Let $\mathcal{D}$ be a bounded open set in $\mathbb{C}$ whose boundary has zero Lebesgue measure. By translation invariance of the Ginibre ensemble, we take the origin to be in the interior of $\mathcal{D}$. Let $\mathcal{D}_0$ be a disk (centred at the origin) which contains $\overline{\mathcal{D}}$ in its interior (where $\overline{\mathcal{D}}$ is the closure of $\mathcal{D}$).

Suppose we have the tolerance Theorem 3.1.2 for a disk. To obtain Theorem 3.1.2 for $\mathcal{D}$, we appeal to the tolerance Theorem 3.1.2 for the disk $\mathcal{D}_0$. Let the number and the sum of the points in $\mathcal{D}$ be N and S respectively, let N_0 and S_0 denote the corresponding quantities for the points in $\mathcal{D}_0$. Define

$$\Sigma := \left\{(\lambda_1, \cdots, \lambda_N) : \sum_{j=1}^{N} \lambda_j = S, \lambda_j \in \mathcal{D}\right\}$$

and

$$\Sigma_0 := \left\{(\lambda_1, \cdots, \lambda_{N_0}) : \sum_{j=1}^{N_0} \lambda_j = S_0, \lambda_j \in \mathcal{D}_0\right\}.$$

The conditional distribution of the vector of points inside $\mathcal{D}_0$, given $\Upsilon_{\text{out}}^{\mathcal{D}_0}$, lives on Σ_0, in fact it has a density f_0 which is positive a.e. with respect to Lebesgue measure on Σ_0. Let there be k points in $\mathcal{D}_0 \setminus \mathcal{D}$ and let their sum be s, clearly we have $N = N_0 - k$ and $S = S_0 - s$. We parametrize Σ by the last $N-1$ co-ordinates. Note that the set $U := \{(\lambda_2, \cdots, \lambda_N) : (S - \sum_{j=2}^{N} \lambda_j, \lambda_2, \cdots, \lambda_N) \in \Sigma\}$ is an open subset of $\mathcal{D}^{N-1}$. Further, we define the set $V := \{(\lambda_1, \cdots, \lambda_k) : \lambda_i \in \mathcal{D}_0 \setminus \mathcal{D}, \sum_{i=1}^{k} \lambda_i = s\}$.

Let the points in $\mathcal{D}_0 \setminus \mathcal{D}$, taken in uniform random order, form the vector $\underline{\mathbf{z}} = (z_1, \cdots, z_k)$. Then we can condition the vector of points in $\mathcal{D}_0$ to have its last k co-ordinates equal to $\underline{\mathbf{z}}$, to obtain the following formula for the conditional density of the vector of points in $\mathcal{D}$ at $(\zeta_1, \cdots, \zeta_N) \in \Sigma$ (with respect to the Lebesgue measure on Σ):

$$f(\zeta_1, \zeta_2, \cdots, \zeta_N) = \frac{f_0(\zeta_1, \zeta_2, \cdots, \zeta_N, z_1, \cdots, z_k)}{\int_U f_0(s - (\sum_{j=2}^N w_j), w_2, \cdots, w_N, z_1, \cdots, z_k) dw_2 \cdots dw_N}. \tag{3.3}$$

It is clear that for a.e. $\underline{\mathbf{z}} \in V$, we have f is positive a.e. with respect to Lebesgue measure on Σ, because the same is true of f_0 on Σ_0.

Let $\underline{\zeta}^0 = (\underline{\zeta}, \underline{\mathbf{z}})$. For estimating f from above and below, recall that a.s. we have

$$m(\Upsilon_{\text{out}}^{\mathcal{D}_0})|\Delta(\underline{\zeta}^0)|^2 \le f_0(\underline{\zeta}^0) \le M(\Upsilon_{\text{out}}^{\mathcal{D}_0})|\Delta(\underline{\zeta}^0)|^2. \tag{3.4}$$

The denominator of the right hand side of (3.3) does not depend on $\underline{\zeta}$, it only depends on the set $\mathcal{D}$ and Υ_{out}. Therefore, to obtain the desired upper and lower bounds on $f(\underline{\zeta})$, it suffices to estimate $|\Delta(\underline{\zeta}^0)|^2$.

For two vectors $\underline{\alpha} = (\alpha_1, \cdots, \alpha_m)$ and $\underline{\beta} = (\beta_1, \cdots, \beta_n)$ we define

$$\Gamma(\underline{\alpha}, \underline{\beta}) = \prod_{1 \le i \le m, 1 \le j \le n} |\alpha_i - \beta_j|.$$

Then we can write $|\Delta(\underline{\zeta}^0)|^2 = |\Gamma(\underline{\zeta}, \underline{\mathbf{z}})|^2 |\Delta(\underline{\mathbf{z}})|^2 |\Delta(\underline{\zeta})|^2$. Further, since $\partial\mathcal{D}$ is of zero Lebesgue measure, therefore a.s. $\Upsilon_{\text{out}} \cap \partial\mathcal{D} = \phi = \Upsilon_{\text{in}} \cap \partial\mathcal{D}$. Hence, there exists $\delta(\Upsilon_{\text{out}}) > 0$ such that Dist $(\overline{\mathcal{D}}, \Upsilon_{\text{out}}) = \delta(\Upsilon_{\text{out}})$, in the sense of the distance between a closed set and a compact set. Hence, if $\underline{\zeta} \in \Sigma$, we have $\delta(\Upsilon_{\text{out}}) \le |z_i - \zeta_j| \le 2$ Radius $(\mathcal{D}_0)$ for all i and j.

Combining all these observations with equations (3.3) and (3.4) we obtain the desired estimates for $f(\underline{\zeta})$.

3.4 Limiting procedure for the Ginibre ensemble

As we already saw in Section 3.3, it suffices to prove Theorem 3.1.1 for $\mathcal{D}$ a disk. To this end, we need to appropriately define the quantities as in the statement of Theorem 3.2.1, and verify that they satisfy the conditions in that theorem. For the Ginibre ensemble, we define the quantities in exactly the same way as in Section 8 of Chapter 2. There we define $\Omega(j) := \underline{\lim}_{k \to \infty} \Omega_{n_k}(j)$. In Proposition 8.1 in Chapter 2 it is shown that $\Omega(j)$ so defined is measurable with respect to $\mathcal{G}_{\text{out}}$ (up to a null set, also see Remark 3.2.1) and $\Omega(j)$-s exhaust Ω^m, as desired.

3.5 Estimates for finite approximations of the GAF

In this section, we recall several estimates from Chapter 2 which would be necessary to establish Theorem 3.1.2. We focus on the event $\Omega_n^{m,\delta}$ which entails that f_n has exactly m

zeroes inside $\mathcal{D}$, and there is at least a δ separation between $\partial\mathcal{D}$ and the outside zeroes. The corresponding event for the GAF zero process has positive probability, so by the distributional convergence $\mathcal{F}_n \to \mathcal{F}$, we have that $\Omega_n^{m,\delta}$ has positive probability (which is bounded away from 0 as $n \to \infty$).

Let us denote the zeroes of f_n (in uniform random order) inside $\mathcal{D}$ by $\underline{\zeta} = (\zeta_1, \cdots, \zeta_m)$ and those outside $\mathcal{D}$ by $\underline{\omega} = (\omega_1, \omega_2, \cdots, \omega_{n-m})$. Let s denote the sum of the inside zeroes. Then the conditional density $\rho_{\underline{\omega}}^n(\underline{\zeta})$ of $\underline{\zeta}$ given $\underline{\omega}$ and s is of the following form, supported on the set $\Sigma_s = \{\underline{\zeta} \in \mathcal{D}^m, \sum_{j=1}^m \zeta_j = s\}$ (see, e.g., [FH99]):

$$\rho_{\underline{\omega},s}^n(\underline{\zeta}) = C(\underline{\omega}, s) \frac{|\Delta(\zeta_1, \ldots, \zeta_m, \omega_1, \ldots, \omega_{n-m})|^2}{\left(\sum_{k=0}^{n} \left| \sigma_k(\underline{\zeta}, \underline{\omega}) \sqrt{\binom{n}{k} k!} \right|^2 \right)^{n+1}} \tag{3.5}$$

where $C(\underline{\omega})$ is the normalizing factor, and for a vector $\underline{v} = (v_1, \cdots, v_N)$ we define

$$\sigma_k(\underline{v}) = \sum_{1 \le i_1 < \cdots < i_k \le N} v_{i_1} \cdots v_{i_k}$$

and for two vectors $\underline{u}$ and $\underline{v}$, $\sigma_k(\underline{u}, \underline{v})$ is defined to be $\sigma_k(\underline{w})$ where the vector $\underline{w}$ is obtained by concatenating the vectors $\underline{u}$ and $\underline{v}$.

Let $(\underline{\zeta}, \underline{\omega})$ and $(\underline{\zeta}', \underline{\omega})$ be two vectors of points (under $\mathcal{F}_n$), such that the sum of the coordinates of $\underline{\zeta}$ and the same quantity for $\underline{\zeta}'$ are equal to s. Then the ratio of the conditional densities at these two vectors is given by

$$\frac{\rho_{\underline{\omega},s}^n(\underline{\zeta}')}{\rho_{\underline{\omega},s}^n(\underline{\zeta})} = \frac{|\Delta(\underline{\zeta}', \underline{\omega})|^2}{|\Delta(\underline{\zeta}, \underline{\omega})|^2} \left(\sum_{k=0}^{n} \left| \frac{\sigma_k(\underline{\zeta}, \underline{\omega})}{\sqrt{\binom{n}{k} k!}} \right|^2 \right)^{n+1} \Bigg/ \left(\sum_{k=0}^{n} \left| \frac{\sigma_k(\underline{\zeta}', \underline{\omega})}{\sqrt{\binom{n}{k} k!}} \right|^2 \right)^{n+1}. \tag{3.6}$$

Proposition 3.5.1. *On $\Omega_n^{m,\delta}$ there are quantities $K(\mathcal{D}, \delta) > 0$ and $\boldsymbol{X}_n(\underline{\omega}) > 0$ such that for any $(\underline{\zeta}, \underline{\zeta}') \in \mathcal{D}^m \times \mathcal{D}^m$ we have*

$$\exp\Big(-2mK(\mathcal{D}, \delta)\boldsymbol{X}_n(\underline{\omega}) \Big) \frac{|\Delta(\underline{\zeta}')|^2}{|\Delta(\underline{\zeta})|^2} \le \frac{|\Delta(\underline{\zeta}', \underline{\omega})|^2}{|\Delta(\underline{\zeta}, \underline{\omega})|^2} \le \exp\Big(2mK(\mathcal{D}, \delta)\boldsymbol{X}_n(\underline{\omega}) \Big) \frac{|\Delta(\underline{\zeta}')|^2}{|\Delta(\underline{\zeta})|^2}$$

where $\boldsymbol{X}_n(\underline{\omega}) = \left| \sum_{\omega_j \in \mathcal{G}_n \cap \mathcal{D}^c} \frac{1}{\omega_j} \right| + \left| \sum_{\omega_j \in \mathcal{G}_n \cap \mathcal{D}^c} \frac{1}{\omega_j^2} \right| + \left(\sum_{\omega_j \in \mathcal{G}_n \cap \mathcal{D}^c} \frac{1}{|\omega_j|^3} \right)$ *and* $\mathbb{E}[\boldsymbol{X}_n(\underline{\omega})] \le c_1(\mathcal{D}, m) < \infty$.

For the remainder of this section we will restrict $\underline{\zeta}$ and $\underline{\zeta}'$ to lie in the same constant-sum hyperplane. Recall the notation that $s = \sum_{i=1}^m \zeta_i$ and

$$\Sigma_s := \{\underline{\zeta} \in \mathcal{D}^m : \sum_{i=1}^{m} \zeta_i = s\}.$$

Let $D(\underline{\zeta},\underline{\omega}) = \left(\sum_{k=0}^{n} \left|\frac{\sigma_k(\underline{\zeta},\underline{\omega})}{\sqrt{\binom{n}{k}k!}}\right|^2\right)$.

When we want to bound the ratio $\left(D(\underline{\zeta}',\underline{\omega})/D(\underline{\zeta},\underline{\omega})\right)^{n+1}$ from above and below, it suffices to show that the quantities

$$\left|\sum_{k=0}^{n} \frac{\overline{\sigma_k(\underline{\zeta},\underline{\omega})}}{\sqrt{\binom{n}{k}k!}} \frac{\sigma_{k-i}(\underline{\omega})}{\sqrt{\binom{n}{k}k!}}\right| \Bigg/ \left(\sum_{k=0}^{n} \left|\frac{\sigma_k(\underline{\zeta},\underline{\omega})}{\sqrt{\binom{n}{k}k!}}\right|^2\right) \tag{3.7}$$

and

$$\left|\sum_{k=0}^{n} \frac{\overline{\sigma_{k-i}(\underline{\omega})}}{\sqrt{\binom{n}{k}k!}} \frac{\sigma_{k-j}(\underline{\omega})}{\sqrt{\binom{n}{k}k!}}\right| \Bigg/ \left(\sum_{k=0}^{n} \left|\frac{\sigma_k(\underline{\zeta},\underline{\omega})}{\sqrt{\binom{n}{k}k!}}\right|^2\right) \tag{3.8}$$

for $m \geq i, j \geq 2$ are bounded above by random variables whose typical size is $O(1/n)$.

The following decomposition of $\sigma_k(\underline{\zeta},\underline{\omega})$ is simple but useful:

$$\sigma_k(\underline{\zeta},\underline{\omega}) = \sum_{r=0}^{m} \sigma_r(\underline{\zeta})\sigma_{k-r}(\underline{\omega}). \tag{3.9}$$

The following expansion of $\sigma_k(\underline{\omega})$ in terms of $\sigma_i(\underline{\zeta},\underline{\omega})$ is more involved:

Proposition 3.5.2. *On the event $\Omega_n^{m,\delta}$ we have, for $0 \leq k \leq n-m$,*

$$\sigma_k(\underline{\omega}) = \sigma_k(\underline{\zeta},\underline{\omega}) + \sum_{r=1}^{k} g_r \sigma_{k-r}(\underline{\zeta},\underline{\omega})$$

where a.s. the random variables g_r are $O(K(\mathcal{D},m)^r)$ as $r \to \infty$, for a deterministic quantity $K(\mathcal{D},m)$ and the constant in O being deterministic and uniform in n and δ.

From Chapter 2 equation (22) in Section 9.1.2 we have, for $l \geq m$,

$$\frac{\sigma_{n-l}(\underline{\omega})}{\sqrt{\binom{n}{n-l}(n-l)!}} = \frac{1}{\xi_n}\left[\xi_l + \eta_l^{(n)}\right]. \tag{3.10}$$

Proposition 3.5.3. *Let $\eta_l^{(n)}$ be as in (3.10) and $\gamma = \frac{1}{8}$. Then $\exists$ positive random variables η_l (independent of n) such that a.s. $\left|\eta_l^{(n)}\right| \leq \eta_l$, and for fixed $l_0 \in \mathbb{N}$ and large enough $M > 0$ we have*

$$(i)\mathbb{P}\left[\eta_l > \frac{M}{l^\gamma} \text{ for some } l \geq 1\right] \leq e^{-c_1 M^2}$$

$$(ii)\mathbb{P}\left[\eta_l > \frac{M}{l^\gamma} \text{ for some } l \geq l_0\right] \leq e^{-c_2 M^2 l_0^{\frac{1}{4}}}$$

where c_1, c_2 are constants that depend on the domain $\mathcal{D}$ and on m.

Define

$$\boldsymbol{E}_n = \sum_{l=0}^{n} |\xi_l|^2, \qquad \boldsymbol{L}_{ij}^{(n)} = \sum_{l=0}^{n-i\wedge n-j} \frac{\overline{\xi_{l+i}}\xi_{l+j}}{\sqrt{(l+i)_i(l+j)_j}} \ ,$$

$$\boldsymbol{M}_{ij}^{(n)} = \sum_{l=0}^{n-i\wedge n-j} \frac{|\xi_{l+i}|\eta_{l+j}}{\sqrt{(l+i)_i(l+j)_j}} \ , \qquad \boldsymbol{N}_{ij}^{(n)} = \sum_{l=0}^{n-i\wedge n-j} \frac{\eta_{l+i}\eta_{l+j}}{\sqrt{(l+i)_i(l+j)_j}} \ .$$

Let

$$\boldsymbol{Y}_n = \sum_{i=2}^{m} \left|\boldsymbol{L}_{0i}^{(n)}\right| + \sum_{i=2}^{m} \boldsymbol{M}_{0i}^{(n)} + \sum_{i,j\geq 2}^{m} \left|\boldsymbol{L}_{ij}^{(n)}\right| + \sum_{i,j\geq 2}^{m} \boldsymbol{M}_{ij}^{(n)} + \sum_{i,j\geq 2}^{m} \boldsymbol{M}_{ji}^{(n)} + \sum_{i,j\geq 2}^{m} \boldsymbol{N}_{ij}^{(n)} \ .$$

In Chapter 2 it has been shown (Section 9.1.2 equation (27)) that

$$1 - K(m,\mathcal{D})\frac{\boldsymbol{Y}_n}{\boldsymbol{E}_n} \leq \frac{D(\underline{\zeta}',\underline{\omega})}{D(\underline{\zeta},\underline{\omega})} \leq 1 + K(m,\mathcal{D})\frac{\boldsymbol{Y}_n}{\boldsymbol{E}_n} \ . \tag{3.11}$$

Regarding $\boldsymbol{Y}_n$ and $\boldsymbol{E}_n$, we have the following estimates:

Proposition 3.5.4. *Given $M > 0$ we have:*
(i) $\mathbb{P}[\boldsymbol{Y}_n \geq M\log M] \leq c(m,\mathcal{D})/M$,
(ii) Given $M > 0$ there exists n_0 such that for $n \geq n_0$ we have $\mathbb{P}[\frac{n}{2} \leq |\mathbb{E}_n| \leq 2n] \geq 1 - \frac{1}{M}$.

These lead to:

Proposition 3.5.5. *Given $M > 0$ large enough, $\exists n_0$ such that for all $n \geq n_0$ the following is true: with probability $\geq 1 - C/M$ we have, on $\Omega_n^{m,\delta}$,*

$$e^{-2K(m,\mathcal{D})M\log M} \leq \left(D(\underline{\zeta}',\underline{\omega})\Big/D(\underline{\zeta},\underline{\omega})\right)^{n+1} \leq e^{2K(m,\mathcal{D})M\log M}$$

for all $\underline{\zeta}' \in \Sigma_s$, where $s = \sum_{i=1}^m \zeta_i$ and $(\underline{\zeta},\underline{\omega})$ is randomly generated from $\mathcal{F}_n$.

Finally, we arrive at

Proposition 3.5.6. *There exist constants $K(m,\mathcal{D},\delta)$ such that given $M > 0$ large enough, we have for $n \geq n_0(m,M,\mathcal{D})$ the following inequalities hold on $\Omega_n^{m,\delta}$, except for an event of probability $\leq c(m,\mathcal{D})/M$:*

$$\exp\left(-K(m,\mathcal{D},\delta)MlogM\right)\frac{|\Delta(\underline{\zeta}'')|^2}{|\Delta(\underline{\zeta}')|^2} \leq \frac{\rho^n_{\underline{\omega},s}(\underline{\zeta}'')}{\rho^n_{\underline{\omega},s}(\underline{\zeta}')} \leq \exp\left(K(m,\mathcal{D},\delta)MlogM\right)\frac{|\Delta(\underline{\zeta}'')|^2}{|\Delta(\underline{\zeta}')|^2}$$

uniformly for all $(\underline{\zeta}',\underline{\zeta}'') \in \Sigma_s \times \Sigma_s$, where $(s,\underline{\omega})$ corresponds to a point configuration picked randomly from $\mathbb{P}[\mathcal{F}_n]$.

The following important estimates deal with inverse power sums of the Gaussian zeroes.

Let $r_0 = \text{radius}\ (\mathcal{D})$. Let φ be a non-negative radial C_c^∞ function supported on $[r_0, 3r_0]$ such that $\varphi = 1$ on $[\frac{3}{2}r_0, 2r_0]$ and $\varphi(r_0 + r) = 1 - \varphi(2r_0 + 2r)$, for $0 \le r \le \frac{1}{2}r_0$. In other words, φ is a test function supported on the annulus between r_0 and $3r_0$ and its "ascent" to 1 is twice as fast as its "descent".

Proposition 3.5.7. *(i) The random variables*

$$S_l(n) := \int \frac{\widetilde{\varphi}(z)}{z^l}\, d[\mathcal{F}_n](z) + \sum_{j=1}^{\infty} \int \frac{\varphi_{2^j}(z)}{z^l}\, d[\mathcal{F}_n](z) = \sum_{\omega \in \mathcal{F}_n \cap \mathcal{D}^c} \frac{1}{\omega^l} \quad (\text{ for } l \ge 1)$$

and

$$\widetilde{S}_l(n) := \int \frac{\widetilde{\varphi}(z)}{|z|^l}\, d[\mathcal{F}_n](z) + \sum_{j=1}^{\infty} \int \frac{\varphi_{2^j}(z)}{|z|^l}\, d[\mathcal{F}_n](z) = \sum_{\omega \in \mathcal{F}_n \cap \mathcal{D}^c} \frac{1}{|\omega|^l} \quad (\text{ for } l \ge 3)$$

have finite first moments which, for every fixed l, are bounded above uniformly in n.

(ii) There exists $k_0 = k_0(\varphi) \ge 1$, uniform in n and l, such that for $k \ge k_0$ the "tails" of $S_l(n)$ and $\widetilde{S}_l(n)$ beyond the disk $2^k \cdot \mathcal{D}$, given by

$$\tau_l^n(2^k) := \sum_{j=k}^{\infty} \int \frac{\varphi_{2^j}(z)}{z^l}\, d[\mathcal{F}_n](z) \ (\text{ for } l \ge 1)$$

$$\text{and} \quad \widetilde{\tau}_l^n(2^k) := \sum_{j=k}^{\infty} \int \frac{\varphi_{2^j}(z)}{|z|^l}\, d[\mathcal{F}_n](z) \ (\text{ for } l \ge 3)$$

satisfy the estimates

$$\mathbb{E}\left[\left|\tau_l^n(2^k)\right|\right] \le C_1(\varphi, l)/2^{kl/2} \quad \text{and} \quad \mathbb{E}\left[\left|\widetilde{\tau}_l^n(2^k)\right|\right] \le C_2(\varphi, l)/2^{k(l-2)/2}.$$

All of the above remain true when $\mathcal{F}_n$ is replaced by $\mathcal{F}$, for which we use the notations S_l , $\widetilde{S}_l$, $\tau_l(2^k)$ and $\widetilde{\tau}_l(2^k)$ to denote the analogous quantities corresponding to $S_l(n)$, $\widetilde{S}_l(n)$, $\tau_l^n(2^k)$ and $\tau_l^n(2^k)$.

Corollary 3.5.8. *For $R = 2^k, k \ge k_0$ as in Proposition 3.5.7. We have $\mathbb{P}[|\tau_l^n(R)| > R^{-l/4}] \le R^{-l/4}$ and $\mathbb{P}[|\widetilde{\tau}_l^n| > R^{-(l-2)/4}] \le R^{-(l-2)/4}$, and these estimates remain true when f_n is replaced with f.*

Proposition 3.5.9. *For each $l \ge 1$ we have $S_l(n) \to S_l$ in probability, and for each $l \ge 3$ we have $\widetilde{S}_l^n \to \widetilde{S}_l$ in probability, and hence we have such convergence a.s. along some subsequence, simultaneously for all l.*

3.6 Limiting procedure for GAF zeroes

In this section, we use the estimates for $\mathcal{F}_n$ to prove Theorem 3.1.2 for a disk $\mathcal{D}$ centred at the origin. We know from Section 3.3 that this is sufficient in order to obtain Theorem 3.1.2 in the general case. We will work in the framework of Section 3.2. More specifically, we will show that the conditions for Theorem 3.2.1 are satisfied, which will give us the desired conclusion.

Notation. Throughout this section, we will use the notation Γ is $O(\delta)$ to imply that the non-negative quantity Γ satisfies $\Gamma \leq c\delta$ where the positive parameter $\delta \to 0$ and c is a **universal constant**, which is (crucially) independent of all the other parameters we introduce (like n and M).

In terms of the notation used in Section 3.2, we have $X^n = \mathcal{F}_n$ and $X = \mathcal{F}$.

3.7 Overview of the limiting procedure

The limiting procedure for the GAF zero process formally involves an appeal to Theorem 3.2.1 - defining the variable ν and the events $\Omega(j)$ and $\Omega_{n_k}(j)$. Further, we need to verify that they indeed satisfy the conditions demanded in Theorem 3.2.1. The definition of ν is fairly straightforward, at least in retrospect after the statement of Theorem 3.1.2. The main challenge in carrying out the above programme is to simultaneously satisfy two conditions demanded in Theorem 3.2.1: measurability of the events with respect to the outside zeroes, and establishing (3.1).

Morally, (3.1) corresponds to a statement that for the poynomial ensembles the ratio of the density of the measure ν with the conditional density of the inside zeroes (given the outside zeroes) is bounded from above and below in a certain uniform sense. In Section 3.5 we mentioned that in order to obtain such bounds (refer to Proposition 3.5.6), we need to bound from above expressions such as $\boldsymbol{Y}_n$ (refer e.g. to (3.11)). However, such expressions involve the coefficients of the Gaussian polynomial, and therefore depend on both the inside and the outside zeroes. Moreover, we would like the criteria defining the events to be somehow consistent, in the sense that having them would imply good bounds (as in (3.1) or Proposition 3.5.6) for all the Gaussian polynomials with large enough degree. This necessitates uniform bounds for the relevant quantities in such a definition (e.g. the tails of their distributions should be small in some uniform sense), but we have uniform bounds only for the inverse powers of the outside zeroes (refer to Proposition 3.5.7).

The above factors make the limiting procedure for the Gaussian zero process to be rather involved. We follow the following route: we first define the relevant events (see Definitions 5 and 6 in Section 3.9). As is necessary, these definitions make the events measurable with respect to the outside zeroes of f (or f_n, as the case may be). But we have to pay the price in that it is not at all clear that these Definitions imply the bounds necessary for obtaining (3.1). Indeed, the bounds are not true on the events $\Omega_{n_k}(j)$ as is, but we will establish that they hold on $\Omega_{n_k}(j) \setminus E$ where $\mathbb{P}(E) = O(\delta_k)$, where δ_k is a decreasing sequence such that

$\sum_k \delta_k < \infty$. This gives (3.1) with the $\vartheta(k;j)$ summand. This deduction can be found in Section 3.10. We further need to show that the $\Omega(j)$-s as defined (see Definition 5) exhaust Ω^m. This is also not clear at all from the definition, and has to be deduced in a similar fashion by perturbing the original events, and then taking care of the differences accrued. This is addressed in Section 3.11. The deductions in Sections 3.10 and 3.11 will involve several of technical propositions. Their proofs, though straightforward (given the estimates in Section 3.5), are somewhat peripheral to the broad picture of the limiting procedure. Therefore, for the sake of clarity, these proofs will be deferred to Section 3.12.

Our analysis will be driven by two basic parameters: $\delta > 0$ (to be thought of as small) varying over a decreasing sequence δ_k with $\sum_k \delta_k < \infty$, and $M > 0$ (to be thought of as large) varying over an increasing sequence $M_j \uparrow \infty$. We will first fix the parameter M and let $\delta \downarrow 0$ along δ_k, and then let $M \uparrow \infty$ along M_j. Many of our technical propositions will be true in a regime where δ is small enough depending on M, but because of the above order in the taking of limits, this does not create any difficulty.

3.8 Notations and Choice of Parameters

In this section, we will introduce some notations and define certain parameters, which will be useful for the subsequent analysis. There will be two basic parameters: δ and M (see below), and all the other parameters will be defined in terms of these two. The motivation for these definitions will only be clear when they are invoked at various places in the sections that follow. The reader might omit the detailed definitions on first reading and refer back when they are used later in the text. The reason we present the definitions together is so that the logical dependence between the various parameters become clear, and there is an organized reference for their future use.

Let us consider parameters $\delta > 0$ (to be thought of as small), and $M > 0$ (to be thought of as large). Let r_0 be the radius of the fixed disk $\mathcal{D}$. This δ bears no relation with the separation parameter δ considered in Section 3.5.

Define, for $j \geq 1$,

$$\psi^{f_n}_{2^j,l} = \int \frac{\varphi_{2^j}(z)}{z^l}\, d[\mathcal{F}_n](z), \qquad \gamma^{f_n}_{2^j,l} = \int \frac{\varphi_{2^j}(z)}{|z|^l}\, d[\mathcal{F}_n](z),$$

$$\psi^{f}_{2^j,l} = \int \frac{\varphi_{2^j}(z)}{z^l}\, d[\mathcal{F}](z), \qquad \gamma^{f}_{2^j,l} = \int \frac{\varphi_{2^j}(z)}{|z|^l}\, d[\mathcal{F}](z).$$

For $j = 0$ replace φ by $\widetilde{\varphi}$ in the above definition.

Define

$$\psi^f_l(k) = \sum_{j=0}^{k} \psi^f_{2^j,l} \quad \gamma^f_l(k) = \sum_{j=0}^{k} \gamma^f_{2^j,l} \quad \psi^f_l = \sum_{j=0}^{\infty} \psi^f_{2^j,l} \quad \Psi^f_l = \sum_{j=0}^{\infty} |\psi^f_{2^j,l}| \quad \gamma^f_l = \sum_{j=0}^{\infty} \gamma^f_{2^j,l}\,.$$

We define $\psi_l^{f_n}(k), \gamma_l^{f_n}(k), \psi_l^{f_n}, \Psi_l^{f_n}$ and $\gamma_l^{f_n}$ as the obvious analogues for f_n (in place of f). Observe that $\psi_l^{f_n} = \sum_{\omega \in \mathcal{D}^c \cap \mathcal{F}_n} \dfrac{1}{\omega^l}$ and $\gamma_l^{f_n} = \sum_{\omega \in \mathcal{D}^c \cap \mathcal{F}_n} \dfrac{1}{|\omega|^l}$.

Let P_l be the l-th Newton polynomial expressing the elementary symmetric function of order l in terms of power sums of order $1, 2, \cdots, l$. That is, for complex numbers $x_1, \cdots, x_n$, let $s_k = \sum_{j=1}^n x_j^k$ and $e_k = \sum_{i_1 < \cdots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k}$. Then $e_l = P_l(s_1, \cdots, s_l)$. Note that as a polynomial of l variables, P_l does not depend on n (see [Sta99], Chapter 7).

Let $h : \mathbb{R}^+ \to \mathbb{R}^+$ be a function such that $\sum_L^{L+h(L)} k^{-1/8} \to 0$ as $L \to \infty$. For the sake of definiteness, we set $h(L) = L^{1/16}$. Let B be such that $\sup_r \frac{K^r}{(r!)^{1/4}} \le B$, as in Proposition 3.5.3.

Let C_δ, L_δ be (large) positive integers depending on δ, to be chosen later.

For a non-negative integer m we will consider functions of C_δ complex variables $(z_1, \cdots, z_{C_\delta})$ given by

$$f_{i,j}^{\delta,m}(z_1, \cdots, z_{C_\delta}) = \left| \sum_{l=1}^{C_\delta} \overline{z_{l+i-m}} z_{l+j-m} l! \right|, 2 \le i \le m, 0 \le j \le m ,$$

$$f_0^{\delta,m}(z_1, \cdots, z_{C_\delta}) = \left(\frac{1}{h(L_\delta)} \sum_{l=L_\delta}^{L_\delta + h(L_\delta)} |z_{l-m}|^2 l! \right) .$$

Recall the notation that $\Omega^{m,1/M}$ is the event that there are exactly m zeroes of f in $\mathcal{D}$ and there is a separation of distance at least $\frac{1}{M}$ between the zeroes of f in $\mathcal{D}^c$ and $\partial\mathcal{D}$. Moreover, $\Omega_n^{m,1/M}$ denotes the analogous event with f replaced by f_n.

With these notations, we make certain choices as follows:

I. We choose an integer L_δ such that

(i) $\sum_{k=L_\delta}^{L_\delta+h(L_\delta)} k^{-1/8} < \delta^2/C_0$ (with C_0 a universal constant as in the proof of Proposition 3.10.3).

(ii) $\left| \dfrac{\sum_{l=L_\delta}^{L_\delta+h(L_\delta)} |\xi_l|^2}{h(L_\delta)} - 1 \right| < 1/2$ with probability $> 1 - \delta^2$.

