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Function theory on open Kähler manifolds

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To my Mother, Wilfrida A. Ogaja

ABSTRACT OF THE DISSERTATION

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The structure of an open complete Riemannian manifold (M^n, g) with nonnegative sectional curvature has been studied extensively and well understood. There are two classical results due to Gromoll-Meyer [9] and Cheeger-Gromol [4]. Gromoll and Meyer proved that a complete open manifold (M^n, g) with positive sectional curvature is diffeormorphic to $\mathbb{R}^n$. On the other hand, Cheeger and Gromoll proved that a complete open manifold (M^n, g) with nonnegative sectional curvature admits a totally geodesic compact submanifold S such that M^n is diffeomorphic to the normal bundle of S in M^n.

It is natural to imagine that these results and many others can easily be attained in Ricci curvature case. However, in this case, there are relatively few structural results except in a lower dimensional case $n = 2$ where all notions of curvature coincide. In [17], Shen proved that a complete open Riemannian manifold with nonnegative Ricci curvature and

maximum volume growth is proper (admits an exhaustion function). Regarding Shen's result, it was observed by Wong and Zhang [21] that a complete open Kähler manifold with positive bisectional curvature and maximum volume growth can be embedded as a complex submanifold in a complex Euclidean space of higher dimension. Their observation is a partial result of a weaker version of Yau's conjecture which states that a complete open Kähler manifold with positive bisectional curvature can be embedded as a complex submanifold in a complex Euclidean space of higher dimension. The original Yau's conjecture [20] states that: a complete open Kähler manifold with positive bisectional curvature is biholomorphic to complex Euclidean space.

Here, we exhibit that a complete open Kähler manifold with positive bisectional curvature can be embedded as a complex submanifold in a complex Euclidean space of higher dimension if the volume of a cone of rays from a fixed base point is asymptotic to the volume of a geodesic ball centered at the same point. The volume growth condition we consider here is weaker than the maximum volume growth condition.

Contents

List of Figures

Chapter 1

Introduction

In this section we present a brief overview of our work. Structure of positively curved manifolds have been extensively studied with analysis of the Busemann function. There are different versions of Busemann functions.

Definition 1.1. *We define the spherical Busemann function as*

$$b_p(x) = \lim_{r\to\infty}\{r - d(x, \partial(B(p,r))\},$$

where $\partial(B(p,r))$ denotes the boundary of a geodesic ball centered at p with radius r.

Definition 1.2. *For a ray γ emanating from a point $p \in M$ we define the Busemann function with respect to a ray γ as*

$$f_\gamma(x) = \lim_{t\to\infty}\{t - d(x, \gamma(t)\}$$

Some texts refer to f_γ as simply a Busemann function.

Definition 1.3. *For a manifold M, a function $f : M \to \mathbb{R}$ is exhaustion if $f^{-1}(-\infty, a]$ is compact for any real number a.*

Interestingly, in a complete open Riemannian manifold, nonnegative sectional curvature does not guarantee that f_γ is exhaustion [18], whereas $b_p(x)$ is exhaustion.

Definition 1.4. *Let M^n be a complete Riemannian n-manifold. Define α_M by*

$$\alpha_M = \lim_{r \to \infty} \frac{Vol[B(p,r)]}{r^n}$$

where $B(p,r)$ denotes a metric ball a round $p \in M$ with radius r. We say that M^n has maximum volume growth if $\alpha_M > 0$.

The proof of the following result can be found in [17].

Theorem 1.5. *(In the proof of Lemma 3.4, [17]) If M is an open complete Riemannian manifold with nonnegative Ricci curvature and maximum volume growth, then b_p is exhaustion for any $p \in M$.*

We extend this theorem by replacing maximum volume growth condition with a weaker volume growth condition (see page 5).

Definition 1.6. *Cone of rays. Let $S_pM \subset T_pM$ be a unit tangent sphere in the tangent space T_pM for a point $p \in M$.*

For any subset $N \subset S_pM$, define

$$C(N) = \{q \in M \mid \textit{there is a minimizing geodesic } \gamma \textit{ from } p \textit{ to } q \textit{ such that} \gamma'(0) \in N\}$$

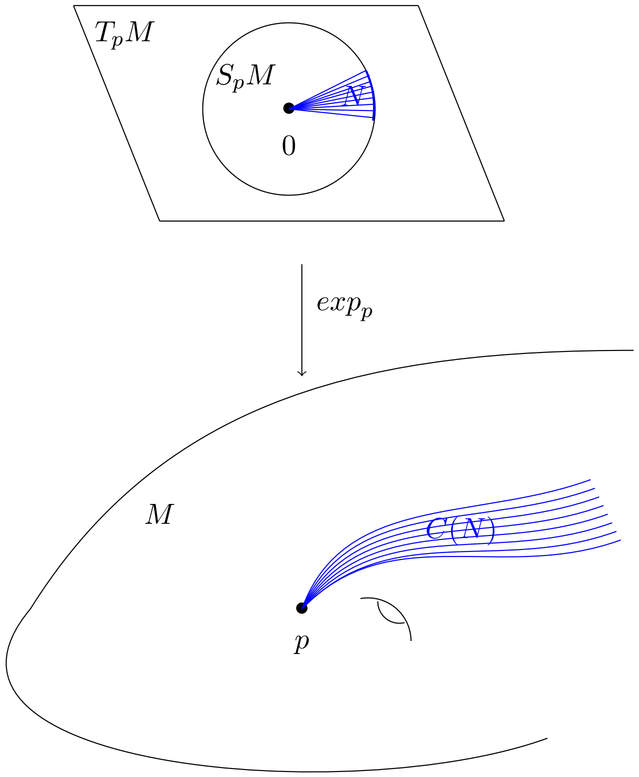

Figure 1.1: Cone of rays

to be the cone over N and let

$$B_N(p, r) = B(p, r) \cap C(N)$$

.

Let $\Sigma = \{v \subset S_pM \mid \exp_p(rv) : [0, \infty) \to M \text{ is a ray}\}$. As denoted above, we have

$$B_\Sigma(p, r) = B(p, r) \cap C(\Sigma).$$

In chapter 4, we establish that if M is a complete open manifold with nonnegative Ricci curvature, then the volume growth condition

$$\lim_{r\to\infty} \frac{Vol(B_\Sigma(p, r))}{Vol(B(p, r))} = 1$$

is weaker than the maximum volume growth condition.

It is essential to note that nonnegative Ricci curvature ensures that our volume growth codition is independent of the base point.

Lemma 1.7. *Let M^n be a complete open manifold with $Ric_M \geq 0$. For a fixed $p_1 \in M$, the volume growth*

$$\lim_{r\to\infty} inf \frac{Vol(B_\Sigma(p, r))}{Vol(B(p_1, r))} = \alpha(n)$$

is independent of the base point $p \in M$.

Proof. Let $p, q \in M$ and $d = d(p, q)$. Then it is clear that $B(p, r) \subset B(q, r + d)$ and $B(q, r) \subset B(p, r + d)$. By Bishop-Gromov volume comparison theorem,

$$\begin{aligned}
\lim_{r\to\infty}\inf\frac{Vol(B_\Sigma(p,r))}{Vol(B(p_1,r))} &\geq \lim_{r\to\infty}\inf\left\{\left[\frac{r}{r+d}\right]^n\frac{Vol(B_\Sigma(p,r+d))}{Vol(B(p_1,r))}\right\} \\
&\geq \lim_{r\to\infty}\inf\left\{\left[\frac{r}{r+d}\right]^n\frac{Vol(B_\Sigma(q,r)}{Vol(B(p_1,r))}\right\} \\
&\geq \lim_{r\to\infty}\left[\frac{r}{r+d}\right]^n\lim_{r\to\infty}\inf\frac{Vol(B_\Sigma(q,r)}{Vol(B(p_1,r))} \\
&\geq \lim_{r\to\infty}\inf\frac{Vol(B_\Sigma(q,r)}{Vol(B(p_1,r))}
\end{aligned}$$

Likewise

$$\lim_{r\to\infty}\inf\frac{Vol(B_\Sigma(p,r))}{Vol(B(p_1,r))} \leq \lim_{r\to\infty}\inf\frac{Vol(B_\Sigma(q,r)}{Vol(B(p_1,r))}$$

Hence

$$\lim_{r\to\infty}\inf\frac{Vol(B_\Sigma(p,r))}{Vol(B(p_1,r))} = \lim_{r\to\infty}\inf\frac{Vol(B_\Sigma(q,r)}{Vol(B(p_1,r))}$$

for any $p, q \in M$.

□

In this paper, we study structures of complete open manifolds with $Ric_{M^n} \geq 0$ and volume growth condition

$$\lim_{r\to\infty}\inf\frac{Vol(B_\Sigma(p,r))}{Vol(B(p_1,r))} > \frac{9^n-1}{9^n}$$

We also study structures of complete open Kӑhler manifolds with positive bisectional curvature and volume growth condition

$$\lim_{r\to\infty} \inf \frac{Vol(B_\Sigma(p,r))}{Vol(B(p_1,r))} > \frac{9^{2n}-1}{9^{2n}}$$

where $n = dim_{\mathbb{C}} M$

Note that for any $p, p_1 \in M$,

$$\lim_{r\to\infty} \sup \frac{Vol(B_\Sigma(p,r))}{Vol(B(p_1,r))} \leq 1$$

We prove the following theorems:

Theorem 1.8. *Let M be a complete open manifold with $Ric_M \geq 0$. Let $\alpha(n) = \frac{9^n-1}{9^n}$ where $n = dim_{\mathbb{R}} M$. If*

$$\alpha(n) < \lim_{r\to\infty} inf \frac{Vol(B_\Sigma(p,r))}{Vol(B(p,r))},$$

then for any $a \in \mathbb{R}$, $b_p^{-1}(a)$ is compact .

Theorem 1.9. *Let M be a complete open Kähler manifold with positive bisectional curvature. If*

$$\alpha(n) < \lim_{r\to\infty} inf \frac{Vol(B_\Sigma(p,r))}{Vol(B(p,r))},$$

where $n = dim_{\mathbb{C}} M$ and $\alpha(n) = \frac{9^{2n}-1}{9^{2n}}$, then M is a Stein manifold.

In chapter 4 we state general results from Riemannian geometry that will be required in the proof of theorem 1.8 and in chapter 5 we state general results from complex geometry that will be required in the proof of theorem 1.9.

In chapters 6 and 7, we provide proofs of theorems 1.8 and 1.9 respectively.

It Remains a challenge to completely remove the volume growth condition or provide a counter example of the Greene-Wu's conjecture (weaker version of Yaus's conjecture). Chen and Zhu proved a result in [6] that points to a possible future direction towards establishing that. They proved that a complete open Kähler manifold with positive bisectional curvature has at least a half-volume growth.

Theorem 1.10. *Let M be a complex n-dimensional complete open Kähler manifold with nonnegative holomorphic bisectional curvature. Suppose also its holomorphic bisectional curvature is positive at least at one point. Then the volume growth of M satisfies*

$$Vol(B(p,r)) \geq \alpha r^n, \quad 1 \leq r < +\infty,$$

where α is some positive constant depending on p and the dimension n.

By having a closer look at the proof of the above theorem in [6], we realize that α is independent of the base point p. A modified version is presented in chapter 5.

Chapter 2

Basic Facts and Definitions

In this chapter we discuss facts, definitions, and ideas that are key to the development of our results. Materials here closely imitate expositions in [23], [3], and [10].

2.1 Notion of Metric, Connection, and Curvature

Definition 2.1. *A Riemannian metric on a smooth manifold M is a 2-tensor field $g \in \mathcal{J}^2(M)$ that is symmetric (i.e $g(X,Y) = g(Y,X)$) and positive definite (i.e., $g(X,Y) > 0$ if $X \neq 0$).*

Thus a Riemannian metric determines an inner product on each tangent space T_pM which is typically written $\langle X, Y\rangle := g(X,Y)$ for $X, Y \in T_pM$. A *Riemannian manifold* is a manifold together with a given Riemannian metric.

If p is a point in a Remannian manifold (M, g), we define the *length* or *norm* of any tangent vector $X \in T_pM$ to be $|X| := \langle X, X\rangle^2$. We define the angle between two nonzero vectors $X, Y \in T_pM$ to be the unique $\theta \in [0, \pi]$ satisfying $\cos\theta = \langle X, Y\rangle/(|X||Y|)$

If $(e_1, ..., e_n)$ is a local frame for TM, and $(\varphi^1, ..., \varphi^n)$ its dual coframe, A Riemannian metric can be locally written as

$$g = g_{ij}\varphi^i \otimes \varphi^j,$$

where $g_{ij} = \langle e_i, e_j \rangle$ is symmetric in i and j and depends smoothly on $p \in M$. In a coordinate frame, g has the form

$$g = g_{ij}dx^i \otimes dx^j. \tag{2.1}$$

Since g_{ij} is symmetric in i and j, (2.1) is equivalent to

$$g = g_{ij}dx^i dx^j$$

Definition 2.2. *Let $\pi : E \to M$ be a vector bundle over a manifold M, and let $\mathcal{E}(M)$ denote the space of smooth sections of E. A connection in M is a map*

$$\nabla : \mathcal{J}(M) \times \mathcal{E}(M) \to \mathcal{E}(M),$$

denoted $(X, Y) \mapsto \nabla_X Y$, satisfying the following properties:

(a) $\nabla_X Y$ is linear over $C^\infty(M)$ in X:

$$\nabla_{fX_1+gX_2}Y = f\nabla_{X_1}Y + g\nabla_{X_2}Y \quad \textit{for } f, g \in C^\infty(M)$$

(b) $\nabla_X Y$ *is linear over* $\mathbb{R}$ *in* Y*:*

$$\nabla_X(aY_1 + bY_2) = a\nabla_X Y_1 + b\nabla_X Y_2 \quad \textit{for } a, b \in \mathbb{R}$$

(c) ∇ *satisfy the following product rule:*

$$\nabla_X(fY) = f\nabla_X Y_1 + (Xf)Y \quad \textit{for } f \in C^\infty(M).$$

$\nabla_X Y$ *is called the covariant derivative of* Y *in the direction of* X*.*

Here, $\mathcal{J}(M)$ *denote a space of vector fields.*

A linear connection on M *is a connection on* TM*, i.e., a map*

$$\nabla : \mathcal{J}(M) \times \mathcal{J}(M) \to \mathcal{J}(M),$$

satisfying properties $(a) - (c)$ *in the definition of a connection above. A linear connection on* M *is often simply called a connection on* M*.*

A linear connection appears in components. Let $\{E_i\}$ be a local frame for TM on an open subset $U \subset M$. For any choices of i and j we can expand $\nabla_{E_i} E_j$ in terms of the same frame.

$$\nabla_{E_i} E_j = \Gamma_{ij}^k E_k \tag{2.2}$$

If the dimension of M is n, n^3 functions Γ_{ij}^k on U are called *Christoffel symbols* with respect to the frame $\{E_i\}$.

Lemma 2.3. *Let ∇ be a linear connection, and let $X, Y \in \mathcal{J}(U)$ be expressed in terms of a local frame by $X = X^i E_i$, $Y = Y^j E_j$. Then*

$$\nabla_X Y = (XY^k + X^i Y^j \Gamma_{ij}^k) E_k \tag{2.3}$$

Proof. By definition rules, we have:

$$\begin{aligned} \nabla_X Y &= \nabla_X (Y^j E_j) \\ &= (XY^j) E_j + Y^j \nabla_{X^i E_i} E_j \\ &= (XY^j) E_j + X^i Y^j \nabla_{E_i} E_j \\ &= XY^j E_j + X^i Y^j \Gamma_{ij}^k E_k \end{aligned}$$

We obtain (2.3) by replacing the dummy index in the first term. □

Example 2.1.1. *On $\mathbb{R}^n$ we define the Euclidean connection by*

$$\overline{\nabla}_X (Y^j \partial_j) = (XY^j) \partial_j \tag{2.4}$$

$\overline{\nabla}_X(Y)$ is just the vector field whose components are just the directional derivatives of the components of Y in the direction of X. Its Christoffel symbols in standard coordinates are all zero.

A linear connection ∇ is said to be *compatible with* a metric $g = \langle \cdot, \cdot \rangle$ if it satisfies the following product rule for all vector fields X, Y, Z.

$$\nabla_X \langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle.$$

Definition 2.4. *A Lie algebra is a real vector space* $\mathfrak{g}$ *equipped a skew-symmetric, bilinear map* $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *such that*

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0, \ \forall X, Y, Z \in \mathfrak{g}$$

The product structure in a Lie algebra is called the *Lie bracket*. The identity above is called the *Jacobi identity*.

The linear connection ∇ has a torsion tensor defined by

$$T_\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y], \text{ for any vector fields } X, Y$$

If $T_\nabla = 0$, then a linear connection ∇ is said to be *torsion-free*.

