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A-PRIORI BOUNDS FOR THE 1-D CUBIC NLS IN NEGATIVE SOBOLEV SPACES

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ABSTRACT. We consider the cubic Nonlinear Schrödinger Equation (NLS) in one space dimension, either focusing or defocusing. We prove that the solutions satisfy a-priori local in time H^s bounds in terms of the H^s size of the initial data for $s \geq -\frac{1}{6}$.

1. INTRODUCTION

The one dimensional cubic Nonlinear Schrödinger equation (NLS)

$$(1) \quad iu_t - u_{xx} \pm u|u|^2 = 0, \quad u(0) = u_0.$$

arises as generic asymptotic equation for modulated wave trains. It has a particularly rich structure: It is Hamiltonian with respect to the symplectic structure

$$\sigma(u, v) = \text{Im} \int u \bar{v} dx$$

and the Hamiltonian

$$\int \frac{1}{2}(u')^2 \pm \frac{1}{4}|u|^4 dx.$$

There are infinitely many conserved quantities. The NLS equation is completely integrable in the sense that there exist Lax pairs for it. The machinery of inverse scattering allows to construct many interesting solutions, among them solitary waves in the focusing case.

The NLS is globally well-posed for initial data $u_0 \in L^2$, and locally in time the solution has a uniform Lipschitz dependence on the initial data in balls.

On the other hand (1) is invariant with respect to the scaling

$$u(x, t) \rightarrow \lambda u(\lambda x, \lambda^2 t)$$

This implies that the scale invariant initial data space for (1) is $\dot{H}^{-\frac{1}{2}}$. Thus one is motivated to ask whether the local well-posedness also holds in negative Sobolev spaces.

The equation (1) is also invariant under the Galilean transformation

$$u(x, t) \rightarrow e^{icx - ic^2t} u(x + 2ct, t)$$

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which corresponds to a shift in the frequency space. As a consequence there is no uniformly continuous dependence on the initial data (see [6], [3]). This is not unexpected; if local uniformly continuous dependence were to hold in any negative Sobolev space, by Galilean invariance and scaling this would imply global in time local in space uniformly continuous dependence on the initial data in L^2 .

What we expect below L^2 is for the cubic NLS to exhibit genuinely nonlinear dynamics, which corresponds to a continuous but not uniformly continuous dependence on the initial data. One may be tempted to think that local well-posedness should hold all the way down to $s = -\frac{1}{2}$. However, such a result is far out of reach for now and we would not even speculate whether it is true or not.

On the other hand, there is another very natural threshold, which is connected to the main motivation of the present paper. In a recent paper Kappeler and Topalov [5] proved that the mKdV equation

$$v_t - v_{xxx} + v_x v^2 = 0, \quad v_0 = v_0$$

on the torus is well-posed for initial data in L^2 . The proof relies on complete integrability of the equation, and it uses the machinery of integrable equations in a fundamental way. One may ask whether the same result holds on the real line, and also whether it is possible to find arguments which do not use the integrable structure.

To connect this problem with the NLS equation we consider modulated wave train solutions v of the form $v = \Re w$ where w is frequency localized in a neighborhood of size h of some large frequency λ . Then w solves the equation $w_t + i\lambda^3 w + 3i\lambda^2(D_x - \lambda)w + 3i\lambda(D_x - \lambda)^2 w + 3i\lambda w|w|^2 \approx O(h^3)w + O(h)w|w|^2 + w^3$. For $h \ll \lambda$ we neglect the first two terms on the right. The w^3 term is non-resonant and is also neglected. Then the substitution

$$w(t, x) = \lambda^{-\frac{1}{2}} e^{-i\lambda^3 t} e^{i\lambda x} u(t, \lambda^{-\frac{1}{2}}(x - 3\lambda^2 t))$$

turns the above equation into (1) with modified constants.

A frequency range of size λ for u turns into a frequency range of size $\mu = \lambda^{\frac{3}{2}}$ for w . By construction this frequency range for w is centered at the origin, but we can use a Galilean transformation to shift it to a dyadic region. We can also easily compute

$$\|u(0)\|_{L^2} = \lambda^{\frac{1}{4}} \|v(0)\|_{L^2} = \mu^{\frac{1}{6}} \|v(0)\|_{L^2}$$

Hence the mKdV equation with initial data in L^2 is similar¹ to the NLS equation with initial data in $H^{-\frac{1}{6}}$. We view the one dimensional NLS equation as

¹We emphasize that this similarity applies only for solutions in a dyadic frequency range. On the other hand in our analysis later in the paper we see that some of the most difficult to control multilinear interactions occur in the case of unbalanced frequencies, where this analogy no longer applies.

a simpler model in the analysis of the KdV equation; this is due to the added Galilean invariance. However, it is also interesting in its own right.

The threshold $s = -\frac{1}{6}$ also arises in several key steps of our analysis later on, having to do with the interaction of high and low frequencies. We are led to

Conjecture 1. *The cubic NLS equation (1) is locally well-posed for initial data in H^s with $s \geq -\frac{1}{6}$.*

To prove this one would need to establish a-priori H^s bounds for the solutions and then prove continuous dependence on the initial data. In this article we solve the easier half of this problem.

Theorem 1. *Let $s \geq -\frac{1}{6}$. For any $M > 0$ there exists $T > 0$ and $C > 0$ so that for any initial data $u_0 \in L^2$ satisfying*

$$\|u_0\|_{H^s} \leq M$$

there exists a solution $u \in C(0, T; L^2)$ to (1) which satisfies

$$\|u\|_{L^\infty H^s} \leq C\|u_0\|_{H^s}$$

While writing this paper the authors have learned that similar results were independently obtained by Christ-Colliander-Tao [2]. Their results apply in the range $s > -\frac{1}{12}$.

We also refer the reader to the work of Vargas-Vega [10] and Grünrock [4] who consider the cubic NLS in alternative function spaces below L^2 , but only in settings where the local Lipschitz dependence on the initial data still holds.

Remark 1.1. *In the process of proving the theorem we actually obtain a better characterization of the solution u , namely we show that u bounded in a space X^s defined in the next section which embeds into $L^\infty H^s$ and has the property that the nonlinear expression $|u|^2 u$ is well defined for $u \in X^s$ with a bound depending only on the H^s norm of the initial data.*

We note that by rescaling the problem reduces to the case of small initial data. Then we take $M = \epsilon$, small and $T = 1$, $C = 2$.

We begin with a dyadic frequency decomposition of the solution u ,

$$u = \sum_{\lambda} u_{\lambda}$$

To measure the H^s norm of u we use the stronger norm than $L^\infty(H^s)$,

$$\|u\|_{L^2 L^\infty H^s}^2 = \sum_{\lambda} \sup_t \lambda^{2s} \|u_{\lambda}(t)\|_{L^2}^2$$

That we can use this instead of the $L^\infty H^s$ norm is a reflection of the fact that there is not much energy transfer between different dyadic frequencies.

To prove the theorem we need two Banach spaces X^s and Y^s , defined in the next section, in order to measure the regularity of the solution u , respectively of the nonlinear term $|u|^2 u$.

The linear part of the argument is given by

Proposition 1.2. *The following estimate holds:*

$$\|u\|_{X^s} \lesssim \|u\|_{l^2 L^\infty H^s} + \|iu_t - \Delta u\|_{Y^s}$$

To estimate the nonlinearity we need a cubic bound,

Proposition 1.3. *Let $-\frac{1}{6} \leq s \leq 0$ and $u \in X^s$. Then $|u|^2 u \in Y^s$ and*

$$\||u|^2 u\|_{Y^s} \lesssim \|u\|_{X^s}^3$$

Finally we need to propagate the H^s norm:

Proposition 1.4. *Let $-\frac{1}{6} \leq s \leq 0$, and u be a solution to (1) with*

$$\|u\|_{l^2 L^\infty H^s} \ll 1.$$

Then we have

$$\|u\|_{l^2 L^\infty H^s} \lesssim \|u_0\|_{H^s} + \|u\|_{X^s}^3.$$

The plan of the paper is as follows. In the next section we motivate and introduce the spaces X^s and Y^s , as well as establish the linear mapping properties in Proposition 1.2. In Section 3 we discuss the linear and bilinear Strichartz estimates for solutions to the linear equation.

The trilinear estimate in Proposition 1.3 is proved in Section 4. Finally in the last section we use a variation of the I-method to construct a quasi-conserved energy functional and compute its behavior along the flow, thus proving Proposition 1.4.

To conclude this section we show that the conclusion of the Theorem follows from the above Propositions. We first note that if $u_0 \in L^2$ then by iteratively solving the equation on small time intervals we obtain a solution u up to time 1, which satisfies

$$(2) \quad iu_t - \Delta u \in L^2$$

This easily implies that $u \in l^2 L^\infty H^s$, and also that $u \in X^s$.

To prove the theorem we use a continuity argument. Let $\varepsilon > 0$ be a small constant and suppose that $\|u_0\|_{H^s(\mathbb{R})} < \varepsilon$. Fix a small threshold δ , $\varepsilon \ll \delta \ll 1$ and denote by A the set

$$A = \{T \in [0, 1]; \|u\|_{l^2 L^\infty H^s([0, T] \times \mathbb{R})} \leq 2\delta, \|u\|_{X^s([0, T] \times \mathbb{R})} \leq 2\delta\}$$

We claim that $A = [0, 1]$. To show this we first observe that $0 \in A$. The norms above increase with T , therefore A is an interval. We show that A is both open and closed in $[0, 1]$.

By (2) it easily follows that the norms in the definition of A are continuous² with respect to T . This implies that A is closed.

Finally let $T \in A$. By Proposition 1.4 we obtain

$$\|u\|_{l^2 L^\infty H^s([0,T] \times \mathbb{R})} \lesssim \varepsilon + \delta^3.$$

Then by Propositions 1.2, 1.3 we obtain

$$\|u\|_{X^s([0,T] \times \mathbb{R})} \lesssim \varepsilon + \delta^3$$

If ε and δ are chosen to be sufficiently small we conclude that

$$\|u\|_{l^2 L^\infty H^s([0,T] \times \mathbb{R})} \leq \delta, \quad \|u\|_{X^s([0,T] \times \mathbb{R})} \leq \delta$$

By the continuity of the norms with respect to T it follows that a neighborhood of T is in A .

Hence $A = [0, 1]$ and the Theorem 1 is proved.

2. THE FUNCTION SPACES

To understand what to expect in terms of the regularity of u we begin with some heuristic considerations. If the initial data u_0 to (1) satisfies $\|u_0\|_{L^2} \leq 1$ then the equation can be solved iteratively using the Strichartz estimates. We obtain essentially linear dynamics, and the solution u belongs to the space $X^{0,1}$ associated to the Schrödinger equation (see the definition in (3) below).

Let $s < 0$. Consider now the same problem but with initial data $u_0 \in H^s$, localized at frequency λ . Then the initial data satisfies $\|u_0\|_{L^2} \lesssim \lambda^{-s}$. By rescaling we conclude that the evolution is still described by linear dynamics up to the time λ^{4s} .

Then it is natural to consider a dyadic decomposition of the solution u

$$u = \sum_{\lambda} u_{\lambda}$$

and to measure the u_{λ} component uniformly in λ^{4s} time intervals. We remark that this is reasonable for as long as there is not much input coming from the higher frequencies. This is the technical point where the $s = -\frac{1}{6}$ threshold arises in our proof.

