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Mathematical Tools and Convergence Results for Dynamics over Networks

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Mathematical Tools and Convergence Results for Dynamics over Networks

A dissertation submitted in partial satisfaction of the
requirements for the degree Doctor of Philosophy

in

Electrical Engineering (Communication Theory and Systems)

by

Rohit Yashodhar Parasnis

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2022

The Dissertation of Rohit Yashodhar Parasnis is approved, and it is acceptable in quality and form for publication on microfilm and electronically.

University of California San Diego

2022

DEDICATION

To आई (*Mom*), बाबा (*Dad*),
and to all my true well-wishers (*a set* $\mathcal{W}$*, whose cardinality is unknown!*)

EPIGRAPH

चरन्मार्गान्विजानाति।

A wanderer (eventually) finds the path.

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Chapter 1, in parts, contains material as it appears in the abstracts of the following papers and research articles: Rohit Parasnis, Massimo Franceschetti and Behrouz Touri, "To-

[1]Inspired by a line from *The Way I Am* by M. Mathers.

wards a Perron-Frobenius Theorem for Strongly Aperiodic Stochastic Chains", *arXiv preprint arXiv:2204.00573* (2022); Rohit Parasnis, Massimo Franceschetti and Behrouz Touri, "On the Convergence Properties of Social Hegselmann–Krause Dynamics," in *IEEE Transactions on Automatic Control* 67.2 (2021): 589-604; Rohit Parasnis, Massimo Franceschetti and Behrouz Touri, "Non-Bayesian Social Learning on Random Digraphs with Aperiodically Varying Network Connectivity", in *IEEE Transactions on Control of Network Systems*, in press (2022); and Rohit Parasnis, Ryosuke Kato, Amol Sakhale, Massimo Franceschetti and Behrouz Touri, "Usefulness of the Age-Structured SIR Dynamics for Modelling COVID-19", *arXiv preprint arXiv:2203.05111* (2022). The dissertation author was the primary investigator and author of these papers and articles.

Chapter 2, in full, is a reprint of the material as it appears in Rohit Parasnis, Massimo Franceschetti, and Behrouz Touri, "Towards a Perron-Frobenius Theorem for Strongly Aperiodic Stochastic Chains", *arXiv preprint arXiv:2204.00573* (2022). The dissertation author was the primary investigator and author of this article.

Chapter 2, in full, is currently being prepared for submission for publication as Rohit Parasnis, Massimo Franceschetti, and Behrouz Touri, "Towards a Perron-Frobenius Theorem for Strongly Aperiodic Stochastic Chains" (the publication venue is to be determined). The dissertation author was the primary investigator and author of this article.

Chapter 3, in full, is a reprint of the material as it appears in Rohit Parasnis, Massimo Franceschetti, and Behrouz Touri, "On the Convergence Properties of Social Hegselmann–Krause Dynamics," in *IEEE Transactions on Automatic Control* 67.2 (2021): 589-604. The dissertation author was the primary investigator and author of this paper.

Chapter 4, in full, is a reprint of the material as it appears in Rohit Parasnis, Massimo Franceschetti, and Behrouz Touri, "Non-Bayesian Social Learning on Random Digraphs with Aperiodically Varying Network Connectivity", in *IEEE Transactions on Control of Network Systems*, in press (2022). The dissertation author was the primary investigator and author of this paper.

Chapter 5, in full, is a reprint of the material as it appears in Rohit Parasnis, Ryosuke Kato, Amol Sakhale, Massimo Franceschetti, and Behrouz Touri, "Usefulness of the Age-Structured SIR Dynamics for Modelling COVID-19", *arXiv preprint arXiv:2203.05111* (2022). The dissertation author was the primary investigator and author of this article.

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PUBLICATIONS

R. Parasnis, M. Franceschetti, B. Touri, "On the Convergence Properties of Social Hegselmann–Krause Dynamics," in *IEEE Transactions on Automatic Control*, vol. 67, no. 2, pp. 589-604, 2022, doi.

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ABSTRACT OF THE DISSERTATION

Mathematical Tools and Convergence Results for Dynamics over Networks

by

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Doctor of Philosophy in Electrical Engineering (Communication Theory and Systems)

University of California San Diego, 2022

Professor Behrouz Touri, Chair
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Mathematical models of networked dynamical systems are ubiquitous - they are used to study power grids, networks of webpages, robotic and sensor networks, and social networks, to name a few. Importantly, most real-world networks are time-varying and are affected by stochastic phenomena such as adversarial attacks and communication link failures. Time-varying networks, therefore, have been under study for a few decades. However, our current understanding of the dynamical processes evolving over such networks is limited. This observation motivates the two-pronged objective of this dissertation: (i) to use theoretical and empirical methods to analyze certain networked dynamical systems that cannot be studied using standard

tools and techniques, and (ii) to develop suitable mathematical techniques for the systematic study of such systems.

As our main contribution resulting from (i), we use the properties of random time-varying networks to provide a rigorous theoretical foundation for the age-structured Susceptible-Infected-Recovered model, a model of epidemic spreading. We then use system identification to show that the age-structured SIR dynamics accurately model the spread of COVID-19 at city and state levels in two different parts of the world – Tokyo and California.

As for our contributions resulting from (ii), we extend two assertions of the Perron-Frobenius theorem to time-varying networks described by strongly aperiodic stochastic chains, thereby widening the applicability of the fundamental result that is foundational to probability theory and to the studies of complex networks, population dynamics, internet search engines, etc. Our results enable us to extend several known results on distributed learning and averaging. Moreover, they promise to advance our understanding of dynamical processes over real-world networks.

As an application of these results, we study non-Bayesian social learning on random time-varying networks that violate the standard connectivity criterion of uniform strong connectivity. In doing so, we also make a methodological contribution: we show how the theory of absolute probability sequences and martingale theory can be combined to analyze nonlinear dynamics that approximate linear dynamics asymptotically in time.

Finally, we study the convergence properties of social Hegselmann-Krause dynamics (which is a variant of the classical Hegselmann-Krause model of opinion dynamics and incorporates state-dependence into distributed averaging). As our main contribution here, we provide nearly tight necessary and sufficient conditions for a given connectivity graph to exhibit unbounded ε-convergence times for such dynamics.

Chapter 1

Introduction

In this chapter, we review a few background concepts that will be used in subsequent chapters, we use some of these concepts to motivate the objectives of this dissertation, and we provide a brief summary for each of the subsequent chapters.

We begin with a few basic concepts from graph theory.

1.1 Graphs and Networks

Given a natural number n, a *time-invariant* graph or a *static graph on n nodes/vertices* is a tuple $G = (V, E)$, where V is a finite set with $|V| = n$, and E is a subset of $V \times V$, where $|\cdot|$ denotes the cardinality of a set and '$\times$' denotes the Cartesian product of two sets. The elements of V are called the *nodes* of G or the *vertices* of G, and the elements of E are called the *edges* of G. For any two nodes $i, j \in V$, node j is said to be a *neighbor* of node i (equivalently, j is *adjacent* to i) if $(i, j) \in E$.

A static graph $G = (V, E)$ is said to be a *directed* graph or a *digraph* if there exists at least one pair of nodes $(i, j) \in E$ such that $(j, i) \notin E$, in which case the edges of G are called *arcs* or *directed edges*. If a graph is not directed, it is said to be *undirected*, in which case its edges are also said to be undirected. Note that the edge set of an undirected graph (V, E) satisfies the following property: for any two nodes $i, j \in V$, we have $(i, j) \in E$ if and only if $(j, i) \in E$.

Suppose now that V is the set of the first n natural numbers, i.e., $V = [n] := \{1, 2, \ldots, n\}$.

Then, by an extension of the terminology introduced above, a *weighted* time-invariant graph on n nodes is a tuple $G = ([n], E, W)$ where $E \subset [n] \times [n]$ and W is a non-negative $n \times n$ matrix such that for any two nodes $i, j \in [n]$, the (i, j)-th entry of W is positive if and only if $(i, j) \in E$, i.e., $w_{ij} > 0$ if and only if $(i, j) \in E$, in which case w_{ij} is called the *weight* of the edge (i, j). Here, the matrix W is called the the *weight matrix* of G. In subsequent chapters, W is also called the *adjacency matrix* of G and thereby denoted by A or A_{adj}.

We now extend all of the above static concepts to their time-varying counterparts. Given a time index t (which may be either discrete, as in $t = 0, 1, 2, \ldots$, or continuous, as in $t \in [0, \infty)$), a *time-varying* graph on n nodes is a tuple $(V, E(t))$ of a static vertex set V that satisfies $|V| = n$ and a time-varying edge set $E(t)$ that satisfies $E(t) \subset V \times V$ for all times t. Similarly, a weighted time-varying graph on n nodes is a tuple $([n], E(t), W(t))$ of the static vertex set $[n]$, a time-varying edge set $E(t) \subset [n] \times [n]$, and a time-varying non-negative weight matrix $W(t)$ such that for all $i, j \in [n]$, we have $w_{ij}(t) > 0$ if and only if $(i, j) \in E(t)$. Similar to time-invariant graphs, time-varying graphs can be either undirected ($(j, i) \in E$ for all $(i, j) \in E$) or directed, regardless of whether they are weighted or unweighted (not weighted).

In this dissertation, we often use the term *network* to refer to a graph, regardless of whether the graph is static or time-varying, directed or undirected, and weighted or unweighted.

1.2 Dynamical Processes over Networks

Let $\mathbb{R}^n$ be the space of all n-dimensional real-valued column vectors. A *continuous-time dynamical process over a network* $G(t) = ([n], E(t), W(t))$ is a function $x : [0, \infty) \to \mathbb{R}^n$ that evolves in time as per an *update rule* that can be expressed by an equation of the form

$$\dot{x}(t) = f(t, x(t), W(t)) \quad \text{for all } t \geq 0, \tag{1.1}$$

where f is a function that has $\mathbb{R}^n$ as its range and satisfies certain regularity conditions, and $x(0)$ is the *initial state* or the initial condition for (1.1). Likewise, a *discrete-time dynamical* *process over a network* $G(t) = ([n], E(t), W(t))$ is a function $x : \{0,1,2,\ldots\} \to \mathbb{R}^n$ that can be described by an update rule of the form

$$x(t+1) = f(t, x(t), W(t)) \quad \text{for all } t \in \{0,1,2,\ldots\}, \tag{1.2}$$

where f and $x(0)$ are as described above. In the context of either of the two dynamics (1.1) and (1.2), we refer to $x(t)$ as the *state* of the network $G(t)$ at time t, and for a generic node index $i \in [n]$, we refer to $x_i(t)$ (the i-th entry of $x(t)$) as the state of the node i at time t. We also refer to dynamical processes over networks as *networked dynamical proceses*.

We now provide a simple example of a class of dynamical processes over networks. Consider a deterministic (non-random) time-invariant network $G = ([n], E, W)$. Suppose the network has a non-random initial state $x(0)$, and suppose the state vector $x(t)$ evolves in time as per one of the following two equations in the absence of a control input,

$$x(t+1) = Ax(t), \quad t = 0,1,2,\ldots \tag{1.3}$$

$$\dot{x}(t) = Ax(t), \quad t \geq 0, \tag{1.4}$$

where A, the *state evolution matrix*, is an $n \times n$ matrix that is determined by W (in the simplest case, we have $A = W$). Then (1.3) and (1.4) define the simplest and most well-understood classes of dynamical processes over networks.

Observe that the dynamics defined by (1.3) and (1.4) are *linear* because the map $z \to Az$ satisfies the properties of homogeneity and superposition, and they are *time-invariant* because the network G (and hence also the state evolution matrix A) is constant in time. They are also deterministic processes since both A and $x(0)$ are assumed to be non-random.

However, most networks in the real world are dynamic rather than static, exhibit non-linear behavior, and are affected by stochastic phenomena such as communication link failures and adversarial attacks. In other words, most real-world network dynamical processes are not time-invariant, non-random, or linear. This motivates us to study certain less-understood dynamics over networks that involve either randomness, non-linearities, temporal dependence, or a combination of some of these properties. In our analysis, we not only characterize the convergence properties (or the long-term evolution) of some of these dynamics, but we also develop mathematical tools that hold promise for advancing our understanding of several other networked dynamical processes that are not studied in this dissertation.

We now categorize the networked dynamical processes studied in this work as follows:

I Dynamics related to *distributed averaging*, a method of information mixing that can be used to model the spread of information in social networks, distributed coordination in sensor networks and robotic networks, etc.

II Dynamics that model epidemic spreading in social networks.

We now introduce each of these categories.

1.3 Distributed Averaging and Related Dynamics

Distributed averaging is a method of information pooling or belief aggregation in *multi-agent systems* or networks of interacting agents that share common objectives, such as networks of temperature sensors used in an industrial plant, or social networks of voters attempting to choose the best political candidates in an election. It is a networked dynamical process in which the state of every node shifts to a weighted average of the neighbors' states in every update period.

Distributed averaging finds applications in distributed optimization [3, 4], distributed parameter estimation and signal processing [5], distributed hypothesis testing [6], networks of power systems [7], decentralized control of robotic networks [8], and opinion dynamics [9, 10]. Hence, a variety of distributed averaging dynamics have been studied till date within different mathematical frameworks [11–13].

We now provide a brief mathematical description of these dynamics.

1.3.1 Distributed Averaging Dynamics

In discrete time, distributed averaging can be described as a dynamical process $\{x(t)\}_{t=0}^{\infty}$ on a network $G(t) = ([n], E(t), W(t))$ with the update rule

$$x(t+1) = A(t, x(t), W(t))x(t) \quad \text{for all } t = 1, 2, \ldots, \tag{1.5}$$

where for each $t \in \{1, 2, \ldots\}$ the state evolution matrix $A(t, x(t), W(t))$ is an $n \times n$ *row-stochastic* matrix, i.e., all the entries of $A(t, x(t), W(t))$ are non-negative and the sum of the entries in every row of $A(t, x(t), W(t))$ equals 1.

In what follows, Chapter 2 develops mathematical tools for studying time-varying distributed averaging dynamics that can be described either by (1.5) after replacing $A(t, x(t), W(t))$ with $A(t) = W(t)$ or by the continuous-time analog of the same dynamics (given by $\dot{x}(t) = A(t)x(t)$). On the other hand, Chapter 3 focuses on a special class of state-dependent dynamics that can be subsumed under (1.5) with $A(t, x(t), W(t)) = A(x(t), W_0)$, where W_0 is a certain binary matrix that remains constant in time.

Chapter 4, by contrast, focuses on a dynamical process that is stochastic and non-linear, but one that is closely related to the distributed averaging dynamics described above. This non-linear process falls under the category of *distributed learning* dynamics, which we introduce below.

1.3.2 Distributed Learning

Distributed learning is a method of implementing hypothesis testing in a decentralized manner across a multi-agent system. To elaborate, consider a network of interacting agents driven by the shared purpose of identifying an unknown *state of the world* or a parameter of interest. Distributed learning then combines the process of belief aggregation or information mixing inherent to distributed averaging with the process of acquiring private signals, measurements, or observations made on the unknown state of the world.

Chapter 4 reviews a few applications of distributed learning as well as a few seminal works on the topic. To see how all of the distributed learning dynamics that we study in Chapter 4 are related to distributed averaging, we note that the former are a special case of the family of networked dynamical processes defined by

$$x_\theta(t+1) = A(t)x_\theta(t) + u(t, x_\theta(t)), \quad \text{for all } t = 0, 1, 2, \ldots \tag{1.6}$$

where $A(t)$ is the adjacency matrix of a random[1] time-varying graph $G(t)$, θ is a parameter that belongs to a finite set Θ, x_θ is the (random) state of the system corresponding to a given value of θ, and the control input $u : \{0, 1, 2, \ldots,\} \times \mathbb{R}^n \to \mathbb{R}^n$ is a random non-linear function satisfying $\lim_{t\to\infty} u(t, x_\theta(t)) = 0$ almost surely for a subset Θ^* of the parameter space Θ. As a result, for sufficiently large values of t we have $x_\theta(t) \approx A(t)x_\theta(t)$ for all $\theta \in \Theta^*$. This means that, for all $\theta \in \Theta^*$, the distributed learning dynamics studied in this dissertation approximate distributed averaging in the limit as time goes to infinity. As we shall see in Chapter 4, this observation enables us to exploit the properties of sequences of row-stochastic matrices in order to derive convergence results on all the distributed learning dynamics analyzed therein.

1.4 Epidemic Spreading in Social Networks

Motivated by the devastating economic and medical impacts of COVID-19 all over the world, Chapter 5 examines certain non-linear dynamical processes that model epidemic spreading in social networks.

The dynamics investigated in Chapter 5 are of two kinds: (a) *age-structured SIR* dynamics, defined by a system of bilinear ODEs, and (b) a time-homogeneous continuous-time Markov chain that describes a stochastic epidemic process occurring over a random, time-varying social network. Since most of the mathematical analysis responsible for our main theoretical results involves a detailed examination of (b), in the remainder of this section we summarize a few

[1] We assume that all random quantities are defined with respect to an underlying probability space $(\Omega, \mathcal{B}, \Pr)$.

prerequisites from the theory of time-homogeneous continuous-time Markov chains (CTMCs).

1.4.1 Prerequisites from the Theory of Time-Homogeneous CTMCs

Suppose we are given a probability space $(\Omega, \mathcal{B}, \Pr)$. We borrow the definition of time-homogeneous CTMCs from [14] and modify it slightly for our purposes.

Definition 1 (Time-Homogeneous CTMCs). *A time-homogeneous CTMC with state space $\mathcal{X}$ is a random process $\{\mathbf{X}(t) : t \geq 0\} \subset \mathcal{X}$ such that*

(a) The paths $\mathbf{X} : [0,\infty) \times \Omega \to \mathcal{X}$ are step functions that are right-continuous with respect to the first argument; and

(b) For any set of times $t_i < t_{i+1} = t_i + \Delta_{i+1}t$ and states $\mathbf{x}_i \in \mathcal{X}$, with $t_0 := 0$, we have

$$\Pr\left(\mathbf{X}\left(t_{k+1}\right) = \mathbf{x}_{k+1} \mid \mathbf{X}\left(t_i\right) = \mathbf{x}_i \;\forall\, i \leq k\right) = \Pr\left(\mathbf{X}\left(\Delta_{k+1}t\right) = \mathbf{x}_{k+1} \mid \mathbf{X}(0) = \mathbf{x}_k\right).$$

For time-homogeneous CTMCs, we define the concept of infinitesimal generators below.

Definition 2 (Infinitesimal Generator). *The infinitesimal generator of a time-homogeneous CTMC $\mathbf{X} := \{\mathbf{X}(t) : t \geq 0\}$ is the function $\mathbf{Q} : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$, defined by*

$$\mathbf{Q}(\mathbf{x},\mathbf{y}) := \lim_{\Delta t \to 0} \frac{\Pr(\mathbf{X}(t+\Delta t) = \mathbf{y} \mid \mathbf{X}(t) = \mathbf{x})}{\Delta t} \quad \text{for all } \mathbf{x},\mathbf{y} \in \mathcal{X} \text{ with } \mathbf{x} \neq \mathbf{y}$$

if this limit exists, in which case we let $\mathbf{Q}(\mathbf{x},\mathbf{x}) := -\sum_{\mathbf{y} \in \mathcal{X} \setminus \{\mathbf{x}\}} \mathbf{Q}(\mathbf{x},\mathbf{y})$ for all $\mathbf{x} \in \mathcal{X}$.

Throughout Chapter 5, we repeatedly use the following two well-known [15] properties of infinitesimal generators for time-homogeneous CTMCs on finite state spaces:

(i) The random time $T := \inf\{t > 0 : \mathbf{X}(t) \neq \mathbf{x}_0\}$ at which the CTMC $\mathbf{X}$ leaves an initial state $\mathbf{x}_0 \in \mathcal{X}$ is exponentially distributed with rate $-\mathbf{Q}(\mathbf{x}_0,\mathbf{x}_0)$ (so that the expected value of T is given by $\mathbb{E}[T] = -\frac{1}{\mathbf{Q}(\mathbf{x}_0,\mathbf{x}_0)}$).

(ii) Given that the CTMC $\mathbf{X}$ has transitioned from a state $\mathbf{x} \in \mathcal{X}$ to another state at time t, the conditional probability that the successor state is $\mathbf{y} \in \mathcal{X} \setminus \{\mathbf{x}\}$ is given by

$$\Pr(\mathbf{X}(t) = \mathbf{y} \mid \mathbf{X}(0) = \mathbf{x}, T = t) = \frac{\mathbf{Q}(\mathbf{x}, \mathbf{y})}{\sum_{\mathbf{z} \in \mathcal{X} \setminus \{\mathbf{x}\}} \mathbf{Q}(\mathbf{x}, \mathbf{z})}.$$

1.5 Outline of the Dissertation

We now summarize the remaining chapters of this dissertation.

1. Chapter 2, which is based on [16], begins with a review of the Perron-Frobenius theorem, a fundamental tool in matrix analysis that has applications to complex networks, population dynamics, social learning, and numerous physical, engineering, and economic phenomena. However, since this theorem and many of its extensions can be applied only to a single matrix at a time, their applications in networked dynamical systems are limited to static networks. To extend the applicability of these results to time-varying networks, Chapter 2 generalizes two assertions of the Perron-Frobenius theorem to sequences as well as continua of row-stochastic matrices. The results reported therein have potential applications in areas such as distributed averaging, optimization, and estimation.

2. Having developed a mathematical tool to analyze variants of distributed averaging, Chapter 3 considers a special class of state-dependent averaging dynamics called social Hegselmann–Krause (HK) dynamics, a variant of the HK model of opinion dynamics where a physical connectivity graph that accounts for the extrinsic factors that could prevent interaction between certain pairs of agents is incorporated. As opposed to the original HK dynamics (which terminates in finite time), we show that for any underlying connected and incomplete graph, under a certain mild assumption, the expected termination time of social HK dynamics is infinity. We then investigate the rate of convergence to the steady state by studying the *ε-convergence time*, i.e., the time required by the social HK system

to enter an ε-neighborhood of the steady state. We provide bounds on the maximum ε-convergence time in terms of the properties of the physical connectivity graph. We extend this discussion and observe that for almost all n, there exists an n-vertex physical connectivity graph on which social HK dynamics may not even ε-converge to the steady state within a bounded time frame. We then provide nearly tight necessary and sufficient conditions for arbitrarily slow merging (a phenomenon that is essential for arbitrarily slow ε-convergence to the steady state). Using the necessary conditions, we show that complete r-partite graphs have bounded ε-convergence times.

3. In Chapter 4, which is based on [17], we study a set of distributed learning dynamics called *non-Bayesian social learning* on random directed graphs, and we show that, under mild connectivity assumptions, all the agents almost surely learn the true state of the world asymptotically in time if the sequence of the associated weighted adjacency matrices belongs to Class $\mathcal{P}^*$ (a broad class of stochastic chains that subsumes uniformly strongly connected chains). We show that uniform strong connectivity, while being unnecessary for asymptotic learning, ensures that all the agents' beliefs converge to a consensus almost surely, even when the true state is not identifiable. We then provide a few corollaries of our main results, some of which apply to variants of the original update rule such as inertial non-Bayesian learning and learning via diffusion and adaptation. Others include extensions of known results on social learning. We also show that, if the network of influences is balanced in a certain sense, then asymptotic learning occurs almost surely even in the absence of uniform strong connectivity.

4. In Chapter 5, which is based on [17], we examine the age-structured SIR model, a variant of the classical Susceptible-Infected-Recovered (SIR) model of epidemic propagation, in the context of COVID-19. In doing so, we provide a theoretical basis for the model, perform an empirical validation, and discover the limitations of the model in approximating arbitrary epidemics. We first establish the differential equations defining the age-structured

SIR model as the mean-field limits of a continuous-time Markov process that models epidemic spreading on a social network involving random, asynchronous interactions. We then show that, as the population size grows, the infection rate for any pair of age groups converges to its mean-field limit if and only if the edge update rate of the network approaches infinity, and we show how the rate of mean-field convergence depends on the edge update rate. We then propose a system identification method for parameter estimation of the bilinear ODEs of our model, and we test the model performance on a Japanese COVID-19 dataset by generating the trajectories of the age-wise numbers of infected individuals in the prefecture of Tokyo for a period of over 365 days. In the process, we also develop an algorithm to identify the different *phases* of the pandemic, each phase being associated with a unique set of contact rates. Our results show a good agreement between the generated trajectories and the observed ones.

Chapter 1, in parts, contains material as it appears in the abstracts of the following papers and research articles: Rohit Parasnis, Massimo Franceschetti and Behrouz Touri, "Towards a Perron-Frobenius Theorem for Strongly Aperiodic Stochastic Chains", *arXiv preprint arXiv:2204.00573* (2022); Rohit Parasnis, Massimo Franceschetti and Behrouz Touri, "On the Convergence Properties of Social Hegselmann–Krause dynamics," in *IEEE Transactions on Automatic Control* 67.2 (2021): 589-604; Rohit Parasnis, Massimo Franceschetti and Behrouz Touri, "Non-Bayesian Social Learning on Random Digraphs with Aperiodically Varying Network Connectivity", in *IEEE Transactions on Control of Network Systems*, in press (2022); and Rohit Parasnis, Ryosuke Kato, Amol Sakhale, Massimo Franceschetti and Behrouz Touri, "Usefulness of the Age-Structured SIR Dynamics for Modelling COVID-19", *arXiv preprint arXiv:2203.05111* (2022). The dissertation author was the primary investigator and author of these papers and articles.

Chapter 2

Towards a Perron-Frobenius Theorem for Strongly Aperiodic Stochastic Chains

2.1 Introduction

Perron-Frobenius theorem is a foundational tool in linear algebra that is central to theory of Markov chain, and has many applications in database systems, complex networks, population dynamics, opinion dynamics, social learning, economic growth and income inequalities, and many other physical, social, and economic phenomena [9, 18–29]. Its strength lies in connecting the limiting behavior of A^k as $k \to \infty$ with the structural (graph-theoretic) pattern of a fixed non-negative matrix A. For example, in the case of Google's PageRank algorithm, A denotes the transition matrix of a Markov chain modelling a web-surfer, and the theory relates the ergodic (long term) behavior of this Markov chain to the centrality of webpages on World Wide Web (WWW).

However, since this theorem and many of its extensions (e.g. [30–33]) apply only to fixed matrices, their applications in understanding networked dynamical systems are limited to static networks. By contrast, most real-world networks are time-varying due to changing connections and communication patterns, temporary link failures, etc. [34–37]. Dynamical processes over such networks are related to *products* of *time-varying* matrices that capture the network structure. Such products are natural generalizations of *powers* of *time-invariant* matrices. This motivates us to study the generalization of the Perron-Frobenius theorem to products of time-varying matrices, as such a generalization would enhance many of the applications of the original result to time-varying settings.

In view of the above, we extend two assertions of the Perron-Frobenius theorem to time-varying networks. Specifically, the original theorem implies that a weighted random walk over a static strongly connected network has a stationary distribution that is (a) unique and (b) positive. Analogously, we establish the generalization of strong connectivity for time-varying networks and show that for lazy random walks over such networks, the *time-varying* extension of stationary distribution (Kolmogorov's absolute probability sequence [38]) is (a) unique and (b) uniformly positive if and only if the network is strongly connected over time in the generalized sense. We believe that this fundamental study will help advance the state-of-the-art understanding of dynamical processes over real-world networked systems.

The chapter is organized as follows. We first present a brief overview of all our main results in Section 2.2. Next, we formulate the problem of interest in Section 2.3 and discuss our main results in Section 2.4. We then provide a few applications of our main results in Section 2.5 and end with a few concluding remarks in Section 5.6. The proofs of all the results are provided in the Appendix.

Other Related Works: This endeavor evolves out of our work on social learning over time-varying networks [39, 40]. Therein, we showed that non-Bayesian social learning can occur asymptotically almost surely over random time-varying networks even if standard connectivity conditions are violated. This result begs the question, "What kinds of network connectivity hinder social learning?". Generalizing the Perron-Frobenius theorem should help answer this question as well as other similar questions in the broader areas of social learning [41–47], distributed optimization [4, 48–52], distributed estimation [5, 53–55], and distributed hypothesis testing [6, 56–59].

Notation: Let $\mathbb{N} := \{1, 2, \ldots\}$ denote the set of natural numbers and let $\mathbb{N}_0 := \mathbb{N} \bigcup \{0\}$. Let $\mathbb{R}$ denote the set of real numbers, let $\mathbb{R}^n$ denote the set of n-dimensional real-valued column vectors, and let $\mathbb{R}^{n\times n}$ denote the set of square matrices with real entries. For a given matrix

$A \in \mathbb{R}^{n\times n}$, we let $a_{ij} = (A)_{ij}$ denote the entry in the i-th row and the j-th column of A.

Let I denote the identity matrix (of the known dimension), let O denote the all-zeros matrix (of the known dimension), and let $\mathbf{0}$ and $\mathbf{1}$ denote the vector (of the known dimension) with all entries equal to zero and the vector with all entries equal to one, respectively.

A vector $v \in \mathbb{R}^n$ is said to be *stochastic* if v is non-negative and $v^T\mathbf{1} = 1$. A matrix $A_0 \in \mathbb{R}^{n\times n}$ is called *row-stochastic* or simply *stochastic* if A_0 is non-negative and if each row of A_0 sums up to 1, i.e., if $A_0 \geq O$ and $A_0\mathbf{1} = \mathbf{1}$. Note that all matrix and vector inequalities are assumed to hold entry-wise. Let $\{A(t)\}_{t=0}^{\infty}$ be a stochastic chain (a sequence of row-stochastic matrices in $\mathbb{R}^{n\times n}$). Then, for any two times $t_1, t_2 \in \mathbb{N}_0$ with $t_1 < t_2$, we use the following notation to denote the backwards matrix product of $\{A(t)\}_{t=0}^{\infty}$ corresponding to the time interval $[t_1, t_2]$.

$$A(t_2 : t_1) := A(t_2 - 1)A(t_2 - 2)\cdots A(t_1)$$

with the convention that $A(t : t) := I$ for all $t \in \mathbb{N}_0$.

For a matrix $A \in \mathbb{R}^{n\times n}$ and a subset $S \subset [n]$, let A_S be the principal sub-matrix of A corresponding to the rows and columns indexed by S. Let $\bar{S} := [n] \setminus S$, and let $A_{S\bar{S}}$ denote the sub-matrix of A corresponding to the rows indexed by S and the columns indexed by $\bar{S}$. For a sequence of matrices $\{A(t)\}_{t=0}^{\infty} \subset \mathbb{R}^{n\times n}$ and times $t_0, t_1 \in \mathbb{N}_0$ satisfying $t_0 \leq t_1$, let $A_S(t_1 : t_0) := (A(t_1 : t_0))_S$ and $A_{S\bar{S}}(t_1 : t_0) := (A(t_1 : t_0))_{S\bar{S}}$.

An unweighted undirected graph with vertex set $[n]$ and edge set E is denoted by $G = ([n], E)$. On the other hand, a weighted time-varying directed graph with vertex set $[n]$, edge set $E(t) \subset [n] \times [n]$, and edge weights $\{w_{ij}(t) : (i, j) \in [n] \times [n]\}$ is denoted by $G(t) = ([n], E(t), W(t))$, where $W(t) \in \mathbb{R}^{n\times n}$ with $(W(t))_{ij} := w_{ij}(t)$, which denotes the edge weight of the node pair $(i, j) \in [n] \times [n]$. We assume that $w_{ij}(t) \neq 0$ if and only if $(i, j) \in E(t)$, i.e., $E(t) = \{(i, j) \in [n] \times [n] : w_{ij}(t) \neq 0\}$. Recall that $G(t)$ is said to be *strongly connected* if, for any two nodes $i, j \in [n]$, there exists a directed path from i to j in $G(t)$.

For a weighted time-varying directed graph $G(t) = ([n], E(t), W(t))$, we let $L(t) =$

$(\ell_{ij}(t))$ denote the weighted *Laplacian* matrix of $G(t)$, defined by $\ell_{ij}(t) = -w_{ij}(t)$ for all $i \neq j$ and $\ell_{ii}(t) = \sum_{j\neq i} w_{ij}(t)$ for all $i \in [n]$. In addition, for a given matrix A_0, we let $\mathcal{G}(A_0) = ([n], \mathcal{E}(A_0), A_0)$ denote the weighted directed graph whose weighted adjacency matrix is A_0, i.e., we let $\mathcal{E}(A_0) = \{(i,j) \in [n] \times [n] : (A_0)_{ij} > 0\}$ and we construct $\mathcal{G}(A_0)$ in such a way that every $(i,j) \in \mathcal{E}(A_0)$ is a directed edge in $\mathcal{G}(A_0)$ with weight $(A_0)_{ij}$.

2.2 Overview of the Main Results

To highlight the main results of this work and before discussing the their connection to the classical Perron-Frobenius Theorem in details, we state our main results here. These results are based on the three definitions provided below. We shall reproduce these definitions in Section and extensively discuss them with regards to the classical Perron-Frobenius Theorem in Section 2.3.

The first definition is an extension of strong aperiodicity [60] to time-varying stochastic chains.

Definition 3 (**Strong Aperiodicity**). *A stochastic chain $\{A(t)\}_{t=0}^{\infty}$ is said to be strongly aperiodic if there exists a constant $\gamma > 0$ such that $a_{ii}(t) \geq \gamma$ for all $i \in [n]$ and all $t \in \mathbb{N}_0$.*

The second object, an *approximate reciprocal chain*, is a central object of this work and we will see shortly its connection to the classical Perron-Frobenius Theorem.

Definition 4 (**Approximate Reciprocity**). *A stochastic chain $\{A(t)\}_{t=0}^{\infty}$ is said to be* approximately reciprocal *if there exist constants $p_0, \beta \in (0,\infty)$ such that the following inequality holds for all $S \subset [n]$ and all times $t_0, t_1 \in \mathbb{N}_0$ that satisfy $t_0 < t_1$.*

$$p_0 \sum_{t=t_0}^{t_1-1} \mathbf{1}^T A_{S\bar{S}}(t)\mathbf{1} \leq \sum_{k=t_0}^{t_1-1} \mathbf{1}^T A_{\bar{S}S}(t)\mathbf{1} + \beta. \tag{2.1}$$

Finally, let us introduce a proper object extending the concept of Perron eigenvector for a stochastic matrix to time-varying chains and a proper extension of positive eigenvector to time-varying setting.

