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FOUNDATIONS OF S-MATRIX THEORY III
THE NORMAL ANALYTIC STRUCTURE

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ABSTRACT

This is the third of a series of reports devoted to a systematic development of S-matrix theory. This report describes the normal analytic structure, which is the property that each scattering function is analytic at all physical points not lying on any positive-α Landau surface corresponding to a connected diagram, and that near most points lying on any such surface it is the limit of an analytic function, taken from directions lying in a specified cone.

A. ANALYTIC FUNCTIONS OF SEVERAL COMPLEX VARIABLES

Analytic functions of several complex variables play a central role in what follows. In this section some basic definitions and properties of these functions are briefly reviewed.

Suppose $z \equiv (z_1, \cdots, z_m)$ is a set of m complex variables. Then a function $f(z)$ that is defined in a full neighborhood of a point $\bar{z}$ is said to be analytic at $\bar{z}$ if and only if there is a power series expansion about $\bar{z}$ that converges to $f(z)$ in some neighborhood of $\bar{z}$. In other words, $f(z)$ is analytic at $\bar{z}$ if and only if for each set $(n_1, \cdots, n_m)$ of m nonnegative integers there is a complex number $a(n_1, \cdots, n_m)$ such that the sum

$$\sum_{n_i=0}^{\infty} a(n_1, \cdots, n_m)(z_1 - \bar{z}_1)^{n_1} \cdots (z_m - \bar{z}_m)^{n_m} , \qquad (A.1)$$

taken in some order, converges to $f(z)$ for all z in some neighborhood of $\bar{z}$.

A basic property of power series expansions is this: If the sum (A.1) taken in some order converges at a point z', then this sum taken in any order converges <u>absolutely</u> on the set

$$\{z: |z_j - \bar{z}_j| < |z'_j - \bar{z}_j|, \quad \text{all } j\} . \qquad (A.2)$$

Moreover, for any set of positive numbers $\epsilon_j > 0$. The sum converges uniformly and absolutely on the set

$$\{z: |z_j - \bar{z}_j| < |z'_j - \bar{z}_j| - \epsilon_j, \quad \text{all } j\} . \qquad (A.3)$$

Furthermore, the function defined by this uniformly and absolutely convergent power series is analytic at all points in (A.3).

Proofs of these basic properties of power series expansions can be found, for example, in Ref. 1, which is an excellent introduction to the theory of functions of several complex variables.

By using the properties just described one can, by the process of analytic continuation,[1] often enlarge the domain on which a function is both defined and analytic. A domain that cannot be so enlarged is called a domain of holomorphy. It is often multisheeted.[1]

A slight generalization of an analytic function is an analytic mapping. Suppose $z \equiv (z_1, \cdots, z_m)$ is a set of m complex variables. And suppose $Z(z) = [Z_1(z), \cdots, Z_M(z)]$ is a set of M complex-valued functions of z. Suppose all of these functions are analytic at all points z in some connected set $\mathcal{U}$. Define $Z(\mathcal{U})$ to be the image of $\mathcal{U}$ under the mapping $Z(z)$:

$$Z(\mathcal{U}) \equiv \{Z: Z = Z(z) \quad \text{for some } z \text{ in } \mathcal{U}\}. \qquad (A.4)$$

Then the mapping from $\mathcal{U}$ to $Z(\mathcal{U})$ defined by $Z(z)$ is called an <u>analytic mapping</u> $Z(z): \mathcal{U} \to Z(\mathcal{U})$.

If $\mathcal{U}$ contains real neighborhoods of real points, and if all real points of $\mathcal{U}$ are mapped into real points, then the analytic mapping $Z(z): \mathcal{U} \to Z(\mathcal{U})$ is called a <u>real analytic mapping</u>.

For analytic functions of a single complex variable the following result is well known: An analytic function of an analytic function is analytic. The generalization of this result to functions of several complex variables is given by the Composition Theorem:[2]

<u>Theorem IIIA.1</u>. Consider an analytic mapping $Z(z): \mathcal{U} \to Z(\mathcal{U})$. Suppose the function $F(Z)$ is analytic at all $Z \in Z(\mathcal{U})$. Then $F \circ Z(z) \equiv F\big(Z(z)\big) = f(z)$ is analytic at all $z \in \mathcal{U}$.

Nonsingular analytic mappings play an important role in what follows. An analytic mapping $Z(z): \mathcal{U} \to Z(\mathcal{U})$ is said to be <u>nonsingular</u> at $\bar{z} \in \mathcal{U}$ if and only if the matrix

$$\partial Z_i(\bar{z})/\partial z_j \equiv \left. \frac{\partial Z_i(z)}{\partial z_j} \right|_{z=\bar{z}} \qquad (A.5)$$

is nonsingular:

$$\mathrm{Det}(\partial Z_i(\bar{z})/\partial z_j) \neq 0 . \qquad (A.6a)$$

Condition (A.6a) is equivalent to the condition that

$$\mathrm{Rank}(\partial Z_i(\bar{z})/\partial z_j) = m = M . \qquad (A.6b)$$

It is also equivalent to the condition that the gradient vectors $\nabla Z_i(\bar{z})$ be linearly independent:

$$\left[\sum_{i=1}^{M} \alpha_i \nabla Z_i(\bar{z}) = 0\right] \Longrightarrow [\text{all } \alpha_i = 0] . \qquad (A.6c)$$

Some texts define a nonsingular analytic mapping by requiring only that $\mathrm{Rank}(\partial Z_i/\partial z_j) = \min(M,m)$. Here, however, condition $M = m$ is also implied.

