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Authors
Rahman, Farzana
Silva, Felipe O.
Farrell, Jay A

Publication Date
2022-05-12

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Farzana Rahman, Felipe O. Silva, and Jay A. Farrell
University of California, Riverside

May 12, 2022

Abstract

This is a reference document. Its purpose is to present the Earth Centered Earth Fixed (ECEF) strapdown inertial navigation mechanization and error state propagation equations.

1 Notation

To clearly distinguish between models and computations, this note uses two different equality symbols. The symbol $\doteq$ indicates that the equation is a model or definition. Models are used to analyze, understand, and physically interpret measurements, often with the goal of designing algorithms to estimate quantities that are of interest (e.g., position). The symbol $=$ indicates that an equation represents an actual algorithmic calculation. When it is necessary to represent the actual and computed versions of a variable, x will represent the actual value, and $\hat{x}$ will represent the computed or estimated value. Vector and matrix variables will be printed in bold font. For a vector $\mathbf{v}$, the notation $[\mathbf{v}]_k$ selects the k-th component of $\mathbf{v}$. For example, $\mathbf{p}^s$ represents the actual position vector for satellite s while $\hat{\mathbf{p}}^s$ represents the computed position vector for satellite s. The frame-of-reference in which a vector is represented will be indicated by a pre-superscript. The main frames-of-reference used herein are: body frame (b); Earth-Centered, Earth-Fixed frame (ECEF) (e); and, local frame (l).

2 ECEF Kinematic Model

The ECEF frame kinematic model, as derived in Section 11.2.2 of [1], is:

$$\begin{bmatrix} {}^e\dot{\mathbf{p}}_r \\ {}^e\dot{\mathbf{v}}_r \\ {}^e_b\dot{\mathbf{R}} \end{bmatrix} \doteq \begin{bmatrix} {}^e\mathbf{v}_r \\ {}^e_b\mathbf{R}\,{}^b\mathbf{f}_{ib} - 2\,{}^e\mathbf{\Omega}_{ie}\,{}^e\mathbf{v}_r + {}^e\mathbf{g} \\ {}^e_b\mathbf{R}\,{}^b\mathbf{\Omega}_{eb} \end{bmatrix} \tag{1}$$

where ${}^e\mathbf{p}_r$ and ${}^e\mathbf{v}_r$ denote the rover position and velocity resolved in ECEF frame, ${}^e_b\mathbf{R}$ represents the rotation matrix transforming a vector from the body frame to ECEF frame, ${}^e\mathbf{\Omega}_{ie}$ is the skew-symmetric matrix computed from the earth rotation rate vector ${}^e\omega_{ie}$. The rotation matrix ${}^e_b\mathbf{R}$ is equivalently represented in quaternion form by ${}^e_b\mathbf{q} \in \Re^4$ for state propagation. The symbol ${}^e\mathbf{g}$ represents the local gravity vector in ECEF frame. Finally, ${}^b\mathbf{\Omega}_{eb}$ is the skew-symmetric matrix form of ${}^b\boldsymbol{\omega}_{eb}$, which represents the angular rate of the b-frame with respect to the e-frame represented in the b-frame. The specific force vector is defined as

$${}^b\mathbf{f}_{ib} \doteq {}^b\mathbf{a}_{ib} - {}^b\mathbf{g} \tag{2}$$

which is the difference between the body frame acceleration with respect to the inertial frame ${}^b\mathbf{a}_{ib}$ and the local gravity vector ${}^b\mathbf{g}$. All vectors are represented in body frame.

3 IMU Measurements

An IMU consists of two triads of orthogonally mounted inertial sensors namely gyroscopes and accelerometers. The measurement model of these sensors are:

$${}^b\mathbf{u}_g(t) \doteq {}^b\boldsymbol{\omega}_{ib}(t) + \mathbf{z}_g(t) \qquad \text{Gyro Model} \tag{3}$$

$${}^b\mathbf{u}_a(t) \doteq {}^b\mathbf{f}_{ib}(t) + \mathbf{z}_a(t) \qquad \text{Accelerometer Model.} \tag{4}$$

The symbol ${}^b\mathbf{u}_g$ denotes the measurement of the angular rate ${}^b\boldsymbol{\omega}_{ib}$ of the body frame with respect to the inertial frame, both of which are represented in body frame. The symbol ${}^b\mathbf{u}_a$ denotes the measurement of the specific force.