(iii) $\sum_{r=1}^\infty \frac{1}{l^{r/16}(r!)^{1/8}} \le \frac{C}{l^{1/16}} < 1/2$ for all $l \ge L_\delta$, (where C is a universal constant as in the proof of Proposition 3.10.3).

(iv) $L_\delta^{1/8} > K$ where $K = K(\mathcal{D}, m)$ as in Proposition 3.5.2.

II. We choose an integer C_δ such that:

(i) $C_\delta > L_\delta + h(L_\delta)$,

(ii) $\sum_{l=C_\delta}^\infty \frac{1}{(l+1)(l+2)} < \delta^6$,

(iii) $\sum_{l=C_\delta}^\infty \frac{1}{l^{1/8}\sqrt{(l+1)(l+2)}} < \delta^4$,

(iv) $e^{-c_2 C_\delta^{1/4}} < \delta^2$, where c_2 is as in Proposition 3.5.3.

III. Let $g(\delta)$ be such that $g(\delta) \to \infty$ as $\delta \to 0$ and

$$\mathbb{P}(\Psi_l^f > g(\delta) - 1) < \delta^2/C_\delta \, , l = 1, 2, \cdots, C_\delta \, ,$$

$$\mathbb{P}(\gamma_3^f > g(\delta) - 1) < \delta^2 \,.$$

IV. Recall that P_l is the l-th Newton polynomial expressing the elementary symmetric function of order l in terms of power sums of order $1, 2, \cdots, l$. Let $f_{i,j}^{\delta,m}$ be the functions defined above. These functions are continuous, and hence uniformly continuous on a compact set. Let $\varepsilon(\delta)$ be such that

$$|f_{i,j}^{\delta,m}(P_1(\underline{w_1})), \cdots, P_{C_\delta}(\underline{w_1})) - f_{i,j}^{\delta,m}(P_1(\underline{w_2})), \cdots, P_{C_\delta}(\underline{w_2}))| < \delta^2 \forall ||\underline{w_1} - \underline{w_2}||_\infty < \varepsilon(\delta)$$

and $||\underline{w_1}||_\infty, ||\underline{w_2}||_\infty \leq g(\delta) + 100$, for each $2 \leq i \leq m, 0 \leq j \leq m$ (here $\underline{w_1}, \underline{w_2}$ are vectors in $\mathbb{C}^{C_\delta}$ and $\| \cdot \|$ is the L^∞ norm on $\mathbb{C}^{C_\delta}$). $\varepsilon(\delta)$ is so chosen that the same inequality should also hold when $f_{i,j}^{\delta,m}$ is replaced by $f_0^{\delta,m}$.

V. Let $R_\delta > 1$ be a large radius (of the form $2^k r_0$ where r_0 is radius of $\mathcal{D}$) such that $\sum_{l \geq 1} R_\delta^{-l/4} < (\varepsilon(\delta) \wedge \delta^2)$. Let $k(\delta)$ be the positive integer such that $R_\delta = 2^{k(\delta)} r_0$.

VI. Let $n(\delta)$ be such that

(i) Except for an event of probability $< \delta^2$, we have $|\psi_{2^k,l}^f - \psi_{2^k,l}^{f_{n(\delta)}}| < \min(\delta^2, \varepsilon(\delta))/(k(\delta)+1)$ for each $0 \leq k \leq k(\delta)$ and each $l \leq C_\delta$, and $|\gamma_{2^k,3}^f - \gamma_{2^k,3}^{f_{n(\delta)}}| < \min(\delta^2, \varepsilon(\delta))/(k(\delta)+1)$ for each $0 \leq k \leq k(\delta)$.

(ii) Except for an event of probability $< \delta^2$, we have $\left| \frac{\sum_{l=1}^{n(\delta)} |\xi_l|^2}{n(\delta)} - 1 \right| < 1/2$.

(iii) Except for an event of probability $< \delta^2$, we have $\left| \frac{P_{\text{in}}^{f_{n(\delta)}}}{|\xi_0|^2} - \frac{P_{\text{in}}^f}{|\xi_0|^2} \right| < \delta^2$ where P_{in}^f is the product of zeroes of f inside $\mathcal{D}$ and $P_{\text{in}}^{f_{n(\delta)}}$ is the analogous quantity for $f_{n(\delta)}$.

(iv) Except on an event of probability $O(\delta^2)$, we have

$$\frac{\left| \displaystyle\sum_{k \leq n(\delta) - C_\delta} \frac{\overline{\sigma_{k-t}(\underline{\omega})} \sigma_{k-i}(\underline{\omega})}{\binom{n(\delta)}{k} k!} \right|}{\displaystyle\sum_{k=0}^{n(\delta)} \frac{|\sigma_k(\underline{\zeta}, \underline{\omega})|^2}{\binom{n(\delta)}{k} k!}} \leq \frac{2\delta^2}{n(\delta)}, \;\; t = 0, 1, 2, \cdots, m, \;\; 2 \leq i \leq m \,.$$

(v) $\mathbb{P}\left(\Omega^{m,1/M} \Delta \Omega_{n(\delta)}^{m,1/M}\right) < \delta^2$.

Remark 3.8.1. *That it is at all possible to find such a $n(\delta)$ as in* ***VI*** *above is not immediate. Conditions VI.(i), VI.(iii) and VI.(v) are ensured by the convergence of the $\mathcal{F}_n$-s to $\mathcal{F}$ on the compact set $2^{k(\delta)}.\mathcal{D}$. VI.(ii) is covered by the law of large numbers. VI.(iv) is addressed in Proposition 3.10.6. Moreover, if we have a sequence $\delta_k \downarrow 0$, it is clear that we can choose $n(\delta_k)$ such that $n(\delta_k) \uparrow \infty$.*

3.9 Proof of Theorem 3.1.2

In this section we provide a proof of Theorem 3.1.2, based on Theorem 3.2.1. We will show how to define the relevant events here; the verification of some of the properties demanded in Theorem 3.2.1 requires substantial amount of work, and will be taken up in subsequent sections.

The notations are from Section 3.8 unless stated otherwise.

Proof of Theorem 3.1.2. Following the notation in Theorem 3.2.1, we begin with Ω^m, $m \geq 0$. The cases $m = 0$ and $m = 1$ are trivial, so we focus on the case $m \geq 2$.

Our candidate for $\nu(\xi, \cdot)$ (refer to Theorem 3.2.1) is the probability measure $Z^{-1}|\Delta(\underline{\zeta})|^2 d\mathcal{L}(\underline{\zeta})$ on Σ_s, where $\Delta(\underline{\zeta})$ is the Vandermonde determinant formed by the coordinates of $\underline{\zeta}$ and $s = S(X_{\text{out}}(\xi))$, $\mathcal{L}$ is the Lebesgue measure on Σ_s and Z is the normalizing factor. Here we recall the definition of $S(X_{\text{out}}(\xi))$ from Theorem 2.1.3 and the definition of Σ_s from Section 3.5.

We now intend to define the events $\Omega(j)$ and $\Omega_{n_k}(j)$ as in Theorem 3.2.1.

Fix two sequences of positive numbers $M_j \uparrow \infty$ and $\delta_k \downarrow 0$ such that $\sum \delta_k < \infty$.

Define $z_l = P_l(\psi_1^f(k(\delta)), \cdots, \psi_l^f(k(\delta)))$.

Definition 5. *For any $M, \delta > 0$ we define the event $\Omega(M;\delta)$ by the following conditions (for clarification of the notations used refer to Section 3.8):*

1. $\Omega^{m,1/M}$ *occurs.*
2. a) $|\psi_s^f(k(\delta))| \leq M$, *for* $s = 1, 2$.
   b) $|\gamma_3^f(k(\delta))| \leq M$.
3. $f_{i,j}^{\delta,m}(z_1, \cdots, z_{C_\delta}) \leq M$, *for* $2 \leq i \leq m$ *and* $0 \leq j \leq m$.
4. $\dfrac{1}{M} \leq f_0^{\delta,m}(z_1, \cdots, z_{C_\delta}) \leq M$.
5. $|\psi_l^f(k(\delta))| \leq g(\delta), l = 1, \cdots, C_\delta$.

Define $\Omega(j) := \underline{\lim}_{k\to\infty} \Omega(M_j; \delta_k)$.

For any given δ, it is clear that $\Omega(M_j;\delta) \subset \Omega(M_{j+1};\delta)$, which implies that $\Omega(j) \subset \Omega(j+1)$. The fact that $\Omega(j)$-s exhaust Ω is by no means clear, and will be proved in Section 3.11.

Our next goal is to define the sequence of events $\Omega_{n_k}(j)$ as in Theorem 3.2.1.

Corresponding to the sequence $\delta_k \downarrow 0$, we have the sequence of positive integers $n_k = n(\delta_k) \uparrow \infty$. Given $M, \delta > 0$, let $z_l' := P_l(\psi_1^{f_{n(\delta)}}(k(\delta)), \cdots, \psi_l^{f_{n(\delta)}}(k(\delta)))$.

Notice that $\Omega(M;\delta)$ was defined in terms of the zeroes of f. In the following definition, we introduce an event $\Omega^0_{n(\delta)}(M)$ which, heuristically, is an analogue of $\Omega(M;\delta)$, but is defined using the zeroes of $f_{n(\delta)}$.

Definition 6. *We define the event* $\Omega^0_{n(\delta)}(M)$ *as:*

1. $\Omega^{m,1/M}_{n(\delta)}$ *occurs .*

2. *a)* $|\psi_s^{f_{n(\delta)}}(k(\delta))| \leq M,$ *for* $s = 1, 2.$

b) $|\gamma_3^{f_{n(\delta)}}(k(\delta))| \leq M.$

3. $f^{\delta,m}_{i,j}(z'_1, \cdots, z'_{C_\delta}) \leq M,$ *for* $2 \leq i \leq m, 0 \leq j \leq m.$

4. $\frac{1}{M} \leq f^{\delta,m}_0(z'_1, \cdots, z'_{C_\delta}) \leq M.$

5. $|\psi_l^{f_{n(\delta)}}(k(\delta))| \leq g(\delta) + \delta^2,$ *for* $l = 1, \cdots, C_\delta.$

We define the event $\Omega_{n_k}(j)$ *to be* $\Omega^0_{n(\delta_k)}(M_j + 1)$.

Clearly, $\Omega_{n_k}(j)$ is measurable with respect to $X^{n_k}_{\text{out}}$. But it is not immediate that $\Omega(j) \subset \underline{\lim}_{k\to\infty} \Omega_{n_k}(j)$, or that (3.1) holds. Both of these will be established in Section 3.10.

Along with Sections 3.10 and 3.11, this shows that Theorem 3.2.1 establishes Theorem 3.1.2. ∎

3.10 $\Omega_{n_k}(j)$ is a good sequence of Events

Our aim in this section is to prove that the events $\Omega_{n_k}(j)$ defined in Section 3.9 satisfy the conditions in Theorem 3.2.1. We state this formally as:

Theorem 3.10.1. *With definitions as in Section 3.9, we have* $\Omega(j) \subset \underline{\lim}_{k\to\infty} \Omega_{n_k}(j)$*, and the* $\Omega_{n_k}(j)$*-s satisfy (3.1).*

We will establish this through a sequence of propositions. Some of the propositions are technical in nature, and we defer their proofs to Section 3.12, so that the main features of the limiting argument are not lost in the details.

We begin with the event $\Omega(M;\delta)$ and show that, except on an event of small probability, this implies the event $\Omega^0_{n(\delta)}(M+1)$. Heuristically, this means replacing quantities defined in terms of f by the corresponding quantities defined in terms of $f_{n(\delta)}$:

Proposition 3.10.2. *For* δ *small enough (depending on* M*), there exists an event* E^1_δ*, with* $\mathbb{P}(E^1_\delta) = O(\delta^2)$*, such that* $\{\Omega(M;\delta) \setminus E^1_\delta\} \subset \Omega^0_{n(\delta)}(M+1)$.

We defer the proof of Proposition 3.10.2 to Section 3.12.

For the rest of this section, the zeroes we discuss are going to be those of $f_{n(\delta)}$, unless otherwise mentioned. $\underline{\zeta}$ and $\underline{\omega}$ will respectively denote the vectors of the zeroes of $f_{n(\delta)}$ inside and outside $\mathcal{D}$, taken in uniform random order. In the next proposition, we prove that in $\sigma_k(\underline{\omega})$ is roughly comparable to $\sigma_k(\underline{\zeta}, \underline{\omega})$ for a range of k that is close to $n(\delta)$ but not too close. We set $l = n(\delta) - k$.

Proposition 3.10.3. *Except on an event of probability* $O(\delta^2)$*, we have* $\frac{1}{2}|\sigma_k(\underline{\zeta},\underline{\omega})| \le |\sigma_k(\underline{\omega})| \le \frac{3}{2}|\sigma_k(\underline{\zeta},\underline{\omega})|$ *for all* $L_\delta \le l \le L_\delta + h(L_\delta)$*, where* $l = n(\delta) - k$ *and* $h(L_\delta) = L_\delta^{1/8}$ *is as in Section 3.8.*

We defer the proof of Proposition 3.10.3 to Section 3.12.

We next define the event $\Omega^1_{n(\delta)}(M)$. Heuristically, this removes the role of $2^{k(\delta)} \cdot \mathcal{D}$ in the definition of $\Omega^0_{n(\delta)}(M)$, and replaces $k(\delta)$ by ∞, that is, the role played by $2^{k(\delta)} \cdot \mathcal{D} \setminus \mathcal{D}$ is now played by $\mathcal{D}^c$.

Definition 7. *We* ***define an event*** $\Omega^1_{n(\delta)}(M)$ *by the following conditions, with* $z''_l = P_l\left(\psi_1^{f_{n(\delta)}}, \cdots, \psi_l^{f_{n(\delta)}}\right)$ *:*

1. $\Omega^{m,1/M}_{n(\delta)}$ *occurs .*
2. *a)* $|\psi_s^{f_{n(\delta)}}| \le M,$ *for* $s = 1, 2.$

   b) $|\gamma_3^{f_{n(\delta)}}| \le M.$
3. $f^{\delta,m}_{i,j}(z''_1, \cdots, z''_{C_\delta}) \le M,$ *for* $2 \le i \le m$ *and* $0 \le j \le m.$
4. $\dfrac{1}{M} \le f^{\delta,m}_0(z''_1, \cdots, z''_{C_\delta}) \le M.$
5. $|\psi_l^{f_{n(\delta)}}| \le g(\delta) + 2\delta^2,$ *for* $l = 1, \cdots, C_\delta.$

We then have

Proposition 3.10.4. $\{\Omega^0_{n(\delta)}(M+1) \setminus E^2_\delta\} \subset \Omega^1_{n(\delta)}(M+2)$*, for an event* E^2_δ *with* $\mathbb{P}(E^2_\delta) = O(\delta^2)$ *and small enough* δ *(depending on* M*).*

We defer the proof of Proposition 3.10.4 to Section 3.12.

The conditions defining the event $\Omega^1_{n(\delta)}(M)$ enables us to obtain certain estimates discussed in the following proposition, the benefits of these will be clear subsequently. We mention that σ_k here is the same as $\sigma_k^{f_{n(\delta)}}$, and P_{in} is the same as $P_{\text{in}}^{f_{n(\delta)}}$.

Proposition 3.10.5. *Set* $z''_l = P_l\left(\psi_1^{f_{n(\delta)}}, \cdots, \psi_l^{f_{n(\delta)}}\right)$.
For all small enough δ *(depending on* M*), we have that the following are true on* $\Omega^1_{n(\delta)}(M)$ *(except on an event of probability* $O(\delta^2)$*):*

- *(i)*
$$\frac{8}{27} f^{\delta,m}_0(z''_1, \cdots, z''_l) \le \frac{|P_{\text{in}}^{f_{n(\delta)}}|^2}{|\xi_0|^2} \le 8 f^{\delta,m}_0(z''_1, \cdots, z''_l). \tag{3.12}$$

- *(ii)*

$$\left(\sum_{k=0}^{n(\delta)} \left| \frac{\sigma_k(\underline{\zeta}, \underline{\omega})}{\sigma_{n(\delta)}(\underline{\omega})} \right|^2 (n(\delta) - k)! \right) \geq \frac{1}{10M} n(\delta) \tag{3.13}$$

We defer the proof of Proposition 3.10.5 to Section 3.12.

Let the bad event in Proposition 3.10.5 (where the conclusions of the proposition do not hold on $\Omega^1_{n(\delta)}(M)$) be denoted by E^3_δ.

Definition 8. *Define* $\Omega^2_{n(\delta)}(M) = \Omega^1_{n(\delta)}(M) \setminus E^3_\delta$.

Proposition 3.10.6. *For all small enough δ, we have, for some constant c,*

$$\mathbb{P}\left(\frac{\left| \displaystyle\sum_{k < n(\delta) - C_\delta} \frac{\overline{\sigma_{k-t}(\underline{\omega})} \sigma_{k-i}(\underline{\omega})}{\binom{n(\delta)}{k} k!} \right|}{\displaystyle\sum_{k=0}^{n(\delta)} \frac{|\sigma_k(\underline{\zeta}, \underline{\omega})|^2}{\binom{n(\delta)}{k} k!}} \leq \frac{2\delta^2}{n(\delta)} \right) \geq 1 - c\delta^2, \ t = 0, 1, 2, \cdots, m; \ 2 \leq i \leq m \, .$$

We defer the proof of Proposition 3.10.6 to Section 3.13.

Let the exceptional event (where the bounds do not hold) in Proposition 3.10.6 be denoted by E^4_δ which clearly has probability $O(\delta^2)$.

Definition 9. *Define* $\Omega^3_{n(\delta)}(M) = \Omega^2_{n(\delta)}(M) \setminus E^4_\delta$.

In the next proposition, we demonstrate that on $\Omega^3_{n(\delta)}(M)$, we indeed have good bounds on ratio of conditional densities along a constant sum submanifold Σ_s.

Proposition 3.10.7. *(a) There is an event E_δ with $\mathbb{P}(E_\delta) = O(\delta^2)$ such that $\{\Omega(M;\delta) \setminus E_\delta\} \subset \Omega^3_{n(\delta)}(M+2)$.*
(b) For a vector of inside and outside zeroes $(\underline{\zeta}, \underline{\omega})$ of $f_{n(\delta)}$ such that $\Omega^m_{n(\delta)}$ occurs, let s denote the sum of the co-ordinates of $\underline{\zeta}$ and let $\Sigma_s = \{\underline{\zeta}' \in \mathcal{D}^m : \sum_{j=1}^m \zeta'_j = s\}$. On the event $\Omega^3_{n(\delta)}(M)$, the ratio of conditional densities $\rho^{n(\delta)}_{\underline{\omega}}(\underline{\zeta}'')/\rho^{n(\delta)}_{\underline{\omega}}(\underline{\zeta}')$ at any two vectors $\underline{\zeta}'', \underline{\zeta}'$ in Σ_s is bounded from above and below by functions of M (uniformly in $n(\delta)$).

We defer the proof of Proposition 3.10.7 to Section 3.12.

Remark 3.10.1. *Let $\widetilde{\Omega}(M;\delta)$ denote the event obtained by demanding conditions 1-4 in the definition of $\Omega(M;\delta)$ (Definition 5). By reversing the arguments in the proofs of Propositions 3.10.2, 3.10.4 and 3.10.7(a), it can be seen that there is an event E'_δ with $\mathbb{P}(E'_\delta) = O(\delta^2)$ such that*

$$\left(\Omega^3_{n(\delta)}(M+2) \setminus E'_\delta\right) \subset \widetilde{\Omega}(M+4;\delta).$$

In view of Proposition 3.10.4 and the Defnitions 8 and 9, the above implies that there is an event E''_δ of probability $O(\delta^2)$ such that

$$\left(\Omega^0_{n(\delta)}(M+1) \setminus E''_\delta\right) \subset \widetilde{\Omega}(M+4;\delta).$$

This will be referred to later in Section 3.11.

Now we are ready to complete the proof of Theorem 3.10.1

Proof of Theorem 3.10.1. Recall that $\Omega(j) = \underline{\lim}_{k\to\infty}\Omega(M_j;\delta_k)$, and $\Omega(M_j;\delta_k) \setminus E^1_\delta \subset \Omega^0_{n(\delta_k)}(M_j+1) = \Omega_{n_k}(j)$ by Proposition 3.10.2. From the fact that $\sum_k \mathbb{P}(E^1_{\delta_k}) \le c\sum_k \delta_k^2 < \infty$ (the last inequality is by choice of δ_k-s) and the Borel Cantelli lemma, it follows that $\Omega(j) \subset \underline{\lim}_{k\to\infty}\Omega_{n_k}(j)$.

To obtain (3.1), we introduce some further notations. On the event $\Omega^m_{n(\delta)}$, let $\gamma_{n(\delta)}(s;\underline{\omega}^{n(\delta)})$ denote the conditional probability measure on the sum s of inside zeroes given the vector of outside zeroes of $f^{n(\delta)}$ to be $\underline{\omega}^{n(\delta)}$, and $\mu_{n(\delta)}(\underline{\zeta};s,\underline{\omega}^{n(\delta)})$ be the conditional measure on the inside zeroes $\underline{\zeta}$ of $f_{n(\delta)}$, given the vector of outside zeroes to be $\underline{\omega}^{n(\delta)}$ and the sum of the co-ordinates of $\underline{\zeta}$ to be s. Let S denote the set of all possible sums of the inside zeroes; clearly S is a bounded open set in $\mathbb{C}$.

From Propositions 3.10.2, 3.10.4 and Definitions 8 and 9, we have $\Omega^0_{n(\delta)}(M+1) \setminus E \subset \Omega^3_{n(\delta)}(M+2)$ where $\mathbb{P}(E) = O(\delta^2)$. From Theorem 3.11.1 (which will be proved in Section 3.11) we have that for large enough M and small enough δ (depending on M and m), we have

$$\mathbb{P}\left(\Omega^0_{n(\delta)}(M+1)\right) \ge \frac{1}{2}\mathbb{P}\left(\Omega^m\right) \ge 100\delta.$$

Recall that both $\Omega^0_{n(\delta)}(M+1)$ and $\Omega^3_{n(\delta)}(M+1)$ subsets of Ω^m.

Hence, heuristically speaking, a 'large part' of $\Omega^0_{n(\delta)}(M+1)$ must be inside $\Omega^3_{n(\delta)}(M+2)$. Put in terms of conditional measures (conditioning successively on $\underline{\omega}^{n(\delta)}$ and then on s), this leads to the following (for all small enough δ) :

$\exists$ a set Ω_{good} (measurable with respect to outside zeroes of $f_{n(\delta)}$), $\Omega_{good} \subset \Omega^0_{n(\delta)}(M+1)$ with

$\mathbb{P}_{n(\delta)}(\Omega^0_{n(\delta)}(M+1) \setminus \Omega_{good}) < \delta$ such that:

for each $\underline{\omega}^{n(\delta)} \in \Omega_{good}$, $\exists$ a measurable set $S_{good}(\underline{\omega}^{n(\delta)}) \subset S$ with

$\gamma(S \setminus S_{good}(\underline{\omega}^{n(\delta)});\underline{\omega}^{n(\delta)}) < \delta$ such that :

for each $s \in S_{good}(\underline{\omega}^{n(\delta)})$ we have $\mu(\underline{\zeta};s,\underline{\omega}^{n(\delta)})(\Sigma_s \setminus H(s)) < \delta$, where $H(s)$ is the set of $\underline{\zeta} \in \Sigma_s$ such that $(\underline{\zeta},\underline{\omega}^{n(\delta)}) \in \Omega^3_{n(\delta)}(M+2)$.

We begin by considering

$$\int_{B\cap\Omega_{good}}\left[\int_{S\cap S_{good}}\left(\int_{A\cap\Sigma_s} d\mu_{n(\delta)}(\underline{\zeta};s,\underline{\omega}^{n(\delta)})\right) d\gamma_{n(\delta)}(s;\underline{\omega}^{n(\delta)})\right] d\mathbb{P}_{n(\delta)}(\underline{\omega}^{n(\delta)})\,. \tag{3.14}$$

Recall that $d\mu_{n(\delta)}$ above has a density with respect to Lebesgue measure on Σ_s. Moreover, by definition of Ω_{good} and S_{good}, we have that for $\underline{\omega}^{n(\delta)} \in \Omega_{good}$ and $s \in S_{good}(\underline{\omega}^{n(\delta)})$, there exists a $\underline{\zeta} \in \Sigma_s$ satisfying $(\underline{\zeta}, s, \underline{\omega}^{n(\delta)}) \in \Omega^3_{n(\delta)}(M+2)$. As a result, on the submanifold Σ_s, the conditional probability measure $d\mu_{n(\delta)}$ above is proportional to the measure $|\Delta(\underline{\zeta})|^2 \, d\mathcal{L}(\zeta)$, the constants of proportionality depending on M and being uniform in δ (refer to Proposition 3.10.7) .

Hence we have,

$$(3.14) \asymp_M \int_{B\cap\Omega_{good}} \left[\int_{S\cap S_{good}} \left(\frac{\int_{A\cap\Sigma_s} |\Delta(\underline{\zeta})|^2 \, d\mathcal{L}(\zeta)}{\int_{\Sigma_s} |\Delta(\underline{\zeta})|^2 \, d\mathcal{L}(\zeta)} \right) d\gamma_{n(\delta)}(s; \underline{\omega}^{n(\delta)}) \right] d\mathbb{P}_{n(\delta)}(\underline{\omega}^{n(\delta)}) \, .$$

Setting $h(A;s) := \frac{\int_{A\cap\Sigma_s} |\Delta(\underline{\zeta})|^2 \, d\mathcal{L}(\zeta)}{\int_{\Sigma_s} |\Delta(\underline{\zeta})|^2 \, d\mathcal{L}(\zeta)}$ we have that the last expression is equal to

$$\int_{B\cap\Omega_{good}} \int_{S\cap S_{good}} h(A;s) d\gamma_{n(\delta)}(s; \underline{\omega}^{n(\delta)}) d\mathbb{P}_{n(\delta)}(\underline{\omega}^{n(\delta)}) \, .$$

Note that the $h(A;s)$ is bounded between 0 and 1, so at the expense of an additive error of size $O(\delta)$ we have

$$(3.14) \asymp_M \int_{B\cap\Omega_{good}} \int_{S} h(A;s) d\gamma_{n(\delta)}(s; \underline{\omega}^{n(\delta)}) d\mathbb{P}_{n(\delta)}(\underline{\omega}^{n(\delta)}) + O(\delta)$$

because $\gamma((S \setminus S_{good}); \underline{\omega}^{n(\delta)}) < \delta$ as long as $\underline{\omega}^{n(\delta)} \in \Omega_{good}$.

The integral above can also be written as

$$\int h(A; s_{n(\delta)}(\xi)) 1\{X^{n(\delta)}_{\text{out}}(\xi) \in B \cap \Omega_{good}\}(\xi) d\mathbb{P}(\xi)$$

where $s_{n(\delta)}(\xi) = \sum_{\zeta_i \in X^{n(\delta)}_{\text{in}}(\xi)} \zeta_i$ and $1\{E\}(\cdot)$ is the indicator function of an event E .

Recalling that $\mathbb{P}(\Omega^0_{n(\delta)}(M+1) \setminus \Omega_{good}) < \delta$, and doing an argument similar to the previous step, we have, at the expense of another additive error of $O(\delta)$

$$(3.14) \asymp_M \int h(A; s_{n(\delta)}(\xi)) 1\{X^{n(\delta)}_{\text{out}}(\xi) \in B \cap \Omega^0_{n(\delta)}(M+1)\}(\xi) d\mathbb{P}(\xi) + O(\delta) \, .$$

We could also complete the S_{good} to S and Ω_{good} to $\Omega^0_{n(\delta)}(M+1)$ in (3.14) straightaway (without going through $h(A;s)$) at the expense of two additive errors of $O(\delta)$ (recall that $\mu_{n(\delta)}$ is a probability measure) to deduce that the integral (3.14)

$$= \int_{B\cap\Omega^0_{n(\delta)}(M+1)} \int_S \left(\int_{A\cap\Sigma_s} d\mu_{n(\delta)}(\underline{\zeta}, s, \underline{\omega}^{n(\delta)}) \right) d\gamma_{n(\delta)}(s; \underline{\omega}^{n(\delta)}) d\mathbb{P}_{n(\delta)}(\underline{\omega}^{n(\delta)}) + O(\delta) \, .$$

We summarize the above computation as:

$$\mathbb{P}(A \cap B \cap \Omega^0_{n(\delta)}(M+1))) \asymp_M \int h(A; s_{n(\delta)}(\xi)) 1\{X^{n(\delta)}_{\text{out}}(\xi) \in B \cap \Omega^0_{n(\delta)}(M+1)\}(\xi) d\mathbb{P}(\xi) + O(\delta) \, . \tag{3.15}$$

We now set $M = M_j$ and $\delta = \delta_k$ and note that $\Omega^0_{n(\delta_k)}(M_j + 1) = \Omega_{n_k}(j)$.

We note that since we send $k \to \infty$ (and hence $\delta_k \to 0$ for fixed j), therefore the criteria in the previous Propositions in this section involving δ being small enough depending on M (e.g. as in Propositions 3.10.2 or 3.10.6 or 3.10.7) is eventually satisfied.

The integral on the right hand side (henceforth abbreviated as r.h.s.) of (3.15) is

$$\int h(A; s_{n_k}(\xi))1\{X^{n_k}_{\text{out}}(\xi) \in B \cap \Omega_{n_k}(j)\}(\xi)d\mathbb{P}(\xi) .$$

Recall that we can consider the event $A \in \mathcal{A}$ as a subset of $\mathcal{D}^m$, and such sets have piecewise smooth boundary. For the definition of $\mathcal{A}$ refer to Section 3.2. For such sets A, we have $h(A; s)$ is continuous in s. But if ω denotes $X_{\text{out}}(\xi)$, then $s_{n_k}(\xi) \to S(\omega)$ a.s. Hence, $\exists \varepsilon_k \downarrow 0$ such that $\mathbb{P}(|s_{n_k}(\xi) - S(\omega)| > \varepsilon_k) \to 0$ as $k \to \infty$. Coupled with the fact that $0 \le h(A; s) \le 1$ this implies that

$$\begin{aligned} &\int h(A; s_{n_k}(\xi))1\{X^{n_k}_{\text{out}}(\xi) \in B \cap \Omega_{n_k}(j)\}(\xi)d\mathbb{P}(\xi) \\ = &\int h(A; S(\omega))1\{X^{n_k}_{\text{out}}(\xi) \in B \cap \Omega_{n_k}(j)\}(\xi)d\mathbb{P}(\xi) + \mathfrak{o}_k(1) \end{aligned}$$

where $\mathfrak{o}_k(1)$ is a quantity which $\to 0$ uniformly in j, as $k \to \infty$. To see the last step, we can write

$$\begin{aligned} h(A; s_{n_k}(\xi)) = h(A; S(\omega)) &+ (h(A; s_{n_k}(\xi)) - h(A; S(\omega)))\, 1[|s_{n_k}(\xi) - S(\omega)| < \varepsilon_k] \\ &+ (h(A; s_{n_k}(\xi)) - h(A; S(\omega)))\, 1[|s_{n_k}(\xi) - S(\omega)| > \varepsilon_k]. \end{aligned}$$

The second term in the right hand side above $\to 0$ a.s. because of continuity of $h(A; s)$ as a function of s, and the expectation of the third term $\to 0$ by choice of ε_k; also remember that $0 \le h(A; s) \le 1$.

It remains to observe that

$$\int h(A; S(\omega))1\{X^{n_k}_{\text{out}}(\xi) \in B \cap \Omega_{n_k}(j)\}(\xi)d\mathbb{P}(\xi) = \int_{B \cap \Omega_{n_k}(j)} \nu(\xi, A)d\mathbb{P}(\xi)$$

where we recall the definition of $\nu(\xi, \cdot)$ from the proof of Theorem 3.1.2 in Section 3.9.

These facts together verify (3.1), and this completes the proof of Theorem 3.10.1. ■

3.11 $\Omega(j)$-s exhaust Ω^m

In this section we prove that for all small enough δ (depending on M) the event $\Omega(M; \delta)$ (almost) contains an event Ω_M of high probability inside Ω^m (the probability will depend on M) . In this way, we would ensure that $\Omega(j) = \underline{\lim}_{k \to \infty} \Omega(M_j; \delta_k)$ tends to be of full measure inside Ω^m as $M \to \infty$. Recall here that Ω^m denotes the event that there are exactly m zeroes of f inside $\mathcal{D}$.

In this section the vectors of the inside and the outside zeroes, denoted respectively by $\underline{\zeta}$ and $\underline{\omega}$, refer to those of $f_{n(\delta)}$ (unless mentioned otherwise).