A basic property of nonsingular analytic mappings is described by the Inverse Mapping Theorem:[3]

<u>Theorem IIIA.2</u>. An analytic mapping $Z(z): \mathcal{U} \to Z(\mathcal{U})$ that is nonsingular at $\bar{z} \in \mathcal{U}$ has a local analytic inverse. That is, there exists an analytic mapping $z(Z): \mathcal{N} \to z(\mathcal{N})$ that satisfies the following conditions: (1) $\bar{z} \in z(\mathcal{N})$; (2) $z(\mathcal{N}) \subset \mathcal{U}$;

(3) $Z \circ z(Z) = Z$ for all Z in $\mathcal{N}$; (4) $z \circ Z(z) = z$ for all z in $z(\mathcal{N})$.

Theorems IIIA.1 and IIIA.2 have the following immediate consequence: Suppose the analytic mapping $Z(z): \mathcal{U} \to Z(\mathcal{U})$ is nonsingular at $\bar{z} \in \mathcal{U}$. Then a function $F(Z)$ is analytic at $\bar{Z} \equiv Z(\bar{z})$ if and only if $f(z) \equiv F \circ Z(z)$ is analytic at $\bar{z}$. That is, the property of a function to be analytic or not analytic is preserved by a nonsingular analytic mapping.

B. ANALYTICITY ON THE MASS SHELL

The normal analytic structure asserts that scattering functions are analytic in certain regions. However, the definition of analyticity given above does not apply directly to scattering functions, because these functions are defined only at points p satisfying the mass-shell and conservation-law constraints. Thus it is necessary to define what it means for a scattering function to be analytic.

Consider a scattering process with a total of n initial and final particles. Let p represent the corresponding set of n four-vectors:

$$p \equiv (p_1^{\mu}, p_2^{\mu}, \cdots, p_n^{\mu}) \qquad \mu = 0,1,2,3 \quad . \tag{B.1}$$

Let the $n + 4$ functions $f_i(p)$ be defined as follows:

$$f_i(p) = p_i^2 - m_i^2 \qquad i = 1,\cdots,n \ , \tag{B.2a}$$

$$f_{n+1+\mu}(p) = \sum_{j=1}^{n} \pm \ p_j^{\mu} \qquad \mu = 0,\cdots,3 \ . \tag{B.2b}$$

Here m_i is the mass of particle i, and

$$\pm \equiv \epsilon_j = \begin{cases} + & \text{if } j \in I \\ - & \text{if } j \in F \end{cases} \ . \tag{B.2c}$$

The I and F are the index sets corresponding to the initial and final particles respectively. Then the <u>mass shell</u> $\mathcal{M}$ is defined by

$$\tilde{\mathcal{M}} = \{p \in C^{4n} : f_i(p) = 0, \quad i = 1,\cdots,n+4\} . \quad (B.3)$$

The space C^{4n} is the space of $4n$ complex variables.

The <u>real mass-shell</u> $\mathcal{M}$ is the subset of $\tilde{\mathcal{M}}$ consisting of its real, positive-energy points:

$$\mathcal{M} = \{p \in R^{4n} : p \in \tilde{\mathcal{M}}, p_j^0 > 0 \quad \text{all } j\} . \quad (B.4)$$

The term "mass-shell" will always mean the (complex) mass-shell $\tilde{\mathcal{M}}$, unless otherwise stated.

A set of points that coincides with the simultaneous zeros of a set of algebraic functions (i.e., polynomials) is called an algebraic variety. Equation (B.3) shows that the mass-shell is an algebraic variety imbedded in C^{4n}.

Suppose $\bar{p}$ is a point on the mass-shell. Suppose $F(p)$ is a function that is defined in some mass-shell neighborhood of $\bar{p}$, but is not defined off the mass-shell. Then $F(p)$ is said to be <u>analytic in the strong sense</u> at $\bar{p}$ if and only if there is a function $G_{\bar{p}}(p)$ that is defined in a full neighborhood of $\bar{p}$, is analytic at $\bar{p}$, and coincides with $F(p)$ in some mass-shell neighborhood of $\bar{p}$. That is, for some neighborhood $\mathcal{N}(\bar{p})$ of $\bar{p}$,

$$F(p) - G_{\bar{p}}(p) = 0 \quad (B.5)$$

for all p in $\mathcal{N}(\bar{p}) \cap \tilde{\mathcal{M}}$.

This definition of analyticity involves an extension of $F(p)$ off the mass shell. An alternative definition that does not involve any extension of $F(p)$ off the mass shell is based on the concept of local analytic coordinates.