The IMU measurements are imperfect, modeled as being corrupted by additive error signals $\mathbf{z}_g(t)$ and $\mathbf{z}_a(t)$. A tutorial discussing methods to develop a state-space model for these errors can be found in [2].

To exemplify the process, the IMU stochastic model assumed herein is

$$\mathbf{z}_g(t) \doteq \mathbf{b}_g(t) + \gamma_g(t) \qquad \text{with} \qquad \dot{\mathbf{b}}_g(t) \doteq \boldsymbol{\lambda}_g \mathbf{b}_g(t) + \boldsymbol{\epsilon}_g(t), \tag{5}$$

$$\mathbf{z}_a(t) \doteq \mathbf{b}_a(t) + \gamma_a(t) \qquad \text{with} \qquad \dot{\mathbf{b}}_a(t) \doteq \boldsymbol{\lambda}_a \mathbf{b}_a(t) + \boldsymbol{\epsilon}_a(t). \tag{6}$$

The symbols $\gamma_g(t)$ and $\gamma_a(t)$ represent Gaussian white random processes with Power Spectral Densities (PSDs) of $\sigma^2_{\gamma_g}$ and $\sigma^2_{\gamma_a}$, respectively. The symbols $\mathbf{b}_g(t)$ and $\mathbf{b}_a(t)$ represent slowly time-varying errors, referred to herein as biases, that are driven by the Gaussian white noise vectors $\boldsymbol{\epsilon}_g(t)$ and $\boldsymbol{\epsilon}_a(t)$ with PSDs of $\sigma^2_{b_g}$ and $\sigma^2_{b_a}$, respectively. This model can closely match the N and B portions of the ASD plot (for details, see [2])). This is sufficient for the studies focusing in INS error accumulation over tens of seconds.

4 INS Time Propagation Model

The INS integrates the (bias compensated) IMU measurements through the vehicle kinematic model to propagate the vehicle state through time at the IMU measurement rate [1, 3, 4]. This section briefly presents the INS kinematic and error models.

4.1 INS Kinematic Model

Based on the ECEF kinematic model of eqn. (1), the INS navigation equations in the ECEF frame are:

$$\begin{bmatrix} {}^e\dot{\hat{\mathbf{p}}}_r \\ {}^e\dot{\hat{\mathbf{v}}}_r \\ {}^e_b\dot{\hat{\mathbf{R}}} \end{bmatrix} = \begin{bmatrix} {}^e\hat{\mathbf{v}}_r \\ {}^e_b\hat{\mathbf{R}}\,{}^b\hat{\mathbf{f}}_{ib} - 2\,{}^e\mathbf{\Omega}_{ie}\,{}^e\hat{\mathbf{v}}_r + {}^e\hat{\mathbf{g}} \\ {}^e_b\hat{\mathbf{R}}\,{}^b\hat{\mathbf{\Omega}}_{eb} \end{bmatrix} \tag{7}$$

where each variable with a hat denotes the INS estimate of the same variable as defined in eqn. (1). For example, ${}^e\hat{\mathbf{p}}_r$ is the INS estimate of ${}^e\mathbf{p}_r$. The symbol ${}^b\hat{\mathbf{\Omega}}_{eb}$ is the skew-symmetric matrix form of ${}^b\hat{\boldsymbol{\omega}}_{eb}$ computed from the gyro measurement as

$${}^b\hat{\boldsymbol{\omega}}_{eb} = ({}^b\mathbf{u}_g - \hat{\mathbf{b}}_g) - {}^b_e\hat{\mathbf{R}}\,{}^e\boldsymbol{\omega}_{ie}, \tag{8}$$

where $\hat{\mathbf{b}}_g$ is the current estimate of the gyro bias. The specific force vector is computed from the IMU measurement as

$${}^b\hat{\mathbf{f}}_{ib} \doteq ({}^b\mathbf{u}_a - \hat{\mathbf{b}}_a), \tag{9}$$

where $\hat{\mathbf{b}}_a$ is the current estimate of the accelerometer bias.

Eqn. (7) is numerically integrated at the IMU sample rate to propagate the position, velocity, and attitude through time. For an overview of the numeric integration process, see Section 5.5 in [5]. The position and velocity integration is well discussed in [6]. The critically important attitude algorithms are well discussed in [7, 8]. The integration starts with a set of initial conditions. Because the initial conditions are not perfect, the IMU is not perfectly calibrated, and the IMU measurements are themselves imperfect, the integration process accumulates error over time. The rate of accumulation is dependent on the quality and calibration of the IMU.