Theorem 3.11.1. *There exists an event Ω_M (with $\mathbb{P}(\Omega_M) \to 1$ as $M \to \infty$) such that for each M sufficiently large, we have $\{\Omega_M \cap \Omega^{m,1/M} \setminus F_\delta\} \subset \Omega(M+3;\delta)$ for all δ small enough (depending on M); here F_δ is an event of probability $O(\delta^2)$. In particular, this implies that $\Omega(j) = \underline{\lim}_{\delta_k \to 0} \Omega(M_j;\delta_k)$ satisfies $\mathbb{P}\left(\Omega^m \setminus \Omega(j)\right) \to 0$ as $j \to \infty$.*

Before we prove Theorem 3.11.1, we will make some technical observations.

Proposition 3.11.2. *The following criteria imply (for all small enough δ, depending on M) that we will be guaranteed* ***condition 3*** *defining $\Omega^0_{n(\delta)}(M+1)$, except on an event of probability $O(\delta^2)$:*

- *(i)* $\dfrac{|P_{\text{in}}^{f_{n(\delta)}}|^2}{|\xi_0|^2} < \sqrt{M}/3\,.$
- *(ii)* $\left|\displaystyle\sum_{k=n(\delta)-C_\delta}^{n(\delta)} \frac{\overline{\sigma_{k-i}(\underline{\omega})}\sigma_{k-j}(\underline{\omega})}{\binom{n(\delta)}{k}k!}\right| \le \dfrac{\sqrt{M}}{|\xi_{n(\delta)}|^2};\ 0 \le i \le m;\ 2 \le j \le m\,.$

We defer the proof of Proposition 3.11.2 to Section 3.12.

Definition 10. *We define an event $\Gamma(\delta, M)$ by the following conditions:*

- *(a)* $|\psi_l^f(k(\delta))| \le M - \dfrac{1}{2};\ l = 1,2$ *and* $\gamma_3^f(k(\delta)) \le M - \frac{1}{2}\,.$
- *(b)* $\dfrac{8}{M} + \kappa\delta^2 \le \dfrac{|P_{\text{in}}^f|^2}{|\xi_0|^2} \le \dfrac{1}{3}\sqrt{M} - \kappa\delta^2$ *where $\kappa > 0$ is a constant to be declared later (see the proof of Proposition 3.11.3).*
- *(c)* $\left|\displaystyle\sum_{l=0}^{C_\delta} \frac{\overline{\xi_{l+i}}\xi_{l+j}}{\sqrt{(l+i)_i(l+j)_j}}\right| \le \dfrac{\sqrt{M}}{4};\ 0 \le i \le m;\ 2 \le j \le m\,.$
- *(d)* $\displaystyle\sum_{l=0}^{C_\delta} \frac{|\xi_{l+i}|}{(l+j)^{1/8}\sqrt{(l+i)_i(l+j)_j}} \le \dfrac{M^{1/4}}{4};\ 0 \le i \le m;\ 2 \le j \le m\,.$
- *(e)* $|\eta_l^{(n(\delta))}| \le \dfrac{M^{1/4}}{l^{1/8}}$ *for all $l \ge 1$.*

With this definition and using Proposition 3.11.2, we have

Proposition 3.11.3. $\Gamma(\delta, M) \cap \Omega^{m,1/M} \subset \Omega^0_{n(\delta)}(M)$ *except on an event F^1_δ of probability $O(\delta^2)$.*

We defer the complete proof of Proposition 3.11.3 to Section 3.12.

Finally, we define our desired event Ω_M.

Definition 11. *We define the event Ω_M to be :*

- (a_0) $|\psi_l^f| \leq M-1, l=1,2; |\gamma_3^f| \leq M-1$.
- (b_0) $\dfrac{8}{M-1} \leq \dfrac{|P_{\text{in}}^f|^2}{|\xi_0|^2} \leq \dfrac{\sqrt{M}}{3} - 1$.
- (c_0) $\left| \sum_{l=0}^{\infty} \dfrac{\overline{\xi_{l+i}}\xi_{l+j}}{\sqrt{(l+i)_i(l+j)_j}} \right| \leq \dfrac{\sqrt{M}}{4} - 1;\ 0 \leq i \leq m;\ 2 \leq j \leq m$.
- (d_0) $\sum_{l=0}^{\infty} \dfrac{|\xi_{l+i}|}{(l+j)^{1/8}\sqrt{(l+i)_i(l+j)_j}} \leq \dfrac{M^{1/4}}{4} - 1;\ 0 \leq i \leq m;\ 2 \leq j \leq m$.
- (e_0) $|\eta_l| < M^{1/4}/l^{1/8} \forall l \geq 1$.

where η_l is as in Proposition 3.5.3.

We are now ready to prove Theorem 3.11.1.

Proof of Theorem 3.11.1. We claim that for fixed M and small enough δ (depending on M), except on an event of probability $O(\delta^2)$, we have $\Omega_M \subset \Gamma(\delta, M)$. This is going to be true because of the choices of the parameters C_δ and $k(\delta)$.

We begin with the event Ω_M. First, we look at (c_0) and (d_0). Condition II(ii) in Section 3.8 implies, via a second moment estimate, that the tail $\left|\sum_{l=C_\delta}^{\infty} \frac{\overline{\xi_{l+i}}\xi_{l+j}}{\sqrt{(l+i)_i(l+j)_j}}\right| < \delta^2$ except with probability $O(\delta^2)$; recall here that we always have $i \vee j \geq 2$. A first moment estimate allows us to draw a similar conclusion about $\sum_{l=C_\delta}^{\infty} \frac{|\xi_{l+i}|}{l^{1/8}\sqrt{(l+i)_i(l+j)_j}}$. Hence, for all small enough δ, we have conditions (c) and (d) defining $\Gamma(\delta, M)$ except on an event of probability $O(\delta^2)$.

Regarding (a_0), the transition from $\psi_l^f(k(\delta))$ to ψ_l^f takes place via assumption V in Section 3.8 on R_δ. Recall from Corollary 3.5.8) that we have

$$\mathbb{P}(|\tau_l(R_\delta)| > R_\delta^{-l/4}) \leq R_\delta^{-l/4} .$$

Combined with the fact that $R_\delta^{-1/4} < \delta^2$ (refer to V, Section 3.8), this enables us to replace ψ_l^f by $\psi_l^f(k(\delta))$ and γ_3^f by $\gamma_3^f(k(\delta))$ in (a_0), implying (for all small enough δ) condition (a) defining $\Gamma(\delta, M)$ except on an event of probability $O(\delta^2)$. For a fixed M, we notice that (b_0) implies (b) as soon as δ is small enough (depending on M). Finally, we look at (e_0). Recall from Proposition 3.5.3 that $|\eta_l^{(n)}| \leq \eta_l$ a.s. for all $l \geq 1$ in order to deduce (e) in the definition of $\Gamma(\delta, M)$ from (e_0) in the definition of Ω_M.

This completes the proof that $\Omega_M \subset \Gamma(\delta, M)$, except for a bad event F_δ^2 of probability $O(\delta^2)$.

Let $F_\delta^3 = F_\delta^1 \cup F_\delta^2$, where we recall the definition of F_δ^1 from Proposition 3.11.3. Clearly $\mathbb{P}(F_\delta^3) = O(\delta^2)$ and we have $\Omega_M \cap \Omega^{m,1/M} \setminus F_\delta^3 \subset \Omega_{n(\delta)}^0(M+1)$.

However, recall from Remark 3.10.1 that $\Omega^0_{n(\delta)}(M+1)$ implies the conditions 1-4 defining $\Omega(M+3;\delta)$, except on an event of probability $O(\delta^2)$. Moreover, the complement of condition 5 defining $\Omega(M+3;\delta)$ itself has probability $< \delta^2$, by choice of $g(\delta)$ (refer to III. in Section 3.8). We combine the last two bad events into F^4_δ.

Let $F_\delta = F^3_\delta \cup F^4_\delta$. Then we have $\mathbb{P}(F_\delta) = O(\delta^2)$ and $\Omega_M \cap \Omega^{m,1/M} \setminus F_\delta \subset \Omega(M+3;\delta)$, as desired.

Finally, to show that $\mathbb{P}(\Omega_M) \to 1$ as $M \to \infty$, we simply observe that $|\psi^f_l|, \gamma^f_3$ and the random variables appearing in conditions (c_0) and (d_0) are random variables with no mass at ∞, and $\frac{|P^f_{\text{in}}|^2}{|\xi_0|^2}$ does not have an atom at 0; the η_l-s (appearing in condition (e_0)) are taken care of by Proposition 3.5.3 part (i).

We can now take liminf over $\delta_k \to 0$ in $\{\Omega_M \cap \Omega^{m,1/M} \setminus F_{\delta_k}\} \subset \Omega(M+3;\delta_k)$. Recall that F_δ is of probability $O(\delta^2)$ and $\sum \delta_k < \infty$. Using the Borel Cantelli lemma we thus obtain, for M bigger than some universal constant,

$$\Omega_M \cap \Omega^{m,1/M} \subset \varliminf_{k\to\infty} \Omega(M+3;\delta_k)\ .$$

Setting $M = M_j - 3$, this implies $\Omega_{M_j-3} \cap \Omega^{m,1/M_j-3} \subset \Omega(j)$. Since this holds for all large enough j, $\mathbb{P}(\Omega_M) \to 1$ and $\mathbb{P}\left(\Omega^m \Delta \Omega^{m,1/M}\right) \to 0$ as $M \to \infty$, we obtain the fact that $\mathbb{P}(\Omega^m \setminus \Omega(j)) \to 0$ as $j \to \infty$. ■

3.12 Proofs of some propositions from Sections 3.10 and 3.11

In this section, we include the proofs of the technical propositions which were deferred in Sections 3.10 and 3.11.

Proof of Proposition 3.10.2

This is essentially a consequence of the choices and observations made in Section 3.8. We also recall Definitions 5 and 6.

First notice that **Condition 1** in Definition 6 is the same event as condition 1 in Definition 5 same except on an event of probability $< \delta^2$, because of the convergence of $\mathcal{F}_n$ to $\mathcal{F}$ on compact sets, see VI.(v) in Section 3.8.

By VI.(i), (outside an event of probability $< \delta^2$) for $\Omega(M;\delta)$, we have $|\psi^{f_{n(\delta)}}_l(k(\delta)) - \psi^f_l(k(\delta))| < \delta^2$ for each $l \leq C_\delta$, and also $|\gamma^{f_{n(\delta)}}_3(k(\delta)) - \gamma^f_3(k(\delta))| < \delta^2$; recall here the definitions of $\psi^{f_{n(\delta)}}_l(k(\delta))$ and $\gamma^{f_{n(\delta)}}_3(k(\delta))$ from Section 3.8. Hence we immediately have **condition 2** in Definition 6; moreover **condition 5** in that definition is also clear from condition 5 defining $\Omega(M;\delta)$.

On $\Omega(M;\delta)$, we have $|\psi^f_l(k(\delta))| < g(\delta)$ for $l \leq C_\delta$. But considering VI.(i), summing over $k \leq k(\delta)$ and applying triangle inequality we have $|\psi^{f_{n(\delta)}}_l(k(\delta)) - \psi^f_l(k(\delta))| < \varepsilon(\delta)$, except for an event of probability $< \delta^2$. Choosing $\underline{w_1} = \{\psi_l^f(k(\delta))\}_{l=1}^{C_\delta}$ and $\underline{w_2} = \{\psi_l^{f_{n(\delta)}}(k(\delta))\}_{l=1}^{C_\delta}$ in IV, we have **condition 3** and **condition 4** in Definition 6 (for all small enough δ, depending on M).

Combining all the bad events in the above discussion, we obtain E_δ^1, whose probability is $O(\delta^2)$.

A moment's thought would convince the reader that the arguments pertaining to conditions 1-4 presented above can be reversed (i.e., the roles of f and $f_{n(\delta)}$ interchanged), dropping an event of probability $O(\delta^2)$ in the process.

Proof of Proposition 3.10.3

We begin by recalling that (refer to Proposition 3.5.2), with $l = n(\delta) - k$, we have

$$\frac{\sigma_k(\underline{\omega})}{\sqrt{\binom{n(\delta)}{k} k!}} = \frac{\sigma_k(\underline{\zeta}, \underline{\omega})}{\sqrt{\binom{n(\delta)}{k} k!}} + \sum_{r=1}^{k} g_r \frac{\sigma_{k-r}(\underline{\zeta}, \underline{\omega})}{\binom{n(\delta)}{k-r}(k-r)!} \frac{1}{\sqrt{(l+1)\cdots(l+r)}}$$

and $\dfrac{\sigma_k(\underline{\zeta}, \underline{\omega})}{\sqrt{\binom{n(\delta)}{k} k!}} = \dfrac{\xi_l}{\xi_{n(\delta)}}$. The last equality follows by applying Vieta's formula to the polynomial $f_{n(\delta)}$. As a result, we have

$$\frac{\sigma_k(\underline{\omega})}{\sigma_k(\underline{\zeta}, \underline{\omega})} = 1 + \sum_{r=1}^{n(\delta)-l} g_r \frac{\xi_{l+r}}{\xi_l} \frac{1}{\sqrt{(l+1)\cdots(l+r)}}\,. \tag{3.16}$$

The aim is to show that the sum over $r \geq 1$ is small with high probability. By the A.M.-G.M. inequality we have

$$\frac{1}{\sqrt{(l+1)\cdots(l+r)}} \leq \frac{1}{l^{r/4}(r!)^{1/4}}\,,$$

and we have already seen in Proposition 3.5.2 that $|g_r| \leq K^r$ where K is a constant that depends on the domain $\mathcal{D}$. As a result, we have

$$\sum_{r=1}^{n(\delta)-l} g_r \frac{\xi_{l+r}}{\xi_l} \frac{1}{\sqrt{(l+1)\cdots(l+r)}} \leq \sum_{r=1}^{n(\delta)-l} \frac{K^r}{l^{r/4}(r!)^{1/4}} \left|\frac{\xi_{l+r}}{\xi_l}\right|\,.$$

Since $l \geq L_\delta$ is big enough such that $K \leq l^{1/8}$ (recall condition I.(iv) from Section 3.8), we need to estimate $\sum_{r=1}^{n(\delta)-l} \frac{1}{l^{r/8}(r!)^{1/4}} \left|\frac{\xi_{l+r}}{\xi_l}\right|$. Since $\left|\frac{\xi_{l+r}}{\xi_l}\right|$ satisfies $\mathbb{P}\left(\left|\frac{\xi_{l+r}}{\xi_l}\right| > x\right) \leq 1/x^2$, we have

$$\mathbb{P}\left(\left|\frac{\xi_{l+r}}{\xi_l}\right| > l^{r/16}(r!)^{1/8}\right) \leq \frac{1}{l^{r/8}(r!)^{1/4}}. \tag{3.17}$$

If $\left|\frac{\xi_{l+r}}{\xi_l}\right| \leq l^{r/16}(r!)^{1/8}$ then for each k, the abolute value of the sum over $r \geq 1$ in (3.16) is $\leq \sum_{r=1}^{\infty} \frac{1}{l^{r/16}(r!)^{1/8}} \leq \frac{C}{l^{1/16}}$ which is $< \frac{1}{2}$ for large enough l (in particular for $l \geq L_\delta$), as we desire (recall the definition of L_δ from Section 3.8). We bound the probability of the complement of this event by a simple union bound over r and see that it is $\le \frac{C_0}{l^{1/8}}$ (refer to (3.17)). This gives us a bound for a fixed l. Now, we want this to be true with high probability for $L_\delta \le l \le L_\delta + h(L_\delta)$. By a union bound of the error probabilities over l in that range, we need to ensure that $\sum_{l=L_\delta}^{L_\delta+h(L_\delta)} \frac{1}{l^{1/8}} < \delta^2/C_0$, which we know is true by the choice of h made in Section 3.8, item I.

Proof of Proposition 3.10.4

Recall Definitions 6 and 7. Observe that $\Omega^1_{n(\delta)}(M)$ differs from $\Omega^0_{n(\delta)}(M)$ in that $\psi_l^{f_{n(\delta)}}(k(\delta))$ is replaced everywhere by $\psi_l^{f_{n(\delta)}}$ (consequently $z'_l = P_l\left(\psi_1^{f_{n(\delta)}}(k(\delta)), \cdots, \psi_l^{f_{n(\delta)}}(k(\delta))\right)$ is replaced by $z''_l = P_l\left(\psi_1^{f_{n(\delta)}}, \cdots, \psi_l^{f_{n(\delta)}}\right)$), $\gamma_3^{f_{n(\delta)}}(k(\delta))$ replaced by $\gamma_3^{f_{n(\delta)}}$ and in condition 5 we have $g(\delta)+\delta^2$ replaced by $g(\delta)+2\delta^2$.

On $\Omega^0_{n(\delta)}(M+1)$, we have $|\psi_l^{f_{n(\delta)}}(k(\delta))| \le M+1$ for $l = 1, 2$ and $|\gamma_3^{k(\delta), f_{n(\delta)}}| \le M+1$. Recall the tail estimates in Corollary 3.5.8. Applying these to $R = R_\delta$ and doing a union bound over l, we get that $|\tau_l(R_\delta)| < \varepsilon(\delta) \wedge \delta^2$ for all l and small enough δ, and also $|\tilde{\tau}_3(R_\delta)| < \delta^2$, except on an event of probability $O(\delta^2)$ (to see this refer to the definition of R_δ in Section 3.8 V). We denote by $(E^2_\delta)'$ the union of the exceptional events so far in this paragraph, and by $(E^2_\delta)''$ the exceptional event in Proposition 3.10.3. Define $E^2_\delta = (E^2_\delta)' \cup (E^2_\delta)''$. Clearly, $\mathbb{P}\left[E^2_\delta\right] = O(\delta^2)$, and $\{\Omega^0_{n(\delta)}(M+1) \setminus E^2_\delta\} \subset \Omega^1_{n(\delta)}(M+2)$.

Observe that the arguments given above, except for those pertaining to condition 5 in the definitions of our events, can be reversed (that is, the roles of $k(\delta)$ and ∞ interchanged), dropping an event of probability $O(\delta^2)$.

Proof of Proposition 3.10.5

The zeroes considered in this proof are those of $f_{n(\delta)}$.

(i) We have $\dfrac{\sigma_k(\underline{\zeta}, \underline{\omega})}{\sqrt{\binom{n(\delta)}{k} k!}} = \frac{\xi_{n(\delta)-k}}{\xi_{n(\delta)}}$. Hence, $\dfrac{|\sigma_k(\underline{\zeta}, \underline{\omega})|^2}{\binom{n(\delta)}{k} k! |\sigma_{n(\delta)}(\underline{\zeta}, \underline{\omega})|^2} = \frac{1}{n(\delta)!} \frac{|\xi_{n(\delta)-k}|^2}{|\xi_0|^2}$.

Multiplying both sides by $P_{\text{in}}^{f_{n(\delta)}}$, and observing that $\dfrac{P_{\text{in}}^{f_{n(\delta)}}}{\sigma_{n(\delta)}(\underline{\zeta}, \underline{\omega})} = \dfrac{1}{\sigma_{n(\delta)-m}(\underline{\omega})}$ we have

$$\frac{|\sigma_k(\underline{\zeta}, \underline{\omega})|^2}{|\sigma_{n(\delta)-m}(\underline{\omega})|^2}(n(\delta)-k)! = |\xi_{n(\delta)-k}|^2 \frac{|P_{\text{in}}^{f_{n(\delta)}}|^2}{|\xi_0|^2} \tag{3.18}$$

On $\Omega^1_{n(\delta)}(M)$, we have $\frac{1}{2}|\sigma_k(\underline{\zeta}, \underline{\omega})| \le |\sigma_k(\underline{\omega})| \le \frac{3}{2}|\sigma_k(\underline{\zeta}, \underline{\omega})|$, which means $\frac{2}{3}|\sigma_k(\underline{\omega})| \le |\sigma_k(\underline{\zeta}, \underline{\omega})| \le$

$2|\sigma_k(\underline{\omega})|$, for $k = n(\delta) - l$ such that $L_\delta \le l \le L_\delta + h(L_\delta)$; refer to Proposition 3.10.3. Hence

$$\frac{4}{9}\frac{1}{h(L_\delta)}\left(\sum_{l=L_\delta}^{L_\delta+h(L_\delta)} \frac{|\sigma_k(\underline{\omega})|^2}{|\sigma_{n(\delta)-m}(\underline{\omega})|^2}(n(\delta)-k)!\right) \le \frac{1}{h(L_\delta)}\left(\sum_{l=L_\delta}^{L_\delta+h(L_\delta)} \frac{|\sigma_k(\underline{\zeta},\underline{\omega})|^2}{|\sigma_{n(\delta)-m}(\underline{\omega})|^2}(n-k)!\right)$$
$$\le 4\frac{1}{h(L_\delta)}\left(\sum_{l=L_\delta}^{L_\delta+h(L_\delta)} \frac{|\sigma_k(\underline{\omega})|^2}{|\sigma_{n(\delta)-m}(\underline{\omega})|^2}(n(\delta)-k)!\right)$$

But by (3.18), the expression in the centre is $\frac{|P_{\text{in}}^{f_{n(\delta)}}|^2}{|\xi_0|^2}\frac{1}{h(L_\delta)}\left(\sum_{L_\delta}^{L_\delta+h(L_\delta)} |\xi_l|^2\right)$. By our choice of L_δ, except on an event of probability $O(\delta^2)$, we have $1/2 < \frac{1}{h(L_\delta)}\left(\sum_{L_\delta}^{L_\delta+h(L_\delta)} |\xi_l|^2\right) < 3/2$. Hence

$$\frac{8}{27}\frac{1}{h(L_\delta)}\left(\sum_{L_\delta}^{L_\delta+h(L_\delta)} \frac{|\sigma_k(\underline{\omega})|^2}{|\sigma_{n(\delta)-m}(\underline{\omega})|^2}(n(\delta)-k)!\right) \le \frac{|P_{\text{in}}^{f_{n(\delta)}}|^2}{|\xi_0|^2}$$

and

$$\frac{|P_{\text{in}}^{f_{n(\delta)}}|^2}{|\xi_0|^2} \le 8\frac{1}{h(L_\delta)}\left(\sum_{L_\delta}^{L_\delta+h(L_\delta)} \frac{|\sigma_k(\underline{\omega})|^2}{|\sigma_{n(\delta)-m}(\underline{\omega})|^2}(n(\delta)-k)!\right)$$

which, taken together, is the same as

$$\frac{8}{27}\frac{1}{h(L_\delta)}\left(\sum_{L_\delta}^{L_\delta+h(L_\delta)} |z''_{l-m}|^2 l!\right) \le \frac{|P_{\text{in}}^{f_{n(\delta)}}|^2}{|\xi_0|^2} \le 8\frac{1}{h(L_\delta)}\left(\sum_{L_\delta}^{L_\delta+h(L_\delta)} |z''_{l-m}|^2 l!\right) \tag{3.19}$$

which gives us (i).

Observe that (3.19) can be re-written as

$$\frac{1}{8}\frac{|P_{\text{in}}^{f_{n(\delta)}}|^2}{|\xi_0|^2} \le f_0^{\delta,m}\left(z''_1,\cdots,z''_{C_\delta}\right) \le \frac{27}{8}\frac{|P_{\text{in}}^{f_{n(\delta)}}|^2}{|\xi_0|^2} \tag{3.20}$$

(ii)On $\Omega^1_{n(\delta)}(M)$, we have $\frac{1}{M} \le f_0^{\delta,m}\left(z''_1,\cdots,z''_{C_\delta}\right) \le M$. So, (3.19) implies $\frac{8}{27M} \le \frac{|P_{\text{in}}^{f_{n(\delta)}}|^2}{|\xi_0|^2}$. Summing (3.18) from 0 to $n(\delta)$, noting that by choice of $n(\delta)$ we have $\sum_{l=0}^{n(\delta)} |\xi_l|^2 > \frac{1}{2}n(\delta)$ (except on an event of probability $< \delta^2$) and using the above lower bound on $P_{\text{in}}^{f_{n(\delta)}}/|\xi_0|^2$, we get the desired lower bound in part (ii).

Proof of Proposition 3.10.6

For $\mathcal{F}_{n(\delta)}$, we recall from (3.10) that $\dfrac{\sigma_{n(\delta)-l}(\underline{\omega})}{\sqrt{\binom{n(\delta)}{n(\delta)-l}(n(\delta)-l)!}} = \dfrac{1}{\xi_{n(\delta)}}\left[\xi_l + \eta_l^{(n(\delta))}\right]$, and also recall that $\displaystyle\sum_{k=0}^{n(\delta)} \frac{|\sigma_k(\underline{\zeta},\underline{\omega})|^2}{\binom{n(\delta)}{k}k!} = \frac{\sum_{l=0}^{n(\delta)}|\xi_l|^2}{|\xi_{n(\delta)}|^2}$. Substituting these, we get

$$\frac{\left|\displaystyle\sum_{k<n(\delta)-C_\delta} \frac{\overline{\sigma_{k-i}(\underline{\omega})}\sigma_{k-i}(\underline{\omega})}{\binom{n(\delta)}{k}k!}\right|}{\displaystyle\sum_{k=0}^{n(\delta)} \frac{|\sigma_k(\underline{\zeta},\underline{\omega})|^2}{\binom{n(\delta)}{k}k!}} = \frac{\left|\displaystyle\sum_{l=C_\delta+1}^{n(\delta)} \frac{1}{\sqrt{(l+i)_i(l+j)_j}}\overline{\left(\xi_{l+i}+\eta_{l+i}^{(n(\delta))}\right)}\left(\xi_{l+j}+\eta_{l+j}^{(n(\delta))}\right)\right|}{\displaystyle\sum_{l=0}^{n(\delta)}|\xi_l|^2} .$$

Set $\boldsymbol{E}_{n(\delta)} = \sum_{l=0}^{n(\delta)}|\xi_l|^2$. Expanding the product in each term of the numerator, we observe that it suffices to upper bound the following quantities:

$$\left|\sum_{l=C_\delta+1}^{n(\delta)} \frac{1}{\sqrt{(l+i)_i(l+j)_j}}\overline{\xi_{l+i}}\xi_{l+j}\right| \Big/ \boldsymbol{E}_{n(\delta)}$$

$$\text{and} \quad \left(\sum_{l=C_\delta+1}^{n(\delta)} \frac{1}{\sqrt{(l+i)_i(l+j)_j}}\,|\xi_{l+i}|\,|\eta_{l+j}^{n(\delta)}|\right) \Big/ \boldsymbol{E}_{n(\delta)}$$

$$\text{and} \quad \left(\sum_{l=C_\delta+1}^{n(\delta)} \frac{1}{\sqrt{(l+i)_i(l+j)_j}}\left|\eta_{l+i}^{n(\delta)}\right|\left|\eta_{l+j}^{n(\delta)}\right|\right) \Big/ \boldsymbol{E}_{n(\delta)} .$$

Recall that $i \vee j \geq 2$ and by convention $(l+i)_i = 1$ for $i = 0$. For the second and the third quantities, we proceed as follows. In Proposition 3.5.3 part (ii) we can set $M = 1$ and $l_0 = C_\delta$ to find that dropping an event of probability $O(\delta^2)$, we have $|\eta_l^{(n(\delta))}| < \frac{1}{l^{1/8}}$ for $l \geq C_\delta$. On the complement of this small event, we can compute the expectation of the numerator in each case. By our choice of C_δ (refer to II. in Section 3.8), this will be $O(\delta^4)$, and by Markov's inequality the numerator is $< \delta^2$ with probability $\geq 1 - c\delta^2$. For the first quantity to be estimated in the display above, we can derive a similar result by using second moments and the Chebyshev's inequality and exploiting our choice of C_δ in Section 3.8. Moreover, by our choice of $n(\delta)$ (refer to VI.(ii) in Section 3.8), we have $\sum_{l=0}^{n(\delta)}|\xi_l|^2 > \frac{1}{2}n(\delta)$ with probability $\geq 1-\delta^2$. Combining all these, we obtain the desired estimate.

Proof of Proposition 3.10.7

All zeroes dealt with in this proof are those of $f_{n(\delta)}$.

Part (a) is clear from the arguments prior to Definition 9; we simply take $E_\delta = \cup_{i=1}^4 E_\delta^i$. These arguments, along with the remarks made at the end of the proofs of Propositions 3.10.2 and 3.10.3 also imply that the inclusion of events in part (a) of this proposition can be reversed except on an event of probability $O(\delta^2)$.

Recall that on $\Omega^3_{n(\delta)}(M)$ we have:

1. (a) $\Omega^{m,1/M}_{n(\delta)}$ occurs. (b) $\gamma_3^{f_{n(\delta)}} \leq M$. (c) $|\psi_l^{f_{n(\delta)}}| \leq M$, for $l = 1, 2$.
2. $f^{\delta,m}_{i,j}(z''_1, \cdots, z''_{C_\delta}) \leq M$ where $z''_l = P_l(\psi_1^{f_{n(\delta)}}, \cdots, \psi_{C_\delta}^{f_{n(\delta)}})$.
3. $\left(\sum_{k=0}^{n(\delta)} |\frac{\sigma_k(\underline{\zeta}, \underline{\omega})}{\sigma_{n(\delta)-2}(\underline{\omega})}|^2 (n(\delta) - k)!\right) \geq \frac{1}{10M} n(\delta)$.

With respect to the last two conditions, we refer the reader to Proposition 3.10.5.

Now, recall from Proposition 3.5.1 and equations (3.7) and (3.5.2) that in order to bound the ratio of conditional densities along the submanifold Σ_s, we need to bound from above:

- **(a)** The ratio of the squared Vandermonde determinants $\frac{|\Delta(\underline{\zeta}',\underline{\omega})|^2}{|\Delta(\underline{\zeta},\underline{\omega})|^2}$.
- **(b)** $\dfrac{\left|\sum_{k=0}^{n(\delta)} \frac{\overline{\sigma_{k-i}(\underline{\omega})}\sigma_{k-j}(\underline{\omega})}{\binom{n(\delta)}{k} k!}\right|}{\sum_{k=0}^{n(\delta)} \frac{|\sigma_k(\underline{\zeta}, \underline{\omega})|^2}{\binom{n(\delta)}{k} k!}}$; $\quad 0 \leq i \leq m;\ 2 \leq j \leq m$.

As bounds we have:

(a) By definition of $\Omega^2_{n(\delta)}(M)$, we have $|\psi_1^{f_{n(\delta)}}|, |\psi_2^{f_{n(\delta)}}|$ and $\gamma_3^{f_{n(\delta)}}$ are bounded by M. By Proposition 3.5.1, this suffices to upper and lower bound the ratio of Vandermondes in terms of M.

(b) We divide the terms as

$$\frac{\left|\sum_{k=0}^{n(\delta)} \frac{\overline{\sigma_{k-i}(\underline{\omega})}\sigma_{k-j}(\underline{\omega})}{\binom{n(\delta)}{k} k!}\right|}{\sum_{k=0}^{n(\delta)} \frac{|\sigma_k(\underline{\zeta}, \underline{\omega})|^2}{\binom{n(\delta)}{k} k!}} \leq \frac{\left|\sum_{k<n-C_\delta} \frac{\overline{\sigma_{k-i}(\underline{\omega})}\sigma_{k-j}(\underline{\omega})}{\binom{n(\delta)}{k} k!}\right|}{\sum_{k=0}^{n(\delta)} \frac{|\sigma_k(\underline{\zeta}, \underline{\omega})|^2}{\binom{n(\delta)}{k} k!}} + \frac{\left|\sum_{k\geq n-C_\delta} \frac{\overline{\sigma_{k-i}(\underline{\omega})}\sigma_{k-j}(\underline{\omega})}{\binom{n(\delta)}{k} k!}\right|}{\sum_{k=0}^{n(\delta)} \frac{|\sigma_k(\underline{\zeta}, \underline{\omega})|^2}{\binom{n(\delta)}{k} k!}}$$

For the sum over the $k < n(\delta) - C_\delta$ terms, we appeal to Proposition 3.10.6 and conclude that the expression corresponding to these terms contributes $\leq 2\delta^2/n(\delta)$ because $\Omega^3_{n(\delta)}(M) \subset (E_\delta^4)^c$.