Local analytic coordinates are defined as follows: Let $\bar{p}$ be a fixed point of $\tilde{\mathcal{M}}$. Let $z = (z_1,\cdots,z_{4n})$ be a set of $4n$ complex variables, and let $\mathcal{S}$ be the $3n-4$ dimensional subspace $\mathcal{S} \equiv \{z: z_{3n-4+i} = 0, \quad i = 1,\cdots,n+4\}$. Suppose there is an analytic mapping $p(z): \mathcal{U} \to p(\mathcal{U})$ that: (1) maps $(z = 0) \in \mathcal{U}$ onto $\bar{p}$; (2) maps $\mathcal{U} \cap \mathcal{S}$ onto $p(\mathcal{U}) \cap \tilde{\mathcal{M}}$; and (3) has an analytic inverse $z(p): p(\mathcal{U}) \to \mathcal{U}$. Then the set of variables $z_1,\cdots,z_{3n-4}$ is said to be a set of <u>local analytic coordinates</u> of the mass-shell at $\bar{p}$. In short, local analytic coordinates of the mass-shell are the coordinates of a flat subspace $\mathcal{S}$ that is mapped locally onto the mass shell by an analytic mapping that has an analytic inverse.

Let $\bar{p}$ be a fixed point on the mass shell. Let $F(p)$ be a function that is defined in a mass-shell neighborhood of $\bar{p}$, but is not defined off the mass shell. Suppose there is an analytic mapping $p(z): \mathcal{U} \to p(\mathcal{U})$ that defines a set of local analytic coordinates of the mass shell at $\bar{p}$. Then the composition $F \circ p(z) \equiv F(p(z)) \equiv f(z)$ defines a function on a full neighborhood of the origin in the space of the local analytic coordinates $(z_1,\cdots,z_{3n-4})$. The function $F(p)$ is said to be <u>analytic in the weak sense</u> at $\bar{p}$ if and only if $f(z)$, considered as a function of the local analytic coordinates, is analytic at $z = 0$. The existence of the analytic inverse, together with Theorem IIIA.1, ensures that this definition is independent of the particular $p(z)$ that defines the local analytic coordinates.

This definition of analyticity applies only to points $\bar{p} \in \tilde{\mathcal{M}}$ such that there is a set of local analytic coordinates of the mass shell at $\bar{p}$. For such points the function $F(p)$ is analytic in the weak sense if and only if it is analytic in the strong sense.

For if $F(p)$ is analytic in the strong sense at $\bar{p}$ then $G_{\bar{p}} \circ p(z) = f(z)$ is analytic at $z = 0$ in the full $4n$ dimensional space, and hence also in the subspace $\mathcal{S}$. Conversely, if $F(p)$ is analytic in the weak sense at $\bar{p}$ then $G_{\bar{p}}(p) \equiv f[z_1(p),\cdots,z_{3n-4}(p),0,\cdots,0]$ is analytic at $\bar{p}$ and coincides with $F(p)$ for p in some mass-shell neighborhood of $\bar{p}$.

For almost every point $\bar{p} \in \widetilde{\mathcal{M}}$ there do exist local analytic coordinates of the mass shell at $\bar{p}$. A sufficient condition is that the $n + 4$ gradient vectors $\nabla f_i(\bar{p}) \equiv [\partial f_i(\bar{p})/\partial p_1^{\,0},\cdots,\partial f_i(\bar{p})/\partial p_n^{\,3}]$ be linearly independent. For in this case one can find $3n - 4$ vectors e_j that combined with the $n + 4$ vectors $\nabla f_i(\bar{p})$ give $4n$ linearly independent vectors. One can then define the $z_i(p)$, $i = 1,\cdots,3n - 4$, to be the linear functions of p that vanish at $\bar{p}$ and satisfy $\nabla z_i(p) = e_i$. And one can define $z_{3n-4+i}(p) = f_i(p)$ for $i = 1,\cdots,n + 4$. Then $z(p)$ is an analytic mapping that is nonsingular at $\bar{p}$, and the inverse mapping Theorem IIIA.2 ensures the existence of an analytic mapping $p(z): \mathcal{U} \to p(\mathcal{U})$ that is the inverse of $z(p): p(\mathcal{U}) \to \mathcal{U}$, that maps $(z = 0) \in \mathcal{U}$ onto $\bar{p}$, and that maps $\mathcal{U} \cap \mathcal{S}$ onto $p(\mathcal{U}) \cap \widetilde{\mathcal{M}}$.

Direct calculation shows that the vectors $\nabla f_i(p)$ are linearly independent at $p \in \widetilde{\mathcal{M}}$ if and only if the n 4-vectors p_i are not all parallel. Accordingly, the <u>restricted mass shell</u> $\widetilde{\mathcal{W}}$ is defined to be the subset of $\widetilde{\mathcal{M}}$ such that the p_j are not all parallel:

$$\widetilde{\mathcal{W}} \equiv \widetilde{\mathcal{M}} - \{p: p_i || p_j \quad \text{for all } (i,j)\} . \tag{B.6}$$

The above definition of analyticity in the weak sense refers only to points $\bar{p} \in \widetilde{\mathcal{W}}$. For a point $\bar{p}$ of $\widetilde{\mathcal{M}} - \widetilde{\mathcal{W}}$ one can introduce the following definition:[5] A function $F(p)$ defined in a mass-shell neighborhood of a point $\bar{p} \in \widetilde{\mathcal{M}} - \widetilde{\mathcal{W}}$, but not defined off $\widetilde{\mathcal{M}}$, is said to be analytic in the weak sense at $\bar{p}$ if and only if for some neighborhood $\mathcal{N}(\bar{p})$ of $\bar{p}$ the function $F(p)$ is continuous in $\mathcal{N}(\bar{p}) \cap \widetilde{\mathcal{M}}$ and analytic in the weak sense at all $p \in \mathcal{N}(\bar{p}) \cap \widetilde{\mathcal{W}}$.