Definition 5 (Uniformly Positive Absolute Probability Sequence). *Let* $\{A(t)\}_{t=0}^{\infty}$ *be a stochastic chain. A sequence of stochastic vectors* $\{\pi(t)\}_{t=0}^{\infty}$ *is said to be a uniformly positive absolute probability sequence for* $\{A(t)\}_{t=0}^{\infty}$ *if*

$$\pi^T(t+1)A(t) = \pi^T(t) \quad \textit{for all } t \in \mathbb{N}_0$$

and if there exists a constant $p^* > 0$ *such that* $\pi(t) \geq p^* \mathbf{1}$ *for all* $t \in \mathbb{N}_0$.

With the introduction of these concepts, we are ready to assert the main results of this work.

Theorem 1 (An Analog of the Positive Eigenvector Assertion of the Perron-Frobenius Theorem). *Suppose* $\{A(t)\}_{t=0}^{\infty} \subset \mathbb{R}^{n \times n}$ *is a strongly aperiodic stochastic chain. Then* $\{A(t)\}_{t=0}^{\infty}$ *has a uniformly positive absolute probability sequence if and only if it is approximately reciprocal.*

As we shall show, this theorem generalizes the following Perron-Frobenius assertion to time-varying matrices: an irreducible non-negative matrix has a positive principal left eigenvector.

Theorem 2 (An Analog of the Uniqueness Assertion of the Perron-Frobenius Theorem). *Let* $\{A(t)\}_{t=0}^{\infty} \subset \mathbb{R}^{n \times n}$ *be a strongly aperiodic stochastic chain that is also approximately reciprocal. Then,* $\{A(t)\}_{t=0}^{\infty}$ *admits a unique absolute probability sequence if and only if its infinite flow graph is connected.*

We shall see that this result extends the following statement to the case of time-varying matrices: an irreducible non-negative matrix has a principal left eigenvector that is unique up to scaling.

We now state the continuous-time analogs of the above results, which are based on the following continuous-time analogs of the above definitions.

Definition 6 (**Uniformly Positive Absolute Probability Sequence in Continuous Time**). *Let $\{A(t)\}_{t\geq 0}$ be a continuous-time stochastic chain and let $\Phi(\cdot,\cdot)$ denote the state transition matrix for the dynamics $\dot{x}(t) = Ax(t)$. Then, a continuum of stochastic vectors $\{\pi(t)\}_{t\geq 0}$ is said to be a uniformly positive absolute probability sequence for $\{A(t)\}_{t\geq 0}$ if*

$$\pi^T(t)\Phi(t,\tau) = \pi^T(\tau)$$

holds for all $t \geq \tau \geq 0$, and if there exists a constant $p^* > 0$ such that $\pi(t) \geq p^* \mathbf{1}$ for all $t \geq 0$.

The following is the natural continuous counterpart of approximate reciprocity.

Definition 7 (**Approximate Reciprocity in Continuous Time**). *A continuous-time stochastic chain $\{A(t)\}_{t\geq 0}$ is said to be approximately reciprocal if there exist $p_0, \beta \in (0,\infty)$ such that*

$$p_0 \int_{t_\ell}^{t_m} \mathbf{1}^T A_{S\bar{S}}(t)\mathbf{1} dt \leq \int_{t_\ell}^{t_m} \mathbf{1}^T A_{\bar{S}S}(t)\mathbf{1} dt + \beta$$

holds for all sets $S \subset [n]$ and for all $\ell, m \in \mathbb{N}_0$ with $\ell \leq m$.

With these definition, we have the following results regarding the continuous-time variations of Theorem 1 and Theorem 2.

Theorem 3 (**Continuous-time Analog of Theorem 1**). *Let $\{A(t)\}_{t\geq 0}$ be a continuous-time stochastic chain that satisfies Assumption 1 (which states that the weights $\{a_{ij}(t) : i \neq j \in [n]\}$, when integrated over certain recurring time intervals, are uniformly bounded). Then $\{A(t)\}_{t\geq 0}$ has a uniformly positive absolute probability sequence if and only if it is approximately reciprocal.*

Theorem 4 (**Continuous-time Analog of Theorem 2**). *Let $\{A(t)\}_{t\geq 0}$ be an approximately reciprocal continuous-time stochastic chain that satisfies Assumption 1 (uniform bound on integral weights). Then $\{A(t)\}_{t\geq 0}$ admits a unique absolute probability sequence if and only if its infinite flow graph is connected.*

The rest of the chapter is on the detailed discussion on these results, their connection to the classical Perron-Frobenius theorem, and their implications.

2.3 Classical Perron-Frobenius Theorem, Approximate Reciprocity, and Absolute Probability

We first review the eigenvector assertions of the original Perron-Frobenius theorem. For this purpose, we need to recall the following property of irreducible matrices, which is often stated as a definition of irreducibility.

Lemma 1 (Section 8.3, [61]). *A non-negative matrix $A_0 \in \mathbb{R}^{n\times n}$ is irreducible if and only if the associated digraph $\mathcal{G}(A_0)$ is a strongly connected directed graph.*

In addition, we need the concepts of *reciprocity* and *infinite flow graphs*, which we reproduce from [62] below.

Definition 8 (**Reciprocity/Cut-balance**). *A stochastic chain $\{A(t)\}_{t=0}^{\infty}$ is said to be cut-balanced or reciprocal if there exists a constant $\alpha \in (0,1)$ such that*

$$\sum_{i\in S}\sum_{j\in \bar{S}} a_{ij}(t) \geq \alpha \sum_{i\in \bar{S}}\sum_{j\in S} a_{ij}(t)$$

holds for all times $t \in \mathbb{N}_0$ and all subsets $S \subset [n]$ and their complements $\bar{S} := [n]\setminus S$. In other words, $\mathbf{1}^T A_{S\bar{S}}(t)\mathbf{1} \geq \alpha \mathbf{1}^T A_{\bar{S}S}(t)\mathbf{1}$ for all $S \subset [n]$ and all $t \in \mathbb{N}_0$.

Intuitively, a stochastic chain is said to be reciprocal if the sequence of associated directed graphs is such that the total influence of any subset S of individuals on the complementary subset $\bar{S}$ is comparable to the total reverse influence of $\bar{S}$ on S, i.e., the ratio of the forward and the backward influences does not vanish in time.

Definition 9 (**Infinite Flow Graph** [62]). *For a stochastic chain $\{A(t)\}_{t=0}^{\infty}$, we define its* infinite flow graph *to be the unweighted undirected graph $G^{\infty} = ([n], E^{\infty})$ with vertex set $[n]$ and edge* *set E^{∞}, where*

$$E^{\infty} := \left\{ \{i,j\} \subset [n] \,\middle|\, \sum_{t=0}^{\infty} (a_{ij}(t) + a_{ji}(t)) = \infty, i \neq j \in [n] \right\}.$$

Intuitively, there exists a link from a node $j \in [n]$ to another node $i \in [n] \setminus \{j\}$ in the infinite flow graph G^{∞} if and only if node j exerts a long-term influence on node i in the time-varying directed graph $\mathcal{G}(A(t))$.

We now observe that for a *static* chain $A(t) = A_0$, the irreducibility of A_0 can be expressed in terms of the connectivity of the infinite flow graph G^{∞} and the reciprocity of $\{A(t)\}_{t=0}^{\infty}$.

Lemma 2. *Let $\{A(t)\}_{t=0}^{\infty}$ be a static stochastic chain with $A(t) = A_0 \in \mathbb{R}^{n\times n}$ for all $t \in \mathbb{N}_0$. Then, A_0 is irreducible if and only if $\{A(t)\}_{t=0}^{\infty}$ is reciprocal and its infinite flow graph G^{∞} is connected.*

We are now ready to recall two eigenvector assertions of the Perron-Frobenius theorem in the context of row-stochastic matrices. Lemma 2 enables us to state these assertions as follows.

Proposition 1 (Eigenvector Assertions of the Perron-Frobenius Theorem for Stochastic Matrices). *Let $\{A(t)\}_{t=0}^{\infty}$ be a static stochastic chain with $A(t) = A_0 \in \mathbb{R}^{n}$ for all $t \in \mathbb{N}_0$. If $\{A(t)\}_{t=0}^{\infty}$ is reciprocal and if its infinite flow graph is connected, then A_0 has a stochastic principal left eigenvector $\pi_0 \in \mathbb{R}^{n}$ that is*

(a) entry-wise positive,

(b) unique.

Note that the original theorem applies to left eigenvectors as well as to right eigenvectors. However, we choose to focus on the former because it is clear that all row-stochastic matrices have $\mathbf{1}$ as their principal right eigenvector. Moreover, our main results are centered on a concept that generalizes the notion of principal left eigenvectors to the case of time-varying row-stochastic matrices. This concept is defined below.

Definition 10 (**Kolmogorov Absolute Probability Sequence** [38]). *Let $\{A(t)\}_{t=0}^{\infty}$ be a stochastic chain. A sequence of stochastic vectors $\{\pi(t)\}_{t=0}^{\infty}$ is said to be an absolute probability sequence for $\{A(t)\}_{t=0}^{\infty}$ if*

$$\pi^T(t+1)A(t) = \pi^T(t) \quad \textit{for all } t \in \mathbb{N}_0.$$

Note that every stochastic chain admits an absolute probability sequence [38]. Moreover, if $\{A(t)\}_{t=0}^{\infty}$ is a static chain with $A(t) = A_0 \in \mathbb{R}^{n\times n}$ for all $t \in \mathbb{N}_0$, then the static sequence $\pi(t) = \pi_0$, where $\pi_0 \in \mathbb{R}^n$ is a stochastic vector satisfying $\pi_0^T A_0 = \pi_0^T$, is an absolute probability sequence for $\{A(t)\}_{t=0}^{\infty}$. Hence, absolute probability sequences are a time-varying analog of stochastic principal left eigenvectors.

This discussion naturally leads to the following question: can we generalize Proposition 1 to the class of non-static stochastic chains (or any sub-class thereof) using the notion of absolute probability sequences? The next section answers this question.

2.4 Results

We first extend Proposition 1 to discrete-time stochastic chains of the form $\{A(t) : t \in \mathbb{N}_0\}$ and then to continuous-time stochastic chains of the form $\{A(t) : t \geq 0\}$.

2.4.1 Discrete Time

To extend the first assertion of Proposition 1 to time-varying matrices (non-static chains), we need to extend the notion of positive principal left eigenvectors to the time-varying case. The following concept, first introduced in [62], offers such an extension.

Definition 11 (**Class $\mathcal{P}^*$** [62]). *We let (Class-)$\mathcal{P}^*$ be the set of all stochastic chains that admit uniformly positive absolute probability sequences, i.e., a sequence of stochastic vectors $\{\pi(t)\}_{t=0}^{\infty}$ for which there exists a $p^* > 0$ such that $\pi(t) \geq p^*\mathbf{1}$ for all $t \in \mathbb{N}_0$. (Note that the absolute probability sequence and the value of p^* may vary from chain to chain).*

It is worth noting that in the context of social learning, if $\{\pi(t)\}_{t=0}^{\infty}$ is an absolute probability sequence for $\{A(t)\}_{t=0}^{\infty}$, then $\pi_i(t)$ denotes the *Kolmogorov centrality* or *social power* of agent i at time t, which quantifies how influential the i-th agent is relative to other agents at time t [62, 63]. In view of Definition 47, this means that a stochastic chain belonging to Class $\mathcal{P}^*$ describes a sequence of influence graphs in which the social power of every agent exceeds a fixed positive threshold p^* at all times.

Since the definition of Class $\mathcal{P}^*$ eludes simple interpretation, we would like to derive necessary and sufficient conditions for a given stochastic chain to belong to Class $\mathcal{P}^*$. To this end, we introduce the idea of *approximate reciprocity*, which is a weaker notion of reciprocity (Definition 8).

Definition 12 (**Approximate Reciprocity**). *A stochastic chain $\{A(t)\}_{k=0}^{\infty}$ is said to be* approximately reciprocal *if there exist constants $p_0, \beta \in (0, \infty)$ such that the following inequality holds for all $S \subset [n]$ and all times $t_0, t_1 \in \mathbb{N}_0$ that satisfy $t_0 < t_1$.*

$$p_0 \sum_{t=t_0}^{t_1-1} \mathbf{1}^T A_{S\bar{S}}(t)\mathbf{1} \le \sum_{k=t_0}^{t_1-1} \mathbf{1}^T A_{\bar{S}S}(t)\mathbf{1} + \beta. \tag{2.2}$$

As it turns out, approximate reciprocity is a necessary condition for a given stochastic chain to belong to Class $\mathcal{P}^*$.

Proposition 2 (**Necessary Conditions for Class $\mathcal{P}^*$**). *Let $\{A(t)\}_{t=0}^{\infty} \subset \mathbb{R}^{n\times n}$ be a stochastic chain. If $\{A(t)\}_{t=0}^{\infty}$ belongs to Class $\mathcal{P}^*$, then $\{A(t)\}_{t=0}^{\infty}$ is approximately reciprocal.*

Interestingly, approximate reciprocity is also a sufficient condition for certain stochastic chains called strongly aperiodic chains to belong to Class $\mathcal{P}^*$. A given stochastic chain is called strongly aperiodic if all the diagonal entries of its matrices are uniformly bounded away from zero.

Definition 13 (**Strong Aperiodicity**). *A stochastic chain $\{A(t)\}_{t=0}^{\infty} \subset \mathbb{R}^{n\times n}$ is said to be strongly aperiodic if there exists a constant $\gamma > 0$ such that $A(t) \ge \gamma I$ for all $t \in \mathbb{N}_0$.*

To connect approximate reciprocity, a property expressed in terms of *sums* of matrix entries, to Class $\mathcal{P}^*$, a concept associated with *products* of matrices, we need the following lemmas that help relate matrix sums to matrix products.

Lemma 3. *Let* $\varepsilon \in (0,1)$ *be given. Then* $1-x \geq e^{-M(\varepsilon)x}$ *for all* $x \in [0,1-\varepsilon]$*, where* $M(\varepsilon) := \frac{1}{1-\varepsilon}\ln\frac{1}{\varepsilon}$.

Lemma 4. *Let* $n,\sigma \in \mathbb{N}$ *and* $i,j \in [n]$ *be given. Let* $\{B(t)\}_{t=0}^{\sigma-1} \subset \mathbb{R}^{n\times n}$ *be a sequence of substochastic matrices and let* $t_L := \max\{t \in \{0,1,\ldots,\sigma-1\} : B_{ji}(t) > 0\}$*. Suppose there exist positive constants* η_i *and* η_j *such that*

$$B_{ii}(t_1 : t_0) \geq \eta_i \quad \textit{if} \quad 0 \leq t_0 \leq t_1 \leq t_L,$$

$$B_{jj}(t_1 : t_0) \geq \eta_j \quad \textit{if} \quad 0 \leq t_0 \leq t_1 \leq \sigma, \textit{ and}$$

$$\sum_{t=0}^{\sigma-1} B_{ji}(t) \geq \delta \quad \textit{for some} \quad \delta \in (0,\eta_j).$$

Then $B_{ji}(\sigma : 0) \geq \frac{1}{2}\eta_i\eta_j\delta$.

In addition to the above lemmas, we need the notion of approximately stochastic chains, which we define below.

Definition 14 (Approximate Stochasticity). *Let* $n \in \mathbb{N}$ *and* $m \in \mathbb{N}\bigcup\{\infty\}$ *be given. A sequence* $\{A(t)\}_{t=0}^{m}$ *of* $n\times n$ *sub-stochastic matrices is said to be* approximately stochastic *if there exists a constant* $\Delta < \infty$ *such that*

$$\sum_{t=0}^{m} \mathbf{1}^T(\mathbf{1} - A(t)\mathbf{1}) \leq \Delta.$$

The constant Δ *will be referred to as the deviation from stochasticity of the sequence* $\{A(t)\}_{t=0}^{m}$*. Note that* $\Delta = 0$ *if* $\{A(t)\}_{t=0}^{m}$ *is a stochastic chain.*

We are now well-equipped to establish approximate reciprocity as a sufficient condition for strongly aperiodic chains to lie in $\mathcal{P}^*$. To do so, we use inductive arguments involving approximately stochastic chains to prove a slightly more general result that asserts that the backward matrix products of the concerned chains can be uniformly lower-bounded by a multiple of the identity matrix. This general result is stated below and proved in the appendix.

Proposition 3. *For every $n \in \mathbb{N}$, there exists a function*

$$\eta_n : (0,1) \times (0,1) \times (0,\infty) \times [0,\infty) \to (0,1)$$

such that the inequality,

$$A(t_1 : t_0) \geq \eta_n(\gamma, p_0, \beta, \Delta) I$$

holds for all $t_0, t_1 \in \mathbb{N}_0, \gamma, p_0 \in (0,1), \beta \in (0,\infty)$, and $\Delta \in [0,\infty)$ whenever $\{A(t)\}_{t=0}^{\infty} \subset \mathbb{R}^{n\times n}$ is a substochastic chain with the following properties.

1. ***(Strong aperiodicity/Feedback property** [62]). $a_{ii}(t) \geq \gamma$ for all $i \in [n]$ and all $t \in \mathbb{N}_0$.*
2. ***(Approximate reciprocity).** For every subset $S \subset [n]$ and $t_0, t_1 \in \mathbb{N}_0$ satisfying $t_0 < t_1$:*

$$p_0 \sum_{t=t_0}^{t_1-1} \mathbf{1}^T A_{S\bar{S}}(t)\mathbf{1} \leq \sum_{t=t_0}^{t_1-1} \mathbf{1}^T A_{\bar{S}S}(t)\mathbf{1} + \beta.$$

3. ***(Approximate stochasticity).** $\{A(t)\}_{t=0}^{\infty}$ is approximately stochastic and Δ is its deviation from stochasticity.*

We now obtain the desired sufficient conditions as a straightforward consequence of the above proposition.

Corollary 1 (Sufficient Conditions for Class $\mathcal{P}^*$). *Suppose $\{A(t)\}_{t=0}^{\infty} \subset \mathbb{R}^{n\times n}$ is a strongly aperiodic stochastic chain, i.e., suppose there exists a $\gamma > 0$ such that $A(t) \geq \gamma I$ for all $t \in \mathbb{N}_0$. If $\{A(t)\}_{t=0}^{\infty}$ is approximately reciprocal, then $\{A(t)\}_{t=0}^{\infty} \in \mathcal{P}^*$.*

As a direct consequence of Corollary 1 and Proposition 2, we obtain the following necessary and sufficient conditions for Class $\mathcal{P}^*$: a strongly aperiodic stochastic chain belongs to $\mathcal{P}^*$ iff it is approximately reciprocal. Since a stochastic chain belongs to Class $\mathcal{P}^*$ iff it has a uniformly positive absolute probability sequence, we obtain Theorem 1, whose statement is reproduced below.

Theorem 1 (An Analog of the Positive Eigenvector Assertion of the Perron-Frobenius Theorem). *Suppose $\{A(t)\}_{t=0}^{\infty} \subset \mathbb{R}^{n\times n}$ is a strongly aperiodic stochastic chain. Then $\{A(t)\}_{t=0}^{\infty}$ has a uniformly positive absolute probability sequence if and only if it is approximately reciprocal.*

Observe how Theorem 1 parallels the first assertion of Proposition 1: the original theorem asserts that for a static network that is reciprocal and whose infinite flow graph is connected (i.e., a network defined by an irreducible matrix), there exists a positive principal left eigenvector. Analogously, Theorem 1 asserts that for a dynamic network that is approximately reciprocal, there exists a uniformly positive absolute probability sequence.

Next, we extend the second assertion of Proposition 1 (the unique eigenvector assertion of Perron-Frobenius theorem). For this purpose, we will need the following lemmas and definitions.

Definition 15 (Ergodicity for Stochastic Chains [64]**).** *A stochastic chain $\{A(t)\}_{t=0}^{\infty} \in \mathbb{R}^{n\times n}$ is said to be ergodic if, for every $t_0 \in \mathbb{N}$, there exists a stochastic vector $\pi(t_0) \in \mathbb{R}^n$ such that*

$$\lim_{t\to\infty} A(t:t_0) = \mathbf{1}\pi^T(t_0).$$

To interpret the above definition, we first observe that in the distributed averaging dynamics $x(t+1) = A(t)x(t)$ with a starting time $t_0 \in \mathbb{N}_0$ and an initial condition $x(t_0) \in \mathbb{R}^n$, we have

$$x(t) = A(t:t_0)x(t_0) \quad \text{for all} \quad t \in \mathbb{N}_0 \tag{2.3}$$

For an ergodic chain, this means that $\lim_{t\to\infty} x(t) = \pi^T(t_0)x(t_0)\mathbf{1}$, which is a *consensus* vector (i.e., all its entries are equal). Therefore, a stochastic chain being ergodic means that it always enables consensus among the nodes of the network, regardless of the starting time t_0 and the starting point $x(t_0)$.

Definition 16 (Infinite Flow Stability [62]). *A stochastic chain $\{A(t)\}_{t=0}^{\infty}$ is said to be infinite flow stable if*

1. *The sequence $\{x(t)\}_{t=t_0}^{\infty}$, which evolves as $x(t+1) = A(t)x(t)$, converges to a limit for all starting times $t_0 \in \mathbb{N}_0$ and all initial conditions $x(t_0) \in \mathbb{R}^n$.*

2. $\lim_{t\to\infty}(x_i(t) - x_j(t)) = 0$ *for all $(i,j) \in E^{\infty}$, where E^{∞} is the edge set of the infinite flow graph of $\{A(t)\}_{t=0}^{\infty}$.*

Put simply, a stochastic chain is infinite flow stable if (a) the states of all the nodes of the corresponding time-varying network converge to a limit asymptotically in time, and (b) if a consensus is necessarily reached among nodes that exert a long-term influence on each other.

Lemma 5. *Suppose $G^{\infty} = ([n], E^{\infty})$, the infinite flow graph of $\{A(t)\}_{t=0}^{\infty}$, is connected. Then $\{A(t)\}_{t=0}^{\infty}$ is ergodic if it is infinite flow stable.*

We are now ready to extend the second assertion of Proposition 1 to dynamic stochastic chains. We thus reproduce the statement of Theorem 2 below.

Theorem 2 (An Analog of the Uniqueness Assertion of the Perron-Frobenius Theorem). *Let $\{A(t)\}_{t=0}^{\infty} \subset \mathbb{R}^{n\times n}$ be a strongly aperiodic stochastic chain that is also approximately reciprocal. Then, $\{A(t)\}_{t=0}^{\infty}$ admits a unique absolute probability sequence if and only if its infinite flow graph is connected.*

Like Theorem 1, Theorem 2 closely parallels an assertion of the Perron-Frobenius theorem. In view of Proposition 1, the original theorem asserts that, if a matrix describes a static network that is reciprocal and whose infinite flow graph is connected, then its principal left eigenvector is unique. Analogously, Theorem 2 asserts that, if a stochastic chain describes a time-varying network that is approximately reciprocal and whose infinite flow graph is connected, then its absolute probability sequence is unique.

Besides, it is worth noting that stochastic chains that are approximately reciprocal and whose infinite flow graphs are connected are a time-varying analog of irreducible matrices (this follows immediately from Lemma 2). Therefore, we shall henceforth use the term *irreducible chains* to refer to such chains.

2.4.2 Continuous Time

We now extend our discrete-time results (Theorems 1 and 2) to continua of row-stochastic matrices, henceforth called continuous-time stochastic chains. Consider the following continuous-time analog of the discrete-time dynamics (2.3).

$$\dot{x}(t) = A(t)x(t) \quad \text{for all } t \geq 0, \tag{2.4}$$

where $A(t) = -L(t)$ is the negative of the Laplacian matrix of a given graph $G(t)$. Throughout this section, we assume

$$\int_{t_1}^{t_2} a_{ij}(t)dt < \infty \quad \text{for all } 0 \leq t_1 < t_2 < \infty. \tag{2.5}$$

It is well-known [65, 66] that under Assumption (2.5), the solution to (2.4) is unique and can be expressed as

$$x(t) = \Phi(t,\tau)x(\tau) \quad \text{for all } t \geq \tau \geq 0, \tag{2.6}$$

where the *state-transition matrix* Φ is the unique solution to the equation continuum

$$\Phi(t,\tau) = I + \int_{\tau}^{t} A(\tau')\Phi(\tau',\tau)d\tau' \quad \text{for all } t \geq \tau \geq 0.$$

It is also known that $\Phi(t,\tau)$ is row-stochastic for all $t \geq \tau \geq 0$, $\Phi(\tau,\tau) = I$ for all $\tau \geq 0$, and

$$\Phi(t_2,t_1) = \Phi(t_2,\tau)\Phi(\tau,t_1) \quad \text{for all } t_2 \geq \tau \geq t_1 \geq 0. \tag{2.7}$$

Therefore, for any sequence of increasing times $\{t_k\}_{k=1}^{\infty} \subset \mathbb{R}_{\geq 0}$, if we let $B(k) := \Phi(t_{k+1},t_k)$ for all $k \in \mathbb{N}_0$, then we have $B(m:\ell) = \Phi(t_m : t_\ell)$ for all $\ell, m \in \mathbb{N}$ with $\ell \leq m$. As a result, an application of Proposition 3 to the stochastic chain $\{B(k)\}_{k=0}^{\infty}$ yields the following result.

Lemma 6. *Let $\Phi(\cdot,\cdot)$ denote the state transition matrix for the dynamics* (2.4) *under the assumption* (2.5)*. Consider now a sequence of increasing times $\{t_k\}_{k=0}^{\infty} \subset \mathbb{R}_{\geq 0}$ and a constant $\gamma > 0$ such that $\Phi(t_{k+1},t_k) \geq \gamma I$ for all $k \in \mathbb{N}_0$. If there exist constants $\tilde{p}_0, \tilde{\beta} \in (0,\infty)$ such that*

$$\tilde{p}_0 \sum_{k=\ell}^{m} \mathbf{1}^T \Phi_{S\bar{S}}(t_{k+1},t_k)\mathbf{1} \leq \sum_{k=\ell}^{m} \mathbf{1}^T \Phi_{\bar{S}S}(t_{k+1},t_k)\mathbf{1} + \tilde{\beta} \tag{2.8}$$

holds for all sets $S \subset [n]$ and for all $\ell, m \in \mathbb{N}_0$ with $\ell \leq m$, then there exists an $\eta > 0$ such that $\Phi(t_m,t_\ell) \geq \eta I$ for all $\ell, m \in \mathbb{N}_0$ satisfying $\ell \leq m$.

It is clear from Lemma 6 above and from Lemma 8 of [67] that the discrete-time chain $\{\Phi(t_{k+1},t_k)\}_{k=0}^{\infty}$ lies in Class $\mathcal{P}^*$ if the approximate reciprocity condition (2.8) and the strong aperiodicity condition $\Phi(t_{k+1},t_k) \geq \gamma I$ are satisfied. As we shall see shortly, the following assumptions ensure that both these conditions are met.

Assumption 1 (**Uniform Bound on Integral Weights** [65])**.** *There exists an $M < \infty$ and an increasing sequence $\{t_k\}_{k=0}^{\infty} \subset \mathbb{R}_{\geq 0}$ such that*

$$\int_{t_k}^{t_{k+1}} a_{ij}(t)dt \leq M$$

for all $k \in \mathbb{N}$ and all $i,j \in [n]$ with $i \neq j$.

Assumption 1 is sufficient to guarantee the strong aperiodicity condition $\Phi(t_{k+1},t_k) \geq \gamma I$ for some $\gamma > 0$ and all $k \in \mathbb{N}_0$. This is evident from the proof of Lemma 8 in [65].

Assumption 2 (Continuous-time Approximate Reciprocity). *There exist $p_0, \beta \in (0, \infty)$ such that*

$$p_0 \int_{t_\ell}^{t_m} \mathbf{1}^T A_{S\bar{S}}(t)\mathbf{1}dt \leq \int_{t_\ell}^{t_m} \mathbf{1}^T A_{\bar{S}S}(t)\mathbf{1}dt + \beta$$

holds for all sets $S \subset [n]$ and for all $\ell, m \in \mathbb{N}_0$ with $\ell \leq m$.

We now establish the required equivalence.

Lemma 7. *Under Assumption 1, Assumption 2 is equivalent to the existence of constants $\tilde{p}_0, \tilde{\beta} \in (0, \infty)$ such that* (2.8) *holds for all sets $S \subset [n]$.*

We now use Lemma 7 to show that approximate reciprocity in continuous time is equivalent to $\{A(t)\}_{t=0}^{\infty}$ belonging to Class $\mathcal{P}^*$. To begin, we first define the continuous-time analogs of absolute probability sequences and Class $\mathcal{P}^*$.

Definition 17 (Continuous-time Absolute Probability Sequence [67]**).** *A continuum of stochastic vectors $\{\pi(t)\}_{t\geq 0}$ is said to be an absolute probability sequence for a continuous-time stochastic chain $\{A(t)\}_{t\geq 0}$ if*

$$\pi^T(t)\Phi(t, \tau) = \pi^T(\tau)$$

holds for all $t \geq \tau \geq 0$, where $\Phi(\cdot, \cdot)$ denotes the state transition matrix for the dynamics (2.4)*.*

Definition 18 (Continuous-time Class $\mathcal{P}^*$ [67]**).** *We let continuous-time Class $\mathcal{P}^*$ be the set of all continuous-time stochastic chains that admit uniformly positive absolute probability sequences, i.e., a continuum of stochastic vectors $\{\pi(t)\}_{t\geq 0}$ such that $\pi(t) \geq p^*\mathbf{1}$ for some scalar $p^* > 0$ and all $t \in \mathbb{N}_0$. (Note that the absolute probability sequence and the value of p^* may vary from chain to chain).*

We are now ready to state the first main result of this section. It is Theorem 3, whose statement is reproduced below.

Theorem 3 (Continuous-time Analog of Theorem 1). *Let $\{A(t)\}_{t\geq 0}$ be a continuous-time* *stochastic chain that satisfies Assumption 1. Then* $\{A(t)\}_{t\geq 0}$ *has a uniformly positive absolute probability sequence if and only if it is approximately reciprocal.*

The next step is to provide a continuous-time analog of Theorem 2. For this purpose, we define the continuous-time analog of the infinite flow graph as follows.

Definition 19 (Infinite Flow Graph in Continuous Time). *For a continuous-time stochastic chain* $\{A(t)\}_{t\geq 0}$*, we define its* infinite flow graph *to be the graph* $G^\infty = ([n], E^\infty)$ *with*

$$E^\infty := \left\{ \{i,j\} \subset [n] \,\middle|\, \int_0^\infty (a_{ij}(t) + a_{ji}(t))\, dt = \infty, i \neq j \in [m] \right\}.$$

We now reproduce the statement of the desired theorem (Theorem 4).

Theorem 4 (Continuous-time Analog of Theorem 2). *Let* $\{A(t)\}_{t\geq 0}$ *be an approximately reciprocal continuous-time stochastic chain that satisfies Assumption 1 (uniform bound on integral weights). Then* $\{A(t)\}_{t\geq 0}$ *admits a unique absolute probability sequence if and only if its infinite flow graph is connected.*

To interpret Theorems 1 - 4, we provide below a series of remarks in which we start from some existing interpretations of Proposition 1, which is based on the classical theorem, and we extend these interpretations to the case of time-varying networks.

Remark 1 (Implications for Markov Chains). *The eigenvector assertions of the Perron-Frobenius theorem can be interpreted as follows: for a time-homogeneous Markov chain with transition probabilities given by an aperiodic irreducible matrix, the probability of visiting any given state converges asymptotically in time to a unique positive value, regardless of the initial probability distribution. Analogously, Theorems 1 and 2 can be interpreted as follows: given a starting time, for a backward-propagating time-non-homogeneous Markov chain with transition probabilities given by strongly aperiodic irreducible chain, the probability of visiting any given state converges asymptotically in time to a unique positive value, regardless of the* *initial probability distribution. Although this limiting probability is a function of the starting time, it is bounded away from zero by a fixed threshold that does not depend on the starting time.*

Remark 2 (**Opinion Dynamics-Based Interpretation**). *In the context of opinion dynamics, the matrix $A(t)$ can be interpreted as the influence matrix at time t, i.e., $a_{ij}(t)$ quantifies the extent to which agent i values agent j's opinion at time t (or equivalently, the extent to which agent j influences agent i at time t). Therefore, an irreducible chain (and hence also an irreducible matrix) describes a network in which every subset of agents influences the complementary subset persistently over the entire course of opinion evolution, which means that there exists no group of elite agents that dominate others forever. Additionally, as mentioned before, absolute probability sequences can be interpreted as quantifying the agents' social powers.*

Therefore, an interpretation of the eigenvector assertions of the original theorem is as follows: in a static social network, the social power of every agent (given by the eigenvector centrality of the corresponding network node) is unique and positive if no subset of agents dominates other agents forever. Analogously, Theorems 1- 2 can be interpreted as follows: in a time-varying social network, the time-varying social power of every agent (given by the Kolmogorov centrality of the corresponding network node) is unique and uniformly positive (lower-bounded by a constant positive threshold) if no subset of agents dominates other agents forever.

Remark 3 (**Implications for Economic Growth**). *Here, we follow the approach taken in [68]. Consider an economy with n sectors of activity, and let x_i denote the activity level in the i-th sector. Then the evolution of activity levels may be expressed as $x(t+1) = A(t)x(t)$, wherein $a_{ij}(t)$ quantifies the number of activity units in Sector i that are required in the next economic cycle as a result of the completion of each activity unit in Sector j. In addition, if $A(t)$ is a static matrix, then its principal left eigenvector gives the long-term economic value of the activity carried out in Sector i during the zeroth time period $(t = 0)$ relative to the long-term values of the activities carried out in other sectors during the zeroth time period.*

Therefore, the original theorem now has the following interpretation: if the activity evolution matrix is static and exhibits the properties of reciprocity and connectivity (via irreducibility), then the long-term economic value of the initial activity in any given sector relative to initial activities in other sectors will be positive and can be determined uniquely. Analogously, our main results imply that, if the activity evolution matrix is time-varying and exhibits the properties of approximate reciprocity and connectivity of the infinite flow graph, then, for any starting time period $t = t_0$ (as opposed to $t = 0$), the long-term economic value of initial activity in any given sector relative to initial activities in other sectors will be uniformly positive and can be uniquely determined.