The curvature R of a linear connection is a $\binom{3}{1}$-tensor field defined by

$$R_{XY}Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

For a given metric $\langle \cdot, \cdot \rangle$ on TM, there is a unique linear connection on M that is torsion-free:

Proposition 2.5. *Given a Riemannian manifold* $(M^n, \langle \cdot, \cdot \rangle)$*, there is a unique connection* ∇ *on* M *that is both torsion free and compatible with* $\langle \cdot, \cdot \rangle$*. This connection is called the Riemannian connection or Levi-Civita connection of the Riemannian manifold.*

As a $\binom{3}{1}$-tensor field, the curvature tensor of a Riemannian connection can in terms of any local frame with on upper and three lower indices. For example, in terms of local coordinates (x^i), R can be written as

$$R = R_{ijk}{}^{l} dx^i \otimes dx^j \otimes dx^k \otimes \partial_l,$$

where the coefficients are defined by

$$R(\partial_i, \partial_j)\partial_k = R_{ijk}{}^{l}\partial_l$$

We define the Riemannian curvature as the covariant for tensor field obtained from the $\binom{3}{1}$-tensor field by lowering the last index. Its action on vector field is defined by

$$R(X, Y, Z, W) = \langle R_{XY} Z, W \rangle$$

and in coordinates it is written as

$$R = R_{ijkl} dx^i \otimes dx^j \otimes dx^k \otimes dx^l$$

where $R_{ijkl} = g_{lm} R_{ijk}{}^{m}$ and $g_{lm} = \langle \partial_l, \partial_m \rangle$. It obeys the following symmetries:

$$\begin{aligned}
R(X, Y, Z, W) &= -R(Y, X, Z, W) \\
R(X, Y, Z, W) &= -R(X, Y, W, Z) \\
R(X, Y, Z, W) &+ R(Z, X, Y, W) + R(Y, Z, X, W) = 0 \\
R(X, Y, Z, W) &= R(Z, W, X, Y)
\end{aligned}$$

Definition 2.6. *Let $\{X, Y\}$ be a basis of a 2-dimensional subspace $P \subset T_pM$, the sectional curvature of the metric in the 2-plane section P is defined by*

$$K(P) = \frac{R(X, Y, Y, X)}{|X|^2|Y|^2 - \langle X, Y\rangle^2}$$

The right hand side is independent of the choice of basis $\{X, Y\}$ of the 2-plane $P \subset T_pM$: suppose $\{Z, W\}$ is another basis of P. Then we have

$$\begin{aligned} Z &= aX + bY \\ W &= cX + dY \end{aligned}$$

for some constants a, b, c, d with $ad - bc \neq 0$. It follows that

$$|Z|^2|W|^2 - \langle Z, W\rangle^2 = (ad - bc)^2(|X|^2|Y|^2 - \langle X, Y\rangle^2)$$

Since R is skew-symmetric in its first two (or las two) positions, we have

$$\begin{aligned} R(Z, W, W, Z) &= (ad - bc)R(X, Y, W, Z) \\ &= (ad - bc)^2 R(X, Y, Y, X) \end{aligned}$$

Thus proving the claim.

Definition 2.7. *Let $\{e_1, ..., e_n\}$ be an orthonormal basis of T_pM. For any $X, Y \in T_pM$, define*

$$r(X,Y) = \sum_{i=1}^{n} R(e_i, X, Y, e_i)$$

Here we are taking the trace of the 4-tensor R at its first and fourth positions. It is clear that $r(X,Y) = r(Y,X)$. So r is a symmetric covariant 2-tensor on M. It is called the Ricci tensor of a Riemannian manifold M. The right hand side of the above formula is independent of the orthonormal basis $\{e_i\}$: suppose $\{e_i'\}$ is another orthonormal basis of T_pM. Then there is an $n \times n$ orthogonal matrix $A = (a_{ij})$ such that $e_i' = \sum_{j=1}^{n} a_{ij} e_j$ for each $1 \leq i \leq n$. We then have

$$\begin{aligned}
\sum_{i=1}^{n} R(e_i', X, Y, e_i') &= \sum_{i=1}^{n} \sum_{j,k=1}^{n} a_{ij} a_{ik} R(e_j, X, Y, e_k) \\
&= \sum_{j,k=1}^{n} \left(\sum_{i=1}^{n} a_{ij} a_{ik} \right) R(e_j, X, Y, e_k) \\
&= \sum_{j,k=1}^{n} \delta_{jk} R(e_j, X, Y, e_k) = \sum_{j=1}^{n} R(e_j, X, Y, e_j)
\end{aligned}$$

For any $0 \neq X \in T_pM$, the Ricci curvature in the direction of X is defined by

$$r(X) = \frac{r(X,X)}{|X|^2}$$

It is just the average value of the sectional curvature for all the 2-plane sections containing X.

The trace of the Ricci tensor r is a scalar valued function on M, which is called the scalar curvature of the Riemannian manifold denoted by s. At a point $p \in M$, the value $s(p)$ is given by

$$s(p) = \sum_{i=1}^{n} r(e_i) = \sum_{1 \le i \neq j \le n} R(e_i, e_j, e_j, e_i)$$

where $\{e_i\}$ *is any orthonormal basis of* T_pM*. The scalar curvature at* p *is the average value of the sectional curvature for all the 2-plane sections in* T_pM*.*

2.2 Geodesics, Rays, and Busemann function

A *curve* in a manifold M is a smooth map $\gamma : I \to M$, where $I \subset \mathbb{R}$ is some interval. If I is a closed bounded interval $[a, b] \subset \mathbb{R}$, then $\gamma : I \to M$ is called a *curve segment*.

Lengths of Curves. If $\gamma : [a, b] \to M$ is a curve segment, we define the lenght of γ as

$$L(\gamma) = \int_a^b |\dot{\gamma}(t)| dt$$

If I has an endpoint, smoothness of γ means that γ extends to a smooth map defined on some open intervals containing I. Here, the notion of smoothness is equivalent to saying the components functions γ^i in any local coordinates have one-sided derivatives of all orders at the endpoint, or having derivatives of all orders that extend continuously to the endpoint. Since γ can always be extended to a smooth curve on a slightly larger open interval and then restrict back to the original after working with it, it suffices to assume whenever convenient that γ is defined on an open interval.

At any time $t \in I$, the *velocity* $\dot{\gamma}(t)$ of γ is defined as the push-forward $\gamma_*(d/dt)$. It acts on functions by

$$\dot{\gamma}(t) f = \frac{d}{dt}(f \circ \gamma)(t).$$

This corresponds to the usual notion of velocity in coordinates. If we write the coordinate representation of γ as $\gamma(t) = (\gamma^1(t), ..., \gamma^n(t))$, then

$$\dot{\gamma}(t) = \dot{\gamma}^i(t)\frac{\partial}{\partial x_i}$$

A vector field along a curve $\gamma : I \to M$ is a smooth map $V : I \to TM$ such that $V(t) \in T_{\gamma(t)}M$ for every $t \in M$. Let $\mathcal{J}(\gamma)$ denote the space of all vector space along γ. A vector field V a long γ is said to be extendible if there exists a vector field $\tilde{V} \in \mathcal{J}(M)$ such that for each $t \in I$, $V(t) = \tilde{V}_{\gamma(t)}$. Not every vector field along a curve γ is extendible. For example, if $\gamma(t_1) = \gamma(t_2)$ but $\dot{\gamma}(t_1) \neq \dot{\gamma}(t_2)$, then $\dot{\gamma}$ is not extendible.

To make sense of a directional derivative along a curve, we have the following.

Lemma 2.8. *Let ∇ be a linear connection on M. For each curve $\gamma : I \to M$, ∇ determines a unique operator*

$$D_t : \mathcal{J}(\gamma) \to \mathcal{J}(\gamma)$$

satisfying the following properties:

(a) Linearity over $\mathbb{R}$:

$$D_t(aV + bW) = aD_tV + bD_tW \quad for\ a, b \in \mathbb{R}.$$

(b) Product rule:

$$D_t(fV) = \dot{f}V + fD_tV \quad for\ f \in C^\infty(I)$$

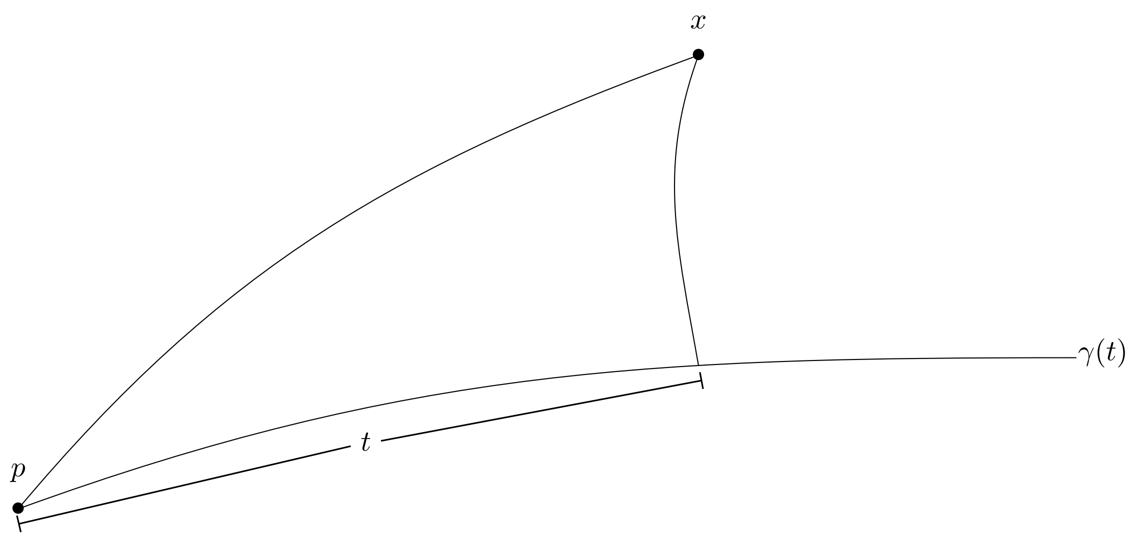

Figure 2.1: f_γ is Lipschitz ($f_\gamma(x) \leq d(p,x)$)

(c) If V is extendible, then for any extension $\tilde{V}$ of V,

$$D_t V(t) = \nabla_{\dot{\gamma}(t)} \tilde{V}.$$

Definition 2.9. *Let M be a manifold with a linear connection ∇, and let γ be a curve in M. The acceleration in γ is the vector field $D_t\dot{\gamma}$ along γ. A curve γ is called a geodesic with respect to ∇ if its acceleration is zero: $D_t\dot{\gamma} = 0$*

Minimizing geodesic: A geodesic γ is minimizing if for any other geodesic with the same endpoints $\tilde{\gamma}$, $L(\gamma) \leq L(\tilde{\gamma})$.

Ray: A ray $\gamma : [0, \infty) \to M$ in M is a minimizing geodesic such that $t_2 - t_1 = d(\gamma(t_2), \gamma(t_1))$ for all $t_2 \geq t_1 \geq 0$.

Recall that $f_\gamma(x) = \lim_{t\to\infty}\{t - d(x, \gamma(t))\}$ for each ray γ. By triangle inequality, $t - d(x, \gamma(t)) \leq d(p, x) < \infty$ (Figure 2.1). So, f_γ is Lipschitz continuous. However, it is not smooth.

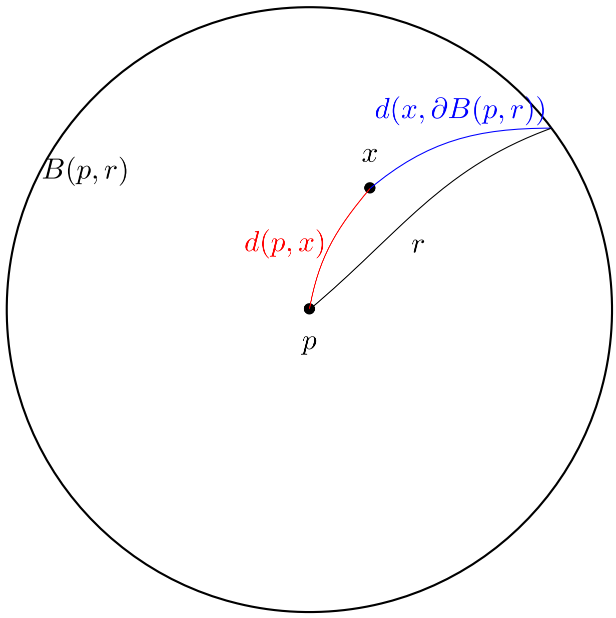

Figure 2.2: b_p is Lipschitz $(r - d(x, \partial B(p,r)) \leq d(p,x))$

Note: It is important to note that for any open complete manifold, there is always at least a ray.

Lemma 2.10. *Let M be a complete open Riemannian manifold. For each $x \in M$ define $B_p^r(x) : \mathbb{R} \to \mathbb{R}$ by $B_p^r(x) = r - d(x, \partial B(p,r))$. $B_p^r(x)$ is non-increasing (with respect to r) for $r \geq d(p,x)$. Here, $\partial B(p,r)$ denotes the boundary of a geodesic ball centered p with radius r.*

Proof. Consider real numbers r_1 and r_2 such that $r_2 \geq r_1 \geq d(x,p)$. Let $x' \in \partial B(p, r_2)$ be such that $d(x,x') = d(x, \partial B(p, r_2))$. Take a minimizing geodesic γ from x to x'. If

$\gamma(t_0) \in \partial B(p, r_1)$, then

$$\begin{aligned} r_1 = d(\gamma(t_0), p) &\geq r_2 - d(\gamma(t_0), \partial B(p, r_2)) \\ &= r_2 - d(x, \partial B(p, r_2)) + t_0, \end{aligned}$$

that implies

$$\begin{aligned} d(x, \partial B(p, r_1)) &\leq d(x, \gamma(t_0)) \\ &= t_0 \\ &\leq d(x, \partial B(p, r_2) - r_2 + r_1 \end{aligned}$$

Hence $B_p^{r_1}(x) \geq B_p^{r_2}(x)$. □

Recall that

$$b_p(x) = \lim_{r \to \infty} \{r - d(x, \partial B(p, r))\}$$

Equivalently, we can verify that b_p is Lipchitz continuous (Figure 2.2). Take $x' \in \partial B(p, r)$ such that $d(x, x') = d(x, \partial B(p, r))$. By triangle inequality,

$$r - d(x, x') \leq d(x, p)$$

Similarly,

$$d(x, x') \leq d(x', p) + d(x, p)$$

i.e

$$-d(x, p) \leq r - d(x, x')$$

Hence $|r - d(x, \partial B(p,r))| \leq d(x,p)$.

2.3 Jacobi fields and Cut Locus

Jacobi field provides a way of describing how geodesics from a given point $p \in M$ spread apart. In particular, the spreading a part is determined by curvature condition.

Definition 2.11. *Let $\gamma : I \to M$ be a geodesic in M. A vector field J along γ is called a $Jacobi\ field$ if it satisfies the equation*

$$D_t^2 J(t) + R(J(t), \dot{\gamma}(t))\dot{\gamma}(t) = 0$$

Proposition 2.12. ***(Existence and Uniqueness of Jacobi Fields)*** *Let $\gamma : I \to M$ be a geodesic in M. Let $a \in I$ and $p = \gamma(a) \in M$. For any pair of vectors $X, Y \in T_pM$, there exists a unique Jacobi field J a long γ such that*

$$J(a) = X, \text{ and } D_t J(a) = Y$$

Proof. Choose an orthonormal basis $\{e_i\}$ for T_pM, and extend it to a parallel frame along all of γ. Writing $J(t) = J^i(t)e_i$, we can express Jacobi equation as

$$\ddot{J}^i + R^i_{jkl} J^j \dot{\gamma}^k \dot{\gamma}^l = 0.$$

This is a linear system of second-order ODEs for the n functions J^i. Making the usual substitution $V^i = \dot{J}^i$ converts it to an equivalent first-order linear system for the $2n$ unknowns $\{J^i, V^i\}$. Then, by existence and uniqueness of linear ODEs, the existence and uniqueness of a solution on the whole interval I with any initial conditions $J^i(a) = X^i$, $V^i(a) = Y^i$ is guaranteed. □

Definition 2.13. *Let $S_pM = \{v \in T_pM : ||v|| = 1\}$ be the unit sphere in T_pM. For any $v \in T_pM$, we define*

$$cut(v) = max\{t : \gamma_v|_{[0,t]} \text{ is minimizing}\}.$$

This defines the cut locus distance function

$$cut : S_pM \to (0, \infty]$$

which is continuous. Let

$$C_p = \{tv : v \in S_pM,\ t \leq cut(v)\}$$

This is a closed subset of T_pM and its boundary ∂C_p is called the cut locus in the tangent space of the point p.

The cut locus of p in M is defined to be image of the cut locus of p in the tangent space under the exponential map at p. Thus, we may interpret the cut locus of p in M as the points in the manifold where the geodesics starting at p stop being minimizing.

2.3.1 Conjugate Points

One application of Jacobi field is to investigate when the exponential map is a local diffeomorphism. We know that if M is complete then exp_p is defined on all of T_pM, and is a local diffeomorphism near 0. However, it may cease to be a local diffeomorphism at points far away.

A good example is by the sphere $S^n(R)$. Each geodesic starting at a given point meet at the antipodal point which is at a distance πR along each geodesic. The exponential map is a diffeomorphism on the geodesic ball $B_{\pi R}(0)$ but fails to be a diffeomorphism on all points on the sphere of radius πR on S_pM.

Definition 2.14. *Consider a geodesic segment γ joining $p, q \in M$. The points p and q are conjugates to each other along γ if there is a Jacobi field along γ vanishing at p and q but not identically zero.*

Remark 2.1. The most important fact about conjugate points is that they are the image of singularities of the exponential map.

A point in a cut locus of $p \in M$ is called a *cut point* of p.