A good candidate for measuring u_{λ} in λ^{4s} time intervals is given by Bourgain's $X^{s,b}$ spaces defined by

$$(3) \quad \|u\|_{X^{s,b}}^2 = \int |\hat{u}(\tau, \xi)|^2 \xi^{2s} (1 + |\tau - \xi^2|)^{2b} d\xi d\tau$$

where the natural choice for b from a scaling standpoint is $b = \frac{1}{2}$. However, this choice leads to logarithmic divergences in estimates, so one commonly uses instead some $b > \frac{1}{2}$ but close to it. We could do this here but it would complicate the bookkeeping and would also not work at $s = -\frac{1}{6}$. For $b = \frac{1}{2}$ one can go one step further and consider dyadic decompositions with respect

²This of course depends on the definition of the X^s norm, but it is straightforward to prove.

to the modulation $\tau - \xi^2$. This leads to the additional homogeneous Besov type norms

$$\|u\|_{\dot{X}^{s, \frac{1}{2}, 1}} = \sum_{\mu} \left(\int_{|\tau - \xi^2| \approx \mu} |\hat{u}(\tau, \xi)|^2 \xi^{2s} |\tau - \xi^2| d\xi d\tau \right)^{\frac{1}{2}}$$

$$\|u\|_{\dot{X}^{s, \frac{1}{2}, \infty}} = \sup_{\mu} \left(\int_{|\tau - \xi^2| \approx \mu} |\hat{u}(\tau, \xi)|^2 \xi^{2s} |\tau - \xi^2| d\xi d\tau \right)^{\frac{1}{2}}$$

Instead in this paper we use the closely related spaces U_{Δ}^2 and V_{Δ}^2 . Spaces of this type have been first introduced in unpublished work of the second author on wave-maps, but in the meantime they have been also used in [7], [1], [8]. They turn out to be useful replacements of $X^{s,b}$ spaces in limiting cases, and they retain the scaling of the corresponding space of homogeneous solutions to the linear equation. We define them and summarize their key properties in what follows.

Definition 2.1. *Let $1 \leq p < \infty$. Then U_{Δ}^p is an atomic space, where atoms are piecewise solutions to the linear equation,*

$$u = \sum_k 1_{[t_k, t_{k+1})} e^{itD_x^2} u_k, \quad \sum_k \|u_k\|_{L^2}^p = 1$$

and $\{t_k\}$ is an arbitrary increasing sequence.

Clearly we have

$$U_{\Delta}^p \subset L^{\infty} L^2$$

In addition, the U_{Δ}^p functions are continuous except at countably many points, and right continuous everywhere.

A close relative is the space V_{Δ}^p of functions with bounded p -variation along the flow:

Definition 2.2. *Let $1 \leq p < \infty$. Then V_{Δ}^p is the space of right continuous functions $u \in L^{\infty}(L^2)$ for which the following norm is finite,*

$$\|u\|_{V_{\Delta}^p}^p = \|u\|_{L^{\infty} L^2}^p + \sup_{\{t_k\} \nearrow} \sum_k \|e^{it_k D_x^2} u(t_k) - e^{it_{k+1} D_x^2} u(t_{k+1})\|_{L^2}^p$$

where the supremum is taken with respect to all increasing sequences $\{t_k\}$.

Conjugation with the Schrödinger group reduces a large part of the study of the spaces V^p and U^p to the scalar case, where we replace the group by the identity.

We have the series of inclusions

$$(4) \quad U_{\Delta}^p \subset V_{\Delta}^p \subset U_{\Delta}^q \subset L^{\infty} L^2, \quad p < q.$$

The inclusion $U^p \subset V^p$ can easily be checked on atoms. The imbedding $V^p \subset U^q$ is a little harder and its proof can be found in Section 5 of [7].

We denote by DU_Δ^p the space of functions

$$DU_\Delta^p = \{(i\partial_t - \partial_x^2)u; u \in U_\Delta^p\}$$

with the induced norm. Then we have the trivial bound

$$(5) \quad \|u\|_{U_\Delta^p} \lesssim \|u(0)\|_{L^2} + \|(i\partial_t - \partial_x^2)u\|_{DU_\Delta^p}$$

Finally, we have the duality relation

$$(6) \quad (DU_\Delta^p)^* = V_\Delta^{p'}.$$

To see this one first verifies the inequality

$$|\int \langle (i\partial_t - \partial_x^2)f, g \rangle_{L^2} dx| \leq \|f\|_{U^p} \|g\|_{V^{p'}}$$

by checking it for atoms f . Secondly, given $L \in (DU_\Delta^p)^*$ we apply it to characteristic functions of intervals, which allows to define a function g with

$$\int \langle (i\partial_t - \partial_x^2)f, g \rangle_{L^2} dt = L((i\partial_t - \partial_x^2)f).$$

An application to suitable atoms shows that $g \in V_\Delta^{p'}$.

Moreover we have the embedding

$$\dot{X}^{0, \frac{1}{2}, 1} \subset U_\Delta^2.$$

To see this it suffices to consider a function the Fourier transform of which is supported in a fixed dyadic annulus. The statement follows now easily. Combined with duality one sees that

$$(7) \quad \dot{X}^{0, \frac{1}{2}, 1} \subset U_\Delta^2 \subset V_\Delta^2 \subset \dot{X}^{0, \frac{1}{2}, \infty}.$$

The U_Δ^p and V_Δ^p spaces behave well with respect to sharp time truncations. Precisely, if I is a time interval and χ_I is its characteristic function then we have the multiplicative mapping properties

$$(8) \quad \chi_I : U_\Delta^p \rightarrow U_\Delta^p, \quad \chi_I : V_\Delta^p \rightarrow V_\Delta^p$$

with uniform bounds with respect to I .

We use a spatial Littlewood-Paley decomposition

$$1 = \sum_{\lambda \geq 1} P_\lambda, \quad u = \sum_{\lambda \geq 1} P_\lambda u = \sum_{\lambda \geq 1} u_\lambda$$

as well as a Littlewood-Paley decomposition with respect to the modulation $\tau - \xi^2$,

$$1 = \sum_{\lambda \geq 1} Q_\lambda$$

Both decompositions are inhomogeneous. It is easy to verify that we have the uniform boundedness properties

$$(9) \quad P_\lambda : U_\Delta^p \rightarrow U_\Delta^p, \quad Q_\lambda : U_\Delta^p \rightarrow U_\Delta^p$$

and similarly for V_Δ^p .

For functions at frequency λ we introduce a minor variation of the U_Δ^2 , respectively V_Δ^2 spaces, which we denote by U_λ^2 , respectively V_λ^2 . Their norms are defined as

$$\|u_\lambda\|_{U_\lambda^2}^2 = \|Q_{\leq \lambda^2} u_\lambda\|_{U_\Delta^2}^2 + \sum_{|I|=\lambda^{-2}, |J|=\lambda^{-1}} \|\chi_I(t)\chi_J(x)Q_{\geq \lambda^2} u_\lambda\|_{U^2}^2,$$

respectively

$$\|u_\lambda\|_{V_\lambda^2}^2 = \|Q_{\leq \lambda^2} u_\lambda\|_{V_\Delta^2}^2 + \sum_{|I|=\lambda^{-2}, |J|=\lambda^{-1}} \|\chi_I(t)\chi_J(x)Q_{\geq \lambda^2} u_\lambda\|_{V^2}^2,$$

Here the time truncation is still sharp, as above. The spatial truncation may be taken sharp or smooth, the two norms are equivalent due to the frequency localization. In the last norm we use the simpler space U^2 (where we replace Δ in U_Δ by zero) instead of U_Δ^2 ; this is also immaterial, the U^2 and U_Δ^2 norms are equivalent at frequency λ and modulation $\geq \lambda^2$.

In doing this the U_Δ^2 norm is slightly weakened, but only in the elliptic region:

$$\|u_\lambda\|_{U_\lambda^2} \lesssim \|u_\lambda\|_{U_\Delta^2}$$

To see this it suffices to consider U_Δ^2 atoms. Steps $t_{k+1} - t_k$ of size larger than λ^{-2} are essentially canceled by the modulation localization operator $Q_{\geq \lambda^2}$, therefore it suffices to restrict ourselves to the λ^{-2} time scale. But on this scale the Schrödinger flow at frequency λ is trivial, i.e. there is no propagation. Thus one obtains the square summability with respect to the λ^{-1} spatial scale.

Since we preserve the duality relation (6) the V_Δ^2 norm is slightly strengthened:

$$\|u_\lambda\|_{V_\lambda^2} \gtrsim \|u_\lambda\|_{V_\Delta^2}$$

The only advantage in using the modified spaces is that they allow us to replace a logarithm of the high frequency by a logarithm of the low frequency in (17), which is needed in order for our proofs to work in the limiting case $s = -\frac{1}{6}$.

We note that the inclusions in (4) as well as the properties (5), (6), (7), (8) and (9) remain valid in the dyadic setting for the modified spaces.

Now we are ready to introduce the function spaces for the solutions u . We set

$$(10) \quad \|u\|_{X^s}^2 = \sum_\lambda \lambda^{2s} \sup_{|I|=\lambda^{4s}} \|\chi_I u_\lambda\|_{U_\lambda^2}^2$$

where we sum over all dyadic integers ≥ 1 with the obvious modification at $\lambda = 1$.

To measure the regularity of the nonlinear term we need

$$(11) \quad \|f\|_{Y^s}^2 = \sum_\lambda \lambda^{2s} \sup_{|I|=\lambda^{4s}} \|\chi_I f_\lambda\|_{DU_\lambda^2}^2$$

Due to (5) we easily obtain the bound in Proposition 1.2.

3. LINEAR AND BILINEAR ESTIMATES

We begin with solutions to the homogeneous equation,

$$(12) \quad iv_t - \Delta v = 0, \quad v(0) = v_0$$

These satisfy the Strichartz estimates:

Proposition 3.1. *Let p, q be indices satisfying*

$$(13) \quad \frac{2}{p} + \frac{1}{q} = \frac{1}{2}, \quad 4 \leq p \leq \infty$$

Then the solution u to (12) satisfies

$$\|v\|_{L_t^p L_x^q} \lesssim \|v_0\|_{L^2}$$

In particular we note the pairs of indices $(\infty, 2)$, $(6, 6)$ and $(4, \infty)$. On occasion it is convenient to interchange the role of the space and time coordinates. Then by interpolating the local smoothing estimate for solutions to (12),

$$\|v_\lambda\|_{L_x^\infty L_t^2} \lesssim \lambda^{-1/2} \|v_{\lambda,0}\|_{L^2}$$

and the maximal function estimate

$$\|v_\lambda\|_{L_x^4 L_t^\infty} \lesssim \lambda^{1/4} \|v_{\lambda,0}\|_{L^2}$$

we obtain

Proposition 3.2. *Let p, q be indices satisfying (13). Then for every solution v to (12) which is localized at frequency λ we have*

$$\|v_\lambda\|_{L_x^p L_t^q} \lesssim \lambda^{\frac{3}{p} - \frac{1}{2}} \|v_{\lambda,0}\|_{L^2}$$

As a straightforward consequence we have

Corollary 3.3. *a) Let p, q be indices satisfying (13). Then*

$$\|v\|_{L_t^p L_x^q} \lesssim \|v\|_{U_\Delta^p}$$

and the same holds with U_Δ^p replaced by V_Δ^2 .

b) In addition, if v is localized at frequency λ then we have

$$\|v\|_{L_x^p L_t^q} \lesssim \lambda^{\frac{3}{p} - \frac{1}{2}} \|v\|_{U_\Delta^p},$$

and the same holds with U_Δ^p replaced by V_Δ^2 if $p > 2$.

c) For v localized at frequency λ the U_Δ^2 and V_Δ^2 norms in (a), (b) can be replaced by U_λ^2 and V_λ^2 .

The proof is straightforward, since it suffices to do it for atoms. In the case of V_Δ^2 we also take advantage of the inclusion $V_\Delta^2 \subset U_\Delta^p$, $p > 2$. The estimate for V_λ^2 and U_λ^2 follows from the embeddings $U_\lambda^2 \subset V_\lambda^2 \subset V_\Delta^2$ for functions at frequency λ .