Remark 4 (**Implications for Population Dynamics**). *Analogous to the economic growth model, we may consider a population comprised of n age groups instead of an economy comprised of n sectors. We may let x_i denote the population size of the i-th age group and further let $a_{ij}(t)$ quantify the fraction of individuals that transition from the j-th age group to the i-th age group in the t-th time period for $i = j + 1$. In addition, we may let $a_{1j}(t)$ quantify the number of births per parent in the j-th age group. In this setting, the entry $\pi_i(t_0)$ of the absolute probability sequence denotes the sensitivity of the long-term total population size to the size that the i-th age group has at time t_0. One may then interpret the original theorem and our results in this context by drawing analogies with the economic growth context discussed above.*

2.5 Applications

We now derive a few corollaries of our main results. Since some of these corollaries apply to random stochastic chains, we first define the relevant terms.

2.5.1 Infinite Flow Stability of Independent Random Chains

The concept of independent random chains is a straightforward extension of the concept of deterministic chains, as per the definition below.

Definition 20 (Independent Random Chain). *A discrete-time stochastic chain* $\{A(t)\}_{t=0}^{\infty}$ *is called an independent random chain if* $\{A(t)\}_{t=0}^{\infty}$ *are all random and independently distributed.*

We now extend the notion of Class $\mathcal{P}^*$ to independent random chains.

Definition 21 (Class $\mathcal{P}^*$ for Independent Random Chains [62]**).** *An independent random chain* $\{A(t)\}_{t=0}^{\infty}$ *is said to belong to Class* $\mathcal{P}^*$ *if the expected chain* $\{\mathbb{E}[A(t)]\}_{t=0}^{\infty}$ *belongs to Class* $\mathcal{P}^*$.

We will also need the notion of *feedback property*, a weak notion of strong aperiodicity for independent random chains.

Definition 22 (Feedback Property for Independent Random Chains [62]). *Let* $\{A(t)\}_{t=0}^{\infty}$ *be an independent random chain. We say that* $\{A(t)\}_{t=0}^{\infty}$ *has the feedback property if there exists a* $\gamma > 0$ *such that* $\mathbb{E}[a_{ii}(t)a_{ij}(t)] \geq \gamma\mathbb{E}[a_{ij}(t)]$ *for all* $t \in \mathbb{N}_0$ *and all distinct* $i, j \in [n]$*, in which case* γ *is called the feedback coefficient.*

In addition, we will use the term *mutual ergodicity*, which we define below.

Definition 23 (Mutual Ergodicity [62]**).** *Let* $\{A(t)\}_{t=0}^{\infty}$ *be a (deterministic or random) stochastic chain. We say that* $i \in [n]$ *and* $j \in [n]$ *are mutually ergodic indices for* $\{A(t)\}_{t=0}^{\infty}$*, which we denote by* $i \leftrightarrow_A j$*, if* $\lim_{t\to\infty}(x_i(t) - x_j(t)) = 0$ *for the dynamics* $x(t+1) = A(t)x(t)$ *started with an arbitrary initial condition* $x(t_0) = x_0$ *(where* $t_0 \in \mathbb{N}_0$ *and* $x_0 \in \mathbb{R}^n$*).*

Based on these concepts, we have the following result.

Corollary 2. *Let* $\{A(t)\}_{t=0}^{\infty}$ *be an independent random chain with feedback property, and suppose the expected chain* $\{\bar{A}(t)\}_{t=0}^{\infty} := \{\mathbb{E}[A(t)]\}_{t=0}^{\infty}$ *is approximately reciprocal. Then,*

(i) $\{A(t)\}_{t=0}^{\infty}$ *is infinite flow stable almost surely (i.e., almost every realization of* $\{A(t)\}_{t=0}^{\infty}$ *is infinite flow stable).*

(ii) For any two indices i *and* j *in* $[n]$*, we have* $i \leftrightarrow_A j$ *iff* $i \leftrightarrow_{\bar{A}} j$*.*

(iii) i and j belong to the same connected component of G^{∞} iff i and j belong to the same connected component of $\bar{G}^{\infty}$, the infinite flow graph of $\{\bar{A}(t)\}_{t=0}^{\infty}$.

Remark 5. *By the definition of infinite flow stability, Assertion (i) of Corollary 2 implies that if $\{A(t)\}_{t=0}^{\infty}$ is an independent random chain with feedback property, then $\lim_{t\to\infty} A(t:t_0)$ exists almost surely for all $t_0 \in \mathbb{N}_0$. In conjunction with Assertions (ii) and (iii), this further implies that, for any two indices $i, j \in [n]$ and an arbitrary starting time $t_0 \in \mathbb{N}_0$, the event that the i-th row of $\lim_{t\to\infty} A(t:t_0)$ equals the j-th row of $\lim_{t\to\infty} A(t:t_0)$ almost surely equals the event that i and j belong to the same connected component of the infinite flow graph of the expected chain $\{\bar{A}(t)\}_{t=0}^{\infty}$.*

2.5.2 Rate of Convergence to Steady State

We now provide a result on the rate of convergence for the dynamics $x(t+1) = A(t)x(t)$ in terms of the quadratic comparison function $V_u(x) = \sum_{i=1}^{m} u_i \left(x_i - u^T x\right)^2$, where u is an arbitrary stochastic vector in $\mathbb{R}^n$.

Corollary 3. *Let $\{A(t)\}_{t=0}^{\infty}$ be an independent random chain with feedback property and feedback coefficient $\gamma > 0$, and suppose the expected chain $\{\bar{A}(t)\}_{t=0}^{\infty}$ is approximately reciprocal. In addition, let $t_q = 0$ for $q = 0$ and let*

$$t_q = \operatorname*{argmin}_{t \geq t_{q-1}+1} \Pr\left(\min_{S\subset[m]} \sum_{t=t_{q-1}}^{t-1} \mathbf{1}^T A_S(t)\mathbf{1} \geq \delta\right) \geq \varepsilon$$

for all $q \geq 1$. Then, for all $q \geq 1$ and all stochastic vectors $u \in \mathbb{R}^n$,

$$\mathbb{E}\left[V_u\left(x\left(t_q\right), t_q\right)\right] \leq \left(1 - \frac{\varepsilon\delta(1-\delta)^2\gamma p^*}{(m-1)^2}\right)^q \mathbb{E}[V_u(x(0), 0)].$$

2.5.3 Implications for Infinite Jet-Flow and Sonin's Jet Decomposition

For a stochastic chain to be ergodic, it is necessary for the chain to possess a property called the *infinite jet-flow* property [69]. In this subsection, our aim is to connect the concept of approximate reciprocity with the infinite jet-flow property and also with the related concept of Sonin's jet decomposition [69, 70]. We first reproduce the required definitions from [69].

Definition 24 (Jet). *For a given set $S \subset [n]$, a jet J in S is a sequence $\{J(t)\}_{t=0}^{\infty}$ of subsets of S. A jet J in S is called proper if $\emptyset \subsetneq J(t) \subsetneq S$ for all $t \in \mathbb{N}_0$. In addition, for a jet J, the jet limit J^* denotes $\lim_{t\to\infty} J(t)$ if it exists, in the sense that the sequence becomes constant after a finite time.*

The following definition quantifies the interaction between two jets.

Definition 25 (Total Interaction between Jets). *For a given stochastic chain $\{A(t)\}_{t=0}^{\infty}$ and any two disjoint jets J^u and J^v in S, the total interaction between the two jets over the time interval $[0, \infty)$, denoted by $U_A(J^u, J^v)$, is defined by*

$$U_A\left(J^u, J^v\right) = \sum_{t=0}^{\infty} \left[\sum_{i \in J^u(t+1)} \sum_{j \in J^v(t)} a_{ij}(t) + \sum_{i \in J^v(t+1)} \sum_{j \in J^u(t)} a_{ij}(t) \right].$$

The next definition captures the idea of long-term non-vanishing interaction between two jets.

Definition 26 (Infinite Jet Flow Property). *A stochastic chain $\{A(t)\}_{t=0}^{\infty}$ is said to have the infinite jet-flow property over a subset S of $[n]$ if, for every proper jet $\{J(t)\}_{t=0}^{\infty}$ in S, we have*

$$U_A(\{J(t)\}_{t=0}^{\infty}, \{[n] \setminus J(t)\}_{t=0}^{\infty}) = \infty.$$

Finally, we reproduce the definition of a weak notion of ergodicity.

Definition 27 (Class Ergodicity). *A stochastic chain $\{A(t)\}_{t=0}^{\infty}$ is called class-ergodic if $\lim_{t\to\infty} A(t:t_0)$ exists for all $t_0 \in \mathbb{N}_0$, in which case $[n]$ can be partitioned into ergodic classes, whereby $i, j \in [n]$ belong to the same ergodic class if $\lim_{t\to\infty}((A(t:t_0))_{ik} - (A(t:t_0))_{jk}) = 0$ for all $k \in [n]$.*

We now have the following result, the second assertion of which is a consequence of Theorem 5.1 in [62] and of Theorem 1 above.

Corollary 4. *Let $\{A(t)\}_{t=0}^{\infty}$ be a strongly aperiodic and an approximately reciprocal stochastic chain. Then*

1. *The infinite jet-flow property holds over each connected component of G^{∞}, the infinite flow graph of $\{A(t)\}_{t=0}^{\infty}$.*

2. *The chain $\{A(t)\}_{t=0}^{\infty}$ is class-ergodic and the connected components of G^{∞} are the ergodic classes of $\{A(t)\}_{t=0}^{\infty}$.*

3. *The connected components of G^{∞} constitute the jet limits in Sonin's jet decomposition [70] of $\{A(t)\}_{t=0}^{\infty}$. These limits are attained in finite time.*

2.5.4 Some Other Applications

We now briefly discuss a few other applications of our main results.

1. *Multiple Consensus:* We say that multiple consensus [71] occurs whenever $\lim_{t\to\infty} x(t)$ exists but is not necessarily a multiple of the consensus vector $\mathbf{1}$, meaning that different entries of $x(t)$ may or may not converge to different limits. An immediate consequence of Theorem 3 above and Theorem 2 of [71] is that multiple consensus always occurs in the continuous-time dynamics $\dot{x}(t) = A(t)x(t)$ if $\{A(t)\}_{t\geq 0}$ is an approximately reciprocal chain that satisfies Assumption 1.

2. *Éminence Grise Coalitions:* In essence, an éminence grise coalition (EGC, [67]) is a subset of the total agent population that has the ability to steer the opinions of all the individuals in the network to a desired consensus asymptotically in time. A direct consequence of Theorem 3 above and Corollary 3 of [67] is as follows: if $\{A(t)\}_{t\geq 0}$ is an approximately reciprocal chain satisfying Assumption 1, then the size of a minimal EGC of a network with dynamics $\dot{x}(t) = A(t)x(t)$ is the number of connected components in the infinite flow graph of $\{A(t)\}_{t\geq 0}$.

3. *Distributed Optimization:* A typical distributed optimization framework consists of a network of n interacting agents with the common objective of minimizing the sum of n convex functions $\{f_i : \mathbb{R}^d \to \mathbb{R}^d\}_{i=1}^n$ subject to the constraint that for each $i \in [n]$, the function f_i is known only to agent i. Notably, [72] provides a continuous-time algorithm for distributed optimization without requiring the associated stochastic chain $\{A(t)\}_{t\geq 0}$ to be *cut-balanced* [62]. However, the results therein are based on an assumption involving an abstract concept called *Class $\mathcal{P}^*$ flows*, the interpretation of which is aided significantly by results such as Theorem 3.

4. *Distributed Learning/Hypothesis Testing:* In a typical distributed learning scenario, there is a set of possible states of the world, of which a subset of states are true. In addition, there is a network of interacting agents whose common objective is to learn the identity of the true state through mutual interaction as well as by performing private measurements on the state of the world. We note that [40] generalizes certain known results on distributed learning to networks described by random, independently distributed time-varying directed graphs. Importantly, the sequence of weighted adjacency matrices of all the networks considered therein are assumed to belong to Class $\mathcal{P}^*$. Hence, along with Definitions 20 and 21, Theorem 1 significantly facilitates our interpretation of the main results of [40].

2.6 Conclusion

We extended two eigenvector assertions of the classical Perron-Frobenius theorem to sequences as well as continua of row-stochastic matrices that satisfy the mild assumption of strong aperiodicity. In the process, we established approximate reciprocity as an equivalent characterization of Class $\mathcal{P}^*$, a special but broad class of stochastic chains that subsumes a few important sub-classes such as cut-balanced (reciprocal) chains, doubly stochastic chains, and uniformly strongly connected chains [62]. We then discussed a few applications of our main results to problems in distributed learning, averaging with strongly quasi-non-expansive maps, etc.

In future, we would like to examine whether it is possible to weaken the assumption of strong aperiodicity while retaining the essence of our main results. We would also like to extend our results to dependent random chains, as such chains are pivotal to the study of real-worlds networks subjected to random phenomena such as communication link failures. As yet another future direction, it would be interesting to attempt an extension of our results to sequences of non-negative matrices that are not necessarily row-stochastic. Such an extension would result in a complete generalization of the eigenvector assertions of the Perron-Frobenius theorem to the case of time-varying matrices.

Nevertheless, we believe that our main results, even in their present form, have the potential to find a significant number of applications other than those discussed above. This belief is rooted in the already wide applicability of the classical theorem.

Appendix

Proof of Lemma 2

Proof. Since $\{A(t)\}_{t=0}^{\infty}$ is a static chain, for any two distinct nodes $i,j \in [n]$, there exists an edge between i and j in G^{∞} if and only if $\sum_{t=0}^{\infty}(a_{ij}(t)+a_{ji}(t)) = \infty$, which holds if and only if $(A_0)_{ij} + (A_0)_{ji} > 0$. Hence, $(i,j) \in E^{\infty}$ iff either $(A_0)_{ij} > 0$ or $(A_0)_{ji} > 0$.

Suppose now that G^{∞} is connected. It follows from the above observation that for every subset of nodes $S \subset [n]$, there exists a pair of nodes $(i,j) \in (S \times \bar{S}) \bigcup (\bar{S} \times S)$ such that $(A_0)_{ij} > 0$. Suppose w.l.o.g. that $(i,j) \in S \times \bar{S}$. If we further assume that $\{A(t)\}_{t=0}^{\infty}$ is reciprocal, this implies that $\mathbf{1}^T A_{\bar{S}S}(t)\mathbf{1} \geq \alpha \mathbf{1}^T A_{S\bar{S}}(t)\mathbf{1} \geq \alpha (A_0)_{ij} > 0$. Hence, there exists a node pair $(p,q) \in \bar{S} \times S$ such that $(A_0)_{pq} > 0$. We have thus shown that for every subset $S \subset [n]$, there exist directed edges from S to $\bar{S}$ as well as from $\bar{S}$ to S in $\mathcal{G}(A_0)$. Therefore, $\mathcal{G}(A_0)$ is strongly connected. It now follows from Lemma 1 that A_0 is irreducible.

On the other hand, suppose we are given that A_0 is irreducible. As a result, $\mathcal{G}(A_0)$ is strongly connected (by Lemma 1). This means that G^{∞} is connected, because the preceding paragraphs argue that $(i,j) \in E^\infty$ if either $(i,j) \in \mathcal{E}(A_0)$ or $(j,i) \in \mathcal{E}(A_0)$. Moreover, $\mathcal{G}(A_0)$ being strongly connected also implies that for every $S \subset [n]$, there exist two pairs of nodes $(i,j) \in S \times \bar{S}$ and $(p,q) \in \bar{S} \times S$ such that $a_{ij}(t) = (A_0)_{ij} > 0$ and $a_{pq}(t) = (A_0)_{pq} > 0$. Therefore, by letting

$$\alpha := \frac{\min_{\ell,m \in [n]:(A_0)_{\ell,m} > 0} (A_0)_{\ell,m}}{n}$$

and by using the fact that $\mathbf{1}^T A_0 \mathbf{1} = \mathbf{1}^T \mathbf{1} = n$, we can easily verify that the inequality given by $\mathbf{1}^T A_{S\bar{S}}(t)\mathbf{1} \geq \alpha \mathbf{1}^T A_{\bar{S}S}(t)\mathbf{1}$ holds for all $t \in \mathbb{N}_0$. Hence, $\{A(t)\}_{t=0}^\infty$ is reciprocal. □

Proof of Proposition 2

Proof. Consider any set $S \subset [n]$, and let $\bar{S} := [n] \setminus S$. Then, there exists a permutation matrix Q such that

$$Q^T A(t) Q = \begin{bmatrix} A_S(t) & A_{S\bar{S}}(t) \\ A_{\bar{S}S}(t) & A_{\bar{S}}(t) \end{bmatrix}$$

for all $t \in \mathbb{N}_0$. Let $\{\pi(t)\}_{t=0}^\infty$ denote an absolute probability sequence for $\{A(t)\}_{t=0}^\infty$. Then one may verify that the corresponding absolute probability sequence for $\{Q^T A(t) Q\}_{t=0}^\infty$ is given by $\{\tilde{\pi}(t)\}_{t=0}^\infty$, where

$$\tilde{\pi}(t) := \begin{bmatrix} \pi_S(t) \\ \pi_{\bar{S}}(t) \end{bmatrix}$$

for all $t \in \mathbb{N}_0$. As a result, the following holds for all $t \in \mathbb{N}_0$.

$$\begin{bmatrix} \pi_S^T(t+1) & \pi_{\bar{S}}^T(t+1) \end{bmatrix} \begin{bmatrix} A_S(t) & A_{S\bar{S}}(t) \\ A_{\bar{S}S}(t) & A_{\bar{S}}(t) \end{bmatrix} = \begin{bmatrix} \pi_S^T(t) & \pi_{\bar{S}}^T(t). \end{bmatrix}$$

The above equation is essentially a pair of two vector equations, the second of which is

$$\pi_S^T(t+1) A_{S\bar{S}}(t) + \pi_{\bar{S}}^T(t+1) A_{\bar{S}}(t) = \pi_{\bar{S}}^T(t).$$

Multiplying each side of this equation by the all-ones vector yields

$$\pi_S^T(t+1)A_{S\bar{S}}(t)\mathbf{1} + \pi_{\bar{S}}^T(t+1)A_{\bar{S}}(t)\mathbf{1} = \pi_{\bar{S}}^T(t)\mathbf{1}. \tag{2.9}$$

On the other hand, the row-stochasticity of $A(t)$ implies that

$$A_{\bar{S}}(t)\mathbf{1} = \mathbf{1} - A_{\bar{S}S}(t)\mathbf{1}. \tag{2.10}$$

Combining (2.9) and (2.10) gives us

$$\pi_S^T(t+1)A_{S\bar{S}}(t)\mathbf{1} + \pi_{\bar{S}}^T(t+1)(\mathbf{1} - A_{\bar{S}S}(t)\mathbf{1}) = \pi_{\bar{S}}^T(t)\mathbf{1}. \tag{2.11}$$

On transposition, we obtain

$$\pi_S^T(t+1)A_{S\bar{S}}(t)\mathbf{1} = \left(\pi_{\bar{S}}^T(t) - \pi_{\bar{S}}^T(t+1)\right)\mathbf{1} + \pi_{\bar{S}}^T(t+1)A_{\bar{S}S}(t)\mathbf{1}$$

Since $\{A(t)\}_{t=0}^{\infty} \in \mathcal{P}^*$, there exists a $p^* > 0$ such that $\pi_S(t+1) \geq p^*\mathbf{1}$. Therefore,

$$\begin{aligned} p^*\mathbf{1}^T A_{S\bar{S}}(t)\mathbf{1} &\leq \left(\pi_{\bar{S}}^T(t) - \pi_{\bar{S}}^T(t+1)\right)\mathbf{1} + \pi_{\bar{S}}^T(t+1)A_{\bar{S}S}(t)\mathbf{1} \\ &\leq \left(\pi_{\bar{S}}^T(t) - \pi_{\bar{S}}^T(t+1)\right)\mathbf{1} + \mathbf{1}^T A_{\bar{S}S}(t)\mathbf{1}. \end{aligned} \tag{2.12}$$

Now, let $k_0, k_1 \in \mathbb{N}$ be any two numbers such that $k_0 < k_1$. Then, summing both the sides of (2.12) over the range $t \in \{k_0, k_0+1, \ldots, k_1-1\}$ yields

$$p^* \sum_{t=k_0}^{k_1-1} \mathbf{1}^T A_{S\bar{S}}(t)\mathbf{1} \leq \left(\pi_{\bar{S}}^T(k_0) - \pi_{\bar{S}}^T(k_1)\right)\mathbf{1} + \sum_{t=k_0}^{k_1-1} \mathbf{1}^T A_{\bar{S}S}(t)\mathbf{1},$$

where we have used a telescoping sum on the right hand side.

Since $\left(\pi_{\bar{S}}^T(k_0) - \pi_{\bar{S}}^T(k_1)\right)\mathbf{1} \le \pi_{\bar{S}}^T(k_0)\mathbf{1} \le \pi^T(k_0)\mathbf{1} = 1$, the above implies that

$$p^* \sum_{t=k_0}^{k_1-1} \mathbf{1}^T A_{S\bar{S}}(t)\mathbf{1} \le 1 + \sum_{t=k_0}^{k_1-1} \mathbf{1}^T A_{\bar{S}S}(t)\mathbf{1},$$

which is the same as (2.2). □

Proof of Lemma 3

Proof. Let $f : [0, 1-\varepsilon] \to \mathbb{R}$ be defined by $f(x) = 1 - x - e^{-M(\varepsilon)x}$. Then $f(0) = 0$. Next, note that $f''(x) = -M(\varepsilon)^2 e^{-M(\varepsilon)x} < 0$ for all $x \in [0, 1-\varepsilon]$, implying that f is concave on its domain. Also, observe that $f(1-\varepsilon) = 0$. Therefore, by Jensen's inequality, for any $x \in [0, 1-\varepsilon]$, we have

$$f(x) = f\left(\frac{x}{1-\varepsilon}(1-\varepsilon) + \left(1 - \frac{x}{1-\varepsilon}\right)\cdot 0\right) \ge \frac{x}{1-\varepsilon} f(1-\varepsilon) + \left(1 - \frac{x}{1-\varepsilon}\right) f(0) = 0.$$

□

Proof of Lemma 4

Proof. We define $N = |\{k \in \{0, \ldots, \sigma - 1\} : B_{ji}(k) > 0\}|$ and use induction on N. For $N = 1$, we have $B_{ji}(k_L) \ge \delta$ and hence

$$B_{ji}(\sigma : 0) \ge B_{jj}(\sigma : k_L + 1) B_{ji}(k_L) B_{ii}(k_L : 0) \ge \eta_j \delta \eta_i, \tag{2.13}$$

which verifies the lemma.

Now, suppose the lemma holds when $N = N_0$ for some $N_0 \in \mathbb{N}$, and consider $N = N_0 + 1$. We define $\varepsilon := B_{ji}(k_L)$, and consider two cases.

If $\varepsilon \ge \delta$, i.e., $B_{ji}(k_L) \ge \delta$, then (2.13) still holds, thereby proving the lemma.

On the other hand, if $\varepsilon < \delta$, then we let $\tilde{B}(k) := B(k)$ for each $k \in \{0, \ldots, \sigma - 1\} \setminus \{k_L\}$, and

$$\tilde{B}(k_L) := B(k_L) - B_{ji}(k_L) e_j e_i^T.$$

In other words $\tilde{B}(k_L)$ is obtained from $B(k_L)$ by setting its $(j,i)^{\text{th}}$ element to zero. Therefore, $\{\tilde{B}(k)\}_{k=0}^{\sigma-1}$ is a sequence of substochastic matrices satisfying

$$|\{k \in \{0,\ldots,\sigma-1\} : \tilde{B}_{ji}(k) > 0\}| = N_0.$$

Next, we have $\tilde{B}_{ii}(k_1 : k_0) = B_{ii}(k_1 : k_0) \geq \eta_i$ whenever $0 \leq k_0 \leq k_1 \leq k_L$. Since the definitions of k_L and $\{\tilde{B}(k)\}_{k=0}^{\sigma-1}$ imply that $\tilde{k}_L := \max\{k \leq \sigma-1 : \tilde{B}_{ij}(k) > 0\} < k_L$, it follows that $\tilde{B}_{ii}(k_1 : k_0) \geq \eta_i$ whenever $0 \leq k_0 \leq k_1 \leq \tilde{k}_L$. Next, note that for all k_0, k_1 satisfying $0 \leq k_0 \leq k_1 \leq \sigma$ and $\{k_0,\ldots,k_1-1\} \not\ni k_L$, we have $\tilde{B}_{jj}(k_1 : k_0) = B_{jj}(k_1 : k_0) \geq \eta_j$ whereas for all k_0, k_1 satisfying $0 \leq k_0 \leq k_L < k_1 \leq \sigma$, we have

$$\tilde{B}_{jj}(k_1 : k_0) \geq B_{jj}(k_1 : k_0) - B_{jj}(k_1 : k_L+1)B_{ji}(k_L)B_{ij}(k_L : k_0) \geq \eta_j - \varepsilon$$

because the substochasticity of $\{B(k)\}$ implies that $\max\{B_{jj}(k_1 : k_L+1), B_{ij}(k_L : k_0)\} \leq 1$. Moreover, $\sum_{k=0}^{\sigma-1} \tilde{B}_{ji}(k) = \sum_{k=0}^{\sigma-1} \tilde{B}_{ji}(k) - B_{ij}(k_L) \geq \delta - \varepsilon$. Thus,

$$\begin{aligned}
&\tilde{B}_{ii}(k_1 : k_0) \geq \eta_i \quad \text{if} \quad 0 \leq k_0 \leq k_1 \leq \tilde{k}_L, \\
&\tilde{B}_{jj}(k_1 : k_0) \geq \tilde{\eta}_j \quad \text{if} \quad 0 \leq k_0 \leq k_1 \leq \sigma, \text{ and} \\
&\sum_{k=0}^{\sigma-1} B_{ji}(k) \geq \tilde{\delta},
\end{aligned}$$

where $\tilde{\eta}_j := \eta_j - \varepsilon > \delta - \varepsilon > 0$ and $\tilde{\delta} := \delta - \varepsilon \in (0, \tilde{\eta}_j)$. Therefore, by our inductive hypothesis, we have $\tilde{B}_{ji}(\sigma : 0) \geq \frac{1}{2}\eta_i\tilde{\eta}_j\tilde{\delta} = \frac{1}{2}\eta_i(\eta_j - \varepsilon)(\delta - \varepsilon)$.

Now, observe that

$$\begin{aligned}
B_{ji}(\sigma : 0) &= \tilde{B}_{ji}(\sigma : 0) + B_{jj}(\sigma : k_L+1)B_{ji}(k_L)B_{ii}(k_L : 0) \\
&\geq \frac{1}{2}\eta_i(\eta_j - \varepsilon)(\delta - \varepsilon) + \eta_j\varepsilon\eta_i \\
&= \frac{1}{2}\eta_i\varepsilon^2 + \frac{1}{2}\eta_i(\eta_j - \delta)\varepsilon + \frac{1}{2}\eta_i\eta_j\delta \geq \frac{1}{2}\eta_i\eta_j\delta,
\end{aligned}$$

where the last inequality holds because $\varepsilon > 0$ and $\eta_j > \delta$. The lemma thus holds for $N = N_0 + 1$ and hence, for all $N \leq \sigma$. $\square$

Proof of Proposition 3

Proof. We use induction on n, the matrix dimension. Consider $n = 1$, suppose that $\gamma, p_0 \in (0,1), \beta \in (0,\infty)$ and $\Delta \in [0,\infty)$ are given, and let $\{A(k)\}_{k=0}^{\infty} = \{a(k)\}_{k=0}^{\infty}$ be a sequence of real numbers satisfying the three properties required by the proposition. Then, by the feedback property of the chain, $\{a_k\}_{k=0}^{\infty}$ is a sequence of scalars in $[\gamma, 1]$. Let $\bar{a}_k := 1 - a_k$ for each $k \in \mathbb{N}_0$. Then $\bar{a}_k \in [0, 1-\gamma]$ for all $k \in \mathbb{N}_0$, and $\sum_{k=0}^{\infty} \bar{a}_k \leq \Delta$ by almost-stochasticity. Hence, for any given $t_0, t_1 \in \mathbb{N}_0$ satisfying $t_0 \leq t_1$,

$$A(t_1 : t_0) = \prod_{k=t_0}^{t_1-1} (1 - \bar{a}_k) \overset{(a)}{\geq} \prod_{k=t_0}^{t_1-1} e^{-M(\gamma)\bar{a}_k} = e^{-M(\gamma)\sum_{k=t_0}^{t_1-1} \bar{a}_k} \geq e^{-M(\gamma)\sum_{k=0}^{\infty} \bar{a}_k} \geq e^{-M(\gamma)\Delta},$$

where (a) is a consequence of Lemma 3. Thus, we may set $\eta_1(\gamma, p_0, \beta, \Delta) = e^{-M(\gamma)\Delta}$. This proves the proposition for $n = 1$.

Now, suppose the proposition holds for all $n \leq q$ for some $q \geq 1$, and consider $n = q+1$. We again suppose that γ, p_0, β and Δ are given, and let $\{A(k)\}_{k=0}^{\infty} \subset \mathbb{R}^{n \times n}$ be a substochastic chain satisfying the required properties. For each $k \in \mathbb{N}_0$, let $v(k) := \mathbf{1} - A(k)\mathbf{1}$ and $v_{\max}(k) := \max_{i \in [n]} (v(k))_i$. Observe that the feedback property and the stub-stochasticity of $A(k)$ together imply that $\mathbf{0} \leq v(k) \leq (1-\gamma)\mathbf{1}$ for all $k \in \mathbb{N}_0$. We also observe that $A(k)\mathbf{1} \geq (1 - v_{\max}(k))\mathbf{1}$ for all $k \in \mathbb{N}_0$. Therefore, for all $0 \leq k_0 \leq k_1 < \infty$, we have

$$\begin{aligned} A(k_1 : k_0)\mathbf{1} = A(k_1-1)\cdots A(k_0+1)A(k_0)\mathbf{1} &\geq A(k_1-1)\cdots A(k_0+1)(1 - v_{\max}(k_0))\mathbf{1} \\ &\overset{(a)}{\geq} \left(\prod_{k=k_0}^{k_1-1} (1 - v_{\max}(k)) \right) \mathbf{1} \\ &\overset{(b)}{\geq} e^{-M(\gamma)\sum_{k=k_0}^{k_1-1} v_{\max}(k)} \mathbf{1} \\ &\geq e^{-M(\gamma)\sum_{k=k_0}^{k_1-1} \mathbf{1}^T v(k)} \overset{(c)}{\geq} e^{-M(\gamma)\Delta}\mathbf{1}, \end{aligned} \quad (2.14)$$

where (a) can be easily shown by induction, (b) is obtained by a repeated application of Lemma 3, and (c) follows from the almost-stochasticity of the chain.

We now construct two chains of substochastic matrices with dimensions smaller than n and then apply our inductive hypothesis to the resulting chains. To this end, let us use $\{\tau_0,\tau_1,\tau_2,\dots,\tau_n\}\subset\mathbb{N}\bigcup\{\infty\}$ to denote the set of times defined by $\tau_0=t_0$ and

$$\tau_l := \inf\left\{\tau\geq\tau_{l-1} : \min_{T\subset[n]}\sum_{k=\tau_{l-1}}^{\tau-1}\mathbf{1}^T A_{T\bar{T}}(k)\mathbf{1}\geq 1\right\}.$$

Further, let $m=\max\{s:\tau_s<\infty\}$ so that $\tau_s=\infty$ iff $s>m$.

Now, consider any $s\in\{0,1,\dots,\min\{m,n-1\}\}$. Then, by the definition of τ_{s+1}, there exists at least one set $T\subset[n]$ such that $\sum_{k=\tau_s}^{\tau_{s+1}-2}\mathbf{1}^T A_{T\bar{T}}(k)\mathbf{1}\leq 1$ (note that this also holds if $m\leq n-1$ and $s=m$, in which case $\tau_{s+1}-2=\infty$). We choose any one such set T and assume that $T=[|T|]$ w.lo.g.[1] We accordingly define the chains $\{B(k)\}_{k=\tau_s}^{\infty}$ and $\{C(k)\}_{k=\tau_s}^{\infty}$ as

$$B(k)=\begin{cases}A_T(k) & \text{if } \tau_s\leq k\leq\tau_{s+1}-1,\\ I_{|T|} & \text{otherwise,}\end{cases}$$

and

$$C(k)=\begin{cases}A_{\bar{T}}(k) & \text{if } \tau_s\leq k\leq\tau_{s+1}-1,\\ I_{|\bar{T}|} & \text{otherwise.}\end{cases}$$

Now, the definition of T implies that $\sum_{k=\tau_s}^{\tau_{s+1}-1}\mathbf{1}^T A_{T\bar{T}}(k)\mathbf{1}\leq 1+n\leq 2n$. Due to approximate reciprocity, it follows that $\sum_{k=\tau_s}^{\tau_{s+1}-1}\mathbf{1}^T A_{\bar{T}T}(k)\mathbf{1}\leq\frac{2n+\beta}{p_0}$. Note that $\sum_{k=\tau_s}^{\tau_{s+1}-1}\mathbf{1}^T A_{T\bar{T}}(k)\mathbf{1}\leq 2n$ also implies that

$$\sum_{k=\tau_s}^{\tau_{s+1}-1}\mathbf{1}^T(\mathbf{1}-A_T(k)\mathbf{1})=\sum_{k=\tau_s}^{\tau_{s+1}-1}\mathbf{1}^T(A_{T\bar{T}}(k)\mathbf{1}+v_T(k))\leq 2n+\Delta.$$

[1]If $T\neq[|T|]$, we can relabel the n coordinates so that $T=[|T|]$.

Similarly, the inequality $\sum_{k=\tau_s}^{\tau_{s+1}-1} \mathbf{1}^T A_{\bar{T}T}(k)\mathbf{1} \leq \frac{2n+\beta}{p_0}$ implies that

$$\sum_{k=\tau_s}^{\tau_{s+1}-1} \mathbf{1}^T(\mathbf{1} - A_{\bar{T}}(k)\mathbf{1}) \leq \frac{2n+\beta}{p_0} + \Delta.$$

Therefore, $\{A_T(k)\}_{k=\tau_s}^{\tau_{s+1}-1}$ and $\{A_{\bar{T}}(k)\}_{k=\tau_s}^{\tau_{s+1}-1}$ are both almost-stochastic sequences. Since I is a stochastic matrix, it follows that $\{B(k)\}_{k=\tau_s}^{\infty}$ and $\{C(k)\}_{k=\tau_s}^{\infty}$ are also almost-stochastic.