Proposition 2.15. *Suppose that $\gamma(t_0)$ is the cut point of $p = \gamma(0)$ along γ. Then:*

a) either $\gamma(t_0)$ is the first conjugate point of $\gamma(0)$ along γ,

b) or there exists a geodesic $\sigma \neq \gamma$ from p to $\gamma(t_0)$ such that $L(\sigma) = L(\gamma)$

Conversely, if (a) and (b) are satisfied, then there exists t in $(0, t_0]$ such that $\gamma(t)$ is the cut point of p along γ.

Chapter 3

Busemann Functions on Complete Open Manifolds with Nonnegative Sectional Curvature

Noncompact complete Riemannian manifolds of nonnegative sectional curvature have been extensively studied by analysis of Buseman function(s).

In the proof of soul theorem [4], Cheeger and Gromoll proved that the Buseman function $b_p(x)$ on a complete open Riemaniann manifold with nonnegative sectional curvature is convex and exhaustion.

Definition 3.1. *A function* $\psi : M \to \mathbb{R}$ *is convex if for any normal geodesic* $\sigma : \mathbb{R} :\to M$ *and any* $\lambda \in [0, 1]$*,* $\psi \circ \sigma((1-\lambda)t_1 + \lambda t_2) \le (1-\lambda)\psi \circ \sigma(t_1) + \lambda\psi \circ \sigma(t_2)$*.*

Definition 3.2. *A function* $f : M \to \mathbb{R}$ *is an exhaustion function if for any* $c \in \mathbb{R}$*,* $f^{-1}((-\infty, c])$ *is a compact subset of* M*.*

Theorem 3.3. *(Theorem 1.10 in [4]) Let M be a complete open manifold with nonnegative sectional curvature. Then the function $f_\gamma : M \to \mathbb{R}$ is convex.*

Definition 3.4. *A subset $S \subseteq M$ is said to be geodesically convex if every point in S can be joined by a minimizing geodesic in M whose image is in S.*

Corollary 3.5. *Let M be a complete open manifold with nonnegative sectional curvature. For $a \in \mathbb{R}$, denote*

$$D_a = \{x \in M : \sup_{\gamma} f_\gamma(x) \leq a\}, \text{ where sup is taken over all rays } \gamma.$$

D_a is geodesically convex.

Proof. Take points $x, y \in D_a$ and let $\sigma : [\alpha, \beta] \to M$ be a minimizing geodesic from x to y ($\sigma(\alpha) = x$ and $\sigma(\beta) = y$). For any ray $\gamma : [0, +\infty) \to M$ we have that $f_\gamma \circ \sigma((1-\lambda)\alpha + \lambda\beta) \leq ((1-\lambda) f_\gamma \circ \sigma(\alpha) + \lambda f_\gamma \circ \sigma(\beta))$. In other words, for any t in $[\alpha, \beta]$, we have that $\sup_{\gamma} f_\gamma(\sigma(t)) \leq a$. Same as saying that the image of $\sigma : [\alpha, \beta] \to M$ is contained in D_a. So, the image of any minimizing geodesic with end points contained in D_a is contained in D_a. Hence D_a is geodesically convex. □

To prove that the Busemann function b_p is exhaustion, it suffices to established that for any $a \in \mathbb{R}$, D_a is a compact subset of M. Here is a short description of a strategy independent of the one originally presented by Cheeger and Gromol:

Theorem 3.6. *Let M be a complete open manifold with nonnegative sectional curvature, then b_p is exhaustion.*

Proof. Assume that D_a is noncompact. Then we can construct a sequence of minimizing geodesics $\{\sigma_i\}$ whose images are contained in D_a such that $\sigma_i(0) = p$ and $\underset{i\to\infty}{\sigma_i} = \sigma$ is a ray. Since D_a is closed, $image(\sigma) \subset D_a$. However, $\sup_{\gamma} f_\gamma(\sigma(t)) = t$ for any $t \geq 0$. A contradiction to the definition of D_a. Thus, D_a must be compact. We have just proved that $b(x) = \sup_{\gamma} f_\gamma$ is exhaustion. For any minimizing geodesic from p to a point on $\partial B(p, r)$ we have that $d(x, \partial B(p, r)) \leq d(x, \gamma(r))$. Thus,

$$b_p(x) = \lim_{r\to\infty} \{r - d(x, \partial B(p, r))\} \geq \sup_\gamma \{f_\gamma(x)\}$$

Since $b(x) \leq b_p(x)$ for any $x \in M$, we have that b_p is exhaustion. □

It is interesting to note that the above result does not hold for a single ray Busemann function f_γ. We can construct examples of complete open manifolds of nonnegative sectional curvature on which f_γ is not exhaustion by simply considering the following theorems by Shiohama [18] page 297.

Theorem 3.7. *Let M^2 be a connected complete open manifold of dimension 2 with a nonnegative Gaussian curvature G. Let $\gamma : [0, \infty) \to M^2$ be a ray with respect to which f_γ is non-exhaustion. Then we have*

$$\int_{M^2} G d\mu \leq \pi$$

Theorem 3.8. *Let M^2 be a connected complete open manifold of dimension 2 with non-negative Gaussian curvature G. Assume that M^2 admits an exhaustion Busemann function f_γ. Then we have*

$$\int_{M^2} G d\mu > \pi$$

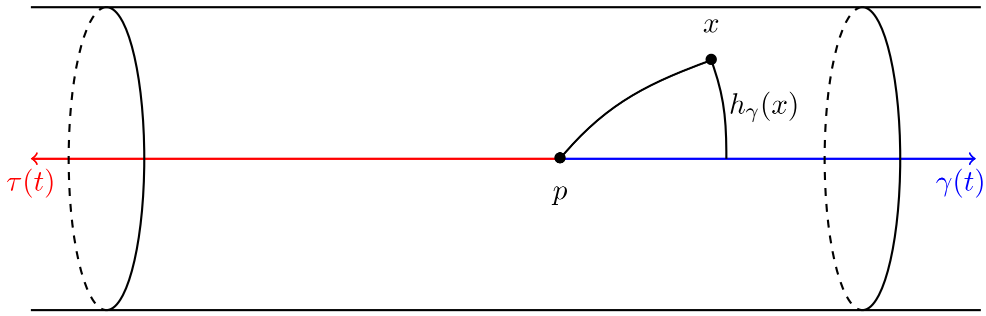

Figure 3.1: $S \times \mathbb{R}$

Example 3.0.1. *Flat Cylinder (Figure 3.1)*

For a point $p \in M^2 = S^2 \times \mathbb{R}$*, there exists only two rays* γ *and* τ*, which are in opposite directions. For any point* $x \in M^2$*, we denote* $h_\gamma(x) = d(x, \gamma)$ *and* $h(x) = d(x, \{\gamma, \tau\})$*.*

By theorem 3.8 above, f_γ *is not an exhaustion function. However,* $b(x) = \sup_\gamma f_\gamma(x)$ *and* $b_p(x)$ *are exhaustion by simply observing that* $d(p, x) - b(x) \leq 2h(x)$ *by triangle inequality, and that* $h(x) \leq \pi$*.*

Chapter 4

Bishop-Gromov Comparison Theorem and Volume Growth Condition

Many important tools and results of Ricci curvature lower bound is from the use of comparison thorems, e.g, Bishop-Gromov volume comparison, Mean curvature comparison, Laplacian comparison, Meyers' theorem, Cheeger-Gromoll's splitting theorem. Here, we will present Bishop-Gromov volume comparison and its' generalized version, Mean curvature comparison, and Laplacian comparison.

Theorem 4.1. ***(Bishop and Gromov volume comparison)***

Let the Ricci curvature of a complete open manifold M satisfies the lower bound

$$Ric_M \geq (n-1)K$$

for a constant $K \in \mathbb{R}$*. Let* M_K^n *be the complete simply connected space of constant sectional curvature* K*. Then for any* $p \in M$ *and* $p_K \in M_K^n$*, the function*

$$\varphi(r) = \frac{Vol(B(p,r))}{Vol(B(p_K,r))}$$

is non-increasing on $(0,\infty)$*.*

Next, we will discuss the volume comparison theorem that generalizes Bishop-Gromov comparison theorem. Some materials in this section follows from exposition in [24].

Let f be a smooth function on M^n. We define its gradient, Hessian, and Laplacian by

$$\langle \nabla f, X \rangle = X(f)$$

,

$$Hess(f)(X,Y) = \langle \nabla_X(\nabla f), Y \rangle$$

and

$$\Delta f = \mathrm{tr}(Hess(f))$$

respectively. For a bilinear form A, we denote $|A|^2 = tr(AA^t)$. Here, we introduce volume comparison theorem from Weitzenböck Formula.

Theorem 4.2. *(The Weitzenböck Formula) Let* M^n, g *be a complete Riemannian manifold. Then for any function* $f \in C^3(M)$*, we have*

$$\frac{1}{2}\Delta|\nabla f|^2 = |Hessf|^2 + \langle \nabla f, \nabla(\Delta f) \rangle + Ric\langle \nabla f, \nabla f \rangle$$

point-wise.

Proof. Fix a point $p \in M$. Let $\{X_i\}_{i=1}^n$ be a local orthonormal frame field such that $\langle X_i, X_j \rangle = \delta_{ij}$ and $\nabla_{X_i} X_j(p) = 0$. Computation at p gives

$$
\begin{aligned}
\frac{1}{2}\Delta|\nabla f|^2 &= \frac{1}{2}\sum_i X_i X_i \langle \nabla f, \nabla f \rangle \\
&= \sum_i X_i \langle \nabla_{X_i} \nabla f, \nabla f \rangle \\
&= \sum_i X_i Hess(f)(X_i, \nabla f) \\
&= \sum_i X_i Hess(f)(\nabla f, X_i)(\text{Hessian is symmetric}) \\
&= \sum_i X_i \langle \nabla_{\nabla_f}(\nabla f), X_i \rangle \\
&= \sum_i \langle \nabla_{X_i} \nabla_{\nabla_f}(\nabla f), X_i \rangle + \sum_i \langle \nabla_{\nabla_f}(\nabla f), \nabla_{X_i} X_i \rangle \\
&= \sum_i \langle \nabla_{X_i} \nabla_{\nabla_f}(\nabla f), X_i \rangle (\text{The other term vanishes at } p) \\
&= \sum_i \langle R(X_i, \nabla f)\nabla f, X_i \rangle + \sum_i \langle \nabla_{\nabla f} \nabla_{X_i} \nabla f, X_i \rangle + \sum_i \langle \nabla_{|X_i, \nabla f|} \nabla f, X_i \rangle
\end{aligned}
$$

The first term is by definition $\text{Ric}(\nabla f, \nabla f)$; the second term is

$$
\begin{aligned}
\sum_i (\nabla f) \langle \nabla_{X_i} \nabla f, X - I \rangle - \langle \nabla_{X_i} \nabla f, \nabla_{\nabla f} X_i \rangle &= (\nabla f) \sum_i \langle \nabla_{X_i} \nabla f, X_i \rangle - 0 \text{ (at } p) \\
&= (\nabla f)(\Delta f) \\
&= \langle \nabla f, \nabla(\Delta f) \rangle,
\end{aligned}
$$

and the third term is

$$
\begin{aligned}
\sum_i Hess(f)([X_i, \nabla f], X_i) &= \sum_i Hess(f)(\nabla_{X_i}\nabla f - \nabla_{\nabla f} X_i, X_i) \\
&= \sum_i Hess(f)(\nabla_{X_i}\nabla f, X_i) - Hess(f)(\nabla_{\nabla f} X_i, X_i) \\
&= \sum_i Hess(f)(\nabla_{X_i}\nabla f, X_i) - 0 \text{ (at } p) \\
&= \sum_i Hess(f)(X_i, \nabla_{X_i}\nabla f) \\
&= \sum_i \nabla_{X_i}\nabla f, \nabla_{X_i}\nabla f\rangle \\
&= [Hess(f)]^2
\end{aligned}
$$

□

This is powerful in the sense that we have a freedom to choose the function. We will choose a distance function. For a fixed $p \in M$ let $r(x) = d(p, x)$ be the distance function from p to x. This is a Lipschitz function and it is smooth except on the cut locus of p. It also satisfies $|\nabla r| = 1$ where it is smooth. We have that $\nabla r = \frac{\partial}{\partial r}$ in general polar coordinates at p. Let $\{e_1, e_2, ..., e_{n-1}\}$ be an orthonormal basis for the geodesic sphere, and denote the mean curvature of the geodesic sphere by $m(r)$ with the outer normal N, where $m(r) = \sum_{i=1}^{n-1} \langle \nabla_{e_i} N, e_i \rangle$. In general polar coordinate, the volume element can be written as $dvol = dr \wedge A_\omega(r) d\omega$ where $d\omega$ is the volume form on the standard S^{n-1}. We suppress the dependence of $A_\omega(r)$ on ω for notational convenience.

Lemma 4.3. *Given a complete Riemannian manifold (M^n, g) and a point $p \in M$, we have $\Delta r = m(r)$ and $m(r) = \frac{A'(r)}{A(r)}$.*

Proof. By definition

$$\begin{aligned}
\Delta r = tr(Hess(r)) &= \sum_{i=1}^{n-1} \langle \nabla_{e_i}(\nabla r), e_i \rangle + \langle \nabla_N(\nabla r), N \rangle \\
&= \sum_{i=1}^{n-1} \langle \nabla_{e_i} N, e_i \rangle + \langle \nabla_N N, N \rangle \\
&= \sum_{i=1}^{n-1} \langle \nabla_{e_i} N, e_i \rangle = m(r)
\end{aligned}$$

This proves the first equation.

Next, consider the map $\varphi : T_pM \to M$ defined by $\varphi(v) = exp_p(rv)$. Let $\{v_1, ..., v_{n-1}\}$ be the orthonormal basis for the unit sphere in T_pM. Then

$$\begin{aligned}
A(r) &= dvol\left(\frac{\partial}{\partial r}, \varphi(v_1), ..., \varphi(v_{n-1})\right) \\
&= dvol\left(\frac{\partial}{\partial r}, dexp_p(rv_1), ..., dexp_p(v_{n-1})\right) \\
&= J_1(r) \wedge J_2(r) \wedge ... \wedge J_{n-1}(r)
\end{aligned}$$

Where $J_i(r) = dexp_p(rv_i)$. Fix r_0. We have

$$\frac{A'(r_0)}{A(r_0)} = \frac{\sum_{i=1}^{n-1} J_1(r_0) \wedge ... \wedge J_i'(r_0) \wedge ... \wedge J_{n-1}(r_0)}{J_1(r_0) \wedge J_2(r_0) \wedge ... \wedge J_{n-1}(r_0)}$$

Let $\overline{J_1}(r), ..., \overline{J}_{n-1}(r)$ be the linear combinations (with constant coefficients) of the $J_i(r)$'s such that $\overline{J}_1(r_0), ..., \overline{J}_{n-1}(r_0)$ form an orthonormal basis. Then

$$
\begin{aligned}
\frac{A'(r_0)}{A(r_0)} &= \frac{\sum_{i=1}^{n-1} J_1(r_0) \wedge ... \wedge J_i'(r_0) \wedge ... \wedge J_{n-1}(r_0)}{J_1(r_0) \wedge J_2(r_0) \wedge ... \wedge J_{n-1}(r_0)} \\
&= \frac{\sum_{i=1}^{n-1} \overline{J}_1(r_0) \wedge ... \wedge \overline{J}_i'(r_0) \wedge ... \wedge \overline{J}_{n-1}(r_0)}{\overline{J}_1(r_0) \wedge \overline{J}_2(r_0) \wedge ... \wedge \overline{J}_{n-1}(r_0)} \\
&= \sum_{i=1}^{n-1} \overline{J}_1(r_0) \wedge ... \wedge \overline{J}_i'(r_0) \wedge ... \wedge \overline{J}_{n-1}(r_0) \\
&= \sum_{i=1}^{n-1} \langle \overline{J}_i'(r_0), \overline{J}_i(r_0) \rangle
\end{aligned}
$$

Let $f_i(t,s) = exp_p(sv_i + t\vec{n})$. Then

$$
J_i(r_0) = dexp_p(r_0 v_i) = \frac{\partial}{\partial s}\Big|_{s=0} f_i(t,s)
$$

and

$$
\begin{aligned}
J_i'(r_0) &= \frac{\partial}{\partial t}\Big|_{t=r_0} \frac{\partial}{\partial s}\Big|_{s=0} f_i(t,s) \\
&= \frac{\partial}{\partial t}\Big|_{s=0} \frac{\partial}{\partial s}\Big|_{t=r_0} f_i(t,s) \\
&= \nabla_{J_i(r_0)} N
\end{aligned}
$$

Therefore, we also have $\overline{J}_i'(r_0) = \nabla_{\overline{J}_i(r_0)} N$

Thus

$$
\begin{aligned}
\frac{A'(r_0)}{A(r_0)} &= \sum_{i=1}^{n-1} \langle \nabla_{\overline{J}_i}(r_0) N, \overline{J}_i(r_0) \rangle \\
&= m(r_0)
\end{aligned}
$$

□

Theorem 4.4. *(Main Comparison Theorem)*

Let (M^n, g) be complete, and assume that $Ric(M) \geq (n-1)H$. Outside the cut locus of p, we have:

1) Laplacian Comparison: $\Delta r \leq \Delta^H r$.

2) Volume Comparison: $\frac{A(r)}{A^H(r)}$ is non-increasing along radial geodesics.