By duality we also obtain

Corollary 3.4. *a) Let p, q be indices satisfying (13). Then*

$$\|v\|_{DU_{\Delta}^2} \lesssim \|v\|_{L_t^{p'} L_x^{q'}}$$

b) In addition, if v is localized at frequency λ then we have

$$\|v\|_{DU_{\Delta}^2} \lesssim \lambda^{\frac{3}{p}-\frac{1}{2}} \|v\|_{L_x^{p'} L_t^{q'}}, \quad \text{for } p > 2.$$

c) For v localized at frequency λ the DU_{Δ}^2 norm in (a), (b) can be replaced by DU_{λ}^2 .

The second type of estimates we use are bilinear:

Proposition 3.5. *Let $\lambda > 0$. Assume that u, v are solutions to (12) which are λ separated in frequency. Then*

$$(14) \quad \|uv\|_{L^2} \lesssim \lambda^{-\frac{1}{2}} \|u_0\|_{L^2} \|v_0\|_{L^2}$$

Proof. In the Fourier space we have

$$\hat{u}(\tau, \xi) = \hat{u}_0(\xi) \delta_{\tau-\xi^2}, \quad \hat{v}(\tau, \xi) = \hat{v}_0(\xi) \delta_{\tau-\xi^2}$$

Then

$$\widehat{(uv)}(\tau, \xi) = \int_{\xi_1+\xi_2=\xi} \hat{u}_0(\xi_1) \hat{v}_0(\xi_2) \delta_{\tau-\xi_1^2-\xi_2^2} d\xi_1$$

which gives

$$\widehat{(uv)}(\tau, \xi) = \frac{1}{2|\xi_1 - \xi_2|} (\hat{u}_0(\xi_1) \hat{v}_0(\xi_2) + \hat{u}_0(\xi_2) \hat{v}_0(\xi_1))$$

where ξ_1 and ξ_2 are the solutions to

$$\xi_1^2 + \xi_2^2 = \tau, \quad \xi_1 + \xi_2 = \xi$$

We have

$$d\tau d\xi = 2|\xi_1 - \xi_2| d\xi_1 d\xi_2$$

therefore we obtain

$$\|uv\|_{L^2} \lesssim \int |\hat{u}_0(\xi_1)|^2 |\hat{v}_0(\xi_2)|^2 |\xi_1 - \xi_2|^{-1} d\xi_1 d\xi_2$$

The conclusion follows. □

As a consequence we obtain

Corollary 3.6. *a) Let u, v be functions which are λ separated in frequency. Then*

$$(15) \quad \|uv\|_{L^2} \lesssim \lambda^{-\frac{1}{2}} \|u\|_{U_{\Delta}^2} \|v\|_{U_{\Delta}^2}$$

b) Let $\lambda \ll \mu$. Then

$$(16) \quad \|u_{\lambda} v_{\mu}\|_{L^2} \lesssim \mu^{-\frac{1}{2}} \|u_{\lambda}\|_{U_{\lambda}^2} \|v_{\mu}\|_{U_{\mu}^2}$$

Again it suffices to prove these estimates for atoms, and then for solutions to the homogeneous Schrödinger equation. But this follows from the L^4 Strichartz estimates and the bilinear estimate of Proposition 3.5.

At a single point in the paper we need a version of (16) with U_λ^2 replaced by V_λ^2 . This is the only place where we use the V_λ^2 modification of V_Δ^2 .

Proposition 3.7. *Let $\lambda \ll \mu$ and $|I| = 1$. Then*

$$(17) \quad \|\chi_I u_\lambda v_\mu\|_{L^2} \lesssim \mu^{-\frac{1}{2}} \ln \lambda \|u_\lambda\|_{V_\lambda^2} \|v_\mu\|_{U_\mu^2}$$

We note that in order to treat the limiting case $s = -\frac{1}{6}$ it is acceptable to lose $\ln \lambda$, but not $\ln \mu$.

Proof. We split u_λ into a low modulation part and a high modulation part,

$$u_\lambda = Q_{\leq \lambda^2} u_\lambda + Q_{\geq \lambda^2} u_\lambda$$

The first term is estimated in V_Δ^2 simply by counting dyadic regions with respect to modulation. The time truncation regularizes the modulation less than 1, so we are left with about $\log \lambda$ regions.

On the other hand for the second term we use the l^2 summability with respect to rectangles of size $\lambda^{-2} \times \lambda^{-1}$. Precisely, via Bernstein's inequality we have

$$\|Q_{\geq \lambda^2} u_\lambda\|_{l^2 L^\infty} \lesssim \lambda^{\frac{1}{2}} \|u_\lambda\|_{V_\lambda^2}$$

It remains to show that

$$\|v_\mu\|_{L^2(R)} \lesssim \lambda^{-\frac{1}{2}} \mu^{-\frac{1}{2}} \|v_\mu\|_{U_\mu^2}$$

where R is a rectangle as above. By the definition of U_μ^2 the problem reduces to the case when v_μ solves the homogeneous Schrödinger equation. But in that case the above inequality is nothing but the classical local smoothing estimate. \square

4. THE CUBIC NONLINEARITY

In this section we prove Proposition 1.3.

For a dyadic frequency λ we estimate the nonlinearity $|u|^2 u$ at frequency λ in a λ^{4s} time interval I . We take a dyadic decomposition of each of the factors and denote the corresponding frequencies by $\lambda_1, \lambda_2, \lambda_3$. We consider several cases:

Case 1. $\lambda_{1,2,3} \lesssim \lambda$. Then the X^s bounds at the λ_j frequencies are localized to time intervals at least as large as I . Hence we use directly the L^6 Strichartz estimates to obtain

$$\begin{aligned} \lambda^s \|\chi_I u_{\lambda_1} \bar{u}_{\lambda_2} u_{\lambda_3}\|_{DU_\lambda^2} &\lesssim \lambda^{3s} \|\chi_I u_{\lambda_1} \bar{u}_{\lambda_2} u_{\lambda_3}\|_{L^2} \\ &\lesssim (\lambda_1 \lambda_2 \lambda_3)^{-s} \lambda^{3s} \|u_{\lambda_1}\|_{X^s} \|u_{\lambda_2}\|_{X^s} \|u_{\lambda_3}\|_{X^s} \end{aligned}$$

The summation with respect to the λ_j 's is straightforward.

Case 2. $\max\{\lambda_1, \lambda_2, \lambda_3\} = \mu \gg \lambda$. In order to have any output at frequency λ we must have at least two λ_j 's of size μ . Hence we can assume that

$$\{\lambda_1, \lambda_2, \lambda_3\} = \{\alpha, \mu, \mu\} \quad \alpha \lesssim \mu$$

We consider two possibilities:

Case 2a. $\alpha \lesssim \lambda \ll \mu$. We begin with the bound

$$(18) \quad \|P_\lambda(\chi_{[0,1]} v_{\lambda_1} \bar{v}_{\lambda_2} v_{\lambda_3})\|_{DU_\lambda^2} \lesssim \mu^{-1} \log \lambda \|v_{\lambda_1}\|_{U_{\lambda_1}^2} \|v_{\lambda_2}\|_{U_{\lambda_2}^2} \|v_{\lambda_3}\|_{U_{\lambda_3}^2}$$

By duality this is equivalent to

$$\left| \int \chi_{[0,1]} v_{\lambda_1} \bar{v}_{\lambda_2} v_{\lambda_3} \bar{v}_\lambda dx dt \right| \lesssim \mu^{-1} \log \lambda \|v_{\lambda_1}\|_{U_{\lambda_1}^2} \|v_{\lambda_2}\|_{U_{\lambda_2}^2} \|v_{\lambda_3}\|_{U_{\lambda_3}^2} \|v_\lambda\|_{V_\lambda^2}$$

which follows from the bilinear L^2 estimate for the factors $u_\alpha u_\mu$ and $\chi_{[0,1]} u_\lambda u_\mu$.

The frequency μ functions are only controlled on μ^{4s} time intervals. Hence we need to use (18) on each such time interval and then sum up the output from about $\lambda^{4s} \mu^{-4s}$ such intervals. For each interval $|J| = \lambda^{-4s}$ we obtain

$$\lambda^s \|P_\lambda(\chi_J u_{\lambda_1} \bar{u}_{\lambda_2} u_{\lambda_3})\|_{DU_\lambda^2} \lesssim \alpha^{-s} \mu^{-2s} \lambda^{5s} \mu^{-4s} \mu^{-1} \log \lambda \|u_{\lambda_1}\|_{X^s} \|u_{\lambda_2}\|_{X^s} \|u_{\lambda_3}\|_{X^s}$$

Then we sum this up with respect to α and μ . This imposes the restriction $s \geq -\frac{1}{6}$ but only due to very large values of μ . We note that we gain almost $1 + 2s$ derivatives in this computation.

Case 2b. $\alpha \gg \lambda$. For later use we summarize the result in this case in the following

Lemma 4.1. *Let I be an interval of length λ^{4s} . Set*

$$f = P_\lambda \chi_I \sum_{\lambda_1, \lambda_2, \lambda_3 \gg \lambda} u_{\lambda_1} \bar{u}_{\lambda_2} u_{\lambda_3}$$

Then we have the estimates

$$\|Q_{\geq \lambda^2} f\|_{X^{0, -\frac{1}{2}, 1}} \lesssim \lambda^{-1-3s} \|u\|_{X^s}^3$$

respectively

$$\|Q_{\leq \lambda^2} f\|_{L_t^1 L_x^2 + \lambda^{-\frac{1}{4}} L_x^{\frac{4}{3}} L_t^1} \lesssim \lambda^{-1-3s} \|u\|_{X^s}^3$$

Remark 4.2. *The same estimates remain true, and in fact become easier, if we replace \bar{u}_{λ_2} by u_{λ_2} .*

We notice that due to the embedding (7) and to Corollary 3.4 Lemma 4.1 implies that

$$\lambda^s \|f\|_{DU_\lambda^2} \lesssim \lambda^{-1-2s} \|u\|_{X^s}^3$$

which is a gain similar to the one in Case 2(a). Then the proof of Proposition 1.3 is concluded.

Proof of Lemma 4.1. To understand the main feature of this case we denote by (τ_i, ξ_i) the frequencies for each factor and by (τ, ξ) the frequency of the output. Then we must have

$$\xi_1 + \xi_3 = \xi_2 + \xi, \quad \tau_1 + \tau_3 = \tau_2 + \tau$$

This yields

$$(\tau_1 - \xi_1^2) - (\tau_2 - \xi_2^2) + (\tau_3 - \xi_3^2) - (\tau - \xi^2) = 2\xi_1\xi_3 - 2\xi_2\xi$$

Since the size of the frequencies $\{\xi, \xi_1, \xi_2, \xi_3\}$ is $\{\lambda, \alpha, \mu, \mu\}$ with $\lambda \ll \alpha \lesssim \mu$ we conclude that

$$|\tau_1 - \xi_1^2| + |\tau_2 - \xi_2^2| + |\tau_3 - \xi_3^2| + |\tau - \xi^2| \gtrsim \alpha\mu, \quad \text{if } \lambda_2 = \mu$$

respectively

$$|\tau_1 - \xi_1^2| + |\tau_2 - \xi_2^2| + |\tau_3 - \xi_3^2| + |\tau - \xi^2| \gtrsim \mu^2, \quad \text{if } \lambda_2 = \alpha$$

This shows that at least one modulation has to be large, namely at least $\alpha\mu$. To take advantage of this we split each factor into a low and a high modulation component. There are several cases to consider:

Case I. This is when we have three small modulations. Then the output has large modulation. Depending on whether the conjugated factor has lower frequency or not we divide this case in three:

Case I(a) Here we consider the first component of f , namely

$$f_1 = \sum_{\lambda \ll \alpha \ll \mu} f_1^{\alpha\mu} = \sum_{\lambda \ll \alpha \ll \mu} P_\lambda(Q_{\ll \alpha\mu}(\chi_I u_\mu) \overline{Q_{\ll \alpha\mu}(\chi_I u_\mu)} Q_{\ll \alpha\mu}(\chi_I u_\alpha))$$

Then $f_1^{\alpha\mu}$ is localized at modulation $\alpha\mu$. We begin with an L^2 bound for the triple product $P_\lambda(v_\mu \bar{v}_\mu v_\alpha)$. We claim that

$$(19) \quad \|P_\lambda(v_\mu \bar{v}_\mu v_\alpha)\|_{L^2} \lesssim \lambda^{\frac{1}{2}} \mu^{-\frac{1}{2}} \|v_\mu\|_{U_\mu^2} \|v_\mu\|_{U_\mu^2} \|v_\alpha\|_{U_\alpha^2}$$

Indeed, using the energy bound for v_μ and the bilinear L^2 bound for $\bar{v}_\mu v_\alpha$ we obtain

$$\|v_\mu \bar{v}_\mu v_\alpha\|_{L^2 L^1} \lesssim \mu^{-\frac{1}{2}} \|v_\mu\|_{U_\mu^2} \|v_\mu\|_{U_\mu^2} \|v_\alpha\|_{U_\alpha^2}$$

Applying P_λ the estimate (19) follows from Bernstein's inequality.