Next, for any subset $U \subset T$, let $\bar{U} := [n] \setminus U$ and $\tilde{U} := T \setminus U$. Then $\{A(k)\}_{k=0}^{\infty}$, being approximately reciprocal, satisfies

$$\begin{aligned}
p_0 \sum_{k=k_0}^{k_1-1} \mathbf{1}^T A_{\tilde{U}U}(k)\mathbf{1} &\leq p_0 \sum_{k=k_0}^{k_1-1} \mathbf{1}^T A_{\bar{U}U}(k)\mathbf{1} \\
&\leq \sum_{k=k_0}^{k_1-1} \mathbf{1}^T A_{U\bar{U}}(k)\mathbf{1} + \beta \\
&= \sum_{k=k_0}^{k_1-1} \mathbf{1}^T A_{U\tilde{U}}(k)\mathbf{1} + \sum_{k=k_0}^{k_1-1} \mathbf{1}^T A_{U\bar{T}}(k)\mathbf{1} + \beta \\
&\leq \sum_{k=k_0}^{k_1-1} \mathbf{1}^T A_{U\tilde{U}}(k)\mathbf{1} + \sum_{k=k_0}^{k_1-1} \mathbf{1}^T A_{T\bar{T}}(k)\mathbf{1} + \beta \leq \sum_{k=k_0}^{k_1-1} \mathbf{1}^T A_{U\tilde{U}}(k)\mathbf{1} + 2n + \beta
\end{aligned}$$

whenever $\tau_s \leq k_0 \leq k_1 \leq \tau_{s+1}$. Since $\mathbf{1}^T B_{U\tilde{U}}(k)\mathbf{1} = 0$ for all $U \subset T$ and $k \geq \tau_{s+1}$, it follows that

$$p_0 \sum_{k=k_0}^{k_1-1} \mathbf{1}^T B_{\tilde{U}U}(k)\mathbf{1} \leq \sum_{k=k_0}^{k_1-1} \mathbf{1}^T B_{U\tilde{U}}(k)\mathbf{1} + 2n + \beta$$

for all $\tau_s \leq k_0 \leq k_1 < \infty$. This shows that $\{B(k)\}_{k=\tau_s}^{\infty}$ is approximately reciprocal (though one of the associated constants is $\beta + 2n$ instead of β). We can similarly show that $\{C(k)\}_{k=0}^{\infty}$ is also approximately reciprocal. It can be easily seen that these two sequences also possess the feedback property. Hence, by our inductive hypothesis, there exist positive constants

$$\eta_B := \min_{r \in [n-1]} \eta_r(\gamma, p_0, \beta + 2n, \Delta + 2n) \text{ and } \eta_C := \min_{r \in [n-1]} \eta_r\left(\gamma, p_0, \beta + \frac{2n+\beta}{p_0}, \Delta + \frac{2n+\beta}{p_0}\right)$$

such that $B(k_1 : k_0) \geq \eta_B I$ and $C(k_1 : k_0) \geq \eta_C I$ for all $k_0, k_1 \in \mathbb{N}_0$ satisfying $\tau_s \leq k_0 \leq k_1 \leq$

τ_{s+1}. By noting that $A_T(k_1:k_0) \geq B(k_1:k_0)$ and $A_{\bar{T}}(k_1:k_0) \geq C(k_1:k_0)$, we observe that $A(k_1:k_0) \geq \eta_{\min} I$ for all $\tau_s \leq k_0 \leq k_1 \leq \tau_{s+1}$, where $\eta_{\min} := \min\{\eta_B, \eta_C\}$. Note that this is true for all $s \in \{0, \ldots, \min\{m, n-1\}\}$ and that the value of $\eta_{\min}$ is independent of s.

We now consider two cases.

Case 1: $m < n$**.** In this case, τ_{m+1} is defined and it equals ∞. Hence, there exists an $s \in \{0, 1, \ldots, m\}$ such that $\tau_s \leq t_1 \leq \tau_{s+1}$. Therefore,

$$A(t_1:t_0) = A(t_1:\tau_s) \cdot A(\tau_s:\tau_{s-1}) \cdots A(\tau_1:\tau_0) \geq \eta_{\min}^{s+1} I \geq \eta_{\min}^{n} I.$$

Case 2: $m = n$**.** In this case, $\tau_n < \infty$, so we either have $t_1 \leq \tau_n$ or $t_1 > \tau_n$.

If $t_1 \leq \tau_n$, then there exists an $s \in \{0, 1, \ldots, n-1\}$ such that $\tau_s \leq t_1 \leq \tau_{s+1}$. Hence, we can proceed as in Case 1. Otherwise, if $t_1 > \tau_n$, we need the following analysis.

For each $s \in \{0, 1, \ldots, n-1\}$, let $\mathcal{G}^{(s)}$ be the directed graph whose adjacency matrix $W^{(s)}$ is given by

$$W_{ij}^{(s)} = \begin{cases} 1, & \text{if } i \neq j \text{ and } \sum_{k=\tau_s}^{\tau_{s+1}-1} A_{ij}(k) \geq \frac{1}{n^2}, \\ 0, & \text{otherwise} \end{cases},$$

for all $i, j \in [n]$. We now claim that $\mathcal{G}^{(s)}$ is a strongly connected graph for each $s \in \{0, \ldots, n-1\}$. In order to prove this claim, suppose it is false for some $s \in \{0, \ldots, n-1\}$. Then, there exists a partition $\{T, \bar{T}\}$ of $[n]$ such that there is no directed link from any node in T to any node in $\bar{T}$ in $\mathcal{G}^{(s)}$. This implies that

$$\sum_{k=\tau_s}^{\tau_{s+1}-1} \mathbf{1}^T A_{\bar{T}T}(k) \mathbf{1} = \sum_{i \in \bar{T}} \sum_{j \in T} \sum_{k=\tau_s}^{\tau_{s+1}-1} A_{ij}(k) < |\bar{T}| \cdot |T| \cdot \frac{1}{n^2} \leq 1,$$

which contradicts the definitions of the times $\tau_0, \ldots, \tau_{n-1}$, thereby proving the claim. Since the weighted adjacency matrix of a strongly connected graph is irreducible, it follows that $W^{(s)}$ is an irreducible matrix for each $s \in \{0, \ldots, n-1\}$.

Thus, for any two indices $i, j \in [n]$, it follows that there exist $r \in [n]$, node indices

$l_0, l_1, \ldots, l_r \in [n-1]$ and time indices $0 \le s_1 \le s_2 \le \cdots \le s_r \le n-1$ such that $l_0 = i$, $l_r = j$, and $W^{(s_1)}_{il_1} = W^{(s_2)}_{l_1 l_2} = \cdots = W^{(s_r)}_{l_{r-1}l_r} = 1$. Moreover, from the definition of $W^{(s)}$, it further follows that

$$\sum_{k=\tau_{s_1}}^{\tau_{s_1+1}-1} A_{il_1}(k) \ge \frac{1}{n^2}, \sum_{k=\tau_{s_2}}^{\tau_{s_2+1}-1} A_{l_1 l_2}(k) \ge \frac{1}{n^2}, \ldots, \sum_{k=\tau_{s_r}}^{\tau_{s_r+1}-1} A_{l_{r-1}j}(k) \ge \frac{1}{n^2}. \tag{2.15}$$

Next, we bound $A_{l_{u-1}l_u}(\tau_{s_u+1} : \tau_{s_u})$ for all $u \in [r]$. On setting $\eta_i = \eta_j = \eta_{\min}$ and $\delta = \min\{\frac{1}{n^2}, \frac{\eta_{\min}}{2}\}$, and then applying Lemma 4 to the sequence $\{A(k)\}_{k=\tau_{s_u}}^{\tau_{s_u+1}}$, we obtain the following for each $u \in [r]$

$$A_{l_{u-1}l_u}(\tau_{s_u+1} : \tau_{s_u}) \ge \frac{1}{2}\eta_{\min}^2 \delta.$$

Now, for any $y \in [n]$ and any two indices $0 \le s < t \le n-1$, we have

$$A_{yy}(\tau_t : \tau_s) \ge \prod_{k=s}^{t-1} A_{yy}(\tau_{k+1} : \tau_k) \ge \prod_{k=s}^{t-1} \eta_{\min} \ge \eta_{\min}^n.$$

Thus,

$$\begin{aligned}
A_{ij}(\tau_n : t_0) &= A_{ij}(\tau_n : \tau_0) \\
&\ge A_{ii}(\tau_{s_1} : \tau_0) A_{i,l_1}(\tau_{s_1+1} : \tau_{s_1}) A_{l_1 l_1}(\tau_{s_2} : \tau_{s_1+1}) A_{l_1 l_2}(\tau_{s_2+1} : \tau_{s_2}) \cdots \\
&\quad \cdots A_{l_{r-1}l_{r-1}}(\tau_{s_r} : \tau_{s_r}) A_{l_{r-1}j}(\tau_{s_r+1}, \tau_{s_r}) A_{jj}(\tau_n : \tau_{s_r+1}) \\
&\ge \left(\eta_{\min}^n \cdot \frac{\eta_{\min}^2 \delta}{2}\right)^r \eta_{\min}^n \ge \eta_D > 0,
\end{aligned}$$

where $\eta_D := \left(\eta_{\min}^n \cdot \frac{\eta_{\min}^2 \delta}{2}\right)^n \eta_{\min}^n$.

Since $i, j \in [n]$ were arbitrary, we have shown that $A(\tau_n : t_0) \ge \eta_D \mathbf{1}\mathbf{1}^T$. From (2.14), it now follows that $A(t_1 : t_0) = A(t_1 : \tau_n) A(\tau_n : t_0) \ge \eta_D A(t_1 : \tau_n) \mathbf{1}\mathbf{1}^T \ge \eta_D e^{-M(\gamma)P} \mathbf{1}\mathbf{1}^T \ge \eta_D e^{-M(\gamma)\Delta} I$.

To summarize, in both Case 1 and Case 2, we have $A(t_1 : t_0) \geq \eta_F I$ where

$$\eta_F := \left(\eta_{\min}^n \cdot \frac{\eta_{\min}^2}{2} \cdot \min\left\{\frac{1}{n^2}, \frac{\eta_{\min}}{2}\right\}\right)^n \eta_{\min}^n e^{-M(\gamma)\Delta} > 0.$$

Since η_F is uniquely determined by γ, p_0, β and Δ, it follows that we can define the function $\eta_n : (0,1) \times (0,1) \times (0,\infty) \times [0,\infty) \to (0,1)$ by the relation $\eta_n(\gamma, p_0, \beta, \Delta) = \eta_F$ while ensuring that

$$A(t_1 : t_0) \geq \eta_n(\gamma, p_0, \beta, \Delta) I$$

for all $0 \leq t_0 \leq t_1 < \infty$ whenever $\{A(k)\}_{k=0}^{\infty}$ satisfies the required properties. Thus, the assertion of the proposition holds for $n = q+1$ and hence, for all $n \in \mathbb{N}$. $\square$

Proof of Corollary 1

Proof. Since $\{A(t)\}_{t=0}^{\infty}$ is a stochastic chain, it satisfies almost stochasticity (with the deviation from stochasticity being $\Delta = 0$). Hence, if $\{A(t)\}_{t=0}^{\infty}$ satisfies (2.2) for all $S \subset [n]$ and all $k_0, k_1 \in \mathbb{N}_0$ with $k_0 < k_1$, then it follows from Proposition 3 that there exists an $\eta > 0$ satisfying

$$A(t_1 : t_0) \geq \eta I$$

for all $t_0, t_1 \in \mathbb{N}_0$ with $t_0 \leq t_1$. This means that $\mathbf{1}^T A(t_1 : t_0) \geq \eta \mathbf{1}^T$ for all $t_1, t_0 \in \mathbb{N}_0$. In light of Lemma 8 of [67], this means that $\{A(t)\}_{t=0}^{\infty} \in \mathcal{P}^*$. $\square$

Proof of Lemma 5

Proof. Since G^∞ is connected, for every pair of indices $(i,j) \in [n] \times [n]$, there exists a path between i and j in G^∞. In other words, there exists an $r \in [n]$ and vertices $\ell_1, \ell_2, \ldots, \ell_r \in [n]$ with $\ell_1 = i$ and $\ell_r = j$ such that $(\ell_1, \ell_2), (\ell_2, \ell_3) \ldots, (\ell_{r-1}, \ell_r) \in E^\infty$. Since $\{A(t)\}_{t=0}^{\infty}$ is also infinite flow stable, this means that $\lim_{t\to\infty}(x_{\ell_k}(t) - x_{\ell_{k+1}}(t)) = 0$ for all $k \in [r-1]$. As a result, we have $\lim_{t\to\infty}(x_i(t) - x_j(t)) = 0$. Since i and j are arbitrary, it follows from Theorem 2.2

in [62] that $\{A(t)\}_{t=0}^{\infty}$ is ergodic. $\square$

Proof of Theorem 2

Proof. From Theorem 1, we know that $\{A(t)\}_{t=0}^{\infty}$ admits a uniformly positive absolute probability sequence, i.e., $\{A(t)\}_{t=0}^{\infty} \in \mathcal{P}^*$. As a result, Theorem 4.4 of [62] implies that $\{A(t)\}_{t=0}^{\infty}$ is infinite flow stable.

Now, suppose that the infinite flow graph of $\{A(t)\}_{t=0}^{\infty}$ is connected. Since the chain is also infinite flow stable, we know from Lemma 5 that the chain is ergodic. It now follows from Theorem 1 of [73] that $\{A(t)\}_{t=0}^{\infty}$ admits a unique absolute probability sequence.

On the other hand, suppose that the infinite flow graph of $\{A(t)\}_{t=0}^{\infty}$ is not connected. Then, by Lemma 3.6 of [62], either there exists an initial condition $(t_0, x(t_0))$ with $t_0 \in \mathbb{N}$ and $x(t_0) \in \mathbb{R}^n$ such that $x(t+1) = A(t)x(t)$ does not converge to a steady state (Case 1: $\lim_{t\to\infty} x(t)$ does not exist), or there exist indices i and j such that $(i,j) \in [n] \times [n]$ and $\limsup_{t\to\infty} |x_i(t) - x_j(t)| > 0$ (Case 2).

In the first case, we know that $\lim_{t\to\infty} A(t : t_0)$ does not exist (because otherwise, $\lim_{t\to\infty} x(t_0) = \lim_{t\to\infty} A(t : t_0)x(t_0)$ would exist). Hence, $\{A(t)\}_{t=0}^{\infty}$ is not ergodic.

Consider now the second case and suppose that $\{A(t)\}_{t=0}^{\infty}$ is ergodic. Then, for every initial condition $(t_0, x(t_0))$, there exists a $\pi(t_0) \in \mathbb{R}^n$ such that $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} A(t : t_0)x(t_0) = \pi^T(t_0)x(t_0)\mathbf{1}$, which implies that $\lim_{t\to\infty} x_l(t) = \lim_{t\to\infty} x_m(t)$ for all $l, m \in [n]$. However, this contradicts the hypothesis of Case 2. Hence, $\{A(t)\}_{t=0}^{\infty}$ cannot be ergodic.

We have thus shown that if the infinite flow graph of $\{A(t)\}_{t=0}^{\infty}$ is not connected, it is not ergodic. It now follows from Theorem 1 in [73] that if the infinite flow graph of $\{A(t)\}_{t=0}^{\infty}$ is not connected, then the chain does not admit a unique absolute probability sequence. $\square$

Proof of Lemma 7

Proof. We first recall from Proposition 7 of [65] that under Assumption 1, there exists a constant $G \in (0,\infty)$ such that

$$G\int_{t_k}^{t_{k+1}} \mathbf{1}^T A_{S\bar{S}}(t)\mathbf{1}dt \leq \mathbf{1}^T \phi_{S\bar{S}}(t_{k+1},t_k)\mathbf{1} \leq n\int_{t_k}^{t_{k+1}} \mathbf{1}^T A_{S\bar{S}}(t)\mathbf{1}dt \tag{2.16}$$

holds for all $k \in \mathbb{N}_0$ and all sets $S \subset [n]$.

Now, suppose Assumption 2 holds. Then, for all $S \subset [n]$ and $\ell, m \in \mathbb{N}_0$ with $\ell \leq m$, we have

$$\begin{aligned}
G\frac{p_0}{n}\sum_{k=\ell}^{m} \mathbf{1}^T \Phi_{S\bar{S}}(t_{k+1},t_k)\mathbf{1} &\leq Gp_0\int_{t_\ell}^{t_{m+1}} \mathbf{1}^T A_{S\bar{S}}(t)\mathbf{1}dt \\
&\overset{(a)}{\leq} G\left(\int_{t_\ell}^{t_{m+1}} \mathbf{1}^T A_{S\bar{S}}(t)\mathbf{1}dt + \beta\right) \\
&\leq \sum_{k=\ell}^{m} \mathbf{1}^T \Phi_{S\bar{S}}(t_{k+1},t_k) + G\beta,
\end{aligned}$$

where (a) follows from Assumption 2. Therefore, (2.8) holds with $\tilde{p}_0 = \frac{Gp_0}{n}$ and $\tilde{\beta} = G\beta$.

Similarly, if we are given that (2.8) holds for all $S \subset [n]$, then we can again use (2.16) to make arguments similar to the preceding ones to show that Assumption 2 holds with $p_0 = \frac{G}{n}\tilde{p}_0$ and $\beta = \frac{\tilde{\beta}}{n}$. □

Proof of Theorem 3

Proof. Suppose $\{A(t)\}_{t\geq 0}$ has a uniformly positive absolute probability sequence, i.e., suppose $\{A(t)\}_{t\geq 0} \in \mathcal{P}^*$. Then we know that $\{\Phi(t_{k+1},t_k)\}_{k=0}^{\infty} \in \mathcal{P}^*$ in discrete time. It follows from Proposition 2 that $\{\Phi(t_{k+1},t_k)\}_{k=0}^{\infty}$ is approximately reciprocal in discrete time, i.e., there exist constants $\tilde{p}_0 > 0$ and $\tilde{\beta} \in (0,\infty)$ such that (2.8) holds for all $S \subset [n]$. Lemma 7 now implies that Assumption 2 holds, which means that $\{A(t)\}_{t\geq 0}$ is approximately reciprocal.

On the other hand, suppose we are given that $\{A(t)\}_{t\geq 0}$ is approximately reciprocal with respect to the increasing sequence of times $\{t_k\}_{k=0}^{\infty} \subset \mathbb{R}_{\geq 0}$. We now show that for any two times $\tau_1, \tau_2 \geq 0$ with $\tau_1 < \tau_2$, the chain $\{A(t)\}_{t \geq 0}$ is also approximately reciprocal with respect to the augmented sequence of times $t_1, t_2, \ldots, t_q, \tau_1, t_{q+1}, \ldots, t_{r-1}, \tau_2, t_r, \ldots$, where $q := \max\{\ell \in \mathbb{N}_0 : t_\ell \leq \tau_1\}$ and $r := \min\{\ell \in \mathbb{N}_0 : t_\ell \geq \tau_2\}$. To this end, we use Assumption 1 to argue that for any set $S \subset [n]$, we have

$$\int_{\tau_1}^{t_{q+1}} \mathbf{1}^T A_{S\bar{S}}(t)\mathbf{1}dt \leq \sum_{i\in[n]} \sum_{j\in[n]\setminus\{i\}} \int_{\tau_1}^{t_{q+1}} a_{ij}(t)dt \leq \sum_{i\in[n]} \sum_{j\in[n]\setminus\{i\}} \int_{t_q}^{t_{q+1}} a_{ij}(t)dt \leq n(n-1)M.$$

Similarly, $\int_{t_q}^{\tau_1} \mathbf{1}^T A_{S\bar{S}}(t)\mathbf{1}dt$, $\int_{t_{r-1}}^{\tau_2} \mathbf{1}^T A_{S\bar{S}}(t)\mathbf{1}dt$, and $\int_{\tau_2}^{t_r} \mathbf{1}^T A_{S\bar{S}}(t)\mathbf{1}dt$ are all upper bounded by $n(n-1)M$. In addition, we have $\int_{t_\ell}^{t_m} \mathbf{1}^T A_{\bar{S}S}(t)\mathbf{1}dt \geq 0$ for all $\ell, m \in \mathbb{N}_0$ with $\ell < m$. As a result, the inequality in Assumption 2 implies that for all $S \subset [n]$ and $\ell < m$, we have

$$p_0 \int_{t'_\ell}^{t'_m} \mathbf{1}^T A_{S\bar{S}}(t)\mathbf{1}dt \leq \int_{t'_\ell}^{t'_m} \mathbf{1}^T A_{\bar{S}S}(t)\mathbf{1}dt + \beta + 2n(n-1)Mp_0,$$

where $\{t'_k\}_{k=0}^{\infty}$ denotes the augmented sequence $t_1, t_2, \ldots, t_q, \tau_1, t_{q+1}, \ldots, t_{r-1}, \tau_2, t_r, \ldots$. Invoking Lemma 7 now shows that the stochastic chain $\{\Phi(t'_{k+1}, t'_k)\}_{k=0}^{\infty}$ is approximately reciprocal in discrete time. Moreover, Assumption 1 (which continues to hold after replacing $\{t_k\}_{k=0}^{\infty}$ with $\{t'_k\}_{k=0}^{\infty}$) and Lemma 8 in [65] together imply that $\{\Phi(t'_{k+1}, t'_k)\}_{k=0}^{\infty}$ is strongly aperiodic. It now follows from Proposition 3 that there exists a constant $\eta > 0$ such that $\Phi(t'_m : t'_\ell) \geq \eta I$ for all $\ell, m \in \mathbb{N}_0$ satisfying $\ell \leq m$. In particular, we have $\Phi(\tau_2 : \tau_1) \geq \eta I$. Since τ_1 and τ_2 are arbitrary, it follows from Lemma 8 of [67] that $\{A(t)\}_{t\geq 0} \in \mathcal{P}^*$. □

Proof of Theorem 4

Proof. Observe that by repeating some of the arguments used to prove Theorem 3, we can show that Assumptions 1 and 2 continue to hold (if only with different constants) even if we augment the sequence $\{t_k\}_{k=0}^{\infty}$ by inserting into it an arbitrary constant $\tau \geq 0$. By Lemma 8 of [65], this further implies that the discrete-time chain $\{\Phi(t'_{k+1} : t'_k)\}_{k=0}^{\infty}$ (where $\{t'_k\}_{k=0}^{\infty}$ denotes the augmented sequence $t_1, t_2, \ldots, \tau, \ldots,$) is strongly aperiodic. In addition, since $\{A(t)\}_{t\geq 0}$ satisfies the uniform bound assumption (Assumption 1) in addition to the condition of approximate reciprocity, we know from Theorem 3 that $\{A(t)\}_{t\geq 0} \in \mathcal{P}^*$. By Definitions 17 and 18, this implies that $\{\Phi(t'_{k+1} : t'_k)\}_{k=0}^{\infty} \in \mathcal{P}^*$ in discrete time. Hence, by Theorem 1, $\{\Phi(t'_{k+1} : t'_k)\}_{k=0}^{\infty}$ is approximately reciprocal.

Now, the infinite flow graph of $\{A(t)\}_{t\geq 0}$ being connected is equivalent to the condition $\int_0^\infty \mathbf{1}^T A_{S\bar{S}}(t)\mathbf{1}dt + \int_0^\infty \mathbf{1}^T A_{\bar{S}S}(t)\mathbf{1}dt = \infty$ being satisfied for all $S \subset [n]$, which, by Proposition 7 of [65], is in turn equivalent to the infinite flow graph of the discrete-time stochastic chain $\{\Phi(t'_{k+1} : t'_k)\}_{k=0}^{\infty}$ being connected. By the strong aperiodicity and the approximate reciprocity of $\{\Phi(t'_{k+1}, t'_k)\}_{k=0}^{\infty}$ (established above), Theorem 2 implies that the connectivity of the infinite flow graph of $\{\Phi(t'_{k+1}, t'_k)\}_{k=0}^{\infty}$ is equivalent to the discrete-time chain admitting a unique absolute probability sequence.

To summarize, the infinite flow graph of $\{A(t)\}_{t\geq 0}$ is connected if and only if $\{\Phi(t'_{k+1} : t'_k)\}_{k=0}^{\infty}$ admits a unique absolute probability sequence, i.e., if and only if the stochastic vectors $\{\pi(t_k)\}_{k=0}^{\infty} \bigcup \{\pi(\tau)\}$ are unique. Since τ is arbitrary, it follows that the infinite flow graph of $\{A(t)\}_{t\geq 0}$ is connected if and only if the absolute probability sequence $\{\pi(\tau)\}_{\tau\geq 0}$ is unique. □

Proof of Corollary 2

Proof. By Lemma 4.2 of [62], $\{A(t)\}_{t=0}^{\infty}$ having feedback property implies that $\{\mathbb{E}[A(t)]\}_{t=0}^{\infty}$ has the strong feedback property (i.e., the expected chain is strongly aperiodic). Since the expected chain is also approximately reciprocal, we know from Theorem 1 that $\{\mathbb{E}[A(t)]\}_{t=0}^{\infty} \in \mathcal{P}^*$. Hence, $\{A(t)\}_{t=0}^{\infty} \in \mathcal{P}^*$. Assertion (i) now follows from Theorem 4.4 of [62] and the remaining assertions follow from Theorem 5.1 of [62]. □

Proof of Corollary 3

Proof. We can repeat the arguments used in the proof of Corollary 2 to show that $\{\mathbb{E}[A(t)]\}_{t=0}^{\infty} \in \mathcal{P}^*$. Therefore, this corollary is a straightforward consequence of Theorem 1 above, Lemma 4.2

of [62], and Theorem 5.2 of [62]. □

Proof of Corollary 4

Proof. The first assertion is a direct consequence of Theorem 1 above and Lemma 2 of [69]. The second assertion follows from Theorem 1 above and from Theorem 5.1 in [62] (see Remark 5 above for a more detailed explanation). The third assertion follows from Theorem 1 above, from Corollary 3 and Theorem 4 of [69], and from the fact that strong aperiodicity implies weak aperiodicity. □

Chapter 2, in full, is a reprint of the material as it appears in Rohit Parasnis, Massimo Franceschetti, and Behrouz Touri, “Towards a Perron-Frobenius Theorem for Strongly Aperiodic Stochastic Chains”, *arXiv preprint arXiv:2204.00573* (2022). The dissertation author was the primary investigator and author of this article.

Chapter 2, in full, is currently being prepared for submission for publication as Rohit Parasnis, Massimo Franceschetti, and Behrouz Touri, “Towards a Perron-Frobenius Theorem for Strongly Aperiodic Stochastic Chains” (the publication venue is to be determined). The dissertation author was the primary investigator and author of this article.

Chapter 3

On the Convergence Properties of Social Hegselmann-Krause Dynamics

3.1 Introduction

With social networks gaining omnipresence and their associated datasets becoming accessible to the public, opinion dynamics has attracted researchers from a range of disciplines in recent times [74]. Besides having social scientific applications such as forecasting election results [75], opinion dynamics models are also used in engineering problems such as *distributed rendezvous in a robotic network* [76].

Among the existing models, confidence-based models form a noteworthy class. In particular, a bounded-confidence model proposed in [77], also known as the Hegselmann-Krause model (referred as the *HK model* from here on), has garnered a lot of interest in the last two decades. Essentially, it models a non-linear time-varying system in which every agent's opinion is either a real number or a real-valued vector, and assumes that every agent has a confidence bound defining their neighborhood (the set of agents influencing them at the given point in time). At every time-step, each agent's belief moves to the arithmetic mean of their neighbors' beliefs.

To cite a few notable results, [78] showed that HK dynamics always converge to a steady state in finite time for every set of initial opinions. Later on, the termination time of the dynamics was studied extensively and it is now known that for a system of n agents having scalar opinions, the maximum termination time is at least $\Omega(n^2)$ and at most $O(n^3)$ [79], [80], [81]. When the opinions are multidimensional, the best known lower and upper bounds are $\Omega(n^2)$ and $O(n^4)$, respectively [79], [82]. Other properties of interest, such as inter-cluster distance and equilibrium stability were studied in [83] and [84].

Even though a number of variants of the HK model have been proposed and analyzed (such as [85–92]), very few models, such as the *social HK model*, proposed in [93], the generalized Deffuant-Weisbuch model proposed in [94], and the *social similarity-based* HK model, proposed in [95], address an important shortcoming that is central to the original model: the assumption that every agent has *access* to every other agent's opinion (regardless of whether or not they are *influenced* by other agents).

Such an assumption is questionable, as on large scales a multitude of extrinsic factors such as geographical separation along with differences in culture, nationality, socio-economic background, etc., may drastically reduce the likelihood of two like-minded individuals contacting each other. To address this issue, the social HK model incorporates a *physical connectivity graph*, denoted by G_{ph}, into the classical HK model. A pair of agents can access each other's opinions if and only if the corresponding vertices are adjacent in G_{ph}.

The social HK model was proposed in [93], which provides a conjecture on the minimum value of the confidence bound required to achieve consensus in the limit as the number of agents goes to infinity. Subsequently, [96] provided an upper bound on the number of time steps in which two agents separated by a minimum distance influence each other. Recently, in [47], we showed that for any incomplete G_{ph} and any continuous probability density function having the state space as its support, the expected termination time of social HK dynamics is infinity.

This result motivates us to investigate the convergence properties of the social HK model in this chapter. We begin by introducing the original HK model, the social HK model, and the associated terminology in Section 5.2. In Section 3.3, we provide the proof of the aforementioned result on the expected termination time of the dynamics. In Section 3.4, we show that the conditional upper bound on the maximum ε-convergence time provided in [47] is applicable to a wider class of initial opinion distributions. In Section 3.5, we show that delaying an event that we call *merging* is the only way to indefinitely delay a social HK system's ε-convergence to the steady state. We then provide a set of sufficient conditions and another set of necessary conditions for arbitrarily slow merging, and use the necessary conditions to show that the ε-convergence time of a complete r-partite graph is bounded. We conclude by observing that these conditions are nearly tight under certain assumptions on the initial opinion distribution, and also provide some future directions.

A subset of the results of this work have also been reported in our conference paper [97], where we discriminate between consensus and non-consensus states, and provide sufficient conditions for a physical connectivity graph to have an unbounded convergence time in each case.

Notation: We denote the set of real numbers by $\mathbb{R}$, the set of positive real numbers by $\mathbb{R}^+$, the set of integers by $\mathbb{Z}$, the set of positive integers by $\mathbb{N}$, and the set $\mathbb{N}\cup\{0\}$ by $\mathbb{N}_0$. We define $[n] := \{1,\ldots,n\}$. We use I to denote the identity matrix (of the known dimension).

We denote the cardinality of a set S by $|S|$, the vector space of column vectors consisting of n-tuples of real numbers by $\mathbb{R}^n$, the ∞-norm in $\mathbb{R}^n$ by $\|\cdot\|_\infty$, and the all-one vector and the all-zero vector in $\mathbb{R}^n$ by $\mathbf{1}_n$ and $\mathbf{0}_n$, respectively, dropping the subscripts when the dimension is clear from the context. For a set S, $\mathbf{1}_S$ denotes $\mathbf{1}_{|S|}$.

An undirected graph on n vertices is $G = (V, E)$ where V or $V(G)$ is the set of vertices and $E = E(G) \subseteq V \times V$ is the set of edges, with $(i, j) \in E$ if and only if (iff) $(j, i) \in E$ for $i, j \in V$. If $|V| = n$, we can label the vertices so that $V = [n]$, without loss of generality (w.l.o.g.). For any vector $w \in \mathbb{R}^n$ and a subset of vertices $V_P \subseteq V$, we let w_P denote the restriction of w to the coordinates specified by V_P. Also, for any $l \in [n]$, let $w_{[l]}$ denote the vector $[w_1 \ldots w_l]^T$. Throughout this work, all the graphs are undirected. We say that i and j are neighbors in G, if $(i, j) \in E$ (and hence, $(j, i) \in E$). The set of neighbors of a node i in G is the set $\mathcal{N}_i := \{j : (i, j) \in E\}$ and the degree of node i is $d_i := |\mathcal{N}_i|$. The adjacency matrix of $G = ([n], E)$ is the $n \times n$ binary matrix A_{adj} where $(A_{\text{adj}})_{ij} = 1$ iff $(i, j) \in E$, and the degree matrix of G is the diagonal matrix D with $D_{ii} = d_i$. We define the normalized adjacency matrix of G to be the matrix $A := D^{-1}A_{\text{adj}}$. The Laplacian of G is defined to be $L := D - A_{\text{adj}}$ and the normalized Laplacian of G is defined to be $N := D^{-1/2}LD^{-1/2} = I - D^{-1/2}A_{\text{adj}}D^{-1/2}$. For two graphs $G_1 = ([n], E_1)$ and $G_2 = ([n], E_2)$ on n vertices, we let $G_1 \cap G_2 = ([n], E_1 \cap E_2)$. For any subscript $_P$, if G_P denotes a graph, then A_P denotes its normalized adjacency matrix. A complete graph (or clique) on n vertices is the graph $K_n := ([n], [n] \times [n])$.

Finally, for any matrix M, we use Null(M) to denote the null space of M.

3.2 Problem Formulation

3.2.1 Original Model

Consider a network of n agents. For each $k \in \mathbb{N}$, let $x_i[k]$ be the opinion of the i^{th} agent at time k. Then the state of the system at time k is defined as $x[k] := [x_1[k]\ x_2[k]\ \ldots\ x_n[k]]^T \in \mathbb{R}^n$. Occasionally, we drop the indexing $[k]$ for the state and its associated quantities when the context makes the time index clear. In the original HK model, at time k, agents i and j are neighbors iff $|x_i[k] - x_j[k]| \le R$, where R, the *confidence bound*, is assumed to be the same for every agent. Thus, the set of neighbors of agent i at time k is:

$$\mathcal{N}_i(x[k]) = \{j \in [n] : |x_i[k] - x_j[k]| \le R\}.$$

Note that $i \in \mathcal{N}_i$ for all $i \in [n]$. Also, i is a neighbor of j iff j is a neighbor of i. Therefore, we can encode all of the information about the influences in the network at time k into an undirected graph, $G_c(x[k])$, which we call the *communication graph* of the network at time k. This n-vertex graph has a link between two vertices iff the corresponding agents are neighbors at time k. Observe that $G_c(x[k])$ always has a self-loop at each vertex at all times. Finally, at every time instant, every agent's opinion shifts to the average of his/her neighbors' current opinions:

$$x_i[k+1] = \frac{\sum_{j \in \mathcal{N}_i(x[k])} x_j[k]}{|\mathcal{N}_i(x[k])|}. \tag{3.1}$$

This being a *bounded confidence* model, it is possible that an agent does not have any neighbor other than himself/herself, in which case, his/her opinion does not change i.e., $x_i[k+1] = x_i[k]$. Such an agent is said to be *isolated*.