4.2 INS Error State Propagation Model

The INS error accumulation process is well understood and can be accurately modeled. Presentation of such models is the purpose of this section. Note that the models of this section are not implemented and integrated through time. Instead, the models allow analysis and prediction of the error accumulations as can be characterized by the error covariance matrix (see eqn. (14)).

Both the INS state vector and IMU calibration parameters can be initialized and corrected in real-time using state estimation methods. These methods are built upon linearized state-space error models as summarized in this section. See [1, 3–5] for derivations.

The linearized INS error state differential equation is:

$$\begin{bmatrix} {}^e\delta\dot{\mathbf{p}} \\ {}^e\delta\dot{\mathbf{v}} \\ {}^e\dot{\boldsymbol{\rho}} \end{bmatrix} \doteq \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} \\ {}^e\mathbf{G} & -2\,{}^e\mathbf{\Omega}_{ie} & {}^e\mathbf{F}_{ib} \\ \mathbf{0} & \mathbf{0} & -{}^e\mathbf{\Omega}_{ie} \end{bmatrix} \begin{bmatrix} {}^e\delta\mathbf{p} \\ {}^e\delta\mathbf{v} \\ {}^e\boldsymbol{\rho} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -{}^b_e\mathbf{R} & \mathbf{0} \\ \mathbf{0} & {}^b_e\mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{z}_a \\ \mathbf{z}_g \end{bmatrix} \tag{10}$$

where ${}^e\delta\mathbf{p} \doteq {}^e\mathbf{p}_r - {}^e\hat{\mathbf{p}}_r$ and ${}^e\delta\mathbf{v} \doteq {}^e\mathbf{v}_r - {}^e\hat{\mathbf{v}}_r$ represent the position and velocity error vectors, and ${}^e\mathbf{F}_{ib}$ is the skew-symmetric matrix computed from ${}^e\mathbf{f}_{ib}$, where ${}^e\mathbf{f}_{ib} = {}^b_e\mathbf{R}\,{}^b\mathbf{f}_{ib}$. The symbol ${}^e\boldsymbol{\rho} \in \Re^3$ is defined as an angle error vector such that ${}^b_e\hat{\mathbf{R}} \doteq {}^b_e\mathbf{R}\,(\mathbf{I} + [{}^e\boldsymbol{\rho}\times])$ (see Section 10.5 in [1]). The symbol ${}^e\mathbf{G}$ represents the Jacobian matrix of the local gravity vector (see Section 11.1.4 in [1]). The symbol $\mathbf{I}$ denotes the identity matrix and each $\mathbf{0}$ represents a matrix of zeros of the appropriate size to conform with its row and column placement.

From eqns. (5-6), the accelerometer and gyroscope bias error propagation equations are

$$\begin{bmatrix} \delta\dot{\mathbf{b}}_g \\ \delta\dot{\mathbf{b}}_a \end{bmatrix} \doteq \begin{bmatrix} \boldsymbol{\lambda}_g\delta\mathbf{b}_g + \boldsymbol{\epsilon}_g \\ \boldsymbol{\lambda}_a\delta\mathbf{b}_a + \boldsymbol{\epsilon}_a \end{bmatrix}. \tag{11}$$

5 Combined INS and IMU Error Model

Sensor fusion can be accomplished by a variety of methods: Kalman filter [9, 10], Extended Kalman filter [11, 12], Unscented Kalman filter [13], maximum a posteriori optimization [14–16]. All of these methods define both a state and an error state. The state estimate is propagated through time by the INS. The error state is estimated in the sensor fusion process at discrete instants of time at which aiding measurements are available. Then the error state is used to correct the state estimate.