3) Mean Curvature Comparison: $m(r) \leq m^H(r)$

(Quantities with superscript are counterparts in the simply connected space with constant sectional curvature H. Equality holds iff all radial sectional curvatures are equal to H)

Proof. First we prove Laplacian Comparison. Let $f(x) = r(x)$ and note that $|\nabla r| = 1$. Out of cut locus of p, we obtain

$$
|Hess(r)|^2 + \frac{\partial}{\partial r}(\Delta r) + Ric\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) = 0
$$

Let $\lambda_1, ..., \lambda_n$ be the eigenvalues of $Hess(r)$. Since the exponential function is radial isometry, one of the eigenvalues, say λ_1 is zero. By the Cauchy-Schwarz inequality, we have

$$|Hess(r)|^2 = \lambda_2^2 + ... + \lambda_n^2 \geq \frac{(\lambda_1 + ... + \lambda_n)^2}{n-1} = \frac{\mathrm{tr}(Hess(f))}{n-1} = \frac{(\Delta)^2}{n-1}$$

Thus, if $Ric \geq (n-1)H$, then

$$\frac{(\Delta)^2}{n-1} + \frac{\partial}{\partial r}(\Delta r) + (n-1)H \leq 0$$

Let $n = \dfrac{(n-1)}{\Delta r}$. Then $\dfrac{u'}{1+Hu^2} \geq 1$. Note that $\Delta r \to \frac{(n-1)}{r}$ when $r \to 0$; thus $u \to r$. Integrating the above inequality gives

$$\Delta r \leq \Delta^H r = \begin{cases} (n-1)\sqrt{H} cot\sqrt{H} r, \text{ for } H > 0, \\ \frac{(n-1)}{r}, \text{ for } H = 0 \\ (n-1)\sqrt{-H} coth\sqrt{-H} r, \text{ for } H < 0 \end{cases}$$

Next, we discuss equality case. If equality holds at r_0, then for any $r \leq r_0$, all the inequalities above become equalities. In particular, the $(n-1)$ eigenvalues of $Hess(r)$ are equal to $\sqrt{N} cot\sqrt{H} r$ (to simplify the constant, we assume $H > 0$. For $H \leq 0$, replace cot by $coth$.) Let X_i be the orthonormal eigenvalues of $Hess(r)$ at r for $i = 1, 2, ..., n$;

Thus

$$\nabla_{X_i} \frac{\partial}{\partial r} = \sqrt{H} cot\sqrt{H} r X_i.$$

Extend X_i in such away that $[X_i, \frac{\partial}{\partial r}] = 0$ at r, then

$$
\begin{aligned}
Sec\left(X, \frac{\partial}{\partial r}\right) &= -\left\langle \nabla_{\frac{\partial}{\partial r}} \nabla_{X_i} \frac{\partial}{\partial r}, X_i \right\rangle \\
&= -\left\langle \nabla_{\frac{\partial}{\partial r}} \left(\sqrt{H} cot \sqrt{H} r\right) X_i, X_i \right\rangle \\
&= H csc^2 \sqrt{H} r - \left\langle \nabla_{\frac{\partial}{\partial r}} X_i, X_i \right\rangle \\
&= H csc^2 \sqrt{H} r - \sqrt{H} cot \sqrt{H} r \left\langle \nabla_{X_i} \frac{\partial}{\partial r}, X_i \right\rangle \\
&= H csc^2 \sqrt{H} r - (\sqrt{H} cot \sqrt{H} r)^2 = H
\end{aligned}
$$

Volume comparison and mean value comparison follows from lemma 4.3. □

For the application of the volume comparison theorem, an integrated form is used.

Lemma 4.5. *Let f,g be two positive functions defined over* $[0, \infty)$. *If* $\frac{f}{g}$ *is non-increasing, then for any* $R > r > 0$, $S > s > 0$, $r > s$, $R > S$, *we have*

$$
\frac{\int_r^R f(t)dt}{\int_s^S f(t)dt} \leq \frac{\int_r^R g(t)dt}{\int_s^S g(t)dt}
$$

Proof. It suffices to show that the function

$$
F(x, y) = \frac{\int_x^y f(t)dt}{\int_x^y g(t)dt}
$$

satisfies $\frac{\partial F}{\partial x} \leq 0$, $\frac{\partial F}{\partial y} \leq 0$. It follows that

$$
\frac{\int_r^R f(t)dt}{\int_r^R g(t)dt} \leq \frac{\int_r^S f(t)dt}{\int_r^S g(t)dt} \leq \frac{\int_s^S f(t)dt}{\int_s^S g(t)dt}
$$

We now compute

$$\begin{aligned}\frac{\partial F}{\partial y} &= \frac{1}{\left(\int_x^y g(t)dt\right)^2}\left(f(y)\int_x^y g(t)dt - g(y)\int_x^y f(t)dt\right)\\ &= \frac{g(y)\int_x^y g(t)dt}{\left(\int_x^y g(t)dt\right)^2}\left(\frac{f(y)}{g(y)} - \frac{\int_x^y f(t)dt}{\int_x^y g(t)dt}\right)\end{aligned}$$

But

$$\frac{f(t)}{g(t)} \geq \frac{f(y)}{g(y)}, \text{ for } x \leq t \leq y$$

Thus

$$\int_x^y f(t)dt \geq \int_x^y \frac{f(y)}{g(y)} \cdot g(t)dt = \frac{f(y)}{g(y)}\int_x^y g(t)dt$$

that is,

$$\frac{f(y)}{g(y)} \leq \frac{\int_x^y f(t)dt}{\int_x^y g(t)dt}$$

which implies $\dfrac{\partial F}{\partial y} \leq 0$.

□

Theorem 4.6. *Let M be a complete open with $Ric \geq (n-1)H$. Let $A^{\Gamma}_{x,y}(p)$ be the set of $q \in M$ such that $x \leq r(q) \leq y$ and any minimal geodesic γ from p to q satisfying $\dot{\gamma} \in \Gamma$. Then*

$$\frac{Vol(A^{\Gamma}_{x,y})}{Vol(A^H(x,y))}$$

is non-increasing. Here, Γ is a measurable subset of S_p^{n-1}.

Proof. Note that

$$Vol(A^{\Gamma}_{(x,y)}) = \int_{\Gamma} d\omega \int_{min\{x,cut(\omega)\}}^{min\{y,cut(\omega)\}} A(r,\omega)dr$$

where $cut(\omega)$ is the distance to the cut locus in the direction $\omega \in S_p^{n-1}$. Sicce $\dfrac{A(r,\omega)}{A^H(r)}$ is non-increasing for any ω and $r < cut(\omega)$, lemma 4.5 implies, for $z \geq y$

$$\frac{\int_{min\{x,cut(\omega)\}}^{min\{y,cut(\omega)\}} A(r,\omega)dr}{\int_{min\{x,cut(\omega)\}}^{min\{y,cut(\omega)\}} A^H(r)dr} \geq \frac{\int_{min\{x,cut(\omega)\}}^{min\{z,cut(\omega)\}} A(r,\omega)dr}{\int_{min\{x,cut(\omega)\}}^{min\{z,cut(\omega)\}} A^H(r)dr}$$

That is

$$\begin{aligned}
\int_{min\{x,cut(\omega)\}}^{min\{y,cut(\omega)\}} A(r,\omega)dr &\geq \frac{\int_{min\{x,cut(\omega)\}}^{min\{y,cut(\omega)\}} A^H(r)dr}{\int_{min\{x,cut(\omega)\}}^{min\{z,cut(\omega)\}} A^H(r)dr} \cdot \int_{min\{x,cut(\omega)\}}^{min\{z,cut(\omega)\}} A(r,\omega)dr \\
&\geq \frac{\int_{x}^{min\{y,cut(\omega)\}} A^H(r)dr}{\int_{x}^{min\{z,cut(\omega)\}} A^H(r)dr} \cdot \int_{min\{x,cut(\omega)\}}^{min\{z,cut(\omega)\}} A(r,\omega)dr \\
&\geq \frac{\int_x^y A^H(r)dr}{\int_x^z A^H(r)dr} \cdot \int_{min\{x,cut(\omega)\}}^{min\{z,cut(\omega)\}} A(r,\omega)dr
\end{aligned}$$

where the last inequality follows from the three possibilities; $cut(\omega) \leq y \leq z$, $y \leq cut(\omega) \leq z$, and $y \leq z \leq cut(\omega)$.

The inequality before that uses the fact that

$$\frac{\displaystyle\int_x^a A^H(r)dr}{\displaystyle\int_x^b A^H(r)dr}$$

is non-increasing for $a < b$. Integrating the above over Γ, we get

$$\begin{aligned} Vol(A^\Gamma_{x,y}) &\geq \frac{\displaystyle\int_x^y A^H(r)dr}{\displaystyle\int_x^z A^H(r)dr} \cdot Vol(A^\Gamma_{x,z}) \\ &= \frac{Vol(A^H(x,y))}{Vol(A^H(x,z))} \cdot Vol(A^\Gamma_{x,z}) \end{aligned}$$

The equality part follows from equality discussion in theorem 4.4.

□

From theorem 4.6 above we deduce the generalized version of Bishop-Gromov volume comparison theorem by letting $H = 0$, $x = 0$, and $y = r$:

Theorem 4.7. *(Generalized version of Bishop and Gromov volume comparison)*
Let M be a complete open manifold with $Ric_M \geq 0$. Then for any measurable subset $N \subset S_pM$ and any $p \in M$, the function

$$\varphi(r) = \frac{Vol(B_N(p,r))}{r^n}$$

is non-increasing on $(0, \infty)$.

Corollary 4.8. *Let M be a complete open manifold with $Ric_M \geq 0$. Then for any measur-*

able subset $N \subset S_pM$*,* $k > 1$*, and* $p \in M$*,*

$$Vol(B_N(p, kr)) \leq k^n Vol(B_N(p, r))$$

For simplicity, let us denote $r(x) = r_p(x)$.

Lemma 4.9. *Let* M *be a complete open manifold. Let* Σ_δ *be a* δ*-neighborhood of* Σ *in* S_pM*.* Σ_δ^c *is a compact subset in* S_pM *since* Σ_δ *is open and* $\Sigma \cap \Sigma_\delta^c = \emptyset$*. So there exist a constant* $r_0 \geq 0$ *such that* $B_{\Sigma_\delta^c}(p, r) \subset B(p, r_0)$ *for any* $r > 0$*.*

Proof. On the contrary, there is a sequence $\{r_n\}$ such that $r_n \to \infty$ and a corresponding sequence of minimizing geodesics $\{\sigma_n\}$ such that σ_n is a minimizing geodesic from p to r_n and $\sigma_n'(0) \in \Sigma_\delta^c$. We can take $\sigma_n(t) = \exp_p(t\sigma_n'(0))$. Since Σ_δ^c is compact in S_pM, the sequence $\{\sigma_n'(0)\}$ converges to $\sigma_\infty'(0) \in \Sigma_\delta^c$. Next, we prove that σ_∞ is a ray. Otherwise, there is an s such that the distance from p to $\exp_p(s\sigma_\infty'(0))$ is less than s. Say, $d(p, \exp_p(s\sigma_\infty'(0))) = s - \varepsilon$. By continuity of $\exp_p$, there exists δ such that $d(\exp_p(s\sigma_\infty'(0)), \exp_p(s\sigma_k'(0))) < \varepsilon$ when $||\sigma_\infty - \sigma_k|| < \delta$, where $\sigma_k \in T_pM$. Then for $r_n \geq s$

$$\begin{aligned} d(p, \exp_p(r_n\sigma_n'(0)) &\leq d(p, \exp_p(s\sigma_\infty'(0)) + d(\exp_p(s\sigma_\infty'(0)), \exp_p(s\sigma_k'(0))) \\ &\quad + d(\exp_p(s\sigma_k'(0)), \exp_p(r_n\sigma_k'(0))) \\ &< (s - \varepsilon) + \varepsilon + (r_n - s) \\ &= r_n \end{aligned}$$

A contradiction. Hence σ_∞ is a ray. This also contradicts the fact that $\sigma'_\infty(0) \in \Sigma_\delta^c$

□

Remark 4.1. Applying theorem 4.7 above, it is easy to show that the maximum volume growth condition implies that

$$\lim_{r\to\infty} \frac{Vol(B_\Sigma(p,r))}{Vol(B(p,r))} = 1 \tag{4.1}$$

given that the manifold is complete open and admits nonnegative Ricci curvature as evidenced in the following lemma and it's corollary. From lemma 4 of [15] we have

Lemma 4.10. *Let M be a complete open manifold with $Ric_M \geq 0$. Suppose that M has a maximum volume growth i.e*

$$\lim_{r\to\infty} \frac{Vol(B_(p,r))}{r^n} = \alpha_M, \ \ \alpha_M > 0$$

then

$$\lim_{r\to\infty} \frac{Vol(B_\Sigma(p,r))}{r^n} = \alpha_M$$

Proof. Since $Vol(B(p,r)) = Vol(B_\Sigma(p,r)) + Vol(B_{\Sigma^c}(p,r))$, it suffices to prove that

$$\lim_{r\to\infty} \frac{Vol(B_{\Sigma^c}(p,r))}{r^n} = 0$$

Given any $\varepsilon > 0$, choose $\delta > 0$ small enough such that the open δ-neighborhood Σ_δ of Σ is such that $Vol_{n-1}(\Sigma\backslash\Sigma_\delta) < \varepsilon$. Here, Vol_{n-1} is the $n-1$-dimensional volume in a unit sphere S_pM. By theorem 9,

$$\frac{Vol(B_{\Sigma_\delta \backslash \Sigma}(p,r))}{r^n}$$

is an increasing function of r. Thus,

$$\begin{aligned}
\frac{Vol(B_{\Sigma_\delta \backslash \Sigma}(p,r))}{r^n} &\leq \lim_{r\to 0} \frac{Vol(B_{\Sigma_\delta \backslash \Sigma}(p,r))}{r^n} \\
&= \frac{Vol_{n-1}(\Sigma_\delta \backslash \Sigma)}{n} < \frac{\varepsilon}{n}
\end{aligned}$$

By lemma 9, there exists a constant $r_0 > 0$ such that $B_{\Sigma_\delta^c}(p,r) \subset B(p,r_0)$ for all $r \geq r_0$ sufficiently large. Therefore

$$\begin{aligned}
\frac{Vol(B_{\Sigma^c}(p,r)}{r^n} &= \frac{Vol(B_{\Sigma_\delta^c}(p,r))}{r^n} + \frac{Vol(B_{\Sigma_\delta \backslash \Sigma}(p,r))}{r^n} \\
&\leq \frac{Vol(B(p,r_0))}{r^n} + \frac{\varepsilon}{n} \\
&\leq \left(\frac{r_0^n Vol_{n-1}(S_pM)}{r^n} + \varepsilon\right)/n < \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary small, we have proved that

$$\frac{Vol(B_{\Sigma^c}(p,r)}{r^n} = 0$$

Hence

$$\lim_{r\to\infty} \frac{Vol(B_\Sigma(p,r))}{r^n} = \alpha_M$$

□

Corollary 4.11. *Let M be a complete open manifold with $Ric_M \geq 0$. Suppose that M has a maximum volume growth. Then equation (4.1) holds.*

Proof. By lemma 11 above

$$\begin{aligned}\lim_{r\to\infty}\frac{Vol(B_\Sigma(p,r))}{Vol(B(p,r))} &= \lim_{r\to\infty}\frac{Vol(B_\Sigma(p,r))}{r^n}\cdot\lim_{r\to\infty}\frac{1}{\frac{Vol(B(p,r))}{r^n}}\\ &= \alpha_M\cdot\frac{1}{\alpha_M}\\ &= 1\end{aligned}$$

□

The converse is not true. In other words, equation (4.1) does not necessarily imply maximum volume growth. We construct a counter example.

4.0.1 Paraboloid in $\mathbb{R}^{n+1}(n \geq 2)$.

In this subsection we investigate the volume growth condition of a paraboloid in $\mathbb{R}^{n+1}(n \geq 2)$ with the purpose of proving that condition (4.1) is weaker than the maximum volume growth codition on a complete open manifold with nonnegative Ricci curvature.