3.2.2 Modification

In the original HK dynamics, if the opinions of any two agents are within a distance of R from each other, then the agents necessarily influence each other. This assumption is relaxed in the social HK model by the introduction of a second graph, as described below.

Let the physical connectivity graph $G_{ph} = ([n], E_{ph})$ be an undirected graph on n-vertices with each vertex representing an agent. Two agents i and j can communicate with each other iff their corresponding vertices are adjacent in G_{ph}. Hence, for two individuals to influence each other's opinions, they not only need to be similarly opinionated but also to be physically connected through G_{ph}. Throughout this chapter, we assume that G_{ph} is connected, time-invariant, and contains all the self-loops, i.e., $(i,i) \in E_{ph}$ for all $i \in [n]$.

Observe that in the special case that G_{ph} is a complete graph, no external restrictions are imposed on the interaction between any two agents. This case, therefore, is equivalent to the well-known original model of the last subsection. However, the social HK generalization starts differing from the original model when there is at least one pair of non-adjacent vertices in G_{ph}, as will be revealed next.

3.2.3 State-Space Representation

Each of the two models discussed above has the following state-space representation:

$$x[k+1] = A(x[k])\,x[k], \tag{3.2}$$

where $A(x[k])$ is the normalized adjacency matrix of $G_{ph} \cap G_c(x[k])$. Thus, $\tilde{G}[k] = G_{ph} \cap G_c(x[k])$ is the effective graph or the *influence graph* at time k. The original HK model is a special case with $G_{ph} = K_n$, which gives $\tilde{G}[k] = G_c(x[k])$.

Note the explicit dependence of the state evolution matrix on the state of the system at time k. It arises from the dependence of the structure of the communication graph on the agents' opinions at the concerned time instant.

Now, let $A_{adj}(x[k])$ denote the adjacency matrix of $\tilde{G}[k]$ and let $D(x[k])$ denote its degree matrix. Then

$$x[k+1] = D^{-1}(x[k]) \cdot A_{adj}(x[k]) \cdot x[k]$$

which can be expressed more compactly as:

$$x[k+1] = D^{-1} A_{adj} x[k]. \tag{3.3}$$

In other words, the state evolution matrix is given by $A = D^{-1} A_{\text{adj}}$. (We drop the dependencies of these matrices on $x[k]$ for notational simplicity).

3.3 Analysis of Termination Time

In this section, we show that social HK dynamics on an incomplete physical connectivity graph may never attain the steady state in finite time. However, as the following result shows, the system is guaranteed to converge to a steady state.

Proposition 4. *Consider the dynamics described by* (3.1) *for any given initial state,* $x_0 \in \mathbb{R}^n$. *Then the limit,* $x_\infty(x_0) := \lim_{k\to\infty} x[k]$ *exists and will be referred to as the steady state of the system corresponding to the initial state* x_0.

Proof. Note that for any trajectory $\{x[k]\}$ of the dynamics, the corresponding sequence of matrices $\{A(x[k])\}$ satisfies (a) $A_{ii}(x[k]) \geq \frac{1}{n}$, as each agent is always its own neighbor, and (b) $A_{ij}(x[k]) \geq \frac{1}{n} A_{ji}(x[k])$ as $G_{ph} \cap G_c(x)$ is undirected for all $x \in \mathbb{R}^n$ and further, we are utilizing uniform neighbor averaging on this graph at each time k. Thus, by Theorem 2 of [98], the limit $x_\infty(x_0)$ exists for all $x_0 \in \mathbb{R}^n$.

□

We now define some quantities in order to make the notion of finite-time termination precise.

Definition 28 (Termination Time)**.** *For an initial state x_0 and a given physical connectivity graph G_{ph}, the termination time $T(G_{ph}, x_0)$ is the time taken by the system to reach the steady state corresponding to x_0, i.e.:*

$$T(G_{ph}, x_0) := \inf\{k \in \mathbb{N} : x[k] = x_\infty(x_0)\}.$$

Next we define the maximum termination time for a given physical connectivity graph.

Definition 29 (Maximum Termination Time)**.** *For a given physical connectivity graph G_{ph}, the* maximum *termination time $T^*(G_{ph})$ is the supremum of termination times over all possible initial states:*

$$T^*(G_{ph}) := \sup_{x_0 \in \mathbb{R}^n} T(G_{ph}, x_0).$$

As a special case, it was shown in [80] and [81] that the maximum termination time of the original HK dynamics satisfies $cn^2 \leq T^*(K_n) \leq Cn^3$ asymptotically as $n \to \infty$ when $d = 1$, for some constants $c, C > 0$.

We now state a few properties of a class of normalized adjacency matrices that appear in the state evolution dynamics (3.2). These properties form the basis of our results.

The following lemma is proven in [82] as well as [47].

Lemma 8. *For any undirected graph $\hat{G}$, the normalized adjacency matrix $\hat{A}$ is similar to $I - \hat{N}$ (where $\hat{N}$ is the normalized Laplacian matrix). As a result, $\hat{A}$ is diagonalizable.*

The next result provides more information about the spectral properties of the adjacency matrix of a graph, if we have mild additional structures on the graph.

Lemma 9. *Let $\hat{G}$ be an undirected and incomplete graph that is connected and has all the self-loops. Then, if the eigenvalues of the normalized adjacency matrix $\hat{A}$ (labeled as $\{\lambda_i\}_{i=1}^n$) are ordered such that $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$, we have $1 = \lambda_1 > |\lambda_2| > 0$. Moreover, $\hat{A}$ has at least one positive eigenvalue besides 1.*

Proof. The first part of the result is proven [47]. Since $\hat{A}$ is a row-stochastic matrix, we have $\lambda_1 = 1$. To show that $\hat{A}$ has a positive eigenvalue besides 1, we have $\sum_{i=1}^n \lambda_i = 1 + \sum_{i=2}^n \lambda_i = \text{Tr}(\hat{A})$, but since $\hat{G}$ is incomplete, $\text{Tr}(\hat{A}) > 1$. Therefore, $\sum_{i=2}^n \lambda_i > 0$ and hence, $\lambda_i > 0$ for some i. □

We are now ready to show that on average, social HK dynamics on an incomplete graph never terminate.

Proposition 5. *Let $x[0]$ be a random vector over $\mathbb{R}^n$ whose distribution induces the Borel-measure μ on $\mathbb{R}^n$ with $\mu(V) > 0$ for any non-empty open set $V \subset \mathbb{R}^n$ (or in other words, the probability density function of $x[0]$ is non-zero almost everywhere). Suppose that G_{ph} is not a complete graph. Then the expected termination time of the dynamics is infinite, i.e., $\mathbb{E}_{x[0]}[T(G_{ph}, x[0])] = \infty$.*

Proof. Let

$$S := \left\{ x \in \mathbb{R}^n : |\max_{i\in[n]} x_i - \min_{j\in[n]} x_j| < R \right\}.$$

Note that S is a nonempty open set in $\mathbb{R}^n$ and hence, $\mu(S) > 0$. Also, whenever $x[0] \in S$, every agent is within the confidence of every other agent and thus $G_c(x[0])$ is an n-clique. Additionally, from the update rule (3.1), it follows that $\max_{i\in[n]} x_i[k]$ is monotonically non-increasing and $\min_{j\in[n]} x_j[k]$ is monotonically non-decreasing. Therefore, the communication graph remains a clique and hence, $\tilde{G}[k] = G_{ph}$ for all $k \in \mathbb{N}$. In this case, the dynamics become linear and time-invariant: $A = A(x[k]) = A(x[0])$ and hence, $x[k] = A^k x[0]$.

Lemma 8 now allows us to use the Spectral Theorem for Diagonalizable Matrices (see [61], page 517) so as to write $A^k = \sum_{j=1}^{n} \lambda_j^k P_j$, where λ_j is the j^{th} eigenvalue of A and P_j is the projector onto the j^{th} eigenspace of A along the range space of $A - \lambda_j I$. Moreover, by Lemma 9 we have $\lambda_1 = 1$ and $|\lambda_j| < 1$ for all $j \in \{2, \ldots, n\}$. Therefore, $A^k = P_1 + \sum_{j=2}^{n} \lambda_j^k P_j$, which implies that $\lim_{k\to\infty} A^k = P_1$.

Now, consider a random initial vector $x[0] = x_0 \sim \mu$. Then by the above discussion, on the set $\{x[0] \in S\}$, we have the following for any $k \in \mathbb{N}$:

$$\begin{aligned} \|x[k] - x_\infty(x[0])\| &= \|A^k x[0] - \lim_{k\to\infty} A^k x[0]\| \\ &= \|\sum_{j=2}^{n} \lambda_j^k P_j x[0]\|, \end{aligned} \tag{3.4}$$

which means that $\|x[k] - x_\infty(x[0])\| = 0$ iff $x[0]$ belongs to the null space of $\sum_{j=2}^{n} \lambda_j^k P_j$.

On the other hand, since $\lambda_2 \neq 0$ by Lemma 9, we have that rank$\left(\sum_{j=2}^{n} \lambda_j^k P_j\right) \geq 1$. This implies that nullity$\left(\sum_{j=2}^{n} \lambda_j^k P_j\right) \leq n-1$. By the continuity of $x[0]$, it follows that Pr$\left(x[0] \in \text{Null}(\sum_{j=2}^{n} \lambda_j^k P_j)\right) = 0$. As a result, (3.4) implies that the event $\{x[k] = x_\infty(x[0])\}$ occurs with zero probability on the set $\{x[0] \in S\}$. Thus, on the set $\{x[0] \in S\}$, the event $\{T(G_{\text{ph}}, x[0]) = \infty\} = \cap_{k=1}^{\infty} \{x[k] \neq x_\infty(x[0])\}$ occurs almost surely. Since Pr$(x[0] \in S) > 0$, it follows that $\mathbb{E}_{x[0]} T(G_{\text{ph}}, x[0]) = \infty$. □

In essence, Proposition 5 states that for any incomplete G_{ph}, there is a continuum of initial states starting from which social HK dynamics never terminate. This shows that HK dynamics over complete graphs are indeed an anomaly.

3.4 Bounds on the Convergence Time

Now that we know that a social HK system may never reach the steady state, the next pertinent question is: how fast does it approach the steady state?

We begin with a few relevant definitions.

Definition 30 (ε-Convergence)**.** *Given a physical connectivity graph G_{ph}, an initial state x_0, and $\varepsilon > 0$, the system is said to have achieved ε-convergence at time $N \geq 0$ if its state lies in the ε-neighborhood of the steady state corresponding to x_0, i.e., $\|x[k] - x_\infty(x_0)\| < \varepsilon$, for all $k \geq N$.*

Based on this, we define the ε-convergence time as:

Definition 31 (ε-Convergence Time)**.** *For a given physical connectivity graph G_{ph}, an initial state $x_0 \in \mathbb{R}^n$, and a given $\varepsilon > 0$, the ε-convergence time $k_\varepsilon(G_{ph}, x_0)$ is the time taken by the system to achieve ε-convergence:*

$$k_\varepsilon(G_{ph}, x_0) := \inf \{N \in \mathbb{N} : \|x[k] - x_\infty(x_0)\| < \varepsilon, \\ \text{for all } k \geq N\}.$$

Similar to T^*, we define k_ε^* to be the supremum of ε-convergence times for all initial states.

Definition 32 (Maximum ε-Convergence Time). *For a given physical connectivity graph G_{ph} and $\varepsilon > 0$, the maximum ε-convergence time $k_\varepsilon^*(G_{ph})$ is the supremum of ε-convergence times over all possible initial opinions:*

$$k_\varepsilon^*(G_{ph}) := \sup_{x_0 \in \mathbb{R}^n} k_\varepsilon(G_{ph}, x_0). \tag{3.5}$$

3.4.1 Lower Bound

We now provide a lower bound on the maximum ε-convergence time $k_\varepsilon^*(G_{\text{ph}})$ in terms of the conductance of G_{ph}. We borrow the definition of conductance from [99].

Let $G = ([n], E)$ be an undirected graph on n vertices. For a subset $S \subset [n]$, let $\partial(S) := \{(i, j) \in E \mid i \in S, j \in \bar{S}\}$, where $\bar{S} = [n] \backslash S$. In words, ∂S represents the set of edges that connect S to the rest of the graph. Further, let $d(S)$ denote the sum of the degrees of the vertices in S. Then we have the following definition.

Definition 33 (Conductance)**.** *The conductance $\phi(G)$ of a graph $G = ([n], E)$ is defined as:*

$$\phi(G) := \min_{\substack{S \subset [n] \\ S \neq \emptyset}} \frac{|\partial(S)|}{\min\Big(d(S), d(\bar{S})\Big)}.$$

The next proposition states that a system whose physical connectivity graph has a low conductance might take a long time to converge to its steady state.

Proposition 6. *For any incomplete graph G_{ph} and any given $\varepsilon > 0$, the maximum ε-convergence time of the social Hegselmann-Krause dynamics, as defined in* (3.5)*, satisfies*

$$k^*_\varepsilon(G_{ph}) > \frac{\log\Big(\frac{\varepsilon\sqrt{2}}{R}\Big)}{\log\Big(1 - 2\phi(G_{ph})\Big)}. \tag{3.6}$$

Proof. Let N denote the normalized Laplacian matrix of G and let $\nu_1 \leq \nu_2 \leq \cdots \leq \nu_n$ denote the eigenvalues of N. Note that they can be ordered so because they are all real numbers by virtue of the symmetry of N. By Lemma 2, we know that the eigenvalues of A are $1-\nu_1 \geq 1-\nu_2 \geq \cdots \geq 1-\nu_n$. By Cheeger's Inequality [100], we have $\nu_2 \leq 2\phi(G)$, which then translates to $|\lambda_2| \geq 1 - 2\phi(G)$ for the eigenvalue of A with the second-largest magnitude.

Now, let $x[0] = v$, where v is an eigenvector of A corresponding to λ_2, that also satisfies $\max_{i=1}^n v_i - \min_{j=1}^n v_j = R$ so that we have $G(x[k]) = G_{ph}$ for all $k \in \mathbb{N}$ as argued earlier. Note that such a choice of v is possible as $v \neq \alpha \mathbf{1}$ for any $\alpha \in \mathbb{R}$. As a result, the state evolution reduces to $x[k] = \lambda_2^k x[0]$ and hence $x_\infty(x[0]) = \lim_{k\to\infty} \lambda_2^k x[0] = 0$ (since $|\lambda_2| < 1$ by Lemma 9). For the system to reach an ε-neighborhood of the steady state, we need $\|\lambda_2^k x[0]\| < \varepsilon$. Together with the lower bound on $|\lambda_2|$, this requires $(1-2\phi(G_{ph}))^k \|x[0]\| < \varepsilon$.

Next, observe that the constraint $\max_i x_i[0] - \min_j x_j[0] = R$ enforces the following:

$$\begin{aligned}\|x[0]\|_2 &\geq \sqrt{(\max_i x_i[0])^2 + (\min_j x_j[0])^2} \\ &= \sqrt{(\min_i x_i[0])^2 + (\min_j x_j[0] + R)^2} \\ &\geq R/\sqrt{2}.\end{aligned}$$

Combining this with the necessary condition derived above, we need to have:

$$(1 - 2\phi(G_{ph}))^k (R/\sqrt{2}) < \varepsilon,$$

and hence,

$$k_\varepsilon(G_{\text{ph}}, v) > \frac{\log\left(\frac{\varepsilon\sqrt{2}}{R}\right)}{\log\left(1 - 2\phi(G_{ph})\right)}.$$

This implies (3.6). □

Remark 5. *Proposition 6 can be used to compute a lower bound on the maximum ε-convergence time in terms of n for graphs whose conductance is known as a function of n. For example, the dumbbell graph on n vertices has $\phi = O\left(\frac{1}{n^2}\right)$ [101] which yields $k_\varepsilon^* = \Omega(n^2)$.*

3.4.2 Upper Bound Applicable to a Class of Initial Opinions

We now show that if the influence graph remains connected and time-invariant until ε-convergence to the steady state, then the said ε-convergence is achieved in $O(n^3 \log n)$ steps.

Proposition 7. *Suppose there exist $\varepsilon > 0$ and an initial state $x_0 \in \mathbb{R}^n$ such that the influence graph, $\tilde{G}[k]$ remains connected and constant in time until ε-convergence is achieved. Then with x_0 as the initial state, the social HK system achieves ε-convergence in $O(n^3 \log n)$ steps.*

Proposition 7 follows immediately from Corollary 5.2 of [102]. See Appendix A for further details.

3.5 Arbitrarily Slow ε-Convergence

The results in the previous section prompt us to ask: What if the initial state does not enable $\tilde{G}[k]$ to remain constant in time? In such cases, the convergence time could be unbounded above if the physical connectivity graph has more than three vertices. In other words, it is possible that $k_\varepsilon^*(G_{ph}) = \infty$.

Here is a relevant example from [96]. Let G_{ph} be the path graph on 4 vertices, let $\varepsilon < R/2$, and let $\mathcal{X} = \left\{[-R, 0, R, -(R-\delta)]^T \text{ for } \delta \in (0, R/2)\right\}$. Then note that for $x[0] \in \mathcal{X}$, we have $x_1[1] = -R/2$, $x_2[1] = 0$, $x_3[1] = R/2$ and $x_4[1] = -(R-\delta)$ because at time 1, the sets of neighbours of the first three agents are $\{1,2\}, \{1,2,3\}$ and $\{2,3\}$ respectively. In $\tilde{G}[1]$, the fourth agent remains disconnected from the first three agents because $R > 2\delta$ and the confidence interval of the fourth agent at time 1 is $[\delta - 2R, \delta]$. By induction, we can show that $x[k] = \left[-R/2^k, 0, R/2^k, -(R-\delta)\right]^T$ as long as the third and the fourth agents remain outside each others' confidence intervals, i.e., as long as $R/2^k + R - \delta > R$, or equivalently, as long as $k < \log_2(R/\delta)$. At time $k = \lceil \log_2(R/\delta) \rceil$, however, agents 3 and 4 become neighbors. Thus, at $k = \lceil \log_2(R/\delta) \rceil$, the influence graph $\tilde{G}[k]$ is a connected graph satisfying $\max_i x_i[k] - \min_j x_j[k] = \max\{R/2^k + R - \delta, 2 \cdot R/2^k\} \leq R$. This implies that $x_\infty(x[0]) = c\mathbf{1}$ for some $c \in \mathbb{R}$. Therefore, ε-convergence requires $|x_i[k] - c| \leq \varepsilon$ for $i \in [n]$. By the triangle inequality, this in turn requires $|x_3[k] - x_4[k]| \leq 2\varepsilon < R$ which is not satisfied for $k < \lceil \log_2(R/\delta) \rceil$. Hence, $k_\varepsilon(G_{\text{ph}}, x[0]) \geq \lceil \log_2(R/\delta) \rceil$. As a result, we have $k_\varepsilon^*(G_{\text{ph}}) \geq \sup_{\delta \in (0, R/2)} \lceil \log_2(R/\delta) \rceil = \infty$.

We can generalize the example above to graphs having more than 4 vertices by choosing the same initial opinions for agents 1 - 4, setting $x_i[0] = x_4[0]$ for $5 \leq i \leq n$, and by repeating the above arguments. Therefore, we may state the following lemma without proof.

Lemma 10. *For every $n \geq 4$, there exists an n-vertex physical connectivity graph G_{ph} such that $k_\varepsilon^*(G_{ph}) = \infty$ for all $\varepsilon \in (0, R/2)$.*

3.5.1 Underlying Phenomenon

In the example leading to Lemma 10, $\tilde{G}[0]$ was a disconnected graph, and we could indefinitely delay the formation of a link between two connected components of this graph (namely, the connected components with vertex sets $\{1,2,3\}$ and $\{4\}$) so as to make $k_\varepsilon(G_{\text{ph}}, x[0])$ arbitrarily large. The next proposition will clarify that for any G_{ph}, this is the only way to make $k_\varepsilon(G_{\text{ph}}, x[0])$ arbitrarily large.

To establish this result, we define two kinds of events that can change the structure of $\tilde{G}[k]$ during opinion evolution.

Definition 34 (Link break). *Let $G_{ph} = (V, E_{ph})$ and let $i, j \in V$. The link (i,j) is said to break at time $k \geq 1$ if i and j are adjacent in $\tilde{G}[k-1]$ but non-adjacent in $\tilde{G}[k]$. Additionally, we let $B(i,j)$ denote the event that the link (i,j) breaks (at the concerned time instant).*

Note that a link $(i,j) \in E_{\text{ph}}$ breaks at time k iff $|x_i[k-1] - x_j[k-1]| \leq R$, and $|x_i[k] - x_j[k]| > R$.

Definition 35 (Merging). *Let $\tilde{G}[k_0 - 1]$ be a disconnected graph for some $k_0 \geq 1$, and let $G_1(x[k]) = (V_1, E_1(x[k]))$ and $G_2(x[k]) = (V_2, E_2(x[k]))$ be two induced subgraphs of $\tilde{G}[k]$ that are disconnected from each other in $\tilde{G}[k]$ at time $k_0 - 1$. Then G_1 and G_2 are said to merge at time k_0 if there exists a pair of agents $(i,j) \in V_1 \times V_2$ such that i and j become neighbors in $\tilde{G}$ at time k_0, i.e., $(i,j) \in \tilde{E}(x[k_0])$.*

Besides merging and link breaks, the only kind of event that can alter the structure of $\tilde{G}$ is the formation of a link between two agents belonging to the same component of this graph. We call these events *intra-component link formations*.

We now borrow from [96] the definition of a Lyapunov function called *energy* and that of a related quantity called *active energy*.

Definition 36 (Energy). *Let $\tilde{G}[k] = (V[k], E[k])$. The energy of the social HK system at time k*

is defined as:

$$\mathcal{E}[k] := \sum_{(i,j)\in E[k]} |x_i[k] - x_j[k]|^2 + \sum_{(i,j)\notin E[k]} R^2.$$

Definition 37 (Active energy)**.** *Let $\tilde{G}[k] = (V[k], E[k])$. The active energy of the social HK system at time k is defined as:*

$$\mathcal{E}_{act}[k] := \sum_{(i,j)\in E[k]} |x_i[k] - x_j[k]|^2.$$

Note that $0 \leq \mathcal{E}[k] \leq 2\binom{n}{2}R^2$ for all $k \in \mathbb{N}$.

Lemma 11. *If $B(i,j)$ occurs at time $k+1$ for some $k \in \mathbb{N}$, then there exist two agents $p,q \in [n]$ such that $p \in \mathcal{N}_i[k]$, $q \in \mathcal{N}_j[k]$, and $|x_p[k] - x_q[k]| > R$.*

Proof. Suppose the lemma is false, i.e., for every pair $(p,q) \in \mathcal{N}_i[k] \times \mathcal{N}_j[k]$, we have $|x_p[k] - x_q[k]| \leq R$. Then

$$\begin{aligned} |x_i[k+1] - x_j[k+1]| &\leq \max\left\{ |\max_{q\in\mathcal{N}_j[k]} x_q[k] - \min_{p\in\mathcal{N}_i[k]} x_p[k]|, |\max_{p\in\mathcal{N}_i[k]} x_p[k] - \min_{q\in\mathcal{N}_j[k]} x_q[k]| \right\} \\ &\leq \max_{(p,q)\in\mathcal{N}_i[k]\times\mathcal{N}_j[k]} |x_p[k] - x_q[k]| \\ &\leq R. \end{aligned}$$

The first inequality stems from the fact that HK dynamics are an averaging dynamics and each agent's opinion at any time instant is bounded by the minimum and the maximum of his/her neighbors' opinions at the previous time instant. The last inequality above implies that agents i and j are neighbors at time $k+1$, thereby contradicting the fact that the link (i,j) breaks at time $k+1$. □

Next, we need to establish that only finitely many link breaks can occur in any opinion evolution process.

Lemma 12. *The total number of link breaks during the entire process of opinion evolution is $O(n^5)$ regardless of the structure of G_{ph} and the initial state $x[0] \in \mathbb{R}^n$.*

Proof. Based on Proposition 1 of [96], we have:

$$\mathcal{E}[k] - \mathcal{E}[k+1] \geq (1 - |\lambda_k|^2)\mathcal{E}_{act}[k] \tag{3.7}$$

for $k \in \mathbb{N}$, where

$$\lambda_k := \{\max |\lambda| : \lambda \neq 1 \text{ is an eigenvalue of } A[k]\},$$

and if we let $d_{\text{eff}}(G)$ be the largest diameter of any connected component of the graph G, we have the lower bound

$$1 - |\lambda_k|^2 \geq \frac{3}{2n^2 d_{\text{eff}}(\tilde{G}[k])} \geq \frac{3}{2n^3}, \tag{3.8}$$

which was derived in [82] (see page 517 of [82]). Here, we derive a lower bound on the active energy. Let $i, j \in [n]$ and suppose $B(i, j)$ occurs at time $k+1$ for some $k \in \mathbb{N}$. Then by Lemma 11, we can find two agents $p, q \in [n]$ such that $p \in \mathcal{N}_i[k]$, $q \in \mathcal{N}_j[k]$, and $|x_p[k] - x_q[k]| > R$. Therefore, by the definition of active energy, we have

$$\begin{aligned} \mathcal{E}_{act}[k] &\geq |x_p[k] - x_i[k]|^2 + |x_i[k] - x_j[k]|^2 + |x_j[k] - x_q[k]|^2 \\ &\geq \frac{1}{3}\left(|x_p[k] - x_i[k]| + |x_i[k] - x_j[k]| + |x_j[k] - x_q[k]|\right)^2 \\ &\geq \frac{1}{3}|x_p[k] - x_q[k]|^2 \\ &> R^2/3, \end{aligned} \tag{3.9}$$

where the second and the third inequalities follow from the Cauchy-Schwarz and the triangle inequalities, respectively.

Combining (3.7), (3.8) and (3.9) yields

$$\mathcal{E}[k] - \mathcal{E}[k+1] \geq \frac{R^2}{2n^3} \tag{3.10}$$

which means that the energy of the network decreases every time a link breaks and the decrement corresponding to each link break is at least $R^2/2n^3$. Since $\mathcal{E}[0] \le n^2R^2$ and $\mathcal{E}[k] \ge 0$ for $k \in \mathbb{N}$, the maximum possible number of link breaks that can ever occur is at most $\frac{n^2R^2}{R^2/2n^3} = O(n^5)$.

□

The next lemma bounds the maximum possible time interval between two consecutive link breaks under the condition that no new link is formed during this interval.

Lemma 13. *Let $G_P = (V_P, E_P)$ be a connected component of $\tilde{G}[k_0]$ at some time $k_0 \ge 0$. Suppose (i) no link break occurs between any two agents of G_P until time $k_1 > k_0$, (ii) one or more link breaks occur within G_P at time k_1, and (iii) no new edge is formed between any node belonging to V_P and any node belonging to $[n]$ during the time interval (k_0, k_1). Then $k_1 - k_0 = O(n^3 \log n)$.*

Proof. Assumptions (i) and (iii) of the lemma imply that G_P remains a connected component of $\tilde{G}[k]$ during (k_0, k_1). It thus follows from Lemma 11 and Assumption (ii) that $D_e := \max_{i \in V_P} x_i[k_1 - 1] - \min_{j \in V_P} x_j[k_1 - 1] > R$. Hence, $D_e > 0$.

Now, consider a hypothetical network $\widehat{\mathcal{N}}$ whose vertex set is V_P, and whose influence graph and state at time k are denoted by $\hat{G}[k]$ and $y[k]$, respectively. Let $D[k] = \max_{i \in V_P} x_i[k] - \min_{j \in V_P} x_j[k]$ and let $c\mathbf{1}_{V_P}$ be the steady state associated with $\widehat{\mathcal{N}}$ corresponding to the initial state $y[0] = x_P[k_0]$ (where x_P denotes the restriction of x to the coordinates specified by V_P). In this case, $\hat{G}[k]$ achieves D_e-convergence latest by time $\Delta := k_1 - 1 - k_0$ because $c \in [\min_i y_i[\Delta], \max_i y_i[\Delta]]$ and hence

$$\begin{aligned}\max_i |y_i[\Delta] - c| &= \max\left(c - \min_i y_i[\Delta], \max_i y_i[\Delta] - c\right) \\ &\le \max_i y_i[\Delta] - \min_i y_i[\Delta] \\ &= \max_i x_{Pi}[\Delta + k_0] - \min_j x_{Pj}[\Delta + k_0] \\ &= D_e.\end{aligned}$$

Moreover, since G_P remains a connected component of $\tilde{G}[k]$ during the interval (k_0, k_1), our choice of $y[0]$ implies that $\hat{G}[k] = G_P$ for $k \in (0, \Delta]$. Thus, by Proposition 7, if there exists an $\varepsilon > 0$ such that $\widehat{\mathcal{N}}$ achieves ε-convergence at time Δ or earlier, then the said ε-convergence occurs in $O(|V_P|^3 \log |V_P|) = O(n^3 \log n)$ steps. Since D_e-convergence occurs latest by time Δ and since $D_e > 0$, we can find an $\varepsilon \in (0, D_e]$ such that $\widehat{\mathcal{N}}$ achieves ε-convergence precisely at time Δ. Hence, $\Delta = O(n^3 \log n)$, which proves the lemma. □

We are now ready to show that merging is unavoidable if we desire arbitrarily slow ε-convergence to the steady state.

Proposition 8. *In social HK dynamics, all the link breaks and intra-component link formations always occur in $O(n^8 \log n)$ time steps. Hence, if there exists $\varepsilon > 0$ such that $k^*_\varepsilon(G_{ph}) = \infty$, then there exists a set $\mathcal{X}_0 \subset \mathbb{R}^n$ such that whenever $x[0] \in \mathcal{X}_0$, merging occurs at least once during the process of opinion evolution.*

Proof. Let $\varepsilon > 0$ be such that $k^*_\varepsilon(G_{\text{ph}}) = \infty$. Consider an arbitrary initial state $x[0] \in \mathbb{R}^n$. We now consider two cases in the evolution of the dynamics. The cases are defined in such a way that merging does not occur in either case.

Case 1: no link formation ever takes places. Then by Lemma 12, we know that at most $O(n^5)$ links break in the opinion evolution process, and by Lemma 13, the maximum possible time interval between two consecutive link breaks is $O(n^3 \log n)$. Therefore, the time at which the last link breaks is at most $O(n^8 \log n)$. After this point in time, the structure of $\tilde{G}[k]$ never changes. Therefore, for any $\varepsilon > 0$, it takes $O(n^2 \log(n) d(G_{\text{ph}}))$ additional time steps to achieve ε-convergence. Hence, $k_\varepsilon(G_{\text{ph}}, x[0]) = O(n^8 \log n) + O(n^2 \log(n) d(G_{\text{ph}})) = O(n^8 \log n)$.

Now, consider Case 2: at least one new link is formed during the opinion evolution but merging never occurs. For $r \in \mathbb{N} \setminus \{0\}$, let t_r denote the time at which the r-th set of simultaneous link breaks occur. Now, suppose an intra-component link formation occurs at a time $k' \in \{t_l, t_l + 1, \ldots, t_{l+1}\}$ for some $l \in \mathbb{N} \setminus \{0\}$. Let (i, j) denote this new link. Since no merging occurs, (i, j) is formed *within* some connected component G' of $\tilde{G}$. Thus, we have

$|x_i[k'-1]-x_j[k'-1]|>R$ and $|x_i[k']-x_j[k']|\leq R$. Also, no link formation or link break during the time interval $[t_l,k'-1]$ implies that G' is a connected component of $\tilde{G}[k]$ for all $k\in[t_l,k'-1]$. In other words, the influence graph has a connected component that remains constant during the time interval $[t_l,k'-1]$. Therefore, it follows from Proposition 7 that $k'-t_l=O(n^2\log n\cdot d(G_1))=O(n^3\log n)$.

Having seen that the time elapsed between a link break and the first intra- component link formation to occur thereafter is $O(n^3\log n)$, one can repeat the arguments of the previous paragraph to show that the time elapsed between two consecutive intra-component link formations too is $O(n^3\log n)$, provided that no link breaks during the elapsed time.

Next, we estimate the maximum number of link formations that can occur in any opinion evolution process. Note that there are at most n^2 links in an n-vertex graph. So, it may appear that at most n^2 link formations can occur. However, every link break gives rise to the possibility of a link formation. Therefore, the maximum number of link formations is at most $n^2+O(n^5)=O(n^5)$. Hence, if all the link breaks and intra-component link formations were to occur one after the other, then by Lemma 12, all of these events would occur in $O(n^5\cdot n^3\log n+n^5\cdot n^3\log n)=O(n^8\log n)$ steps. On the other hand, it is also possible that some of these events occur simultaneously, so that the last of them occurs even sooner. After all the link breaks and link formations, however, the structure of $\tilde{G}$ remains constant and ε-convergence occurs in $O(n^3\log n)$ additional steps. Thus, $k_\varepsilon(G_{\text{ph}},x[0])=O(n^8(\log n))$ in Case 2.

Finally, since $k_\varepsilon^*(G_{\text{ph}})=\infty$, there exists a set of initial states $\mathcal{X}_0\subset\mathbb{R}^n$ that do not belong to the above two cases, i.e., a merging event occurs during the evolution of the dynamics started at those initial states. $\square$

3.5.2 Sufficient Conditions for Arbitrarily Slow Merging

Since the results of the previous subsection imply that arbitrarily slow merging between two components of G_{ph} is necessary as well as sufficient for arbitrarily slow ε-convergence, it is essential to analyze the concept of arbitrarily slow merging in order to better understand the latter concept. In view of this, we provide conditions on the components of G_{ph} that ensure that the time of merging of the corresponding components of the influence graph is unbounded.