Hepp[6] has remarked that analyticity in the weak sense on the mass shell, defined in this way, is equivalent to analyticity in the strong sense, due to a theorem of Oka.[7] However, to avoid the arbitrariness associated with extensions off the mass shell we shall always use the weak definition.

C. THE CLUSTER DECOMPOSITION

The cluster decomposition of the transition function $S(p)$ is defined by

$$S(p) = \sum_{\kappa} \alpha_\kappa \prod_{s=1}^{N_\kappa} S_1(p_{\kappa s}) . \qquad (C.1)$$

The sum runs over the different partitions κ of the set of variables p into disjoint subsets $p_{\kappa s}$. The term corresponding to the partition κ is the product of a phase factor α_κ--which will be defined presently--with a product of factors $S_1(p_{\kappa s})$, one for each of the subsets $p_{\kappa s}$ into which p is separated by the partition κ.

The functions $S_1(p_{\kappa s})$ occurring on the right-hand side of (C.1) are defined inductively by (C.1) itself. Thus if $\kappa = 1$ labels the trivial partition into one set $(p_{11} = p)$, then $S_1(p_{11}) = S_1(p)$ is defined by

$$\alpha_1 S_1(p) = S(p) - \sum_{\kappa=2}^{N_\kappa} \alpha_\kappa \prod_{s=1}^{N_\kappa} S_1(p_{\kappa s}) . \qquad (C.2)$$

This equation defines $S_1(p)$ in terms of $S(p)$ and the $S_1(p_{\kappa s})$ for the proper subsets $p_{\kappa s}$ of p.

The function $S_1(p)$ defined by (C.2) is called the <u>connected part</u> of $S(p)$. The function $S_c(p)$ defined by

$$S_1(p) = (2\pi)^4 \delta(\Sigma \pm p_j) S_c(p) , \qquad (C.3)$$

is called a <u>scattering function</u>. The $\pm$ in (C.3) is defined in (B.2).

The cluster decomposition is essentially a definition of the scattering functions. Its importance resides in the special properties enjoyed by the scattering functions defined in this way. These properties are described in subsequent sections.

It was noted in Chapter I that each particle of a scattering process is associated with a triad (p_j, μ_j, t_j). The variable t_j is the type index; it specifies whether particle j is a proton or electron, or positron, etc. The variable μ_j is the spin index; it specifies the component of the spin of particle j along a specified axis. The variable p_j is a positive-energy mass-shell four-vector associated with particle j.

The argument p in $S(p)$ is an abbreviation for an ordered set of triads:

$$S(p) = S(p'; p'') \qquad (C.4a)$$

where

$$p' = (p_{i_1}, \mu_{i_1}, t_{i_1}; p_{i_2}, \mu_{i_2}, t_{i_2}; \cdots) \qquad (C.4b)$$

and

$$p'' = (p_{f_1}, \mu_{f_1}, t_{f_1}; p_{f_2}, \mu_{f_2}, t_{f_2}; \cdots) . \qquad (C.4c)$$

Here the indices i_j and f_j label initial and final particles respectively. The argument $p_{\kappa s}$ in $S_1(p_{\kappa s})$ is, similarly, an ordered set of triads.

The partition κ in (C.1) is actually a partition of the ordered set of triads that make up p into the ordered subsets of triads $p_{\kappa s}$. The triads in the various subsets $p_{\kappa s}$ can, for definiteness, be ordered the same way as they are ordered in p. And the subsets $p_{\kappa s}$ themselves can, for definiteness, be ordered so that the first triads in the various subsets have the same order as they do in p. These stipulations simply ensure that there is no double counting: terms that differ only by the order of the triads within one or more of the subsets $p_{\kappa s}$, and/or by the ordering of the subsets $p_{\kappa s}$ themselves are not included as separate contributions to (C.1).

The phase factor α_κ is determined by the order of the triads in p with respect to the order of the triads in the sets $p_{\kappa s}$. The factor α_κ is unity if

$$p' = (p'_{\kappa_1}, p'_{\kappa_2}, \cdots, p'_{\kappa N_\kappa}) \qquad (C.5a)$$

and

$$p'' = (p''_{\kappa 1}, p''_{\kappa 2}, \cdots, p''_{\kappa N_\kappa}) \; . \qquad (C.5b)$$

In general,

$$\alpha_\kappa = (-1)^N , \qquad (C.5c)$$

where N is the number of reorderings of "fermion variables" needed to bring the variables of p into the order specified by the right-hand sides of (C.5a) and (C.5b). The concept of "fermion variables" introduced here is now discussed.