Some materials in this subsection closely follow expositions from pages 6,7, and 8 in [5]. Let H be a hypersurface in $\mathbb{R}^{n+1}$ with a smooth parametrization $f : D \to H$ defined by

$$f(x_1, x_2, \cdots, x_n) = (x_1, \cdots, x_n, r(x_1, \cdots, x_n)),$$

where $D \subset \mathbb{R}^{n+1}$. The standard basis $\{\partial_1, \cdots, \partial_n\}$ for the tangent space T_pH at each point

$p = (x_1, \cdots, x_n)$ is given by

$$\partial_i = \frac{\partial f}{\partial x_i} = (0, \cdots, 0, 1, 0, \cdots, 0, r_i),$$

where $r_i = \dfrac{\partial r}{\partial x_i}$. We then have that the unit normal vector is defined by

$$N = \frac{(-r_1, \cdots, -r_n, 1)}{\sqrt{r_1^2 + \cdots + r_n^2 + 1}}$$

Second fundamental form and shape operator. Second fundamental form and shape operator makes it easier to compute curvature tensors of hypersurfaces. Second fundamental form of H denoted by

$$\Pi : T(H) \times T(H) \to \mathcal{N}(H)$$

is defined as $\Pi(X, Y) := (\tilde{\nabla}_X Y)^{\perp}$, where $T(H)$ is a tangent bundle of H and $\mathcal{N}(H)$ is a normal bundle of H and $\tilde{\nabla}$ is a Riemannian connection on $\mathbb{R}^{n+1}$. Considering N, we can replace Π by a simpler scalar-valued form h

$$h(X, Y) = \langle \Pi(X, Y), N \rangle.$$

Since N is a unit vector,

$$\Pi(X, Y) = h(X, Y)N$$

So by definition,

$$
\begin{aligned}
h_{i,j} = h(\partial_i, \partial_j) &= \tilde{\nabla}_{\partial_i}\partial_j \cdot N \\
&= \frac{\partial^2 f}{\partial_{x_i}\partial_{x_j}} \cdot N \\
&= \frac{r_i r_j}{\sqrt{r_1^2 + \cdots + r_n^2 + 1}},
\end{aligned}
$$

where $r_{i,j} = \partial_i \partial_j r$. Raising one index of h, we get a shape operator s of M characterized by

$$\langle sX, Y \rangle = h(X, Y),$$

for all $X, Y \in T(H)$. Because h is symetric, s is a self adjoint linear endomorphism on $T_p(H)$, that is

$$\langle sX, Y \rangle = \langle X, sY \rangle,$$

for all $X, Y \in T(H)$.

Computing Riemannian curvature tensor of a hypersurface by shape operator. Using s we can compute Riemannian curvature tensor of a hypersurface H as follows. The Weingarten equation states that if $X, Y \in T(H)$ and $N \in \mathcal{N}(H)$ are arbitrary extended to $\mathbb{R}^{n+1}$ then

$$\langle \tilde{\nabla}_X N, Y \rangle = -\langle N, \Pi(X, Y) \rangle$$

at every point of H. So,

$$s\partial_i = -\frac{\partial N}{\partial x_i} \tag{4.2}$$

Also by denoting $s\partial_i = s_i^j \partial_j$, we have

$$s\partial_i = (s_i^1, \cdots, s_i^n, \sum_j s_i^j r_j) \tag{4.3}$$

Let $b = r_1^2 + \cdots + r_n^2 + 1$. Then from equations (4.2) and (4.3), we obtain

$$s_i^j = -b^{-\frac{3}{2}} r_j \sum_k r_k r_{k,i} + b^{-\frac{1}{2}} r_{j,i}. \tag{4.4}$$

We can also write the shape operator s in matrix form:

$$\begin{aligned} h(\partial_i, \partial_k) = h_{ik} &= \langle s\partial_i, \partial_k \rangle \\ &= \langle s_i^j \partial_i, \partial_k \rangle \\ &= s_i^j g_{jk} \end{aligned}$$

So

$$s_i^j = g^{ik} h_{ik} \tag{4.5}$$

and we can rewrite (4.4) as

$$s_i^j = \sum_k (-b^{\frac{3}{2}} r_j r_k + b^{-\frac{1}{2}} \delta_{jk}) r_{k,i}$$

then by symmetry of h and equation (4.5) we have that $g^{ik} = -b^{-1} r_j r_k + \delta_{jk}$, where δ_{jk} is the Kronecker symbol.

For any $X, Y, Z, W \in T_p(H)$, we define Gauss equation as

$$\overline{R}(X,Y,Z,W) = R(X,Y,Z,W) - \langle \Pi(X,W), \Pi(Y,Z)\rangle + \langle \Pi(X,Z), \Pi(Y,W)\rangle$$

in $\mathbb{R}^{n+1}$, $\overline{R} = 0$. Hence

$$R(X,Y,Z,W) = -h(X,Z)h(Y,W) + h(X,W)h(Y,Z)$$

and

$$R(X,Y,Y,X) = h(X,X)h(Y,Y) - h(X,Y)^2$$

that is

$$R(\partial_i, \partial_j, \partial_k, \partial_l) = h(\partial_i, \partial_l)h(\partial_j, \partial_k) - h(\partial_i, \partial_k)h(\partial_j, \partial_l)$$

and

$$R_{ijkl} = h_{il}h_{jk} - h_{ik}h_{jl} = b^{-1}(r_{i,l}r_{j,k} - r_{i,k}r_{j,l})$$

Paraboloid. Consider a hypersurface $f : \mathbb{R}^n \to M \subset \mathbb{R}^{n+1}$ defined by $f(x_1, \cdots, x_n) = (x_1, \cdots, x_n, r(x_1, \cdots, x_n))$, where $r = x_1^2 + \cdots + x_n^2$. Then $r_i = 2x_i$, $r_{i,j} = 2\delta_{ij}$ and

$b = r_1^2 + \cdots + r_n^2 + 1 = 4x_1^2 + \cdots + 4x_i^2 + 1$. The curvature tensor becomes

$$R_{ijkl} = 4b^{-1}(\delta_{il}\delta_{jk} - r_{ik}r_{jl})$$

and

$$R_{ijji} = 4b^{-1}(\delta_{ii}\delta_{jj} - \delta_{ij}\delta_{ji}) = \frac{4}{b}$$

At $p = (x_1, \cdots, x_n)$ let $X \in T_pM$ be a vector. Construct an orthonormal basis $\{E_i\} \subset T_pM$ such that $\frac{X}{|X|} = E_1$. Let

$$X = a_1\partial_1 + \cdots + a_n\partial_n$$

and

$$E_i = e_1^i\partial_1 + \cdots + e_n^i\partial_n$$

then

$$R(E_i, X, X, E_i) = h(X, X)h(E_i, E_i) - h(X, E_i).$$

Since h is bilinear,

$$R(E_i, X, X, E_i) = \frac{4}{b}(a_i^2 + \cdots + a_n^2)((e_1^i)^2 + \cdots + (e_1^i)^2) - \frac{4}{b}(a_1e_1^i + \cdots + a_ne_n^i)$$

Note that $\langle X, E_i \rangle = 0$ implies $a_1 e_1^i + \cdots + a_n e_n^i = 0$. Thus $R(E_i, X, X, E_i) > 0$ and

$$Ricci(X) = \sum_{i=1}^{n} R(E_i, X, X, E_i) > 0$$

The vertex 0 of a paraboloid M has an empty cut locus. Assume otherwise, that is, let $\gamma : [0, a] \to M$ be a minimizing geodesic such that $\gamma(0) = 0$ and $\gamma(a) = p$, where p is a cut point of 0. We consider two cases.

Case I: p is a conjugate point of 0 a long a geodesic γ. Choose an orthonormal basis $\{e_i\}_{i=1,\cdots,n}$ of $T_0 H$ where $e_1 = \dot{\gamma}(0)$ which can be extended to parallel orthonormal fields $\dot{\gamma}(t), e_2(t), \cdots, e_n(t)$ along γ. Let J be a nontrivial Jacobi field a long γ such that $J(0) = 0 = J(p)$. Then $J(t) = \sum_{i=1}^{n} a_i(t) e_i(t)$ for smooth functions $a_i \in C^\infty$, $i = 1, \cdots, n$.

$$\begin{aligned}
R(J, \dot{\gamma})\dot{\gamma} &= \sum_{i=1}^{n} \langle R(J, \dot{\gamma})\dot{\gamma}, e_i \rangle e_i \\
&= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_j \langle R(e_j, \dot{\gamma})\dot{\gamma}, e_i \rangle e_i \right) \\
&= \sum_{j=1}^{n} a_j \langle R(e_j, \dot{\gamma})\dot{\gamma}, e_1 \rangle e_1 + \sum_{i=2}^{n} \left(\sum_{j=1}^{n} a_j \langle R(e_j, \dot{\gamma})\dot{\gamma}, e_i \rangle e_i \right) \\
&= 0 + \sum_{i=2}^{n} a_i e_i \frac{4}{b} = \frac{4}{b} \sum_{i=2}^{n} a_i e_i
\end{aligned}$$

Also note that

$$\frac{d^2}{dt^2} J = \sum_{i=1}^{n} a_i(t)'' e_i(t)$$

so

$$\sum_{i=1}^{n} a_i(t)'' e_i(t) + \sum_{i=2}^{n} a_i(t) e_i(t) \frac{4}{b} = 0. \tag{4.6}$$

By Symmetry of curvature tensor,

$$\begin{aligned}
\frac{d^2}{dt^2}\langle J, \dot{\gamma} \rangle &= \langle D_t^2 J, \dot{\gamma} \rangle \\
&= -\langle R(J, \dot{\gamma})\dot{\gamma}, \dot{\gamma} \rangle \\
&= -R(J, \dot{\gamma}, \dot{\gamma}, \dot{\gamma}) = 0
\end{aligned}$$

Thus $\langle J, \dot{\gamma} \rangle$ is a linear function of t or a constant. By construction, $\langle J, \dot{\gamma} \rangle = a_1(t)$, and from initial conditions of Jacobi field J, $a_1(0) = a_1(p) = 0$. Thus $a_1(t) = 0$ for any t. Hence equation (4.6) becomes

$$\sum_{i=2}^{n} a_i(t)'' e_i(t) + \sum_{i=2}^{n} a_i(t) e_i(t) \frac{4}{b} = 0.$$

i.e

$$\frac{d^2}{dt^2} J + J \frac{4}{b} = 0. \tag{4.7}$$

Applying ODE to equation (4.7) above, we obtain a unique solution $J(t)$.

Since M is complete there exists a ray τ from $0 \in M$. Let $\{f_i\}_{i=1,\cdots,n}$ be an orthonormal basis of T_0M where $f_1 = \dot{\tau}(0)$ and can be extended to parallel orthonormal fields $\dot{\tau}(t)$, $f_2(t), \cdot\cdot$

$\cdot, f_n(t)$ along τ. Let J be a Jacobi field along τ normal to τ. Then

$$J(t) = \sum_{i=2}^{n} b_i f_i(t)$$

and

$$\frac{d^2}{dt^2}J + \frac{4}{b}J = 0 \tag{4.8}$$

Note that $\frac{4}{b}$ is constant on each metric circle $S_0(t) = \{p \in \mathbb{R}^n : d(0,p) = t\}$ for $t \in [0, \infty)$. Since γ and τ are minimizing geodesics within $(0, a]$, there is a unique solution satisfying both equations (4.7) and (4.8). This contradicts that τ is a ray. Hence there is no conjugate point of 0 a long γ. In other words, the vertex 0 has no conjugate points. Hence 0 is a pole.

Case II. There is another minimizing geodesics $\tau : [0, a] \to M$ such that $\gamma \neq \tau$ in $(0, a)$, $\gamma(0) = \tau(0)$, and $\gamma(a) = \tau(a)$. Since 0 is a pole and M is diffeomorphic to $\mathbb{R}^n$, there is no such geodesic τ.

From cases I and II above, the vertex 0 of a paraboloid $M \subset \mathbb{R}^{n+1}$ has an empty cut locus. Thus volume growth condition (4.1) holds at 0 and extends to other points by lemma 1.7.

Maximum volume growth. Next, we establish that a paraboloid $M \subset \mathbb{R}^{n+1}$ doesn't admit maximum volume growth condition. Let $r > 0$ and a point $p \in M$. Denote

$$R(p, r) = \{\gamma(r) : \ \gamma \ a \ ray \ from \ p\}$$

We define $\mathcal{D}(p, r) = \sup_{\mathcal{B}} diam(\mathcal{B})$, where the supremum is taken over all bounded components $\mathcal{B}$ of $M \backslash \overline{B(p, r)}$, with $\mathcal{B} \cap R(p, r) \neq \emptyset$. The following is a special case of lemma 4.1 in [16].

Lemma 4.12. *Let* $f : \mathbb{R}^+ \to \mathbb{R}^+$ *be a function defined by* $f(z) = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$, *where* $f(z) = \sqrt{z}$. $f(z) = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$ *defines a surface of revolution in* $\mathbb{R}^{n+1}$ *with coordinates* $(x_1, x_2, \cdots, x_n, z)$. *The volume growth of the surface(paraboloid) is at most* $r^{\frac{n+1}{2}}$.

Proof. Let $r = d(0.(x_1, x_2, \cdots, x_n, z))$. It follows that $r = \int_0^z \sqrt{1 + [f'(z)]^2} dz$. By definition $f(z) \leq \mathcal{O}(z^{\frac{1}{2}})$. Then $f'(z) \leq \mathcal{O}(z^{-\frac{1}{2}}) \leq \mathcal{O}(1)$. Hence

$$z \leq r \leq Cz \tag{4.9}$$

for some constants $C > 0$. Clearly $\mathcal{D}(0, r) \leq 2\pi f(z) = \mathcal{O}(z^{\frac{1}{2}})$. Hence by equation(4.9), $\mathcal{D}(0, r) \leq \mathcal{O}(r^{\frac{1}{2}})$. Thus the volume of a ball centered at 0 is at most $r^{\frac{n+1}{2}}$. □

Remark 4.2. From lemma 4.14 above, if $M = \{(x_1, x_2, \cdots, x_n, z) : z = x_1^2 + x_2^2 + \cdots + x_n^2\}$, then M does not admit the maximum volume growth condition. However, since the point $p = (0, 0, \cdots, 0)$ has an empty cut locus, equation (4.1) is satisfied at p. By lemma 1.7, we have the same conclusion at other points as well.

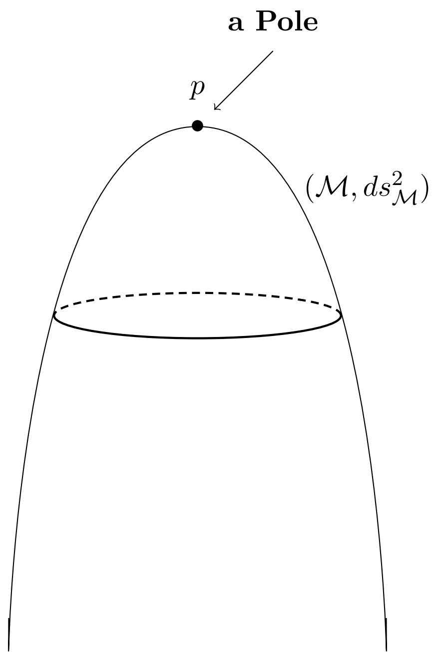

Figure 4.1: A paraboloid in $\mathbb{R}^{n+1}$

Chapter 5

Complex Manifolds

Definition 5.1. *Let M be a topological space which is connected, Hausdorff and has countable base. It is called complex manifold of (complex) dimension n if there exists an open covering $\{U_a\}_{a \in A}$, and for each $a \in A$ a homeomorphism f_a from U_a onto an open subset $D_a \in \mathbb{C}^n$, such that for any pair $a, b \in A$ with $U_{ab} = U_a \cap U_b \neq \emptyset$, the mapping $f_a \circ f_b^{-1}$ is a biholomorphism (i.e., homeomorphism that is two way holomorphic) between $f_b(U_{ab})$ and $f_a(U_{ab})$.*

5.0.1 The Almost Complex Structure

Let M^n be a complex manifold of dimension n. Since $C^n \cong \mathbb{R}^{2n}$, and biholomorphism are diffeomorphisms, M is also differentiable manifold of (real) dimension $2n$. It is denoted by $M_{\mathbb{R}}$ and it is called the *the underlying differentiable manifold* of the complex manifold. The complex manifold M is called a *complex structure* on $M_{\mathbb{R}}$.

Write $N^{2n} = M_{\mathbb{R}}$. The complex structure on N induces a splitting of the complexification of the tangent bundle $TN^{\mathbb{C}} = TN \otimes_{\mathbb{R}} \mathbb{C}$ into the sum of complex subbundles of equal rank.

We can denote this splitting by $TN^{\mathbb{C}} = TM^{(1,0)} \oplus TM^{(0,1)}$. We can describe this splitting in coordinate neighborhood as follows.

Let $\{z_1, ..., z_n\}$ be a local holomorphic coordinate in a neighborhood U of $p \in M$. Denote $z_i = x_j + iy_j$. Then $(x_1, ..., x_n, y_1, ..., y_n)$ is a smooth(real) coordinate in U, and $\left\{\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial y_n}\right\}$ gives a local frame of the tangent bundle TN. Denote by

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) \text{ and } \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$$

for each $1 \leq j \leq n$. Then $TM^{(1,0)}$ is the complex subbundle of $TN^{\mathbb{C}}$ spanned by $\{\frac{\partial}{\partial z_1}, ..., \frac{\partial}{\partial z_n}\}$, while $TM^{(0,1)}$ is spanned by $\{\frac{\partial}{\partial \overline{z}_1}, ..., \frac{\partial}{\partial \overline{z}_n}\}$.

A bundle isomorphism $J : TN \to TN$ defined by

$$J\frac{\partial}{\partial x_j} = \frac{\partial}{\partial y_j}, \quad J\frac{\partial}{\partial y_j} = -\frac{\partial}{\partial x_j}$$

for each $1 \leq j \leq n$ can be linearly extended over $\mathbb{C}$ to an isomorphism on $TN^{\mathbb{C}}$, and still denoted by J, where

$$J\frac{\partial}{\partial z_j} = i\frac{\partial}{\partial z_j}, \quad J\frac{\partial}{\partial \overline{z}_j} = -i\frac{\partial}{\partial \overline{z}_j}$$

for each $1 \leq j \leq n$. Definition of J is independent of local coordinates, so J is a bundle map from TN onto TN.