These conditions can be motivated informally as follows. Pick an arbitrary connected component of G_{ph} and partition it into two induced subgraphs, G_{P_0} and G_{Q_0}. Let $1, 2, \ldots, l$ be the nodes of G_{P_0} that are adjacent to one or more nodes of G_{Q_0} in G_{ph} (see Fig. 3.1). We call $[l]$

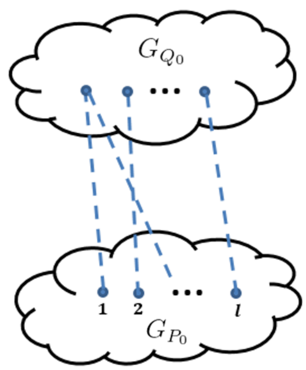

Figure 3.1. Potential Neighbors of G_{Q_0} in G_{P_0}

the set of *potential neighbors of* G_{Q_0} *in* G_{P_0} because only these nodes of G_{P_0} have the potential to influence G_{Q_0}. Now, suppose that at time 0, all the agents of G_{Q_0} have the same opinion $R-\delta$, where δ is a small positive quantity. Then their opinions will remain constant in time as long as G_{P_0} and G_{Q_0} are disconnected in the influence graph. To ensure that G_{P_0} and G_{Q_0} are disconnected in $\tilde{G}[0]$, let us set all the initial opinions of $[l]$ to some sufficiently negative values (see Fig. 3.2). This can be done by setting $x_1[0] = v_1, x_2[0] = v_2, \ldots, x_l[0] = v_l$, where v is an eigenvector of the normalized adjacency matrix A_{P_0} of G_{P_0} such that $v_1, \ldots, v_l$ are all negative. Now, if the corresponding eigenvalue λ is in $(0,1)$, the opinions of G_{P_0} will monotonically increase and approach 0 as time passes, whereas those of G_{Q_0} will remain constant in time (at least initially). At a point in time k_M, the opinion of some potential neighbor of G_{Q_0} exceeds the

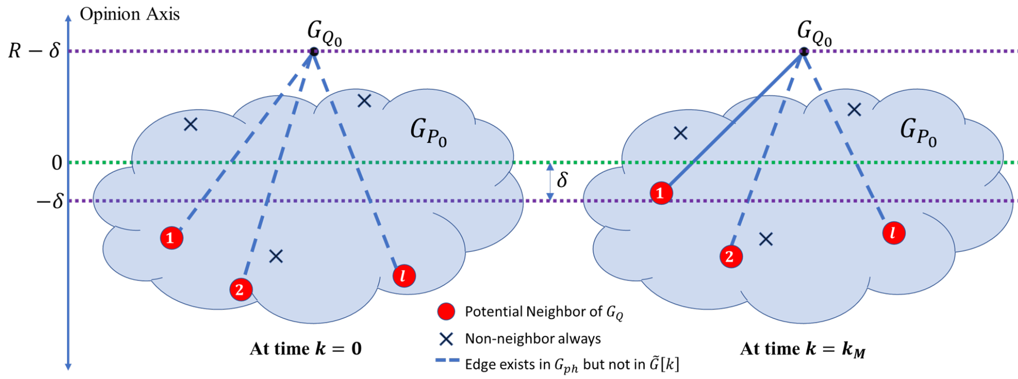

Figure 3.2. Illustration for the Proof of Proposition 9

confidence threshold $-\delta$ (see Fig. 3.2) where G_{Q_0} merges with G_{P_0}. Such an argument shows that we can make k_M arbitrarily large by choosing a sufficiently small δ.

To formalize the above discussion, we need some additional notation. Let V_P and V_Q be the vertex sets of G_{P_0} and G_{Q_0}, respectively, and let $G_P[k] = (V_P, E_P[k])$ and $G_Q[k] = (V_Q, E_Q[k])$ denote the corresponding induced subgraphs of the influence graph $\tilde{G}[k]$. Also, for a given initial state $x_0 \in \mathbb{R}^n$, let $k_M(x_0, V_P, V_Q)$ denote the time at which G_P and G_Q merge for the first time. If no merging occurs, then we set $k_M(x_0, V_P, V_Q) = \infty$. We then have the following result.

Proposition 9. *Let G_{P_0} and G_{Q_0} be two vertex-disjoint induced subgraphs of G_{ph} (as described above), and let $[l] \subset V_P$ be the set of potential neighbors of G_{Q_0} in G_{P_0} (as shown in Fig. 3.1). Suppose the following conditions hold:*

1. *G_{P_0} is a connected graph.*
2. *A_{P_0} has an eigenpair (λ, v) such that $\lambda \in (0,1)$, and $v_1, v_2, \ldots, v_l$ are all positive or all negative (which means that the entries of v corresponding to the potential neighbors of G_{Q_0} within G_{P_0} are of the same sign).*

Then there exists a set $\mathcal{X}_M \subset \mathbb{R}^n$ such that $k_M(x_0, V_P, V_Q) < \infty$ holds for all $x_0 \in \mathcal{X}_M$ but $\sup_{x_0 \in \mathcal{X}_M} k_M(x_0, V_P, V_Q) = \infty$, which means that G_P and G_Q merge if the dynamics start in $\mathcal{X}_M$, but the merging time is unbounded.

Proof. W.l.o.g., let $V_P = \{1, 2, \dots, p\}$ and $V_Q = \{p+1, p+2, \dots, p+q\}$ for some $p, q \in [n]$. Scale v (outlined in condition (2) of the proposition) suitably so as to satisfy $v_i < 0$ for $1 \le i \le l$ and $\max_i v_i - \min_j v_j \le R$. Let $v_0 := -\max_{i \in [l]} v_i$. Consider $\mathcal{X}_M = \{z \in \mathbb{R}^n : z_P = v, z_Q = (R - \delta)\mathbf{1}_q$ for some $\delta \in (0, v_0)\}$. Then observe that if $x[0] \in \mathcal{X}_M$, then $\delta \in (0, v_0)$ ensures that $\tilde{G}[0]$ is a disjoint union of G_{P_0} and G_{Q_0} (and possibly some other connected components). This is because all the potential neighbours of G_Q in G_P, i.e., the nodes $1, 2, \dots l$, are outside the confidence interval $[-\delta, 2R - \delta]$ of every agent in V_Q, and because $\max_i x_{Pi}[0] - \min_j x_{Pj}[0] \le R$ implies that G_{P_0} is an induced subgraph of $\tilde{G}[0]$. Also, note that $\max_i x_{Pi}[0] - \min_j x_{Pj}[0] \le R$ enforces $x_P[1] = A_{P_0} x[0] = \lambda v$.

Now, $\lambda > 0$ implies that $x_{Pi}[1] = \lambda v_i < 0$ for $i \in [l]$. Therefore, G_P and G_Q are also disconnected from each other in $\tilde{G}[1]$ provided $\delta < \lambda v_0$. Similarly, for $k \ge 1$, we have $x_P[k] = \lambda^k v$ implying that G_P and G_Q remain disconnected from each other as long as $\delta < \lambda^k v_0$, i.e., for $k < \log_{1/\lambda}(v_0/\delta)$. However, since $\lambda < 1$, a time is reached when $k = \lceil \log_{1/\lambda}(v_0/\delta) \rceil$ and consequently, merging occurs, i.e., $k_M(x[0], V_P, V_Q) = \lceil \log_{1/\lambda}(v_0/\delta) \rceil < \infty$ because the agent having the opinion $\max_{i \in [l]} x_{Pi}[k] = -\lambda^k v_0$ enters the confidence interval $[-\delta, 2R - \delta]$ of its potential neighbor(s) in G_Q. Therefore, $\sup_{x[0] \in \mathcal{X}_M} k_M(x[0], V_P, V_Q) = \sup_{\delta \in (0, v_0)} \lceil \log_{1/\lambda}(v_0/\delta) \rceil = \infty$. □

3.5.3 Necessary Conditions for Arbitrarily Slow Merging

Having seen a set of sufficient conditions for arbitrarily slow merging, we now try to derive a set of necessary conditions for a pair of subgraphs of the influence graph to exhibit this property. To be specific, we seek to identify conditions on G_{ph} to indefinitely delay the merging of $G_P[k]$ and $G_Q[k]$ for some induced subgraphs G_{P_0} and G_{Q_0} of G_{ph}?

The sufficient conditions in Proposition 9 provide a good starting point for this. For simplicity, let us suppose that Condition 1 of this proposition holds (i.e., G_{P_0} is connected), and let us focus only on Condition 2. Recall that this condition serves two purposes:

1. At time 0, the sign pattern of v allows the nodes $1, \dots, l$ (the potential neighbors of G_Q) to be outside the confidence interval $[-\delta, 2R-\delta]$ of G_Q as well as to *lie on the same side* of this confidence interval.

2. In the limit as $k \to \infty$, the requirement $\lambda \in (0,1)$ ensures that the opinions of these nodes *approach 0 monotonically*.

As a result, merging occurs exactly when the opinions of the potential neighbors of G_Q move sufficiently close to 0 (i.e., cross the threshold $-\delta$). In this manner, Condition 2 guarantees the occurrence of the desired merging event while simultaneously allowing us to delay this event indefinitely.

This discussion motivates us to ask: if Condition 2 is violated, is arbitrarily slow merging still possible? Can it happen by some other means? As per the intuition described below, the answer is likely to be 'no'.

Intuition

Suppose Condition 2 of Proposition 9 is violated. For simplicity, let us consider a special case: let us assume that all the agents of G_{Q_0} have the same initial opinion, say R, so that the state of $G_Q[k]$ is fixed until the merging time $k_M = k_M(x[0], V_P, V_Q)$. In addition, let us assume that the initial state $x_P[0]$ of G_P can be expressed as a linear combination of the eigenvectors of A_{P_0} corresponding to eigenvalues having distinct magnitudes, i.e., $x_P[0] = c_0\mathbf{1} + \sum_{j=2}^{m} v_j$ for some $c_0 \in \mathbb{R}$, where $\{(\lambda_j, v_j)\}_{j=2}^{m}$ is a set of eigenpairs of A_{P_0} satisfying $|\lambda_2| > \cdots > |\lambda_m|$. Finally, suppose $G_P[k]$ remains constant and connected until it merges with G_Q. Then, observe that the state of G_Q evolves as $x_P[k] = c_0\mathbf{1} + \sum_{j=2}^{m} \lambda_j^k v_k$ for $k \in [0, k_M)$, implying that the opinion of every agent of G_P moves closer to c_0 with time (because $|\lambda_j| < 1$ by Lemma 9).

Now, the confidence interval of every agent of $G_Q[0]$ is $[0, 2R]$. So, if c_0 is very far from $[0, 2R]$, then G_P may never merge with G_Q (because $x_P[k]$ moves closer to $c_0\mathbf{1}$ with time). On the other hand, if c_0 is close to the midpoint R of $[0, 2R]$, then merging is likely to occur when

G_P achieves R-convergence to $c_0\mathbf{1}$. So, by Proposition 7, k_M is likely to be bounded in this case. Therefore, we assume that either $|c_0 - 0|$ or $|c_0 - 2R|$ is very small.

Next, since the eigenvalues have distinct magnitudes, we have $\sum_{j=2}^{m} \lambda_j^k v_j \approx \lambda_2^k v_2$ for sufficiently large k, i.e., the eigenvector v_2 *dominates* the difference vector $x_P[k] - c_0\mathbf{1}$. Since we are examining the boundedness of merging time k_M, k_M can be assumed to be much greater than the time around which v_2 dominates the difference vector. Thus, we may ignore the effects of other eigenvectors on $x_P[k]$.

With these approximations, if $\lambda_2 > 0$, then the violation of Condition 2 of Proposition 9 is likely to imply that $(x_P[k])_{[l]}$ has entries *on either side* of c_0 (i.e., it has some entries that are less than c_0 as well as some entries that are greater than c_0). Since c_0 is very close to either 0 or $2R$ (say, c_0 is close to 0), then this means that at least a few of the nodes $1, \ldots, l$ are likely to be within the confidence interval $[0, 2R]$ of G_Q (because their opinions are greater than c_0), implying that merging occurs as soon as v_2 becomes dominant.

On the other hand, if $\lambda_2 < 0$, then every non-zero entry of $x_P[k] - c_0\mathbf{1} \approx (-1)^k |\lambda_2|^k v_2$ keeps flipping its sign as k increases, meaning that the potential neighbors' opinions keep oscillating about c_0. Once again, since c_0 is very close to 0 or $2R$, this suggests that G_P will merge with G_Q as soon as v_2 begins to dominate.

Hence, the only way to indefinitely delay merging is to indefinitely delay the emergence of v_2 as the dominant vector. One way to do so is to scale down v_2 appropriately so that its entries becomes negligible relative to those of some other eigenvector v_j (where $j \geq 3$). However, this would result in v_j dominating $x_P[k] - c_0\mathbf{1}$ for small values of k, in which case we can simply repeat the above arguments to show that the domination of v_j would lead to immediate merging.

The above discussion suggests that arbitrarily slow merging is possible only if Condition 2 of Proposition 9 is violated. We state this formally in Lemma 16. However, in order to prove this lemma, we need to extend our intuition to account for more general initial conditions.

Extension to More General Initial States

Above, we argued that if Condition 2 of Proposition 9 is violated, and if there exists a dominant eigenpair (λ, v), then the sign of λ and the sign pattern of v would possibly imply that some potential neighbors of G_Q would have opinions that are less than c_0, and some whose opinions would exceed c_0 (both of which happen only periodically if $\lambda < 0$). Thus, both $(-\infty, c_0)$ and (c_0, ∞) get occupied by the opinions of $[l]$. This insight suggested that the merging time is bounded, which is enabled by the fact that the eigenvalues are distinct in magnitude and hence, we have just one dominant eigenvector and ignore the rest.

For an arbitrary $x_P[0] \in \mathbb{R}^{|V_P|}$, we have $x_P[0] = c_0\mathbf{1} + \sum_{i=2}^{\tau} a_i v_i$ for some $c_0 \in \mathbb{R}$, $\tau = |V_P|$ and $a_i \in \{0, 1\}$, where (λ_i, v_i) are the eigenpairs of A_{P_0} ordered as $|\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_\tau|$ but with possible repetitions in these absolute values. In this setting, v_2 may not necessarily dominate $x_P[k] - c_0\mathbf{1}$ for large k. This is because the following cases may arise.

Case 1: $\lambda_2 = \lambda_3 = \cdots = \lambda_m$ *for some* $m \leq \tau$*, and* $|\lambda_m| > |\lambda_{m+1}|$ *if* $m < \tau$. In this case, we can simply combine $v_2, \ldots, v_m$ into a single eigenvector $v_2' = \sum_{j=2}^{m} a_j v_j$, and observe that v_2' dominates $x_P[k] - c_0\mathbf{1}$ for large k.

Case 2: $\lambda_2 = \cdots = \lambda_s = -\lambda_{s+1} = \cdots = -\lambda_m$ *for some* $s < m \leq \tau$*, and* $|\lambda_m| > |\lambda_{m+1}|$ *if* $m < \tau$. W.l.o.g. suppose $\lambda_2 > 0$. Then we can combine the contributions of $v_2, \ldots, v_s$ by setting $v_2^+ := \sum_{j=2}^{s} a_j v_j$. Similarly, we set $v_2^- := \sum_{j=s+1}^{m} a_j v_j$, and observe that for sufficiently large k, we have $x_P[k] - c_0\mathbf{1} \approx \lambda_2^k v_2'$ for even k, whereas $x_P[k] - c_0\mathbf{1} \approx \lambda_2^k v_2''$ for odd k, where $v_2' := v_2^+ + v_2^-$ and $v_2'' := v_2^+ - v_2^-$.

We use such combinations of the eigenvectors of A_{P_0} to rigorously address the problem at hand. In addition, we need to consider the transient phases of opinion evolution during which combinations of some other eigenvectors (those corresponding to smaller $|\lambda_i|$) may dominate. To address this, the comparison between $\|(v_2)_{[l]}\|, \ldots, \|(v_\tau)_{[l]}\|$ becomes important. For these purposes, we introduce the *Elimination Method* below.

The Elimination Method

Let $\{\lambda_i\}_{i=1}^{\tau} \subset \mathbb{R}$, $\{u_{\boldsymbol{i}}\}_{i=1}^{\tau} \subset \mathbb{R}^n$ and $l \in [n]$ be fixed, and let $S[k] := \sum_{i=1}^{\tau} \lambda_i^k u_{\boldsymbol{i}}$ for each $k \in \mathbb{N}$, so that $S[k]$ has the same form as $x_P[k] - c_0\mathbf{1}$ had in the preceding subsection. Then the Elimination Method entails the following:

1. Find a minimal set of real numbers, $\{\mu_i\}_{i=1}^{m}$, such that $\{\lambda_i\}_{i=1}^{n} = \{\pm\mu_i\}_{i=1}^{m} \cup \{0\}$, and $\mu_1 > \cdots > \mu_m > 0$. Distinguishing between the magnitudes of $\{\lambda_i\}_{i=1}^{n}$ helps us identify the dominant vector combinations.

2. For each $i \in [m]$, find $\nu_i, \sigma_i \in [\tau]$ satisfying $\lambda_{\nu_i} = \mu_i = -\lambda_{\sigma_i}$, and define $u_{\boldsymbol{i}}^+ := u_{\boldsymbol{\nu}_i}$ and $u_{\boldsymbol{i}}^- := u_{\boldsymbol{\sigma}_i}$. If no such ν_i (respectively, σ_i) exists, then set $u_{\boldsymbol{i}}^+ = 0$ (respectively, $u_{\boldsymbol{i}}^- = 0$).

3. For $i \in [m]$, define $\alpha_i = \max_{j \in [l]} |u_{\boldsymbol{ij}}^+ + u_{\boldsymbol{ij}}^-|$ and $\zeta_i = \max_{j \in [l]} |u_{\boldsymbol{ij}}^+ - u_{\boldsymbol{ij}}^-|$. Further, define $v_{\boldsymbol{i}}$ by: $v_{\boldsymbol{i}} := \left(u_{\boldsymbol{i}}^+ + u_{\boldsymbol{i}}^-\right)/\alpha_i$ if $\alpha_i \neq 0$ and $v_{\boldsymbol{i}} := 0$ otherwise. Likewise, let $z_{\boldsymbol{i}} := \left(u_{\boldsymbol{i}}^+ - u_{\boldsymbol{i}}^-\right)/\zeta_i$ if $\zeta_i \neq 0$ and let $z_{\boldsymbol{i}} := 0$ otherwise. This is analogous to defining v_2' and v_2'' (as above), finding the infinity-norms and normalizing them with respect to the evaluated norms.

4. If $\alpha_i = \zeta_i = 0$, discard μ_i from $\{\mu_j\}_{j=1}^{m}$, decrement the value of m by 1, and re-enumerate $\{\mu_j\}_{j=1}^{m}$ so that $\mu_1 > \mu_2 > \cdots > \mu_m$. This helps us remove redundant terms from the summation.

The following observations illustrate the rationale behind the above method. First, we have the following for every $j \in [l]$:

$$S_j[k] = \sum_{i=1}^{m} \alpha_i \mu_i^k v_{\boldsymbol{ij}} \quad \forall\, k \in \mathbb{N} : k \text{ is even,}$$
$$S_j[k] = \sum_{i=1}^{m} \zeta_i \mu_i^k z_{\boldsymbol{ij}} \quad \forall\, k \in \mathbb{N} : k \text{ is odd.} \tag{3.11}$$

Hence, the vectors, $\{v_{\boldsymbol{i}}\}_{i=1}^{m}$ and $\{z_{\boldsymbol{i}}\}_{i=1}^{m}$ will be called even-k vectors and odd-k vectors respectively. Second, $\{(\mu_i, v_{\boldsymbol{i}})\}_{i=1}^{m}$ and $\{(\mu_i, z_{\boldsymbol{i}})\}_{i=1}^{m}$ behave like sets of eigenpairs with distinct eigenvalue magnitudes. Third, if we restrict our attention to the first l entries alone, then $v_{\boldsymbol{1}},\ldots,v_{\boldsymbol{m}}$ and $z_{\boldsymbol{1}},\ldots,z_{\boldsymbol{m}}$ have the same 'strength', because $\|(z_{\boldsymbol{i}})_{[l]}\| = \|(v_{\boldsymbol{i}})_{[l]}\| = 1$ for each $i \in [m]$. Thus, the scaling of these vectors is completely captured by $\{\alpha_i\}$ and $\{\zeta_i\}$. Finally, note that at least one of $\{\alpha_i, \zeta_i\}$ is comparable to $\|(u_{\boldsymbol{i}}^+)_{[l]}\|$ because Triangle Inequality yields

$$\|(2u_{\boldsymbol{i}}^+)_{[l]}\| \le \|(\alpha_i v_{\boldsymbol{i}})_{[l]}\| + \|(\zeta_i z_{\boldsymbol{i}})_{[l]}\| = \alpha_i + \zeta_i. \tag{3.12}$$

Let us now put the Elimination Method in the context of arbitrarily slow merging. Recall the notation of the previous subsection, and suppose $x_j[k] - c_0 = S_j[k]$ for all $j \in V_P$. Then, going by our earlier arguments, in order to show that Condition 2 of Proposition 9 is necessary for arbitrarily slow merging, it seems enough to show that $v_{\boldsymbol{r}}$ and $z_{\tilde{\boldsymbol{r}}}$ collectively have significant positive and negative elements, where $v_{\boldsymbol{r}}$ (respectively, $z_{\tilde{\boldsymbol{r}}}$) is the vector that dominates $S[k]$ for even k (respectively, for odd k). This idea motivates the following lemma, which plays a key role in the proof of Lemma 15.

Lemma 14. *Suppose we have performed the Elimination Method on $\{S_j[k]\}_{j=1}^l$ (defined above). Also, suppose that for every pair $(\lambda_i, u_{\boldsymbol{i}})$ satisfying $\lambda_i > 0$ and $(u_{\boldsymbol{i}})_{[l]} \neq 0$, we have $\max_{p\in[l]} u_{\boldsymbol{ip}} > 0$, $\min_{p\in[l]} u_{\boldsymbol{ip}} < 0$ and $|\max_{p\in[l]} u_{\boldsymbol{ip}}|/|\min_{q\in[l]} u_{\boldsymbol{iq}}| \ge \gamma_0$, where $\gamma_0 \in \mathbb{R}^+$ is a constant. Further, for each $i \in [m]$, let $p(i) := \arg\max_{j\in[l]} v_{\boldsymbol{ij}}$, and $\tilde{p}(i) := \arg\max_{j\in[l]} z_{\boldsymbol{ij}}$. Then for every $i \in [m]$, we have*

$$\max\left(v_{\boldsymbol{ip}(i)}, z_{\boldsymbol{i}\tilde{p}(i)}\right) \ge \hat{\gamma}_0,$$

where $\hat{\gamma}_0 := \frac{\gamma_0}{2+\gamma_0}$. Furthermore, if $v_{\boldsymbol{ip}(i)} < \hat{\gamma}_0$ (respectively, $z_{\boldsymbol{i}\tilde{p}(i)} < \hat{\gamma}_0$), then $\zeta_i/\alpha_i \ge \hat{\gamma}_0$ (respectively $\alpha_i/\zeta_i \ge \hat{\gamma}_0$).

Proof. Consider any $i \in [m]$ and let $q(i) := \arg\min_{j=1}^l v_{\boldsymbol{ij}}$ and $\tilde{q}(i) := \arg\min_{j=1}^l z_{\boldsymbol{ij}}$.

Now, two possibilities arise: either $(u_{\boldsymbol{i}}^+)_{[l]} = 0$ or $(u_{\boldsymbol{i}}^+)_{[l]} \neq 0$. If $(u_{\boldsymbol{i}}^+)_{[l]} = 0$, then $(v_{\boldsymbol{i}})_{[l]} = -(z_{\boldsymbol{i}})_{[l]}$. Hence, either $v_{\boldsymbol{ip}(i)} \ge |v_{\boldsymbol{iq}(i)}|$ or $z_{\boldsymbol{i}\tilde{p}(i)} \ge |z_{\boldsymbol{i}\tilde{q}(i)}|$. Since it turns out that $\max\left(\max_{f\in[l]}|v_{\boldsymbol{if}}|, \max_{f\in[l]}|z_{\boldsymbol{if}}|\right) = 1 \ge \hat{\gamma}_0$ due to the Elimination Method, we infer that

$\max\left(v_{\boldsymbol{i}p(i)}, z_{\boldsymbol{i}\tilde{p}(i)}\right) \geq \hat{\gamma}_0$.

On the other hand, if $(u_{\boldsymbol{i}}^+)_{[l]} \neq 0$, then $\mu_i = \lambda_{\nu_i}$ for some $\nu_i \in [\tau]$. Hence, $\max_{f\in[l]} u_{\boldsymbol{i}f}^+ > 0$ and $|\max_{f\in[l]} u_{\boldsymbol{i}f}^+|/|\min_{f\in[l]} u_{\boldsymbol{i}f}^+| \geq \gamma_0$. In the light of Lemma 19, this implies that either

$$\alpha_i v_{\boldsymbol{i}p(i)} > 0 \text{ and } |v_{\boldsymbol{i}p(i)}| \geq \hat{\gamma}_0 |v_{\boldsymbol{i}q(i)}|, \tag{3.13}$$

or

$$\zeta_i z_{\boldsymbol{i}\tilde{p}(i)} > 0 \text{ and } |z_{\boldsymbol{i}\tilde{p}(i)}| \geq \hat{\gamma}_0 |z_{\boldsymbol{i}\tilde{q}(i)}|. \tag{3.14}$$

If (3.13) holds, then as a result of the Elimination Method, we have $1 = \max_{f\in[l]} |v_{\boldsymbol{i}f}| = \max\left(|v_{\boldsymbol{i}p(i)}|, |v_{\boldsymbol{i}q(i)}|\right)$. Since $\alpha_i \geq 0$, this means that either $|v_{\boldsymbol{i}p(i)}| = 1 \geq \hat{\gamma}_0$, or $|v_{\boldsymbol{i}p(i)}| \geq \hat{\gamma}_0 |v_{\boldsymbol{i}q(i)}| = \hat{\gamma}_0$. Thus, $|v_{\boldsymbol{i}p(i)}| \geq \hat{\gamma}_0$ in either subcase. Similarly, (3.14) leads to the conclusion that $|z_{\boldsymbol{i}\tilde{p}(i)}| \geq \hat{\gamma}_0$.

For the second part, suppose $v_{\boldsymbol{i}p(i)} < \hat{\gamma}_0$. Then $z_{\boldsymbol{i}\tilde{p}(i)} \geq \hat{\gamma}_0$ by the first assertion. We now consider two cases.

Case (a): $\max_{f\in[l]} |v_{\boldsymbol{i}f}| = 0$, implying that $\alpha_i = 0$. By (3.14), $\zeta_i \neq 0$. Thus, $\zeta_i/\alpha_i = \infty > \hat{\gamma}_0$.

Case (b): $\max_{f\in[l]} |v_{\boldsymbol{i}f}| > 0$. Then $\max_{f\in[l]} |v_{\boldsymbol{i}f}| = 1$ due to the Elimination Method. Furthermore, $\hat{\gamma}_0 < 1$ by the definitions of γ_0 and $\hat{\gamma}_0$. Now, the assumption $v_{\boldsymbol{i}p(i)} < \hat{\gamma}_0$ and the facts $\max_{f\in[l]} |v_{\boldsymbol{i}f}| = 1$, $\hat{\gamma}_0 < 1$ and $\max_{f\in[l]} |v_{\boldsymbol{i}f}| = \max(|v_{\boldsymbol{i}p(i)}|, |v_{\boldsymbol{i}q(i)}|)$ together imply that $|v_{\boldsymbol{i}q(i)}| = 1$. Consequently, $v_{\boldsymbol{i}p(i)} < \hat{\gamma}_0 |v_{\boldsymbol{i}q(i)}|$. Therefore, by Lemma 19,

$$\begin{aligned} \zeta_i z_{\boldsymbol{i}\tilde{p}(i)} &\geq \frac{\gamma_0 |\alpha_i v_{\boldsymbol{i}q(i)}| - \max\left(\alpha_i v_{\boldsymbol{i}p(i)}, 0\right)}{\gamma_0 + 1} \\ &= \frac{\gamma_0 \alpha_i - \max\left(\alpha_i v_{\boldsymbol{i}p(i)}, 0\right)}{\gamma_0 + 1}. \end{aligned}$$

If $v_{\boldsymbol{i}p(i)} \leq 0$, then the above yields:

$$\frac{\zeta_i}{\alpha_i} \geq \frac{\gamma_0}{(\gamma_0+1)z_{\boldsymbol{i}\tilde{p}(i)}} \geq \frac{\gamma_0}{\gamma_0+1} > \hat{\gamma}_0$$

because $0 < z_{\boldsymbol{i}\tilde{p}(i)} \leq \max_{f\in[l]} |z_{\boldsymbol{i}f}| = 1$. On the other hand, if $v_{\boldsymbol{i}p(i)} > 0$, then

$$\frac{\zeta_i}{\alpha_i} \geq \frac{\gamma_0 - v_{\boldsymbol{i}p(i)}}{\gamma_0+1} > \frac{\gamma_0 - \hat{\gamma}_0}{\gamma_0+1} = \hat{\gamma}_0.$$

□

When Lemma 14 is applied to $(\lambda_i, u_{\boldsymbol{i}})$ and $(\lambda_i, -u_{\boldsymbol{i}})$, it effectively means that under conditions similar to the violation of Condition 2 of Proposition 9, $v_{\boldsymbol{i}}$ and $z_{\boldsymbol{i}}$ collectively have significant positive and negative elements.

A Preliminary Result

We are now ready to make the above arguments precise and prove that Condition 2 of Proposition 9 is nearly necessary for arbitrarily slow merging. We first state this result formally.

Lemma 15. *For every initial state $x[0] \in \mathbb{R}^n$, let $G_P[k] = G_P(x[k]) = (V_P, E_P(x[k]))$ and $G_Q[k] = G_Q(x[k]) = (V_Q, E_Q(x[k]))$ be two vertex-disjoint induced subgraphs of $\tilde{G}[k]$ such that G_{P_0}, the subgraph of G_{ph} induced by V_P, is connected. Also, let $\mathcal{X}$ denote the set of all $x[0] \in \mathbb{R}^n$ satisfying assumptions below:*

(a). All the agents of $G_Q[0]$ have the same opinion value, i.e., $x_i[0] = x_Q$ for all $i \in V_Q$, where $x_Q \in \mathbb{R}$ is constant in time but depends on $x[0]$.

(b). $k_M(x[0], V_P, V_Q) < \infty$, i.e., G_P and G_Q indeed merge.

(c). $G_P[k]$ is connected and constant until time k_M.

Furthermore, for some $l \in \mathbb{R}$, let $[l]$ index the set of nodes of $G_P[0]$ that are adjacent to one or more nodes of $G_Q[0]$ in the graph G_{ph}, as shown in Fig. 3.3.

Now, suppose $\sup_{x[0]\in\mathcal{X}} k_M(x[0]) = \infty$*. Then* A_{P_0} *has an eigenpair* (λ, v) *such that* $0 < \lambda < 1$*,* $v_i \neq 0$ *for some* $i \in [l]$*, and* $v_i v_j \geq 0$ *for all* $i, j \in [l]$*.*

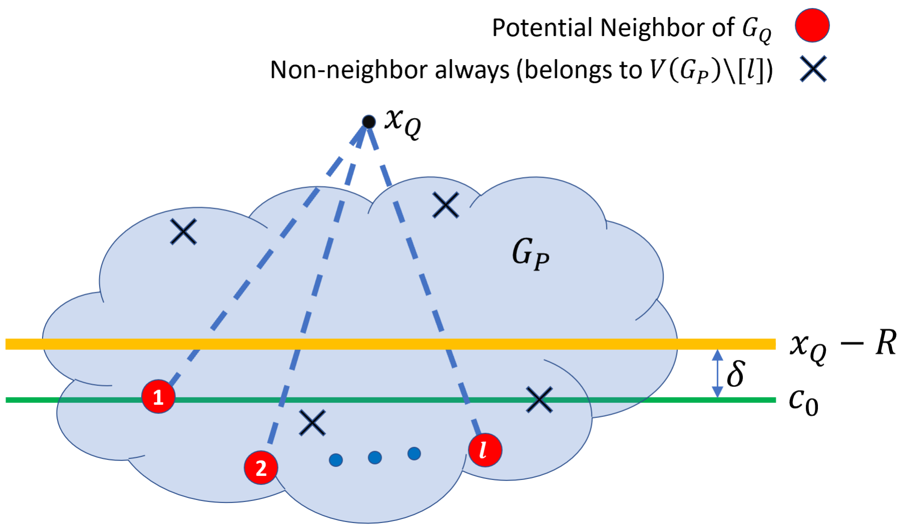

Figure 3.3. Illustration for the Proof of Lemma 15

We prove this lemma by showing that it is mathematically equivalent to Lemma 16, a purely technical result. To this end, suppose all the agents of G_Q have the same initial opinion $x_Q = R$, and that Assumption (b). of Lemma 15 holds. Then, we can express the opinion evolution of G_P during the interval $[0, k_M)$ as $x_P[k] = c_0\mathbf{1} + \sum_{i=1}^{\tau} \lambda_i^k u_{\boldsymbol{i}}$, where $c_0 \in \mathbb{R}$, and $\{(\lambda_i, u_{\boldsymbol{i}})\}_{i=1}^{\tau} \cup \{(1, \mathbf{1})\}$ are the eigenpairs of A_{P_0}. Now, as per Fig. 3.3, the lower confidence threshold of G_Q is $x_Q - R$. Thus, G_P and G_Q merge at time k_M iff the following hold: $x_P[k] < (x_Q - R)\mathbf{1}$ for all $k < k_M$, and $x_j[k_M] \geq x_Q - R$ for some $j \in [l]$. These relations are equivalent to:

$$\sum_{i=1}^{\tau} \lambda_i^k u_{\boldsymbol{i}j} < \delta \quad \text{for all } 1 \leq k < k_M \text{ and all } j \in [l], \tag{3.15}$$

and

$$\sum_{i=1}^{\tau} \lambda_i^{k_M} u_{\boldsymbol{i}t} \geq \delta \quad \text{for some } t \in [l], \tag{3.16}$$

where $\delta := x_Q - R - c_0$ (see Fig. 3.3). Note that δ may or may not be positive. Also, for any i, the eigenvector $u_{\boldsymbol{i}}$ need not even be unique up to a scaling factor because λ_i can be a repeated eigenvalue of A_{P_0}. Nevertheless, $u_{\boldsymbol{i}}$ is constrained to belong to U_i, the λ_i-eigenspace of A_{P_0}.

In light of the above, arbitrarily slow merging is equivalent to $\sup \mathcal{K} = \infty$, where $\mathcal{K}$ is the set of all merging times that satisfy the above relations. This motivates the next lemma.