For the expression corresponding to the $k \geq n(\delta) - C_\delta$ terms, we clear $n(\delta)!$ from numerator and denominator, and divide both by $\left|\sigma_{n(\delta)-m}(\underline{\omega})\right|^2$ (note that $n(\delta) - m$ is the largest k for which $\sigma_k^{n(\delta)}(\underline{\omega})$ can be non-zero when $\Omega_{m,M}^{n(\delta)}$ occurs). We thus reduce ourselves to upper bounding

$$\frac{\left|\sum_{k \geq n(\delta) - C_\delta} \frac{\overline{\sigma_{k-i}(\underline{\omega})}\sigma_{k-j}(\underline{\omega})}{|\sigma_{n(\delta)-m}(\underline{\omega})|^2}(n(\delta)-k)!\right|}{\sum_{k=0}^{n(\delta)} \frac{|\sigma_k(\underline{\zeta},\underline{\omega})|^2}{|\sigma_{n(\delta)-m}(\underline{\omega})|^2}(n(\delta)-k)!}$$

From the definitions in Section 3.8, it follows that
$\sigma_{k-i}(\underline{\omega})/\sigma_{n(\delta)-m}(\underline{\omega}) = z''_q := P_q(\psi_1^{f_{n(\delta)}}, \cdots, \psi_q^{f_{n(\delta)}})$ with $q = n(\delta) - k + i - m$.

To be more explicit, recall that $\psi_j^{f_{n(\delta)}}$ equals the sum of the j-th inverse powers of the zeroes of $f_{n(\delta)}$ that are outside $\mathcal{D}$ (which form the co-ordinates of the vector $\underline{\omega}$ here). By definition of the j-th Newton polynomial P_j, z''_j equals the elementary symmetric function of order j formed by the inverses of the co-ordinates of $\underline{\omega}$. Using the fact $\sigma_j(\underline{\omega})$ is the elementary symmetric function of order j formed by the co-ordinates of the vector $\underline{\omega}$ (which is of dimension $n(\delta) - m$ here), simple algebra gives us the equality $\sigma_{k-i}(\underline{\omega})/\sigma_{n(\delta)-m}(\underline{\omega}) = z''_q := P_q(\psi_1^{f_{n(\delta)}}, \cdots, \psi_q^{f_{n(\delta)}})$ with $q = n(\delta) - k + i - m$.

The numerator above is then simply $f_{i,j}^{\delta,m}(z''_1, \cdots, z''_{C_\delta})$. We can then combine the bounds in conditions 2 and 3 (laid down at the beginning of this proof) to deduce that

$$\left|\sum_{k \geq n(\delta) - C_\delta} \frac{\overline{\sigma_{k-i}(\underline{\omega})}\sigma_{k-j}(\underline{\omega})}{|\sigma_{n(\delta)-m}(\underline{\omega})|^2}(n(\delta)-k)!\right| \Big/ \left(\sum_{k=0}^{n(\delta)} \frac{|\sigma_k(\underline{\zeta},\underline{\omega})|^2}{|\sigma_{n(\delta)-m}(\underline{\omega})|^2}(n(\delta)-k)!\right) \leq 10M^2/n(\delta)$$

In view of (b) above and (3.7),(3.5.2) and (3.11) (also refer to equation (14) in Chapter 2 for greater clarification), we have

$$1 - 11K(m,\mathcal{D})M^2/n(\delta) \leq \frac{D(\zeta',\omega)}{D(\underline{\zeta},\underline{\omega})} \leq 1 + 11K(m,\mathcal{D})M^2/n(\delta)$$

on $\Omega_{n(\delta)}^3(M)$.

Recalling (3.6), we deduce from the above bounds that there exist positive functions $U_1(M)$ and $L_1(M)$ (which do not depend on $n(\delta)$) such that if $(\underline{\zeta},\underline{\omega}) \in \Omega_{n(\delta)}^3(M)$ then

$$\exp\left(-L_1(M)\right) \leq \frac{\rho_{\underline{\omega}}^{n(\delta)}(\underline{\zeta}')}{\rho_{\underline{\omega}}^{n(\delta)}(\underline{\zeta})} \leq \exp\left(U_1(M)\right)$$

where $\underline{\zeta}'$ is any other vector in Σ_s, the constant sum submanifold corresponding to $\underline{\zeta}$. From here, we estimate the ratio at any two vectors $\underline{\zeta}', \underline{\zeta}''$ belonging to Σ_s a.s.:

$$\exp\left(-L(M)\right) \leq \frac{\rho_{\underline{\omega}}^{n(\delta)}(\underline{\zeta}'')}{\rho_{\underline{\omega}}^{n(\delta)}(\underline{\zeta}')} \leq \exp\left(U(M)\right) \tag{3.21}$$

where $L(M) = U(M) = L_1(M) + U_1(M)$.

Proof of Proposition 3.11.2

First recall that by definition of $n(\delta)$, we have $\frac{1}{2} \leq \frac{\sum_{l=0}^{n(\delta)} |\xi_l|^2}{n(\delta)} \leq \frac{3}{2}$ except on a set of probability $< \delta^2$. Also recall the notation

$$z_l'' = P_l(\psi_1^{f_{n(\delta)}}, \cdots, \psi_l^{f_{n(\delta)}}) = \sum_{\omega_{i_j} \neq \omega_{i_k} \in \mathcal{D}^c \cap \mathcal{F}_{n(\delta)}} \frac{1}{\omega_{i_1} \cdots \omega_{i_l}} .$$

Moreover, we have $\frac{|\sigma_k(\underline{\zeta}, \underline{\omega})|^2}{\binom{n(\delta)}{k} k!} = \left| \frac{\xi_{n(\delta)-k}}{\xi_{n(\delta)}} \right|^2$. As a result, condition (ii) implies

$$\frac{\left| \sum_{k=n(\delta)-C_\delta}^{n(\delta)} \frac{\overline{\sigma_{k-i}(\underline{\omega})} \sigma_{k-j}(\underline{\omega})}{\binom{n(\delta)}{k} k!} \right|}{\sum_{k=0}^{n(\delta)} \frac{|\sigma_k(\underline{\zeta}, \underline{\omega})|^2}{\binom{n(\delta)}{k} k!}} \leq 2\sqrt{M}/n(\delta) .$$

On the left hand side, we can replace $n(\delta)!$ by $\left|\sigma_{n(\delta)-m}(\underline{\omega})\right|^2$ both in the numerator and the denominator, and combined with the observation that $\sigma_k(\underline{\omega})/\sigma_{n(\delta)-m}(\underline{\omega}) = z''_{n(\delta)-k-m}$ this would lead to (switching from k to the new variable $l = n(\delta) - k$)

$$\frac{\left| \sum_{l=0}^{C_\delta} \overline{z''_{l+i-m}} z''_{l+j-m} l! \right|}{\sum_{k=0}^{n(\delta)} \frac{|\sigma_k(\underline{\zeta}, \underline{\omega})|^2}{|\sigma_{n(\delta)-m}(\underline{\omega})|^2} (n(\delta) - k)!} \leq 2\sqrt{M}/n(\delta).$$

Observe that the numerator is simply $f_{i,j}^{\delta,m}(z_1'', \cdots, z_{C_\delta}'')$ However, summing (3.18) from 0 to $n(\delta)$ and noting that by choice of $n(\delta)$ we have $\sum_{l=0}^{n(\delta)} |\xi_l|^2 < \frac{3}{2} n(\delta)$ (except on an event of probability $< \delta^2$), we have

$$\sum_{k=0}^{n(\delta)} \frac{|\sigma_k(\underline{\zeta}, \underline{\omega})|^2}{|\sigma_{n(\delta)-m}(\underline{\omega})|^2} (n(\delta) - k)! \leq \frac{3}{2} n(\delta) \frac{|P_{\text{in}}|^2}{|\xi_0|^2} .$$

Using this and condition (i) in the statement of Proposition 3.11.2, we have $f_{i,j}^{\delta,m}(z_1'', \cdots, z_{\mathbb{C}_\delta}'') \leq M$. However, we need a similar inequality with $f_{i,j}^{\delta,m}(z_1', \cdots, z_{\mathbb{C}_\delta}')$. The choice of $k(\delta)$ and the tail estimates in Proposition 3.5.7 and Corollary 3.5.8 enable us to replace $\psi_l^{f_{n(\delta)}}$ by $\psi_l^{f_{n(\delta)}}(k(\delta))$, and z_l'' by $z_l' = P_l(\psi_1^{f_{n(\delta)}}(k(\delta)), \cdots, \psi_{C_\delta}^{f_{n(\delta)}}(k(\delta)))$, with an additive error of size $O(\delta^2)$, except on an event of probability $O(\delta^2)$. This is on similar lines to the switch from z_l' to z_l'' we did in defining $\Omega_{n(\delta)}^1(M)$. Tallying all these, we obtain **condition 3** on $\Omega_{n(\delta)}^0(M+1)$, for all small enough δ depending on M.

Proof of Proposition 3.11.3

First recall Definitions 10 and 6. We start with $\Gamma(\delta, M) \cap \Omega^{m,1/M}$. The choice of $n(\delta)$ (condition VI.(v)) in Section 3.8 ensures that except on an event of probability $< \delta^2$, we have that $\Omega^{m,1/M}_{n(\delta)}$ occurs, thereby verifying **condition 1** in Definition 6. By choice of $k(\delta)$ and $n(\delta)$, criterion (a) defining $\Gamma(\delta, M)$ ensures **condition 2** in Definition 6, except on an event of probability $O(\delta^2)$ for δ small enough (depending on M). The complement of **condition 5** in Definition 6 is itself an event of probability $O(\delta^2)$, by choice of $g(\delta), k(\delta)$ and $n(\delta)$.

The choice of $n(\delta)$ ensures that $\left| |P^f_{\text{in}}|^2/|\xi_0|^2 - |P^{f_{n(\delta)}}_{\text{in}}|^2/|\xi_0|^2 \right| < \delta^2$ except on an event of probability $O(\delta^2)$. Recall condition (b) defining $\Gamma(\delta, M)$ and equation (3.20). Let us recall the notations $z''_l = P_l(\psi^{f_{n(\delta)}}_1, \cdots, \psi^{f_{n(\delta)}}_l)$ and $z'_l = P_l(\psi^{k(\delta),f_{n(\delta)}}_1, \cdots, \psi^{k(\delta),f_{n(\delta)}}_l)$ The choice of $k(\delta)$ (condition V) in Section 3.8 enables us to replace $f^{\delta,m}_0(z''_1, \cdots, z''_{C_\delta})$ by $f^{\delta,m}_0(z'_1, \cdots, z'_{C_\delta})$ (except on an event of probability $O(\delta^2)$), on similar lines to the switch from z'_l to z''_l in the definition of $\Omega^1_{n(\delta)}(M)$. The upshot of all this is that except on an event of probability $O(\delta^2)$ we have that on $\Gamma(\delta, M)$ the following is true:

$$f^{\delta,m}_0(z'_1, \cdots, z'_{C_\delta}) \geq f^{\delta,m}_0(z''_1, \cdots, z''_{C_\delta}) - \left| f^{\delta,m}_0(z'_1, \cdots, z'_{C_\delta}) - f^{\delta,m}_0(z''_1, \cdots, z''_{C_\delta}) \right|$$

$$\geq \frac{1}{8} \frac{|P^{f_{n(\delta)}}_{\text{in}}|^2}{|\xi_0|^2} - \left| f^{\delta,m}_0(z'_1, \cdots, z'_{C_\delta}) - f^{\delta,m}_0(z''_1, \cdots, z''_{C_\delta}) \right|$$

$$\geq \frac{1}{8} \frac{|P^f_{\text{in}}|^2}{|\xi_0|^2} - \frac{1}{8} \left| \frac{|P^{f_{n(\delta)}}_{\text{in}}|^2}{|\xi_0|^2} - \frac{|P^f_{\text{in}}|^2}{|\xi_0|^2} \right| - \left| f^{\delta,m}_0(z'_1, \cdots, z'_{C_\delta}) - f^{\delta,m}_0(z''_1, \cdots, z''_{C_\delta}) \right|$$

$$\geq \frac{1}{8} \frac{|P^f_{\text{in}}|^2}{|\xi_0|^2} - \kappa_1 \delta^2 \tag{3.22}$$

Similarly except on an event of probability $O(\delta^2)$, we have that on $\Gamma(\delta, M)$ the following is true:

$$f^{\delta,m}_0(z'_1, \cdots, z'_{C_\delta}) \leq \frac{27}{8} \frac{|P^f_{\text{in}}|^2}{|\xi_0|^2} + \kappa_2 \delta^2 \tag{3.23}$$

Here κ_1 and κ_2 are two positive constants. In condition (b) defining $\Gamma(\delta, M)$, we choose κ to be $\max(\kappa_1, \kappa_2, 1)$. The last pair of inequalities (3.22) and (3.23), along with condition (b) defining $\Gamma(\delta, M)$ gives us **condition 4** in Definition 6, for all M sufficiently large (in fact, as soon as $\sqrt{M} \geq 9/8$).

Thus, the only condition in Definition 6 that remains to be dealt with is condition 3.

We have already seen how criteria (i) and (ii) in Proposition 3.11.2 suffice to imply **condition 3** in Definition 6, except on an event of probability $O(\delta^2)$. Criterion (i) is implied by the upper bound in condition (b) defining $\Gamma(\delta, M)$, applied along with the inequality

$\left|\frac{|P_{\mathrm{in}}^{f_{n(\delta)}}|^2}{|\xi_0|^2} - \frac{|P_{\mathrm{in}}^{f}|^2}{|\xi_0|^2}\right| < \delta^2$ (which holds everywhere except on an event of probability $< \delta^2$ by definition of $n(\delta)$; recall that $\kappa \geq 1$ by its definition).

Thus, it remains to show that conditions labeled by (c),(d) and (e) defining $\Gamma(\delta, M)$ together imply criterion (ii) in Proposition 3.11.2. To this end, we refer to Section 3.5 where we obtain expressions of $\sigma_k(\underline{\omega})$ as expansions in $\xi_l/\xi_{n(\delta)}$ (as in (3.10)). From such expansions, we find that in order to establish criterion (ii), it suffices to show (here we switch to the variable $l = n(\delta) - k$)

$$\frac{1}{|\xi_{n(\delta)}|^2}\left|\sum_{l=0}^{C_\delta}\left(\frac{\overline{\xi}_{l+i} + \overline{\eta}_{l+i}^{(n(\delta))}}{\sqrt{(l+i)_i}}\right)\left(\frac{\xi_{l+j} + \eta_{l+j}^{(n(\delta))}}{\sqrt{(l+j)_j}}\right)\right| \leq \frac{\sqrt{M}}{|\xi_{n(\delta)}|^2} \, .$$

Recall from Proposition 3.5.3 that except with probability $O(\delta^2)$, we have for all $l \geq 1, |\eta_l^{(n(\delta))}| \leq |\eta_l| \leq l^{-1/8}$. It is then a straightforward calculation using the triangle inequality that indeed (c), (d) and (e) defining $\Gamma(\delta, M)$ imply the above inequality, for $0 \leq i \leq m; 2 \leq j \leq m$.

3.13 Proof of Theorem 3.2.1

We claim that it suffices to show that for every j_0 , we have for any $A \in \mathcal{A}$ and any Borel set B in $\mathcal{S}_{\mathrm{out}}$

$$\mathbb{P}\left((\underline{\zeta}(X_{\mathrm{in}}) \in A) \cap (X_{\mathrm{out}} \in B) \cap \Omega(j_0)\right) \asymp_{j_0} \int_{X_{\mathrm{out}}^{-1}(B)\cap\Omega(j_0)} \nu(\xi, A) d\mathbb{P}(\xi) \, . \tag{3.24}$$

First, note that (3.24) implies the conclusion (3.2) of Theorem 3.2.1 a.s. on the event $\Omega(j_0)$ with the quantities m and M depending only on j_0. To see this, let $\mathcal{X}$ be the sigma-algebra generated by X_{out} , and let $\mathcal{X}(j_0)$ be the collection of sets formed by intersecting the sets of $\mathcal{X}$ with $\Omega(j_0)$. So, every set in $\mathcal{X}(j_0)$ is of the form $X_{\mathrm{out}}^{-1}(B) \cap \Omega(j_0)$ for some Borel set $B \subset \mathcal{S}_{\mathrm{out}}$. We can then consider the finite measure space $(\Omega(j_0), \mathcal{X}(j_0), \mathbb{P})$. For fixed $A \in \mathcal{A}$ and any Borel set B in $\mathcal{S}_{\mathrm{out}}$ we have

$$\mathbb{P}\left((\underline{\zeta}(X_{\mathrm{in}}) \in A) \cap (X_{\mathrm{out}} \in B) \cap \Omega(j_0)\right) = \int_{X_{\mathrm{out}}^{-1}(B)\cap\Omega(j_0)} \rho(X_{\mathrm{out}}(\xi), A) d\mathbb{P}(\xi) \, . \tag{3.25}$$

We can now compare the (3.24) and (3.25) for fixed $A \in \mathcal{A}$ and all Borel sets B in $\mathcal{S}_{\mathrm{out}}$. Considering this on the finite measure space $(\Omega(j_0), \mathcal{X}(j_0), \mathbb{P})$, we obtain obtain (3.2) for a.s. $\xi \in \Omega(j_0)$ and a fixed $A \in \mathcal{A}$. The constants m and M obtained are the same as those appearing in (3.24) for this j_0. Here we use the fact that all the sides in (3.2) are measurable with respect to X_{out}. We conclude that for a.s. $\xi \in \Omega(j_0)$, (3.2) is simultaneously true for all sets $A \in \mathcal{A}$, because $\mathcal{A}$ is a countable collection of sets. From here, we use Proposition 2.6.1 to obtain (3.2) for all Borel sets A in $\mathcal{D}^m$ and a.s. $\xi \in \Omega(j_0)$. But the $\Omega(j)$-s exhaust Ω^m, so we have the conclusion of Theorem 3.2.1 holding a.e. $\xi \in \Omega^m$. The constants however now depend on $\omega = X_{\text{out}}(\xi)$, because for any given ξ, we simply take the constants coming out of (3.24) for the minimal j_0 for which $\xi \in \Omega(j_0)$; recall here that $\Omega(j_0)$ is measurable with respect to X_{out} .

Now we focus on proving (3.24). For any Borel set B in $\mathcal{S}_{\text{out}}$, given $\varepsilon > 0$ we can find a $\widetilde{B} \in \mathcal{B}$ such that $\mathbb{P}\left(\left(X_{\text{out}}^{-1}(B) \cap \Omega(j_0)\right) \Delta X_{\text{out}}^{-1}(\widetilde{B})\right) < \varepsilon$. This can be seen by considering the push forward probability measure $(X_{\text{out}})_*\mathbb{P}$ on $\mathcal{S}_{\text{out}}$; recall that $\Omega(j_0)$ is also measurable with respect to X_{out}. The aim of this reduction is to exploit the fact that as $k \to \infty$, we have $1_{\widetilde{B}}(X_{\text{out}}^{n_k}) \to 1_{\widetilde{B}}(X_{\text{out}})$ a.s.

We start from (3.1) applied to $A, \widetilde{B}, j_0$ (where $A \in \mathcal{A}$) and intend to derive (3.24) for A, B, j_0. We want to show that both sides of (3.1) converge to the appropriate quantities in (3.24) as $k \to \infty$ and $\varepsilon \to 0$.

In what follows, we will denote by $\mathbb{P}_h$ the non-negative finite measure on Ξ obtained by setting $d\mathbb{P}_h(\xi) = h(\xi)d\mathbb{P}(\xi)$ where $h : \Xi \to [0, 1]$ is a measurable function. We note that for any event E, we have $0 \leq \mathbb{P}_h(E) \leq \mathbb{P}(E) \leq 1$. For the rest of this proof, the value of j_0 is held fixed.

We start with $\mathbb{P}_h[(X_{\text{in}}^{n_k} \in A) \cap (X_{\text{out}}^{n_k} \in \widetilde{B}) \cap \Omega_{n_k}(j_0)]$. This is equal to

$$\mathbb{P}_h\left[(X_{\text{in}} \in A) \cap (X_{\text{out}} \in \widetilde{B}) \cap \Omega_{n_k}(j_0)\right] + o_k(1; \widetilde{B})$$

where $o_k(1; \widetilde{B})$ stands for a quantity that tends to 0 as $k \to \infty$ for fixed $\widetilde{B}$. This step uses the fact that $1_{\widetilde{B}}(X_{\text{out}}^{n_k}(\xi)) \to 1_{\widetilde{B}}(X_{\text{out}}(\xi))$ a.s. The last expression above equals

$$\mathbb{P}_h\left[(X_{\text{in}} \in A) \cap ((X_{\text{out}} \in B) \cap \Omega(j_0)) \cap \Omega_{n_k}(j_0)\right] + o_\varepsilon(1) + o_k(1; \widetilde{B})$$

where $o_\varepsilon(1)$ denotes a quantity that tends to 0 uniformly in k as $\varepsilon \to 0$. Observe that $(X_{\text{out}})^{-1}(B) \cap \Omega(j_0) \subset \Omega(j_0) \subset \underline{\lim}_{k\to\infty} \Omega_{n_k}(j_0)$ and $\mathbb{P}\left(\underline{\lim}_{k\to\infty} \Omega_{n_k}(j_0) \Delta \left(\bigcap_{l\geq k} \Omega_{n_l}(j_0)\right)\right) = \mathfrak{o}_k(1)$ where $\mathfrak{o}_k(1)$ denotes a quantity such that $\mathfrak{o}_k(1) \to 0$ as $k \to \infty$, uniformly in B and ε. Hence we have

$$\mathbb{P}_h\left[(X_{\text{in}} \in A) \cap (X_{\text{out}} \in B) \cap \Omega(j_0) \cap \Omega_{n_k}(j_0)\right]$$
$$= \mathbb{P}_h\left[(X_{\text{in}} \in A) \cap (X_{\text{out}} \in B) \cap \Omega(j_0) \cap \left(\bigcap_{l\geq k} \Omega_{n_l}(j_0)\right) \cap \Omega_{n_k}(j_0)\right] + \mathfrak{o}_k(1) \; .$$

But $\left(\bigcap_{l\geq k} \Omega_{n_l}(j_0)\right) \cap \Omega_{n_k}(j_0) = \left(\bigcap_{l\geq k} \Omega_{n_l}(j_0)\right)$ and

$$\mathbb{P}_h\left[(X_{\text{in}} \in A) \cap (X_{\text{out}} \in B) \cap \Omega(j_0) \cap \left(\bigcap_{l\geq k} \Omega_{n_l}(j_0)\right)\right]$$
$$= \mathbb{P}_h\left[(X_{\text{in}} \in A) \cap (X_{\text{out}} \in B) \cap \Omega(j_0) \cap \underline{\lim}_{k} \Omega_{n_k}(j_0)\right] + \mathfrak{o}_k(1).$$

But again, $(X_{\text{out}} \in B) \cap \Omega(j_0) \subset \underline{\lim}_k \Omega_{n_k}(j_0)$, so the upshot of all this is that

$$\mathbb{P}_h\Big[(X_{\text{in}}^{n_k} \in A) \cap (X_{\text{out}}^{n_k} \in \widetilde{B}) \cap \Omega_{n_k}(j_0)\Big] = \mathbb{P}_h\Big[(X_{\text{in}} \in A) \cap (X_{\text{out}} \in B) \cap \Omega(j_0)\Big] + o_k(1; \widetilde{B})$$

$$+\mathfrak{o}_k(1) + o_\varepsilon(1).$$

We apply the above reduction to the left hand side of (3.1) with $h \equiv 1$ and to the right hand side of (3.1) with $h(\xi) = \nu(\xi, A)$, and obtain

$$\mathbb{P}\Big[(X_{\text{in}} \in A) \cap (X_{\text{out}} \in B) \cap \Omega(j_0)\Big] \asymp_{j_0} \left(\int_{(X_{\text{out}})^{-1}(B)\cap\Omega(j_0)} \nu(\xi, A) d\mathbb{P}(\xi)\right) + o_\varepsilon(1) + o_k(1; \widetilde{B}) + \mathfrak{o}_k(1)$$

$$+\vartheta(k; j_0)\ .$$

Recall from the statement of Theorem 3.2.1 that $\vartheta(k; j_0) \to 0$ as $k \to \infty$. First letting $k \to \infty$ in the above (with $\widetilde{B}$ held fixed) and then $\varepsilon \to 0$ (i.e., letting $\widetilde{B} \to B$ in the sense that $\mathbb{P}(\widetilde{B} \Delta B) \to 0$) in the above, we obtain (3.24).

This completes the proof of Theorem 3.2.1.

Chapter 4

Continuum Percolation for Gaussian zeroes and Ginibre eigenvalues

4.1 Introduction

Let Π be a simple point process in Euclidean plane. We place open disks of the same radius r around each point of Π, and say that two points are neighbours if the corresponding disks overlap. Two points in Π are connected if there is a sequence of neighbouring points of Π that include these two points. We can then study the statistical properties of the maximal connected components (referred to as "clusters") of the points of Π. Of particular interest are the infinite cluster(s). This is the basic setting of the continuum percolation model, also referred to as the Boolean model.

It is clear from an easy coupling argument that the probability that an infinite cluster exists is an increasing function of the radius of the disks. We say that there is a non-trivial critical radius if there exists an $0 < r_c < \infty$ such that the probability of having an infinite cluster is zero when $0 < r < r_c$ and the same probability is is strictly positive when $r_c < r < \infty$. For $r_c < r < \infty$, one can ask whether the infinite cluster is unique. For point processes which are ergodic under the action of translations, the event that there is an infinite cluster is translation-invariant, and therefore its probability is either 0 or 1. Similarly, the number of infinite clusters is a translation-invariant random variable, and therefore a.s. a constant.

In this chapter we focus on the two main natural examples of repelling point processes on the plane: the Ginibre ensemble, arising as weak limits of certain random matrix eigenvalues, and the Gaussian zero process arising as weak limits of zeroes of certain random polynomials. The latter process will be abbreviated as the GAF zero process. For details on these models, see Chapter 1.

In [BY11] (see Corollary 3.7 and the discussion thereafter) it has been shown that there exists a non-zero and finite critical radius for the Ginibre ensemble.

In this chapter we prove the following theorems:

Theorem 4.1.1. *In the Boolean percolation model on the Ginibre ensemble, a.s. there is exactly one infinite cluster in the supercritical regime.*

Theorem 4.1.2. *In the Boolean percolation model on the GAF zero process, there exists a non-zero and finite critical radius. Moreover, in the supercritical regime, a.s. there is exactly one infinite cluster.*

Continuum percolation is well-studied in theoretical and applied probability, as a model of communication networks, disease-spreading through a forest, and many other phenomena. This model, also referred to as the Gilbert disk model or the Boolean model, is almost as old as the more popular discrete bond percolation theory. It was introduced by Gilbert in 1961 [Gil61]. In the subsequent years, it has been studied extensively by different authors, such as [Ha88], [Mo94],[MR96] and [Pe03], among others. Closely related models such as random geometric graphs, random connection models, face percolation in random Voronoi tessellations have also been studied. For a detailed discussion of continuum percolation and related models, we refer the reader to [MR96] and [BoRi09]. For further details on point processes, we refer to [DV97].

Much of the literature so far has focused on studying continuum percolation where the underlying point process Π is either a Poisson process or a variant thereof. Most of these models exhibit some kind of spatial independence. This property is extremely useful in the study of continuum percolation on these models. E.g., the spatial independence enables us to carry over Peierls type argument from discrete percolation theory for establishing phase transitions in the existence of infinite clusters, or Burton and Keane type arguments in order to prove uniqueness of infinite clusters.

While the Poisson process is the most extensively studied point process, the spatial independence built into it makes it less effective as a model for many natural phenomena. This makes it of interest to study point processes with non-trivial spatial correlation, particularly those where the points exhibit repulsive behaviour. On the complex plane, the main natural examples of translation-invariant point processes exhibiting repulsion are the Ginibre ensemble and the Gaussian zero process. The latter process is also known as the Gaussian analytic function (GAF) zero process. The former arises as weak limits of eigenvalues of (non-Hermitian) random matrices, while the latter arises as weak limits of zeros of Gaussian polynomials. For precise definitions of these processes we refer the reader to Chapter 1.

The Ginibre ensemble was introduced by the physicist Ginibre [Gin65] as a physical model based on non-Hermitian random matrices. In the mathematics literature it has been studied by [RV07] and [Kr06] among others. The Gaussian zero process also has been studied in either field, see, e.g. [BoBL92], [STs1-04],[STs2-06],[STs3-05],[NSV07], [FH99]. We refer the reader to [NS10] for a survey. These models are distinguished elements in broader classes of repulsive point processes. For example, the Ginibre ensemble is essentially the unique determinantal process on the plane whose kernel $K(z, w)$ is holomorphic in the first variable, and conjugate holomorphic in the second ([Kr06]). The Gaussian zero process is essentially unique (up to scaling) among the zero sets of Gaussian power series in that its distribution is invariant under translations ([STs1-04]). For an exposition on both the processes, we refer the reader to [HKPV10].

The strong spatial correlation present in the above models severely limits the effectiveness of standard independence-based arguments from the Poisson setting while studying continuum percolation. Our aim in this chapter is to study continuum percolation on the two natural models of repulsive point processes mentioned above, and establish the basic results. Namely, there is indeed a non-trivial critical radius, and the infinite cluster is unique when we are in the supercritical regime.

While the spatial independence of the Poisson process is not available in these models, we observe that this obstacle can be largely overcome if we can obtain detailed understanding of spatial conditioning in these point processes. Such understanding has been obtained in Chapter 2, where it has been shown that for a given domain $\mathcal{D}$, the point configuration outside $\mathcal{D}$ determines a.s. the number of points in $\mathcal{D}$ (in the Ginibre ensemble) and their number and the centre of mass (in the Gaussian zeroes ensemble), and "nothing further". For a precise statement of the results, we refer the reader to the Theorems 4.10.1, 4.10.2, 4.11.1 and 4.11.2 quoted in this chapter. In the present work, we demonstrate that along with certain estimates on the strength of spatial dependence, this understanding is sufficient to overcome the problem of lack of independence, and answer the basic questions in continuum percolation on these two processes.

For determinantal point processes in Euclidean space, it is known that a non-trivial critical radius exists, see, e.g. [BY11]. This covers the Ginibre ensemble. The uniqueness of the infinite cluster (in the supercritical regime), however, was not known, and this is proved in Section 4.10. For the Gaussian zero process, both the existence of a non-trivial critical radius and the uniqueness of the infinite cluster (when one exists) are new results, and are established in Sections 4.5 and 4.11 respectively.

In the case of the GAF zero process, while proving our main results we derive new estimates for hole probabilities. Let $B(0;R)$ be the disk with centre at the origin and radius R. The hole probability for $B(0;R)$ is the probability $p(R) = \mathbb{P}\left(B(0;R) \text{ has no GAF zeroes }\right)$. It has been studied in detail in [STs3-05], and culminated in the work of Nishry [Nis10] where he obtained the precise asymptotics as $R \to \infty$. It turns out that as $R \to \infty$ we have $-\log p(R)/R^4 \to c$ where $c > 0$ is a constant. In this chapter, however, we need to understand hole probabilities for much more general sets than disks.

Let us divide the plane into $\theta \times \theta$ squares given by the grid $\theta\mathbb{Z}^2$. Let $\Gamma(L)$ be a connected set comprising of L such squares. We prove that for $\theta \geq \theta_0$ (an absolute constant), there is a quantity $c(\theta) > 0$ such that $\mathbb{P}\left(\Gamma(L) \text{ has no GAF zeroes }\right) \leq \exp\left(-c(\theta)L\right)$. The techniques generally used in the literature to study hole probabilities, e.g. Offord-type estimates, do not readily apply to this situation. Instead, we exploit a certain 'almost independence' property of GAF, and combine it with a Cantor set type construction to obtain the desired result.

4.2 The Boolean model

Let Π be a point process in $\mathbb{R}^2$ whose one-point and two-point intensity measures are absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^2$ and $\mathbb{R}^2 \times \mathbb{R}^2$ respectively. We say two points x, y of Π are *neighbours* of each other if $\|x - y\|_2 < 2r$. Equivalently, we can place open disks of radius r around each point; then two points are connected if and only the corresponding disks intersect. Two points x, y of Π are *neighbours* if $\exists$ a finite sequence of points $x_0, x_1, \cdots, x_n \in \Pi$ such that $x_0 = x, x_n = y$ and x_{j+1} is the neighbour of x_j for $0 \leq j \leq n - 1$.

This is the Boolean percolation model on the point process Π with radius r, denoted by $X(\Pi, r)$.