For a real vector (or vector field) X on N, $X - iJ$ is a vector in $TM^{(1,0)}$, and any vector in $TM^{(1,0)}$ is in this form for some real tangent vector X. Thus the map $X \mapsto X - iJX$ defines an isomorphism (over $\mathbb{R}$) between TN and $TM^{(1,0)}$. Sections of $TN^{\mathbb{C}}$ ($TM^{(1,0)}$, *or* $TM^{(0,1)}$) are called complex fields (*of type* (1,0) or (0,1)) on M. A complex field is of type $(1,0)$ if and only if it is in the form $X - iJX$ for some real vector field X.

Definition 5.2. *An endomorphism J of a tangent bundle of a differentiable manifold N satisfying $J^2 = -I$ is called an almost complex structure on N. Here, I denotes the identity map of TN*

5.0.2 K$\ddot{a}$hler manifold

Let M^n be a complex manifold. A Hermitian metric on M is simply a Hermitian metric defined on the holomorphic tangent bundle TM. That is, a covariant 2-tensor which is Hermitian symmetric and positive definite everywhere, i.e

$$h = \sum_{i,j=1}^{n} h_{i\bar{j}} dz_i d\overline{z_j}$$

where $(z_1, z_2, \cdots, z_n)$ is a local holomorphic coordinate, $h = (h_{i\bar{j}})$ is an nxn matrix of smooth functions which are Hermitian symmetric and positive definite. The real part $g = \mathrm{Re}(h)$ of h, is a symmetric, positive definite covariant 2-tensor. So, g is a Riemannian metric on the underlying smooth manifold $M_{\mathbb{R}}$ of M. Thus a Hermitian manifold is always a Riemannian manifold. Conversely, if we start with a Riemannian metric g on $M_{\mathbb{R}}$, then it is (coming from) a Hermitian metric iff $g(JX, JY) = g(X, Y)$ for any real vector fields X, and Y, where J is the almost complex structure on M. Under a local holomorphic coordinate $(z_1, z_2, \cdots, z_n)$, if we write $z_k = x_k + x_{n+k}$, then $(x_1, x_2, \cdots, x_{2n})$ is a local diffentiable coordinate in $M_{\mathbb{R}}$. By writing

$$g = \sum_{a,b=1}^{2n} g_{ab} dx_a dx_b$$

we have that

$$g = (g_{ab}) = \begin{bmatrix} A & B \\ -B & A \end{bmatrix},$$

where $h = (h_{i\bar{j}}) = A + \sqrt{-1}B$

Hermitian metric is a Kähler metric if it's torsion, $T = d\varphi + {}^t\theta \wedge \varphi$ vanishes, where $\theta = \partial h h^{-1}$ and φ is the dual of a local holomorphic frame of a tangent space.

Suppose (E, h) is a Hermitian vector bundle over a complex manifold M^n and u and v are sections in E, then we define a curvature tensor $\Omega_{u\bar{v}}$ by

$$\Omega_{u\bar{v}} = \sum_{i,j,k=1}^{r} \Omega_{ik} h_{k\bar{j}} u_i v_j$$

where $\Omega = d\theta - \theta \wedge \theta$, $\Omega = (\Omega_{ij})_{1 \leq i,j \leq n}$, $u = \sum u_i e_i$, and $v = \sum v_i e_i$ under a frame $\{e_1, \cdots, e_r\}$ of E. For sections u, v of E and $(1,0)$ type tangent vectors X, Y in M, we denote

$$R_{X\overline{Y}u\bar{v}} = \Omega_{u\bar{v}}(X, \overline{Y})$$

Restricting the curvature to the tangent bundle of M^n, i.e $E = T^{1,0}M$, it becomes a covariant 4-tensor

$$R_{X\overline{Y}Z\overline{W}} = \Omega_{Z\overline{W}}(X, \overline{Y})$$

The normalized curvature in the direction of X and Z

$$B(X, Z) = \frac{R_{X\overline{X}Z\overline{Z}}}{|X|^2|Z|^2}$$

is called the bisectional curvature of h in the direction of X and Z, while

$$H(X) = B(X,X) = \frac{R_{X\overline{X}X\overline{X}}}{|X|^4}$$

is called the holomorphic sectional curvature of h in the direction X.

This R is not the Riemannian curvature tensor of the Riemannian metric $\mathrm{Re}(h)$ in general, unless h is Kähler. The following is a discussion of Kähler case:

Let (M^n, h) be a Kähler manifold. Denote by $\langle , \rangle$ the underlying Riemannian metric $\mathrm{Re}(h)$, and by ∇, R the Riemannian connection and Riemannian curvature respectively. The curvature tensor R of h is just the linear extension (over $\mathbb{C}$) of the Riemannian curvature tensor R of the metric $\mathrm{Re}(h)$. This justify the use of same symbol.

The Kählerness of the metric means $\langle Ju, Jv\rangle = \langle u, v\rangle$ and $\nabla_u(Jv) = J\nabla_u v$ for any two real vector fields u and v, where J is the almost complex structure of M. The Riemannian curvature tensor satisfies $R(u,v,Jz,Jw) = R(u,v,z,w)$ because of $R(u,v)Jz = JR(u,v)z$ by definition of R and commutativity of ∇_* with J. In practice, it is more convenient to write R in terms of its complex components.

When we extend $g = \langle , \rangle$ linearly over $\mathbb{C}$ to $TM_{\mathbb{R}} \bigotimes \mathbb{C} = TM \bigoplus \overline{TM}$, then $\langle , \rangle$ becomes a complex bilinear form and $h(X,Y) = 2\langle X, \overline{Y}\rangle$, $\langle X, Y\rangle = \langle \overline{X}, \overline{Y}\rangle$ for any $X, Y \in TM$. For the linearly extended Riemannian curvature tensor R over $\mathbb{C}$ to a quadrilinear map on $TM \bigoplus \overline{TM}$, the only non-trivial components of R are $R(X, \overline{Y}, Z, \overline{W})$ for X, Y, Z, W in

TM. By First Bianchi identity, we get

$$\begin{aligned} R(X,\overline{Y},Z,\overline{W}) &= R(Z,\overline{Y},X,\overline{W}) \\ &= R(X,\overline{W},Z,\overline{Y}) \end{aligned}$$

If $X = \frac{1}{\sqrt{2}}(u - iJu)$ and $X = \frac{1}{\sqrt{2}}(v - iJv)$, by Bianchi identity, we have

$$\begin{aligned} R(X,\overline{X},Y,\overline{Y}) &= -R(u,Ju,v,Jv) \\ &= R(v,u,Ju,Jv) + R(Ju,v,u,Jv) \\ &= R(v,u,u,v) + R(Ju,v,v,Ju) \end{aligned}$$

Therefore when X, Y are non-zero,

$$B(X,Y) = \frac{|Ju \wedge v|^2}{|u|^2|v|^2} K(Ju \wedge v) + \frac{|u \wedge v|^2}{|u|^2|v|^2} K(u \wedge v)$$

and $H(X) = B(X,X) = K(u \wedge Ju)$ where K is the sectional curvature:

$$K(Ju \wedge v) = \frac{R(Ju,v,v,Ju)}{|Ju \wedge v|^2}$$

and

$$K(u \wedge v) = \frac{R(u,v,v,u)}{|u \wedge v|^2}$$

From this we deduce that for a Kähler manifold, the bisectional curvature is "dominated" by the sectional curvature in the sense that B will be positive (negative,nonpositive, or nonnegative) if K is so. While the holomorphic sectional curvature just the sectional curvature in a 2-plane $\pi \in T_pM_{\mathbb{R}}$ such that $J\pi = \pi$ (i.e a holomorphic plane section).

The Ricci curvature tensor of Kähler manifold (M, h) is defined to be the trace of R: $r(X, \overline{Y}) = \sum_{i=1}^{n} R(X, \overline{Y}, e_i, e_i)$ for any unitary frame $\{e_i\}$. We use the same letter to denote value

$$r(X) = \frac{1}{|X|^2} r(X, \overline{X}) = \sum_{i=1}^{n} B(X, e_i)$$

called the Ricci curvature of h in the direction of $X \neq 0$. So, r is the average value of B which implies that the bisectional curvature dominates the Ricci curvature.

The scalar curvature s of a Kähler manifold is defined by

$$s = 2 \sum_{i=1}^{n} r(e_i)$$

Definition 5.3. *If f is of class C^2, then f is plurisubharmonic iff the hermitian matrix $L_f = (\alpha_{ij})$ called Levi matrix with entries $\alpha_{ij} = \frac{\partial^2 f}{\partial z_i \partial \overline{z_j}}$ is positive semidefinite.*
Equivalently, a C^2-function f is plurisubharmonic iff $\sqrt{-1}\partial\overline{\partial} f$ is a nonnegative (1,1) form.

Next, we exhibit a modified version of theorem 1.9. We simply adjust the proof presented by Chen and Zhu in [6].

Theorem 5.4. *Let M be a complex n-dimensional complete open Kähler manifold with nonnegative bisectional curvature. Suppose also its bisectional curvature is positive at a point $x_0 \in M$. Then the volume growth of M satisfies*

$$Vol(B(p, r)) \geq \alpha r^n$$

for all $p \in M$ and $1 \leq r < +\infty$, where α is a positive constant depending on x_0 and the dimension n.

Schoen and Yau constructed a cut-off function [14] as follows (see Lemma 2.1 in [6]).

Lemma 5.5. *Suppose M is an n-dimensional complete Riemannian with nonnegative Ricci curvature. Then there exists a constant $C(n) > 0$, depending only on the dimension n, such that for any $p \in M$ and any number $0 < r < +\infty$, there exists a smooth function $\varphi_r \in C^\infty(M)$ satisfying*

$$e^{-C(n)(1+\frac{d(x,p)}{r})} \leq \varphi_r(x) \leq e^{-(1+\frac{d(x,p)}{r})},$$

$$|\nabla \varphi_r(x)| \leq \frac{C(n)}{r}\varphi_r(x),$$

and

$$|\Delta \varphi_r(x)| \leq \frac{C(n)}{r^2}\varphi_r(x).$$

for $x \in M$ where $d(x,p)$ is the geodesic distance between x and p.

Proof of Theorem 5.4. The Busemann function b_p is strictly plurisubharmonic in a neighborhood of x_0. Note also that nonnegative bisectional curvature implies nonnegative Ricci curvature. So, by Lemma 5.5 above, for a fixed $p \in M$, we get the cut-off point function φ_r for any $r > 0$. The following construction is found in [6] (Proof of Theorem 1, page 73); fix a small positive number δ and a large positive number r. Let $\rho : \mathbb{R}^{2n} \to \mathbb{R}$ be a nonnegative smooth function supported in the unit ball centered at the origin of $\mathbb{R}^{2n}$, with

$$\int_{\mathbb{R}^{2n}} \rho(v)dv = 1$$

Set

$$\rho_\varepsilon(v) = \frac{1}{\varepsilon^{2n}}\rho(\frac{v}{\varepsilon}), \quad v \in \mathbb{R}^{2n},\ \varepsilon > 0,$$

and

$$(\eta * \rho_\varepsilon)(x) = \int_{\mathbb{R}^{2n}} \rho_\varepsilon(v)\eta(\exp_x(v))dv, \quad \text{for} \quad x \in M.$$

Clearly $\eta * \rho_\varepsilon$ is smooth and converges to η uniformly on compact sets as $\varepsilon \to 0$. It is also well-known that $\eta * \rho_\varepsilon$ is also plurisubharmonic on any compact subset as $\varepsilon > 0$ becomes small enough. Denote by ω the Kähler form of M. We compute

$$\begin{aligned}
&\int\limits_{\{\varphi_r>\delta\}} (\varphi_r - \delta)^n(\sqrt{-1})^n(\partial\overline{\partial}(\eta * \rho_\varepsilon)^n \\
&= - \int\limits_{\{\varphi_r>\delta\}} n(\varphi_r - \delta)^{n-1}(\sqrt{-1})^n\partial\varphi_r \wedge \overline{\partial}(\eta * \rho_\varepsilon) \wedge (\partial\overline{\partial}(\eta * \rho_\varepsilon))^{n-1} \\
&\leq \int\limits_{\{\varphi_r>\delta\}} \frac{2nC(2n)}{r}(\varphi_r - \delta)^{n-1}\varphi_r(\sqrt{-1})^{n-1}(\partial\overline{\partial}(\eta * \rho_\varepsilon))^{n-1} \wedge \omega \\
&\leq \int\limits_{\{\varphi_r>\delta\}} \frac{(2nC(2n))^2}{r^2}(\varphi_r - \delta)^{n-2}\varphi_r^2(\sqrt{-1})^{n-2}(\partial\overline{\partial}(\eta * \rho_\varepsilon))^{n-2} \wedge \omega^2 \\
&\cdots \\
&\leq \int\limits_{\{\varphi_r>\delta\}} \frac{(2nC(2n))^n}{r^n}\varphi_r^n\omega^n. \qquad (5.1)
\end{aligned}$$

Since η is strictly plurisubharmonic in a neighborhood of x_0, by letting $\varepsilon \to 0$ and the $\delta \to 0$, we know that there is a positive number c_0, depending on x_0 and the dimension n, such that for $r \geq 1$,

$$
\begin{aligned}
0 < c_0 &\leq \int_M \varphi_r^n (\sqrt{-1})^n (\partial\overline{\partial} b_p)^n \\
&\leq \frac{2nC(2n)}{r^n} \int_M \varphi_r^n \omega^n \\
&\leq \frac{2nC(2n)}{r^n} \int_M \varphi_r \omega^n \qquad (5.2)
\end{aligned}
$$

By Lemma 5.5, we have

$$
\begin{aligned}
\int_M \varphi_r \omega^n &\leq \int_{B(p,r)} e^{-(1+\frac{d(x,p)}{r})} \omega^n + \sum_{k=0}^{\infty} \int_{B(p,2^{k+1}r)\setminus B(p,2^k r)} e^{-(1+\frac{d(x,p)}{r})} \omega^n \\
&\leq Vol(B(p,r)) + \sum_{k=0}^{\infty} e^{-2^k} (2^{k+1})^{2n} Vol(B(p,r)) \\
&\leq CVol(B(p,r)) \qquad (5.3)
\end{aligned}
$$

where C is a positive constant depending only on the dimension n, $C = 1 + \sum_{k=0}^{\infty} e^{-2^k} (2^{k+1})^{2n}$. Since p is arbitrary, by combining equations (5.2) and (5.3), we get

$$
Vol(B(p,r) \geq \alpha r^n
$$

for all $p \in M$ and $1 \leq r < \infty$, where $\alpha = \frac{c_0}{2nC(2n)C}$ as desired. □

Definition 5.6. *A complex manifold M^n is* ***Stein*** *if and only if it can be embedded as a closed complex submanifold in $\mathbb{C}^N$, $n < N$.*

For any open neighborhood $V \subset \mathbb{C}^N$ (Figure 4.1), $V \cap M^n = \{f_1 = 0, ..., f_{N-n} = 0\}$, where $f_1, ..., f_{N-n}$ are holomorphic in V and $\left(\frac{\partial f_i}{\partial z_j}\right)_{\substack{1 \leq j \leq n \\ 1 \leq i \leq N-n}}$ has rank $N - n$.

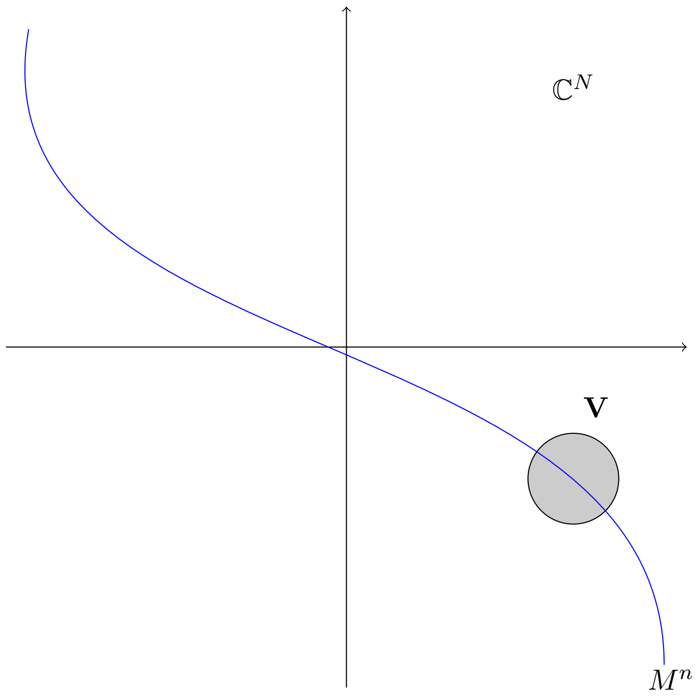

Figure 5.1: Stein Manifold

Stein manifolds can also be defined intrinsically. That is, let $M^n \subset \mathbb{C}^N$ be such a manifold, and denote by $\mathcal{O}(M)$ the ring of global holomorphic functions on M^n. Then M satisfies the following:

1. M is *holomorphically convex*, i.e for any compact $K \subset M$, the set $\widehat{K} := \{x \in M \mid |f(x)| \leq \sup_K |f|, \forall f \in \mathcal{O}(M)\}$ is also compact.

2. Given any two distinct points x and $y \in M$, there exists $f \in \mathcal{O}(M)$ such that $f(x) \neq f(y)$.

3. Given any $x \in M$, there exists $f_1, \cdots, f_n$ in $\mathcal{O}(M)$ such that $(f_1, \cdots, f_n)$ gives a holomorphic coordinate in a neighborhood of x.