To use (19) in order to bound f_1 we decompose each factor u_μ , respectively u_α with respect to time intervals of length μ^{4s} , respectively α^{4s} and apply (19) for each combination. The contributions of μ^{4s} separated intervals is negligible since the kernel of $Q_{\ll \alpha\mu}$ decays rapidly on the $(\alpha\mu)^{-1}$ timescale. Hence there are about $\lambda^{4s} \mu^{-4s}$ contributions to add up. We obtain

$$\|f_1^{\mu\alpha}\|_{L^2} \lesssim \lambda^{4s} \mu^{-4s} \alpha^{-s} \mu^{-2s} \lambda^{\frac{1}{2}} \mu^{-\frac{1}{2}} \|u_\mu\|_{X^s} \|u_\mu\|_{X^s} \|u_\alpha\|_{X^s}$$

Since $f_1^{\mu\alpha}$ has modulation $\alpha\mu$ this gives

$$\|f_1^{\mu\alpha}\|_{X^{0, -\frac{1}{2}, 1}} \lesssim \lambda^{\frac{1}{2} + 4s} \mu^{-1 - 6s} \alpha^{-\frac{1}{2} - s} \|u_\mu\|_{X^s} \|u_\mu\|_{X^s} \|u_\alpha\|_{X^s}$$

The summation with respect to the dyadic indices α and μ is straightforward provided that $s \geq -\frac{1}{6}$. We obtain

$$\|f_1\|_{X^{0,-\frac{1}{2},1}} \lesssim \lambda^{-1-3s} \|u\|_{X^s}^3$$

Case I(b) The second component of f is

$$f_2 = \sum_{\lambda \ll \alpha \ll \mu} f_2^{\alpha\mu} = \sum_{\lambda \ll \alpha \ll \mu} P_\lambda(Q_{\ll\mu^2}(\chi_I u_\mu) \overline{Q_{\ll\mu^2}(\chi_I u_\alpha)} Q_{\ll\mu^2}(\chi_I u_\mu))$$

Then $f_2^{\alpha\mu}$ is localized at modulation μ^2 . We can still use (19) since the location of the complex conjugates does not matter. Hence $f_2^{\alpha\mu}$ satisfies the same L^2 bound as $f_1^{\alpha\mu}$. However, because of the larger modulation we obtain a better $X^{0,-\frac{1}{2},1}$ bound, namely

$$\|f_2^{\alpha\mu}\|_{X^{0,-\frac{1}{2},1}} \lesssim \lambda^{\frac{1}{2}+4s} \mu^{-\frac{3}{2}-6s} \alpha^{-s} \|u_\mu\|_{X^s} \|u_\mu\|_{X^s} \|u_\alpha\|_{X^s}$$

After summation with respect to α and μ we obtain the same bound for f_2 as for f_1 ; the difference is that the summation can be carried out for $s \geq -\frac{3}{14}$.

Case I(c) The third component of f is

$$f_3 = \sum_{\lambda \ll \mu} f_3^\mu = \sum_{\lambda \ll \mu} P_\lambda(Q_{\ll\mu^2}(\chi_I u_\mu) \overline{Q_{\ll\mu^2}(\chi_I u_\mu)} Q_{\ll\mu^2}(\chi_I u_\mu))$$

Then $f_3^{\alpha\mu}$ is localized at modulation μ^2 . We claim that (19) still holds. To prove this we first observe that in order for the output to be at low frequency λ , two of the frequencies $\xi_1, -\xi_2, \xi_3$ must be μ separated. Then we use the bilinear L^2 bound for those two factors, and the energy bound for the third.

By (19) we obtain as in Case I(a)

$$\|f_3^\mu\|_{X^{0,-\frac{1}{2},1}} \lesssim \lambda^{\frac{1}{2}+4s} \mu^{-\frac{3}{2}-7s} \|u_\mu\|_{X^s} \|u_\mu\|_{X^s} \|u_\mu\|_{X^s}$$

and the summation with respect to μ can be carried out for $s \geq -\frac{3}{14}$.

Case II. This is when at least one factor has large modulation. Depending on which factor has large modulation and on whether the conjugated factor has lower frequency or not we divide this case in six:

Case II(a). Here we consider

$$f_4 = \sum_{\lambda \ll \alpha \ll \mu} f_4^{\alpha\mu} = \sum_{\lambda \ll \alpha \ll \mu} P_\lambda(\chi_I Q_{\gtrsim\alpha\mu}(\chi_I u_\mu) \overline{u_\mu} u_\alpha) + P_\lambda(\chi_I u_\mu \overline{Q_{\gtrsim\alpha\mu}(\chi_I u_\mu)} u_\alpha)$$

The two terms are similar, so we restrict our attention to the first one. Our starting point is the bound

$$(20) \quad \|Q_{\gtrsim\alpha\mu} v_\mu \overline{v_\mu} v_\alpha\|_{L^1} \lesssim \alpha^{-\frac{1}{2}} \mu^{-1} \|v_\mu\|_{U_\mu^2} \|v_\mu\|_{U_\mu^2} \|v_\alpha\|_{U_\alpha^2}$$

which is obtained from the L^2 estimate for the first factor and a bilinear L^2 estimate for the remaining product.

Low modulation output: By Bernstein's inequality (20) implies

$$(21) \quad \|Q_{\gtrsim\alpha\mu} v_\mu \overline{v_\mu} v_\alpha\|_{L^1 L^2} \lesssim \lambda^{\frac{1}{2}} \alpha^{-\frac{1}{2}} \mu^{-1} \|v_\mu\|_{U_\mu^2} \|v_\mu\|_{U_\mu^2} \|v_\alpha\|_{U_\alpha^2}$$

Summing up (21) over $\lambda^{4s}\mu^{-4s}$ time intervals of length μ^{4s} we obtain

$$\|f_4^{\alpha\mu}\|_{L^1L^2} \lesssim \alpha^{-s}\mu^{-2s}\lambda^{4s}\mu^{-4s}\lambda^{\frac{1}{2}}\alpha^{-\frac{1}{2}}\mu^{-1}\|u_\mu\|_{X^s}\|u_\mu\|_{X^s}\|u_\alpha\|_{X^s}$$

For $s \geq -\frac{1}{6}$ we can sum this up with respect to α and λ to obtain

$$\|f_4\|_{L^1L^2} \lesssim \lambda^{-1-3s}\|u\|_{X^s}^3$$

Intermediate modulation output: Consider now the $X^{0,-\frac{1}{2},1}$ estimate at modulations $\lambda^2 \leq \sigma \leq \alpha\mu$. From (20) and Bernstein's inequality we obtain

$$(22) \quad \|Q_\sigma P_\lambda(Q_{\gtrsim\alpha\mu}v_\mu\overline{v_\mu}v_\alpha)\|_{L^2} \lesssim (\lambda\sigma)^{\frac{1}{2}}\alpha^{-\frac{1}{2}}\mu^{-1}\|v_\mu\|_{U_\mu^2}\|v_\mu\|_{U_\mu^2}\|v_\alpha\|_{U_\alpha^2}$$

The kernel of Q_σ is rapidly decaying off diagonal on the σ^{-1} scale. Then in estimating the sum over μ^{4s} intervals there is a gain coming from the fact that we only need square summability with respect to intervals of size σ^{-1} . We consider two cases.

a) If $\sigma^{-1} < \mu^{4s}$ then we need square summability with respect to intervals of size μ^{4s} so we obtain

$$\|Q_\sigma f_4^{\alpha\mu}\|_{L^2} \lesssim \alpha^{-s}\mu^{-2s}\lambda^{2s}\mu^{-2s}(\lambda\sigma)^{\frac{1}{2}}\alpha^{-\frac{1}{2}}\mu^{-1}\|u_\mu\|_{X^s}\|u_\mu\|_{X^s}\|u_\alpha\|_{X^s}$$

or equivalently

$$\|Q_\sigma f_4^{\alpha\mu}\|_{X^{0,-\frac{1}{2},1}} \lesssim \lambda^{\frac{1}{2}+2s}\alpha^{-\frac{1}{2}-s}\mu^{-1-4s}\|u_\mu\|_{X^s}\|u_\mu\|_{X^s}\|u_\alpha\|_{X^s}$$

Adding up with respect to σ yields

$$\sum_{\sigma=\max\{\lambda^2,\mu^{-4s}\}}^{\alpha\mu} \|Q_\sigma f_4^{\alpha\mu}\|_{X^{0,-\frac{1}{2},1}} \lesssim \lambda^{\frac{1}{2}+2s}\alpha^{-\frac{1}{2}-s}\mu^{-1-4s} \ln \mu \|u_\mu\|_{X^s}\|u_\mu\|_{X^s}\|u_\alpha\|_{X^s}$$

and now the summation with respect to α and μ is straightforward for $s > -\frac{1}{4}$.

b) If $\sigma^{-1} > \mu^{4s}$ then we need square summability with respect to intervals of size σ^{-1} so we obtain

$$\|Q_\sigma f_{21}^{\alpha\mu}\|_{L^2} \lesssim \alpha^{-s}\mu^{-2s}\lambda^{2s}\sigma^{-\frac{1}{2}}\mu^{-4s}(\lambda\sigma)^{\frac{1}{2}}\alpha^{-\frac{1}{2}}\mu^{-1}\|u_\mu\|_{X^s}\|u_\mu\|_{X^s}\|u_\alpha\|_{X^s}$$

or equivalently

$$\|Q_\sigma f_{21}^{\alpha\mu}\|_{X^{0,-\frac{1}{2},1}} \lesssim \sigma^{-\frac{1}{2}}\lambda^{\frac{1}{2}+2s}\alpha^{-\frac{1}{2}-s}\mu^{-1-6s}\|u_\mu\|_{X^s}\|u_\mu\|_{X^s}\|u_\alpha\|_{X^s}$$

Adding up with respect to σ gives

$$\sum_{\sigma=\lambda^2}^{\mu^{-4s}} \|Q_\sigma f_4^{\alpha\mu}\|_{X^{0,-\frac{1}{2},1}} \lesssim \lambda^{-\frac{1}{2}+2s}\alpha^{-\frac{1}{2}-s}\mu^{-1-6s}\|u_\mu\|_{X^s}\|u_\mu\|_{X^s}\|u_\alpha\|_{X^s}$$

In this case the summation with respect to α, μ gains $-2 + 4s$ derivatives, which is better result than needed, but the summation requires $s \geq -\frac{1}{6}$.

