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Nonlinear norm-observability notions and stability of switched systems

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Abstract

This paper proposes several definitions of "norm-observability" for nonlinear systems and explores relationships among them. These observability properties involve the existence of a bound on the norm of the state in terms of the norms of the output and the input on some time interval. A Lyapunov-like sufficient condition for norm-observability is also obtained. As an application, we prove several variants of LaSalle's stability theorem for switched nonlinear systems. These results are demonstrated to be useful for control design in the presence of switching as well as for developing stability results of Popov type for switched feedback systems.

Keywords: nonlinear system, observability, switched system, LaSalle's stability theorem.

1 Introduction

For *linear* time-invariant systems with outputs, there are several equivalent ways to define observability. A standard approach is through distinguishability, which is the property that different initial conditions produce different outputs. This is equivalent to 0-distinguishability, which says that nonzero initial conditions produce nonzero outputs. The state of an observable linear system can be reconstructed from the output measurements on a time interval of arbitrary length by inverting the observability Gramian.

In the *nonlinear* context, various definitions of observability are no longer equivalent, and in general nonlinear observability is not as completely understood. In particular, the distinguishability concept has a natural counterpart for nonlinear systems, but does not lend itself to an explicit state reconstruction procedure as readily as in the linear case. In fact, it is well known that recovering the state of a nonlinear system from its output, even asymptotically by means of a dynamic observer, is a difficult task. Also, the

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choice of inputs plays a nontrivial role in reconstructing the state. These issues are discussed in books such as [20, 22, 34].

Instead of building an observer, however, it is sometimes sufficient for control purposes (although still far from being trivial) to construct a "norm estimator", i.e., to obtain an upper bound on the norm of the state using the output; see [36] for a discussion and references. This motivates us to address the concept of "norm-observability", which concerns the ability to determine such a bound rather than the precise value of the state.

Another notion related to observability is *detectability*, which for linear systems means that all solutions producing zero outputs decay to zero. In [36], a variant of detectability for nonlinear systems (called "output-to-state stability") is defined as the property that the state is bounded in terms of the supremum norm of the output, modulo a decaying term depending on initial conditions. This turns out to be a very useful and natural property, which is dual to input-to-state stability (ISS). For systems with inputs, one combines the inputs with the outputs and arrives at the concept of "input-output-to-state stability" which has been studied in [23, 36].

In Section 2 we present several possible definitions of norm-observability for nonlinear systems with inputs, which involve a bound on the norm of the state in terms of the norms of the output and the input on some (arbitrarily small) time interval. We establish implications and equivalences among these notions in Section 3. We demonstrate, in particular, that the length of the time interval can affect the existence of a state bound. Our formulation is related to the developments of [23, 36] and helps establish a link between observability and detectability in the nonlinear context. In fact, one of our definitions is obtained directly from the notion of input-output-to-state stability by imposing an additional requirement which says, loosely speaking, that the term describing the effects of initial conditions can be chosen to decay arbitrarily fast. In the spirit of [36], we derive a Lyapunov-like sufficient condition for this property in Section 4.

A problem of interest which serves as a motivation for studying these concepts is to extend LaSalle's invariance principle to switched systems. As shown in [17], a switched linear system is globally asymptotically stable if each subsystem possesses a weak Lyapunov function nonincreasing along its solutions and is observable with respect to the derivative of this function, as long as one imposes a suitable non-chattering assumption on the switching signal and a coupling assumption on the multiple Lyapunov functions. This can be viewed as an invariance-like principle for switched linear systems. We generalize this result to switched nonlinear systems in Section 5, exploiting the norm-observability notions introduced here. Stability theorems of LaSalle type for switched systems are useful in situations where it is natural—or even necessary—to work with weak Lyapunov functions. In Section 6, we discuss several specific examples of such situations arising in the context of stability analysis as well as control design in the presence of switching.

Preliminary versions of our findings for systems without inputs were stated in [18]. The more general treatment presented here relies on a novel result about boundedness of reachable sets for systems with inputs, which is proved in the Appendix.

2 Observability properties

Consider the system

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x,u) \end{aligned} \tag{1}$$

where u is a measurable locally essentially bounded disturbance or control input taking values in a set $\mathcal{U} \subset \mathbb{R}^m$ containing the origin, $f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^n$ is a locally Lipschitz function with $f(0,0) = 0$, and

$h : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^p$ is a continuous function with $h(0,0) = 0$. We assume that this system has the *unboundedness observability* property, which means that for each trajectory that becomes unbounded in a finite time T, the output becomes unbounded as $t \to T$ (in the sense that $\limsup_{t\to T} |y(t)| = \infty$). The unboundedness observability property is strictly weaker than *forward completeness*, which is the property that each trajectory is defined for all $t \geq 0$; see, e.g., [3]. We also assume unboundedness observability of the backward-in-time dynamics $\dot{x} = -f(x,u)$, $y = h(x,u)$. We will denote by $\|z\|_J$ the (essential) supremum norm of a signal z on an interval $J \subset [0,\infty)$. The standard Euclidean norm will be denoted by $|\cdot|$ and the corresponding induced matrix norm by $\|\cdot\|$.

Our first observability definition involves a bound on the norm of the initial state in terms of the norms of the output and the input on an arbitrarily small time interval. We will say that the system (1) is *small-time initial-state norm-observable* if the following property holds[1]:

$$\forall\, \tau > 0 \quad \exists\, \gamma, \chi \in \mathcal{K}_\infty \text{ such that } \; |x(0)| \leq \gamma(\|y\|_{[0,\tau]}) + \chi(\|u\|_{[0,\tau]}) \quad \forall\, x(0),\, u. \tag{2}$$

Rather than bounding the initial state in terms of the future output and input on an interval, we can bound the state at the end of an interval in terms of the past output and input on that interval. Let us say that the system (1) is *small-time final-state norm-observable* if

$$\forall\, \tau > 0 \quad \exists\, \gamma, \chi \in \mathcal{K}_\infty \text{ such that } \; |x(\tau)| \leq \gamma(\|y\|_{[0,\tau]}) + \chi(\|u\|_{[0,\tau]}) \quad \forall\, x(0),\, u. \tag{3}$$

We now define a different pair of observability properties, similar to the above, as follows. Let us say that the system (1) is *large-time initial-state norm-observable* if

$$\exists\, \tau > 0,\, \gamma, \chi \in \mathcal{K}_\infty \text{ such that } \; |x(0)| \leq \gamma(\|y\|_{[0,\tau]}) + \chi(\|u\|_{[0,\tau]}) \quad \forall\, x(0),\, u. \tag{4}$$

Note that the only difference between the conditions (2) and (4) is that in the former the length τ of the time interval can be arbitrary, while the latter requires the inequalities to hold for at least one positive τ (of course, they will then also hold for all larger values of τ). For linear systems these two properties are known to be equivalent, but for nonlinear systems this is in general not true, as we will see below.

As before, we can bound the state in terms of past output and input rather than future output and input. We will say that the system (1) is *large-time final-state norm-observable* if

$$\exists\, \tau > 0,\, \gamma, \chi \in \mathcal{K}_\infty \text{ such that } \; |x(\tau)| \leq \gamma(\|y\|_{[0,\tau]}) + \chi(\|u\|_{[0,\tau]}) \quad \forall\, x(0),\, u. \tag{5}$$

The above terminology is prompted by the one used in the controllability literature [15].

Continuing along the same lines, we can impose a bound on the initial state in terms of the output and the input on the semi-infinite time interval. Let us say that the system (1) is *infinite-time norm-observable* if

$$\exists\, \gamma, \chi \in \mathcal{K}_\infty \text{ such that } \; |x(0)| \leq \gamma(\|y\|_{[0,\infty)}) + \chi(\|u\|_{[0,\infty)}) \quad \forall\, x(0),\, u. \tag{6}$$

In [36], the authors define the property of input-output-to-state stability, which is a variant of detectability and is characterized by an inequality of the form

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma(\|y\|_{[0,t]}) + \chi(\|u\|_{[0,t]}) \quad \forall\, x(0),\, u,\, t \geq 0 \tag{7}$$

[1]Recall that a function $\alpha : [0,\infty) \to [0,\infty)$ is said to be of class $\mathcal{K}$ if it is continuous, strictly increasing, and $\alpha(0) = 0$. If α is also unbounded, then it is said to be of class $\mathcal{K}_\infty$. A function $\beta : [0,\infty) \times [0,\infty) \to [0,\infty)$ is said to be of class $\mathcal{KL}$ if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each fixed $t \geq 0$ and $\beta(r,t)$ decreases to 0 as $t \to \infty$ for each fixed $r \geq 0$. We will use the shorthand notation $\alpha \in \mathcal{K}_\infty$, $\beta \in \mathcal{KL}$. We will exploit the simple fact that for every class $\mathcal{K}$ function α and arbitrary positive numbers $r_1, r_2, \ldots, r_k$ we have $\alpha(r_1 + \cdots + r_k) < \alpha(kr_1) + \cdots + \alpha(kr_k)$.

where $\beta \in \mathcal{KL}$ and $\gamma, \chi \in \mathcal{K}_\infty$. Strengthening this notion, we say that the system (1) is *small-time-$\mathcal{KL}$ norm-observable* if for every $\varepsilon > 0$ and every function $\nu \in \mathcal{K}$ there exist functions $\beta \in \mathcal{KL}$ and $\gamma, \chi \in \mathcal{K}_\infty$ such that the inequality (7) holds along all solutions and, moreover, we have

$$\beta(r, \varepsilon) \leq \nu(r) \quad \forall\, r \geq 0. \tag{8}$$

The condition (8) can be interpreted as saying that β can be chosen to decay arbitrarily fast in the second argument, because ε can be arbitrarily small and ν can grow arbitrarily slowly while $\beta(r, 0) \geq r$. (In general, this is achieved at the expense of choosing γ and χ to be sufficiently large.)