To make (C.5c) well defined each variable (p_j, μ_j, t_j) of p must be identified as either a fermion variable or not a fermion variable. The normal analytic structure entails that each triad can be so identified, and moreover that this identification depends only on the type index: all triads having a given type index are fermion variables, or none are. The normal analytic structure thus

entails that each type index be classified as Fermi or non-Fermi. Particles of the Fermi type are called fermions; particles of the non-Fermi type are called bosons.

The normal connection between spin and statistic identifies the fermions as the particles of half-odd-integral spin. This connection will be derived later from a combination of analyticity properties and Lorentz invariance requirements. However, it is not regarded as part of the normal analytic structure: the normal analytic structure allows half-odd-integral spin particles to be bosons, and integral-spin particles to be fermions.

The restriction to Fermi and Bose statistics implicit in the rules described above arises from some assumptions that go beyond macrocausality alone. This is discussed in Chapter IV.

In the remainder of this chapter the order of the variables will be fixed. And the variables μ_j and t_j will be fixed, though arbitrary. Thus to avoid needless notation these variables will be suppressed, and p will become simply the set of intial and final four-vector p_j. In general the spin and type variables are suppressed when they are not relevant to the matters being discussed.

D. THE POSITIVE-α RULE

The normal analytic structure is a set of analytic properties of scattering functions. These properties are specified by the positive-α rule, which is described in this section, and the plus $i\epsilon$ rule, which is described in the next section.

The positive-α rule is the assertion that scattering functions are analytic at all real mass-shell points not lying on any Landau surface $\mathcal{L}(D_c^+)$: i.e.,

$$S_c(p) \text{ is analytic at all } p \in \mathcal{M} - \mathcal{L}_c^+ , \tag{D.1}$$

where

$$\mathcal{L}_c^+ = \bigcup_{D_c^+} \mathcal{L}(D_c^+) . \tag{D.2}$$

Here D_c^+ is a connected positive-α Landau diagram, and $\mathcal{L}(D_c^+)$ is the Landau surface corresponding to D_c^+. Landau diagrams and the surfaces corresponding to them are described in the following paragraphs.

A Landau diagram D^σ is a collection of directed line segments L_j and point vertices V_r. Each line L_j is associated with a particle-type index t_j, and hence with a mass $m_j \equiv m(t_j)$. Each line L_j has either an initial point L_j^- in $\{V_r\}$, or a final point L_j^+ in $\{V_r\}$, or both. In this latter case L_j^+ and L_j^- lie at different points V_r. The topological structure of a Landau diagram D^σ is defined by the set of type indices t_j, and by the coefficients ϵ_{jr} specified by

$$\epsilon_{jr} = \begin{cases} +1 & \text{if } L_j^+ = V_r \\ -1 & \text{if } L_j^- = V_r \\ 0 & \text{otherwise .} \end{cases} \tag{D.3a}$$

The lines L_j are classified as initial lines $(j \in I)$, final lines $(j \in F)$, and internal lines $(j \in \text{Int})$ by means of the rule

$$\sum_r \epsilon_{jr} = \begin{cases} -1 & \text{for } j \in I \\ +1 & \text{for } j \in F \\ 0 & \text{for } j \in \text{Int} . \end{cases} \tag{D.3b}$$

The initial and final lines are called the external lines, and the union of I and F is the set $\text{Ext} \equiv I \cup F$.

Each internal line L_j of a Landau diagram D^σ has a "sign" $\sigma_j = \pm$. The superscript σ on D^σ specifies this set of signs. In particular, D^+ represents a Landau diagram having all $\sigma_j = +$, and D^- represents a Landau diagram having all $\sigma_j = -$. Any diagram D^+ or D^- is required to have at least one internal line; otherwise the sign on $D^\pm$ would have no meaning. The significance of these signs σ_j will be explained presently.

Landau diagrams are not the same as the causal diagrams described in Sec. C of Chapter II. A causal diagram D is a space-time structure whereas a Landau diagram D^σ is merely a topological structure.

A Landau diagram can, however, have the same topological structure as a causal diagram: a Landau diagram D^σ is said to be topologically equivalent to a causal diagram D if the indices t_j and coefficients ϵ_{jr} are the same for D^+ and D.

For each Landau diagram D^σ there is a corresponding set of Landau equations $\mathcal{E}(D^\sigma)$. To obtain these equations each line L_j of D^σ is associated with a four-vector p_j. And each internal line of D^σ is associated also with a complex number α_j. Then the Landau equations $\mathcal{E}(D^\sigma)$ consist of the mass-shell constraints

$$p_j^2 - m_j^2 = 0 \quad \text{all } j \in \text{Int} \cup \text{Ext} , \tag{D.4a}$$

the conservation-law constraints

$$\sum_{\text{all } j} p_j \epsilon_{jr} = 0 \quad \text{all } r \in \text{Ver} , \tag{D.4b}$$

the α-conditions

$$\sigma_j \alpha_j > 0 \quad \text{all } j \in \text{Int} , \tag{D.4c}$$

and the loop equations

$$\sum_{j \in \text{Int}} \alpha_j p_j n_{j\ell} = 0 \quad \text{all } \ell \in L . \tag{D.4d}$$

The indices $\ell \in L$ label the directed closed loops that can be formed on the internal lines of D^σ, and $n_{j\ell}$ is the number of times loop ℓ passes along line L_j in the plus direction (i.e., from L_j^- to L_j^+) minus the number of times loop ℓ passes along L_j in the minus direction (i.e., from L_j^+ to L_j^-).