Connectivity as defined above is an equivalence relation, and the maximal connected components are called *clusters*. The size of a cluster is the number of points of Π in that cluster. We say that the model percolates if there is at least one infinite cluster. We say that $x_0 \in \mathbb{R}^2$ is connected to the infinity if there is a point $x \in \Pi$ such that $\|x - x_0\|_2 < r$ and x belongs to an infinite cluster. The probability of having an infinite cluster and that of the origin being connected to infinity both depend on the parameter r. The trivial coupling obtained from the inclusion of a disk of radius r' inside a disk of radius r with the same centre (for $r' < r$) shows that both of these probabilities are non-decreasing in r.

Notation 2. *Let $\Lambda(r)$ denote the number of infinite clusters when the disks are of radius r.*

Definition 12. *The point process Π is said to have a critical radius $0 < r_c < \infty$ if $\Lambda(r) = 0$ a.s. when $0 < r < r_c$ and $\mathbb{P}(\Lambda(r) > 0) > 0$ when $r_c < r < \infty$.*

For any point process Π in $\mathbb{R}^2$, the group of translations of $\mathbb{R}^2$ acts in a natural way on Π: a translation T takes the point $x \in \Pi$ to $T(x)$, the resulting point process being denoted $T_*\Pi$. The process Π is said to be translation invariant if $T_*\Pi$ has the same distribution as Π for all translations T. The process is said to be ergodic under translations if this action is ergodic.

For any translation invariant point process, the probability of the origin being connected to infinity is the same as that for any $x \in \mathbb{R}^2$, so by a simple union bound over $x \in \mathbb{R}^2$ with rational co-ordinates, the probability of having an infinite cluster is positive if and only if the probability of the origin being connected to infinity is positive.

Clearly, $\Lambda(r)$ is a translation-invariant random variable. If the distribution of Π is ergodic under translations, $\Lambda(r)$ is a.s. a non-negative integer constant. In particular, the probability of having at least one infinite cluster is either 0 or 1.

4.3 The underlying graph

Consider the Boolean model with radius r on a point process Π in $\mathbb{R}^2$. By the *underlying graph* $\mathfrak{g}$ of this model we mean the graph whose vertices are the points of Π and two vertices

x, y are neighbours iff $\|x - y\|_2 < 2r$. By $\Phi(\mathfrak{g})$ we denote the subset of $\mathbb{R}^2$ formed by the union of the points of Π and straight line segments drawn between two such points whenever their mutual distance is less than $2r$. Since the two point intensity measure of Π is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^2 \times \mathbb{R}^2$, therefore the probability that there are two points of Π at a mutual distance $2r$ is 0. Hence, if a take a large open disk D, then there exists an $\varepsilon > 0$ such that for each point x of Π in D we have $\widetilde{B}(x; \varepsilon) \subset D$ and x can be moved to any new position in the open disk $\widetilde{B}(x; \varepsilon)$ without changing the connectivity properties of $\mathfrak{g}$. In other words, let Φ map each point of the configuration inside $\mathcal{D}$ to any point in its ε neighbourhood, and let it map every other point of the confugration to itself. Then $\Phi(x)$ and $\Phi(y)$ are neighbours if and only if x and y are neighbours.

4.4 Discrete approximation and the critical radius

The first step in our study of continuum percolation will be to relate our events of interest to events defined with respect to a grid, so that the problem becomes amenable to techniques similar to the ones that are effective in studying percolation in discrete settings.

Definition 13. *Let $\theta > 0$ be a parameter, to be called base length, and consider the grid formed by $\theta\mathbb{Z}^2$ (which includes the horizontal and vertical edges connecting the points of $\theta\mathbb{Z}^2$). Each $\theta \times \theta$ closed square (including the interior) whose vertices are the points of $\theta\mathbb{Z}^2$ will be referred to as a standard square. Two (distinct) standard squares are said to be neighbours if their boundaries intersect. So, each standard square has 8 neighbours.*

Notation 3. *For $x \in \mathbb{R}^2$ and $R > 0$, we will denote by $\widetilde{B}(x; R)$ the open disk with centre x and radius R.*

We define W_R, the box of size R, to be the set $W_R := \{x \in \mathbb{R}^2 : \|x\|_\infty = R\}$.

For a subset $K \subset \mathbb{R}^2$, we will denote by $\overline{K}$ the topological closure of K.

Definition 14. *Fix a radius $r > 0$ and a base length $\theta > 0$.*

A continuum path γ of length n is defined to be a piecewise linear curve whose vertices are given by the sequence of points $x_j \in \Pi, 1 \leq j \leq n$ such that x_{i+1} is a neighbour of x_i for $1 \leq i \leq n - 1$.

For a continuum path γ with vertices $\{x_1, \cdots, x_n\}$, we denote by $S(\gamma)$ the set $\bigcup_{i=1}^n \widetilde{B}(x_i; r)$.

A lattice path Γ of length n is defined to be a sequence of standard squares $\{X_j\}_{j=1}^n$ such that X_{i+1} is a neighbour of X_i for $1 \leq i \leq n - 1$. A lattice path $\{X_i\}_{i=1}^n$ is said to be non-repeating if $X_i \neq X_j$ for $i \neq j$.

For a lattice path $\Gamma = \{X_1, \cdots, X_n\}$, we denote by $V(\Gamma)$ the set $\bigcup_{i=1}^n X_i$.

We say that a continuum path γ connects the origin to W_R if $0 \in S(\gamma)$ and $S(\gamma) \cap W_R \neq \varphi$. For $R \in \mathbb{Z}_+$, we say that a lattice path Γ connects the origin to $W_{R\theta}$ if $0 \in V(\Gamma)$ and $V(\Gamma) \cap W_{R\theta} \neq \varphi$.

With these notions in hand, we are ready to state:

Proposition 4.4.1. *Consider the Boolean percolation model $X(\Pi, r)$. Let the base length $\theta = r/\sqrt{8}$. Suppose, for some $L \in \mathbb{Z}_+$, there exists a non-repeating lattice path $\Gamma = \{X_1, \cdots, X_n\}$ that connects 0 to $W_{L\theta}$ with each X_i containing at least one point in Π. Then there exists a continuum path γ that connects 0 to $W_{L\theta}$.*

Proof. The result follows from the fact that with $\theta = r/\sqrt{8}$, disks of radius r centred at any two points in adjacent standard squares intersect with each other. This is true because the maximum possible distance between two points in adjacent standard squares is $\theta\sqrt{8}$. ∎

Proposition 4.4.2. *Fix a base length θ and an integer $k \geq 0$. For any $0 < r < \theta/18k$ the following happens: Suppose in $X(\Pi, r)$ there exists a continuum path γ connecting 0 to $W_{L\theta}$ (where $L \in \mathbb{Z}_+$). Then there exists a non-repeating lattice path Γ connecting 0 to $W_{L\theta}$ such that each standard square in Γ contains $\geq k$ points $\in \Pi$.*

Proof. Let r be a radius such that $k < \theta/18r$, and let γ be a continuum path with vertices $\{x_i\}_{i=1}^n$ connecting 0 to $W_{L\theta}$. A finite lattice path Γ_1 is said to be contained in another finite lattice path Γ_2, denoted by $\Gamma_1 \subset \Gamma_2$, if $V(\Gamma_1) \subset V(\Gamma_2)$.

Now, consider the set Ξ of all finite lattice paths Γ (non-repeating or otherwise), $0 \in V(\Gamma)$, such that each standard square in Γ contains $\geq k$ points. Clearly, $\subset$ is a partial order on Ξ. Moreover, Ξ is non-empty, because γ must reach L^∞ distance θ from the origin, and in doing so must have at least $\theta/2r$ points $\in \Pi$. The 4 standard squares whose closures contain the origin contain these $\theta/2r$ points, so at least one of them must have at least $\theta/8r \geq k$ points in Π.

Let Γ be a maximal element in Ξ under $\subset$. If Γ connects 0 to $W_{L\theta}$ then we are done. Otherwise, we define the *surround* $\Sigma(\Gamma)$ of Γ as the union of all standard squares which are neighbours of the standard squares in Γ and are contained in the unbounded component of the complement of Γ. Since γ connects 0 to $W_{L\theta}$, therefore γ intersects $\partial\Sigma(\Gamma) \setminus V(\Gamma)$. Let j be the least index $\in [n]$ such that the line segment $(x_{j-1}, x_j]$ intersects $\partial\Sigma(\Gamma) \setminus V(\Gamma)$. Since $r < \theta$, we must have $x_{j-1} \in \text{ Int } (\Sigma(\Gamma))$, where $\text{ Int } (H)$ denotes the interior of a set H. Let σ be a standard square in $\Sigma(\Gamma)$ such that σ contains x_{j-1}. Consider the continuum path γ' with vertices $\{x_j, x_{j-1}, \cdots, x_i\}$ where i is the largest index $\leq j-1$ such that $x_i \in \Gamma$. In other words, we trace the vertices of γ backwards from x_j until we are in Γ. Now the part of γ' contained in σ and its neighbouring standard squares (that are in $\Sigma(\Gamma)$) is of length at least θ, therefore it has at least $\theta/2r$ points $\in \Pi$ contained in these squares. But

$$\text{Total number of such squares (including } \sigma)$$

$$\leq 1 + \text{ number of standard squares neighbouring } \sigma = 9.$$

Therefore we have $\theta/2r$ points $\in \Pi$ contained in ≤ 9 squares in $\Sigma(\Gamma)$. Therefore at least one square σ' in $\Sigma(\Gamma)$ has at least $\theta/18r \geq k$ points of γ. Let $\Gamma = \{X_i\}_{i=1}^N$ and let σ' be a neighbour of $X_j \in \Gamma$. We define a new lattice path $\Gamma' \in \Xi$ by

$$\Gamma' = \{X_1, X_2 \cdots, X_N, X_{N-1}, X_{N-2} \cdots, X_j, \sigma\},$$

that is, by backtracking along Γ until we reach X_j and then appending σ at the end. Clearly, $S(\Gamma') \supset S(\Gamma)$ as a proper subset, contradicting the maximality of Γ.

Since the procedure described above must terminate after finitely many steps because $W_{L\theta}$ is a compact set, a maximal element Γ of Ξ must connect 0 to $W_{L\theta}$. Such a lattice path may not be non-repeating. However, we can erase the loops in Γ in the chronological order to obtain a non-repeating lattice path of the desired kind that connects 0 to $W_{L\theta}$. ■

In the next theorem, we provide some general conditions under which there exists a non-trivial critical radius for the Boolean percolation model

Theorem 4.4.3. *Let Π be a translation invariant and ergodic point process with the property that for any connected set Γ of L standard squares (with base length θ) the following are true:*

- *(i)For large enough θ, we have*

  $$\mathbb{P}\left[\Gamma \text{ contains no points } \in \Pi\right] \leq \exp\left(-c_1(\theta)L\right)$$

  with $c_1(\theta) \to \infty$ as $\theta \to \infty$

- *(ii)For large enough θ, we have*

  $$\mathbb{P}\left[\text{ Each standard square in } \Gamma \text{ has at least } k \text{ points } \in \Pi\right] \leq \exp\left(-c_2(\theta,k)L\right)$$

  with $\lim_{\theta\to\infty}\lim_{k\to\infty} c_2(\theta,k) = \infty$.

In the Boolean percolation model $X(\Pi, r)$ on such a Π, let r denote the radius of each disk and let $\Lambda(r)$ denote the number of infinite clusters. Then there exists $0 < r_c < \infty$ such that for $0 < r < r_c$, we have $\Lambda(r) = 0$ a.s. and for $r_c < r < \infty$ we have $\Lambda(r) > 0$ a.s.

Proof. The proof follows a Peierl's type argument from the classical bond percolation theory, after appropriate discretization using Propositions 4.4.1 and 4.4.2. We first note that by translation invariance, it suffices to show that $\mathbb{P}\left[0 \text{ is connected to } \infty \text{ with radius } r\right] > 0$ or $= 0$ respectively in order to show that $\Lambda(r) > 0$ or $= 0$ a.s.

We want to show that for small enough r, there is no continuum path connecting 0 to ∞. Consider possible base lengths θ so large that our hypothesis (ii) is valid. Fix base length θ and k a positive integer large enough such that $2\log 3 - c_2(\theta,k) < 0$ where c_2 is as in (ii). We call a non-repeating lattice path Γ to be $k-$full if each standard square in Γ contains $\geq k$ points $\in \Pi$. By condition (ii), if there are L distinct standard squares in Γ, then the probability of Γ being k-full $\leq \exp\left(-c_2(\theta,k)L\right)$. Since each standard square has ≤ 9 neighbours, therefore the number of non-repeating lattice paths Γ containing 0 and having L standard squares $\leq 9^L$. So,

$$\mathbb{P}\Bigg[\text{There is a k-full lattice path of length } L \text{ containing the origin}\Bigg]$$

$$\leq \exp\left((2\log 3 - c_2(\theta, k))L\right)$$

The right hand side is summable in positive integers L, hence by Borell-Cantelli lemma,

$$\mathbb{P}\left[\text{ There exists a k-full lattice path connecting the origin to } W_{L\theta} \text{ for all } L \in \mathbb{Z}_+\right] = 0$$

If there was a continuum path γ connecting 0 to ∞, then for any integer $t > 0$ there will be a continuum path connecting 0 to $W_{t\theta}$. We now appeal to Proposition 4.4.2 for this k and find an r small enough such that for any continuum path γ connecting 0 to the box $W_{t\theta}$ we can find a k-full lattice path Γ connecting 0 to $W_{t\theta}$. But we have already seen that a.s. there are only finitely many k-full lattice paths, which gives us a contradiction, and proves that there is no continuum path connecting 0 to ∞, with probability 1.

By translation invariance, this proves that for small enough r, we have $\Lambda(r) = 0$ a.s.

Next, we want to show that for large enough r, with positive probability there exists a continuum path connecting 0 to ∞. Fix a radius r in the Boolean model. The event that there exists no continuum path from 0 to ∞, implies by Proposition 4.4.1 that (choosing the base length to be θ as in Proposition 4.4.1 with $\theta = r/\sqrt{8}$) there exists $L \in \mathbb{Z}_+$ such that there is no lattice path connecting the origin to $W_{L\theta}$. The last statement implies that there exists a circuit of standard squares surrounding the origin such that the interiors of the standard squares in this circuit do not contain any point from Π. Therefore, it suffices to prove that the probability of this event can be made < 1 by choosing r sufficiently large.

To this end, we recall that the number of circuits of standard squares containing the origin and consisting of L distinct standard squares is $\exp(cL)$ for some constant $c > 0$. For details on this, we refer the reader to [BoRi09], Chapter 1, proof of Lemma 2.

The probability that a specific circuit of standard squares surrounding the origin and containing L standard squares is empty $\leq \exp(-c_1(\theta)L)$ when base length is θ, which follows from condition (i) in the present theorem. Therefore,

$$\mathbb{P}\left[\text{ There exists an empty circuit surrounding the origin }\right] \leq \sum_{L=1}^{\infty} e^{c(d)L} e^{-c_1(\theta)L} \qquad (4.1)$$

Now, by choosing r large enough, we can make θ large enough (by Proposition 4.4.1), so that condition (i) would imply that the right hand side of (4.1) is less than 1. This completes the proof that when r is large enough, 0 is connected to ∞ with positive probability. ■

Remark 4.4.1. *Theorem 4.4.3 carries over verbatim to d dimensions instead of 2, with standard squares replaced by d dimensional standard cubes, whose definition is analogous. The proof works on similar lines.*

4.5 Critical radius for Gaussian zeros

In this section we aim to study the Boolean model on the planar GAF zero process. First of all, we will prove an estimate on hole probabilities and overcrowding probabilities in the

Gaussian zero ensemble, which is taken up in Section 4.6. It will subsequently be used to prove the existence of critical radius for the Boolean percolation model on Gaussian zeroes in Section 4.7.

4.6 Exponential decay of hole and overcrowding probabilities

The main goal of this section is to prove the following estimate on the hole and overcrowding probabilities of connected sets composed of standard squares:

Theorem 4.6.1. *Let Γ be a connected set composed of L standard squares of side length θ. Let E and F_k denote the events that that each standard square in Γ has no zeroes and has $\geq k$ zeroes respectively. Then, for θ bigger than some universal constant, we have :*

$$(i)\mathbb{P}\left[E\right] \leq \exp\left(-c_1(\theta)L\right) \qquad (ii)\mathbb{P}\left[F_k\right] \leq \exp\left(-c_2(\theta,k)L\right)$$

where $c_1(\theta) \to \infty$ as $\theta \to \infty$ and $\lim_{\theta\to\infty} \lim_{k\to\infty} c_2(\theta,k) = \infty$

We will perform a certain Cantor type construction which will be used in proving Theorem 4.6.1. For the rest of this section, the symbol "log" denotes logarithm to the base 2. We will first perform the construction for a straight line, and then take a product of the construction along the two axes which will result in a similar construction for a square.

We consider the normalized GAF $f^*(z) = e^{-\frac{1}{2}|z|^2} f(z)$. We will make use of the following almost independence theorem from [NS]:

Theorem 4.6.2. *Let F be a GAF. There exists numerical constant $A > 1$ with the following property. Given a family of compact sets K_j in $\mathbb{C}$ with diameters $d(K_j)$, let $\rho_j \geq \sqrt{\log(3 + d(K_j))}$. Suppose that $A\rho_j$-neighbourhoods of the sets K_j are pairwise disjoint. Then*

$$F^* = F_j^* + G_j^* \text{ on } K_j$$

where F_j are independent GAFs and for a positive numerical constant C we have

$$\mathbb{P}\left\{\max_{K_j} |G_j^*| \geq e^{-\rho_j^2}\right\} \leq C \exp[-e^{\rho_j^2}]$$

Define the function $h(x) = 2A\log(x/2)$, where A is as in Theorem 4.6.2.

Our construction will be parametrized by two parameters: $\theta > 0$ and $0 < \lambda < 1$. We will think of θ to be large enough and λ to be small enough; the exact conditions demanded of θ and λ will be described as we proceed along the construction. It turns out that the resulting choice of θ and λ can be made to be uniform in all the other variables in the construction (like the length L), and it suffices to take λ smaller than some universal constant and θ large enough, depending on λ. To begin with, we demand that θ be so large that $C\exp[-e^{(\log\theta)^2}] < 1$, for C as in Theorem 4.6.2, and

$$\sqrt{\log(3 + x\theta\sqrt{2})} < \theta \log x \text{ for all } x \geq 1. \tag{4.2}$$

A Cantor type construction: straight line

Let Γ_0 be the straight line segment $[0, L\theta]$ (or a horizontal translate thereof). An obvious analogue for this construction can be carried out for a similar line segment aligned along the vertical axis.

We start with Γ_0. Let Δ_0 be the line segment $\subset \Gamma_0$ of length $\theta h(L)$ such that $(\Delta_0)^c \cap \Gamma_0$ consists of two line segments $\mathcal{L}$ and $\mathcal{R}$ of equal length $(L\theta - \theta h(L))/2$, situated respectively to the left and right of Δ_0. Set $\Gamma_1^1 = \mathcal{L}$, $\Gamma_1^2 = \mathcal{R}$ and $\Gamma_1 = \Gamma_1^1 \cup \Gamma_1^2$. We proceed inductively as follows. For $N = \lceil \log \lambda L \rceil > j \geq 1$ where $0 < \lambda < 1$ is to be fixed later, consider $\Gamma_j^i, 1 \leq i \leq 2^j$. Let Δ_j^i be the segment $\subset \Gamma_j^i$ of length $\theta h(L/2^j)$ such that $\mathcal{L}_i, \mathcal{R}_i \subset \Gamma_j^i$ are segments of equal length, located to the left and right respectively of Δ_j^i. It will be clear from the arguments towards the end of this section that with N as above and λ chosen appropriately, we must have length(Δ_j^i)$= \theta h(L/2^j) <$ length(Γ_j^i) for each i and j; therefore the last step in the construction makes sense. Each of $\mathcal{L}_i, \mathcal{R}_i$ has length $\big(\text{ Length } (\Gamma_j^i) - \theta h(L/2^j)\big)/2$.

Doing this for each $1 \leq i \leq 2^j$ we have a collection of 2^j pairs of line segments $\mathcal{L}_i$ and $\mathcal{R}_i$, with a natural ordering among them along the horizontal axis from the left to the right. Following this order, we denote these segments $\Gamma_{j+1}^i, 1 \leq i \leq 2^{j+1}$. Finally, we define $\Gamma_{j+1} = \cup_{i=1}^{2^{j+1}} \Gamma_{j+1}^i$. We call $\Delta_j = \cup_i \Delta_j^i$ as the "removed" portion in the j-th round and Γ_{j+1} to be the "surviving" portion after the j-th round.

This completes our construction for a straight line segment Γ_0 of length $L\theta$. Before moving on, let us make some observations about the above construction. Recall that $N = \lceil \log \lambda L \rceil$, so in the end there are 2^N disjoint segments, and $\lambda L \leq 2^N \leq 2\lambda L$. The length of each Γ_j^i is clearly bounded above by $L\theta/2^j$, and the length of Δ_j is bounded above by $2^j \theta h(L/2^j)$ We upper bound the total length of the removed portion $\cup_{j=0}^{N-1} \Delta_j$ by

$$\sum_{j=0}^{N-1} \text{Length}(\Delta_j) = \sum_{j=1}^{N} 2^{j-1} \cdot 2A\theta \log(L/2^j) = 2A(2^N - 1)\theta \log L - 2A\theta(N-1)2^N - 2A\theta$$

$$= 2^{N+1} A\theta (\log L - N) - 2A\theta \log L + 2^{N+1} A\theta - 2A\theta \leq 4A \left(\lambda \log \frac{1}{\lambda}\right) L\theta + 4A\lambda L\theta$$

where in the last step we have used

$$\lambda L \leq 2^N \leq 2\lambda L. \tag{4.3}$$

By choosing λ small enough (less than some universal constant), we can ensure that the total length of the removed portion is $\leq \frac{1}{2} L\theta$. In particular, this means that for each i and j, we have length$(\Delta_j^i) <$ length(Γ_j^i), justifying that the construction leaves us with a non-trivial surviving portion at each step.

In Γ_N, each final surviving segment Γ_N^i is of equal length, and since the length of $\Gamma_N \geq \frac{1}{2}L\theta$, therefore each Γ_N^i is of length $\geq \frac{1}{2^{N+1}}L\theta \geq \frac{\theta}{4\lambda}$.

Notice that $A\rho_j$-neighbourhoods of Γ_j^i are disjoint where $\rho_j = \theta \log(L/2^j)$.

A Cantor type construction: square

We now describe an extension of the Cantor type construction in Section 4.6 which applies to a square B_0 of dimension $L\theta \times L\theta$. For each round, we will describe the connected components surviving at the end of that round.

We begin by noting that $B_0 = \Gamma_{0,1} \times \Gamma_{0,2}$ where $\Gamma_{0,i}$ are straight lines of length $L\theta$ along horizontal and vertical directions respectively. We perform the construction for a straight line segment on each of $\Gamma_{0,1}, \Gamma_{0,2}$ and let the surviving set at the end of round j be denoted by $\Gamma_{j,1}$ and $\Gamma_{j,2}$ respectively. Then the surviving set at the end of round j for B_0 is given by $B_j := \Gamma_{j,1} \times \Gamma_{j,2}$. Now B_j clearly contains 2^{2j} connected components (which are in fact squares of side length $\leq \theta L/2^j$); call these $B_j^i, 1 \leq i \leq 2^{2j}$, and number them in any order. By the arguments in Section 4.6, it is clear that the final set B_N has area at least $\frac{1}{4}\theta^2 L^2$, has 4^N connected components which are squares of side $\geq \theta/4\lambda$. For λ smaller than some absolute constant, each of these connected components contains at least one standard square of side θ. Moreover

$$\text{Euclidean Dist}(B_j^i, B_j^{i'}) \geq L^\infty\text{-Dist}(B_j^i, B_j^{i'}) \geq 2A\theta \log(L/2^j) \text{ for all } i \neq i', \text{ for each } j.$$

Then with $\rho_j = \theta \log(L/2^j)$, we have that the $A\rho_j$-neighbourhoods of the B_j^i-s are disjoint, and by choice of θ in equation (4.2) we have $\rho_j \geq \sqrt{\log(3 + \text{ Diam } (B_j^i))}$. In obtaining the last assertion, we use the fact that Diam $(B_j^i) \leq \theta L\sqrt{2}/2^j$, recall (4.2) and (4.3) and choose λ such that $\frac{1}{2\lambda} \geq 1$. The last condition is to ensure that for each $j \leq N$ we have $x \geq 1$ as in (4.2).

Functional decomposition in the Cantor construction

We can consider the sets B_j^i to be the vertices of a tree $\mathcal{T}$ of depth N where each vertex has 4 children (except at depth N). The children of the vertex B_j^i are the vertices $B_{j+1}^{i'}$ where $B_{j+1}^{i'}$ are obtained by applying the $j+1$-th level of the construction in Section 4.6 to B_j^i.

Corresponding to the tree $\mathcal{T}$, we can perform a decomposition of the normalized GAF f^* using Theorem 4.6.2. We start with f^*, which we also call f_0^*. We apply Theorem 4.6.2 to the compact sets $B_1^i, 1 \leq i \leq 4$ to obtain i.i.d. normalized GAF s $f_{1,i}^*$ and corresponding errors $g_{1,i}^*$. These are the functions corresponding to the first level of the tree. At the next level, we perform a similar decomposition on each $f_{1,i}^*$ to obtain $f_{2,j}^*$ and $g_{2,j}^*, 1 \leq j \leq 4^2$. So, on B_2^i we have $f^* = f_{2,i}^* + g_{2,i}^* + g_{1,i'}^*$ where $B_2^i \subset B_1^{i'}$ We continue this decomposition recursively until we reach level N in $\mathcal{T}$. At level N we have $f^* = f^*_{N,i} + G^*_i$ on $B^i_N, 1 \le i \le 2^N$ where the $f^*_{N,i}$ i.i.d. normalized GAFs. The G^*_i are the cumulative errors given by $G^*_i = \sum_{k=1}^N g^*_{k,n(k,i)}$ where $n(k,i)$ are such that $B^i_N \subset B^{n(k,i)}_k$. The $g^*_{j,i}$-s are not independent. However, for any two distinct vertices $B^i_j, B^{i'}_j$ at the same level j in $\mathcal{T}$, the errors corresponding to the descendants of B^i_j and those of $B^{i'}_j$ are independent. Thus at level j, the 4^j functions $g_{j,i}$ can be grouped into 4^{j-1} groups $\mathfrak{J}_{j,i''}, 1 \le i'' \le 4^{j-1}$ (each group consisting of 4 functions whose vertices have the same parent at level $j-1$ in $\mathcal{T}$). Thus, the index i'' in $\mathfrak{J}_{j,i''}$ can be thought to be varying over the vertices in $\mathcal{T}$ at level $j-1$. Clearly, $\mathfrak{J}_{j,i}$ and $\mathfrak{J}_{j,i'}$ are independent sets of functions for $i \neq i'$. We call $\mathfrak{J}_{j,i}$ to be "good" if each $g_{j,k} \in \mathfrak{J}_{j,i}$ satisfies $\{\max_{B^k_j} |g^*_{j,k}| \le e^{-\rho_j^2}\}$, otherwise we call it "bad". Recall from Theorem 4.6.2 (and a simple union bound) that $\mathbb{P}\{\mathfrak{J}_{j,i} \text{ is bad}\} \le C\exp[-e^{\rho_j^2}]$.

Set $p_j = C\exp[-e^{\rho_j^2}]$ as above. Denote by b_j the number of $\mathfrak{J}_{j,i}$ at level j which are not good. By a simple large deviation bound, we have, for any $0 < x_j < 1$,

$$\mathbb{P}(b_j > x_j \cdot 4^{j-1}) \le \exp\left(-4^{j-1} I_j\right) \tag{4.4}$$

where $I_j = x_j \ln \frac{x_j}{p_j} + (1-x_j)\ln\frac{1-x_j}{1-p_j}$ (for reference, see [DZ98] Theorem 2.1.10).

We set $x_j = 1/4^{N-j+1}$ whereas $p_j = C\exp\left(-e^{\rho_j^2}\right)$, and

$$\rho_j = \theta \log(L/2^j) = \rho_N + (N-j)\theta$$

Further, $\theta \log \frac{1}{2\lambda} \le \rho_N \le \theta \log\frac{1}{\lambda}$ (recall that $N = \lceil \log \lambda L \rceil$). Combining all these facts, we have

$$-x_j \ln p_j = \left[\exp\left(\theta^2\left(\frac{1}{\theta}\rho_N + (N-j)\right)^2\right) - \ln C\right] \Big/ 4^{N-j+1}$$

By choosing θ larger than and λ smaller than certain absolute constants, we can make the numerator of the above expression $\ge 2\theta 4^{2(N-j+1)}$ for all $N \ge 1$ and $1 \le j \le N$. Since $|x_j| \le 1/4$ for each $1 \le j \le N$, we have $|x_j \ln x_j| \le \frac{1}{4}\ln 4$. Also, for θ bigger than and λ smaller than some absolute constants, we have $c_1 \le \left|(1-x_j)\ln\frac{1-x_j}{1-p_j}\right| \le c_2 \forall j \le N$ where c_1 and c_2 are two positive constants. The upshot of all this is that by choosing θ larger than a constant we can make $I_j \ge \theta 4^{N-j+1}$ for all j, where we recall that $I_j = x_j \ln\frac{x_j}{p_j} + (1-x_j)\ln\frac{1-x_j}{1-p_j}$.

Hence we have

$$\mathbb{P}(b_j > x_j \cdot 4^{j-1}) \le \exp\left(-\theta 4^{j-1} 4^{N-j+1}\right) = \exp\left(-\theta 4^N\right) \le \exp\left(-\theta\lambda^2 L^2\right)$$

We denote by Ω the event $\{b_j \ge x_j \cdot 4^j \text{ for some } j \le N\}$. By a union bound over $1 \le j \le N$, we have $\mathbb{P}(\Omega) \le N\exp(-\theta\lambda^2L^2) \le \exp(-c_2(\theta)L^2)$ when θ is large enough, depending on λ. Here we recall again that $N = \lceil \log \lambda L \rceil$.

We call G^i_N to be "good" if each summand $g^*_{k,n(k,i)}$ in $G^*_i = \sum_{k=1}^N g^*_{k,n(k,i)}$ belongs to good $\mathfrak{J}$-s. Now, each bad $\mathfrak{J}$ at level j gives rise to 4^{N-j+1} bad G^i_N-s at level N. Outside the event

Ω, there are at most $x_j 4^{j-1}$ bad $\mathfrak{J}$-s at level j, leading to $x_j 4^N$ bad G_i^*-s. But $\sum_{j=1}^N x_j < 1/2$, hence except with probability $\leq \exp\left(-c_2(\theta)L^2\right)$, we have $\geq \frac{1}{2}4^N \geq \frac{1}{2}\lambda^2 L^2$ good G_i^*-s. For any good G_i^*, we have, for θ larger than and λ smaller than absolute constants,

$$\sup_{B_N^i}|G_i^*| \leq \sum_{k=1}^N \sup_{B_k^{n(k,i)}}|g_{k,n(k,i)}^*| \leq \sum_{k=1}^N e^{-\rho_k^2} \leq 2e^{-\rho_N^2} \leq e^{-5\theta^2}.$$

Let the final set of surviving connected square segments be denoted as $\Upsilon(B)$.

A variant of the Cantor type construction for a square

Here we will discuss a variant of the construction for the $L\theta \times L\theta$ square, which is as follows. We begin with a square $B'(L)$, each side of length $2L\theta + 2\lceil A \log L \rceil \theta$. In the first step, we remove the horizontal and vertical strips corresponding to the central segments of length $2\lceil A \log L \rceil \theta$ on each side of the square, with A as in Theorem 4.6.2. This leaves us with four squares of side length $L\theta$ each. We can parametrize the four $L\theta \times L\theta$ squares as $B_{00}, B_{10}, B_{01}, B_{11}$, where an increase the first subscript denotes an increase in the horizontal co-ordinate of the centre of the square and an increase in the second subscript denotes the same for the vertical co-ordinate of the centre. We now repeat the construction for an $L\theta \times L\theta$ square for each of $B_{ij}, 0 \leq i, j \leq 1$. Let $\Upsilon(B_{ij})$ denote the final surviving set for the construction on B_{ij}. The final surviving set for $B'(L)$ is denoted by $\Upsilon = \cup_{i,j=0}^1 \Upsilon(B_{ij})$. The normalized GAF f^* on $B'(L)$ has an analogous decomposition on similar lines as the $L\theta \times L\theta$ square case. Further, with probability $\leq C \exp(-e^{(\theta \log L)^2})$, all the errors g^* due to the first step of the construction (from $B'(L)$ to B_{ij}) are $\leq e^{-(\theta \log L)^2}$.