In the same spirit as before, we introduce a variant of the above property by requiring that (8) hold for all $\nu \in \mathcal{K}$ and at least one positive ε (it will then hold for all larger values of ε). Namely, we will say that the system (1) is *large-time-$\mathcal{KL}$ norm-observable* if there exists an $\varepsilon > 0$ such that for every function $\nu \in \mathcal{K}$ there exist functions $\beta \in \mathcal{KL}$ and $\gamma, \chi \in \mathcal{K}_\infty$ for which the conditions (7) and (8) are satisfied.

Remark 1 One could also define local variants of the above observability properties, by allowing the functions γ and χ to be undefined (or take the value ∞) outside some bounded intervals. The corresponding definitions can also be rewritten in more familiar ε, δ terms. For example, the local variant of the small-time initial-state norm-observability property (2) would be

$$\forall\, \tau > 0,\, \varepsilon > 0 \quad \exists\, \delta > 0 \text{ such that } \|y\|_{[0,\tau]}, \|u\|_{[0,\tau]} \leq \delta \Rightarrow |x(0)| \leq \varepsilon.$$

Such a weaker observability notion (for systems with no inputs) was considered in [39]. □

State bounds provided by the above definitions involve the norms of the output and the input on some time interval of positive length. An alternative way to define observability, which is known to work for linear systems, is to demand that the state be bounded in terms of the instantaneous values of the output, the input, and a suitable number of their derivatives. We do not study such observability properties here. Notions of this kind are frequently encountered in the nonlinear control literature (see, e.g., [30, 13, 37, 27]).

For the system with no inputs

$$\begin{aligned} \dot{x} &= f(x) \\ y &= h(x) \end{aligned} \tag{9}$$

one can define corresponding observability notions by simply dropping the terms depending on u from the right-hand sides; see [18] for details. In this way one also obtains stronger versions of the above observability notions for the system (1), which require that the state be bounded in terms of the output only, uniformly over all inputs. These settings fit into our more general framework; we will return to this topic at the end of Section 3.

3 Implications and equivalences

In this section we study the relationships among the observability properties introduced in Section 2.

Technical lemmas

The following lemma is proved in the Appendix.

Lemma 1 *For every $\tau > 0$ there exist functions $\nu_f, \gamma_f, \chi_f \in \mathcal{K}_\infty$ such that along all solutions of the system (1) we have*

$$|x(t_2)| \leq \nu_f(|x(t_1)|) + \gamma_f(\|y\|_{[t_1,t_2]}) + \chi_f(\|u\|_{[t_1,t_2]}) \tag{10}$$

for each pair of times t_1, t_2 satisfying $t_1 \geq 0$ and $t_2 \in [t_1, t_1 + \tau]$.

We also need the backward-in-time counterpart.

Lemma 2 *For every $\tau > 0$ there exist functions $\nu_b, \gamma_b, \chi_b \in \mathcal{K}_\infty$ such that along all solutions of the system (1) we have*

$$|x(t_1)| \le \nu_b(|x(t_2)|) + \gamma_b(\|y\|_{[t_1,t_2]}) + \chi_b(\|u\|_{[t_1,t_2]})$$

for each pair of times t_1, t_2 satisfying $t_1 \ge 0$ and $t_2 \in [t_1, t_1 + \tau]$.

Small-time norm-observability

It is useful and simple to give several slight reformulations of the small-time initial-state norm-observability property. By time-invariance, the condition (2) can be equivalently expressed as

$$\forall\, \tau > 0 \quad \exists\, \gamma, \chi \in \mathcal{K}_\infty \text{ such that } \ |x(t)| \le \gamma(\|y\|_{[t,t+\tau]}) + \chi(\|u\|_{[t,t+\tau]}) \quad \forall\, x(0),\, u,\, t \ge 0 \tag{11}$$

or, after taking the supremum over $t \in [t_1, t_2]$ for arbitrary $t_2 \ge t_1 \ge 0$, as

$$\forall\, \tau > 0 \quad \exists\, \gamma, \chi \in \mathcal{K}_\infty \text{ such that } \ \|x\|_{[t_1,t_2]} \le \gamma(\|y\|_{[t_1,t_2+\tau]}) + \chi(\|u\|_{[t_1,t_2+\tau]}) \quad \forall\, x(0),\, u,\, t_2 \ge t_1 \ge 0. \tag{12}$$

This last condition includes (2) as a special case (just let $t_1 = t_2 = 0$), and so it is easy to see that (2), (11), and (12) are equivalent.

Similar remarks apply to small-time final-state norm-observability. By time-invariance, (3) is equivalent to

$$\forall\, \tau > 0 \quad \exists\, \gamma, \chi \in \mathcal{K}_\infty \text{ such that } \ |x(t)| \le \gamma(\|y\|_{[t-\tau,t]}) + \chi(\|u\|_{[t-\tau,t]}) \quad \forall\, x(0),\, u,\, t \ge \tau. \tag{13}$$

Taking the supremum over $t \in [t_1, t_2]$ for arbitrary $t_2 \ge t_1 \ge \tau$, we arrive at

$$\forall\, \tau > 0 \quad \exists\, \gamma, \chi \in \mathcal{K}_\infty \text{ such that } \ \|x\|_{[t_1,t_2]} \le \gamma(\|y\|_{[t_1-\tau,t_2]}) + \chi(\|u\|_{[t_1-\tau,t_2]}) \quad \forall\, x(0),\, u,\, t_2 \ge t_1 \ge \tau \tag{14}$$

and this includes (3) as a special case.

We show next that all the variants of small-time norm-observability introduced in Section 2 are in fact equivalent, and that they are also equivalent to another property which involves a bound on the norm of the state on an interval in terms of the norms of the output and the input on the same interval.

Proposition 3 *The following statements are equivalent:*

1. *The system (1) is small-time initial-state norm-observable.*
2. *The system (1) is small-time final-state norm-observable.*
3. *The system (1) satisfies the condition*

$$\forall\, \tau > 0 \quad \exists\, \gamma, \chi \in \mathcal{K}_\infty \textit{ such that } \ \|x\|_{[t_1,t_2]} \le \gamma(\|y\|_{[t_1,t_2]}) + \chi(\|u\|_{[t_1,t_2]}) \quad \forall\, x(0),\, u,\, t_1 \ge 0,\, t_2 \ge t_1 + \tau. \tag{15}$$

4. *The system (1) is small-time-$\mathcal{KL}$ norm-observable.*

Proof. The equivalences $1 \Leftrightarrow 2 \Leftrightarrow 3$ follow rather easily from Lemmas 1 and 2. Indeed, (2) implies (3) in view of Lemma 1, as can be seen from the inequalities

$$\begin{aligned} |x(\tau)| &\le \nu_f(|x(0)|) + \gamma_f(\|y\|_{[0,\tau]}) + \chi_f(\|u\|_{[0,\tau]}) \\ &\le \nu_f\big(\gamma(\|y\|_{[0,\tau]}) + \chi(\|u\|_{[0,\tau]})\big) + \gamma_f(\|y\|_{[0,\tau]}) + \chi_f(\|u\|_{[0,\tau]}) \\ &\le \bar{\gamma}(\|y\|_{[0,\tau]}) + \bar{\chi}(\|u\|_{[0,\tau]}) \end{aligned}$$

where $\bar{\gamma}(r) := \nu_f(2\gamma(r)) + \gamma_f(r)$ and $\bar{\chi}(r) := \nu_f(2\chi(r)) + \chi_f(r)$. This proves that $1 \Rightarrow 2$. The converse implication follows from Lemma 2 in the same manner. The fact that (12) implies (15) is deduced similarly with the help of Lemma 1, while the converse is straightforward. Since (12) is one of the equivalent formulations of small-time initial-state norm-observability, this proves that $1 \Leftrightarrow 3$.

We now turn to proving the more interesting fact that the first three properties are equivalent to the last one. Suppose that the system is small-time-$\mathcal{KL}$ norm-observable. Pick an arbitrary $\tau > 0$. By Lemma 2, there exist class $\mathcal{K}_\infty$ functions ν_b, γ_b and χ_b such that

$$|x(0)| \le \nu_b(|x(\tau)|) + \gamma_b(\|y\|_{[0,\tau]}) + \chi_b(\|u\|_{[0,\tau]}) \tag{16}$$

for all $x(0)$ and all u. By small-time-$\mathcal{KL}$ observability, for $\varepsilon := \tau$ and

$$\nu(r) := \frac{1}{3}\nu_b^{-1}(r/2) \tag{17}$$

there exist functions $\beta \in \mathcal{KL}$ and $\gamma, \chi \in \mathcal{K}_\infty$ such that the conditions (7) and (8) hold. In particular, (8) and (17) give

$$\nu_b(3\beta(|x(0)|, \tau)) \le |x(0)|/2 \tag{18}$$

while (16) and (7) with $t = \tau$ yield

$$\begin{aligned} |x(0)| &\le \nu_b\big(\beta(|x(0)|, \tau) + \gamma(\|y\|_{[0,\tau]}) + \chi(\|u\|_{[0,\tau]})\big) + \gamma_b(\|y\|_{[0,\tau]}) + \chi_b(\|u\|_{[0,\tau]}) \\ &\le \nu_b(3\beta(|x(0)|, \tau)) + \nu_b(3\gamma(\|y\|_{[0,\tau]})) + \nu_b(3\chi(\|u\|_{[0,\tau]})) + \gamma_b(\|y\|_{[0,\tau]}) + \chi_b(\|u\|_{[0,\tau]}). \end{aligned}$$

Using (18), we conclude that

$$|x(0)| \le \bar{\gamma}(\|y\|_{[0,\tau]}) + \bar{\chi}(\|u\|_{[0,\tau]})$$

where $\bar{\gamma}(r) := 2\nu_b(3\gamma(r)) + 2\gamma_b(r)$ and $\bar{\chi}(r) := 2\nu_b(3\chi(r)) + 2\chi_b(r)$. Therefore, the condition (2) holds, and so we have shown that $4 \Rightarrow 1$.