The Landau surface corresponding to a Landau diagram D^σ is defined as follows: Let the set of vectors p_j associated with the initial and final lines of D^σ be denoted by p:

$$p \equiv (p'; p'') \tag{D.5a}$$

$$p' \equiv (p_{i_1}, p_{i_2}, \cdots) \quad i_j \in I \tag{D.5b}$$

$$p'' \equiv (p_{f_1}, p_{f_2}, \cdots) \quad f_j \in F . \tag{D.5c}$$

Then the real Landau surface $\mathcal{L}(D^\sigma)$ is the set of real positive-energy points p such that for some set of real positive-energy vectors p_i and real numbers α_i, for $i \in \text{Int}$, the Landau equations corresponding to D^σ are soluble:

$$\mathcal{L}(D^\sigma) \equiv \{p: \mathcal{E}(D^\sigma) \text{ with all } p_j \text{ real, all } \alpha_i \text{ real, and all } p_j^0 > 0\} . \tag{D.6}$$

The subscript c on D_c means that the diagram is connected: any two points on a diagram D_c can be joined by a continuous path that lies in the diagram. The plus sign on D_c^+ means that all the α_j are positive [See (D.4c)]. Thus D_c^+ is called a connected positive-α diagram.

Landau surfaces were originally introduced by Landau,[8] in his study of the analyticity properties of Feynman-diagram functions. Their importance in the present context arises from the Coleman-Norton theorem:[9]

Theorem III.D.1. A point p lies on $\mathcal{L}(D_c^+)$ if, and only if there is a connected causal diagram D_c that: (1) is topologically equivalent to D_c^+; and (2) has its external lines L_j directed along the vectors p_j defined by p: i.e., $L_j \subset \Gamma^{u_j}(p_j)$ for some u_j, for all $j \in$ Ext.

This theorem follows immediately from the identification of the vectors $\alpha_j p_j$ defined by the Landau equations with the vectors Δ_j of the topologically equivalent causal diagram. With this identification the loop equations become just the condition that the sum of the space-time displacements around any closed loop of the causal diagram is zero. Thus if a connected causal diagram D_c topologically equivalent to D_c^+ exists, then the loop equation can all be satisfied. Alternatively, if the loop equations can all be satisfied then the vectors $\Delta_j \equiv \alpha_j p_j$ will fit together to form a connected space-time diagram D_c that is topologically equivalent to D_c^+. The remaining Landau equations are identical to those required for causal diagrams.

The surface $\mathcal{L}_c^+$ has a simple structure. To describe this structure it is convenient to introduce the surfaces $\mathcal{L}_0(D^\sigma)$. The surface $\mathcal{L}_0(D^\sigma)$ is the subset of $\mathcal{L}(D^\sigma)$ obtained by deleting from it two sets of points. The first of these sets, $\mathcal{M}_0$, is the set of points $p \in \mathcal{M}$ such that two or more of the initial p_j are parallel or two or more of the final p_j are parallel:

$$\mathcal{M}_0 \equiv \{p \in \mathcal{M}: p_i || p_j \text{ for some pair } (i,j) \text{ in } (I,I) \cup (F,F)\} . \qquad (D.7)$$

The second deleted set consist of those points that satisfy not only the conditions (D.6) that define $\mathcal{L}(D^\sigma)$, but also the conditions

obtained from these by replacing some, but not all, of the conditions $\sigma_j\alpha_j > 0$ by $\alpha_j = 0$. Thus

$$\mathcal{L}_0(D^\sigma) \equiv \mathcal{L}(D^\sigma) - \mathcal{M}^0 - \bigcup_{\substack{\text{Int}' \subset \text{Int} \\ \text{Int}' \neq \emptyset \\ \text{Int}' \neq \text{Int}}} \mathcal{L}(D^\sigma; \text{Int}') \qquad (D.8)$$

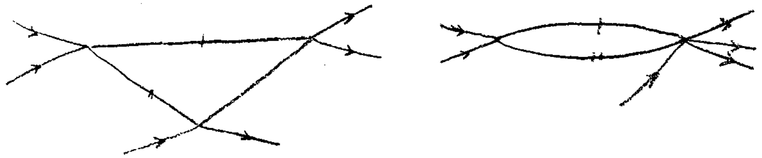
where $\mathcal{L}(D^\sigma; \text{Int}')$ is defined in the same way as $\mathcal{L}(D^\sigma)$, except that the conditions $\sigma_j\alpha_j > 0$ are replaced by $\alpha_j = 0$ for $j \in$ Int'. The set $\emptyset$ is the empty set.

The surfaces $\mathcal{L}_0(D_c^+)$ have three important properties. The first is that every point of $\mathcal{L}_c^+ - \mathcal{M}_0$ lies on some $\mathcal{L}_0(D_c^+)$:

$$\mathcal{L}_c^+ - \mathcal{M}_0 \equiv \bigcup_{D_c^+} \mathcal{L}_0(D_c^+) . \qquad (D.9)$$

This conclusion follows from the definitions (D.2) and (D.8), plus the fact that $\mathcal{L}(D_c^\sigma; \text{Int}')$ lies on $\mathcal{L}_0(D_c^{\sigma'})$, where $D_c^{\sigma'}$ is the diagram obtained from D_c^σ by first contracting to points the lines L_j with $j \in$ Int', then equating the vertices V_r that are thus brought into coincidence, and finally removing all lines that then begin and end at the same point.