Grauert established that if a complex manifold admits a smooth strictly plurisubharmonic exhaustion function then it is a Stein manifold.

Theorem 5.7 (H. Grauert)**.** *Let M be a complex manifold. Suppose there exists a C^∞ strictly plurisubharmonic exhaustion function $f : M \to \mathbb{R}$. Then M is a Stein manifold.*

The proof of Grauert is hard. However, it's converse is easy to prove.

Theorem 5.8. *Let $M^n \subset \mathbb{C}^N$ be a Stein manifold, then there exists a C^∞ strictly plurisubharmonic exhaustion function.*

Proof. Let $\rho : \mathbb{C}^n \to \mathbb{R}$ be a distance function i.e $\rho(z) = |z|^2 = |z_1|^2 + ... + |z_k|^2 : \mathbb{C}^n \to \mathbb{R}$. By direct computation, $\partial\overline{\partial}\rho$ is positive definite i.e ρ is strictly plurisubharmonic. The restriction of ρ on M, $\rho|_M : M \to \mathbb{R}$ is again strictly plurisubharmonic. It is clear that ρ is exhaustion. And then restriction, $\rho|_M$ is also an exhaustion.

□

Chapter 6

Complete Open Manifolds with nonnegative Ricci curvature

In this chapter we investigate structures of a complete open manifold with nonnegative Ricci curvature. It is natural to expect that the rich results in the case of sectional curvature are easily attainable in the case of Ricci curvature. However, in the latter case, there are relatively few structural results except in a lower dimensional case $n = 2$ where all notions of curvature coincide. The Busemann function is one of the most useful tool in studying the structure of positively curved complete open manifolds. In the former case ($sec \geq 0$), it is an easy consequence from Topogonov comparison theorem that b_p is convex. It then easily follows that the sublevel sets of b_p are convex and compact. This argument doesn't work for the latter case ($Ric \geq 0$) because it is only known that b_p is subharmonic. Since the volume expansion is dependent on the *Ricci* curvature tensor, the natural idea is to apply volume growth comparison theorem. As mentioned in chapter 1 (Theorem 1.5), Shen proved in [17] that if M is a complete open manifold with a nonnegative Ricci curvature and maximum volume growth, then for any point $p \in M$, b_p is exhaustion. By remark 4.1, we extend this theorem by replacing the maximum volume growth condition with a weaker condition in theorem 1.8.

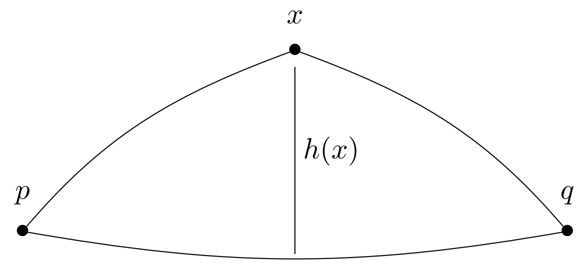

Figure 6.1: Excess function

Theorem 1.8 states: Let M be a complete open manifold with $\mathrm{Ric}_M \geq 0$. Let $\alpha(n) = \frac{9^n-1}{9^n}$ where $n = dim_{\mathbb{R}}M$. If

$$\alpha(n) < \lim_{r\to\infty} \inf \frac{Vol(B_\Sigma(p,r))}{Vol(B(p,r))},$$

then for any $a \in \mathbb{R}$, $b_p^{-1}(a)$ is compact .

Proof. (Proof of Theorem 1.8)

Proving by contradiction, we assume that $b^{-1}(a)$ is non-compact and then show that the assumed volume growth condition is not true.

We define the excess function for two points p, q as

$$e_{p,q} = d(p,x) + d(x,q) - d(p,q).$$

By the triangle inequality (Figure 6.1), we have that

$$e_{p,q}(x) \leq 2h(x) \tag{6.1}$$

Denote $r_p(x) = d(p,x)$. Assume that the minimizing geodesic between p and q is part of a ray emanating from p. Now, taking the limit of inequality (6.1) as q goes to infinity, we

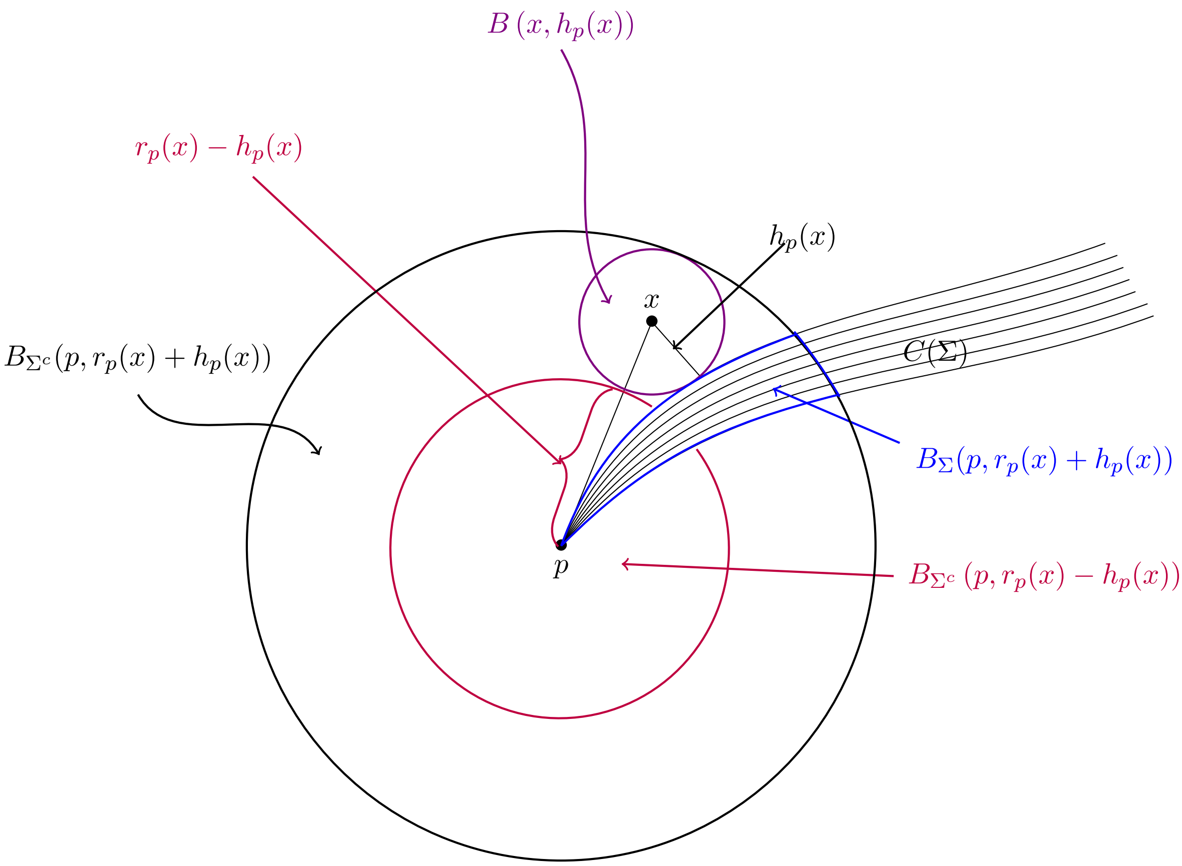

Figure 6.2: $B(x, h_p(x)) \subset B_{\Sigma^c}(p, r_p(x) + h_p(x)) \backslash B_{\Sigma^c}(p, r_p(x) - h_p(x))$

end up with the following inequality

$$r_p(x) - \lim_{t\to\infty}\{t - d(x, \gamma(t))\} \leq 2h_\gamma(x), \tag{6.2}$$

where $h_\gamma(x)$ is a distance from x to a ray γ emenating from p. Since

$$r_p(x) - b_p(x) \leq r_p(x) - \lim_{t\to\infty}\{t - d(x, \gamma(t))\}$$

for each ray γ emanating from p, inequality (6.2) implies that

$$r_p(x) - b_p(x) \leq 2h_\gamma(x) \tag{6.3}$$

Let $h_p(x) = d(x, Rp)$, where R_p is a union of rays emanating from p. Since inequality (6.3) holds for any ray γ, we have that

$$r_p(x) - b_p(x) \leq 2h_p(x) \tag{6.4}$$

Recall that

$$\Sigma = \{v \subset S_pM \mid \exp_p(rv) : [0, \infty) \to M \text{ is a ray}\}.$$

and

$$C(\Sigma) \cap C(\Sigma^c) = \emptyset.$$

For any $r > 0$ and $p \in M$ we have that

$$B_\Sigma(p, r) \cap B_{\Sigma^c}(p, r) = \emptyset$$

Observe that $B(x, h_p(x)) \subset C(\Sigma^c)$. It then follows that

$$B(x, h_p(x)) \subset B_{\Sigma^c}(p, r_p(x) + h_p(x))$$

See figure 6.2.

Since b_p is exhaustion whenever h_p is bounded, we assume that h_p is unbounded. Due to boundlessness of $b_p^{-1}(a)$, we can construct a diverging sequence $\{x_m\} \subset b_p^{-1}(a)$. Consequently, $\{h_p(x_m)\}$ is a divergence sequence.

For simplicity let us denote $h_m = h_p(x_m)$ and $r_m = r(x_m)$. By the generalized version of Bishop-Gromov volume comparison theorem,

$$\frac{Vol(B_{\Sigma^c}(p, r_m - h_m))}{Vol(B_{\Sigma^c}(p, r_m + h_m))} \geq \left[\frac{r_m - h_m}{r_m + h_m}\right]^n = \left[\frac{1 - \frac{h_m}{r_m}}{1 + \frac{h_m}{r_m}}\right]^n$$

From figure 6.2 above, we have

$$B(x_m, h_m) \subset B_{\Sigma^c}(p, r_m + h_m) \backslash B_{\Sigma^c}(p, r_m - h_m) \tag{6.5}$$

and

$$\begin{aligned} Vol(B(x_m, h_m) &\leq Vol(B_{\Sigma^c}(p, r_m + h_m)) - Vol(B_{\Sigma^c}(p, r_m - h_m)) \\ &\leq \left\{1 - \left[\frac{1 - \frac{h_m}{r_m}}{1 + \frac{h_m}{r_m}}\right]^n\right\} Vol(B_{\Sigma^c}(p, r_m + h_m)) \\ &\leq Vol(B_{\Sigma^c}(p, 3h_m + a)) \end{aligned} \tag{6.6}$$

The last inequality is due to the fact that $h \leq r$ and

$$r_p(x) - b_p(x) \leq 2h_p(x)$$

In particular

$$r_p(x) + h(x) \leq 3h(x) + a, \text{ when } x \in b_p^{-1}(a)$$

Now, denote $r_1(x) = d(x_1, x)$. By triangle inequality and equation (6.4),

$$\begin{aligned} \lim_{m\to\infty} \sup \frac{r_1(x_m)}{h_p(x_m)} &\leq \lim_{m\to\infty} \sup \frac{r_1(p)}{h_p(x_m)} + \lim_{l\to\infty} \sup \frac{r_p(x_m)}{h_p(x_m)} \\ &\leq 2 \end{aligned} \tag{6.7}$$

Also note that

$$B(x_1, h_m) \subset B(x_m, h_m + r_1(x_m)) \tag{6.8}$$

By volume comparison theorem we obtain

$$Vol(B(x_m, h_m)) \geq \left[\frac{h_m}{h_m + r_1(x_m)}\right]^n Vol(B(x_m, h_m + r_1(x_m)) \tag{6.9}$$

For simplicity we denote $f_p(r) = Vol(B(p, r))$ for a fixed $p \in M$. From inequality (6.7), (6.8), and (6.9), we have

$$
\begin{aligned}
&\lim_{m\to\infty} \quad \inf \frac{Vol(B(x_m,h_m))}{f_p(h_m)} \\
\geq & \quad \lim_{m\to\infty} \inf \left[\left(\frac{h_m}{(h_m + r_1(x_m))} \right)^n \frac{Vol(B(x_m, h_m + r_1(x_m)))}{f_p(h_m)} \right] \\
\geq & \quad \lim_{m\to\infty} \inf \left[\left(\frac{h_m}{(h_m + r_1(x_m))} \right)^n \frac{Vol(B(x_1, h_m))}{f_p(h_m)} \right] \\
\geq & \quad \lim_{m\to\infty} \inf \frac{1}{\left(1 + \frac{r_1(x_m)}{h_m}\right)^n} \lim_{m\to\infty} \inf \frac{Vol(B(x_1,h_m))}{f_p(h_m)} \\
\geq & \quad \frac{1}{3^n} \qquad (6.10)
\end{aligned}
$$

The last inequality is due to the fact that the volume growth

$$
\lim_{m\to\infty} \inf \frac{Vol(B(x_1,h_m))}{f_p(h_m)}
$$

is independent of the base point x_1.

From inequalities (6.6), (6.10), and the volume comparison theorem, we have

$$
\begin{aligned}
\frac{1}{3^n} \quad \leq & \quad \lim_{m\to\infty} \inf \frac{Vol(B(x_m,h_m))}{f_p(h_m)} \\
\leq & \quad \lim_{m\to\infty} \inf \frac{Vol(B_{\Sigma^c}(p, 3h_m + a))}{f_p(h_m)} \\
\leq & \quad 3^n \lim_{m\to\infty} \inf \frac{Vol(B_{\Sigma^c}(p, h_m))}{f_p(h_m)} \qquad (6.11)
\end{aligned}
$$

Which leads to the inequality

$$
\lim_{m\to\infty} \inf \frac{Vol(B_{\Sigma^c}(p,h_m))}{f_p(h_m)} \geq \frac{1}{9^n} \qquad (6.12)
$$

However, the equation

$$Vol(B(p,r) = Vol(B_{\Sigma}(p,r)) + Vol(B_{\Sigma^c}(p,r))$$

and the volume growth condition assumption imply that

$$\lim_{r\to\infty} \inf \frac{Vol(B_{\Sigma^c}(p,r))}{f_p(r)} < \frac{1}{9^n} \tag{6.13}$$

which contradicts inequality (6.12). Hence $b_p^{-1}(a)$ must be compact.

□

Example 6.0.1. *Consider* $\mathcal{M} = \{(x_1, x_2, \cdots, x_n, z) : z = x_1^2 + x_2^2 + \cdots + x_n^2\} \subset \mathbb{R}^{n+1}$*.* $(\mathcal{M}, ds^2_{\mathcal{M}})$ *is a complete open manifold with positive sectional curvature as shown in chapter 4. Here,* ds^2_M *is an induced Euclidean metric. For* $0 \neq q \in \mathcal{M}$*, let* $D_r(q)$ *be a geodesic ball of radius* r *around* q*. Consider a smooth function* $f : \overline{D_r(q)} \to \mathbb{R}$*. For a small neiborhood* U *of* $\overline{D_r(q)}$*, there exists a smooth function* $h : \mathcal{M} \to \mathbb{R}$ *such that* $h|_{\overline{D_r(q)}} = f$ *and supp* $h \subset U$*. For* $\varepsilon > 0$*, denote* $\mathcal{M}_\varepsilon = (\mathcal{M}, ds^2_{\mathcal{M}} + \varepsilon h ds^2_{\mathcal{M}})$*. We can choose* ε *small enough such that the curvature remains positive throughout* $\mathcal{M}$ *and an extension* $\gamma : [0, \infty) \to \mathcal{M}_\varepsilon$ *of a minimal geodesic from* $0 = p$ *to* q *leaves* $D_r(q)$ *forever and intersects one of the previously a ray at a point. It follows that the point* p *is no longer a pole. Since only rays intersecting and neighboring* $D_r(q)$ *are affected in a new manifold, for* $0 < \alpha < 1$*, we can vary* $r > 0$ *and* $\varepsilon > 0$ *such that*

$$\lim_{r\to\infty} inf \frac{Vol(B_{\Sigma}(p,r))}{Vol(B(p,r))} = \alpha$$

Neither any other point is a pole in this metric.

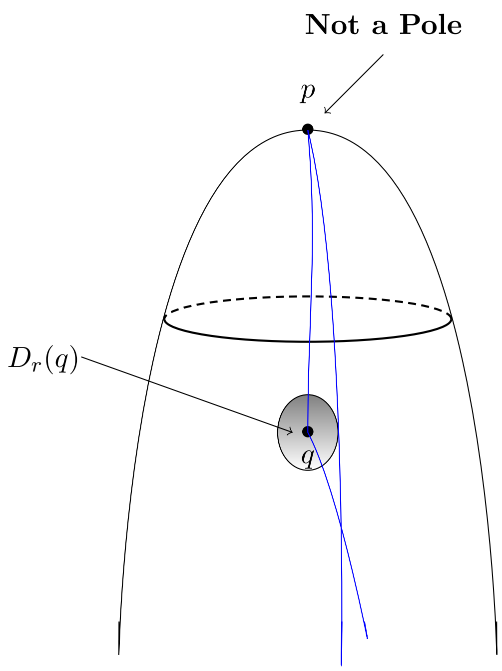

Figure 6.3: Metric pertubation of a paraboloid ($\mathcal{M}_\varepsilon = (\mathcal{M}, ds^2_{\mathcal{M}} + \varepsilon h ds^2_{\mathcal{M}})$)

Chapter 7

On the Structure of a Complete Open K$\ddot{a}$hler Manifolds

In this section, we present the proof of theorem 1.9.