Finally, we prove that $2 \Rightarrow 4$, using the formulation of small-time final-state norm-observability provided by the condition (13). Pick arbitrary $\varepsilon > 0$ and $\nu \in \mathcal{K}$. Let $\tau := \varepsilon/2$. Applying (13), we conclude that there exist functions $\gamma, \chi \in \mathcal{K}_\infty$ such that all trajectories satisfy

$$|x(t)| \le \gamma(\|y\|_{[t-\varepsilon/2,t]}) + \chi(\|u\|_{[t-\varepsilon/2,t]}) \le \gamma(\|y\|_{[0,t]}) + \chi(\|u\|_{[0,t]}) \quad \forall\, t \ge \varepsilon/2. \tag{19}$$

Also, Lemma 1 gives

$$|x(t)| \le \nu_f(|x(0)|) + \gamma_f(\|y\|_{[0,t]}) + \chi_f(\|u\|_{[0,t]}) \quad \forall\, t \in [0, \varepsilon/2] \tag{20}$$

for some $\nu_f, \gamma_f, \chi_f \in \mathcal{K}_\infty$. Choose a function $\beta \in \mathcal{KL}$ such that for all $r \ge 0$ we have

$$\beta(r, \varepsilon/2) \ge \nu_f(r) \tag{21}$$

and

$$\beta(r, \varepsilon) = \nu(r/2). \tag{22}$$

One possible definition is

$$\beta(r, t) := \max\Big\{2\nu_f(r)\frac{\varepsilon - t}{\varepsilon}, \nu(r/2)\frac{2\varepsilon}{t + \varepsilon}\Big\}.$$

The condition (8) automatically holds because of (22). Now, for $t \in [0, \varepsilon/2]$ we can use (20) and (21) to write

$$\begin{aligned} |x(t)| &\le \nu_f(|x(0)|) + \gamma_f(\|y\|_{[0,t]}) + \chi_f(\|u\|_{[0,t]}) \\ &\le \beta(|x(0)|, \varepsilon/2) + \gamma_f(\|y\|_{[0,t]}) + \chi_f(\|u\|_{[0,t]}) \\ &\le \beta(|x(0)|, t) + \gamma_f(\|y\|_{[0,t]}) + \chi_f(\|u\|_{[0,t]}). \end{aligned}$$

Combining this with (19), we deduce that for all $t \geq 0$ we have the inequality

$$|x(t)| \leq \beta(|x(0)|, t) + \bar{\gamma}(\|y\|_{[0,t]}) + \bar{\chi}(\|u\|_{[0,t]})$$

where $\bar{\gamma} := \gamma_f + \gamma$ and $\bar{\chi} := \chi_f + \chi$. Therefore, the condition (7) holds. □

In view of Proposition 3, we will drop the unnecessary qualifiers and refer to each of the properties appearing in its statement as *small-time norm-observability*.

Remark 2 In the sequel, we will encounter a restricted version of small-time-$\mathcal{KL}$ norm-observability, which is limited to functions $\nu \in \mathcal{K}$ that are bounded from below by a linear function with a positive slope. A close examination of the proof of Proposition 3 (the implication 4 ⇒ 1) reveals that this weaker form of small-time-$\mathcal{KL}$ norm-observability is equivalent to the other small-time observability properties if we can take the function ν_b in Lemma 2 to be linear (or with a linear upper bound). This condition on ν_b holds true for every linear system and also for every nonlinear system with a uniform exponential bound on the growth of solutions. □

Large-time norm-observability

Our treatment of large-time norm-observability parallels that of small-time norm-observability. By time-invariance, large-time initial-state norm-observability defined by (4) is equivalent to

$$\exists\, \tau > 0,\ \gamma, \chi \in \mathcal{K}_\infty \text{ such that } |x(t)| \leq \gamma(\|y\|_{[t,t+\tau]}) + \chi(\|u\|_{[t,t+\tau]}) \quad \forall\, x(0),\ u,\ t \geq 0. \tag{23}$$

Taking the supremum over $t \in [t_1, t_2]$, we can further rewrite this as

$$\exists\, \tau > 0,\ \gamma, \chi \in \mathcal{K}_\infty \text{ such that } \|x\|_{[t_1,t_2]} \leq \gamma(\|y\|_{[t_1,t_2+\tau]}) + \chi(\|u\|_{[t_1,t_2+\tau]}) \quad \forall\, x(0),\ u,\ t_2 \geq t_1 \geq 0. \tag{24}$$

The condition (4) is a special case of (24), and we easily see that (4), (23), and (24) are equivalent.

Similarly, large-time final-state norm-observability defined by (5) can be equivalently rewritten as

$$\exists\, \tau > 0,\ \gamma, \chi \in \mathcal{K}_\infty \text{ such that } |x(t)| \leq \gamma(\|y\|_{[t-\tau,t]}) + \chi(\|u\|_{[t-\tau,t]}) \quad \forall\, x(0),\ u,\ t \geq \tau \tag{25}$$

or, taking the supremum over $t \in [t_1, t_2]$, as

$$\exists\, \tau > 0,\ \gamma, \chi \in \mathcal{K}_\infty \text{ such that } \|x\|_{[t_1,t_2]} \leq \gamma(\|y\|_{[t_1-\tau,t_2]}) + \chi(\|u\|_{[t_1-\tau,t_2]}) \quad \forall\, x(0),\ u,\ t_2 \geq t_1 \geq \tau. \tag{26}$$

The following result is established by the same arguments (for a given τ) as Proposition 3.

Proposition 4 *The following statements are equivalent:*

1. *The system* (1) *is large-time initial-state norm-observable.*
2. *The system* (1) *is large-time final-state norm-observable.*
3. *The system* (1) *satisfies the condition*

$$\exists\, \tau > 0,\ \gamma, \chi \in \mathcal{K}_\infty \text{ such that } \|x\|_{[t_1,t_2]} \leq \gamma(\|y\|_{[t_1,t_2]}) + \chi(\|u\|_{[t_1,t_2]}) \quad \forall\, x(0),\ u,\ t_1 \geq 0,\ t_2 \geq t_1 + \tau. \tag{27}$$

4. *The system* (1) *is large-time-$\mathcal{KL}$ norm-observable.*

We will henceforth refer to each of these properties as *large-time norm-observability*.

Infinite-time norm-observability

In the same way as before, we can rewrite the condition (6) as

$$\exists\,\gamma,\chi \in \mathcal{K}_\infty \text{ such that } \ |x(t)| \le \gamma(\|y\|_{[t,\infty)}) + \chi(\|u\|_{[t,\infty)}) \quad \forall\, x(0),\, u,\, t \ge 0 \tag{28}$$

or as

$$\exists\,\gamma,\chi \in \mathcal{K}_\infty \text{ such that } \ \|x\|_{[t_1,t_2]} \le \gamma(\|y\|_{[t_1,\infty)}) + \chi(\|u\|_{[t_1,\infty)}) \quad \forall\, x(0),\, u,\, t_2 \ge t_1 \ge 0. \tag{29}$$

In particular, taking $t_1 = 0$ and $t_2 = \infty$ in the last formula, we arrive at

$$\exists\,\gamma,\chi \in \mathcal{K}_\infty \text{ such that } \ \|x\|_{[0,\infty)} \le \gamma(\|y\|_{[0,\infty)}) + \chi(\|u\|_{[0,\infty)}) \quad \forall\, x(0),\, u \tag{30}$$

thus recovering precisely the *strong observability* property as defined in [32]. However, this terminology is not suitable here because, as we will see next, infinite-time norm-observability is actually the weakest among all the observability notions studied in this paper.

Relationships among the different notions

The following theorem is the main result of this section.

Theorem 5 *The only implications that hold among the observability properties defined above for the system* (1) *are:*

small-time norm-observability

$$\boxed{\underbrace{(2) \Leftrightarrow (11) \Leftrightarrow (12)}_{\text{initial-state}} \Leftrightarrow \underbrace{(3) \Leftrightarrow (13) \Leftrightarrow (14)}_{\text{final-state}} \Leftrightarrow (15) \Leftrightarrow \underbrace{\forall\,\varepsilon:\ (7)\&(8)}_{\mathcal{KL}}}$$

$$\Downarrow$$

large-time norm-observability

$$\boxed{\underbrace{(4) \Leftrightarrow (23) \Leftrightarrow (24)}_{\text{initial-state}} \Leftrightarrow \underbrace{(5) \Leftrightarrow (25) \Leftrightarrow (26)}_{\text{final-state}} \Leftrightarrow (27) \Leftrightarrow \underbrace{\exists\,\varepsilon:\ (7)\&(8)}_{\mathcal{KL}}}$$

$$\Downarrow$$

infinite-time norm-observability $\boxed{(6) \Leftrightarrow (28) \Leftrightarrow (29) \Leftrightarrow (30)}$

PROOF. We have already established all the equivalences for each property. The implications shown for the different properties are apparent from the definitions. It remains to prove that the converse implications do not hold.

We first give a counterexample showing that large-time norm-observability does not imply small-time norm-observability. It is enough to do this for the case of no inputs. Consider the system (9) with $x \in \mathbb{R}$ and f and h both odd functions satisfying

$$f(0) = 0, \qquad f(x) = 1 \quad \text{if } \ x \ge 1$$

and

$$h(x) = x \quad \text{if} \;\; x \in [0,1], \qquad h(x) = 0 \quad \text{if} \;\; x \in [2,3], \qquad h(x) = x - 3 \quad \text{if} \;\; x \geq 4$$

respectively (extended arbitrarily to those x for which their values are not specified; see Figure 1, left). Then it is straightforward to verify that the condition (4) is satisfied with $\tau = 3$ and γ the identity function (omitting u). Indeed, for $x(0) \in [0,1]$ we have $y(0) = x(0)$, whereas for $x(0) > 1$ we have $x(3) > 4$ hence $y(3) = x(3) - 3 = x(0)$; the arguments for $x(0) < 0$ are similar. On the other hand, the condition (2) does not hold for any $\tau \leq 1$, because for $x(0) = 2$ we have $y(t) \equiv 0$ on $[0,1]$.