High modulation output: Here we estimate the output at modulations $\sigma \gg \alpha\mu$. In order to obtain such an output at least one of the factors must have modulation at least σ . Without any restriction in generality we assume that this is the first factor, as the other cases are considerably simpler. This

gives a trivial improvement in the L^2 bound for the first factor, so instead of (22) we have

$$(23) \quad \|Q_\sigma P_\lambda(Q_{\gtrsim\alpha\mu} v_\mu \overline{v_\mu} v_\alpha)\|_{L^2} \lesssim (\lambda\sigma)^{\frac{1}{2}} \sigma^{-\frac{1}{2}} \mu^{-\frac{1}{2}} \|v_\mu\|_{U_\mu^2} \|v_\mu\|_{U_\mu^2} \|v_\alpha\|_{U_\alpha^2}$$

Then the bound in case (a) above is replaced by

$$\|Q_\sigma f_4^{\alpha\mu}\|_{L^2} \lesssim \alpha^{-s} \mu^{-2s} \lambda^{2s} \mu^{-2s} (\lambda\sigma)^{\frac{1}{2}} \sigma^{-\frac{1}{2}} \mu^{-\frac{1}{2}} \|u_\mu\|_{X^s} \|u_\mu\|_{X^s} \|u_\alpha\|_{X^s}$$

or equivalently

$$\|Q_\sigma f_{21}^{\alpha\mu}\|_{X^{0, -\frac{1}{2}, 1}} \lesssim \lambda^{\frac{1}{2}+2s} \alpha^{-s} \mu^{-\frac{1}{2}-4s} \sigma^{-\frac{1}{2}} \|u_\mu\|_{X^s} \|u_\mu\|_{X^s} \|u_\alpha\|_{X^s}$$

which has better summability with respect to large σ .

Case II(b). This is when the low frequency factor has high modulation. We consider terms of the form

$$f_5 = \sum_{\lambda \ll \alpha \ll \mu} f_5^{\alpha\mu} = \sum_{\lambda \ll \alpha \ll \mu} P_\lambda(\chi_I u_\mu \overline{u_\mu} Q_{\gtrsim\alpha\mu}(\chi_I u_\alpha))$$

Depending on the relative size of α and μ we divide the problem into two sub-cases:

Case II(b)-1. $\lambda\mu \leq \alpha^2$. By orthogonality we can assume that the two u_μ factors are frequency localized in α separated intervals of length α . Then we use the bilinear L^2 bound for their product and the L^2 bound for the high modulation factor to obtain a weaker analogue of (20), namely

$$(24) \quad \|P_\lambda(v_\mu \overline{v_\mu} Q_{\gtrsim\alpha\mu} v_\alpha)\|_{L^1} \lesssim \alpha^{-1} \mu^{-\frac{1}{2}} \|v_\mu\|_{U_\mu^2} \|v_\mu\|_{U_\mu^2} \|v_\alpha\|_{U_\alpha^2}$$

Low modulation output. Compensating for the weaker bound (24), in this case there is an improvement in the summation over time intervals. We decompose the λ^{4s} time interval I in two steps. First we split it into $\lambda^{4s} \alpha^{-4s}$ time intervals of length α^{4s} , which gives a $\lambda^{4s} \alpha^{-4s}$ factor in the summation. Secondly we split each α^{4s} time interval into $\alpha^{4s} \mu^{-4s}$ time intervals of length μ^{4s} . Since the frequency α factor is square summable with respect to this partition, by Cauchy-Schwartz this gives only an $\alpha^{2s} \mu^{-2s}$ factor in the summation. We obtain

$$\|f_5^{\alpha\mu}\|_{L^1} \lesssim \alpha^{-s} \mu^{-2s} \lambda^{4s} \alpha^{-2s} \mu^{-2s} \alpha^{-1} \mu^{-\frac{1}{2}} \|u_\mu\|_{X^s} \|u_\mu\|_{X^s} \|u_\alpha\|_{X^s}$$

which by Bernstein's inequality implies that

$$\|Q_{\leq \lambda^2} f_5^{\alpha\mu}\|_{L^1 L^2} \lesssim \lambda^{\frac{1}{2}+4s} \alpha^{-1-3s} \mu^{-\frac{1}{2}-4s} \|u_\mu\|_{X^s} \|u_\mu\|_{X^s} \|u_\alpha\|_{X^s}$$

This is summable with respect to large μ only if $s \geq -\frac{1}{8}$. However, the restriction $\lambda\mu \leq \alpha^2$ improves the summation. Assuming $s < -\frac{1}{8}$ the μ summation yields

$$\sum_{\alpha \leq \mu \leq \lambda^{-1} \alpha^2} \|Q_{\leq \lambda^2} f_5^{\alpha\mu}\|_{L^1 L^2} \lesssim \lambda^{1+8s} \alpha^{-2-11s} \|u\|_{X^s}^2 \|u_\alpha\|_{X^s}$$

which is summable with respect to α for $s > -\frac{2}{11}$.

Intermediate modulation output. $\lambda^2 \leq \sigma \leq \alpha\mu$. Here we argue as in Case II(a) but using (24) instead of (20). From (24) and Bernstein's inequality we obtain

$$(25) \quad \|Q_\sigma P_\lambda(v_\mu \overline{v_\mu} Q_{\gtrsim \alpha\mu} v_\alpha)\|_{L^2} \lesssim (\lambda\sigma)^{\frac{1}{2}} \alpha^{-1} \mu^{-\frac{1}{2}} \|v_\mu\|_{U_\mu^2} \|v_\mu\|_{U_\mu^2} \|v_\alpha\|_{U_\alpha^2}$$

We split this again depending on σ but also taking into account the improved summability up to the α^{4s} time scale, as discussed above for the case of low modulation output.

a) If $\sigma^{-1} \leq \mu^{4s}$ then due to the square integrability of u_α in each $\lambda^{4\alpha}$ time interval we have an interval summation factor $\lambda^{2s} \alpha^{-2s}$. Hence

$$\|Q_\sigma f_5^{\alpha\mu}\|_{L^2} \lesssim \alpha^{-s} \mu^{-2s} \lambda^{2s} \alpha^{-2s} (\lambda\sigma)^{\frac{1}{2}} \alpha^{-1} \mu^{-\frac{1}{2}} \|u_\mu\|_{X^s} \|u_\mu\|_{X^s} \|u_\alpha\|_{X^s}$$

which yields

$$\|Q_\sigma f_5^{\alpha\mu}\|_{X^{0, -\frac{1}{2}, 1}} \lesssim \lambda^{\frac{1}{2}+2s} \alpha^{-1-3s} \mu^{-\frac{1}{2}-2s} \|u_\mu\|_{X^s} \|u_\mu\|_{X^s} \|u_\alpha\|_{X^s}$$

The summation with respect to σ , α and μ is straightforward for $s \geq -\frac{1}{4}$.

b) The case $\lambda^2 < \sigma < \mu^{-4s}$ is somewhat worse because the kernel of Q_σ decays only on the σ^{-1} scale which is now larger than μ^{4s} . Hence inputs from μ^{4s} time intervals within each σ^{-1} time interval are no longer orthogonal. This yields a weaker interval summation factor, namely $\lambda^{2s} \alpha^{-2s} \sigma^{-\frac{1}{2}} \mu^{2s}$. Hence

$$\|Q_\sigma f_5^{\alpha\mu}\|_{L^2} \lesssim \alpha^{-s} \mu^{-2s} \lambda^{2s} \alpha^{-2s} \sigma^{-\frac{1}{2}} \mu^{-2s} (\lambda\sigma)^{\frac{1}{2}} \alpha^{-1} \mu^{-\frac{1}{2}} \|u_\mu\|_{X^s} \|u_\mu\|_{X^s} \|u_\alpha\|_{X^s}$$

which yields

$$\|Q_\sigma f_5^{\alpha\mu}\|_{X^{0, -\frac{1}{2}, 1}} \lesssim \sigma^{-\frac{1}{2}} \lambda^{\frac{1}{2}+2s} \alpha^{-1-3s} \mu^{-\frac{1}{2}-4s} \|u_\mu\|_{X^s} \|u_\mu\|_{X^s} \|u_\alpha\|_{X^s}$$

The summation with respect to σ is straightforward:

$$\sum_{\sigma=\lambda^2}^{\mu^{-4s}} \|Q_\sigma f_5^{\alpha\mu}\|_{X^{0, -\frac{1}{2}, 1}} \lesssim \lambda^{-\frac{1}{2}+2s} \alpha^{-1-3s} \mu^{-\frac{1}{2}-4s} \|u_\mu\|_{X^s} \|u_\mu\|_{X^s} \|u_\alpha\|_{X^s}$$

However, in the α and μ summation we need to use the restriction $\lambda\mu \leq \alpha^2$ exactly as in the case of low modulation output.

High modulation output: Here we estimate the output at modulations $\sigma \gg \alpha\mu$. Then we can assume that the last factor has modulation at least σ therefore it satisfies a better L^2 bound, which leads to

$$(26) \quad \|Q_\sigma P_\lambda(v_\mu \overline{v_\mu} Q_{\gtrsim \alpha\mu} v_\alpha)\|_{L^2} \lesssim \lambda^{\frac{1}{2}} \alpha^{-\frac{1}{2}} \|v_\mu\|_{U_\mu^2} \|v_\mu\|_{U_\mu^2} \|v_\alpha\|_{U_\alpha^2}$$

Then instead of the estimate in case (a) above we obtain

$$\|Q_\sigma f_5^{\alpha\mu}\|_{L^2} \lesssim \alpha^{-s} \mu^{-2s} \lambda^{2s} \alpha^{-2s} \lambda^{\frac{1}{2}} \alpha^{-\frac{1}{2}} \|u_\mu\|_{X^s} \|u_\mu\|_{X^s} \|u_\alpha\|_{X^s}$$

or equivalently

$$\|Q_\sigma f_5^{\alpha\mu}\|_{X^{0, -\frac{1}{2}, 1}} \lesssim \lambda^{\frac{1}{2}+2s} \alpha^{-\frac{1}{2}-3s} \mu^{-2s} \sigma^{-\frac{1}{2}} \|u_\mu\|_{X^s} \|u_\mu\|_{X^s} \|u_\alpha\|_{X^s}$$

and hence

$$\|Q_{\gg \alpha\mu} f_5^{\alpha\mu}\|_{X^{0, -\frac{1}{2}, 1}} \lesssim \lambda^{-1/2+2s} \alpha^{-1-3s} \mu^{-\frac{1}{2}-4s} \|u_\mu\|_{X^s} \|u_\mu\|_{X^s} \|u_\alpha\|_{X^s}.$$

The condition $\lambda\mu \leq \alpha^2$ is again needed.

Case II(b)-2: $\lambda\mu > \alpha^2$. Then the arguments in the previous case fail to provide enough decay in order to insure summability for very large μ .

Low modulation output. In this case we are able to establish the following improvement of (24),

$$(27) \quad \|Q_{<\lambda^2} P_\lambda(v_\mu \bar{v}_\mu Q_{\gtrsim\alpha\mu} v_\alpha)\|_{L^1 L^2} \lesssim \mu^{-1} \|v_\mu\|_{U_\mu^2} \|v_\mu\|_{U_\mu^2} \|v_\alpha\|_{U_\alpha^2}$$

The rest of the analysis is similar to the computation in Case II(b)-1. The only difference is that here we gain an extra factor of $\alpha(\lambda\mu)^{-\frac{1}{2}} \leq 1$, which improves the summation for large μ .