Lemma 16. *Let $\{\lambda_i\}_{i=1}^{\tau}$ and $\{U_i\}_{i=1}^{\tau}$ be such that $|\lambda_i| < 1$ and U_i is a linear subspace of $\mathbb{R}^l$ for each $i \in [\tau]$, where $l \in \mathbb{N}$. Further, suppose $\lambda_f > 0$ for some $f \in [\tau]$. Let $\mathcal{K}$ be the set of all $k_M \in \mathbb{N}$ such that* (3.15) *and* (3.16) *hold for some $\delta \in \mathbb{R}$ and some $(u_{\boldsymbol{1}}, \ldots, u_{\boldsymbol{\tau}}) \in \prod_{i=1}^{\tau} U_i$. If $\sup \mathcal{K} = \infty$, then there exists a $d \in [\tau]$ such that $\lambda_d > 0$, and there exists a corresponding non-zero vector $v \in U_d$ such that $v_i v_j \geq 0$ for all $i, j \in [l]$.*

To put it simply, Lemma 16 asserts that if we wish to have arbitrarily slow merging between an evolving network G_P and a static network G_Q, then Condition 2 of Proposition 9 must be satisfied, except for the possible presence of zeros in the concerned eigenvectors.

Proof of Lemma 16

We first sketch the proof outline below:

1. Suppose that $\sup \mathcal{K} = \infty$ but the implication of the lemma does not hold. Then, for any $f \in [\tau]$ such that $\lambda_f > 0$, every $u_{\boldsymbol{f}} \in U_f$ has significant positive and negative elements (by Lemma 18). We use this observation later on.

2. As time goes by, only one of the component vectors of $\sum_i \lambda_i^k u_{\boldsymbol{i}}$ remains significant. Typically, this is the one corresponding to the largest $|\lambda_i|$. However, this is not necessary because of possible repetitions in $\{|\lambda_i|\}_{i=1}^{\tau}$ and because the norms of $\{u_{\boldsymbol{i}}\}_{i=1}^{\tau}$ may not be comparable to each other. To resolve these two issues, we perform the Elimination method and obtain $x_j[k] - c_0 = S_j[k]$, where $S_j[k]$ is given by (3.11).

3. To use the assumption $\sup \mathcal{K} = \infty$ effectively, we define ρ_{s1}, ρ_{s2}, and a few more auxiliary quantities in terms of $\{\alpha_i\}_{i=1}^{m}, \{v_{\boldsymbol{i}}\}_{i=1}^{m}, \{z_{\boldsymbol{i}}\}_{i=1}^{m}$ and $\{\zeta_i\}_{i=1}^{m}$, and consider an increasing

sequence of merging times $\{k_M^{(h)}\}_{h=1}^{\infty}$. We then try to identify the most dominant vector among $\{v_{\boldsymbol{i}}\}_{i=1}^m$ by identifying an index r such that for $h \in \mathbb{N}$:

$$\alpha_r^{(h)} \geq \frac{\alpha_i^{(h)}}{\rho_{r1}^{(h)}} \text{ for some } \rho_{r1}^{(h)} > 0 \text{ and all } 1 \leq i < r, \text{ and}$$

$$\alpha_r^{(h)} \geq \rho_{r2}^{(h)} \alpha_i^{(h)} \text{ for some } \rho_{r2}^{(h)} > 0 \text{ and all } r < i \leq m,$$

where $\lim_{h\to\infty} \rho_{r1}^{(h)} = 0$ but $\lim_{h\to\infty} \rho_{r2}^{(h)} > 0$. The rationale is that for large h and large k, the relations $\rho_{r2}^{(h)} > 0$ and $\mu_r^k \gg \cdots \gg \mu_m^k$ guarantee the domination of $v_{\boldsymbol{r}}$ over $v_{\boldsymbol{r+1}}, \ldots, v_{\boldsymbol{m}}$, whereas $\rho_{r1}^{(h)} \approx 0$ ensures that $v_{\boldsymbol{r}}$ dominates $v_{\boldsymbol{1}}, \ldots, v_{\boldsymbol{r-1}}$. Similarly, we identify $z_{\tilde{\boldsymbol{r}}}$, the most dominant vector among $\{z_{\boldsymbol{i}}\}_{i=1}^m$.

4. We then use (3.15) and (3.16) to show that, along the sequence $\{k_M^{(h)}\}_{h=1}^{\infty}$, the greatest positive entry $v_{\boldsymbol{rp}}$ of $v_{\boldsymbol{r}}$ is eventually insignificant. Similarly, the greatest positive entry $z_{\tilde{\boldsymbol{r}}\tilde{\boldsymbol{p}}}$ of $z_{\tilde{\boldsymbol{r}}}$ is eventually insignificant.

5. We use the above observation along with Lemma 14 to reach a contradiction.

We now prove the lemma by following this proof outline.

Proof. Suppose that $\sup \mathcal{K} = \infty$ and that for every $d \in [\tau]$ such that $\lambda_d > 0$, every vector $v \in U_d \setminus \{0\}$ has both positive and negative entries.

Step 1: If $\lambda_d > 0$, then by Lemma 18 and the above assumption, there exists a positive constant γ_d that lower bounds the ratios $|\max_{p\in[l]} v_p|/|\min_{q\in[l]} v_q|$ and $|\min_{p\in[l]} v_p|/|\max_{q\in[l]} v_q|$ for all $v \in U_d \setminus \{0\}$. Hence, the positive constant $\gamma_0 := \min_{d\in[\tau]:\lambda_d>0} \gamma_d$ lower bounds these ratios for every $d \in [\tau]$ for which $\lambda_d > 0$. Thus, every vector lying in $U_d \setminus \{0\}$ has significant positive and negative entries.

Step 2: We now perform the Elimination Method so as to obtain the values of $\{\alpha_i\}_{i=1}^m$, $\{\zeta_i\}_{i=1}^m$, $\{v_{\boldsymbol{i}}\}_{i=1}^m$ and $\{z_{\boldsymbol{i}}\}_{i=1}^m$ for which $S_j[k] = \sum_{i=1}^{\tau} \lambda_i^k u_{\boldsymbol{ij}}$ satisfies (3.11). Here we make a few observations. First, $\mu_i < 1$ for all $i \in [m]$ because $|\lambda_i| < 1$ for all $i \in [m]$. Second, if k_M is even and $k_M \geq 4$, then (3.15) and (3.16) imply that $\alpha_i > 0$ for some $i \in [m]$. W.l.o.g., we assume that k_M is even from now onwards.

Step 3: We let $\hat{\gamma}_0 := \gamma_0/(2+\gamma_0)$, and for each $s \in [m]$ that satisfies $\alpha_s > 0$, we define the following quantities:

$$\rho_{s1} = \max_{i \leq s-1} \frac{\alpha_i}{\alpha_s}, \quad \rho_{s2} = \min_{i \geq s} \frac{\alpha_s}{\alpha_i},$$

$$p(s) \in \arg\max_{j \in [l]} v_{\boldsymbol{s}j}, \quad q(s) \in \arg\min_{j \in [l]} v_{\boldsymbol{s}j},$$

$$v_{\boldsymbol{s}0} = \begin{cases} v_{\boldsymbol{s}p(s)}, & \text{if } v_{\boldsymbol{s}p(s)} \geq \hat{\gamma}_0 \\ 1, & \text{otherwise} \end{cases}, \text{ and}$$

$$\tau_s = \frac{v_{\boldsymbol{s}0}}{\max_{r \in [l]} \left(\sum_{i=s}^{m} |v_{\boldsymbol{i}r}| \right)}.$$

Similarly, for each $s \in [m]$ satisfying $\zeta_s \neq 0$, we define:

$$\tilde{\rho}_{s1} = \max_{i \leq s-1} \frac{\zeta_i}{\zeta_s}, \quad \tilde{\rho}_{s2} = \min_{i \geq s} \frac{\zeta_s}{\zeta_i},$$

$$\tilde{p}(s) \in \arg\max_{j \in [l]} z_{\boldsymbol{s}j}, \quad \tilde{q}(s) \in \arg\min_{j \in [l]} z_{\boldsymbol{s}j},$$

$$z_{\boldsymbol{s}0} = \begin{cases} z_{\boldsymbol{s}\tilde{p}(s)}, & \text{if } z_{\boldsymbol{s}\tilde{p}(s)} \geq \hat{\gamma}_0 \\ 1, & \text{otherwise} \end{cases}, \text{ and}$$

$$\tilde{\tau}_s = \frac{z_{\boldsymbol{s}0}}{\max_{r \in [l]} \left(\sum_{i=s}^{m} |z_{\boldsymbol{i}r}| \right)}.$$

We also let $\rho_{11} = 0$ if $\alpha_1 > 0$ and $\rho_{m2} = 1$ if $\alpha_m > 0$. Similarly, $\tilde{\rho}_{11} = 0$ if $\zeta_1 > 0$ and $\tilde{\rho}_{m2} = 1$ if $\zeta_m > 0$.

With the above, we can easily show that $\tau_s \in [\frac{\hat{\gamma}_0}{n}, 1]$.

We now analyze the evolution of the quantities defined above as $k_M \to \infty$. Consider any sequence, $\{y^{(h)}\}_{h=1}^{\infty} = \{(u_{\boldsymbol{1}}^{(h)}, \ldots, u_{\boldsymbol{\tau}}^{(h)}, \delta^{(h)})\}_{h=1}^{\infty}$, of variables associated with an increasing and unbounded sequence of solutions $\{k_M^{(h)}\}_{h=1}^{\infty} \subset \mathcal{K}$. Since $m < \infty$, there exists an index

$M_e \in [m]$ and a subsequence $\{y^{(h_g)}\}_{g=1}^{\infty}$ of the original sequence $\{y^{(h)}\}_{h=1}^{\infty}$ such that $M_e \in \arg\max_{i\in[m]} \alpha_i^{(h_g)}$ (where $\alpha_i^{(h)} := \alpha_i(y^{(h)})$), for all $g \in \mathbb{N}$. Pick such a subsequence and relabel it as $\{y^{(h)}\}_{h=1}^{\infty}$, so that $0 \leq \alpha_i^{(h)}/\alpha_{M_e}^{(h)} \leq 1$ for all $i \in [m]$. Now that $\{\alpha_i^{(h)}/\alpha_{M_e}^{(h)}\}_{h=1}^{\infty}$ is bounded for each $i \in [m]$, we may assume (by passing to yet another subsequence if necessary) that $\eta_i := \lim_{h\to\infty} \alpha_i^{(h)}/\alpha_{M_e}^{(h)}$ exists for each $i \in [m]$.

Now, let $r = \min\{i \in [m] : \eta_i > 0\}$. Then $\eta_b = 0$ for $b \in [r-1]$ and hence, $\lim_{h\to\infty} \rho_{r1}^{(h)} = 0$ whereas $\lim_{h\to\infty} \rho_{r2}^{(h)} = \eta_r > 0$. Thus, for sufficiently large h, we have:

$$\eta_r/2 < \rho_{r2}^{(h)} \leq 3\eta_r/2, \tag{3.17}$$

$k_M^{(h)}$ is large enough, and $\rho_{r1}^{(h)}$ is small enough (as will be made precise later). Moreover, since $\mu_r^k \gg \mu_{r+1}^k \gg \cdots \gg \mu_m^k$ for large k, we observe that $v_{\boldsymbol{r}}$ dominates $S_j[2k]$ for large k.

Step 4: Our next goal is to show that

$$v_{\boldsymbol{r}p(r)}^{(h)} < \hat{\gamma}_0. \tag{3.18}$$

for sufficiently large h. To this end, we restrict our focus to even values of k, assume that h is sufficiently large, and drop the superscript $^{(h)}$ to reduce clutter in notation. Since the detailed proof of (3.18) is tedious, we relegate it to Appendix C and only present the informal argument below.

Suppose $v_{\boldsymbol{rp}} \geq \tilde{\gamma}_0$, implying that $v_{\boldsymbol{rp}}$ is significant. Then for even $k_M^{(h)}$ and positive δ, (3.16) would imply that $\mu_r^{k_M^{(h)}} v_{\boldsymbol{rp}} \geq \delta$, which would mean that $v_{\boldsymbol{rp}} \gg \delta$ because $k_M^{(h)} \gg 1$ for large h. However, this likely implies that $\sum_{i=1}^{\tau} \lambda_i^k u_{\boldsymbol{ip}} \approx \mu_r^k v_{\boldsymbol{rp}} \geq \mu_r^{k_M} v_{\boldsymbol{rp}} \geq \delta$ for large $k < k_M$, thereby violating (3.15). Thus, our assumption that $v_{\boldsymbol{rp}} \geq \tilde{\gamma}_0$ was wrong. This proves (3.18).

Next, we establish the following odd-k analog of (3.18):

$$z_{\tilde{\boldsymbol{r}}\tilde{p}(\tilde{r})} < \hat{\gamma}_0, \tag{3.19}$$

where we have defined $M_o := \arg\max_{i\in[m]} \zeta_i^{(h)}$, $\tilde{\eta}_i := \lim_{h\to\infty} \zeta_i^{(h)}/\zeta_{M_o}^{(h)}$, and $\tilde{r} := \min\{i \in [m] : \tilde{\eta}_i > 0\}$ analogous to M_e, η_i, and r, respectively. Please see 3.6.3 for further details.

Step 5: Note that (3.18), (3.19), and Lemma 14 imply that $r \neq \tilde{r}$. We may assume that $r < \tilde{r}$ because the case $r > \tilde{r}$ can be handled similarly. Then by the definition of $\tilde{r}$, we have $\lim_{h\to\infty} \zeta_r^{(h)}/\zeta_{\tilde{r}}^{(h)} = 0$. Furthermore, by applying Lemma 14 to both r and $\tilde{r}$, we obtain $\min(\alpha_{\tilde{r}}^{(h)}/\zeta_{\tilde{r}}^{(h)}, \zeta_r^{(h)}/\alpha_r^{(h)}) \geq \hat{\gamma}_0$. Therefore,

$$\begin{aligned} \lim_{h\to\infty} \frac{\alpha_{\tilde{r}}^{(h)}}{\alpha_{M_e}^{(h)}} &= \lim_{h\to\infty} \frac{\alpha_{\tilde{r}}^{(h)}}{\zeta_{\tilde{r}}^{(h)}} \cdot \frac{\zeta_{\tilde{r}}^{(h)}}{\zeta_r^{(h)}} \cdot \frac{\zeta_r^{(h)}}{\alpha_r^{(h)}} \cdot \frac{\alpha_r^{(h)}}{\alpha_{M_e}^{(h)}} \\ &\geq \hat{\gamma}_0 \cdot \infty \cdot \hat{\gamma}_0 \cdot \eta_r = \infty \end{aligned}$$

because $\eta_r > 0$ by our definition of r. But this contradicts the definition of M_e, thereby proving the lemma. □

By proving Lemma 16, we have effectively proven Lemma 15. So, the next step is to extend Lemma 15. To this end, we eliminate the assumption that the opinions of G_Q remain constant until the desired merging event occurs. To analyze the resulting scenario, we make a few observations:

1. Since we now allow both G_P and G_Q to have changing opinions, we not only need to focus on the potential neighbors of G_Q in G_P, but also on the potential neighbors of G_P in G_Q. Let us assume for simplicity that $\{V_P, V_Q\}$ is a partition of $[n]$, the vertex set of G_{ph}. Then, from the viewpoint of G_{ph}, the potential neighbors are precisely the *boundary* nodes of G_P and G_Q.

2. The merging time $k_M(x[0], V_P, V_Q)$ depends not on the absolute opinions of the agents but rather on the differences between the opinions of the agents of G_P and G_Q. So, it is sufficient to observe the opinion evolution of G_P from the *reference frame* of the boundary nodes of G_Q. To elaborate, suppose for simplicity that there are exactly b boundary nodes in G_{P_0} and G_{Q_0} each, and that there are exactly b boundary edges $\{(i_e, j_e)\}_{e=1}^{b} \subset V_P \times V_Q$

between the two subgraphs. Then, instead of analyzing $\{x_{i_e}[k]\}_{e=1}^b$, the absolute opinions of the nodes $\{i_e\}_{e=1}^b$, we may analyze $\{x_{i_e}[k]-x_{j_e}[k]\}_{e=1}^b$, their opinions relative to $\{j_e\}_{e=1}^b$. Now, observe that we can express the vector $[x_{i_1}[k],\ldots,x_{i_b[k]}]^T$ as a linear combination of the eigenvectors of A_{P_0} after restricting these eigenvectors to the coordinates specified by $\{i_e\}_{e=1}^b$. We can do the same for $[x_{j_1}[k],\ldots,x_{j_b[k]}]^T$. Hence, we can express the relative opinion vector $[x_{i_1}[k]-x_{j_1}[k],\ldots,x_{i_b}[k]-x_{j_b}[k]]^T$ as a linear combination of the eigenvectors of A_{P_0} and those of A_{Q_0} after restricting them to the coordinates specified by the boundary nodes. This motivates the concept of boundary-restricted eigenvectors, which we formally define below.

Suppose $G_{P_0}=(V_P,E_{P_0})$ and $G_{Q_0}=(V_Q,E_{Q_0})$ are two induced subgraphs of G_{ph} such that $V_P\cap V_Q=\emptyset$. Let $\{(i_e,j_e)\}_{e=1}^b\subset V_P\times V_Q$ be the set of boundary edges of $\{G_{P_0},G_{Q_0}\}$ in G_{ph} (i.e., the set of edges connecting G_{P_0} with G_{Q_0} in G_{ph}), and let $\{1\}\cup\{\lambda_d\}_{d=1}^m$ be the union of the spectra of A_{P_0} and A_{Q_0} (such that $\lambda_d\neq 1$ for all $d\in[m]$). Further, for each $d\in[m]$, let $U_d(P)$ (respectively, $U_d(Q)$) be the eigenspace of λ_d with respect to A_{P_0} (respectively, A_{Q_0}) if λ_d is an eigenvalue of A_{P_0} (respectively, A_{Q_0}), and let $U_d(P)=\{0\}$ (respectively, $U_d(Q)=\{0\}$), otherwise. Finally, for each $d\in[m]$, let $f_{br}^P(u):=[u_{i_1}\ \ldots\ u_{i_b}]^T$ for all $u\in U_d(P)$, and let $f_{br}^Q(w):=[w_{j_1}\ \ldots\ w_{j_b}]^T$ for all $w\in U_d(Q)$. Note that the dimensions of $f_{br}^P(u)$ and $f_{br}^Q(w)$ equal b for all $u\in U_d(P)$ and $w\in U_d(Q)$.

Definition 38. *For each $d\in[m]$, the boundary-restricted eigenspace of λ_d associated with $\{G_{P_0},G_{Q_0}\}$ is the set $\hat{U}_d^{PQ}:=\hat{U}_d^P+\hat{U}_d^Q$, where $\hat{U}_d^P:=\text{span}(\{f_{br}^P(v):v\in U_d(P)\})$ and $\hat{U}_d^Q:=\text{span}(\{f_{br}^Q(v):v\in U_d(Q)\})$. For any $v\in\hat{U}_d^{PQ}$, we refer to v as a boundary-restricted eigenvector of $\{G_{P_0},G_{Q_0}\}$ corresponding to the eigenvalue λ_d, and we refer to (λ_d,v) as a boundary-restricted eigenpair of $\{G_{P_0},G_{Q_0}\}$.*

Finally, we let $k_M(x[0])$ denote the time at which $G_P[k]$ and $G_Q[k]$ merge for the first time.

Proposition 10. *For every initial state $x[0]\in\mathbb{R}^n$, let $G_P[k]=G_P(x[k])=(V_P,E_P(x[k]))$ and*

$G_Q[k] = G_Q(x[k]) = (V_Q, E_Q(x[k]))$ be the induced subgraphs of $\tilde{G}[k]$ corresponding to V_P and V_Q, respectively, and let $\mathcal{X}$ denote the set of all $x[0] \in \mathbb{R}^n$ satisfying the assumptions below:

(i). $k_M(x[0]) < \infty$, i.e., G_P indeed merges with G_Q.

(ii). $G_P[k]$ and $G_Q[k]$ are connected graphs for $0 \le k < k_M(x[0])$.

(iii). No link breaks occur in $G_P[k]$ or $G_Q[k]$ until time $k_M(x[0])$.

Suppose $\sup_{x[0]\in\mathcal{X}} k_M(x[0]) = \infty$. Then there exists an index $d \in [m]$ and a corresponding boundary-restricted eigenpair $(\lambda_d, \hat{v})$ of $\{G_{P_0}, G_{Q_0}\}$ such that $\lambda_d \in (0,1)$, $\hat{v}_{[b]} \neq 0$, and $\hat{v}_e \hat{v}_f \geq 0$ for all $e, f \in [b]$.

Proof. Since no link break occurs within $G_P[k]$ or $G_Q[k]$ until they merge, both of them remain connected in $\tilde{G}[k]$ until $k_M(x[0])$. Moreover, since $\sup_{x[0]\in\mathcal{X}} k_M(x[0]) = \infty$, and because all the intra-component link formations taking place in $\tilde{G}[k]$ occur in $O(n^8 \log n)$ steps as per Proposition 8, we may choose an $x[0] \in \mathbb{R}^n$ such that $k_M(x[0])$ is large enough, and $G_P[k]$ and $G_Q[k]$ both remain constant and connected during a time interval $[k_c, k_M(x[0]))$ for some $k_c < k_M(x[0])$. As a result, we may further assume that the sub-networks of G_{ph} corresponding to G_{P_0} and G_{Q_0} achieve $R/4$-convergence to their respective consensus states at some time $k_R \in [k_c, k_M(x[0]))$. We now shift the origin of our time axis to k_R, thus obtaining $G_P[0] = G_{P_0}$ and $G_Q[0] = G_{Q_0}$.

Now, we express the initial states of G_P and G_Q as

$$x_P[0] = c_P \mathbf{1}_P + \sum_{d=1}^{m} \beta_d^P v_{\boldsymbol{d}}^P$$
$$x_Q[0] = c_Q \mathbf{1}_Q + \sum_{d=1}^{m} \beta_d^Q v_{\boldsymbol{d}}^Q,$$

where $c_P, c_Q \in \mathbb{R}$ depend on our choice of $x_P[0]$ and $x_Q[0]$, and the vectors are chosen such that for each $d \in [m]$, $v_{\boldsymbol{d}}^P$ (respectively, $v_{\boldsymbol{d}}^Q$) is an eigenvector of A_{P_0} (respectively, A_{Q_0}) corresponding to λ_d iff λ_d is an eigenvalue of A_{P_0} (respectively, A_{Q_0}) and $v^P_{\boldsymbol{d}} = 0$ (respectively, $v^Q_{\boldsymbol{d}} = 0$) otherwise. This is possible because A_{P_0} and A_{Q_0} are diagonalizable by Lemma 8. In addition, we assume that $\{v^P_{\boldsymbol{d}}\}_{d=1}^m \setminus \{0\}$ and $\{v^Q_{\boldsymbol{d}}\}_{d=1}^m \setminus \{0\}$ are bases of eigenvectors for A_{P_0} and A_{Q_0}, respectively.

Next, let $\{(i_e, j_e)\}_{e=1}^b \subset V_P \times V_Q$ enumerate the set of boundary edges of $\{G_{P_0}, G_{Q_0}\}$ in G_{ph}. Note that assumption (3.5.3) requires $|x_{i_e}[k] - x_{j_e}[k]| > R$ for all $e \in [b]$ and $0 \le k < k_M := k_M(x[0])$, and $|x_{i_t}[k_M] - x_{j_t}[k_M]| \le R$ for some $t \in [b]$. Now, for a given $e \in [b]$, we could either have

$$x_{i_e}[k] - x_{j_e}[k] > R, \text{ or} \tag{3.20}$$

$$x_{j_e}[k] - x_{i_e}[k] > R \tag{3.21}$$

for a particular $k \in [0, k_M)$. Suppose (3.20) holds at some $k_1 \in [0, k_M)$ and (3.21) at some $k_2 \in [0, k_M)$. Then $\max(x_{i_e}[k_1] - x_{i_e}[k_2], x_{j_e}[k_2] - x_{j_e}[k_1]) > R$. But this contradicts the assumption that both G_P and G_Q have achieved $R/4$-convergence to their respective consensus states at time 0. Therefore, for a given $e \in [b]$, if (3.20) holds for some $k \in [0, k_M)$, then it must hold for all $k \in [0, k_M)$. Similarly, we can show that for a given $k \in [0, k_M)$, if (3.20) holds for some $e \in [b]$, then it must hold for all $e \in [b]$. The same applies to (3.21). Hence, w.l.o.g., we assume (3.21) for all $e \in [b]$ and all $0 \le k < k_M$.

Now, for each $d \in [m]$, let $\hat{v}_{\boldsymbol{d}} := [\hat{v}_{\boldsymbol{d}1} \ \ldots \ \hat{v}_{\boldsymbol{d}b}]^T$, where $\hat{v}_{\boldsymbol{d}e} = \beta^P_d v_{\boldsymbol{d}i_e} - \beta^Q_d v_{\boldsymbol{d}j_e}$ for $e \in [b]$. Further, let $\delta := c_Q - R - c_P$. With these definitions and the assumption given by (3.21), we can express assumption (3.5.3) of the proposition as:

$$\sum_{d=1}^m \lambda_d^k \hat{v}_{\boldsymbol{d}e} < \delta \quad \text{for all } 0 \le k < k_M \text{ and all } e \in [b],$$

and $\sum_{d=1}^m \lambda_d^{k_M} \hat{v}_{\boldsymbol{d}t} \ge \delta$ for some $t \in [b]$. Since $|\lambda_d| < 1$ for all $d \in [m]$ and $\max_d \lambda_d > 0$ by Lemma 9, and since $(\hat{v}_{\boldsymbol{1}}, \ldots, \hat{v}_{\boldsymbol{m}}) \in \prod_{d=1}^m \hat{U}_d^{PQ}$, the assertion of Proposition 10 now follows immediately from Lemma 16. □

3.5.4 Graphs with Finite Maximum ε-Convergence Time

In this section, we show that the ε-convergence time of a complete r-partite graph is bounded. We first define complete r-partite graphs below.

Definition 39 (Complete r-Partite Graph). *An undirected graph $G=(V,E)$ is said to be a complete r-partite graph if its vertex set, V admits a partition $\{V_1,\dots,V_r\}$ such that $(i,j)\in E$ iff $(i,j)\notin \cup_{l=1}^{r} V_l^2$.*

We now characterize the eigenvectors of the normalized adjacency matrix of a complete r-partite graph with self-loops. To this end, suppose $G=([n],E)$ is a complete r-partite graph for some $n\in\mathbb{N}$. Let G have all the n self-loops, let $A\in\mathbb{R}^{n\times n}$ be the normalized adjacency matrix of G, and let $n_i:=|V_i|\geq 1$ for $i\in[r]$. For each $i\in[r]$, let $V_i=\{N_{i-1}+1,\dots,N_i\}$, where $N_j:=\sum_{i=1}^{j} n_i$ for $j\in[r]$ and $N_0:=0$. Finally, we define a matrix $B\in\mathbb{R}^{r\times r}$ by:

$$B_{ij}:=\begin{cases}(n-n_i+1)^{-1} & \text{if } j=i\\ n_j(n-n_i+1)^{-1} & \text{if } j\neq i\end{cases},$$

and let $\{w^{(i)}\}_{i=1}^{q}$ be an eigenvector basis for B with $\{\lambda^{(i)}\}_{i=1}^{q}$being the corresponding eigenvalues. We then have the following lemma.

Lemma 17. *The matrices A and B (as described above) have the following properties:*

(i) For each $i\in[r]$ such that $n_i\geq 2$ and each $t\in\{2,\dots,n_i\}$, the vector $v^{(i,t)}\in\mathbb{R}^n$, defined as:

$$v_j^{(i,t)}:=\begin{cases}+1, & \textit{if } j=N_{i-1}+1\\ -1, & \textit{if } j=N_{i-1}+t\\ 0, & \textit{otherwise}\end{cases},$$

is an eigenvector of A corresponding to $1/(n-n_i+1)$. Moreover, the set $U_1 := \{v^{(i,t)} : 2 \le t \le n_i, i \in [r]\}$ is a set of linearly independent vectors.

(ii) For each $i \in [q]$, the vector $\tilde{v}^{(i)} \in \mathbb{R}^n$, defined as $\tilde{v}^{(i)}_p = w^{(i)}_j$ for all $p \in V_j$ and $j \in [r]$, is an eigenvector of A corresponding to $\lambda^{(i)}$.

(iii) The eigenvectors of B span $\mathbb{R}^r$, i.e., $q = r$.

(iv) If $\lambda^{(i)} \neq 1$, then $\lambda^{(i)} \le 0$ for all $i \in [r]$.

(v) $U := \cup_{j=1}^{r}\{v^{(j,t)} : 2 \le t \le n_j\} \cup \{\tilde{v}^{(i)}\}_{i=1}^{r}$ is an eigenvector basis for A.

Proof. Observe that for all $p \in [r]$, the degree of each vertex in V_p, with its self-loop counted, is $n - n_p + 1$. Hence, given $i \in [r]$, for all $p \in [r] \setminus \{i\}$ and $j \in V_p$, we have:

$$\begin{aligned}(Av^{(i,t)})_j &= (n-n_p+1)^{-1}\left(v^{(i,t)}_{N_{i-1}+1} + v^{(i,t)}_{N_{i-1}+t}\right)\\ &= 0 = (n-n_p+1)^{-1} v^{(i,t)}_j.\end{aligned}$$

Next, if $j = N_{i-1}+1$, then

$$\begin{aligned}(Av^{(i,t)})_j &= (n-n_i+1)^{-1} v^{(i,t)}_{N_{i-1}+1} + 0 \cdot v^{(i,t)}_{N_{i-1}+t}\\ &= (n-n_i+1)^{-1} v^{(i,t)}_j.\end{aligned}$$

Similarly, $(Av^{(i,t)})_j = v^{(i,t)}_j/(n-n_i+1)$ also holds for $j = N_{i-1}+t$. Finally, for $j \in V_i \setminus \{N_{i-1}+1, N_{i-1}+t\}$, we have $(Av^{(i,t)})_j = A_{jj} \cdot 0 + \sum_{s \in V \setminus V_i} A_{js} \cdot 0 = v^{(i,t)}_j/(n-n_i+1)$. So, for each $i \in [r]$ and each $t \in \{2, \ldots, n_i\}$, $v^{(i,t)}$ is an eigenvector of A corresponding to $\frac{1}{n-n_i+1}$. By taking linear combinations, we can easily see that $\{v^{(i,t)} : 2 \le t \le n_i, i \in [r]\}$ are linearly independent vectors. This proves (i).

As for (ii), for any $j \in [r]$ and $p \in V_j$, we have:

$$
\begin{aligned}
(A\tilde{v}^{(i)})_p &= \sum_{s=1}^{n} A_{ps}\tilde{v}_s^{(i)} \\
&= \frac{1}{n-n_j+1}\cdot\tilde{v}_p^{(i)} + \sum_{l\in[r]\setminus\{j\}}\left(\sum_{m\in V_l}\frac{1}{n-n_j+1}\cdot\tilde{v}_m^{(i)}\right) \\
&= (n-n_j+1)^{-1}\cdot w_j^{(i)} + \sum_{l\in[r]\setminus\{j\}} n_l(n-n_j+1)^{-1}\cdot w_l^{(i)} \\
&= \sum_{l=1}^{r} B_{jl}w_l^{(i)} \\
&= (Bw^{(i)})_j = (\lambda^{(i)}w^{(i)})_j = \lambda^{(i)}\tilde{v}_p^{(i)}.
\end{aligned}
$$

To prove (iii), note that $B = D_1SD_2$, where $D_1 := \text{diag}((n-n_1+1)^{-1},\ldots,(n-n_r+1)^{-1})$, $D_2 := \text{diag}(n_1,\ldots,n_r)$, and S is the symmetric $r\times r$ matrix given by:

$$
S_{ij} = \begin{cases} \frac{1}{n_i} & \text{if } j = i \\ 1 & \text{if } j \neq i \end{cases}.
$$

Now, observe that the commutativity of diagonal matrices allows us to express D_1SD_2 as $D_A(D_BSD_B)D_A^{-1}$, where $D_A := (D_1D_2^{-1})^{\frac{1}{2}}$ and $D_B := (D_1D_2)^{\frac{1}{2}}$. Thus, the matrix $B = D_A(D_BSD_B)D_A^{-1}$ is similar to the symmetric matrix D_BSD_B and hence, its eigenvectors span $\mathbb{R}^r$, i.e., $q = r$.

As for (iv), for any $\lambda^{(i)} \neq 1$, we know that $D^{\frac{1}{2}}\tilde{v}^{(i)}$ is an eigenvector of $D^{\frac{1}{2}}AD^{-\frac{1}{2}}$ which is a symmetric matrix as per Lemma 1 of [47]. Hence, $\{D^{\frac{1}{2}}\tilde{v}^{(i)}\}_{i=1}^r$ is an orthogonal set. Since $\mathbf{1} \in \{\tilde{v}^{(i)}\}_{i=1}^r$, this implies that

$$
\mathbf{1}^T D\tilde{v}^{(i)} = 0 \text{ if } i \in [r] \text{ and } \lambda^{(i)} \neq 1, \tag{3.22}
$$

thereby forcing each $\tilde{v}^{(i)}$ to have both positive and negative entries. Now, pick any $i \in [r]$ for which $\lambda^{(i)} \neq 1$, and let $s \in [r]$ be the index such that $w_j^{(i)} \geq 0$ for $j \in [s]$ and $w_j^{(i)} < 0$ otherwise (we can always label the vertices suitably so that such an s exists). Then (3.22) implies that $1 \leq s \leq r-1$. Consequently, we have the following relations:

$$\lambda^{(i)}|w_1^{(i)}| = \frac{|w_1^{(i)}| + \sum_{j=2}^{s} n_j |w_j^{(i)}| - \sum_{j=s+1}^{r} n_j |w_j^{(i)}|}{n - n_1 + 1}$$
$$-\lambda^{(i)}|w_{s+1}^{(i)}| = \frac{\sum_{j=1}^{s} n_j |w_j^{(i)}| - \sum_{j=s+2}^{r} n_j |w_j^{(i)}| - |w_{s+1}^{(i)}|}{n - n_{s+1} + 1}.$$

On the basis of this, we have the following for $\lambda^{(i)} \notin \{0,1\}$:

$$\begin{aligned} 0 &< (n - n_1 + 1)|w_1^{(i)}| + (n - n_{s+1} + 1)|w_{s+1}^{(i)}| \\ &= -\frac{(n_1 - 1)|w_1^{(i)}| + (n_{s+1} - 1)|w_{s+1}^{(i)}|}{\lambda^{(i)}}, \end{aligned}$$

implying $\lambda^{(i)} < 0$ because $n_1, n_{s+1} \geq 1$ by assumption.