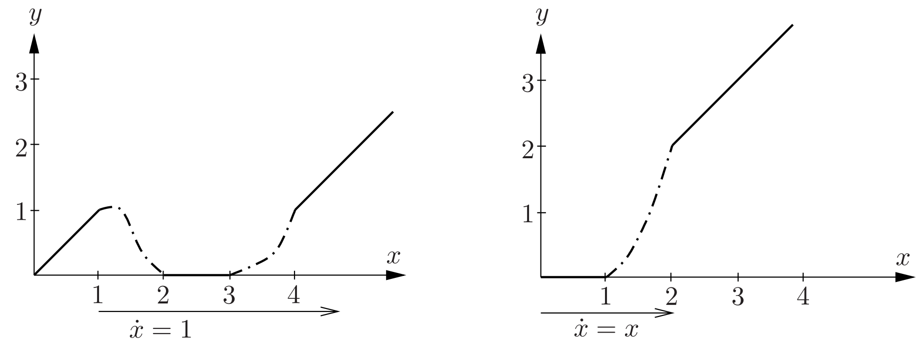

Figure 1: The two counterexamples

As a counterexample showing that infinite-time norm-observability does not imply large-time norm-observability, consider again the system (9) with $x \in \mathbb{R}$ and f and h both odd functions such that

$$f(x) = x \quad \text{if} \;\; x \in [0,2]$$

(the behavior of f for $x > 2$ is unimportant) and

$$h(x) = 0 \quad \text{if} \;\; x \in [0,1], \qquad h(x) \in [0,2] \quad \text{if} \;\; x \in [1,2], \qquad h(x) = x \quad \text{if} \;\; x \geq 2$$

respectively (see Figure 1, right). Then the condition (4) cannot hold for any $\tau > 0$, no matter how large, since for every $\tau > 0$ we can find a sufficiently small $x(0) > 0$ such that along the corresponding solution we have $\|y\|_{[0,\tau]} = 0$. On the other hand, (30) is satisfied because we have $\|x\|_{[0,\infty)} = \|y\|_{[0,\infty)}$. (In fact, by a proper choice of f we can ensure that these norms are finite for all initial conditions.) $\square$

Remark 3 It is easy to see that each of the observability properties considered here implies 0-distinguishability under the zero input: the only subset of $\ker h(\cdot, 0)$ which is invariant under the zero input is $\{0\}$. Note that the converse does not hold. As an example, consider the scalar system $\dot{x} = x$, $y = \arctan x$. It is clearly 0-distinguishable (in fact, distinguishable: the output map is invertible), but x blows up while y stays bounded. $\square$

Remark 4 For linear systems, all of the above properties are equivalent to the usual observability. For the properties expressed by the conditions (2)–(6) this can be easily shown using the observability Gramian. For (small-time-)$\mathcal{KL}$ norm-observability this fact is less obvious, and can be viewed as a generalization of the *squashing lemma* from [29] which is a refinement of the well-known result about arbitrary pole placement by output injection. This lemma says that if (C, A) is an observable pair, then for every $\varepsilon > 0$ and every $\delta > 0$ there exist a $\lambda > 0$ and an output injection matrix K such that we have $\|e^{(A+KC)t}\| \leq \delta e^{-\lambda(t-\varepsilon)}$, which implies $\|e^{(A+KC)\varepsilon}\| \leq \delta$. Therefore, in the linear case the function β in (7) can be chosen to satisfy $\beta(r, \varepsilon) \leq \delta r$ with δ arbitrarily small. This means that the condition (8) can be fulfilled if and only if the function ν is restricted to be bounded from below by a linear function with a positive slope. We know from

Remark 2 that for linear systems, this restricted form of small-time-$\mathcal{KL}$ norm-observability is equivalent to the other small-time observability properties, because we can take the function ν_b in Lemma 2 to be linear. □

Remark 5 It is interesting to compare our findings with the results reported in [31] for discrete-time systems. For example, the counterparts of initial-state observability and final-state observability in discrete time are not equivalent. To see why, it is enough to consider a system whose output map is zero and whose state becomes zero after one step, regardless of the input. (The difference with the above continuous-time setting is that backward-in-time dynamics in this example are not well defined.) □

Uniform observability and systems with no inputs

Now, consider the observability notions obtained by omitting the term of the form $\chi(\|u\|_J)$ from the right-hand sides of the inequalities (2)–(7) and demanding that the inequalities hold for the system (1) uniformly over all inputs or, for the system without inputs (9), simply over all trajectories (cf. the end of Section 2). Then the same implications among the resulting properties as those in Theorem 5 remain valid, provided that $\mathcal{U}$ is a compact set and $f(0,u) = 0$ for all $u \in \mathcal{U}$ when inputs are present. This is proved by the same arguments with suitable minor modifications, using appropriate versions of Lemmas 1 and 2 which are mentioned in Remark 12 in the Appendix; compactness of $\mathcal{U}$ is required for these lemmas to hold. An explicit treatment of (forward and backward complete) systems with no inputs can be found in [18]. The terms *small-time norm-observability* and *large-time norm-observability* will also be used for the corresponding variants of the observability notions for systems with no inputs. When the system with inputs (1) satisfies these observability properties without the input-dependent terms on the right-hand sides, we will call it *uniformly small-time norm-observable* or *uniformly large-time norm-observable*.

4 Lyapunov functions

One advantage of defining observability via the formulas (7) and (8) lies in the fact that this leads to characterizations of small-time and large-time norm-observability in terms of Lyapunov-like inequalities, as we now show.

Proposition 6 *Consider the system* (1). *Suppose that for every $\varepsilon > 0$ and every $\nu \in \mathcal{K}$ there exist a C^1 function $V : \mathbb{R}^n \to \mathbb{R}$, class $\mathcal{K}_\infty$ functions α_1, α_2 and ρ, and a positive definite continuous function $\alpha_3 : [0,\infty) \to [0,\infty)$ such that we have*

$$\alpha_1(|x|) \le V(x) \le \alpha_2(|x|) \tag{31}$$

and

$$|x| \ge \rho\left(\left|\begin{pmatrix} y \\ u \end{pmatrix}\right|\right) \Rightarrow \frac{\partial V}{\partial x} f(x,u) \le -\alpha_3(V(x)) \tag{32}$$

and moreover

$$\eta^{-1}(\eta(r) + \varepsilon) \le \alpha_1 \circ \nu \circ \alpha_2^{-1}(r) \quad \forall\, r \ge 0 \tag{33}$$

where η is defined by[2]

$$\eta(r) := -\int_1^r \frac{ds}{\alpha_3(s)}.$$

Then the system is small-time norm-observable. If the above conditions hold for some $\varepsilon > 0$ and every $\nu \in \mathcal{K}$, then the system is large-time norm-observable.

[2]Decreasing α_3 near zero if necessary, we can assume with no loss of generality that $\lim_{r\to 0^+} \eta(r) = \infty$; see [28, proof of Lemma 4.4]. We thus use the conventions $\eta(0) = \infty$ and $\eta^{-1}(\infty) = 0$ which are consistent with continuity of η.

Proof. Let $\varepsilon > 0$ and $\nu \in \mathcal{K}$ be given and suppose that all the conditions in the statement are satisfied. Following the proofs of [28, Lemma 4.4] and [36, Lemma 11], we conclude that all solutions of (1) satisfy the inequality (7) with

$$\beta(r,t) = \alpha_1^{-1} \circ \eta^{-1}(\eta \circ \alpha_2(r) + t) \tag{34}$$

and

$$\gamma(r) = \alpha_1^{-1} \circ \alpha_2 \circ \rho(r).$$

To derive the condition (8), simply write (33) for $\alpha_2(r)$ in place of r and substitute it into (34) written for $t = \varepsilon$. □

An informal interpretation of Proposition 6 is that the system is small-time norm-observable if there exists a positive definite radially unbounded function V which decays along solutions whenever $|x|$ is sufficiently large compared to $|y|$ and $|u|$ and, moreover, this decay rate—described by the function α_3—can be made arbitrarily fast by a proper choice of V. (The "gain margin" function ρ, on the other hand, may have to be increased in order to achieve this; note that the extra condition (33) does not involve ρ.) To better understand the role of α_3, note that if α_3 grows rapidly, then the graph of η is "flat", and consequently the function $\eta^{-1}(\eta(\cdot) + \varepsilon)$ is small. In fact, this function is approximated, up to the first-order term in ε, by $r - \alpha_3(r)\varepsilon$. It is straightforward to obtain a counterpart of Proposition 6 for the case when the observability notions do not involve inputs, by simply dropping u from the formula (32).

Example 1 Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - x_2 - \varphi(x_1) \\ y &= x_2 \end{aligned}$$

where the function φ satisfies the sector condition

$$|\varphi(x_1)| \le \delta |x_1|, \quad \delta \in (0,1). \tag{35}$$

Systems of this form are frequently encountered as models of mass-spring systems and electrical circuits. Let us rewrite this system in the "output-injected" form

$$\begin{aligned} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} &= A_k \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} 0 \\ \varphi(x_1) \end{pmatrix} + Ky \\ y &= x_2 \end{aligned} \tag{36}$$

where

$$A_k := \begin{pmatrix} 0 & 1 + 4\lambda_k^2 \\ -1 & -1-k \end{pmatrix}, \quad K := \begin{pmatrix} -4\lambda_k^2 \\ k \end{pmatrix}, \quad \lambda_k := \frac{k+1}{4}$$

and k is an arbitrary positive number. It is straightforward to check that the derivative of the positive definite quadratic function

$$V_k := x_1^2 + 2\lambda_k x_1 x_2 + (1 + 2\lambda_k^2) x_2^2$$

along solutions of the system (36) is given by

$$\dot{V}_k = -2\lambda_k V_k - 2\lambda_k x_1 \varphi(x_1) + (2\lambda_k k - 8\lambda_k^2) x_1 x_2 - 2(1 + 2\lambda_k^2) x_2 \varphi(x_1) + 2(k - 2\lambda_k + 2\lambda_k^2 k - 6\lambda_k^3) x_2^2. \tag{37}$$