7.0.1 Yau's conjecture

Yau's Uniformization conjecture: Let (M, ds^2) be a complete open Kähler manifold with positive bisectional curvature. Then M is biholomorphic to $\mathbb{C}^n$, $n = \dim_{\mathbb{C}} M$.

Weaker version (Greene-Wu, Siu, Yau): Let M be a complete open Kähler manifold with positive bisectional curvature. Then M is Stein.

The most recent partial result of Yau's uniformization conjecture is stated as follows.

Theorem 7.1. *(Gang Liu [12], Lee-Tam [11], 2017) Let M be a complete open Kähler manifold with nonnegative bisectional curvature and maximum volume growth. Then M is biholomorphic to $\mathbb{C}^n$*

In [21], Wong and Zhang partially proved a weaker version.

Theorem 7.2. *(Wong-Zhang) Let M be a complete open Kähler manifold with positive bisectional curvature. If M admits a maximum volume growth then M is Stein.*

Definition 7.3. *A real-valued function $g : M \to \mathbb{R}$ is said to support f at $p \in M$ if and only if g is continuous near p, $g \leq f$ and $g(p) = f(p)$.*

Definition 7.4. *Define*

$$Sf(p) = \lim_{r\to 0} \; inf \frac{2(dimM)}{r^2} \left\{ \int_{\partial B(p,r)} f \star d\sigma_p^r - f(p) \right\}$$

where $\star$ is the usual star operator in Hodge theory and σ_x^r is the Green's function of the ball $B(p,r)$. Recall that σ_x^r is a fundamental solution of the Laplace-Beltrami operator ∇ with singularity at x (i.e $\nabla \sigma_x^r = \delta_x$) which vanishes on $\partial B(p,r)$.

Let M be a Kähler manifold and $f|_L$ be the restriction of f to a 1-dimensional complex submanifold L through p. Then $S(f|_L)(p)$ is defined via induced Kähler metric on L, and by definition,

$$Pf(p) = \inf_L S(f|_L)(p)$$

where the infinum is taken over all 1-dimensional complex submanifold L of M through p.

Lemma 7.5. *[Lemma 3 [22]] f is strictly plurisubharmonic if for some positive function k on M, $Pf \geq k$.*

Lemma 7.6. *[Lemma 4 [22]] If g supports f at p, then $Pf(p) \geq Pg(p)$. If f is supported at every point of M by a strictly plurisubharmonic then f is strictly plurisubharmonic.*

Lemma 7.7. *[Lemma 1 [22]] If f is C^2 at p, then $Pf = 4\min_X \partial\overline{\partial} f(X, X)$ where X runs through all unit vectors tangent vectors of type $(1, 0)$ at p.*

Greene and Wu [22, Theorem A part (c)] proved the following result:

Theorem 7.8. *If M is a complete open Kähler manifold with positive bisectional curvature then b_p is strictly plurisubharmonic.*

Idea of the proof: It suffices to prove that $Pb_p(x) \geq 0$. Recall that $B_p^r(x) = r - d(x, \partial B(p, r))$ and that $B_p^r(x) \to b_p(x)$ uniformly. Consider a ball B of radius s containing x. Let $y \in \partial B(p, r)$ a point satisfying $d(x, y) = d(x, \partial B(p, r)) = r$. Define $f : B \to \mathbb{R}$ by $f = r - d(\cdot, y)$. Since $f(x) = B_p^r(x)$, f supports B_p^r at x.

The next move is to find a C^∞ function $g : B \to \mathbb{R}$ that supports f at x.

Let $\gamma : [0, r] \to M$ be a minimizing geodesic from x to a point in $\partial B(p, r)$ such that $|\dot{\gamma}| = 1$. Let A be a ball of radius s in T_pM. Define the following ($\dim_{\mathbb{R}}$)-parameter variation of γ, namely $k : [0, r] \times A \to M$ such that if $X \in A$ and $X(t)$ is the parallel translate of X a long γ to $\gamma(t)$, $k(t, X) = exp_{\gamma(t)}[(1 - \frac{t}{r})X(t)]$.

Note that since parallel translation preserves the complex structure tensor J, $J[X(t)] = (JX)(t)$ for every $X \in A$. This plays a role in the following summary of the properties of k

i) k is C^∞

ii) $k(t, 0) = \gamma(t)$ for all $t \in [0, r]$, where 0 is the point of origin in T_pM.

iii) $k(0, X) = exp_pX$ for every $X \in A$.

iv) $k(r, X) = y$ for evry $x \in A$.

v) if $X \in T_pM$ and $sX \in A$ for every $s \in (-a, a)$. Then the variation of γ given by $[0, r] \times (-a, a) \to M$ such that $(t, s) \mapsto k(t, sX)$ is a variation that induces the vector field a long $(1 - \frac{t}{r})X(t)$ along γ.

vi) The variation of γ given by $(t, s) \mapsto k(t, sX)$ and $(t, s) \mapsto k(t, sJX)$ induces a vector field along γ such that J applied to the former yields the latter.

Now define $g : B \cap exp_x A \to \mathbb{R}$ by $g(exp_x A) = r -$ [length of the curve $t \mapsto k(t, X)$]. g is C^∞ and supports b_p at $x(= exp_x 0)$. It is enough to prove that there exists a sequence ε_r, ε_r depends only on B and $\varepsilon_r \to 0$ as $r \to \infty$ such that the minimum eigenvalue of $\partial\overline{\partial}g$ at x exceed ε_r. Since M is Kähler and g is C^∞, the eigenvalues of $\partial\overline{\partial}g$ can be calculated from those of the Hessian $D^2 g$ in the following way:

$$\partial\overline{\partial}g(X_0, \overline{X_0}) = D^2 g(X, X) + D^2 g(JX, JX),$$

where $X_0 = X + \sqrt{-1}JX$ and D^2 is the covariant differential operator.

Let X be any unit vector in T_pM. It follows that JX is also a unit vector in T_pM. Let $\eta_1 : (-a, a) \to B$ and $\eta_2 : (-a, a) \to B$ be geodesics such that $\eta_1(s) = exp_x(sX)$ and $\eta_2(s) = exp_x(sJX)$. In particular, $\eta_1(0) = X$, $\eta_2(0) = JX$. Then $D^2 g(X, X) = (g \circ \eta_1)''(0)$ and $D^2 g(JX, JX) = (g \circ \eta_2)''(0)$. From the definition of g, $-(g \circ \eta_1)''(0)$ is nothing but the second variation of arclength of the family of curves $t \mapsto k(t, sX)$, $s \in (-a, a)$, similarly for $-(g \circ \eta_2)''(0)$.

Thus if $V(t)$ and $JV(t)$ are the vector fields a long γ given by $V(t) = \left(1 - \frac{t}{r}\right) X(t)$ and $JV(t) = \left(1 - \frac{t}{r}\right) JX(t)$, then properties (v) and (vi) of k and the second variation of arc-length formula give:

$$D^2g(X,X) = \int_0^r [R(\dot\gamma, V, \dot\gamma, V) - \langle \dot V, \dot V\rangle + \{\langle V, \dot\gamma\rangle'\}dt$$

$$D^2g(JX,JX) = \int_0^r [R(\dot\gamma, JV, \dot\gamma, JV) - \langle J\dot V, J\dot V\rangle + \{J\langle V, \dot\gamma\rangle'\}dt$$

where the prime denotes $\frac{d}{dt}$. Now, let $P_1 = \text{span}\{\dot\gamma, V\}$ and $P_2 = \text{span}\{\dot\gamma, JV\}$ at each $\gamma(t)$. Since $|V| = 1 - \frac{t}{r}$, $R(\dot\gamma, V, \dot\gamma, V) + R(\dot\gamma, JV, \dot\gamma, JV) = (1 - \frac{t}{r})B(P_1, P_2)$, where B is the bisectional cuvarture. Moreover, $\langle \dot V, \dot v\rangle = \langle J\dot V, J\dot v\rangle = \frac{1}{r^2}$. Hence

$$\partial\overline{\partial}g(X_0, \overline{X_0}) \geq \frac{-2}{r} + \int_0^l \left(1 - \frac{t}{r}\right) Bdt$$

For r large enough, $\partial\overline{\partial}g(X_0, \overline{X_0}) > 0$.

By Lemmas 7.5, 7.6, and 7.7, b_p is strictly plurisubharmonic.

□

From a similar argument, we also have that for a fixed ray γ, f_γ is also strictly plurisubharmonic given the same conditions.

Since in Kähler manifold positive bisectional curvature implies positive Ricci curvature, we can deduce theorem 1.9 from theorem 1.8. Here we list some results that leads to the proof of theorem 1.9.

Let the sheaf of germs of continuous real-valued functions on a C^∞ Riemannian manifold M be denoted by ζ.

Definition 7.9. *a C^0 fine topology on the set $\Gamma(\zeta, M)$ of continuous functions on M is the topology generated by sets*

$$\{F \in \Gamma(\zeta, M) || f(p) - F(p)| < g(p), p \in M\},$$

where $f \in \Gamma(\zeta, M)$, $g \in \Gamma(\zeta, M)$ and g is positive everywhere on M.

Definition 7.10. *A C^∞ function f on a Riemannian manifold M is subharmonic if Δf is nonnegative everywhere on M. If $X_1, ..., X_n$ is an orthonormal frame in the tangent space T_pM of M at a point p, then*

$$\Delta f|_p = \sum_{i=1}^{n} D^2 f(X_i, X_i)$$

where $D^2 f(X_i, X_i)$ = second derivative of f at p along the geodesic through p having tangent vector X_i

Remark 7.1. From definition above, it follows that a plurisubharmonic function is necessarily a subharmonic function.

Definition 7.11. *A compact-open topology on $\Gamma(\zeta, M)$ is the topology generated by the sets*

$$\{F \in \Gamma(\zeta, M) || f(p) - F(p)| < \varepsilon, p \in K\}$$

where $f \in \Gamma(\zeta, M)$, ε is a positive real number, and K is a compact subset of M.

The following two results are due to Greene and Wu in [8].

Theorem 7.12. *(Pg 67 in [8])*

A C^∞ subharmonic exhaustion functions are dense in the compact-open topology in the continuous subharmonic exhaustion functions.

Theorem 7.13. *(Pg 80 in [8])*

If M is a complex manifold then the C^∞ strictly plurisubharmonic functions are dense in the continuous strictly plurisubharmonic functions in the C^0 fine topology.

For convenience, we restate theorem 1.9: Let M be a complete open Kähler manifold with positive bisectional curvature. If

$$\alpha(n) < \lim_{r\to\infty} \inf \frac{Vol(B_\Sigma(p,r))}{Vol(B(p,r))},$$

where $n = \dim_{\mathbb{C}} M$ and $\alpha(n) = \frac{9^{2n}-1}{9^{2n}}$, then M is a Stein manifold.

Proof. (Proof of theorem 1.9) As stated in page 80, in Kähler manifold, positive bisectional curvature implies that the Ricci curvature is positive as well. Let $p \in M$. Then by theorem 1.8 and theorem 7.8, b_p is an exhaustion function and a continuous strictly plurisubharmonic function. By theorem 7.13 above, it follows that there exists a C^∞ strictly plurisubharmonic function $g : M \to \mathbb{R}$ such that $b_p(x) < g(x)$. Since g dominantes an exhaustion function b_p at every point, g itself is exhaustion. So, $g(x)$ is smooth, exhaustion, and strictly plurisubharmonic. Hence M is Stein.

□

Chapter 8

Application to Topology

Let $H_k(M,\mathbb{Z})$ denote the k-th singular homology group of M with integer coefficients. It is well know that if M is a complete proper Riemannian n-dimensional manifold with $Ric_M \geq 0$, then using Morse theorem, M has the homotopy type of a CW complex with cells each of dimension $\leq n-2$ and $H_i(M,\mathbb{Z}) = 0$, $i \geq n-1$ [16], [13].

As an application of theorem 1.8, we have the following result.

Theorem 8.1. *Let (M,g) be a complete manifold with $Ric_M \geq 0$. If*

$$\alpha(n) < \lim_{r\to\infty} inf \frac{Vol(B_\Sigma(p,r))}{Vol(B(p,r))}$$

where $n = dim_{\mathbb{R}} M$ and $\alpha(n) = \frac{9^n-1}{9^n}$, then M has the homotopy type of a CW complex with cells each of dimension $\leq n-2$. In particular, $H_i(M,\mathbb{Z}) = 0$, $i \geq n-1$

It is well established that if M is a Stein manifold of n-dimension, then the homology groups $H_k(M,\mathbb{Z})$ are zero if $k > n$ and $H_n(M,\mathbb{Z})$ is torsion free [1, theorem 1], [2]. As an application of theorem 1.9, we have the following result.

Theorem 8.2. *Let M be a complete open Kähler manifold with positive bisectional curvature. If*

$$\alpha(n) < \lim_{r\to\infty} inf \frac{Vol(B_\Sigma(p,r))}{Vol(B(p,r))},$$

where $n = dim_{\mathbb{C}} M$ and $\alpha(n) = \frac{9^{2n}-1}{9^{2n}}$. Then

$$H_k(M,\mathbb{Z}) = 0, \;\; for \; k > n$$

and $H_n(M,\mathbb{Z})$ is torsion free

Bibliography

[1] Andreoti A. and T. Frankel, The Lefchetz theorem on hyperplane sections. *Annals of Math.*69 (1959) , 713 − 717

[2] Andreoti A. and R. Narasimhan, A topological property of Runge pairs. *Annals of Math.*76 (1962) , 499 − 509

[3] Manfredo Perdigão do Carmo, Riemannian Geometry, *Birkhäuser Boston*, 1992

[4] J.Cheeger and D. Gromoll, On the structure of complete manifolds of nonnegative curvature, *Ann. of Math.* (2) 96 (1972), 413 − 443

[5] T. Chen, H. Yang, S. Yang, Beltrami system and 1-Quasiconformal embeddings in higher dimensions. *arXiv* : 1709.07324*v*1

[6] Bing-Long Cheng and Xi-Ping Zhu, Volume growth and curvature decay of positively curved, Kähler manifolds, *Pure and Applied Mathematics Quarterly Vol.*1 *No.*1 (2005), 68 − 108

[7] Greene R.E. and Wu H., On Kähler manifolds of positive bisectional curvature and a theorem of Hartogs. *Abh.Math. Sem. Univ. Hamburg* 47(1978), 171 − 185

[8] Greene R.E. and Wu H., C^{∞} approximation of convex, subharmonic, and plurisubharmonic functions, $Annales\ scientifiques\ de\ L'E.N.S.$ 4e $S\tilde{e}rie,\ tome$ 12, $no.$(1979), $47-84$

[9] D.Gromoll and W.T. Meyer, complete open manifolds of positive curvature, $Ann,\ of\ Math.$(2) 96 (1969), $75-90$.

[10] John M. Lee, Riemannian Manifolds: and introduction to curvature, $Springer-Verlag\ New\ York$, 1997

[11] M.-C. Lee, L.-F. Tam, On existence and curvature estimate of Ricci flow. $arXiv$: 1702.02667 (2017)

[12] Gang Liu, On Yau's uniformization conjecture. $arXiv$: 1606.08958 (2017)

[13] J. Milnor, Morse theory, $Notes\ by\ M.\ Spivak\ and\ R.\ Wells,\ Princeton\ University\ Press,\ Princeton\ NJ$ 1963.

[14] Schoen, R. and Yau, S. T., Lectures on differential geometry, $in\ Conference\ Proceedings\ and\ Lecture\ Notes\ in\ Geometry\ and\ Topology,\ Vol.$1, $International\ Press\ Publications$, (1994).

[15] D.Ordway, B. Stephens, and D. G. Yang, Large volume growth and finite topological type. $Proceedings\ of\ AMS\ Vol.$128, $No.$4 (1999), 1191-1196

[16] Zhongmin Shen and Guofang Wei, Volume growth and finite topology type, $Proceedings\ of\ Symposia\ in\ Pure\ Math.\ Vol.$54 (1993) ,$part$ 3

[17] Zhongmin Shen, Complete manifolds with nonnegative Ricci curvature and large volume growth, $Invent.\ math.$125, (1996) $393-404$

[18] Katsuhiro Shiohama, Busemann Functions and Total Curvature, *Inventiones math.* 53, (1979) 281 – 297

[19] C. Shormani, Busemann functions on manifolds with lower bounds on Ricci curvature and minimal volume growth, *J.Differential Geom.* 48 (1998) 557 – 585

[20] Yau, S.-T, A review of complex differential geometry, *Proc. Sympos. Pure Math.*, 52 *Part* 2 (1991) 619 – 625.

[21] Bun Wong and Qi S. Zhang, Refined gradient bounds, poisson equations and some applications to open Kähler manifolds. *Asian J. Math Vol.*7, *No.* 3 (2003) 337–364.

[22] H. Wu, An elementary method in the study of nonnegative curvature, *Acta. Math.* 142 (1979) 57–78, *MR* 0512212, *Zbl*0403.53022.

[23] Fangyang Zheng, Complex Differential Geometry, *AMS/IP studies in advanced mathematics*, *Volume* 18, 2000.

[24] Shunhui Zhu, The comparison geometry of Ricci curvature, *Comparison Geometry MSRI Publications Vol.*30 (1997) 221 – 262.