To prove (27) we only use the L^2 bound for v_α . Then, using the atomic decomposition for each of the two v_μ factors, we conclude that it suffices to prove (27) in the case when both v_μ factors solve the linear equation. By orthogonality we can assume that both are frequency localized in α intervals which are α separated. Then we use the L^2 bound for the product of $u_\mu \bar{u}_\mu$,

$$\|v_\mu \bar{v}_\mu\|_{L^2} \lesssim \alpha^{-\frac{1}{2}} \|v_\mu\|_{U_\mu^2} \|v_\mu\|_{U_\mu^2}$$

However, due to the frequency localization we also obtain that the product is Fourier localized in a thin rectangle R of size $\alpha^2/\mu \times \alpha\mu$ at slope μ^{-1} . Next we consider the product

$$(v_\mu \bar{v}_\mu) \cdot (Q_{\gtrsim\alpha\mu} v_\alpha)$$

which we view as a product of two L^2 functions with different Fourier localizations. The product is only estimated in a Fourier rectangle of size $\lambda \times \lambda^2$, therefore by orthogonality it suffices to estimate the product assuming that both factors are Fourier localized in rectangles R_1, R_2 of similar size. The intersection $R_0 = R \cap R_1$ is a shorter rectangle of size $\alpha^2/\mu \times \lambda^2$. Our assumption $\alpha^2 < \lambda\mu$ insures that R_0 is essentially vertical. But by Bernstein's inequality we have the pointwise bound

$$\|g\|_{L^2 L^\infty} \lesssim \alpha\mu^{-\frac{1}{2}} \|g\|_{L^2}, \quad \text{supp } \hat{g} \subset R_0$$

therefore (27) follows.

Intermediate modulation output, $\lambda^2 < \sigma \lesssim \alpha\mu$. Then a similar argument applies. R_0 has size $\alpha^2/\mu \times \sigma$, which yields the pointwise bound

$$\|g\|_{L^\infty} \lesssim \alpha\mu^{-\frac{1}{2}} \sigma^{\frac{1}{2}} \|g\|_{L^2}, \quad \text{supp } \hat{g} \subset R_0$$

This in turn leads to

$$(28) \quad \|Q_\sigma P_\lambda(v_\mu \bar{v}_\mu Q_{\gtrsim\alpha\mu} v_\alpha)\|_{L^2} \lesssim \sigma^{\frac{1}{2}} \alpha^{\frac{1}{2}} \mu^{-1} \|v_\mu\|_{U_\mu^2} \|v_\mu\|_{U_\mu^2} \|v_\alpha\|_{U_\alpha^2}$$

which is again an improvement of $\alpha(\lambda\mu)^{-\frac{1}{2}}$ over the similar computation in Case II(b)-1.

Large modulation output, $\sigma \gg \alpha\mu$. Then we can assume that the third factor has modulation at least σ . Also R has size $\alpha^2/\mu \times \alpha\mu$, therefore

$$\|g\|_{L^\infty} \lesssim \alpha^{\frac{3}{2}} \|g\|_{L^2}, \quad \text{supp } \hat{g} \subset R$$

which implies that

$$(29) \quad \|Q_\sigma P_\lambda(v_\mu \overline{v_\mu} Q_{\gtrsim \sigma} v_\alpha)\|_{L^2} \lesssim \sigma^{-\frac{1}{2}} \alpha \|v_\mu\|_{U_\mu^2} \|v_\mu\|_{U_\mu^2} \|v_\alpha\|_{U_\alpha^2},$$

an improvement of at least $\alpha(\lambda\mu)^{-\frac{1}{2}}$ over (26). The conclusion follows in a similar fashion.

Case II(c). This is when the low frequency factor is conjugated but does not have high modulation. We consider terms of the form

$$f_6 = \sum_{\lambda \ll \alpha \lesssim \mu} f_6^{\alpha\mu} = \sum_{\lambda \ll \alpha \lesssim \mu} P_\lambda(\chi_I Q_{\gtrsim \mu^2}(\chi_I u_\mu) \overline{u_\alpha} u_\mu)$$

If $\alpha \ll \mu$ then the last two factors are μ separated in frequency. But even if $\alpha \approx \mu$, in order for the final output to be at frequency λ the two last factors must still be μ separated. Then we can use the trilinear bound

$$(30) \quad \|P_\lambda(Q_{\gtrsim \alpha\mu} v_\mu \overline{v_\alpha} v_\mu)\|_{L^1} \lesssim \alpha^{-1/2} \mu^{-1} \|v_\mu\|_{U_\mu^2} \|v_\mu\|_{U_\mu^2} \|v_\alpha\|_{U_\alpha^2},$$

obtained by estimating in L^2 the first factor and the remaining product.

The constants here are better than the ones in Case II(a), and the rest of the argument proceeds as there without any significant changes.

Case II(d). This is when the low frequency factor is conjugated and has high modulation. We consider terms of the form

$$f_7 = \sum_{\lambda \ll \alpha \ll \mu} f_7^{\alpha\mu} = \sum_{\lambda \ll \alpha \ll \mu} P_\lambda(\chi_I u_\mu \overline{Q_{\gtrsim \mu^2}(\chi_I u_\alpha)} u_\mu)$$

In order for the final output to be at frequency λ the two frequency μ factors must still be μ separated. This leads to the trilinear bound

$$(31) \quad \|P_\lambda(v_\mu \overline{Q_{\gtrsim \alpha\mu} v_\alpha} v_\mu)\|_{L^1} \lesssim \mu^{-\frac{3}{2}} \|v_\mu\|_{U_\mu^2} \|v_\mu\|_{U_\mu^2} \|v_\alpha\|_{U_\alpha^2},$$

and the argument is completed again as in Case II(a) but with better constants.

Case II(e). This is when all frequencies are equal and the conjugated factor has high modulation. We consider terms of the form

$$f_8 = \sum_{\lambda \ll \mu} f_8^\mu = \sum_{\lambda \ll \mu} P_\lambda(\chi_I u_\mu \overline{Q_{\gtrsim \mu^2}(\chi_I u_\mu)} u_\mu)$$

In some sense this is the worst case because we cannot enforce any frequency separation between the two unconjugated factors. We still want to gain some power of μ in order to have summability for large μ . At least to some extent we can do this by the lateral Strichartz estimates in Corollary 3.3(bc) to obtain

$$(32) \quad \|v_\mu \overline{Q_{\gtrsim \mu^2} v_\mu} v_\mu\|_{L_x^{\frac{4}{3}} L_t^1} \lesssim \mu^{-\frac{5}{4}} \|v_\mu\|_{U_\mu^2} \|v_\mu\|_{U_\mu^2} \|v_\mu\|_{U_\mu^2}$$

This is done for instance by using the $L_x^\infty L_t^2$ bound for one v_μ factor, respectively the $L_x^4 L_t^\infty$ for the other v_μ factor.

Low modulation output: After summation with respect to μ^{4s} time intervals (32) gives

$$\|Q_{<\lambda^2} P_\lambda(u_\mu \overline{Q_{\gtrsim \mu^2} u_\mu u_\mu})\|_{L_x^{\frac{4}{3}} L_t^1} \lesssim \lambda^{4s} \mu^{-4s} \mu^{-3s} \mu^{-\frac{5}{4}} \|u_\mu\|_{X^s}^3$$

which is easily summed up with respect to μ for $s \geq -\frac{5}{28}$.

Intermediate modulation output: $\lambda^2 < \sigma \leq \mu^2$. From (32) combined with Bernstein's inequality we obtain

$$(33) \quad \|Q_\sigma P_\lambda(v_\mu \overline{Q_{\gtrsim \mu^2} v_\mu v_\mu})\|_{L^2} \lesssim \sigma^{\frac{1}{2}} \lambda^{\frac{1}{4}} \mu^{-\frac{5}{4}} \|v_\mu\|_{U_\mu^2} \|v_\mu\|_{U_\mu^2} \|v_\mu\|_{U_\mu^2}$$

Adding this up with respect to μ^{4s} time intervals yields

$$\|Q_\sigma P_\lambda(u_\mu \overline{Q_{\gtrsim \mu^2} u_\mu u_\mu})\|_{L^2} \lesssim \lambda^{\frac{1}{4}+4s} \sigma^{\frac{1}{2}} \mu^{-\frac{5}{4}-7s} \|u_\mu\|_{X^s}^3$$

or equivalently

$$\|Q_\sigma P_\lambda(u_\mu \overline{Q_{\gtrsim \mu^2} u_\mu u_\mu})\|_{X^{0, -\frac{1}{2}, 1}} \lesssim \lambda^{\frac{1}{4}+4s} \mu^{-\frac{5}{4}-7s} \|u_\mu\|_{X^s}^3$$

which is easily summed up with respect to σ and μ for $s > -\frac{5}{28}$.

Finally, the output modulations which are larger than μ^2 are treated as in the first case.

High modulation output: $\sigma \gg \mu^2$. Without any restriction in generality we assume that the second factor has modulation at least σ . Instead of (33) we get

$$(34) \quad \|Q_\sigma P_\lambda(v_\mu \overline{Q_{\gtrsim \sigma} v_\mu v_\mu})\|_{L^2} \lesssim \lambda^{\frac{1}{4}} \mu^{-\frac{1}{4}} \|v_\mu\|_{U_\mu^2} \|v_\mu\|_{U_\mu^2} \|v_\mu\|_{U_\mu^2}$$

Adding this up with respect to $\lambda^{4s} \mu^{-4s}$ time intervals yields

$$\|Q_\sigma P_\lambda(u_\mu \overline{Q_{\gtrsim \sigma} u_\mu u_\mu})\|_{L^2} \lesssim \lambda^{\frac{1}{4}+4s} \mu^{-\frac{1}{4}-7s} \|u_\mu\|_{X^s}^3$$

or equivalently

$$\|Q_\sigma P_\lambda(u_\mu \overline{Q_{\gtrsim \mu^2} u_\mu u_\mu})\|_{X^{0, -\frac{1}{2}, 1}} \lesssim \lambda^{\frac{1}{4}+4s} \sigma^{-\frac{1}{2}} \mu^{-\frac{1}{4}-7s} \|u_\mu\|_{X^s}^3$$

The summation with respect to σ and μ requires again $s > -\frac{5}{28}$. □

5. THE ENERGY CONSERVATION

It remains to study the conservation of the H^s energy. We first set

$$E_0(u) = \langle A(D)u, u \rangle$$

For the straight H^s energy conservation it suffices to take

$$a(\xi) = (1 + \xi^2)^s$$

However in order to gain the uniformity in t required by (10) we need to allow a slightly larger class of symbols.

Definition 5.1. Let $s \in \mathbb{R}$ and $\epsilon > 0$. Then S_ϵ^s is the class of spherically symmetric symbols with the following properties:

(i) symbol regularity,

$$|\partial^\alpha a(\xi)| \lesssim a(\xi)(1 + \xi^2)^{-\alpha/2}$$

(ii) decay at infinity,

$$s \leq \frac{\ln a(\xi)}{\ln(1 + \xi^2)} \leq s + \epsilon, \quad s - \epsilon \leq \frac{d \ln a(\xi)}{d \ln(1 + \xi^2)} \leq s + \epsilon$$

Here ϵ is a small parameter.