Using this formula and the inequality (35), it can be easily shown that given an arbitrary number $c > 0$, we can choose sufficiently large numbers k and d to have

$$|x_1| \ge d|x_2| \Rightarrow \dot{V}_k \le -cV_k.$$

For each k, the function $V = V_k$ satisfies the conditions (31) and (32) with $\alpha_1(r) = a_1 r^2$ and $\alpha_2(r) = a_2 r^2$, for some $a_2 > a_1 > 0$, and $\alpha_3(r) = cr$ so that $\eta^{-1}(\eta(r) + \varepsilon) = e^{-c\varepsilon} r$. By enforcing a sufficiently large c, we can satisfy the condition (33) if and only if ν is bounded from below by a linear function with a positive slope. In view of Remark 2, this implies that the system is small-time norm-observable, because it is clear from (35) and (37) that the state of (36) admits a uniform exponential growth bound. □

We remark also that the condition (32) with $\alpha_3(r) = cr$ leads naturally to the design of a dynamic state-norm estimator for (1). This is a system with inputs y and u and state z such that for some $\beta \in \mathcal{KL}$ and $\mu \in \mathcal{K}_\infty$, the inequality $|x(t)| \leq \beta(|x(0)| + |z(0)|, t) + \mu(|z(t)|)$ holds along all solutions; see [36]. Rewriting (32) as $\dot{V} \leq -cV + \alpha_4(|y|) + \alpha_5(|u|)$ for suitable $\alpha_4, \alpha_5 \in \mathcal{K}_\infty$, we are led to the norm estimator $\dot{z} = -cz + \alpha_4(|y|) + \alpha_5(|u|)$. If c can be made arbitrarily large, then $V(x(t)) - z(t)$ converges to 0 arbitrarily fast. On the other hand, one can loosely interpret the small-time norm-observability property as providing an arbitrarily fast norm estimator, obtained directly from the definitions rather than constructed using Lyapunov functions.

5 Stability of switched systems: theorems of LaSalle type

Systems with no inputs

Consider the system $\dot{x} = f(x)$, $x \in \mathbb{R}^n$. One version (in fact, a special case) of the well-known LaSalle's invariance principle[3] can be stated as follows. If there exists a positive definite, radially unbounded, continuously differentiable (C^1) function $V : \mathbb{R}^n \to \mathbb{R}$ whose derivative along solutions satisfies $\frac{\partial V}{\partial x} f(x) \leq -W(x) \leq 0$ for all x, and if moreover the largest invariant set contained in the set $\{x : W(x) = 0\}$ is equal to $\{0\}$, then the system is globally asymptotically stable. The second condition can be regarded as observability (0-distinguishability) with respect to the auxiliary output $y := -W(x)$; see, e.g., [5] for a discussion of this relationship. Here and below, the negative sign is used for convenience, so that $y \geq 0$.

In this section we extend the above result to switched systems. This generalizes the earlier work on switched linear systems reported in [17]. Some remarks on relationships to other LaSalle-like theorems available in the literature are provided afterwards.

Consider the family of systems

$$\dot{x} = f_p(x), \quad p \in \mathcal{P} \tag{38}$$

where $\mathcal{P}$ is a finite index set and $f_p : \mathbb{R}^n \to \mathbb{R}^n$ is a locally Lipschitz function for each $p \in \mathcal{P}$. We impose the following two assumptions on each of these systems, which parallel the assumptions for the traditional LaSalle's theorem stated above. The first assumption is the existence of a weak (i.e., nonstrictly decreasing) Lyapunov function, and the second one is observability with respect to an auxiliary output defined exactly as shown earlier for the case of a single system (however, instead of 0-distinguishability we require the stronger small-time norm-observability property; see Remark 3 in Section 3.)

Assumption 1. For each $p \in \mathcal{P}$ there exist a positive definite radially unbounded C^1 function $V_p : \mathbb{R}^n \to \mathbb{R}$ and a nonnegative definite continuous function $W_p : \mathbb{R}^n \to \mathbb{R}$ such that

$$\frac{\partial V_p}{\partial x} f_p(x) \leq -W_p(x) \quad \forall\, x.$$

Assumption 2. For each $p \in \mathcal{P}$ the system

$$\begin{aligned} \dot{x} &= f_p(x) \\ y &= W_p(x) \end{aligned} \tag{39}$$

[3] See, e.g., [22, Section 4.3]. Although we make reference to LaSalle [24], the particular result stated here was proved earlier by Barbashin and Krasovskii [7].

is small-time norm-observable. For our present purposes, the most convenient way to state this property is via the condition (11), which in the absence of inputs reads

$$\forall\,\tau > 0 \quad \exists\,\gamma \in \mathcal{K}_\infty \text{ such that } |x(t)| \le \gamma(\|y\|_{[t,t+\tau]}) \quad \forall\, x(0),\, t \ge 0. \tag{40}$$

Since $\mathcal{P}$ is a finite set, there is no loss of generality in taking γ to be independent of p.

Remark 6 In view of Remark 3 and the standard LaSalle's theorem cited earlier, Assumptions 1 and 2 imply that all systems in the family (38) are globally asymptotically stable. □

We now consider the *switched system*

$$\dot{x} = f_\sigma(x) \tag{41}$$

where $\sigma : [0,\infty) \to \mathcal{P}$ is a piecewise constant *switching signal*, continuous from the right. We denote by t_i, $i = 1, 2, \ldots$ the consecutive discontinuities of σ (the *switching times*). With regard to this switched system, two more assumptions are needed. The first one is a rather mild non-chattering requirement on σ, which will be further examined below.

Assumption 3. If there are infinitely many switching times, there exists a $\tau > 0$ such that for every $T \ge 0$ we can find a positive integer i for which $t_{i+1} - \tau \ge t_i \ge T$. In other words, we persistently encounter intervals of length at least τ between switching times.

Our last assumption imposes a condition on the evolution of the functions V_p, $p \in \mathcal{P}$ at switching times, of the type typically encountered in results involving multiple Lyapunov functions (see [12, 16, 26]). It is trivially satisfied in the case of a common weak Lyapunov function, i.e., when the functions V_p, $p \in \mathcal{P}$ are the same.

Assumption 4. For each $p \in \mathcal{P}$ and every pair of switching times $t_i < t_j$ such that $\sigma(t_i) = \sigma(t_j) = p$, we have

$$V_p(x(t_j)) \le V_p(x(t_{i+1})). \tag{42}$$

In other words, the value of V_p at the beginning of each interval on which $\sigma = p$ does not exceed the value of V_p at the end of the previous such interval (if one exists).

Remark 7 Note that when the switching is controlled by a supervisor, Assumption 4 can be explicitly incorporated into the switching logic so that it holds by construction. For example, suppose that for some times $\bar{t} > t_i$ and indices $p, q \in \mathcal{P}$ we have $\sigma(t_i) = p$, $\sigma(\bar{t}) = q$, and $V_p(x(\bar{t})) > V_p(x(t_{i+1}))$. Then a switch to $\sigma = p$ at time $t_j := \bar{t}$ is disallowed because it would violate (42). However, the system $\dot{x} = f_q(x)$ is globally asymptotically stable by Remark 6. Thus we know that if σ is held fixed at q, then $x(t)$ will decay to 0 and there will be another time $t_j > \bar{t}$ at which (42) will hold. From that time onward, a switch to $\sigma = p$ can be enabled. Note that by ensuring that the switching is not too fast, we also simultaneously enforce Assumption 3 (see below for more details on this issue). Supervisory control algorithms with switchings triggered by values of Lyapunov functions, although with a different purpose, are considered in [14, 2]. □

Theorem 7 *Under Assumptions 1–4 the switched system* (41) *is globally asymptotically stable.*

Proof. Stability of the origin in the sense of Lyapunov follows from Assumptions 1 and 4 and the finiteness of $\mathcal{P}$ by virtue of [9, Theorem 2.3]. Now, take an arbitrary solution of (41). Our goal is to prove that it converges to 0. We are assuming that there are infinitely many switching times, for otherwise the result immediately follows from Remark 6. In light of Assumption 3 and the fact that $\mathcal{P}$ is finite, we can pick an infinite subsequence of switching times $t_{i_1}, t_{i_2}, \ldots$ such that the corresponding intervals $[t_{i_j}, t_{i_j+1})$,

$j = 1, 2, \ldots$ have length no smaller than some fixed $\tau > 0$ and the value of σ on all these intervals is the same, say, $q \in \mathcal{P}$. Let us denote the union of these intervals by $\mathcal{Q}$ and consider the auxiliary function

$$y_{\mathcal{Q}}(t) := \begin{cases} W_q(x(t)) & \text{if } t \in \mathcal{Q} \\ 0 & \text{otherwise} \end{cases}$$

In view of Assumptions 1 and 4, for every $t \geq 0$ we have

$$\int_0^t y_{\mathcal{Q}}(s)ds \leq V_q(x(t_{i_1})) - V_q(x(t)) \leq V_q(x(t_{i_1})).$$

Since $y_{\mathcal{Q}}$ is nonnegative by Assumption 1, we see that $y_{\mathcal{Q}} \in \mathcal{L}_1$.

We proceed to prove[4] that $y_{\mathcal{Q}}(t) \to 0$ as $t \to \infty$. Suppose that this is not true. Then there exist an $\varepsilon > 0$ and an infinite sequence of times $s_1, s_2, \ldots$ such that the values $y_{\mathcal{Q}}(s_1)$, $y_{\mathcal{Q}}(s_2)$, $\ldots$ are bounded away from zero by at least ε. It follows from the definition of $y_{\mathcal{Q}}$ that the times $s_1, s_2, \ldots$ necessarily belong to $\mathcal{Q}$. Assumption 1 guarantees that x remains bounded, hence $\dot{x}$ is also bounded and so $y_{\mathcal{Q}}$ is uniformly continuous on $\mathcal{Q}$. Therefore, we can find a $\delta > 0$ such that each s_i is contained in some interval of length δ on which $y_{\mathcal{Q}}(t) \geq \varepsilon/2$ (recall that the length of each interval in $\mathcal{Q}$ is bounded from below by τ). This contradicts the assertion proved earlier that $y_{\mathcal{Q}} \in \mathcal{L}_1$, thus indeed $y_{\mathcal{Q}}(t) \to 0$.