Hence, on similar lines to Section 4.6, we can conclude that for λ smaller than an absolute constant and θ large enough (depending on λ), in each configuration $\Upsilon(B_{ij})$ at least $\frac{1}{2}4^N \geq \frac{1}{2}\lambda^2 L^2$ surviving components are good, except on an event Ω with $\mathbb{P}(\Omega) \leq \exp\left(-c(\theta)L^2\right)$. On each good component in the final surviving set, the accumulated error G^* satisfies $\max|G^*| \leq e^{-5\theta^2}$.

Proof of Theorem 4.6.1

Suppose we have a connected set Γ of standard squares of base length θ and consisting of L standard squares. Then there is a square B of side length $L\theta$, consisting of L^2 standard squares of base length θ, such that $\Gamma \subset B$. As in Section 4.6, we form a square $B'(L)$ of side length $2L\theta + 2\lceil A \log L \rceil \theta$, consisting of standard squares, such that B sits inside $B'(L)$ as the square B_{00}. Let the final Cantor set on $B'(L)$. after applying the construction of Section 4.6, be $\Upsilon = \cup_{i,j=0}^1 \Upsilon(B_{ij})$. Recall that the connected components of Υ are squares of side length $\geq \theta/4\lambda$, and each such component contains at least one standard square of side θ. Moreover, any two of the $\Upsilon(B_{ij})$s are isometrically isomorphic with each other under an appropriate horizontal and/or vertical translation by $(L + 2\lceil A \log L \rceil)\theta$. We select one standard square from each connected component in $\Upsilon(B_{00})$, and in the other $\Upsilon(B_{ij})$s we select those standard squares which correspond under the above isomorphisms to the ones chosen in $\Upsilon(B_{00})$. The resulting union of standard squares for each B_{ij} will be denoted by $\Upsilon'(B_{ij})$ and we define $\Upsilon' = \cup\Upsilon'(B_{ij})$. So, given any standard square σ in $\Upsilon'(B_{00})$, there are four isomorphic copies corresponding to σ in Υ' under the translations mentioned above, with one copy in each $\Upsilon'(B_{ij})$.

Denote by $\Upsilon' + (m, n)$ the translate in $\mathbb{R}^2$ of the set Υ' by the vector $(m\theta, n\theta)$. Let $\mathcal{I}$ be the set of such translates of Υ' by (m, n) in the range $0 \leq m, n < L + 2\lceil A \log L \rceil$. Given any standard square σ in $\Upsilon'(B_{00})$, under horizontal or vertical translations by $L+2\lceil A \log L\rceil$ there are four isomorphic copies corresponding to σ in Υ' (one in each $\Upsilon'(B_{ij})$) and it is easy to check under the action of $\mathcal{I}$, any given square in $B'(L)$ (and hence any given square in Γ) is covered by exactly one of these. The total number of such σ (identifying copies isomorphic in the above sense) is $4^N \geq \lambda^2 L^2$, while $|\mathcal{I}| = (L+2\lceil A \log L\rceil)^2$. Hence, choosing a translate from $\mathcal{I}$ uniformly at random, the probability that a particular standard square in Γ is covered $\geq \lambda^2 L^2/(L+2\lceil A \log L\rceil)^2 \geq \kappa = \kappa(\lambda) > 0$. Hence the expected fraction of Γ covered by a random translate of Υ' is also $\geq \kappa$. This implies that there exists a translate $T(\Upsilon')$ of Υ' (chosen from $\mathcal{I}$) such that it covers at least a fraction κ of the standard squares in Γ. The same translation gives rise to a translate $T(\Upsilon)$ of Υ.

We call a constituent standard square of $T(\Upsilon')$ to be "good" if the corresponding G_k^* in $T(\Upsilon(B_{ij}))$ (from the Cantor type construction applied to the square $T(B'(L))$) is good as defined in Section 4.6. But in each $T(\Upsilon(B_{ij}))$, we have that except on an event Ω such that $\mathbb{P}(\Omega) < e^{-c(\theta)L^2}$, we have at least $\frac{1}{2}$ of the G_k^*-s to be good. Let $\Gamma(i), 1 \leq i \leq L$ denote the standard squares in Γ. Call a standard square to be "empty" or "full" according as it contains respectively 0 or $\geq k$ points in $\mathcal{F}$. Call Γ "empty" or "full" if all standard squares in Γ are empty or full. In what follows, we treat the state "empty", but in all steps it can be replaced by the state "full".

We observe that

$$\{\Gamma \text{ is empty }\} \subset \Omega \cup \{ \text{ Some subset of } \lfloor \kappa L \rfloor \text{ standard squares in } \Gamma,$$

$$\text{namely those in } T(\Upsilon') \cap \Gamma, \text{ are good and empty }\}$$

We have, via a union bound,

$$\mathbb{P}(\Gamma \text{ is empty}) \leq \mathbb{P}(\Omega) + \sum_{\mathcal{S}} \mathbb{P}\left(\bigcap_{\{\Gamma_{i_k}\} \in \mathcal{S}} \{\Gamma_{i_k} \text{ is empty and good }\} \right)$$

where the last summation is over $\mathcal{S}$ which is the collection of all possible subsets $\{\Gamma_{i_k}\}$ of $\lfloor \kappa L \rfloor$ standard squares in Γ such that $\Gamma_{i_k} \in T(\Upsilon')$ for all k. Since there are at most 2^L subsets of standard squares in Γ, it suffices to show that for any fixed $\{\Gamma_{i_k}\} \in \mathcal{S}$, we have for large enough θ

$$\mathbb{P}\left(\bigcap_{\{\Gamma_{i_k}\} \in \mathcal{S}} \{\Gamma_{i_k} \text{ is empty and good }\} \right) \leq \exp\left(-c_1(\theta) L\right) \tag{4.5}$$

$$\mathbb{P}\left(\bigcap_{\{\Gamma_{i_k}\}\in\mathcal{S}}\{\Gamma_{i_k} \text{ is full and good }\}\right) \leq \exp\left(-c_2(\theta,k)L\right) \tag{4.6}$$

where $c_1(\theta)\to\infty$ as $\theta\to\infty$ and $\lim_{\theta\to\infty}\lim_{k\to\infty} c_2(\theta,k)=\infty$.

Let A_{i_k} denote the event that $\{\Gamma_{i_k}$ is empty and good $\}$. Recall that Γ_{i_k} being empty implies that $f^*|_{\Gamma_{i_k}}$ does not have any zeros, and Γ_{i_k} being good implies that $\max_{\Gamma_{i_k}}|G^*_{i_k}|\leq e^{-5\theta^2}$, where $G^*_{i_k}$ are the cumulative errors in the cantor set construction, as estimated in Section 4.6.

Define A'_{i_k} to be the event that $f^*_{i_k}|_{\Gamma_{i_k}}$ does not have any zeros. Here $f^*_{i_k}$ are the final independent normalized GAFs obtained in the Cantor set construction. Clearly, the events A'_{i_k} are independent.

We will show that $A_{i_k}\subset A'_{i_k}\cup\Omega_{i_k}$, where the Ω_{i_k}-s are independent events with $\mathbb{P}(\Omega_{i_k})< e^{-c\theta}$. To this end, we note that on Γ_{i_k}, we have $f^*=f^*_{i_k}+G^*_{i_k}$, and also $\max_{\Gamma_{i_k}}|G^*_{i_k}|\leq e^{-5\theta^2}$. Applying Corollary 4.6.4 to the square Γ_{i_k}, we deduce that except for a bad event Ω_{i_k} of probability $\leq e^{-c\theta}$, we have $|f^*_{i_k}|>e^{-5\theta^2}$ on $\partial\Gamma_{i_k}$. Hence the equation $f^*=f^*_{i_k}+G^*_{i_k}$ on Γ_{i_k} along with Rouche's theorem implies that f^* and $f^*_{i_k}$ have the same number of zeros in Γ_{i_k}. So, on $A_{i_k}\cap\Omega^c_{i_k}$ we have that A'_{i_k} holds, in other words $A_{i_k}\subset A'_{i_k}\cup\Omega_{i_k}$, as desired. The Ω_{i_k}-s are independent since Ω_{i_k} is defined in terms of $f^*_{i_k}$ which are independent normalized GAFs.

Therefore we can write, for a fixed $\{\Gamma_{i_k}\}\in\mathcal{S}$

$$\mathbb{P}\left(\bigcap_k A_{i_k}\right)\leq\mathbb{P}\left(\bigcap_k\left(A'_{i_k}\cup\Omega_{i_k}\right)\right)=\prod_k\mathbb{P}\left(A'_{i_k}\cup\Omega_{i_k}\right)$$

But it is not hard to see that for the state "empty" we have $\mathbb{P}\left(A'_{i_k}\cup\Omega_{i_k}\right)\leq\mathbb{P}\left(A'_{i_k}\right)+\mathbb{P}\left(\Omega_{i_k}\right)\leq e^{-c(\theta)}$ where $c(\theta)\to\infty$ as $\theta\to\infty$. It is also easy to see that if we consider the state "full" instead of "empty" $\mathbb{P}(A'_{i_k})\leq e^{-c(\theta,k)}$ where $c(\theta,k)\to\infty$ as $k\to\infty$ for fixed θ, and $\mathbb{P}(\Omega_{i_k})\leq e^{-c\theta}$. Therefore we have $\mathbb{P}\left(A'_{i_k}\cup\Omega_{i_k}\right)\leq\exp\left(-c_2(\theta,k)\right)$ where $\lim_{\theta\to\infty}\lim_{k\to\infty}c_2(\theta,k)=\infty$. This proves equations (4.5) and (4.6) and hence completes the proof of the theorem.

Lower bound on the size of f^*

Our goal in this section is to establish that with large probability, the size of a normalized GAF on the perimeter of a circle (or a square) cannot be too small. Of course, there is a trade-off between the "largeness" of the probability and "smallness" of the GAF, depending on the radius of the circle or the side length of the square. Such estimates, along with Rouche's theorem, would be useful in replacing $f^*|_{\Gamma_{i_k}}$ with the independent $f^*_{i_k}$ on "good" Γ_{i_k}-s in Section 4.6.

Proposition 4.6.3. *Let us consider a disk $\mathcal{D}$ of radius $R > 1$, and let $\nu > 2$. Then*

$$\mathbb{P}\left(|f^*(z)| \le e^{-\nu R^2} \text{ for some } z \in \partial\mathcal{D}\right) \le e^{-C(\nu)R} \tag{4.7}$$

for a $C(\nu) > 0$. Here $f^*(z) = e^{-\frac{1}{2}|z|^2} f(z)$ where f is the standard planar GAF.

Proof. First, we show that we can take $\mathcal{D}$ to be centred at the origin. Let w be the centre of $\mathcal{D}$ and let $\mathcal{D}'$ be the disk of the same radius as $\mathcal{D}$ centred at the origin. Then the random field $|f^*(z)|$ for $z \in \mathcal{D}$ can be re-parametrized as the random field $\exp\left(-\frac{1}{2}|z+w|^2\right)|f(z+w)|$ for $z \in \mathcal{D}'$. The latter can be written as $e^{-\frac{1}{2}|z|^2}\exp\left(-\Re(z\overline{w}) - \frac{1}{2}|w|^2\right)|f(z+w)|$. But it is clear from a simple covariance computation that for any fixed w the random fields $f(z)$ and $f_w(z) = \exp\left(-z\overline{w} - \frac{1}{2}|w|^2\right) f(z+w)$ have the same distribution, and so do $|f(z)|$ and $|f_w(z)| = \exp\left(-\Re(z\overline{w}) - \frac{1}{2}|w|^2\right)|f(z+w)|$. In other words, the random field $\{|f^*(z)|\}_{z\in\mathbb{C}}$ has a translation-invariant distribution. Hereafter we assume that $\mathcal{D}$ is centred at the origin.

We want to show that $|f^*(z)| \ge e^{-\nu R^2}$, or equivalently, $|f(z)| \ge e^{-(\nu-1/2)R^2}$ on $\partial\mathcal{D}$ except on an event with probability exponentially small in R. We choose $\eta = \lceil 2\pi R^3 e^{(\nu+2)R^2}\rceil$ equi-spaced points $\{z_j\}_{j=1}^{\eta}$ on $\partial\mathcal{D}$. Then

$$\mathbb{P}\left(|f^*(z_j)| \le 2e^{-\nu R^2}\right) \le ce^{-2\nu R^2} \text{ for all } j$$

since $f^*(z)$ is a standard complex Gaussian for each fixed z. Consider the event $\Omega_1 = |f^*(z_j)| > 2e^{-\nu R^2}$ for all $j \le N$. By a union bound over the $\{z_j\}$-s we have for $R > 1$

$$\mathbb{P}(\Omega_1^c) \le cR^3 e^{-(\nu-2)R^2} \le e^{-c(\nu)R^2}$$

for some $c(\nu) > 0$.

$f'(z)$ is a centred analytic Gaussian process on $\mathbb{C}$ with covariance $\mathrm{Cov}(f'(z), f'(w)) = z\overline{w}e^{z\overline{w}}$. Set $\sigma_R^2 = \max_{2\cdot\mathcal{D}} \mathrm{Var}\left(f'(z)\right) = 4R^2e^{4R^2}$. We use Lemma 2.4.4 from [HKPV10] and apply it to the Gaussian analytic function $f'(2Rz)$. Using this, we obtain

$$\mathbb{P}\left(\mathrm{Max}_{\mathcal{D}}|f'(z)| > t\right) \le 2\exp\left(-\frac{t^2}{8\sigma_R^2}\right)$$

Setting $t = R^{3/2}e^{2R^2}$ in the above we get

$$\mathbb{P}\left(\mathrm{Max}_{\mathcal{D}}|f'(z)| > R^{3/2}e^{2R^2}\right) \le 2e^{-\frac{1}{32}R}$$

Let Ω_2 denote the event $\{\mathrm{Max}_{\mathcal{D}}|f'(z)| \le R^{3/2}e^{2R^2}\}$. The distance between any two consecutive z_j-s is $\le 2\pi R/\eta \le R^{-2}e^{-(\nu+2)R^2}$. Hence, on Ω_2 we have, via the mean value theorem, $|f(z) - f(z_j)| < R^{-1/2}e^{-\nu R^2}$ for any point $z \in \partial\mathcal{D}$ where z_j is the nearest point to z among $\{z_j\}_{j=1}^{\eta}$. As a result, for $R > 1$ we have on $\Omega_1 \cap \Omega_2$

$$|f(z)| \ge |f(z_j)| - |f(z) - f(z_j)| \ge 2e^{-(\nu-1/2)R^2} - R^{-1/2}e^{-\nu R^2} \ge e^{-(\nu-1/2)R^2}$$

For $R > 1$ we have $\mathbb{P}(\Omega_1 \cap \Omega_2) \ge 1 - e^{-C(\nu)R}$ for some constant $C(\nu) > 0$, as desired. ∎

Corollary 4.6.4. *Let us consider a square $\mathcal{T}$ of side length $S > 1$, and let $\nu > 1$. Then*

$$\mathbb{P}\left(|f^*(z)| \leq e^{-\nu S^2} \text{ for some } z \in \partial\mathcal{T}\right) \leq e^{-C(\nu)S} \tag{4.8}$$

for some constant $C(\nu) > 0$. Here $f^*(z) = e^{-\frac{1}{2}|z|^2} f(z)$ where f is the standard planar GAF.

Proof. We can follow the proof of Proposition 4.6.3. The maximum of $|f'|$ on $\mathcal{T}$ is bounded above by the maximum of $|f'|$ on the circumscribing circle of $\mathcal{T}$, whose radius is $S/\sqrt{2}$ and for which we can apply Lemma 2.4.4 from [HKPV10]. ∎

4.7 Proof of Theorem 4.1.2: existence of critical radius

We simply observe that Theorem 4.6.1 proves that the criteria outlined in Proposition 4.4.3 are valid for $\mathcal{F}$, thereby establishing that a critical radius exists for $\mathcal{F}$.

4.8 Uniqueness of infinite cluster

In this section we will prove that in the supercritical regime for the Boolean percolation models $(\mathcal{G}, r)$ and $(\mathcal{F}, r)$, a.s. there is exactly one infinite cluster.

4.9 An approach to uniqueness

We will first describe a proposition which has important implications regarding such uniqueness for a translation invariant point process Π.

Proposition 4.9.1. *Let $r > r_c$ for the Boolean percolation model $X(\Pi, r)$, where Π is a translation invariant point process on $\mathbb{R}^2$, and $0 < r_c < \infty$ is the critical radius. For $R > 0$ let B_R denote the set $\{x \in \mathbb{R}^2 : \|x\|_\infty \leq R\}$. Then the following event has probability 0:*

$$E(R) = \Big\{ \text{There is an infinite cluster } C' \text{ with the property that } C' \cap (B_R)^c \text{ contains}$$

$$\text{at least three infinite clusters and such that there is at least one point from } \Pi \text{ in } C' \cap B_R \Big\}$$

The proof of the above proposition can be found in [MR96], proof of Theorem 3.6. The event $E(R)$ from Proposition 4.9.1 corresponds to the event $E^0(N)$ there.

A general approach to a proof that a.s. there cannot be inifnitely many infinite clusters is to show that such an event would imply $E(R)$ would occur for some R.

4.10 Uniqueness of infinite clusters: Ginibre ensemble

In this section we prove that in $X(\mathcal{G}, r)$ with $r > r_c$, we have $\Lambda(r) = 1$ a.s.

To this end, we would need to have an understanding of the conditional distribution of the points of $\mathcal{G}$ inside a domain given the points outside. This has been obtained in Chapter 2 Theorems 1.1 and 1.2. We state these results below.

Let $\mathcal{D}$ be a bounded open set in $\mathbb{C}$ whose boundary has zero Lebesgue measure, and let $\mathcal{S}_{\text{in}}$ and $\mathcal{S}_{\text{out}}$ denote the Polish spaces of locally finite point configurations on $\mathcal{D}$ and $\mathcal{D}^c$ respectively. $\mathcal{G}_{\text{in}}$ and $\mathcal{G}_{\text{out}}$ respectively denote the point processes obtained by restricting $\mathcal{G}$ to $\mathcal{D}$ and $\mathcal{D}^c$.

Theorem 4.10.1. *For the Ginibre ensemble, there is a measurable function $N : \mathcal{S}_{\text{out}} \to \mathbb{N} \cup \{0\}$ such that a.s.*

$$\textit{Number of points in } \mathcal{G}_{\text{in}} = N(\mathcal{G}_{\text{out}}) .$$

Let the points of $\mathcal{G}_{\text{in}}$, taken in uniform random order, be denoted by the vector $\underline{\zeta}$. Let $\rho(\Upsilon_{\text{out}}, \cdot)$ denote the conditional measure of $\underline{\zeta}$ given $\mathcal{G}_{\text{out}} = \Upsilon_{\text{out}}$. Since a.s. the length of $\underline{\zeta}$ equals $N(\mathcal{G}_{\text{out}})$, we can as well assume that each measure $\rho(\Upsilon_{\text{out}}, \cdot)$ is supported on $\mathcal{D}^{N(\Upsilon_{\text{out}})}$.

Theorem 4.10.2. *For the Ginibre ensemble, $\mathbb{P}[\mathcal{G}_{\text{out}}]$-a.s. $\rho(\mathcal{G}_{\text{out}}, \cdot)$ and the Lebesgue measure $\mathcal{L}$ on $\mathcal{D}^{N(\mathcal{G}_{\text{out}})}$ are mutually absolutely continuous.*

We are now ready to prove Theorem 4.1.1.

Proof of Theorem 4.1.1. Let r be such that $\Lambda(r) > 0$ a.s. In what follows, we will repeatedly use the fact that if there are two points $x, y \in \mathbb{R}^2$ at Euclidean distance d, then there can be connected to each other by $(1 + \lceil d/2r \rceil)$ overlapping open disks of radius r, such that the centres of no two disks are exactly at a distance $2r$.

We will first deal with the case where a.s. $\Lambda(r) > 1$ but finite. A similar argument will show that if $3 \le \Lambda(r) \le \infty$ then the event $E(R)$ as in Proposition 4.9.1 occurs, with a suitable choice of R. This would rule out the possibility $\Lambda(r) = \infty$, and complete the proof.

We argue by contradiction, and let if possible $1 < \Lambda(r) < \infty$ a.s. Let $\mathcal{D}_1 \subset \mathcal{D}_2$ be two concentric open disks centred at the origin and respectively having radii $R_1 < R_2$. Recall the definition of the underlying graph $\mathfrak{g}$ from Section 4.3. Let $\boldsymbol{E}$ be the event that:

- (i) There are two infinite clusters $\mathcal{C}_1$ and $\mathcal{C}_2$ in the underlying graph $\mathfrak{g}$ such that $\mathcal{C}_1 \cap \mathcal{D}_1 \neq \emptyset \neq \mathcal{C}_2 \cap \mathcal{D}_1$ (in the sense that there is at least one vertex from each $\mathcal{C}_i$ in $\mathcal{D}_1$) .
- (ii) There exists a finite cluster $\mathcal{C}_3$ of vertices of $\mathfrak{g}$ which has $\ge 1 + \lceil 2R_1/r \rceil$ vertices such that $\mathcal{C}_3 \subset \operatorname{Int}(\mathcal{D}_2 \setminus \mathcal{D}_1)$, where $\operatorname{Int}(A)$ is the interior of the set A.

It is not hard to see that the event $\boldsymbol{E}$ depends on the parameters R_1 and R_2, and when R_1 and R_2 are large enough, we have $\mathbb{P}(\boldsymbol{E}) > 0$. Fix such disks $\mathcal{D}_1$ and $\mathcal{D}_2$. We denote the configuration of points outside $\mathcal{D}_2$ by ω and those inside $\mathcal{D}_2$ by ζ. Let the number of points in $\mathcal{D}_2$ be denoted by $N(\omega)$. Any two points of Π inside $\mathcal{D}_1$ are at most at a Euclidean distance of $2R_1$, and hence can be connected by at most $(1 + \lceil R_1/r \rceil)$ open disks of radius r such that the centres of no two disks are exactly at a distance $2r$. We define an event $\boldsymbol{E}'$ as follows: corresponding to every configuration (ζ, ω) in $\boldsymbol{E}$, we define a new configuration (ζ', ω) where ζ' is obtained by moving $(1 + \lceil R_1/r \rceil)$ points of $\mathcal{C}_3$ to the interior of $\mathcal{D}_1$ and placing them such that in the new underlying graph $\mathfrak{g}'$ (for definition see Section 4.3) the clusters $\mathcal{C}_1$ and $\mathcal{C}_2$ become connected with each other.

Similar to the observations made in Section 4.3, we can move each point in ζ' in a sufficiently small disk around itself, resulting in new configurations (ζ'', ω) such that the connectivity properties of $\mathfrak{g}'$ as well as the number of points in $\mathcal{D}_2$ remain unaltered. The event $\boldsymbol{E}'$ consists of all such configurations (ζ'', ω) as (ζ, ω) varies over all configurations in $\boldsymbol{E}$. Observe that for each ω, the set of configurations $\{\zeta'' : (\zeta'', \omega) \in \boldsymbol{E}'\}$ constitutes an open subset of $\mathcal{D}^{N(\omega)}$, when considered as a vector in the usual way. Since $\mathbb{P}(\boldsymbol{E}) > 0$, by Theorem 4.10.2 applied to the domain $\mathcal{D}_2$, we also have $\mathbb{P}(\boldsymbol{E}') > 0$. But on $\boldsymbol{E}'$, there is one less infinite cluster than on $\boldsymbol{E}$. This gives us the desired contradiction, and proves that $\mathbb{P}(1 < \Lambda(r) < \infty) = 0$.

Had it been the case $\Lambda(r) \geq 3$ a.s., observe that an argument analogous to the previous paragraph can be carried through with three instead of two infinite clusters ($\mathcal{C}_1$ and $\mathcal{C}_2$ above). The end result would be that with positive probability we can connect all the three clusters with each other. If $\Lambda(r) = \infty$ a.s. then we carry out the above argument with three of the infinite clusters, and observe that the event $E(R)$ as in Proposition 4.9.1 occurs with a set B_R where $R > R_2$, on the modified event analogous to $\boldsymbol{E}'$ above. This proves that $\mathbb{P}(\Lambda(r) = \infty) = 0$. ■

4.11 Uniqueness of infinite clusters: Gaussian zeroes

In this section we prove that in $X(\mathcal{F}, r)$ with $r > r_c$, we have $\Lambda(r) = 1$ a.s.

To this end, we would need to have an understanding of the conditional distribution of the points of $\mathcal{F}$ inside a domain given the points outside. This has been obtained in Chapter 2 Theorems 1.3 and 1.4. $\mathcal{F}_{\text{in}}$ and $\mathcal{F}_{\text{out}}$ respectively denote the point processes obtained by restricting $\mathcal{F}$ to $\mathcal{D}$ and $\mathcal{D}^c$ respectively. We state these results below. Some of the notation is from Section 4.10.

Theorem 4.11.1. *For the GAF zero ensemble,*
(i) There is a measurable function $N : \mathcal{S}_{\text{out}} \to \mathbb{N} \cup \{0\}$ such that a.s.

$$\textit{Number of points in } \mathcal{F}_{\text{in}} = N(\mathcal{F}_{\text{out}}).$$

(ii) There is a measurable function $S : \mathcal{S}_{\text{out}} \to \mathbb{C}$ such that a.s.

$$\textit{Sum of the points in } \mathcal{F}_{\text{in}} = S(\mathcal{F}_{\text{out}}).$$

Define the set

$$\Sigma_{S(\mathcal{F}_{\text{out}})} := \{\underline{\zeta} \in \mathcal{D}^{N(\mathcal{F}_{\text{out}})} : \sum_{j=1}^{N(\mathcal{F}_{\text{out}})} \zeta_j = S(\mathcal{F}_{\text{out}})\}$$

where $\underline{\zeta} = (\zeta_1, \cdots, \zeta_{N(\mathcal{F}_{\text{out}})})$.

Since a.s. the length of $\underline{\zeta}$ equals $N(\mathcal{F}_{\text{out}})$, we can as well assume that each measure $\rho(\Upsilon_{\text{out}}, \cdot)$ gives us the distribution of a random vector in $\mathcal{D}^{N(\Upsilon_{\text{out}})}$ supported on $\Sigma_{S(\Upsilon_{\text{out}})}$.

Theorem 4.11.2. *For the GAF zero ensemble, $\mathbb{P}[\mathcal{F}_{\text{out}}]$-a.s. $\rho(\mathcal{F}_{\text{out}}, \cdot)$ and the Lebesgue measure $\mathcal{L}_{\Sigma}$ on $\Sigma_{S(\mathcal{F}_{\text{out}})}$ are mutually absolutely continuous.*

We are now ready to prove Theorem 4.1.2.

Proof of Theorem 4.1.2 : uniqueness of infinite cluster . The proof follows the contour of Section 4.10, with extended arguments to take care of the fact that for $\mathcal{F}$ there are two conserved quantities for local perturbations of the zeros inside a disk : their number and their sum, unlike $\mathcal{G}$ where only the number of points is conserved.

We first show that it cannot be true that a.s. $1 < \Lambda(r) < \infty$. We argue by contradiction, and let if possible $1 < \Lambda(r) < \infty$ a.s. We will define events $\boldsymbol{E}$ and $\boldsymbol{E}'$ in analogy to Proposition 4.10 such that on $\boldsymbol{E}'$ there one less infinite cluster than on $\boldsymbol{E}$ and $\mathbb{P}(\boldsymbol{E}) > 0$ and $\mathbb{P}(\boldsymbol{E}') > 0$.

Let $\mathcal{D}_1, \mathcal{D}_2$ and $\mathcal{D}_3$ be two concentric open disks centred at the origin and respectively having radii $R_1 < R_2 < R_3$.

Let $\boldsymbol{E}$ be the event that :

(i)$\mathcal{C}_1 \cap \mathcal{D}_1 \neq \emptyset \neq \mathcal{C}_2 \cap \mathcal{D}_1$ for two infinite clusters $\mathcal{C}_1$ and $\mathcal{C}_2$ (in the sense that there is at least one vertex from each $\mathcal{C}_i$ inside $\mathcal{D}_1$)

(ii)$\exists$ a cluster $\mathcal{C}_3$ of vertices of the underlying graph $\mathfrak{g}$ which has $\geq n = 1 + \lceil R_1/r \rceil$ vertices such that $\mathcal{C}_3 \subset$ Int $(\mathcal{D}_2 \setminus \mathcal{D}_1)$

(iii)$\exists$ a cluster $\mathcal{C}_4 \subset$ Int $(\mathcal{D}_3 \setminus \mathcal{D}_2)$ with $\geq n' = \lceil 2R_2 n \rceil$ vertices (where n is as in (ii) above) such that

$$\text{Euclidean dist(vertices in } \mathcal{C}_4, \text{vertices in } \mathfrak{g} \setminus \mathcal{C}_4) > 10$$

(recall that $\mathfrak{g}$ denotes the underlying graph, for definition see Section 4.3).

It is not hard to see that $\mathbb{P}(\boldsymbol{E}) > 0$ when $R_i, i = 1, 2, 3$ are large enough. Fix such disks $\mathcal{D}_i, i = 1, 2, 3$. We denote the configuration of points of $\mathcal{F}$ outside $\mathcal{D}_3$ by ω and those inside $\mathcal{D}_3$ by ζ. Let the number of points in $\mathcal{D}_3$ be denoted by $N(\omega)$ and let their sum be $S(\omega)$.

We start with a configuration (ζ, ω) in $\boldsymbol{E}$. With the vertices inside $\mathcal{D}_2$, we first perform the same operations as in the proof of Theorem 4.1.1 . However, in $\mathcal{F}$, unlike in $\mathcal{G}$, we need to further ensure that the sum of the points inside $\mathcal{D}_3$ remain unchanged at $S(\omega)$ in order to stay absolutely continuous. We note that due to the operations already performed on the points inside $\mathcal{D}_2$, the sum of the points inside $\mathcal{D}_3$ has changed by at most $2R_2 n$, since $\leq n$ points have been moved and each of them can move by at most $2R_2$ which is the diameter of $\mathcal{D}_2$. We observe that we can compensate for this by translating each point in $\mathcal{C}_4$ by an distance ≤ 1 in an appropriate direction. Due to the separation condition in (iii) in the definition of $\boldsymbol{E}$, this does not change the connectivity properties of any vertex in $\mathfrak{g} \setminus \mathcal{C}_4$.

By the observations made in Section 4.3, we can move each point in ζ' in a sufficiently small disk around itself, resulting in new configurations (ζ'', ω) such that the connectivity properties of $\mathfrak{g}'$ as well as the number of the points in $\mathcal{D}_3$ remain unaltered. The event $\boldsymbol{E}'$ consists of all such configurations (ζ'', ω) as (ζ, ω) varies over all configurations in $\boldsymbol{E}$. Observe that for each ω, the set of configurations $\{\zeta'' : (\zeta'', \omega) \in \boldsymbol{E}'\}$ constitutes an open subset of $\mathcal{D}_3^{N(\omega)}$, when considered as a vector in the usual way. Hence its intersection with $\Sigma_{S(\omega)}$ is an open subset of $\Sigma_{S(\omega)}$. Since $\mathbb{P}(\boldsymbol{E}) > 0$, by Theorem 4.11.2 applied to the domain $\mathcal{D}_3$, we also have $\mathbb{P}(\boldsymbol{E}') > 0$. But in $\boldsymbol{E}'$, there is one less infinite cluster than in $\boldsymbol{E}$. This gives us the desired contradiction, and proves that $\mathbb{P}(1 < \Lambda(r) < \infty) = 0$.

We take care of the case $\Lambda(r) = \infty$ as we did in the proof of Theorem 4.1.1. Had it been the case $\Lambda(r) \geq 3$ a.s., an argument analogous to the previous paragraph can be carried through with three instead of two infinite clusters ($\mathcal{C}_1$ and $\mathcal{C}_2$ above), with the end result that with positive probability we can connect all the three infinite clusters with each other. If $\Lambda(r) = \infty$ a.s. then we carry out the above argument with three of the infinite clusters, and observe that the event $E(R)$ in Proposition 4.9.1 occurs on the modified event (analogous to $\boldsymbol{E}'$ above) with a set B_R where $R > R_3$. This proves that $\mathbb{P}(\Lambda(r) = \infty) = 0$. ■

Chapter 5

Determinantal processes and completeness of random exponentials

5.1 Introduction

Any locally finite point set $\Lambda \subset \mathbb{R}$ gives us a set of functions

$$\mathcal{E}_\Lambda := \{e_\lambda : \lambda \in \Lambda\} \subset L^2[-\pi, \pi],$$

where $e_\lambda(x) = e^{i\lambda x}$, i being the imaginary unit. The following question is classical:

Question 1. *Does $\mathcal{E}_\Lambda$ span $L^2[-\pi, \pi]$?*

An equivalent terminology found in the literature to describe the fact that $\mathcal{E}_\Lambda$ spans $L^2[-\pi, \pi]$ is that $\mathcal{E}_\Lambda$ is complete in $L^2[-\pi, \pi]$. When Λ is deterministic, this a well studied problem in the literature. In the case where Λ is random, that is, Λ is a point process, much less is known. For any ergodic point process Λ, it can be easily checked that the event in question has a 0-1 law.