5.1 State and Error State Vector Definitions

The variables to be estimated include the vehicle state vector $\mathbf{p}_r$, $\mathbf{v}_r$, and $\mathbf{q} \in \Re^4$; and, the IMU calibration parameters $\mathbf{b}_g$ and $\mathbf{b}_a$. These variables are organized into the state vector

$$\boldsymbol{x}(t) \doteq [\mathbf{p}_r(t)^T, \mathbf{v}_r(t)^T, {}^e_b\mathbf{q}(t)^T, \mathbf{b}_a(t)^T, \mathbf{b}_g(t)^T]^T \in \Re^{16}.$$

Each of these symbols and parameters have already been defined in previous sections. The estimated state vector is

$$\hat{\boldsymbol{x}}(t) \doteq [\hat{\mathbf{p}}_r(t)^T, \hat{\mathbf{v}}_r(t)^T, {}^e_b\hat{\mathbf{q}}(t)^T, \hat{\mathbf{b}}_a(t)^T, \hat{\mathbf{b}}_g(t)^T]^T \in \Re^{16}.$$

Because both ${}^e_b\mathbf{q}$ and ${}^e_b\hat{\mathbf{q}}$ are unit vectors, the error between them can be represented by the attitude error vector ${}^e\boldsymbol{\rho} \in \Re^3$. Therefore the error state vector is defined as

$$\delta\boldsymbol{x}(t) = [{}^e\delta\mathbf{p}(t)^T, {}^e\delta\mathbf{v}(t)^T, {}^e\boldsymbol{\rho}(t)^T, \delta\mathbf{b}_g(t)^T, \delta\mathbf{b}_a(t)^T]^T$$

which is a vector in $\Re^{15}$.

Before the aiding measurement arrives at time t_k, the INS has a prior state estimate $\hat{\boldsymbol{x}}_k^- = \hat{\boldsymbol{x}}^-(t_k)$, computed by integration of eqn. (7). The sensor fusion algorithm uses that prior and the measurement at t_k to produce an error state estimate $\delta\hat{\boldsymbol{x}}_k^+$, which corrects the prior to produce the posterior state estimate $\hat{\boldsymbol{x}}_k^+$. The posterior state estimate $\hat{\boldsymbol{x}}_k^+$ provides the initial condition from which the INS integrates the vehicle state until the time of the next aiding measurement.

5.2 INS Error State Time Propagation Model

The state $\hat{\boldsymbol{x}}(t)$ is integrated through time using eqn. (7) and the IMU data as shown in eqns. (8-9). The full error state-space model is obtained by combining eqns. (10) and (11):

$$\begin{bmatrix} {}^e\delta\dot{\mathbf{p}} \\ {}^e\delta\dot{\mathbf{v}} \\ {}^e\dot{\boldsymbol{\rho}} \\ \delta\dot{\mathbf{b}}_g \\ \delta\dot{\mathbf{b}}_a \end{bmatrix} \doteq \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ {}^e\mathbf{G} & -2\,{}^e\boldsymbol{\Omega}_{ie} & {}^e\mathbf{F}_{ib} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -{}^e\Omega_{ie} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{\lambda}_g & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{\lambda}_a \end{bmatrix} \begin{bmatrix} {}^e\delta\mathbf{p} \\ {}^e\delta\mathbf{v} \\ {}^e\boldsymbol{\rho} \\ \delta\mathbf{b}_g \\ \delta\mathbf{b}_a \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -{}^b_e\mathbf{R} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & {}^b_e\mathbf{R} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \gamma_a \\ \gamma_g \\ \boldsymbol{\epsilon}_g \\ \boldsymbol{\epsilon}_a \end{bmatrix}. \tag{12}$$

where $\boldsymbol{\omega}(t) = \begin{bmatrix} \gamma_a(t)^\top & \gamma_g(t)^\top & \boldsymbol{\epsilon}_g(t)^\top & \boldsymbol{\epsilon}_a(t)^\top \end{bmatrix}^\top$.

Using standard methods (see Section 4.7 in [1]), the linear stochastic differential equation in (12) can be converted to an equivalent discrete-time model:

$$\delta\boldsymbol{x}_{k+1} \doteq \boldsymbol{\Phi}_k\,\delta\boldsymbol{x}_k + \boldsymbol{\omega}_k \tag{13}$$

where $\boldsymbol{\Phi}_k$ is the discrete-time state transition matrix, $\boldsymbol{\omega}_k$ is the discrete-time noise vector equivalent to the integral of the continuus-time noise vector $\boldsymbol{\omega}(t)$ through the kinematic model, and $cov(\boldsymbol{\omega}_k) = \mathbf{Q}_k$. From eqn. (13), the equation for propagating the error state covariance matrix $\mathbf{P}_k$ through time is

$$\mathbf{P}_{k+1} = \boldsymbol{\Phi}_k\,\mathbf{P}_k\,\boldsymbol{\Phi}_k^\top + \mathbf{Q}_k. \tag{14}$$

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