To show that $x(t)$ converges to 0, we invoke Assumption 2. Applying the condition (40) with $t = t_{i_j}$, $j = 1, 2, \ldots$ and using the above analysis, we conclude that $x(t_{i_j}) \to 0$ as $j \to \infty$. It then follows from stability of the origin in the sense of Lyapunov that $x(t) \to 0$ as needed. □

Remark 8 If we strengthen Assumption 4 by assuming that $V_\sigma(x)$ is nonincreasing in time, then the finiteness requirement on $\mathcal{P}$ can be dropped, provided that γ in (40) is still independent of p. This applies in particular to the case of a common weak Lyapunov function. The proof proceeds along the above lines but works with $y := W_\sigma(x)$ rather than $y_{\mathcal{Q}}$ (cf. the linear version in [17, Theorem 4]). □

Remark 9 The above theorem asserts global asymptotic stability (in the standard sense of this term as applied to time-varying systems) for each fixed switching signal σ satisfying Assumption 3. While stability is uniform over all such σ, asymptotic convergence is not. For linear systems, the convergence is uniform over a smaller class of switching signals (obtained by imposing an upper bound on the separation between consecutive intervals of length at least τ in Assumption 3), provided that Assumption 4 is also strengthened; see [17] for details. □

Remark 10 It is clear from the above proof that a local version of the small-time norm-observability condition (40), defined as explained in Remark 1 and in [39], is sufficient for Theorem 7 to hold. In [39], a similar argument was used to establish that a suitably defined passivity property implies closed-loop stability under negative output feedback for switched control systems. □

One way to satisfy Assumption 3 is to demand that consecutive switching times be separated by some positive *dwell time* τ_D. A less severe condition is provided by the following concept, introduced in [19]. The switching signal σ is said to have *average dwell time* $\tau_{AD} > 0$ if the number of its discontinuities on an arbitrary interval (t_1, t_2), which we denote by $N_\sigma(t_2, t_1)$, satisfies

$$N_\sigma(t_2, t_1) \leq N_0 + \frac{t_2 - t_1}{\tau_{AD}} \tag{43}$$

for some $N_0 > 0$. Under suitable conditions, a useful class of hysteresis-based switching logics is known to guarantee the average dwell time property [19, 26].

[4]The argument that follows mimics the standard proof of the so-called Barbalat's lemma, which cannot be directly applied in the present case because y is not continuous.

Lemma 8 *If σ has average dwell time τ_{AD}, then Assumption 3 holds with an arbitrary τ in the interval $(0, \tau_{AD})$.*

PROOF. Suppose the contrary: there exist numbers $\tau \in (0, \tau_{AD})$ and $T \geq 0$ such that for $t \geq T$ all intervals between switching times have length smaller than τ. In this case, for every positive integer k the interval $(T, T + k\tau)$ must contain at least k switching times. The formula (43) then implies

$$k \leq N_0 + \frac{k\tau}{\tau_{AD}}$$

but this is clearly false for sufficiently large k. □

Note that the average dwell time τ_{AD} in the above result can be arbitrarily small, as long as it exists. If τ_{AD} is known, then we can relax Assumption 2 by requiring only that the system (39) be large-time norm-observable with $\tau < \tau_{AD}$. Accordingly, if this system is known to be large-time norm-observable but not small-time norm-observable, then a variant of Theorem 7 can be established under a suitable slow switching condition. We thus introduce the following modified versions of Assumptions 2 and 3.

Assumption 2′. For each $p \in \mathcal{P}$ the system (39) is large-time norm-observable, which can be stated as:

$$\exists\, \tau > 0,\ \gamma \in \mathcal{K}_\infty \text{ such that } \ |x(t)| \leq \gamma(\|y\|_{[t,t+\tau]}) \quad \forall\, x(0),\ t \geq 0. \tag{44}$$

Assumption 3′. If there are infinitely many switching times, for every $T \geq 0$ we can find a positive integer i for which $t_{i+1} - \tau \geq t_i \geq T$. Here τ is the number provided by (44), which we take to be independent of p (this is no loss of generality since $\mathcal{P}$ is a finite set).

The following result is proved by the same arguments as Theorem 7.

Theorem 9 *Under Assumptions 1, 2′, 3′, and 4 the switched system* (41) *is globally asymptotically stable.*

A different version of LaSalle's invariance principle for systems with switching events has appeared in [38, Theorem 1]. That result states that if for a given hybrid system with a finite number of discrete states one can find a function of both the continuous and the discrete state which is nonincreasing along solutions, then all bounded solutions approach the largest invariant set inside the set of states where the instantaneous change of this function is zero. The proof proceeds along the same lines as the standard argument for continuous time-invariant systems, using the notion of invariance suitably adapted to deal with hybrid systems. See also [25] for a similar result.

Theorems 7 and 9 apply to a different class of systems than the hybrid systems studied in [38], because in the present setting the switching is not assumed to be state-dependent. Of course, one way in which a switched system of the type considered here may arise is from a hybrid system by means of an abstraction procedure (i.e., when details of the switching mechanism are neglected). In this case, our observability assumptions would serve as sufficient conditions for the largest invariant set mentioned earlier to be the origin, since they guarantee that along a nonzero solution the output cannot remain identically zero on any interval between switching times (cf. Remark 3). Note, however, that we do not require the existence of a single function nonincreasing along solutions, and instead work with multiple weak Lyapunov functions satisfying Assumption 4. This aspect of the results presented above—namely, that they rely to a large extent on separate conditions regarding the individual systems being switched—also sets them apart from LaSalle-like theorems available in the literature for certain classes of time-varying and other systems. (On the other hand, the conclusions provided by results such as Theorem 1 of [38] are stronger and closer in spirit to those of the classical LaSalle's theorem.)

Systems with inputs

There are several ways to extend Theorem 7 to systems with inputs. We collect them in the statement of the next theorem. The proof follows along the same lines as the proof of Theorem 7 and is omitted. Theorem 9 can be generalized in a completely analogous fashion.

Consider the family of systems

$$\dot{x} = f_p(x, u), \quad p \in \mathcal{P} \tag{45}$$

where $\mathcal{P}$ is a finite index set, $u \in \mathcal{U} \subset \mathbb{R}^m$, and $f_p : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^n$ is locally Lipschitz for each $p \in \mathcal{P}$. We write the corresponding switched system as

$$\dot{x} = f_\sigma(x, u) \tag{46}$$

where σ is a switching signal with switching times t_i, $i = 1, 2, \dots$ as before.

Theorem 10

A. *Let $\mathcal{U}$ be a compact set. Suppose that for each $p \in \mathcal{P}$ we have $f_p(0, \cdot) \equiv 0$, there exist a positive definite radially unbounded C^1 function $V_p : \mathbb{R}^n \to \mathbb{R}$ and a nonnegative definite continuous function $W_p : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$ such that*

$$\frac{\partial V_p}{\partial x} f_p(x, u) \le -W_p(x, u) \quad \forall\, x, u$$

and the system

$$\begin{aligned} \dot{x} &= f_p(x, u) \\ y &= W_p(x, u) \end{aligned} \tag{47}$$

is uniformly small-time norm-observable in the sense of Section 3 (so that, e.g., the condition (40) *holds uniformly over inputs). Let the switching signal σ be such that Assumptions 3 and 4 are satisfied. Then the switched system* (46) *has the following properties.*

Uniform stability: *For every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every initial condition satisfying $|x(0)| \le \delta$ and every input, we have $|x(t)| \le \varepsilon$ for all $t \ge 0$.*

Global attractivity: *For all initial conditions and all uniformly continuous inputs, we have $x(t) \to 0$ as $t \to \infty$.*

B. *Suppose that for each $p \in \mathcal{P}$ there exist functions V_p and W_p as in A and the system* (47) *is small-time norm-observable, e.g., in the sense of* (11)*. Let the switching signal σ be such that Assumptions 3 and 4 are satisfied. Then the switched system* (46) *has the same uniform stability property as in A and the following property.*

Asymptotic gain: *There exists a function $\bar{\chi} \in \mathcal{K}_\infty$ such that for all initial conditions and all bounded uniformly continuous inputs we have*

$$\limsup_{t \to \infty} |x(t)| \le \limsup_{t \to \infty} \bar{\chi}(|u(t)|). \tag{48}$$

C. *Suppose that for each $p \in \mathcal{P}$ there exist a positive definite radially unbounded C^1 function $V_p : \mathbb{R}^n \to \mathbb{R}$, a nonnegative definite continuous function $W_p : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$, and a continuous function $U_p : \mathcal{U} \to \mathbb{R}$ with $U_p(0) = 0$ such that*

$$\frac{\partial V_p}{\partial x} f_p(x, u) \le -W_p(x, u) + U_p(u) \quad \forall\, x, u$$

and the system

$$\dot{x} = f_p(x,u)$$
$$y = W_p(x,u)$$

is small-time norm-observable. Let the switching signal σ be such that Assumptions 3 and 4 are satisfied. Then the switched system (46) *has the following properties.*

Stability under zero input: *For every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every initial condition satisfying $|x(0)| \le \delta$, the corresponding solution of the system $\dot{x} = f_\sigma(x,0)$ satisfies $|x(t)| \le \varepsilon$ for all $t \ge 0$.*

Asymptotic gain under bounded-energy inputs: *There exists a function $\bar{\chi} \in \mathcal{K}_\infty$ such that for all initial conditions and all bounded uniformly continuous inputs satisfying $\int_0^\infty U_p(u(s))ds < \infty$ for all $p \in \mathcal{P}$, the condition* (48) *holds.*

Remark 11 The uniform continuity assumption on the inputs can be dropped when W_p does not depend on u. For differentiable u, this assumption means that $\dot{u}$ is bounded; such assumptions on disturbance inputs have been explored in the literature (see, e.g., [21]). □

6 Stability of switched systems: examples

Since each system in the family (38) is globally asymptotically stable by Remark 6, it does admit a Lyapunov function (strictly decreasing along nonzero solutions). There is a multitude of results on stability of switched systems which employ such multiple Lyapunov functions [12, 16, 26]. The precise conditions on the evolution of these functions at switching times are usually milder than Assumption 4. Under suitable slow switching requirements such as the existence of a sufficiently large (average) dwell time, it is sometimes possible to prove that such conditions automatically hold, thereby deducing global asymptotic stability of the switched system from global asymptotic stability of individual subsystems (see [19, 26]). However, in the nonlinear context these results rely on additional, often restrictive assumptions on the Lyapunov functions, and the required bounds on the switching rate are much more conservative than Assumption 3′ and especially than Assumption 3. Thus Theorems 7 and 9 provide a useful alternative approach to stability analysis of switched systems. (It must be noted, though, that unless the functions V_p, $p \in \mathcal{P}$ are strictly decreasing along nonzero solutions, Assumption 4 does not automatically follow from any slow switching condition.)