In this chapter we provide a complete answer to Question 1 in the case where Λ is the continuum sine kernel process (see Section 5.2 for a precise definition):

Theorem 5.1.1. *When Λ is a realisation of the continuum sine kernel process on $\mathbb{R}$, almost surely $\mathcal{E}_\Lambda$ spans $L^2[-\pi, \pi]$.*

Similar questions can be asked in higher dimensions as well. On $\mathbb{C}$, we consider the analogous question with Λ coming from the Ginibre ensemble (see Section 5.2 for a precise definition). An exponential function here is defined as $\mathcal{E}_\lambda(z) = e^{\overline{\lambda} z}$ and the natural space to study completeness is the Fock-Bargmann space. The latter space is the closure of the set of polynomials (in one complex variable) in $L^2(\gamma)$ where γ is the standard complex Gaussian measure on $\mathbb{C}$, having the density $\frac{1}{\pi}e^{-|z|^2}$ with respect to the Lebesgue measure. In this case we prove:

Theorem 5.1.2. *When Λ is a realisation of the Ginibre ensemble on $\mathbb{C}$, a.s. $\mathcal{E}_\Lambda$ spans the Fock-Bargmann space. Equivalently, a.s. in Λ the following happens: if there is a function f in the Fock-Bargmann space which vanishes at all the points of Λ, then $f \equiv 0$.*

All these questions are specific realisations of the following completeness question that was asked of any determinantal process by Peres and Lyons in 2009.

Consider a determinantal point process π in a space Ξ equipped with a background measure μ. Let π correspond to a projection onto the subspace $\mathcal{H}$ of $L^2(\mu)$ in the usual way, for details, see Section 5.2 and also [HKPV10], [Sos00] and [Ly03]. Let $K(\cdot,\cdot)$ be the kernel of the determinantal process, which is also the integral kernel corresponding to the projection onto $\mathcal{H}$. Consequently, $\mathcal{H}$ is a reproducing kernel Hilbert space, with the kernel $K(\cdot,\cdot)$. Let $\{x_i\}_{i=1}^{\infty}$ be a sample from π. Clearly, $\{K(x_i,\cdot)\}_{i=1}^{\infty} \subset \mathcal{H}$. In 2009, Lyons and Peres asked the following question:

Question 2. *Is the random set of functions $\{K(x_i,\cdot)\}_{i=1}^{n}$ complete in $\mathcal{H}$ a.s.?*

The answer to Question 2 is trivial in the case where $\mathcal{H}$ is finite dimensional (say the dimension is N), there it follows simply from the fact that the matrix $(K(x_i,x_j))_{i,j=1}^{N}$ is a.s. non-singular on one hand, and it is the Gram matrix of the vectors $\{K(x_i,\cdot)\} \subset \mathcal{H}$ on the other. In the case where Ξ is a countable space, this was first proved for spanning forests by Morris [Mor03]. Subsequently, this has been answered in the affirmative for any discrete determinantal process by Lyons, see [Ly03]. However, in the continuum (e.g. when $\Xi = \mathbb{R}^d$ and $\mathcal{H}$ is infinite dimensional), the answer to Question 2 is unknown.

In this chapter, we answer Question 2 in the affirmative for rigid determinantal processes. Let π be a translation invariant determinantal point process with determinantal kernel $K(\cdot,\cdot)$ on $\mathbb{R}^d$ with a background measure μ that is mutually absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$, such that $K(\cdot,\cdot)$ is also the projection kernel for the subspace $\mathcal{H}$ that is canonically associated with π. Let the map $x \to K(x,\cdot)$ be continuous as a map from $\mathbb{R}^d \to \mathcal{H}$.

Theorem 5.1.3. *Let Π be rigid, in the sense that for any ball B, the point configuration outside B a.s. determines the number of points N_B of π inside B. Then $\{K(x,\cdot) : x \in \Pi\}$ is a.s. complete in $\mathcal{H}$.*

To see the correspondence between Question 2 and Theorems 5.1.1 - 5.1.2, we can make appropriate substitutions for the kernels and spaces in Theorem 5.1.3, for details see Section 5.2. The statement of Theorem 5.1.1 is equivalent to Question 2 (with an affirmative answer) under Fourier conjugation. The statement in Theorem 5.1.2 involving the vanishing of functions on Λ is a result of the fact that the Fock-Bargmann space is a reproducing kernel Hilbert space, with the reproducing kernel and background measure being the same as the determinantal kernel and the background measure of the Ginibre ensemble.

Completeness (in the appropriate Hilbert space) of collections of exponential functions indexed by a point configuration is a well-studied theme, for a classic reference see the survey by Redheffer [Re77]. However, most of the classical results deal with deterministic point configurations, and are often stated in terms of some sort of density of the underlying point set. E.g., one crucial parameter is the Beurling Malliavin density of the point configuration, for details, see [Ly03] Definition 7.13 and the ensuing discussion there. Typically, the results are of the following form : if the relevant density parameter is supercritical, then the exponential system is complete, and if it is subcritical, it is incomplete. E.g., see Beurling and Malliavin's theorem, stated as Theorem 71 in [Re77].

However, it is not hard to check that the point configurations of our interest almost surely exhibit the critical density in terms of the classical results. The critical density, in most cases, turns out to be equal to the one-point intensity for negatively associated ergodic point processes (including the homogeneous Poisson process). In this chapter we have chosen the normalizations for our models such that the one-point intensity (and hence the critical density) is equal to 1. The critical cases in the deterministic setting are much harder to handle. E.g., in $L^2[-\pi,\pi]$, $\{e_\lambda : \lambda \in \mathbb{Z}\}$ is a complete set of exponentials, but $\{e_\lambda : \lambda \in \mathbb{Z} \setminus \{0\}\}$ is incomplete. The study of completeness problems for random point configurations is much more limited, let alone in the critical case. Nevertheless, when the densities are super or subcritical, we can either invoke the results in the deterministic setting (e.g., Theorem 71 in [Re77]), or there is existing literature (see citeCLP Theorems 1.1 and 1.2 or [SU97]). However, at critical densities, which is the case we are interested in, much less is known. To the best of our knowledge, the only known case is that of a perturbed lattice, where completeness was established under some regularity conditions on the (random) perturbations, see [CL97] Theorem 5. Our result in Theorem 5.1.3 answers this question for natural point processes, like the Ginibre or the sine kernel, which are not i.i.d. perturbations of $\mathbb{Z}$.

There are other natural examples of determinantal point processes for which similar questions are not amenable to our approach. E.g., one can consider Question 2 for the zero process of the hyperbolic Gaussian analytic function, where the answer, either way, is unknown, because this determinantal point process is not rigid (see [HS10]).

En route proving Theorem 5.1.3, we provide a quantitative expression for the negative association property for determinantal point processes in the continuum, which is relevant for our purposes. In the discrete setting, a complete theorem to this effect was has been proved in [Ly03]. However, we could not locate such a result in the literature on determinantal point processes in the continuum. In Theorem 5.1.4, we establish negative association for the number of points for determinantal point processes under minimal regularity hypotheses.

Definition 15. *We call two non-negative real valued random variables X and Y negatively associated if for any real numbers r and s we have*

$$\mathbb{P}\big((X > r) \cap (Y > s) \big) \leq \mathbb{P}\big(X > r\big)P\big(Y > s\big).$$

This is equivalent to the complementary condition

$$\mathbb{P}\big((X \leq r) \cap (Y \leq s) \big) \leq \mathbb{P}\big(X \leq r\big)\mathbb{P}\big(Y \leq s\big).$$

In the theory of determinantal point processes on $\mathbb{R}^d$, by a standard kernel we mean a Hermitian kernel $\mathbb{K}(x, y)$ which is continuous as a function from $\mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ and is a non-negative trace class contraction when viewed as an integral operator from $L^2(\mu) \to L^2(\mu)$, where μ is the background measure. For further details, we refer the interested reader to [HKPV10].

Theorem 5.1.4. *For any determinantal point process with a standard kernel on $\mathbb{R}^d$ and the background measure μ absolutely continuous with respect to the Lebesgue measure, the numbers of points in two disjoint Borel sets are negatively associated random variables, in the sense of Definition 15.*

In addition to the completeness questions for random exponentials defined with respect to natural determinantal processes, we also answer two questions asked by Lyons and Steif in [LySt03]. First, we give a little background on a certain class of stationary determinantal processes on $\mathbb{Z}^d$ studied in [LySt03].

Let f be a function $\mathbb{T}^d \to [0, 1]$. Then multiplication by f is a non-negative contraction operator from $L^2(\mathbb{T}^d)$ to itself. Under Fourier conjugation, this gives rise to a non-negative contraction operator $Q : \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$. This, in turn, gives rise to a determinantal point process $\mathbb{P}^f$ on $\mathbb{Z}^d$ in a canonical way, for details see [Ly03]. For a point configuration ω drawn from the distribution of $\mathbb{P}^f$, we denote by $\omega(\underline{k})$ the indicator function of having a point at $\underline{k} \in \mathbb{Z}^d$ in the configuration ω. We denote by ω_{out} the configuration of points on $\mathbb{Z}^d \setminus \mathbf{0}$ obtained by restricting ω to $\mathbb{Z}^d \setminus \mathbf{0}$, where $\mathbf{0}$ denotes the origin in $\mathbb{Z}^d$. For a point process π on a space Ξ, we denote by $[\pi]$ the (random) counting measure obtained from a realisation of π. The k-point intensity functions of a point process, when it exists, will be denoted by $\rho_k, k \geq 1$. For the processes $\mathbb{P}^f$, all intensity functions exist. Moreover, the translation invariance of $\mathbb{P}^f$ implies that $\rho_k(x_1 + x, \cdots, x_k + x) = \rho_k(x_1, \cdots, x_k)$ for all $x, x_1, \cdots, x_k \in \mathbb{Z}^d$, in particular, ρ_1 is a constant $\in [0, 1]$.

In the paper [LySt03] it was conjectured that all determinantal processes obtained in this way are insertion and deletion tolerant, meaning that both $\mathbb{P}[\omega(\mathbf{0}) = 1|\omega_{\text{out}}] > 0$ and $\mathbb{P}[\omega(\mathbf{0}) = 0|\omega_{\text{out}}] > 0$. We answer this question in the negative, showing that for f which is the indicator function of an interval, this is not true.

Theorem 5.1.5. *Let f be the indicator function of an interval $I \subset \mathbb{T}$. Then there exists a measurable function*

$$N : \textit{Point configurations on } \mathbb{Z} \setminus \mathbf{0} \to \mathbb{N} \cup \{0\}$$

such that a.s. we have $\omega(\mathbf{0}) = N(\omega_{\text{out}})$. Consequently, the events $\{\mathbb{P}[\omega(\mathbf{0}) = 1|\omega_{\text{out}}] = 0\}$ and $\{\mathbb{P}[\omega(\mathbf{0}) = 1|\omega_{\text{out}}] = 0\}$ both have positive probability (in ω_{out}).

We end by answering another question from [LySt03], where we demonstrate that "almost all" functions f can be reconstructed from the distribution $\mathbb{P}^f$.

Theorem 5.1.6. *Define $\mathcal{E}$ to be the set of functions*

$$\mathcal{E} := \{f \in L^\infty(\mathbb{T}) : 0 \le f(x) \le 1 \text{ for almost every } x \in \mathbb{T}\}.$$

Then $\mathbb{P}^f$ determines f up to translation and flip, except possibly for a meagre set of functions in the L^∞ topology on $\mathcal{E}$.

For any (as opposed to "almost all") function f, we prove that $\mathbb{P}^f$ determines the value distribution of f.

Proposition 5.1.7. *For any $f \in \mathcal{E}$, $\mathbb{P}^f$ determines the value distribution of f. This is true for $\mathbb{P}^f$ defined on $\mathbb{Z}^d$ for any $d \ge 1$.*

5.2 Definitions

In this section, we give precise descriptions of the models under study.

A determinantal point process on a space Ξ with background measure μ and kernel $\mathbb{K} : \Xi \times \Xi \to \mathbb{C}$, is a point process whose n-point intensity functions (with respect to the measure $\mu^{\otimes n}$) is given by

$$\rho_n(x_1, \cdots, x_n) = \det \left(\mathbb{K}(x_i, x_j)_{i,j=1}^n\right).$$

The kernel $\mathbb{K}$ induces an integral operator on $L^2(\mu)$, which must be locally trace class and, additionally, a positive contraction. An interesting class of examples is obtained when the integral operator given by $\mathbb{K}$ is a projection onto a subspace $\mathcal{H}$ of $L^2(\mu)$.

Our first example is the Ginibre ensemble, which is obtained as above with $K(z,w) = e^{\bar{z}w}, d\mu(z) = e^{-|z|^2} d\mathcal{L}(z)$ and $\mathcal{H}$ is the Fock-Bargmann space $\subset L^2(\mu)$ (here $\mathcal{L}$ is the Lebesgue measure on $\mathbb{C}$). For every n, we can consider an $n \times n$ matrix of i.i.d. complex Gaussian entries. The Ginibre ensemble arises as the weak limit (as $n \to \infty$) of the point process given by the eigenvalues of this matrix. The continuum sine kernel process is given by $K(x,y) = \frac{\sin \pi(x-y)}{\pi(x-y)}, \mu =$ the Lebesgue measure on $\mathbb{R}$, $\mathcal{H} =$ the Fourier conjugates of the set of L^2 functions supported on $[-\pi, \pi]$. The continuum sine kernel process arises as the bulk limit of the eigenvalues of the Gaussian Unitary Ensemble (GUE).

For greater details on these processes, see [HKPV10].

5.3 Completeness of random function spaces

In this Section, we prove Theorem 5.1.3. Due to the connections discussed in the introduction, this will automatically establish Theorems 5.1.1 and 5.1.2. On the way, we provide a proof of Theorem 5.1.4.

Before the main theorem, we establish a preparatory result.

Proposition 5.3.1. *Suppose* Π *is a rigid point process, in the sense that for any ball* B, *the point configuration outside* B *a.s. determines the number of points* N_B *of* Π *inside* B. *Assume that for any ball* B, $\mathbb{E}[N_B] < \infty$. *Let* $A(r)$ *denote the closed annulus of thickness* r *around* B, *and let* $\Pi|_A(r)$ *denote the point process in* $A(r)$ *obtained by restricting* Π *to* $A(r)$. *Then we have*

$$\mathbb{E}\left[N_B \middle| \Pi|_{A(r)}\right] \to N_B \tag{5.1}$$

a.s. as $r \to \infty$.

Proof. This follows from Levy's 0-1 law and the convergence of the Doob's martingales $M_r := \mathbb{E}\left[N_B|\Pi|_{A(r)}\right]$, $r \geq 0$ of N_B, as $r \to \infty$. Note that $M_\infty = N_B$ because Π is rigid. ■

Proof of Theorem 5.1.4. We will use an appropriate discretization argument to invoke the result for the discrete case (see Theorem 8.1, [Ly03]), and pass to the continuum limit.

Let A, B be disjoint Borel sets in $\mathbb{R}^d$ and r, s be two non-negative integers. We intend to prove

$$\mathbb{P}\left((N(A) \leq r) \cap (N(B) \leq s)\right) \leq \mathbb{P}\left(N(A) \leq r\right)\mathbb{P}\left(N(B) \leq s\right) \tag{5.2}$$

First of all, we can assume the sets A, B to be contained in a compact d-dimensional cube $\mathfrak{D}$, the general case can easily be deduced from this by considering $A \cap \mathfrak{D}$ and $B \cap \mathfrak{D}$ and letting $\mathfrak{D} \uparrow \mathbb{R}^d$; the probabilities in (5.2) are converge to the appropriate limits under this procedure.

On $L^2(\mu_{|\mathfrak{D}})$ we have $\mathbb{K}$ is again a standard kernel and $\mu_{|\mathfrak{D}}$ is clearly absolutely continuous with respect to Lebesgue measure restricted to $\mathfrak{D}$. Therefore, by Mercer's theorem we have an eigenvector expansion for the kernel

$$\mathbb{K}(x,y) = \sum_{\lambda_i \downarrow 0} \lambda_i \phi_i(x)\overline{\phi_i(y)} \tag{5.3}$$

where ϕ_i are the eigenvectors and λ_i the corresponding eigenvalues for $\mathbb{K}$ acting from $L^2(\mu_{|\mathfrak{D}})$ to itself. For brevity, from here on we will drop the subscript $\mathfrak{D}$ and pretend that μ is a measure supported on $\mathfrak{D}$.

Both sides of (5.2) are continuous in the kernel $\mathbb{K}$, in the sense that if $\mathbb{K}_n$ is a sequence of standard kernels converging to $\mathbb{K}$ as continuous functions (i.e., in the sup norm) on $\mathfrak{D} \times \mathfrak{D}$, then the corresponding probabilities in (5.2) converge to that of $\mathbb{K}$. This follows from the fact that the convergence of $\mathbb{K}_n$-s to $\mathbb{K}$ in the sup norm implies the convergence in distribution of the corresponding determinantal point processes. This enables us to replace $\mathbb{K}$ by finite truncations of the expansion (5.3). Since $\mathbb{K}$ is a standard kernel, we have $0 < \lambda_i \leq 1$. However, for technical reasons we will assume that $\lambda_i < 1$ for all i. The general case can be easily recovered by taking limits $\lambda_i \uparrow 1$ for the eigenvalue 1, this can again be justified by the regularity of (5.2) under $\mathbb{K}$.

We have thus reduced to the case when $\mathbb{K}(x,y) = \sum_{i=1}^{N} \lambda_i \phi_i(x)\overline{\phi_i(y)}$ where N is a positive integer and $0 < \lambda_i < 1$, without loss of generality say $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$. Moreover, the background measure lives on $\mathfrak{D}$.

For a positive integer m, we divide each side of $\mathfrak{D}$ into m equal parts, and index the resulting sub-cubes by $\mathfrak{D}_i; i = 1, 2, \cdots, n = m^d$. We define a measure μ_n on $[n]$ by setting $\mu_n(i) = \mu(\mathfrak{D}_i)$. If the centre of $\mathfrak{D}_i$ is x_i, then we define a function $\mathbb{K}_n : [n] \times [n] \to \mathbb{C}$ by $\mathbb{K}_n(i,j) = \mathbb{K}(x_i, x_j)\sqrt{\mu_n(i)\mu_n(j)}$.

We claim that for n large enough, $\mathbb{K}_n$ as defined above is a non-negative contraction on ℓ_n^2 (where ℓ_n^2 is the space of n-tuples of complex numbers equipped with the standard L_2 metric $\|\cdot\|_2$). To see this, consider the quadratic form $Q(a_1, \cdots, a_n) = \sum_{i,j=1}^n a_i \overline{a_j} \mathbb{K}_n(i,j)$. Clearly, the matrix $\mathbb{K}_n$ is Hermitian, and we intend to show that $|Q(a_1, \cdots, a_n)| \le 1$ when $\|(a_1, \cdots, a_n)\|_2 \le 1$. Observe that the a_i for which $\mu_n(i) = 0$ do not affect Q, so we can scale a_i by $\sqrt{\mu_n(i)}$ and consider the form $Q'(a_1', \cdots, a_n') = \sum_{i,j=1}^n a_i' \overline{a_j'} \mathbb{K}_n(i,j)\sqrt{\mu_n(i)\mu_n(j)}$. We want to maximise $|Q'(a_1', \cdots, a_n')|$ over $\sum_{i=1}^n |a_i'|^2 \mu_n(i) \le 1$. We compare $Q'(a_1', \cdots, a_n')$ with the integral $I(a') = \int\int_{\mathfrak{D}\times\mathfrak{D}} a'(x)\mathbb{K}(x,y)\overline{a'(y)}d\mu(x)d\mu(y)$, where $a'(x) = \sum_{i=1}^n a_i' \chi_{\mathfrak{D}_i}(x)$. Here $\chi_{\mathfrak{D}_i}$ is, as usual, the indicator function of $\mathfrak{D}_i$. For any $\delta > 0$, by taking n large enough, we can ensure that $\sup_{(x,y)\in\mathfrak{D}_i\times\mathfrak{D}_j} |\mathbb{K}(x,y) - \mathbb{K}(x_i, x_j)| < \delta$, hence

$$|I(a') - Q'(a_1', \cdots, a_n')| \le \delta \left(\sum_{i=1}^n |a_i'| \mu_n(i) \right)^2 \le \delta \left(\sum_{i=1}^n |a_i'|^2 \mu_n(i) \right) \left(\sum_{i=1}^n \mu_n(i) \right) \le \delta\mu(\mathfrak{D})$$

The last step in the above computation uses the Cauchy Schwarz inequality. Further, note that since the operator norm of $\mathbb{K}$ as an integral operator on $L^2(\mu \times \mu)$ is $\lambda_1 < 1$, therefore, $|I(a')| \le \lambda_1 < 1$, because the norm of the function $x \to a'(x)$ in $L_2(\mu)$ is ≤ 1. Hence, for large enough n, we can have δ small enough such that

$$|Q'(a_1', \cdots, a_n')| \le I(a') + \delta\mu(\mathfrak{D}) \le \lambda_1 + \delta\mu(\mathfrak{D}) \le 1.$$

Hence for n large enough, $\mathbb{K}_n$ is a contraction.

On the other hand, $(K(x_i, x_j))_{i,j=1}^n$ is a non-negative definite matrix, a fact clear from the representation (5.3) of $\mathbb{K}$ with $\lambda_i \ge 0$. Since $\mathbb{K}_n(i,j) = \mathbb{K}(x_i, x_j)\sqrt{\mu_n(i)\mu_n(j)}$, therefore the matrix $(\mathbb{K}_n(i,j))_{i,j=1}^n$ is also non-negative definite, being the Hadamard product of the two non-negative definite matrices $(K(x_i, x_j))_{i,j=1}^n$ and $\left(\sqrt{\mu_n(i)\mu_n(j)}\right)_{i,j=1}^n$. This proves that the matrix $\mathbb{K}_n$ is indeed a non-negative contraction.

Also, notice that it follows from the definition of $\mathbb{K}_n$ that, for $n \ge N$, the rank of $\mathbb{K}_n$ is (at most) N, because the rank of $\mathbb{K}$ is N. In particular, the rank of $\mathbb{K}_n$ remains fixed as n grows; this fact will be referred to later.

Thus, we can consider a discrete determinantal process $\mathbb{P}_n$ on $[n]$ given by the contraction $\mathbb{K}_n$ (acting on $\ell^2([n])$). From Theorem 8.1 [Ly03] we know that such a process satisfies

$$\mathbb{P}_n\left((N(S_1) \le r) \cap (N(S_2) \le s)\right) \le \mathbb{P}_n\left(N(S_1) \le r\right) \mathbb{P}_n\left(N(S_2) \le s\right) \tag{5.4}$$

where S_1 and S_2 are any two disjoint subsets $[n]$ and $N(S_1)$ is the number of indices $\in S_1$ obtained in a random sample from $\mathbb{P}_n$.

To obtain the final result, we fix two disjoint subsets A and B of $\mathfrak{D}$ which are finite unions of dyadic subcubes of $\mathfrak{D}$. It suffices to prove (5.2) for such sets because of the regularity of both sides of (5.2) under approximations $A_k \uparrow A$. Now, for large enough dyadic partition of size n (such that A and B can be expressed as the union of dyadic sub cubes from this partition), we consider the corresponding determinantal process $\mathbb{P}_n$.

Let us denote by A_n and B_n the set of indices corresponding to the sub cubes (in the dyadic partition of size n) of the sets A and B respectively. As before, let $N(A_n)$ denote the number of indices in A_n obtained in a random sample from $\mathbb{P}_n$. Applying the inequality (5.4), we can write

$$\mathbb{P}_n\left((N(A_n) \leq r) \cap (N(B_n) \leq s)\right) \leq \mathbb{P}_n\left(N(A_n) \leq r\right) \mathbb{P}_n\left(N(B_n) \leq s\right).$$

All that remains to show is that the random variable $(N(A_n), N(B_n))$ (under $\mathbb{P}_n$) converges in distribution to the random variable $(N(A), N(B))$ (under $\mathbb{P}$).

To this end, we use Laplace functionals. For any non-negative reals t_1 and t_2, we consider the quantity $\Psi_n(t_1, t_2) = \mathbb{E}[\exp(-t_1 N(A_n) - t_2 N(B_n))]$. We will invoke Theorem 1.2 and Remark 2.2 in [ShTa03]. As per the notations used in [ShTa03], we consider the case $\alpha = -1$, (which corresponds to determinantal point processes), $R = [n]$, λ to be the counting measure on $[n]$, K to be $\mathbb{K}_n$, $f(j) = t_1 1_{A_n}(j) + t_2 1_{B_n}(j)$ and $\phi(j) = (1 - e^{-f(j)})$.

We then appeal to Theorem 1.2 and Remark 2.2 in [ShTa03], and notice that $\mathbb{E}[\exp(-t_1 N(A_n) - t_2 N(B_n))]$ can be expressed as a series (as in Remark 2.2 of [ShTa03]). We also notice that this series in our case is in fact a finite series, because all terms of degree $> N$ are 0 since the rank of $\mathbb{K}_n$ as a matrix is $\leq N$.

Also observe that the k-th term of the series is infact a Riemann sum which converges to

$$\frac{(-1)^k}{k!} \int_{R^k} \prod_{i=1}^{k} \psi(x_k) \mathrm{Det}\left[(K(x_i, x_j))\right] d\mu(x_1) \cdots d\mu(x_n),$$

where $R = \mathbb{R}^d$ and $\psi(x) = (1 - e^{-f(x)})$, with $f(x) = t_1 1_A(x) + t_2 1_B(x)$.

Thus, we have $\mathbb{E}[\exp(-t_1 N(A_n) - t_2 N(B_n))] \to \mathbb{E}[\exp(-t_1 N(A) - t_2 N(B))]$ as $n \to \infty$. Since all the random variables in the last expression are non-negative integer valued, therefore we have $(N(A_n), N(B_n)) \to (N(A), N(B))$ in distribution, as desired.

Since we have (5.4) for each finite n, therefore in the limit as $n \to \infty$ we have (5.2).

This completes the proof of the theorem.

■

We are now ready to establish the main Theorem 5.1.3.

Proof of Theorem 5.1.3. Let $\mathcal{H}_0$ be the random closed subspace of $\mathcal{H}$ generated by the functions $\{K(x, \cdot) : x \in \Pi\}$ inside $L^2(\mu)$. We wish to show that a.s. we have $\mathcal{H}_0 = \mathcal{H}$. It suffices to prove that $K(\mathbf{0}, \cdot) \in \mathcal{H}_0$ a.s., where $\mathbf{0}$ denotes the origin in $\mathbb{R}^d$. Because, by translation invariance, this would imply that a.s. $K(q, \cdot) \in \mathcal{H}_0$ simultaneously for all $q \in \mathbb{R}^d$

with all co-ordinates rational. Then the continuity of the map $x \to K(x,\cdot)$ would imply that $K(x,\cdot) \in \mathcal{H}_0$ simultaneously for all $x \in \mathbb{R}^d$. This implies that $\mathcal{H}_0 = \mathcal{H}$ a.s.

For the points $\{x_i\}_{i=1}^n \in \mathbb{R}^d$, let $\mathcal{D}(x_1,\cdots,x_n)$ denote $\text{Det}\left[(K(x_i,x_j))_{i,j=1}^n\right]$. Then, if we consider the function $K(z,\cdot)$ as a vector in the Hilbert space $\mathcal{H}$, the squared norm of the projection of $K(x,\cdot)$ on to the orthogonal complement of Span $\{K(x_i,\cdot), 1 \le i \le n\}$ is given by the ratio $\frac{\mathcal{D}(x,x_1,\cdots,x_n)}{\mathcal{D}(x_1,\cdots,x_n)}$. But this is also equal to the conditional intensity $p(x|x_1,\cdots,x_n)$ of Π at x given that $\{x_1,\cdots,x_n\} \subset \Pi$.

Fix an $\epsilon > 0$. For $r > 0$, denote by B_r the ball of radius r centred at $\mathbf{0}$. Let ω_R be the set of points of Π in $B_R \setminus B_\epsilon$. For any feasible realisation Υ_R of ω_R we have

$$\mathbb{E}[\text{ Number of points in } B_\epsilon|\Upsilon_R \subset \Pi] = \int_{B_\epsilon} p(x|\Upsilon_R)d\mu(x) \tag{5.5}$$

We now proceed with the left hand side in (5.5) as

$$\begin{aligned}
&\mathbb{E}[\text{ Number of points in } B_\epsilon|\Upsilon_R \subset \Pi]\\
&= \sum_{k=1}^{\infty} \mathbb{P}[\text{ There are } \ge k \text{ points in } B_\epsilon|\Upsilon_R \subset \Pi]\\
&\le \sum_{k=1}^{\infty} \mathbb{P}[\text{ There are } \ge k \text{ points in } B_\epsilon|\omega_R = \Upsilon_R] \text{ (from negative association, see Theorem 5.1.4)}\\
&= \mathbb{E}[\text{ Number of points in } B_\epsilon|\omega_R = \Upsilon_R]
\end{aligned}$$

The inequality above involving negative association follows by applying Theorem 5.1.4 to the point process obtained by conditioning Π to contain Υ_R, which is a determinantal process on $\mathbb{R}^d$ having a standard kernel, see [ShTa03] Corollary 6.6. To elaborate this point, we consider the determinantal point process Π' (with a standard kernel) obtained by conditioning Π to contain Υ_R. Applying Theorem 5.1.4 to Π', we get that

$$\mathbb{P}(\text{ There are } \ge k \text{ points of } \Pi' \text{ in } B_\epsilon \cap \text{ There is no point of } \Pi' \text{ in } B_R \setminus B_\epsilon)$$

$$\ge \mathbb{P}(\text{ There are } \ge k \text{ points of } \Pi' \text{ in } B_\epsilon)\,\mathbb{P}(\text{ There is no point of } \Pi' \text{ in } B_R \setminus B_\epsilon).$$

Since Υ_R is obtained as a realization of the full point configuration of Π in $B_R \setminus B_\epsilon$, therefore a.s. in Υ_R, we have $\mathbb{P}(\text{ There is no point of } \Pi' \text{ in } B_R \setminus B_\epsilon) > 0$. Hence the last inequality can be rephrased as

$$\mathbb{P}(\text{ There are } \ge k \text{ points of } \Pi' \text{ in } B_\epsilon|\text{ There is no point of } \Pi' \text{ in } B_R \setminus B_\epsilon)$$

$$\ge \mathbb{P}(\text{ There are } \ge k \text{ points of } \Pi' \text{ in } B_\epsilon).$$

Under the conditioning $\Upsilon_R \subset \Pi$, the event that there is no point of Π' in $B_R \setminus B_\epsilon$ corresponds to $\omega_R = \Upsilon_R$. This gives us the desired inequality.

By Proposition 5.3.1, we have that given any $\delta > 0$, we can find R_δ such that except on an event Ω_1^δ of probability $< \delta$, we have

$$|\mathbb{E}[\text{Number of points in } B_\epsilon | \omega_{R_\delta}] - N_{B_\epsilon}| < \delta.$$

But, except on an event Ω_2, measurable with respect to ω_{out} and of probability $O(\epsilon^d)$, we have $N_{B_\epsilon} = 0$.

Hence, except on the event $\Omega_1^\delta \cup \Omega_2$, we have

$$\int_{B_\epsilon} p(x|\Upsilon_{R_\delta}) d\mu(x) \leq \delta$$

Letting $\delta \downarrow 0$ along a summable sequence, we deduce that a.s. on $(\Omega_2)^c$ we have

$$\varliminf_{\delta \to 0} \int_{B_\epsilon} p(x|\Upsilon_{R_\delta}) d\mu(x) = 0.$$

This is true because, by the Borel Cantelli lemma, the event that Ω_1^δ occurs infinitely often (along this summable sequence of δ-s) has probability zero.