A diagram $D^{\sigma'}$ obtained in this way from D^σ is called a <u>contraction</u> of D^σ. An example is shown in Fig. IIID.1.



D^{σ} $\qquad$ $D^{\sigma'}$

Fig. III.D.1. A diagram D^{σ} and a contraction $D^{\sigma'}$ of D^{σ}.

The second important property of the surfaces $\mathcal{L}_0(D_c^+)$ is that only a finite number of different diagrams D_c^+ give surfaces $\mathcal{L}_0(D_c^+)$ that enter any bounded region $\mathcal{R}$ of p space:

Theorem IIID.2. Suppose $\mathcal{R}$ is a bounded region in p space. Suppose the mass spectrum of stable particles excludes zero, and has no accumulation point. Then only a finite number of different Landau diagrams D_c^+ give surfaces $\mathcal{L}_0(D_c^+)$ that intersect $\mathcal{R}$.

This theorem is proved in Ref. 10.

The third important property of the surfaces $\mathcal{L}_0(D_c^+)$ is that each surface $\mathcal{L}_0(D_c^+)$ can be defined locally by an equation $\phi(z) = 0$, where $\phi(z)$ is locally analytic with nonzero gradient:

Theorem IIID.3. Let D_c^+ be any connected positive-α diagram. Suppose $\bar{p}$ lies on $\mathcal{L}_0(D_c^+)$. Then $\bar{p}$ lies on $\mathcal{W}$ and hence there exist sets $z \equiv (z_1,\cdots,z_{3n-4})$ of local real analytic coordinates of the mass shell at $\bar{p}$. For any such set there is a real neighborhood of the origin $\mathcal{U} \subset \mathcal{R}^{3n-4} \subset \mathcal{S}$ and a real analytic mapping $\phi(z): \mathcal{U} \to \phi(\mathcal{U}) \subset \mathcal{R}^1$ such that

(a) $\nabla \phi(z) \equiv (\partial \phi(z)/\partial z_1,\cdots,\partial \phi(z)/\partial z_{3n-4})$ is nonzero for every z in $\mathcal{U}$, and (D.10a)

(b) $z[\mathcal{L}_0(D_c^+)] \cap \mathcal{U} = \{z: \phi(z) = 0,\ z \in \mathcal{U}\}$. (D.10b)

This theorem says that $\mathcal{L}_0(D_c^+)$ is a subset of the restricted real mass-shell $\mathcal{W}$ and that--considered as a subset of $\mathcal{W}$-it is defined locally as the set of zeros of a real analytic function $\phi(z)$. And this function has a locally nonvanishing gradient. The proof is given in Ref. 11.

This function $\phi(z)$ is real-valued for z in the real set $\mathcal{U}$. The sign of $\phi(z)$ is fixed by the following sign convention:

The sign of $\phi(z)$ is fixed so that a formal increase $m_j \to m_j(1 + \epsilon)$ of the masses of all the internal lines L_j of D_c^+ shifts $\mathcal{L}(D_c^+)$ into the region $\phi(z) > 0$.

This sign convention is used in the definition of the plus $i\epsilon$ rule given in the next section.

E. THE PLUS $i\epsilon$ RULE

The plus $i\epsilon$ rule is the assertion that the scattering function $S_c(p)$ near any point $\bar{p} \in \mathcal{L}_c^+ - \mathcal{M}_0$ is the limit of a function that is analytic in the intersection of a mass-shell neighborhood of $\bar{p}$ with a specified cone. This cone is essentially the intersection of the upper-half planes $\text{Im}\,\phi_g > 0$ corresponding to the various surfaces $\mathcal{L}_0(D_{cg}^+)$ that contain $\bar{p}$.

The precise statement of the plus $i\epsilon$ rule is as follows:

Let $\bar{p}$ be any point on $\mathcal{L}_c^+ - \mathcal{M}_0$. Let $(z,\cdots,z_{3n-4}) = z$ be any set of local real analytic coordinates of the mass shell at $\bar{p}$. And let $\epsilon > 0$ be any positive number. Then there is a positive number $\delta \equiv \delta(\epsilon) > 0$, and an open cone $Y_\epsilon^+(\bar{p})$ such that:

(1) $S_c\big(p(z)\big)$ is analytic in

$$\{z = x + iy\colon |x| < \delta,\ |y| < \delta,\ y \in Y_\epsilon^+(\bar{p})\}\ , \qquad \text{(E.1)}$$

(2) $$S_c\big(p(x)\big) = \lim_{\substack{y\to 0\\ y\in Y_\epsilon^+(\bar{p})}} S_c\big(p(x+iy)\big) \qquad \text{(E.2)}$$

for all x in

$$\{x\colon |x| < \delta,\ p(x) \notin \mathcal{L}_c^+\}\ ,$$

(3) Equation (E.2) holds in the distribution sense on the set $|x| < \delta$. (See below.)