The usefulness of Theorems 7 and 9 stems in part from the fact that it is sometimes easier to find weak Lyapunov functions nonincreasing along solutions and satisfying Assumption 4 (or even a common weak Lyapunov function for a given family of systems) than to find strictly decreasing Lyapunov functions satisfying suitable conditions on their evolution at switching times (or, in particular, a common Lyapunov function). The following example illustrates this point.

Example 2 Consider the two systems

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - x_2 - \varphi(x_1)\end{aligned} \qquad\qquad \begin{aligned}\dot{x}_1 &= -x_2 \\ \dot{x}_2 &= x_1 - x_2 + \varphi(x_1)\end{aligned}$$

where φ satisfies the condition (35). We know from Example 1 that the first system is small-time norm-observable with respect to the output $y = x_2$. The same is true for the second system, since it is obtained from the first one by the coordinate transformation $x_2 \to -x_2$. It is not hard to check that these two systems do not possess a common Lyapunov function; indeed, otherwise the sum of the two right-hand sides would be an asymptotically stable vector field (see, e.g., [26, Corollary 2.3]), which it is not. On the other hand, the function

$$V(x) := \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \int_0^{x_1} \varphi(z)dz \tag{49}$$

serves as a common weak Lyapunov function, with derivative along solutions of each system being $\dot{V} = -x_2^2$. Thus Assumption 4 is trivially satisfied. It is clear that small-time norm-observability with respect to $y = x_2$ implies small-time norm-observability with respect to $y = x_2^2$ (just modify the function γ in the definitions of Section 2 by composing it with the square root function). Therefore, Assumptions 1 and 2 are fulfilled, and the switched system generated by these two subsystems is globally asymptotically stable for every switching signal satisfying Assumption 3. □

An important problem which leads one to consider weak common Lyapunov functions, and thus provides further motivation for the results of Section 5, is the problem of feedback stabilization for systems with switching dynamics. A positive definite C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ is said to be a *common control Lyapunov function* for the family of control systems (45) if

$$\inf_u \left\{ \frac{\partial V}{\partial x} f_p(x,u) \right\} < 0 \quad \forall\, x \neq 0,\, p \in \mathcal{P}$$

(in the case of a single system, this reduces to the standard control Lyapunov function concept [6, 33]). If the above property holds with the nonstrict inequality instead of the strict one, then we say that V is a *common weak control Lyapunov function* for the family (45). We now demonstrate that for certain families of control-affine systems, common control Lyapunov functions do not exist—while there may exist common weak control Lyapunov functions.

Proposition 11 *Suppose that the right-hand sides of the systems from* (45) *take the form*

$$f_p(x,u) = \bar{f}_p(x) + G(x)u, \quad p \in \mathcal{P} \tag{50}$$

where $1 \leq m < n$ *(m and n are the input and state dimensions) and* $\operatorname{rank} G(0) = m$*. Then there is no common control Lyapunov function for the family* (45) *if*

$$\operatorname{co}\{\bar{f}_p(x),\, p \in \mathcal{P}\} \bigcap \operatorname{range} G(x) \neq \emptyset \quad \forall\, x \neq 0 \tag{51}$$

(here co *denotes convex hull).*

Proof (sketch). Fix an arbitrary $x \neq 0$. The condition (51) implies that we have $\sum_{i=1}^k \lambda_{p_i} \bar{f}_{p_i}(x) = G(x)l$ for some vector $l \in \mathbb{R}^m$, indices $p_1, \dots, p_k \in \mathcal{P}$, and nonnegative numbers $\lambda_{p_1}, \dots, \lambda_{p_k}$ satisfying $\lambda_{p_1} + \cdots + \lambda_{p_k} = 1$. Suppose that the family (45), (50) admits a common control Lyapunov function V. Then for each p, there exists a control value u_p such that $(\partial V/\partial x)(\bar{f}_p(x) + G(x)u_p) < 0$. Combining the above formulas, we obtain $(\partial V/\partial x)G(x)(l + \sum_{i=1}^k \lambda_{p_i} u_{p_i}) < 0$. Since $x \neq 0$ was arbitrary, this means that V is a control Lyapunov function for the system $\dot{x} = G(x)u$, which is known to contradict Brockett's necessary condition for feedback stabilizability [11] (see [35] for details). □

The condition (51) clearly holds when $\operatorname{co}\{\bar{f}_p(x),\, p \in \mathcal{P}\}$ contains the origin for all $x \neq 0$, which would happen, e.g., if $\bar{f}_p(x) = p\bar{f}(x)$ for some $\bar{f}$ and $\mathcal{P} \subset \mathbb{R}$ contains both positive and negative values. Letting

$$\bar{f}_p(x) := p \begin{pmatrix} x_2 \\ -x_1 - \varphi(x_1) \end{pmatrix}, \qquad G(x) := \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \mathcal{P} := \{-1, 1\} \tag{52}$$

and applying the feedback law $u = -x_2$, we recover the two systems considered in Example 2. Proposition 11 confirms that those two systems do not have a common Lyapunov function. More generally, it implies that the same is true for two systems obtained from the pair of control systems defined by (45), (50) and (52) using an arbitrary pair of feedback laws (it is not necessary to use the same feedback law for both systems).[5] On the other hand, a common weak control Lyapunov function for this pair of control systems does exist and is given by (49).

The next example is a variation on Example 2 and also builds on Example 1.

Example 3 Consider the two systems

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - x_2 - \varphi_1(x_1)\end{aligned} \qquad\qquad \begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - x_2 - \varphi_2(x_1)\end{aligned}$$

where φ_1 and φ_2 satisfy (35). Define the functions

$$V_i(x) := \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \int_0^{x_1} \varphi_i(z)dz, \quad i = 1, 2.$$

Then Assumptions 1 and 2 can be checked as in Example 2, and the switched system generated by these two subsystems is globally asymptotically stable for every switching signal satisfying Assumptions 3 and 4. The latter assumption holds, for example, if the switching is constrained to occur on the set $\{x : \int_0^{x_1} \varphi_1(z)dz = \int_0^{x_1} \varphi_2(z)dz\} \supset \{x : x_1 = 0\}$; this condition still leaves considerable freedom in defining a specific switching rule. If the switching is controlled by a supervisor, then Assumptions 3 and 4 can be directly incorporated into the switching logic (see Remark 7). □

Functions $V : \mathbb{R}^n \to \mathbb{R}$ consisting of a quadratic form plus an integral of a nonlinearity—such as the ones considered in Examples 2 and 3 above—frequently arise as (weak) Lyapunov functions in absolute stability theory, which studies nonlinear feedback systems of the form

$$\dot{x} = Ax - b\varphi(c^T x), \quad x, b, c \in \mathbb{R}^n. \tag{53}$$

Consider the transfer function $g(s) = c^T(sI - A)^{-1}b$ and assume that it has no pole-zero cancellations. If g has one pole at 0 and the rest in the open left half-plane and the function $(1 + as)g(s)$ is positive real for some $a \geq 0$, then there exists a positive definite matrix P (independent of φ) such that the derivative of the function $V(x) = x^T Px + a\int_0^{c^T x} \varphi(z)dz$ along solutions of (53) satisfies $\dot{V} \leq -c^T x\varphi(c^T x)$. Observability of the pair (c^T, A) combined with LaSalle's theorem can now be used to conclude that the system (53) is globally asymptotically stable for every function φ that satisfies the sector condition $0 < y\varphi(y)$ for all $y \neq 0$. (This is one version of Popov's stability criterion; see, e.g., [10] for details.)

With the help of Theorem 7, we can then study stability of the switched system generated by a finite family of systems of the form (53), with the same linear part but different nonlinearities:

$$\dot{x} = Ax - b\varphi_p(c^T x), \quad p \in \mathcal{P}. \tag{54}$$

Under the conditions of Popov's criterion cited above, the corresponding weak Lyapunov functions take the form $V_p(x) = x^T Px + a\int_0^{c^T x} \varphi_p(z)dz$. An argument similar to the one employed in Example 1 shows that the system

$$\begin{aligned}\dot{x} &= Ax - b\varphi_p(c^T x) \\ y &= c^T x\varphi_p(c^T x)\end{aligned}$$

[5]This statement remains valid if the second component of the vector used to define $\bar{f}_p(x)$ in (52) is replaced by 0.

is small-time norm-observable for each $p \in \mathcal{P}$ provided that the functions φ_p, $p \in \mathcal{P}$ satisfy the slightly strengthened sector condition

$$\forall\, p \in \mathcal{P} \quad \exists\, \rho_p \in \mathcal{K}_\infty,\, k_p > 0 \text{ such that } \ \rho_p(|y|) \le y\varphi_p(y) \le k_p|y|^2 \quad \forall\, y. \tag{55}$$

As in Example 3, we see that Assumption 4 is satisfied whenever the switching is confined to the set of equal "potential energies"

$$\left\{ x : \int_0^{c^T x} \varphi_p(z)dz = \int_0^{c^T x} \varphi_q(z)dz \quad \forall\, p, q \in \mathcal{P} \right\} \supset \{x : c^T x = 0\}. \tag{56}$$

We have established the following corollary of Theorem 7 (a variant corresponding to Theorem 9 is also straightforward to obtain).