We compute the derivative of E_0 along the flow,

$$\frac{d}{dt} E_0(u) = R_4(u) = 2\Re \langle iA(D)u, |u|^2 u \rangle$$

We write R_4 as a multilinear operator in the Fourier space,

$$R_4(u) = 2\Re \int_{P_4} ia(\xi_1)\hat{u}(\xi_1)\hat{u}(\xi_2)\overline{\hat{u}(\xi_3)\hat{u}(\xi_4)}d\sigma$$

where

$$P_4 = \{\xi_1 + \xi_2 - \xi_3 - \xi_4 = 0\}$$

This can be symmetrized,

$$R_4(u) = \frac{1}{2}\Re \int_{P_4} i(a(\xi_1) + a(\xi_2) - a(\xi_3) - a(\xi_4))\hat{u}(\xi_1)\hat{u}(\xi_2)\overline{\hat{u}(\xi_3)\hat{u}(\xi_4)}d\sigma$$

Following a variation of the I -method, see Tao [9]-3.9 and references therein, we seek to cancel this term by perturbing the energy, namely by

$$E_1(u) = \int_{P_4} b_4(\xi_1, \xi_2, \xi_3, \xi_4)\hat{u}(\xi_1)\hat{u}(\xi_2)\overline{\hat{u}(\xi_3)\hat{u}(\xi_4)}d\sigma$$

To determine the best choice for B we compute

$$\begin{aligned} \frac{d}{dt} E_1(u) &= \int_{P_4} ib_4(\xi_1, \xi_2, \xi_3, \xi_4)(\xi_1^2 + \xi_2^2 - \xi_3^2 - \xi_4^2)\hat{u}(\xi_1)\hat{u}(\xi_2)\overline{\hat{u}(\xi_3)\hat{u}(\xi_4)}d\sigma \\ &+ R_6(u) \end{aligned}$$

where $R_6(u)$ is given by

$$R_6(u) = 4\Re \int_{\xi_1 + \xi_2 - \xi_3 - \xi_4 = 0} ib_4(\xi_1, \xi_2, \xi_3, \xi_4)\widehat{|u|^2 u}(\xi_1)\hat{u}(\xi_2)\overline{\hat{u}(\xi_3)\hat{u}(\xi_4)}d\sigma$$

To achieve the cancellation of the quadrilinear form we define b_4 by

(35)

$$b_4(\xi_1, \xi_2, \xi_3, \xi_4) = -\frac{a(\xi_1) + a(\xi_2) - a(\xi_3) - a(\xi_4)}{\xi_1^2 + \xi_2^2 - \xi_3^2 - \xi_4^2}, \quad (\xi_1, \xi_2, \xi_3, \xi_4) \in P_4$$

Summing up the result of our computation, we obtain

$$(36) \quad \frac{d}{dt}(E_0(u) + E_1(u)) = R_6(u)$$

In order to estimate the size of $E_1(u)$ and of R_6 we need to understand the size and regularity of b . A-priori b is only defined on the diagonal P_4 . However, in order to separate variables easier it is convenient to extend it off diagonal in a smooth way.

Proposition 5.2. *Assume that $a \in S_c^s$ with $s + \epsilon \leq 0$. Then for each dyadic $\lambda \leq \alpha \leq \mu$ there is an extension of b_4 from the diagonal set*

$$\{(\xi_1, \xi_2, \xi_3, \xi_4) \in P_4, |\xi_1| \approx \lambda, |\xi_3| \approx \alpha, |\xi_2|, |\xi_4| \approx \mu\}$$

to the full dyadic set

$$\{|\xi_1| \approx \lambda, |\xi_3| \approx \alpha, |\xi_2|, |\xi_4| \approx \mu\}$$

which satisfies the size and regularity conditions

$$(37) \quad |\partial_1^{\beta_1} \partial_2^{\beta_2} \partial_3^{\beta_3} \partial_4^{\beta_4} b_4(\xi_1, \xi_2, \xi_3, \xi_4)| \lesssim a(\lambda) \alpha^{-1} \mu^{-1} \lambda^{-\beta_1} \alpha^{-\beta_2} \mu^{-\beta_3 - \beta_4}$$

Here the implicit constants are independent of λ, α, μ but may depend on the β_j 's.

Proof. We first note that on P_4 we have the factorization

$$\xi_1^2 + \xi_2^2 - \xi_3^2 - \xi_4^2 = 2(\xi_1 - \xi_3)(\xi_1 - \xi_4)$$

along with all versions of it due to the symmetries of P_4 . We consider several cases:

(a) $\lambda \ll \alpha \leq \mu$. Then the extension of b_4 is defined using the formula

$$b_4(\xi_1, \xi_2, \xi_3, \xi_4) = -\frac{a(\xi_1) + a(\xi_2) - a(\xi_3) - a(\xi_4)}{2(\xi_1 - \xi_3)(\xi_1 - \xi_4)}$$

and its size and regularity properties are straightforward since $|\xi_1 - \xi_3| \approx \alpha$ and $|\xi_1 - \xi_4| \approx \mu$.

(b) $\lambda \approx \alpha \ll \mu$. Then the extension of b_4 is defined using the formula

$$b_4(\xi_1, \xi_2, \xi_3, \xi_4) = -\frac{a(\xi_1) - a(\xi_3)}{2(\xi_1 - \xi_3)(\xi_1 - \xi_4)} - \frac{a(\xi_2) - a(\xi_4)}{2(\xi_4 - \xi_2)(\xi_1 - \xi_4)}$$

Now only $|\xi_1 - \xi_4| \approx \mu$ is an elliptic factor, while the remaining quotients exhibit suitable cancellation properties.

(c) $\lambda \approx \alpha \approx \mu$. Then the extension of b_4 is defined by

$$b_4(\xi_1, \xi_2, \xi_3, \xi_4) = -\frac{a(\xi_1) + a(\xi_2) - a(\xi_1 + \xi_2 - \xi_4) - a(\xi_4)}{2(\xi_1 - \xi_4)(\xi_2 - \xi_4)}$$

To see that this is a smooth function on the appropriate scale we write it in the form

$$\begin{aligned} b_4(\xi_1, \xi_2, \xi_3, \xi_4) &= \frac{1}{2(\xi_2 - \xi_4)} \left(\frac{a(\xi_4) - a(\xi_1)}{\xi_1 - \xi_4} - \frac{a(\xi_2) - a(\xi_2 + \xi_1 - \xi_4)}{\xi_1 - \xi_4} \right) \\ &= \frac{q(\xi_4, \xi_1) - q(\xi_4 + (\xi_2 - \xi_4), \xi_1 + (\xi_2 - \xi_4))}{2(\xi_2 - \xi_4)} \end{aligned}$$

where q is the smooth function

$$q(\xi, \eta) = \frac{q(\xi) - q(\eta)}{\xi - \eta}$$

□

The contribution of E_1 to the energy is easy to control,

Proposition 5.3. *Assume that $a \in S_\epsilon^s$ with $-\frac{1}{2} < s - \epsilon < s + \epsilon \leq 0$. Then*

$$(38) \quad |E_1(u)| \lesssim E_0(u)^2$$

We note that the threshold $s = -\frac{1}{2}$ in the proposition is consistent with the scaling.

Proof. We organize the four frequencies ξ_1, ξ_2, ξ_3 and ξ_4 in dyadic regions of size $\lambda \leq \alpha \leq \mu = \mu$. The pointwise bound on b is all we need for the proof since in the Fourier space one sees that only the size of the Fourier transform matters. For a function u we define \tilde{u} by $\hat{\tilde{u}} = |\hat{u}|$. We obtain

$$\begin{aligned} |E_1(u)| &\lesssim \sum_{\lambda \leq \alpha \leq \mu = \mu} |E_1(u_\lambda, u_\alpha, u_\mu, u_\mu)| \\ &\lesssim \sum_{\lambda \leq \alpha \leq \mu = \mu} a(\lambda) \alpha^{-1} \mu^{-1} \|\tilde{u}_\lambda \tilde{u}_\alpha \tilde{u}_\mu \tilde{u}_\mu\|_{L^1} \\ &\lesssim \sum_{\lambda \leq \alpha \leq \mu = \mu} a(\lambda) \alpha^{-1} \mu^{-1} \|\tilde{u}_\lambda\|_{L^\infty} \|\tilde{u}_\alpha\|_{L^\infty} \|\tilde{u}_\mu\|_{L^2}^2 \\ &\lesssim \sum_{\lambda \leq \alpha \leq \mu = \mu} \lambda^{\frac{1}{2}} a(\lambda) \alpha^{-\frac{1}{2}} \mu^{-1} \|\tilde{u}_\lambda\|_{L^2} \|\tilde{u}_\alpha\|_{L^2} \|\tilde{u}_\mu\|_{L^2}^2 \\ &\lesssim E_0(u)^2 \sum_{\lambda \leq \alpha \leq \mu} \frac{\lambda a(\lambda)}{\alpha a(\alpha) \mu^2 a^2(\mu)} \end{aligned}$$

where at the last step we have used Cauchy-Schwartz with respect to all parameters. Since $s - \epsilon > -\frac{1}{2}$ it follows that the function $\lambda a(\lambda)$ increases polynomially with respect to λ therefore the last sum is finite.

□

The more difficult result we need to prove is

Proposition 5.4. *Assume that $a \in S_\epsilon^s$ with $s + \epsilon \leq 0$ and $s \geq -\frac{1}{6}$. Then we have*

$$(39) \quad \left| \int_0^1 R_6(u) dx dt \right| \lesssim \|u\|_{X^s}^6$$

We note that the restriction on the symbol a above is very mild. This is because, as one can see in the proof below, there is always a low frequency gain in the estimates. The main condition $s \geq -\frac{1}{6}$ arises in the summation with respect to high frequency factors.

Proof. We consider a full dyadic decomposition and express the above integral in the Fourier space as a sum of terms of the form

$$I = \int_0^1 \int_{P_6} b_4(\xi_1, \xi_2, \xi_3, \xi_0) \hat{u}_{\lambda_1}(\xi_1) \overline{\hat{u}_{\lambda_2}(\xi_2)} \hat{u}_{\lambda_3}(\xi_3) P_{\lambda_0}(\overline{\hat{u}_{\lambda_4}(\xi_4)} \hat{u}_{\lambda_5}(\xi_5) \overline{\hat{u}_{\lambda_6}(\xi_6)}) d\xi dt$$

where

$$P_6 = \{\xi_1 + \xi_3 + \xi_5 = \xi_2 + \xi_4 + \xi_6\}, \quad \xi_0 = \xi_1 - \xi_2 + \xi_3$$

Since b is smooth in each variable on the corresponding dyadic scale we can expand it in a rapidly convergent Fourier series and separate the variables. Hence from here on we replace b by the pointwise bound given in Proposition 5.2. There are two cases to consider:

Case 1: $\lambda_0 \ll \lambda_{4,5,6}$. Then for the frequency λ_0 factor we use Lemma 4.1. We denote

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_0) = (\lambda, \alpha, \mu, \mu), \quad \lambda \leq \alpha \leq \mu$$

and

$$f_{\lambda_0} = \sum_{\lambda_{4,5,6} \gg \lambda_0} P_{\lambda_0}(\overline{\hat{u}_{\lambda_4}(\xi_4)} \hat{u}_{\lambda_5}(\xi_5) \overline{\hat{u}_{\lambda_6}(\xi_6)})$$

We also recall the bound for b_4 , namely

$$|b_4| \lesssim a(\lambda) \alpha^{-1} \mu^{-1}$$

Case 1(a) $\lambda_0 = \mu$. We consider the three possible terms in f_{λ_0} . For the $L^1 L^2$ term we bound u_λ, u_α in L^∞ and u_μ in $L_x^\infty L_t^2$. We also sum up with respect to μ^{4s} time intervals. This yields

$$\begin{aligned} |I| &\lesssim \mu^{-4s} \lambda^{-s} \alpha^{-s} \mu^{-s} \lambda^{\frac{1}{2}} \alpha^{\frac{1}{2}} \mu^{-1-3s} a(\lambda) \alpha^{-1} \mu^{-1} \|u\|_{X^s}^6 \\ &= \lambda^{\frac{1}{2}-s} a(\lambda) \alpha^{-s-\frac{1}{2}} \mu^{-2-8s} \|u\|_{X^s}^6 \end{aligned}$$

The summation with respect to λ, μ and α is straightforward if $s > -\frac{1}{4}$.

For the $L_x^{\frac{4}{3}} L_t^1$ term in f we bound u_λ, u_α in L^∞ and u_μ in $L^4 L^\infty$. This yields

$$|I| \lesssim \mu^{-4s} \lambda^{-s} \alpha^{-s} \mu^{-s} \lambda^{\frac{1}{2}} \alpha^{\frac{1}{2}} \mu^{\frac{1}{4}} \mu^{-\frac{5}{4}-3s} a(\lambda) \alpha^{-1} \mu^{-1} \|u\|_{X^s}^6$$

which gives the same outcome as in the previous case.