The cone $Y_\epsilon^+(\bar{p})$ is defined by

$$Y_\epsilon^+(\bar{p}) \equiv \{y\colon t\cdot y > \epsilon|t|\,|y| \quad \text{for all nonzero } t \in Y(\bar{p})\}, \qquad (\beta.3\text{a})$$

where

$$Y(\bar{p}) \equiv \left\{t\colon t = \sum_{g\in G(\bar{p})} \lambda_g \nabla\phi(0),\quad \lambda_g \geq 0\right\}\ , \qquad \text{(E.3b)}$$

and

$$G(\bar{p}) \equiv \{g\colon \bar{p} \in \mathcal{L}_0(D_{cg}^+)\}\ . \qquad \text{(E.3c)}$$

Here $\phi_g(z)$ is the function introduced in Sec. D:

$$\{\phi_g(x) = 0\} \bigcap \{|x| < \delta,\ \ y = 0\} \qquad \text{(E.4)}$$

$$= \{z[\mathcal{L}_0(D_{cg}^+)]\} \bigcap \{|x| < \delta,\ \ y = 0\}\ .$$

The vectors x and y are real $(3n - 4)$-dimensional vectors:

$$x \equiv \text{Re}\ z\ , \qquad \text{(E.5a)}$$

and

$$y \equiv \text{Im}\ z\ . \qquad \text{(E.5b)}$$

The vector t is a real $(3n - 4)$-dimensional vector, with components

$$t_j = \sum_g \lambda_g \left.\frac{\partial\phi(z)}{\partial z_j}\right|_{z=0} \qquad \text{(E.6)}$$

The inner product $t\cdot y$ is the usual inner product

$$t\cdot y = \sum_{j=1}^{3n-4} t_j\, y_j\ . \qquad \text{(E.7)}$$

As ϵ approaches zero the cone $Y_\epsilon^+(\bar{p})$ approaches the intersection of the half-space $\{y: y\cdot\nabla\phi_g(0) > 0\}$. However, the number $\delta = \delta(\epsilon)$ may also approach zero, in which case the neighborhood $\{|x| < \delta, |y| < \delta\}$ would shrink to a point, as ϵ goes to zero.

If for some point $\bar{p} \in \mathcal{L}_c^+ - \mathcal{M}_0$ the intersection over $g \in G(\bar{p})$ of the half-spaces $\{y: y\cdot\nabla\phi_g(0) > 0\}$ is empty, then the plus $i\epsilon$ rule for that point is also empty: The rule does not ensure that the scattering function near this point $\bar{p}$ can be represented as a limit of an analytic function. Such point $\bar{p}$ do in fact exist, but they are rare.

To show that these points are rare one needs the following result:

<u>Theorem IIIE.1</u>. If two surfaces $\mathcal{L}_0(D_{ca}^+)$ and $\mathcal{L}_0(D_{cb}^+)$ coincide in some real neighborhood of a point $\bar{p} \in \mathcal{L}_0(D_{ca}^+)$ then the two gradients $\nabla\phi_a(0)$ and $\nabla\phi_b(0)$ are parallel, not antiparallel.

This theorem is proved in Ref. 11. It means that the set $Y_\epsilon^+(\bar{p})$ can be empty, for all $\epsilon > 0$, only if $\bar{p}$ lies on the intersection of two or more locally noncoincident surfaces $\mathcal{L}_0(D_c^+)$. Thus the set of all such points $\bar{p}$ has dimension at least one less than that of $\mathcal{L}_c^+$, and at least two less than that of $\mathcal{W}$.

Point (3) of the plus $i\epsilon$ rule asserts that (E.2) holds in the distribution sense on $|x| < \delta$. This means that for any test function $\phi(x)$ that vanishes outside $|x| < \delta$, and that has continuous partial derivatives of all orders, the following identity holds:

$$\int S_c\big(\bar{p}(x)\big)\, \phi(x)\, dx = \lim_{\substack{y\to 0 \\ y\in Y_\epsilon^+(\bar{p})}} \int S_c\big(\bar{p}(x+iy)\big)\, \phi(x)\, dx .$$

That is, these weighted integrals of $S_c\big(p(x)\big)$ can be expressed as limits of integrals along contours that lie in the domain of analyticity of the function $S_c\big(p(z)\big)$.

The surface $\mathcal{L}_c^+$ separates $\mathcal{M}$ into a collection of disjoint sectors. The positive-α rule asserts that in each of these sectors the function $S_c(p)$ is analytic. The plus $i\epsilon$ rule then ensures that these analytic functions in the different sectors are parts of one single analytic function. That is, the analytic functions $S_c\big(p(x)\big)$ in the different sectors of $\{x: |x| < \delta, p(x) \notin \mathcal{L}_c^+\}$ are boundary values of one single analytic function $S_c\big(p(z)\big)$, and hence the invertibility of the functions $p(z)$ allows one to join together the analytic functions $S_c(p)$ defined in the various sectors into one single analytic function $S_c(p)$. This fact that the scattering functions in the different sectors are parts of one single analytic function $S_c(p)$ is a key ingredients of S-matrix theory.

FOOTNOTES AND REFERENCES

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