Proposition 12 *Consider the family of systems* (54)*, where $\mathcal{P}$ is a finite index set and the functions φ_p, $p \in \mathcal{P}$ satisfy the sector condition* (55)*. Assume that the transfer function $g(s) = c^T(sI - A)^{-1}b$ has no pole-zero cancellations, one pole at 0, and all other poles in the open left half-plane, and that the function $(1 + as)g(s)$ is positive real for some $a \ge 0$. Then the switched system $\dot{x} = Ax - b\varphi_\sigma(c^T x)$ is globally asymptotically stable if the switching signal σ satisfies Assumption 3 and x belongs to the first set in* (56) *at every switching time.*

7 Conclusions and remarks on related notions

We introduced and studied several norm-observability notions for nonlinear systems with inputs and outputs, in a set-up compatible with the input-to-state stability framework. Their application to LaSalle-like stability analysis of switched systems was described and illustrated through examples. Potential implications for control design in the presence of switching were briefly sketched and remain to be investigated. We end the paper with some remarks about related observability notions. For simplicity, we consider the system with no inputs (9), although the extension to the general case of (1) is immediate.

Integral versions

Motivated by the integral versions of ISS and OSS (see [4]), we can modify the definitions by replacing supremum norms on the outputs with integral norms. For example, in place of small-time initial-state norm-observability for the system (9), consider the following property:

$$\forall\, \tau > 0 \quad \exists\, \alpha, \gamma \in \mathcal{K}_\infty \text{ such that } \ \alpha(|x(0)|) \le \int_0^\tau \gamma(|y(s)|)ds \quad \forall\, x(0).$$

Other notions can be obtained similarly. Using the integral versions of Lemmas 1 and 2 established in [3, Section 2.6], one can show that the relationships proved in Section 3 remain valid for these new notions. It is also not hard to see that the integral versions of the small-time and large-time properties are stronger than their supremum-norm counterparts (just apply the mean value theorem), while this is not the case for the infinite-time property (as can be shown by a counterexample). Unfortunately, these integral observability notions do not seem to be as useful for establishing LaSalle-like stability theorems for switched systems, unless the integral gain function γ happens to be the identity.

Incremental versions

In order to come up with definitions that more closely relate to the traditional notions of observability (i.e., the possibility of having a finite time estimate of the actual value of the state), one needs to resort to incremental observability notions. By incremental notions, we mean inequalities involving norms of differences of states, inputs and/or outputs rather than their absolute values. In particular, the incremental notion of small-time initial-state observability for the system (9) would read as follows:

$$\forall \tau > 0 \quad \exists \gamma \in \mathcal{K}_\infty \text{ such that } \ |x_1(0) - x_2(0)| \leq \gamma(\|y_1 - y_2\|_{[0,\tau]}) \quad \forall x_1(0), x_2(0)$$

where $y_i(\cdot)$ is the output corresponding to the trajectory with the initial state $x_i(0)$, $i = 1, 2$. In this more general context, norm-observability as discussed elsewhere in this paper becomes a special case, in which an arbitrary trajectory is compared with the zero (equilibrium) trajectory. This line of thought has been followed for instance in [36, Section 5], where a definition of incremental detectability is provided, or in [1, Section 6.B], where a notion of incremental stability after output injection is considered. Generally speaking, results which are based only on manipulation of estimates carry over to the incremental set-up; results which involve some sort of compactness are usually much harder to derive, unless a priori bounds on trajectories are explicitly assumed. Treatment of these notions in detail is outside the scope of the present paper and we leave it open for future investigations. A related observability concept for linear hybrid systems was studied in [8].

Appendix: Proof of Lemma 1

Fix an arbitrary $\tau > 0$. We first need to show boundedness of reachable sets for the system (1) in time τ, starting from a compact set, for bounded inputs, and bounded outputs. For $\eta \geq 0$, let

$$\mathcal{R}(\eta) := \big\{x(t) : 0 \leq t \leq \tau,\, |x(0)| \leq \eta,\, \|y\|_{[0,t]} \leq \eta,\, \|u\|_{[0,t]} \leq \eta\big\}$$

where $x(\cdot)$ denotes a solution of (1) with initial condition $x(0)$ and input $u(\cdot)$, and $y(\cdot)$ is the corresponding output.

Lemma 13 *The set $\mathcal{R}(\eta)$ is bounded for every $\eta \geq 0$.*

Proof. Let $\eta \geq 0$ be given. Introduce the auxiliary system[6]

$$\dot{z} = f(z, v)\phi_\eta(|h(z, v)|), \quad z \in \mathbb{R}^n \tag{57}$$

where $\phi_\eta : \mathbb{R} \to [0, 1]$ is some smooth function such that $\phi_\eta(r) = 1$ for $r \leq \eta$ and $\phi_\eta(r) = 0$ if and only if $r \geq \eta + 1$. The set $\mathcal{R}(\eta)$ is contained in the reachable set

$$\big\{z(t) : 0 \leq t \leq \tau,\, |z(0)| \leq \eta,\, \|v\|_{[0,t]} \leq \eta\big\}$$

for the system (57). If we show that the system (57) is forward complete, then we will be done, because Proposition 5.1 from [28] says that reachable sets for forward complete systems, in bounded time, starting from a compact set, and for bounded inputs are bounded.

[6] Existence of solutions for this system is guaranteed if the output map h is locally Lipschitz. Even if it is not, one can prove the validity of all that follows by working with a suitable approximation of h; see [3, Remark 2.15].

Take a solution $z(\cdot)$ of (57) corresponding to some initial condition $z(0)$ and some input $v(\cdot)$, and suppose that $|z(s)| \to \infty$ as s approaches some time $S > 0$. We now construct a new input $\bar{v}$ by "collapsing" time intervals on which $\phi_\eta(|h(z,v)|) = 0$. Let $\psi(s) := \int_0^s I_\eta(r)dr$, where

$$I_\eta(r) := \begin{cases} 1, & |h(z(r), v(r))| < \eta + 1 \\ 0, & |h(z(r), v(r))| \geq \eta + 1 \end{cases}$$

This defines a continuous mapping ψ from the interval $[0, S)$ onto an interval $[0, \bar{S})$, where $\bar{S} \leq S$. For $s \in [0, \bar{S})$, let $\bar{v}(s) := v(\bar{s})$ where $\bar{s} := \min\{r \in [0, S) : \psi(r) = s\}$. Let $\bar{z}(\cdot)$ be the solution of the system (57) corresponding to the same initial condition $z(0)$ and this new input $\bar{v}(\cdot)$. Since $\bar{z}$ is obtained from z by removing time intervals on which z is constant, the two trajectories are the same up to this time reparameterization, and in particular, $|\bar{z}(s)| \to \infty$ as $s \to \bar{S}$.

Now, consider

$$\varphi(s) := \int_0^s \phi_\eta(|h(\bar{z}(r), \bar{v}(r))|dr.$$

Since $\phi_\eta(|h(\bar{z}(r), \bar{v}(r))|) > 0$ for almost all $r \in [0, \bar{S})$ by construction, φ is a strictly increasing function from $[0, \bar{S})$ to some interval $[0, T)$, where actually $T \leq \bar{S}$ because $\phi_\eta \leq 1$ everywhere. For $t \in [0, T)$, let $x(t) := \bar{z}(\varphi^{-1}(t))$. Then $x(\cdot)$ is absolutely continuous and satisfies $\dot{x} = f(x, u)$, where $u(t) := \bar{v}(\varphi^{-1}(t))$. Since $|\bar{z}(s)| \to \infty$ as $s \to \bar{S}$, we have $|x(t)| \to \infty$ as $t \to T$. The system (1) has the unboundedness observability property by assumption, and so the output $h(x, u)$ must become unbounded as $t \to T$. But we have $h(x(t), u(t)) = h(\bar{z}(\varphi^{-1}(t)), \bar{v}(\varphi^{-1}(t)) = h(\bar{z}(s), \bar{v}(s))$ and the norm of the last expression does not exceed $\eta + 1$, which is a contradiction. □

The above argument is an extension of the proof of Lemma 2.1 in [3], which establishes the corresponding property of reachable sets for systems whose output maps do not depend on the inputs. We now proceed with the proof of Lemma 1. For $r \geq 0$, define

$$\hat{\alpha}(r) := \sup\big\{|x| : x \in \mathcal{R}(r)\big\}.$$

The above function is well defined since the supremum is bounded for each r by virtue of Lemma 13. We have $\hat{\alpha}(0) = 0$ because $f(0,0) = 0$ and $h(0,0) = 0$. Therefore, we can choose some class $\mathcal{K}_\infty$ function α such that $\hat{\alpha}(r) \leq \alpha(r)$ for all $r \geq 0$. This gives

$$|x(t)| \leq \alpha\big(\max\{|x(0)|, \|y\|_{[0,t]}, \|u\|_{[0,t]}\}\big) \leq \alpha(|x(0)|) + \alpha(\|y\|_{[0,t]}) + \alpha(\|u\|_{[0,t]}) \quad \forall\, t \in [0, \tau]. \tag{58}$$

Since the system is time-invariant, the lemma holds with $\nu_f = \gamma_f = \chi_f := \alpha$. □

Remark 12 A similar construction is used in the proof of Lemma 2.2 in [3]. In the case when the system is forward complete, the condition involving y can be omitted from the definition of the reachable set $\mathcal{R}(\eta)$. The reachable set is still bounded due to Proposition 5.1 from [28]. Thus for forward complete systems, y disappears from (58) and Lemma 1 holds without the y-dependent term in (10). Alternatively, if inputs are not present or take values in a compact set, then the condition involving u can be omitted from the definition of $\mathcal{R}(\eta)$. In this case, assuming that $f(0) = 0$ when there are no inputs or $f(0, u) = 0$ for all u when there are inputs, we see that Lemma 1 holds without the u-dependent term in (10). In particular, for forward complete systems with no inputs the inequality (10) reduces to just $|x(t_2)| \leq \nu_f(|x(t_1)|)$; this special case of the result is a straightforward consequence of the continuous dependence of solutions on initial conditions and the presence of the equilibrium at the origin. The same remarks apply to Lemma 2. □

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