By Fatou's lemma, this implies that on $(\Omega_2)^c$ we have

$$\int_{B_\epsilon} \varliminf_{\delta \to 0} p(x|\Upsilon_{R_\delta}) d\mu(x) = 0.$$

This implies that $\varliminf_{\delta \to 0} p(x|\Upsilon_{R_\delta}) = 0$ for almost every $x \in B_\epsilon$ on the event $(\Omega_2)^c$. By our previous discussion, at the beginning of the proof, regarding the connection between the squared norms of projections and conditional intensities, this means that on Ω_2^c, $K(x, \cdot) \in \mathcal{H}_0$ for a dense set of $x \in B_\epsilon$. By the continuity of the map $x \to K(x, \cdot)$, we have $K(\mathbf{0}, \cdot) \in \mathcal{H}_0$ on the event Ω_2^c. Letting $\epsilon \to 0$ (which implies $\Omega_2 \downarrow \phi$), we have $K(\mathbf{0}, \cdot) \in \mathcal{H}_0$ with probability one. ■

5.4 Rigidity and Tolerance for certain determinantal point processes

Let f be a function $\mathbb{T}^d \to [0, 1]$. Then multiplication by f is a non-negative contraction operator from $L^2(\mathbb{T}^d)$ to itself. Under Fourier conjugation, this gives rise to a non-negative contraction opertaor $Q : \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$. This, in turn, gives rise to a determinantal point process $\mathbb{P}^f$ on $\mathbb{Z}^d$ in a cannonical way, for details see [Ly03]. For a point configuration ω drawn from the distribution of $\mathbb{P}^f$, we denote by $\omega(\underline{k})$ the indicator function of having a point at $\underline{k} \in \mathbb{Z}^d$ in the configuration ω. We denote by ω_{out} the configuration of points on $\mathbb{Z}^d \setminus \mathbf{0}$ obtained by restricting ω to $\mathbb{Z}^d \setminus \mathbf{0}$, where $\mathbf{0}$ denotes the origin. For a point process π on a space Ξ, we denote by $[\pi]$ the (random) counting measure obtained from a realisation of π. The k-point intensity functions of a point process, when it exists, will be denoted by $\rho_k, k \geq 1$.

For the processes $\mathbb{P}^f$, all intensity functions exist. Moreover, translation invariance of $\mathbb{P}^f$ implies that $\rho_k(x_1+x,\cdots,x_k+x)=\rho_k(x_1,\cdots,x_k)$ for all $x,x_1,\cdots,x_k\in\mathbb{Z}^d$, in particular, ρ_1 is a constant in $[0,1]$.

In this Section we discuss the insertion and deletion tolerance question from [LySt03], principally the proof of Theorem 5.1.5.

We will use the following general observation for determinantal point processes given by a projection kernel:

Proposition 5.4.1. *Consider a determinantal point process* Π *in a locally compact space* Ξ *with determinantal kernel* $K(\cdot,\cdot)$ *and background measure* μ*, such that* K *is idempotent as an integral operator on* $L^2(\mu)$*. Let* ψ *be a compactly supported function on* Ξ*. Then*

$$\mathrm{Var}\left[\int \psi\, d[\Pi]\right]=\frac{1}{2}\iint |\psi(x)-\psi(y)|^2|K(x,y)|^2\, d\mu(x)d\mu(y) \tag{5.6}$$

Proof. This follows from the determinantal formula for the two point intensity function of Π and the idempotence of K. ∎

Proof of Theorem 5.1.5. We will approach this question by estimating the variance of linear statistics of $\mathbb{P}^f$. A similar approach has been used in [GP12] to obtain rigidity behaviour for the Ginibre ensemble and the zero process of the standard planar Gaussian analytic function.

Let φ be a C_c^∞ function on $\mathbb{R}$ which is $\equiv 1$ in a neighbourhood of the origin. Viewed as a function on $\mathbb{Z}$, φ is compactly supported and $=1$ at the origin. Let φ_L be defined by $\varphi_L(x)=\varphi(x/L)$. Since f is the indicator function of an interval $I\subset\mathbb{T}$, therefore the determinantal kernel K of $\mathbb{P}^f$ is an idempotent on $\ell^2(\mathbb{Z})$. Applying Proposition 5.4.1 with $\Pi=\mathbb{P}^f$, $\Xi=\mathbb{Z}$, $\mu=$ the counting measure on $\mathbb{Z}$ and $\psi=\varphi_L$ we get

$$\mathrm{Var}\left[\int \varphi_L\, d[\mathbb{P}^f]\right]=\frac{1}{2}\sum_{i,j\in\mathbb{Z}}|\varphi(\frac{i}{L})-\varphi(\frac{j}{L})|^2|\hat{f}(i-j)|^2. \tag{5.7}$$

Observe that translating the interval $I\subset\mathbb{T}$ leaves the measure $\mathbb{P}^f$ invariant, so, without loss of generality, we take I to be corresponding to the interval $[-a,a]$ where $\mathbb{T}$ is parametrized as $(-\pi,\pi]$ and $0<a<\pi$. Then $\hat{f}(k)=c(a)\mathrm{sin}ak/k$ where $c(a)$ is a constant. This implies that, for some constant $c>0$, we have

$$\mathrm{Var}\left[\int \varphi_L\, d[\mathbb{P}^f]\right]=c\sum_{i,j\in\mathbb{Z}}|\varphi(\frac{i}{L})-\varphi(\frac{j}{L})|^2\left(\sin^2 a(i-j)\right)|i-j|^{-2}. \tag{5.8}$$

This, in turn, implies (using $|\mathrm{sin}\theta|\le 1$) that

$$\mathrm{Var}\left[\int \varphi_L\, d[\mathbb{P}^f]\right]\le c\sum_{i,j\in\mathbb{Z}}|\varphi(\frac{i}{L})-\varphi(\frac{j}{L})|^2|(i-j)/L|^{-2}L^{-2}. \tag{5.9}$$

Hence we have

$$\varlimsup_{L\to\infty} \mathrm{Var}\left[\int \varphi_L \, d[\mathbb{P}^f]\right] \leq c \int\int \left(\frac{\varphi(x)-\varphi(y)}{x-y}\right)^2 d\mathcal{L}(x)d\mathcal{L}(y) \tag{5.10}$$

where $\mathcal{L}$ denotes the Lebesgue measure on $\mathbb{R}$.

For any $\mathbb{C}_c^1$ functions ψ_1, ψ_2 on $\mathbb{R}$, we define the form

$$\Lambda(\psi_1,\psi_2) = \int\int \frac{(\psi_1(x)-\psi_1(y))(\psi_2(x)-\psi_2(y))}{(x-y)^2} d\mathcal{L}(x)d\mathcal{L}(y). \tag{5.11}$$

It is known that $\Lambda(\psi,\psi)$ is related to the $H^{1/2}$ norm of ψ.

A simple calculation shows that for any $\lambda > 0$, we have $\Lambda((\psi_1)_\lambda, (\psi_2)_\lambda) = \Lambda(\psi_1,\psi_2)$. In particular, this implies that $\Lambda(\psi_1, (\psi_2)_\lambda) = \Lambda((\psi_1)_{1/\lambda}, \psi_2)$. Further, we will see in Proposition 5.4.3 that $\Lambda(\psi, \psi_{\lambda^{-1}}) \to 0$ as $\lambda \to 0$.

For an integer $n > 0$, let $0 < \lambda = \lambda(n) < 1$ be such that for φ as above, $|\Lambda(\varphi, \varphi_{\lambda^{-i}})| \leq 1/2^i$ for $1 \leq i \leq n$. Such a choice can be made because of the observations in the previous paragraph. Define $\Phi^n = \left(\sum_{i=1}^n \varphi_{\lambda^{-i}}\right)/n$. Note that $\Phi^n \equiv 1$ in a neighbourhood of $\mathbf{0}$ in $\mathbb{R}$, and the same is true for all scalings Φ_L^n of Φ^n whenever $L \geq 1$.

Let $L = L(n) > 1$ be such that

$$\mathrm{Var}\left[\int \Phi_L^n \, d[\mathbb{P}^f]\right] \leq c\Lambda(\Phi^n, \Phi^n) + \frac{1}{n}.$$

But $\Lambda(\Phi^n,\Phi^n) = \frac{1}{n^2}\left(\sum_{i,j=1}^n \Lambda(\varphi_{\lambda^{-i}}, \varphi_{\lambda^{-j}})\right)$. Observe that

$$\Lambda(\varphi_{\lambda^{-i}}, \varphi_{\lambda^{-j}}) = \Lambda(\varphi, \varphi_{\lambda^{-|i-j|}}) \leq 2^{-|i-j|}.$$

This implies that

$$\sum_{i,j=1}^n \Lambda(\varphi_{\lambda^{-i}}, \varphi_{\lambda^{-j}}) \leq C(\varphi)n.$$

Hence $\mathrm{Var}\left[\int \Phi_L^n \, d[\mathbb{P}^f]\right] \leq C(\varphi)/n$.

By the Borel-Cantelli lemma, we have, as $n \to \infty$,

$$\left|\int \Phi_L^{2^n} \, d[\mathbb{P}^f] - \mathbb{E}[\int \Phi_L^{2^n} \, d[\mathbb{P}^f]]\right| \to 0. \tag{5.12}$$

But $\int \Phi_L^{2^n} \, d[\mathbb{P}^f] = \omega(\mathbf{0}) + \int_{\mathbb{Z}\setminus\mathbf{0}} \Phi_L^{2^n} \, d[\mathbb{P}^f]$, and the second term can be evaluated if we know ω_{out}. $\mathbb{E}[\int \Phi_L^{2^n} \, d[\mathbb{P}^f]]$ can also be computed explicitly in terms of the first intensity measure of $\mathbb{P}^f$. This implies that from (5.12), we can deduce the value of $\omega(\mathbf{0})$ by letting $n \to \infty$.

Thus, ω_{out} a.s. determines the value of $\omega(\mathbf{0})$. Since both the events $\omega(\mathbf{0}) = 0$ and $\omega(\mathbf{0}) = 1$ occur with positive probability, therefore the events $\{\mathbb{P}[\omega(\mathbf{0}) = 1|\omega_{\text{out}}] = 0\}$ and $\{\mathbb{P}[\omega(\mathbf{0}) = 0|\omega_{\text{out}}] = 0\}$ both have positive probability (in ω_{out}). ∎

Remark 5.4.1. *For f which is the indicator function of a finite, disjoint union of intervals $\subset \mathbb{T}$, we have $|\hat{f}(k)| \leq c/|k|$, hence the same argument and the same conclusion as Theorem 5.1.5 holds for such f.*

Remark 5.4.2. *A similar argument shows that, in fact, for any finite set $S \subset \mathbb{Z}$, the point configuration of $\mathbb{P}^f$ restricted to S^c a.s. determines the number of points of $\mathbb{P}^f$ in S, when f is the indicator function of an interval.*

A similar class of determinantal point processes in the continuum can be obtained by considering L^2 functions $f : \mathbb{R}^d \to [0.1]$. The multiplication operator M_f defined by such a function f is clearly a contraction on $L^2(\mathbb{R}^d)$. By considering the Fourier conjugate of such an operator, we get another contraction on $L^2(\mathbb{R}^d)$, which gives us a translation invariant determinantal point process $\mathbb{P}^f$ in $\mathbb{R}^d$. One of the most important examples of such a point process is the sine kernel process on $\mathbb{R}$, which is defined by the determinantal kernel $\frac{\sin\pi(x-y)}{\pi(x-y)}$ with the Lebesgue measure on $\mathbb{R}$ as the background measure. Here the relevant function f is the indicator function of the interval $[-\pi, \pi]$. For details, see [AGZ09]. More generally we can consider the indicator function of any measurable subset of $\mathbb{R}$, which will give us a projection operator on $L^2(\mathbb{R})$, and hence a determinantal point process corresponding to a projection kernel. In this setting, we have a continuum analogue of Theorem 5.1.5, which says that whenever f is the indicator of a finite union of compact intervals in $\mathbb{R}$, we have that $\mathbb{P}^f$ is a rigid process.

Theorem 5.4.2. *Let $f : \mathbb{R} \to [0, 1]$ be an indicator function of a finite union of compact intervals. Then the determinantal point process $\mathbb{P}^f$ in $\mathbb{R}$ is "rigid" in the following sense. Let $U \subset \mathbb{R}$ be a bounded interval, and let ω be the point configuration sampled from the distribution $\mathbb{P}^f$. Define the restricted point configurations $\omega_{\text{in}} = \omega_{|U}$ and $\omega_{\text{out}} = \omega_{|U^c}$. Let $|\omega_{\text{in}}|$ be the number of points of ω in U. Then there exists a measurable function*

$$N : \text{ Point configurations in } U^c \to \mathbb{N} \cup \{0\}$$

such that a.s. we have $|\omega_{\text{in}}| = N(\omega_{\text{out}})$. This holds true for all bounded intervals U. In particular, the continuum sine kernel process is "rigid" in the above sense.

Proof. By translation invariance, it suffices to take U to be centred at the origin. Let φ be a C_c^∞ function which is $\equiv 1$ in a neighbourhood of U. Then we have

$$\text{Var}\left[\int \varphi \, d[\mathbb{P}^f]\right] = \frac{1}{2} \int \int |\varphi(x) - \varphi(y)|^2 |\hat{f}(x-y)|^2 d\mathcal{L}(x) d\mathcal{L}(y). \tag{5.13}$$

But for any compact interval $[a, b]$ we have $|\hat{1}_{[a,b]}(\xi)| \leq c|\xi|^{-1}$, hence we have

$$\text{Var}\left[\int \varphi \, d[\mathbb{P}^f]\right] \leq C \int \int \left(\frac{\varphi(x) - \varphi(y)}{x - y}\right)^2 d\mathcal{L}(x) d\mathcal{L}(y). \tag{5.14}$$

Recall the form $\Lambda(\cdot,\cdot)$ as in (5.11). Proposition 5.4.3 implies that $\Lambda(\varphi,\varphi_{\lambda^{-1}}) \to 0$ as $\lambda \to 0$. For an integer $n > 0$, let $0 < \lambda < 1$ be such that for φ as above, $|\Lambda(\varphi,\varphi_{\lambda^{-i}})| \leq 1/2^i$ for $1 \leq i \leq n$. Define $\Phi^n = \left(\sum_{i=1}^n \varphi_{\lambda^{-i}}\right)/n$. We have $\Phi^n \equiv 1$ in a neighbourhood of U in $\mathbb{R}$. Due to our choice of λ, we have $\operatorname{Var}\left(\int \Phi^n d[\mathbb{P}^f]\right) = \Lambda(\Phi^n,\Phi^n) = O(1/n)$. From here, we proceed on similar lines to the proof of Theorem 5.1.5 and deduce the existence of N as prescribed in the statement of Theorem 5.4.2. ∎

We now prove Proposition 5.4.3, which will complete the proof of Theorem 5.1.5.

Proposition 5.4.3. *For a C_c^1 function φ, we have $\Lambda(\varphi,\varphi_{\lambda^{-1}}) \to 0$ as $\lambda \to 0$.*

Proof. We begin with the expression

$$\Lambda(\varphi,\varphi_{\lambda^{-1}}) = \int\int \frac{(\varphi(x)-\varphi(y))(\varphi(\lambda x)-\varphi(\lambda y))}{(x-y)^2}\, d\mathcal{L}(x)d\mathcal{L}(y). \tag{5.15}$$

Fix $a > 0$. Let K be the support of φ. We define the function

$$\gamma(x,y) = \begin{cases} \|\varphi'\|_\infty^2 & \text{if } x \text{ or } y \in K \text{ and } |x-y| \leq a \\ \frac{4\|\varphi\|_\infty^2}{(x-y)^2} & \text{if } x \text{ or } y \in K \text{ and } |x-y| > a \\ 0 & \text{otherwise} \end{cases} \tag{5.16}$$

For $0 < \lambda < 1$, the integrand in (5.15) is bounded in absolute value from above pointwise by $\gamma(x,y)$. To see this, note that the integrand in (5.15) is non-zero only on the set $S = \{(x,y) : x \text{ or } y \in K\}$. On S, we bound the integrand from above as follows: for $(x,y) \in S$ such that $|x-y| \leq a$ we use $|\varphi(x)-\varphi(y)| \leq \|\varphi'\|_\infty |x-y|$, for other $(x,y) \in S$ we use $|\varphi(x)-\varphi(y)| \leq 2\|\varphi\|_\infty$.

Since K is a compact set, we have

$$\int\int \gamma(x,y)\, d\mathcal{L}(x)d\mathcal{L}(y) < \infty, \tag{5.17}$$

where we use the fact that $\int_{|t|>a} \frac{1}{t^2} dt = \frac{1}{a}$.

(5.17) enables us to use the dominated convergence theorem and let $\lambda \to 0$ in the integrand of (5.15), whence $\Lambda(\varphi,\varphi_{\lambda^{-1}}) \to 0$. ∎

Next we show that in any dimension d, whenever f is not the indicator of a subset of $\mathbb{T}^d$, $\operatorname{Var}\left[\int \varphi_L d[\mathbb{P}^f]\right]$ blows up at least like L^d as $L \to \infty$.

Proposition 5.4.4. *Let $f : \mathbb{T}^d \to [0,1]$ not equal the indicator function of some subset of $\mathbb{T}^d$ (up to Lebesgue-null sets). Then $\operatorname{Var}\left[\int \varphi_L d[\mathbb{P}^f]\right] = \Omega(L^d)$ as $L \to \infty$.*

Proof. Let $\rho_2(\cdot,\cdot)$ be the two point intensity function of $\mathbb{P}^f$, given by the formula

$$\rho_2(i,j) = \det\begin{pmatrix} \hat{f}(0) & \hat{f}(i-j) \\ \hat{f}(j-i) & \hat{f}(0) \end{pmatrix}$$

Let λ_d denote the normalized Lebesgue measure on $\mathbb{T}^d$. Using the above formula, we can write the variance in question as:

$$\begin{aligned}
\mathrm{Var}\left[\int \varphi_L d[\mathbb{P}^f]\right] &= \mathbb{E}\left(\int \varphi_L d[\mathbb{P}^f]\right)^2 - \left(\mathbb{E}\left[\int \varphi_L \, d[\mathbb{P}^f]\right]\right)^2 \\
&= \sum_i \varphi_L(i)^2 \hat{f}(0) + \sum_{i,j} \varphi_L(i)\varphi_L(j)\left(\hat{f}(0)^2 - |\hat{f}(i-j)|^2\right) - \sum_{i,j} \varphi_L(i)\varphi_L(j)\hat{f}(0)^2 \\
&= \sum_i \varphi_L(i)^2 \left(\hat{f}(0) - \sum_j |\hat{f}(i-j)|^2\right) + \sum_{i,j}\left(\varphi_L(i)^2 - \varphi_L(i)\varphi_L(j)\right)|\hat{f}(i-j)|^2 \\
&= \left(\hat{f}(0) - \sum_k |\hat{f}(k)|^2\right)\left(\sum_k \varphi_L(k)^2\right) + \frac{1}{2}\left(\sum_{i,j} |\varphi_L(i) - \varphi_L(j)|^2 \, |\hat{f}(i-j)|^2\right) \\
&\geq \left(\hat{f}(0) - \sum_k |\hat{f}(k)|^2\right)\left(\sum_k \varphi_L(k)^2\right) \\
&= \left(\int_{\mathbb{T}^d} f(x) d\lambda_d(x) - \int_{\mathbb{T}^d} f(x)^2 d\lambda_d(x)\right)\left(\sum_k \varphi_L(k)^2\right).
\end{aligned}$$

In the last step we have used Parseval's identity: $\sum_k |\hat{f}(k)|^2 = \int f(x)^2 d\lambda_d(x)$. Note that since $0 \leq f \leq 1$, we have $\left(\int_{\mathbb{T}^d} f(x)d\lambda_d(x) - \int_{\mathbb{T}^d} f(x)^2 d\lambda_d(x)\right) \geq 0$ with strict inequality holding if and only if f is not the indicator of some subset of $\mathbb{T}^d$ (upto Lebesgue null sets). Finally, observe that as $L \to \infty$ we have

$$\frac{1}{L^d}\left(\sum_k \varphi_L(k)^2\right) = \sum_k \frac{1}{L^d}\varphi\left(\frac{k}{L}\right)^2 \to \|\varphi\|_2^2.$$

This completes the proof of the proposition. ■

5.5 $\mathbb{P}^f$ determines f

In this Section we provide the proofs of Theorem 5.1.6 and Proposition 5.1.7

Proof of Theorem 5.1.6. Define $\mathfrak{F}$ to be the subset of functions of $\mathcal{E}$ satisfying the following conditions:

- (i) Either $\hat{f}(k) \neq 0 \forall k \in \mathbb{Z}$, or f is a trigonometric polynomial of degree N, and $\hat{f}(k) \neq 0$ for all $|k| \leq N$.
- (ii) For every $n \geq 3$ (and $n \leq$ the degree N in the case of f being a trigonometric polynomial), we have $\mathrm{Arg}(\hat{f}(n)) - \mathrm{Arg}(\hat{f}(n-1))$ does not differ from $\mathrm{Arg}(\hat{f}(2)) -$

$\mathrm{Arg}(\hat{f}(1))$ by an integer multiple of π. Further, $\mathrm{Arg}(\hat{f}(2)) - 2\mathrm{Arg}(\hat{f}(1))$ is not an integer multiple of π. Here we consider Arg to be a number in $(-\pi, \pi]$.

We claim that the complement of $\mathfrak{F}$ in $\mathcal{E}$, denoted by $\mathcal{G} := \mathcal{E} \setminus \mathfrak{F}$, is a meagre subset of $\mathcal{E}$, and for $f \in \mathfrak{F}$, we have $\mathbb{P}^f$ determines f up to the rotation and flip. A meagre subset of a topological space is a set which can be expressed as a countable union of nowhere dense sets.

To show that $\mathcal{G}$ is meagre, we will show that it is a subset of a countable union of nowhere dense sets. Indeed, we can write $\mathcal{G}$ as:

$$\mathcal{G} \subset \cup_i A_i \cup_{n \geq 3} B_n \cup C$$

where

$$A_i := \{f \in \mathcal{E} : \hat{f}(i) = 0\},$$

$$B_n := \{f \in \mathcal{E} : \text{ Either } \hat{f}(k) = 0 \text{ for } k = 1, 2, n, n-1 \text{ or}$$

$$\mathrm{Arg}(\hat{f}(n)) - \mathrm{Arg}(\hat{f}(n-1)) = \mathrm{Arg}(\hat{f}(2)) - \mathrm{Arg}(\hat{f}(1)) + t\pi,\ t \text{ an integer with } |t| \leq 4\},$$

$$C := \{\hat{f}(1) = 0 \text{ or } \hat{f}(2) = 0 \text{ or } \mathrm{Arg}(\hat{f}(2)) - 2\mathrm{Arg}(\hat{f}(1)) = t\pi,\ t \text{ an integer with } |t| \leq 3\}.$$

It is not hard to see that each A_i and B_n are closed sets in $L^\infty(\mathbb{T})$, being defined by closed conditions on finitely many co-ordinates of the Fourier expansion (observe that $|\hat{f}(n)| \leq \|f\|_\infty$ for each n, hence $f \to \hat{f}(n)$ is a continuous linear functional on $L^\infty(\mathbb{T})$). The same holds true for C. It is also clear that none of the A_i-s or B_n-s or C contain any $L^\infty(\mathbb{T})$ ball (since even small perturbations in the relevant Fourier cofficients can lead us outside these sets), showing that they are nowhere dense. All these combine to prove that $\mathcal{G}$ is a meagre subset of $L^\infty(\mathbb{T})$.

Let f be a function in $\mathfrak{F}$. We begin with the Fourier expansion f:

$$f(x) = \sum_{-\infty}^{\infty} a_j e^{ijx} \tag{5.18}$$

where i is the imaginary unit.

We make the following observations about the expansion (5.18). First, f is real valued implies

$$a_{-j} = \overline{a_j} \text{ for all } j. \tag{5.19}$$

Secondly, letting $f_\xi = f(x + \xi) = \sum_{j=-\infty}^{\infty} a_j(\xi) e^{ijx}$ where $\xi \in (-\pi, \pi]$ and the addition is in $\mathbb{T}$, we have

$$a_j(\xi) = a_j e^{ij\xi}. \tag{5.20}$$

Finally, setting $\widetilde{f}(x) = f(-x) = \sum_{j=-\infty}^{\infty} \widetilde{a}_j e^{ijx}$, we have

$$\widetilde{a}_j = \overline{a_j}. \tag{5.21}$$

We want to recover the coefficients a_j (up to the symmetries (5.19), (5.20) and (5.21)) from the measure $\mathbb{P}^f$. We observe that the class of functions $\mathfrak{F}$ is preserved under these symmetries.

In particular, a trigonometric polynomial remains a trigonometric polynomial of the same degree and the same regularity property as demanded in the definition of the class $\mathfrak{F}$.

To recover f, we begin by observing that $a_0 = \rho_1$, and is therefore determined by $\mathbb{P}^f$. Further,

$$\rho_2(0,n) = \begin{vmatrix} a_0 & a_n \\ a_{-n} & a_0 \end{vmatrix} = a_0^2 - |a_n|^2. \tag{5.22}$$

This implies that $\mathbb{P}^f$ determines $|a_n|$ for all n.

Recall that $a_1 \neq 0$ for $f \in \mathfrak{F}$ (unless f is a constant function). Using the symmetry (5.20), we choose a_1 to be a positive real number, equal to its absolute value which is determined by $\mathbb{P}^f$. If $f \in \mathfrak{F}$ is a trigonometric polynomial of degree 1, then we are done. Else, we proceed as follows.

For any integer n, we have

$$\rho_3(0,1,n) = \begin{vmatrix} a_0 & a_1 & a_n \\ a_{-1} & a_0 & a_{n-1} \\ a_{-n} & a_{-(n-1)} & a_0 \end{vmatrix} \tag{5.23}$$

Expanding the right hand side along the first row, we have

$$\rho_3(0,1,n) = a_0 \begin{vmatrix} a_0 & a_{n-1} \\ a_{-(n-1)} & a_0 \end{vmatrix} - a_1 \begin{vmatrix} a_{-1} & a_{n-1} \\ a_{-n} & a_0 \end{vmatrix} + a_n \begin{vmatrix} a_{-1} & a_0 \\ a_{-n} & a_{-(n-1)} \end{vmatrix}.$$

Expanding the 2×2 determinants, we can simplify the above equation to

$$\rho_3(0,1,n) = 2a_1 \Re(a_n \overline{a_{n-1}}) + g(a_0, a_1, |a_{n-1}|, |a_n|), \tag{5.24}$$

where $g(w,x,y,z)$ is a polynomial in four complex variables. Since all quantities in (5.24) except $\Re(a_n \overline{a_{n-1}})$ are known and $a_1 > 0$, we deduce that $\mathbb{P}^f$ determines $\Re(a_n \overline{a_{n-1}})$ for all integers n.

For $n = 2$, we have $a_{n-1} = a_1$, and this implies that $\mathbb{P}^f$ in fact determines $\Re(a_2)$. Since $|a_2|$ is also known, this implies that $\mathbb{P}^f$ determines $|\Im(a_2)|$, which is non-zero because of the assumption $\mathrm{Arg}(\hat{f}(2)) - 2\mathrm{Arg}(\hat{f}(1))$ is not an integer multiple of π, $\mathrm{Arg}(\hat{f}(1))$ being 0 because $\hat{f}(1)$ is real. Using the symmetry (5.21), we choose a_2 such that $\Im(a_2) > 0$.

We have now spent all the symmetries present in the problem, and our goal is to show that all the other a_n-s ($n \geq 0$) are determined exactly by $\mathbb{P}^f$. a_n for $n < 0$ can then be found using the symmetry (5.19).

To this end, we apply induction. Suppose $n \geq 3$ and we know the values of $a_k, 0 \leq k \leq n-1$. Since $f \in \mathfrak{F}$, therefore all such a_k are non-zero. Computing $\rho_3(0,2,n)$ along similar lines to $\rho_3(0,1,n)$ we obtain

$$\rho_3(0,2,n) = 2\Re(a_n \overline{a_2} \overline{a_{n-2}}) + g(a_0, a_2, |a_{n-2}|, |a_n|), \tag{5.25}$$

where g is as in (5.24).

If $|a_n| = 0$, then we deduce that f is a trigonometric polynomial of degree $n-1$, because $f \in \mathfrak{F}$ (recall condition (i) defining $\mathfrak{F}$). In that case, we have already determined f. Else, $a_n \neq 0$, and we proceed as follows.

Since we already know $|a_n|$, we need only to determine $\mathrm{Arg}(a_n)$. For any complex number $z \neq 0$, $|z|$ and $\Re(z)$ determines $\mathrm{Arg}(z)$ (considered as a number in $(-\pi, \pi]$) up to sign. Hence, if we know $\Re(z\overline{z_1})$ and $\Re(z\overline{z_2})$ for two non-zero complex numbers z_1 and z_2 such that $\mathrm{Arg}(z_1)$ does not differ from $\mathrm{Arg}(z_2)$ by an integer multiple of π, then this data would be sufficient to determine $\mathrm{Arg}(z)$. But this is precisely the situation we have in our hands, with $z = a_n$, $z_1 = a_{n-1}$ and $z_2 = a_2 a_{n-2}$. None of them is 0 by condition (i) defining $\mathfrak{F}$, and the condition on the difference of arguments of z_1 and z_2 follows from condition (ii) defining $\mathfrak{F}$. This enables us to determine $\mathrm{Arg}(a_n)$, and hence a_n.

This completes the proof. ■

Remark 5.5.1. *The argument used in proof of Theorem 5.1.6 can also be used to recover the function f if all its Fourier coefficients are real. E.g., if f is the indicator function of an interval A in $(-\pi, \pi]$, then we can "rotate" f (symmetry 5.20) so that 0 is in the centre of the interval A. Then all the Fourier coefficients of f are real, and we can identify the interval A. This argument will also work for any set A which has a point of reflectional symmetry when looked upon as a subset of $\mathbb{T}$.*

Proof of Proposition 5.1.7. Recall that the harmonic mean of a function $f : \mathbb{T}^d \to \mathbb{R}$ is defined as

$$\mathrm{HM}(f) = \left(\int_{\mathbb{T}^d} \frac{d\lambda_d(x)}{f(x)} \right)^{-1},$$

where λ_d is the Lebesgue measure on $\mathbb{T}^d$. The fact that we know the distribution $\mathbb{P}^f$ implies that we know the distribution $\mathbb{P}^{tf}$ for any $0 < t < 1$, e.g. by performing an independent site percolation with survival probability t on $\mathbb{P}^f$. By taking complements, that is by considering the point process of the excluded points (see [LySt03] for more details), this implies that we know the distribution $\mathbb{P}^{1-tf}$. But this enables us to recover the harmonic mean $\mathrm{HM}(1-tf)$ of the function $1 - tf$ by the formula

$$\mathrm{HM}(1-tf) = \mathrm{Sup}\{p \in [0,1] : \mu_p \preccurlyeq_{\mathrm{f}} \mathbb{P}^{1-tf}\}.$$

Here μ_p is the standard site percolation on $\mathbb{Z}$ with survival probability p, and $\mu_p \preccurlyeq_{\mathrm{f}} \mathbb{P}^{1-tf}$ means that $\mathbb{P}^{1-tf}$ is uniformly insertion tolerant at level p, that is, $\mathbb{P}^{1-tf}[\omega(0) = 1 | \omega_{\mathrm{out}}] \geq p$ a.s. in ω_{out}. For details, we refer to Definition 5.15 and Theorem 5.16 in [LySt03]. But

$$\mathrm{HM}(1-tf)^{-1} = \int_{\mathbb{T}^d} \frac{d\lambda_d(x)}{1 - tf(x)}.$$

By expanding the integral on the right as a power series in t, we can recover $\int_{\mathbb{T}^d} f^k(x)\, d\lambda_d(x)$ for each $k \geq 1$. But we have

$$\int_{\mathbb{T}^d} f^k(x)\, d\lambda_d(x) = \int_0^1 \xi^{k-1} \nu_f(\xi) d\mathcal{L}(\xi),$$

where $\mathcal{L}$ is the Lebesgue measure on $\mathbb{R}$, and ν_f is the value distribution of f, given by

$$\nu_f(\xi) = \lambda_d(\{x \in \mathbb{T}^d : f(x) \geq \xi\}).$$

Thus we have all the moments of the measure $\nu_f(\xi)d\mathcal{L}(\xi)$. Since $\nu_f(\xi)d\mathcal{L}(\xi)$ is a compactly supported measure on the interval $[0, 1]$, its Fourier transform is finite everywhere, and the moments enable us to compute the Fourier transform. The Fourier transform determines the measure, and hence ν_f, uniquely. This enables us to recover the value distribution of f. ■

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