For the L^2 part of f at modulation $\sigma \gg \mu^2$ we note that at least one other factor must also have modulation at least σ . We bound that factor in L^2 and the other two in L^∞ to obtain

$$\begin{aligned} |I| &\lesssim \mu^{-4s} \lambda^{-s} \alpha^{-s} \mu^{-s} \alpha^{\frac{1}{2}} \mu^{\frac{1}{2}} \mu^{-1-3s} a(\lambda) \alpha^{-1} \mu^{-1} \|u\|_{X^s}^6 \\ &= a(\lambda) \lambda^{-s} \alpha^{-s-\frac{1}{2}} \mu^{-\frac{3}{2}-8s} \|u\|_{X^s}^6 \end{aligned}$$

which is then summed with respect to λ, μ and α provided that $s > -\frac{1}{5}$.

Case 1(b) $\lambda_0 = \alpha \ll \mu$. This case is simpler; As a consequence of Lemma 4.1 and of Bernstein's inequality we have the L^2 type bounds

$$(40) \quad (\alpha \mu)^{-\frac{1}{2}} \|Q_{\leq \alpha \mu} f_{\lambda_0}\|_{L^2} + \|Q_{\gg \alpha \mu} f_{\lambda_0}\|_{X^{0, -\frac{1}{2}, 1}} \lesssim \lambda_0^{-1-3s} \|u\|_{X^s}^3$$

which is all that we need in the sequel.

For the low modulation part of f_{λ_0} we use the first part of (40). By orthogonality we can localize the frequency μ factors to α intervals. Then we use the bilinear L^2 estimate for $u_\lambda u_\mu$ and the pointwise bound for the other u_μ factor. This gives

$$\begin{aligned} |I| &\lesssim \alpha^{-2s} \mu^{-2s} \lambda^{-s} \mu^{-2s} \alpha^{\frac{1}{2}} \mu^{-\frac{1}{2}} \alpha^{-1-3s} (\alpha\mu)^{\frac{1}{2}} a(\lambda) \alpha^{-1} \mu^{-1} \|u\|_{X^s}^6 \\ &= a(\lambda) \lambda^{-s} \alpha^{-1-5s} \mu^{-1-4s} \|u\|_{X^s}^6 \end{aligned}$$

The factor $\alpha^{-2s} \mu^{-2s}$ above comes from summation over small time intervals. This is better than the earlier μ^{-4s} factor because $Q_{\leq \alpha\mu} f_{\lambda_0}$ is square integrable on the better α^{4s} time scale. This is summable with respect to λ , α and μ if $s \geq -\frac{2}{9}$.

For the L^2 part of f_{λ_0} at modulation $\sigma \gg \alpha\mu$ we note that at least one other factor must also have modulation at least σ . We bound that factor in L^2 and the other two in L^∞ to obtain

$$|I| \lesssim \alpha^{-2s} \mu^{-2s} \lambda^{-s} \mu^{-2s} \alpha \alpha^{-1-3s} a(\lambda) \alpha^{-1} \mu^{-1} \|u\|_{X^s}^6$$

which is the same result as above. Note that only an $\alpha^{\frac{1}{2}}$ factor is lost in the pointwise bound for u_μ due to the additional frequency localization to an interval of size α .

Case 1(c) $\lambda_0 = \lambda \ll \alpha$. For the part of f_{λ_0} with modulation $\lesssim \alpha\mu$ we bound $u_\alpha u_\mu$ in L^2 and the other u_μ in L^∞ . This works even if $\alpha \approx \mu$ as two of the μ sized frequencies must be μ separated. We obtain

$$\begin{aligned} |I| &\lesssim \lambda^{-2s} \mu^{-2s} \alpha^{-s} \mu^{-2s} \alpha^{\frac{1}{2}} \mu^{-\frac{1}{2}} \lambda^{-1-3s} (\alpha\mu)^{\frac{1}{2}} a(\lambda) \alpha^{-1} \mu^{-1} \|u\|_{X^s}^6 \\ &= a(\lambda) \lambda^{-5s-1} \alpha^{-s} \mu^{-1-4s} \|u\|_{X^s}^6 \end{aligned}$$

which can be summed up for $s \geq -\frac{1}{5}$.

If we consider the part of f_{λ_0} with modulation $\sigma \gg \alpha\mu$ then another factor must have modulation at least σ . We bound that factor in L^2 and the other two in L^∞ as in Case 1(b).

Case 2: $\lambda_0 \gtrsim \min\{\lambda_4, \lambda_5, \lambda_6\}$. Without any restriction in generality we assume that

$$\lambda_1 \leq \lambda_2 \leq \lambda_3, \quad \lambda_4 \leq \lambda_5 \leq \lambda_6$$

Then we must have

$$\lambda_4 \leq \lambda_0 \leq \lambda_3$$

We can distribute P_{λ_0} to each factor and also assume that the λ_5, λ_6 factors have frequency spread at most λ_0 .

Denote

$$\{\lambda_1, \lambda_2, \lambda_3, \lambda_0\} = \{\lambda, \alpha, \mu, \mu\}, \quad \lambda \leq \alpha \leq \mu$$

Case 2a: $\lambda_0 = \mu$.

Case 2a(i): $\lambda_5 = \lambda_6 \gg \mu$. We use the bilinear L^2 estimate for the products $u_\lambda u_{\lambda_5}$ and $u_{\lambda_4} u_{\lambda_6}$ and the L^∞ bound for u_λ , u_α and add up with respect to

λ_6^{-4s} time intervals. We obtain

$$\begin{aligned} |I| &\lesssim \lambda_6^{-4s} \lambda^{-s} \alpha^{-s} \mu^{-2s} \lambda_6^{-2s} \lambda^{\frac{1}{2}} \alpha^{\frac{1}{2}} \lambda_6^{-1} a(\lambda) \alpha^{-1} \mu^{-1} \prod \|u_{\lambda_j}\|_{X^s} \\ &\lesssim a(\lambda) \lambda^{\frac{1}{2}-s} \alpha^{-s-\frac{1}{2}} \mu^{-2s-1} \lambda_6^{-1-6s} \prod \|u_{\lambda_j}\|_{X^s} \end{aligned}$$

which we sum easily with respect to the parameters λ_j subject to the restrictions above. We note that the summation with respect to $\lambda_5 = \lambda_6$ requires imposes the tight restriction $s \geq -\frac{1}{6}$.

Case 2a(ii): $\lambda_5 \leq \lambda_6 = \mu$ and $\alpha \ll \mu$. Then we use the pointwise bound for u_λ , the bilinear L^2 estimate for $u_\alpha u_\mu$ and the L^6 Strichartz estimate for the remaining two factors; finally, we sum up with respect to μ^{-4s} time intervals. We obtain

$$\begin{aligned} |I| &\lesssim \mu^{-4s} \lambda^{-s} \alpha^{-s} \mu^{-4s} \lambda^{\frac{1}{2}} \mu^{-\frac{1}{2}} a(\lambda) \alpha^{-1} \mu^{-1} \|u_\lambda\|_{X^s} \|u_\alpha\|_{X^s} \|u_\mu\|_{X^s} \|u\|_{X^s}^3 \\ &\lesssim a(\lambda) \lambda^{\frac{1}{2}-s} \alpha^{-s-1} \mu^{-\frac{3}{2}-8s} \|u_\lambda\|_{X^s} \|u_\alpha\|_{X^s} \|u_\mu\|_{X^s} \|u\|_{X^s}^3 \end{aligned}$$

The summation with respect to λ , α and μ requires $s \geq \frac{3}{16}$.

Case 2a(iii): $\lambda_5 \leq \lambda_6 = \mu$ and $\alpha = \mu$. Then we use the L^6 Strichartz estimate for all the factors to obtain

$$|I| \lesssim a(\lambda) \mu^{-4s} \mu^{-6s} \mu^{-2} \|u\|_{X^s}^6 = a(\lambda) \mu^{-10s-2} \|u\|_{X^s}^6$$

Case 2b: $\lambda_0 = \alpha \ll \mu$.

Case 2b(i): $\lambda_5 = \lambda_6 \gg \mu$. Then we use the bilinear L^2 estimate for $u_\mu u_{\lambda_5}$ and $u_\mu u_{\lambda_6}$ and L^∞ for u_λ , u_{λ_4} and add up with respect to λ_6^{4s} time intervals. We obtain exactly the same bound as in Case 2a(i).

Case 2b(ii): $\lambda_6 \leq \lambda_7 \leq \mu$. Then we are in the same situation as in Case 2a(ii).

Case 2c: $\lambda_0 = \lambda \ll \mu$. Then we can argue in the same way as in Case 2b. \square

The final step in the paper is to use Proposition 5.4 in order to conclude the proof of Proposition 1.4. We have

$$\|u_0\|_{H^s}^2 = \sum_\lambda \lambda^{2s} \|u_{0\lambda}\|_{L^2}^2$$

Then the following result is straightforward:

Lemma 5.5. *There is a sequence $\{\beta_\lambda\}$ with the following properties:*

- (i) $\lambda^{2s} \|u_{0\lambda}\|_{L^2}^2 \leq \beta_\lambda \|u_0\|_{H^s}^2$.
- (ii) $\sum \beta_\lambda \lesssim 1$.
- (iii) β_λ is slowly varying in the sense that

$$|\log_2 \beta_\lambda - \log_2 \beta_\mu| \leq \frac{\epsilon}{2} |\log_2 \lambda - \log_2 \mu|$$

The sequence β_λ is easy to produce. One begins with the initial guess

$$\beta_\lambda^0 = \frac{\lambda^{2s} \|u_{0\lambda}\|_{L^2}^2}{\|u_0\|_{H^s}^2}$$

which satisfies (i) and (ii) but might not be slowly varying. To achieve (iii) we mollify β_λ^0 on the dyadic scale and set

$$\beta_\lambda = \sum_{\mu} 2^{-\frac{\epsilon}{2} |\log_2 \lambda - \log_2 \mu|} \beta_\mu^0$$

The sequence β_λ will play the role of frequency localized energy threshold. Precisely, we assume that

$$(41) \quad \|u\|_{l^2 L^\infty L^2} \ll 1.$$

and we will show that

$$(42) \quad \sup_t \lambda_0^s \|u_{\lambda_0}(t)\|_{L^2} \lesssim \beta_{\lambda_0}^{\frac{1}{2}} (\|u_0\|_{H^s} + \|u\|_{X^s}^3)$$

which by (ii) implies the conclusion of Proposition 1.4.

In order to prove (42) for some frequency λ_0 we define the sequence

$$a_\lambda = \lambda^{2s} \max\{1, \beta_{\lambda_0}^{-1} 2^{-\epsilon |\ln \lambda - \ln \lambda_0|}\}$$

We obtain using the slowly varying condition (iii)

$$\sum_{\lambda} a(\lambda) \|u_{0\lambda}\|_{L^2}^2 \lesssim \sum_{\lambda} \lambda^{2s} \|u_{0\lambda}\|_{L^2}^2 + 2^{-\frac{\epsilon}{2} |\ln \lambda - \ln \lambda_0|} \lambda^{2s} \beta_{\lambda_0}^{-1} \|u_{0\lambda}\|_{L^2}^2 \lesssim \|u_0\|_{H^s}^2$$

Correspondingly we find a function $a(\xi) \in S_\epsilon^s$ so that

$$a(\xi) \approx a(\lambda), \quad |\xi| \approx \lambda$$

From (41) we obtain $\sup_t E_0(u(t)) \ll 1$. Now we use the energy estimates in Proposition 5.4 for this choice of a . By Proposition 5.3 the E_1 component of the energy is controlled by E_0 , so we obtain

$$\left(\sum_{\lambda} a(\lambda) \|u_{\lambda}(t)\|_{L^2}^2 \right)^{\frac{1}{2}} \lesssim \|u_0\|_{H^s} + \|u\|_{X^s}^3$$

which at $\lambda = \lambda_0$ gives (42).

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