For part (v), note that U_1 and $\{\tilde{v}^{(i)}\}_{i=1}^{r}$ are linearly independent sets by assertions (i) and (ii). Also, observe that $\text{span}\{\tilde{v}^{(i)} \mid i \in [r]\} = \text{span}^{\perp}\{v^{(j,t)} \mid j \in [r], t \in \{2, \ldots, n_j\}\}$ because $\tilde{v}^{(i)T} v^{(j,t)} = w_j^{(i)} \times 1 + w_j^{(i)} \times -1 = 0$. Finally, noting that $|U| = \sum_{j=1}^{r}(n_j - 1) + r = \sum_{j=1}^{r} n_j = n$, we conclude that U is an eigenvector basis for A. □

Remark 6. *Points (1), (4) and (5) of Lemma 17, along with the fact that eigenspaces are linear, imply that every eigenpair* (λ, v) *of* A *that satisfies* $\lambda \in (0,1)$, *corresponds to some* $i \in [r]$ *such that* $|V_i| \geq 2$ *and* $v_s = 0$ *for all* $s \notin V_i$. *Furthermore,* $\sum_{s \in V_i} v_s = 0$ *for such an* i.

We are now well equipped to establish our main result.

Proposition 11. *Let* $n \in \mathbb{N}$ *and* $\varepsilon > 0$ *be given, and let* $G_{ph} = ([n], E_{ph})$ *be a complete* r*-partite graph for some* $r \in [n]$. *Then* $k_{\varepsilon}^{*}(G_{ph}) < \infty$.

Proof. If G_{ph} is a complete 1-partite graph, then $E_{\text{ph}} = \emptyset$ and hence $k_{\varepsilon}^{*}(G_{\text{ph}}) = 0$. On the other hand, if $r = n$, then $G_{\text{ph}} = K_n$. In this case, $k_{\varepsilon}^{*}(G_{\text{ph}}) = O(n^3) < \infty$ by [80] and [47]. Therefore, we assume $1 < r < n$ hereafter.

Suppose $k_\varepsilon(G_{\text{ph}}) = \infty$. From Proposition 8, we know that arbitrarily slow convergence happens only in the presence of arbitrarily slow merging and that all the other structural changes in $\tilde{G}[k]$ occur in $O(n^8 \log n)$ steps. Hence, it suffices to show that no two connected components of the influence graph can take an arbitrarily long period of time to merge, under the assumption that no link breaks occur.

For this purpose, let $V_1, \ldots, V_r$ be the r parts of G_{ph}, and let $V_P, V_Q \subset [n]$ be any two disjoint sets. Further, let $\{(i_e, j_e)\}_{e=1}^b \subset V_P \times V_Q$ be the set of boundary edges connecting G_{P_0} and G_{Q_0} in G_{ph}, and let $\{1\} \cup \{\lambda_d\}_{d=1}^m$ be the union of the sets of eigenvalues of A_{P_0} and A_{Q_0} (such that $\lambda_d \neq 1$ for all d). Now, since G_{ph} is a complete r-partite graph, it follows that G_{P_0} and G_{Q_0} are also complete p-partite and q-partite graphs for some $p, q \in [r]$, and their parts are given by the partitions $\{V_P \cap V_i\}_{i=1}^r \setminus \{\emptyset\}$ and $\{V_Q \cap V_i\}_{i=1}^r \setminus \{\emptyset\}$, respectively.

Next, for each initial state $x[0] \in \mathbb{R}^n$, let $G_P[k] = G_P(x[k]) = (V_P, E_P(x[k]))$ and $G_Q[k] = G_Q(x[k]) = (V_Q, E_Q(x[k]))$ be disconnected from each other in $\tilde{G}[k] = \tilde{G}(x[k])$ until they merge at time $k_M(x[0], V_P, V_Q)$. As per our earlier reasoning, we may restrict our attention to the subset $\mathcal{X}(V_P, V_Q) \subset \mathbb{R}^n$ of initial states for which (i) $k_M(x[0], V_P, V_Q) < \infty$, i.e., merging occurs, (ii) no link breaks occur within $G_P[k]$ or $G_Q[k]$ until they merge, i.e., for $k \leq k_M(x[0], V_P, V_Q)$, and (iii) both $G_P[k]$ and $G_Q[k]$ are connected graphs for $k \leq k_M(x[0], V_P, V_Q)$.

Now, suppose $\sup_{x[0] \in \mathcal{X}(V_P, V_Q)} k_M(x[0], V_P, V_Q) = \infty$. Then Proposition 10 implies that there exists a $d \in [m]$ with $\lambda_d \in (0,1)$ and a corresponding vector $v \in \hat{U}_d^{PQ}$ satisfying $v_e \neq 0$ for some $e \in [b]$ and $v_f v_g \geq 0$ for all $f, g \in [b]$. Since $v_e = u_{i_e} + w_{j_e}$ for some $u \in U_d(P)$ and $w \in U_d(Q)$, we have either $u_{i_e} \neq 0$ or $w_{j_e} \neq 0$. W.l.o.g., we assume $u_{i_e} > 0$ (and hence that (λ_d, u) is an eigenpair of A_{P_0}). Now, let $\rho \in [r]$ and $\sigma \in [r]$ denote the indices for which $i_e \in V_{P\rho} := V_P \cap V_\rho$ and $j_e \in V_{Q\sigma} := V_Q \cap V_\sigma$. Then observe that $\rho \neq \sigma$ because $(i_e, j_e) \in E_{\text{ph}}$. Also, by Remark 6, $\lambda_d \in (0,1)$ implies that $\sum_{s \in V_{P\rho}} u_s = 0$. Hence, there exists another node $z \in V_{P\rho}$ such that $u_z < 0$. Now, two cases arise: either $|V_{Q\sigma}| = 1$ or $|V_{Q\sigma}| \geq 2$.

Consider Case 1: $|V_{Q\sigma}| = 1$, i.e., $V_{Q\sigma} = \{j_e\}$. Now, if λ_d is not an eigenvalue of A_{Q_0}, then $U_d = \{0\}$, which means $w = 0$. Hence, $w_{j_e} = 0$. Otherwise, by Remark 6, Lemma 17 requires $w_{j_e} = 0$ because $\lambda_d > 0$ and $|V_{Q\sigma}| < 2$. Thus, $w_{j_e} = 0$ is true whenever $|V_{Q\sigma}| = 1$. Moreover, $\rho \neq \sigma$ implies that $(z, j_e) \in E_{\text{ph}}$. Since $z \in V_P$ and $j_e \in V_Q$, we may denote z by i_f and j_e by j_f so that (z, j_e) is the f-th boundary edge, (i_f, j_f), for some $f \in [b]$. But now, $v_f = u_{i_f} + w_{j_f} = u_z + w_{j_e} = u_z < 0$, whereas $v_e = u_{i_e} > 0$. As a result, $v_e v_f < 0$, thus violating the requirement that $v_f v_g \geq 0$ for all $f, g \in [b]$.

On the other hand, in Case 2: $|V_{Q\sigma}| \geq 2$, both $w_{j_e} = 0$ and $w_{j_e} \neq 0$ are possible subcases. If $w_{j_e} = 0$, then we simply repeat the arguments of the previous paragraph to show that $v_e v_f < 0$ for some $f \in [b]$. So, assume $w_{j_e} \neq 0$. Then (λ_d, w) is necessarily an eigenpair of A_{Q_0}. Therefore, the requirement $\sum_{s \in V_{Q\sigma}} w_s = 0$ of Lemma 17 implies $w_y w_{j_e} < 0$ for some $y \in V_{Q\sigma}$. First, suppose $w_{j_e} > 0$ and $w_y < 0$. Then, $\rho \neq \sigma$ implies that $(z, y) \in E_{\text{ph}}$ and hence that (z, y) is a boundary edge. By denoting (z, y) as the f-th boundary edge (i_f, j_f) for some $f \in [b]$, we have $v_f = u_{i_f} + w_{j_f} = u_z + w_y < 0$. However, we still have $v_e = u_{i_e} + w_{j_e} > 0$, implying that $v_e v_f < 0$. Now, assume $w_{j_e} < 0$ and $w_y > 0$. Then, by denoting the boundary edges (z, j_e) and (i_e, y) as (i_α, j_α) and (i_β, j_β), respectively for some $\alpha, \beta \in [b]$, we have $v_\alpha = u_z + w_{j_e} < 0$ and $v_\beta = u_{i_e} + w_y > 0$. This implies that $v_\alpha v_\beta < 0$. Thus, the requirement $v_f v_g \geq 0$ for all $f, g \in [b]$ is violated in Case 2 as well.

Hence, $\sup_{x[0] \in \mathcal{X}(V_P, V_Q)} k_M(x[0], V_P, V_Q) < \infty$. Note that this applies to every selection of $V_P \subset V$ and $V_Q \subset V$ such that $V_P \cap V_Q = \emptyset$. Moreover, since the number of such choices of V_P and V_Q is finite, we conclude that no merging event can be delayed indefinitely in the social HK dynamics on the given G_{ph}. This completes the proof. □

3.6 Conclusion and Future Directions

In this chapter, we have investigated the convergence properties of the social HK model of opinion dynamics. We have shown that for certain physical connectivity graphs, we cannot even guarantee ε-convergence to the steady state within a bounded time-frame, much less termination in finite time. In addition, we have shown that complete r-partite graphs have bounded ε-convergence times. Moreover, we can observe that the necessary and sufficient conditions provided by Proposition 9 and Lemma 15 are nearly tight (i.e., tight under the assumption $v_i v_j \neq 0$, in addition to the other assumptions made by these two results). However, finding a set of necessary and sufficient conditions for arbitrarily slow merging (and thereby for arbitrarily slow ε-convergence) that are tight in the most general case, remains an interesting open problem. Also open is the problem of finding other classes of graphs that have bounded ε-convergence times.

Appendices

3.6.1 Proof of Proposition 7

Proof. Let ε and x_0 be as described above, and let $x[0] = x_0$. Then observe that the state evolution until ε-convergence can be expressed as $x[k] = A^k x_0$, where A is the weighted adjacency matrix of $\tilde{G}[0]$. This is equivalent to the equal-neighbor, time-invariant, bidirectional model of distributed averaging defined in [102]. Moreover, $\tilde{G}[0]$ is a connected graph. Therefore, invoking Corollary 5.2 of [102] completes the proof of the proposition. □

3.6.2 Some Technical Lemmas

Lemma 18. *Consider a vector subspace $U \subset \mathbb{R}^n$ such that for every $v \in U \setminus \{0\}$, we have $v_i v_j < 0$ for some $i, j \in [l]$. Further, define $\phi : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ as*

$$\phi(v) = \min \left(\left| \frac{\min_{i \in [l]} v_i}{\max_{i \in [l]} v_i} \right|, \left| \frac{\max_{i \in [l]} v_i}{\min_{i \in [l]} v_i} \right| \right).$$

Then there exists a constant $\gamma > 0$ such that $\phi(v) \geq \gamma$ for all $v \in U \setminus \{0\}$.

Proof. Since $\phi(\lambda v) = \phi(v)$ for all $\lambda \in \mathbb{R} \setminus \{0\}$ and $v \in \mathbb{R}^n \setminus \{0\}$, it suffices to prove the lemma for $v \in D := U \cap \{v \in \mathbb{R}^n : \|v\| = 1\}$.

Observe that D is the intersection of the unit ball and a vector subspace of $\mathbb{R}^n$. Hence, it is a compact set. Next, since ϕ is continuous on D, we know that $\inf_{v \in D} \phi(v)$ is attained because D is a compact set, i.e., $U^* := \arg\min_{v \in D} \phi(v)$ exists and is well defined. Hence, for all $v \in D$ and any $u \in U^*$, we have $\phi(v) \geq \min_{v \in D} \phi(v) = \phi(u) > 0$, which follows from the assumption of the lemma enforcing $\max_{i \in [l]} u_i > 0 > \min_{i \in [l]} u_i$. □

Lemma 19. *Let $v \in \mathbb{R}^l$ satisfy $v_i v_j < 0$ for some $i, j \in [l]$, and let $\gamma \in \mathbb{R}$ be any constant satisfying $0 < \gamma \leq |\max_i v_i| / |\min_i v_i|$. Then for any $u \in \mathbb{R}^l$, either*

$$\max_i (v+u)_i > 0 \text{ and } \left| \frac{\max_i (v+u)_i}{\min_i (v+u)_i} \right| \geq \gamma' \tag{3.23}$$

or

$$\max_i (v-u)_i > 0 \text{ and } \left| \frac{\max_i (v-u)_i}{\min_i (v-u)_i} \right| \geq \gamma' \tag{3.24}$$

where $\gamma' := \frac{\gamma}{\gamma+2}$. Moreover, if (3.23) *holds and $\min_i (v-u)_i < 0$, then*

$$\max_i (v+u)_i \geq \frac{\gamma |\min_i (v-u)_i| - \max(0, \max_i (v-u)_i)}{\gamma + 1}. \tag{3.25}$$

Proof. We first prove that either (3.23) or (3.24) holds. Before we begin, observe that $\gamma > 0$ implies $\gamma' < \min(1, \gamma)$.

Now, suppose neither (3.23) nor (3.24) is true. However, we know that $\max_i (v+u)_i + \max_i (v-u)_i \geq \max_i [(v+u)+(v-u)]_i = 2 \max_i v_i > 0$. As a result, either $\max_i (v+u)_i > 0$ (in which case $|\max_i (v+u)_i| < \gamma' |\min_i (v+u)_i|$), or $\max_i (v-u)_i > 0$ (in which case $|\max_i (v-u)_i| < \gamma' |\min_i (v-u)_i|$). By implication, there exists a constant $\varepsilon \in (0, \gamma')$ such that $P_1 \leq \varepsilon M_1$ and $P_2 \leq \varepsilon M_2$, where we define $P_1 := \max(0, \max_i (v+u)_i)$, $P_2 := \max(0, \max_i (v-u)_i)$, $M_1 := |\min_i (v+u)_i|$ and $M_2 := |\min_i (v-u)_i|$.

Now, three cases arise.

Case 1: $(\min_i(v+u)_i)(\min_i(v-u)_i) \neq 0$ and either $\min_i(v+u)_i > 0$ or $\min_i(v-u)_i > 0$. Suppose $\min_i(v+u)_i = M_1 > 0$. Then $\max_i(v+u)_i/\min_i(v+u)_i \geq 1 \geq \gamma'$, thus contradicting the inequality $P_1 \leq \varepsilon M_1$ and thereby proving the first part of the lemma. The subcase $\min_i(v-u)_i > 0$ is handled similarly.

Case 2: Either $\min_i(v+u)_i = 0$ or $\min_i(v-u)_i = 0$. Suppose $\min_i(v+u)_i = M_1 = 0$. If $\max_i(v+u)_i > 0$, then we have $\varepsilon M_1 = 0 < P_1$, which again results in a contradiction and establishes the first part of the lemma. On the other hand, if $\max_i(v+u)_i = 0$, then it follows that $u = -v$. Consequently, the assumptions made by the lemma lead to the following: $\max_i(v-u)_i = 2\max_i v_i > 0$, and $\max_i(v-u)_i \geq 2\gamma|\min_i v_i| = \gamma|\min_i(v-u)_i| > \gamma'|\min_i(v-u)_i|$. These inequalities establish (3.24) and hence prove the first assertion of the lemma. The subcase $\min_i(v-u)_i = 0$ is handled similarly.

Case 3: $\min_i(v+u)_i = -M_1 < 0$ and $\min_i(v+u)_i = -M_2 < 0$. Observe that

$$\begin{aligned} 2\max_i v_i &= \max_i\{(v+u)+(v-u)\}_i \\ &\leq \max_i(v+u)_i + \max_i(v-u)_i \\ &\overset{(a)}{\leq} P_1 + P_2 \leq \varepsilon(M_1+M_2). \end{aligned} \tag{3.26}$$

Also,

$$\begin{aligned} 2\min_i v_i &= \min_i\{(v+u)+(v-u)\}_i \\ &\leq \max_i(v+u)_i + \min_i(v-u)_i \\ &\overset{(b)}{\leq} P_1 - M_2 \leq \varepsilon M_1 - M_2. \end{aligned} \tag{3.27}$$

Similarly,

$$2\min_i v_i \leq \varepsilon M_2 - M_1. \tag{3.28}$$

Within this case, two subcases arise. Subcase 1: Suppose both $\varepsilon M_1 - M_2 < 0$ and $\varepsilon M_2 - M_1 < 0$. In other words, $\varepsilon < \eta < 1/\varepsilon$, where we define $\eta := M_1/M_2$. Consider the inequality $\varepsilon M_1 - M_2 < 0$ first. Along with (3.27), it implies: $|\min_i v_i| \geq 0.5|M_2 - \varepsilon M_1|$. Likewise, (3.26) and the assumption $\max_i v_i > 0$ imply: $|\max_i v_i| \leq 0.5\varepsilon(M_1 + M_2)$. Combining these inequalities with the assumption $\max_i v_i \geq \gamma|\min_i v_i|$ yields:

$$\frac{\varepsilon(M_1 + M_2)}{M_2 - \varepsilon M_1} \geq \gamma. \tag{3.29}$$

Similarly, the subcase inequality $\varepsilon M_2 - M_1 < 0$, leads to:

$$\frac{\varepsilon(M_1 + M_2)}{M_1 - \varepsilon M_2} \geq \gamma. \tag{3.30}$$

We express (3.29) and (3.30) in terms of η as (3.31)-(a) and (3.31)-(b) respectively:

$$\frac{\gamma - \varepsilon}{\varepsilon(1+\gamma)} \overset{(a)}{\leq} \eta \overset{(b)}{\leq} \frac{\varepsilon(1+\gamma)}{\gamma - \varepsilon}, \tag{3.31}$$

which is possible only if $\gamma - \varepsilon \leq \varepsilon(1+\gamma)$, i.e., only if $\varepsilon \geq \frac{\gamma}{\gamma+2} = \gamma'$. This contradicts that $\varepsilon \in (0, \gamma')$, thus establishing the first assertion of the lemma.

Finally, we have Subcase 2: $\varepsilon M_2 - M_1 \geq 0$ or $\varepsilon M_1 - M_2 \geq 0$. We assume the former w.l.o.g. Then $\eta \leq \varepsilon < \gamma' < 1 < 1/\varepsilon$. Hence $\varepsilon M_1 - M_2 < 0$, implying (3.31)-(a) again. On eliminating η by using the observation $\eta \leq \varepsilon$, we obtain $(\gamma+1)\varepsilon^2 + \varepsilon - \gamma \geq 0$. Since ε is positive by assumption, this inequality requires $\varepsilon \geq \frac{\gamma}{\gamma+1} \geq \gamma'$ which contradicts $\varepsilon \in (0, \gamma')$, thereby proving that either (3.23) or (3.24) holds.

For the second part, given that (3.23) holds, we have $\max_i(v+u)_i = P_1 > 0$. Note that if $P_1 \geq M_2 = |\min_i(v-u)_i|$, then (3.25) follows from $\gamma > 0$. So, suppose that $P_1 < M_2$. Then (3.27)-(b) implies that $2|\min_i v_i| \geq M_2 - P_1$. Likewise, $\max_i v_i > 0$ and (3.26)-(a) together imply that $2|\max_i v_i| \leq P_1 + P_2$. Combining these inequalities with the lemma assumption

$\max_i v_i \geq \gamma|\min_i v_i|$ yields $\frac{P_1+P_2}{M_2-P_1} \geq \gamma$, rearranging which we obtain:

$$P_1 \geq \frac{\gamma M_2 - P_2}{\gamma + 1}$$

which is equivalent to (3.25). □

3.6.3 Proofs of (3.18) and (3.19)

Proof of (3.18)

We will assume that (3.18) is false, and show that $v_{\boldsymbol{r}}$ dominates other vectors for a range of values of k. To begin, let $p = p(r)$, suppose $v_{\boldsymbol{rp}} \geq \hat{\gamma}_0$ so that $v_{\boldsymbol{r}0} = v_{\boldsymbol{rp}}$ and assume that k is even. Then, by (3.11):

$$S_j[k] = \sum_{i=1}^{r-1} \alpha_i \mu_i^k v_{\boldsymbol{ij}} + \alpha_r \mu_r^k v_{\boldsymbol{rj}} + \sum_{i=r+1}^{m} \alpha_i \mu_i^k v_{\boldsymbol{ij}}, \tag{3.32}$$

and by (3.15), this implies:

$$\sum_{i=1}^{r-1} \alpha_i \mu_i^k v_{\boldsymbol{ij}} + \alpha_r \mu_r^k v_{\boldsymbol{rj}} + \sum_{i=r+1}^{m} \alpha_i \mu_i^k v_{\boldsymbol{ij}} < \delta, \tag{3.33}$$

for k in the range $2 \leq k < k_M$ and $j \in [l]$.

Now, for any $j \in [l]$, we can show that:

$$\left| \sum_{i=1}^{r-1} \alpha_i \mu_i^k v_{\boldsymbol{ij}} + \sum_{i=r+1}^{m} \alpha_i \mu_i^k v_{\boldsymbol{ij}} \right| \tag{3.34}$$

$$\leq r \rho_{r1} \alpha_r \mu_1^k + (\rho_{r2} \tau_r)^{-1} \alpha_r \mu_{r+1}^k v_{\boldsymbol{r}0}. \tag{3.35}$$

Next, we identify a range of k over which the contribution from μ_r dominates the contributions from both μ_{r+1} and μ_1. Let $k_{re} := \max(0, 2\lceil 0.5 \log_{(\mu_r/\mu_{r+1})}(40/\eta_r \tau_r)\rceil)$ and $k_r' := 2\lfloor 0.5 \log_{(\mu_1/\mu_{r+1})}(\frac{v_{\boldsymbol{r}0}}{r \rho_{r1} \rho_{r2} \tau_r})\rfloor$. Then, for ρ_{r1} small enough, the bounds on τ_r and (3.17) ensure that $k_{re} \leq k'_r < \infty$, and

$$r\rho_{r1}\alpha_r\mu_1^k \leq (\rho_{r2}\tau_r)^{-1}\alpha_r\mu_{r+1}^k v_{\boldsymbol{r}0} \text{ for } k \leq k'_r, \tag{3.36}$$

Furthermore, the definition of k_{re} and (3.17) imply that

$$(\rho_{r2}\tau_r)^{-1}\alpha_r\mu_{r+1}^k v_{\boldsymbol{r}0} \leq 0.05\alpha_r\mu_r^k v_{\boldsymbol{r}0} \text{ for } k \geq k_{re}. \tag{3.37}$$

Combining (3.34), (3.36) and (3.37) yields:

$$\left|\sum_{i=1}^{r-1}\alpha_i\mu_i^k v_{\boldsymbol{ij}} + \sum_{i=r+1}^{m}\alpha_i\mu_i^k v_{\boldsymbol{ij}}\right| \leq 0.1\alpha_r\mu_r^k v_{\boldsymbol{r}0} \tag{3.38}$$

for $k_{re} \leq k \leq k'_r$ and $j \in [l]$. Thus, *if* (3.18) *fails, then the contribution of the dominant vector* $v_{\boldsymbol{r}}$ *is much greater than the combined contributions of other even-*k *vectors when* $k_{re} \leq k \leq k'_r$. Now, (3.38), the assumption $v_{\boldsymbol{rp}} \geq \hat{\gamma}_0$, and (3.33) at $j = p$ together result in the following:

$$\begin{aligned}\delta &> \alpha_r\mu_r^k v_{\boldsymbol{rp}} - \left|\sum_{i=1}^{r-1}\alpha_i\mu_i^k v_{\boldsymbol{ij}} + \sum_{i=r+1}^{m}\alpha_i\mu_i^k v_{\boldsymbol{ip}}\right| \\ &\geq 0.9\alpha_r\mu_r^k v_{\boldsymbol{r}0} \text{ for } k_{re} \leq k \leq k'_r.\end{aligned} \tag{3.39}$$

By (3.32), (3.38), and (3.39), we have:

$$S_j[k] \leq 1.1\alpha_r\mu_r^k v_{\boldsymbol{r}0} \leq 1.1\alpha_r\mu_r^{k_{re}+\log_{1/\mu_r}(11/9)} v_{\boldsymbol{r}0} < \delta, \tag{3.40}$$

for all $j \in [l]$ and $k \in \left[k_{re} + \log_{1/\mu_r}(11/9), \min(k'_r, k_M)\right]$.

In particular,

$$S_j^{(h)}[k_m^{(h)}] \leq 1.1\alpha_r^{(h)}\mu_r^{k_m^{(h)}} v_{\boldsymbol{r}0} < \delta^{(h)}, \tag{3.41}$$

where $k_m^{(h)} := \min(k_r'^{(h)}, k_M^{(h)})$. On the other hand, for (3.16) to hold for an arbitrarily large k_M, we need $v_{\boldsymbol{rp}}$ to be *much greater* than δ so as to compensate for the corresponding (small) value of $\mu_r^{k_M}$. This leads to a contradiction. To elaborate, let $t^{(h)} \in [l]$ be the index satisfying (3.16). Then, for every $h \geq h_0$, (3.16), (3.41), and (3.39) imply:

$$\sum_{i=1}^{m} \alpha_i^{(h)} u_{\boldsymbol{it}^{(h)}}^{(h)} \left(\mu_i^{k_M^{(h)}} - \mu_i^{k_m^{(h)}} \right) > 0.9\alpha_r^{(h)} \mu_r^{k_{re}} v_{\boldsymbol{r}0} - 1.1\alpha_r^{(h)} \mu_r^{k_m^{(h)}} v_{\boldsymbol{r}0}. \tag{3.42}$$

Division by $\alpha_r^{(h)}$ and rearranging the terms yield:

$$1.1\mu_r^{k_m^{(h)}} v_{\boldsymbol{r}0} + \sum_{i=1}^{m} \frac{\alpha_i^{(h)}}{\alpha_r^{(h)}} u_{\boldsymbol{it}^{(h)}} \left(\mu_i^{k_M^{(h)}} - \mu_i^{k_m^{(h)}} \right) > 0.9\mu_r^{k_{re}} v_{\boldsymbol{r}0}. \tag{3.43}$$

However, the left-hand side of (3.43) tends to zero as $h \to \infty$ (since $\eta_r > 0$) because the fact that $\lim_{h\to\infty} \rho_{r1}^{(h)} = 0$ implies that $\lim_{h\to\infty} k_r'^{(h)} = \infty$ and in turn that $\lim_{h\to\infty} k_m^{(h)} = \infty$, whereas the right-hand side remains positive. This contradicts our assumption on $v_{\boldsymbol{rp}}$, thus proving (3.18).

Proof of (3.19)

Proof. By (3.17), (3.18) and Lemma 14, we have

$$\zeta_r^{(h)} \geq \hat{\gamma}_0 \alpha_r^{(h)} > 0 \tag{3.44}$$

for $h \geq h_0$. Therefore, analogous to M_e, η_i for $i \in [m]$, and r, we define $M_o := \arg\max_{i\in[m]} \zeta_i^{(h)}$ for $h \geq h_0$, $\tilde{\eta}_i := \lim_{h\to\infty} \zeta_i^{(h)}/\zeta_{M_o}^{(h)}$ for $i \in [m]$, and $\tilde{r} := \min\{i \in [m] : \tilde{\eta}_i > 0\}$, respectively (by passing to a subsequence of $\{y^{(h)}[0]\}_{h=1}^{\infty}$ if necessary).

Suppose now that (3.19) is false. Note that we did not use the assumption that $k_M^{(h)}$ is even until (3.41). Thus, if $z_{\tilde{\boldsymbol{r}}\tilde{p}(\tilde{r})} \geq \hat{\gamma}_0$ holds, then similar to $\delta^{(h)} > 0.9\alpha_r^{(h)} \mu_r^{k_{re}} v_{\boldsymbol{r}0}$, we have:

$$\delta^{(h)} > 0.9\zeta_{\tilde{r}}^{(h)} \mu_{\tilde{r}}^{k_{ro}} z_{\tilde{\boldsymbol{r}}0}, \tag{3.45}$$

where $k_{ro} := \max(0, 2\lceil 0.5 \log_{(\mu_{\tilde{r}}/\mu_{\tilde{r}+1})}(40/\tilde{\eta}_{\tilde{r}}\tilde{\tau}_{\tilde{r}})\rceil)$. On the other hand, $S_{t^{(h)}}[k_M^{(h)}] \geq \delta^{(h)}$ implies:

$$\delta^{(h)} \leq \sum_{i=1}^{m} \alpha_i^{(h)} \mu_i^{k_M^{(h)}} v_{\boldsymbol{i}j} \leq \alpha_{M_e}^{(h)} \mu_1^{k_M^{(h)}} m \tag{3.46}$$

since $|v_{\boldsymbol{i}j}| \leq 1$. Then, (3.45) and (3.46) result in:

$$\frac{\alpha_{M_e}^{(h)}}{\zeta_{\tilde{r}}^{(h)}} \geq \frac{0.9 \mu_r^{k_{ro}} z_{\boldsymbol{r}0}}{m} \left(\frac{1}{\mu_1}\right)^{k_M^{(h)}},$$

implying that $\lim_{h\to\infty}(\alpha_{M_e}^{(h)}/\zeta_{\tilde{r}}^{(h)}) = \infty$. Hence:

$$\begin{aligned} \lim_{h\to\infty} \frac{\zeta_r^{(h)}}{\alpha_r^{(h)}} &= \lim_{h\to\infty} \frac{\zeta_r^{(h)}}{\zeta_{M_o}^{(h)}} \cdot \frac{\zeta_{M_o}^{(h)}}{\zeta_{\tilde{r}}^{(h)}} \cdot \frac{\zeta_{\tilde{r}}^{(h)}}{\alpha_{M_e}^{(h)}} \cdot \frac{\alpha_{M_e}^{(h)}}{\alpha_r^{(h)}} \\ &= \tilde{\eta}_r \cdot \tilde{\eta}_{\tilde{r}}^{-1} \cdot 0 \cdot \eta_r^{-1} = 0. \end{aligned}$$

However, this contradicts (3.44). Therefore, (3.19) holds.

□

Chapter 3, in full, is a reprint of the material as it appears in Rohit Parasnis, Massimo Franceschetti, and Behrouz Touri, “On the Convergence Properties of Social Hegselmann–Krause dynamics,” in *IEEE Transactions on Automatic Control* 67.2 (2021): 589-604. The dissertation author was the primary investigator and author of this paper.

Chapter 4

Non-Bayesian Social Learning on Random Digraphs with Aperiodically Varying Network Connectivity

4.1 Introduction

The advent of social media and internet-based sources of information such as news websites and online databases over the last few decades has significantly influenced the way people learn about the world around them. For instance, while learning about political candidates or the latest electronic gadgets, individuals tend to gather relevant information from internet-based information sources as well as from the social groups they belong to.

To study the impact of social networks and external sources of information on the evolution of individuals' beliefs, several models of social dynamics have been proposed during the last few decades (see [9] and [10] for a detailed survey). Notably, the manner in which the agents update their beliefs ranges from being naive as in [103], wherein an agent's belief keeps shifting to the arithmetic mean of her neighbors' beliefs, to being fully rational (or Bayesian) as in the works [104] and [43]. For a survey of results on Bayesian learning, see [45].

However, as argued in [105] and in several subsequent works, it is unlikely that real-world social networks consist of fully rational agents because not only are Bayesian update rules computationally burdensome, but they also require every agent to understand the structure of the social network they belong to, and to know every other agent's history of private observations. Therefore, the seminal paper [105] proposed a non-Bayesian model of social learning to model agents with limited rationality (agents that intend to be fully rational but end up being only partially rational because they have neither the time nor the energy to analyze their neighbors' beliefs critically). This model assumes that the world (or the agents' object of interest) is described by a set of possible states, of which only one is the true state. With the objective of identifying the true state, each agent individually performs measurements on the state of the world and learns her neighbors' most recent beliefs in every state. At every new time step, the agent updates her beliefs by incorporating her own latest observations in a Bayesian manner and others' beliefs in a naive manner. With this update rule, all the agents almost surely learn the true state asymptotically in time, without having to learn the network structure or others' private observations.

Notably, some of the non-Bayesian learning models inspired by the original model proposed in [105] have been shown to yield efficient algorithms for distributed learning (for examples see [6, 106–113], and see [114] for a tutorial). Furthermore, the model of [105] has motivated research on decentralized estimation [115], cooperative device-to-device communications [116], crowdsensing in mobile social networks [117], manipulation in social networks [118], impact of social networking platforms, social media and fake news on social learning [119, 120], and learning in the presence of malicious agents and model uncertainty [121].

It is also worth noting that some of the models inspired by [105] have been studied in fairly general settings such as the scenario of infinitely many hypotheses [109], learning with asynchrony and crash failures [113], and learning in the presence of malicious agents and model uncertainty [121].

However, most of the existing non-Bayesian learning models make two crucial assumptions. First, they assume the network topology to be deterministic rather than stochastic. Second, they describe the network either by a static influence graph (a time-invariant graph that indicates whether or not two agents influence each other), or by a sequence of influence graphs that are *uniformly strongly connected*, i.e., strongly connected over time intervals that occur periodically.

By contrast, real-world networks are not likely to satisfy either assumption. The first assumption is often violated because real-world network structures are often subjected to a variety of random phenomena such as communication link failures. As for the second assumption, the influence graphs underlying real-world social networks may not always exhibit strong connectivity properties, and even if they do, they may not do so periodically. This is because there might be arbitrarily long deadlocks or phases of distrust between the agents during which most of them value their own measurements much more than others' beliefs. This is possible even when the agents know each other's beliefs well.

This dichotomy motivates us to extend the model of [105] to random directed graphs satisfying weaker connectivity criteria. To do so, we identify certain sets of agents called *observationally self-sufficient* sets. The collection of measurements obtained by any of these sets is at least as useful as that obtained by any other set of agents. We then introduce the concept of γ-epochs which, essentially, are periods of time over which the underlying social network is adequately well-connected. We then derive our main result: under the same assumptions as made in [105] on the agents' prior beliefs and observation structures, if the sequence of the weighted adjacency matrices associated with the network belongs to a broad class of random stochastic chains called Class $\mathcal{P}^*$, and if these matrices are independently distributed, then our relaxed connectivity assumption ensures that all the agents will almost surely learn the truth asymptotically in time.

The contributions of this chapter are as follows:

1. ***Criteria for Learning on Random Digraphs:*** Our work extends the earlier studies on non-Bayesian learning to the scenario of learning on random digraphs, and as we will show, our assumption of recurring γ-epochs is weaker than the standard assumption of uniform strong connectivity. Therefore, our main result identifies a set of sufficient conditions for almost-sure asymptotic learning that are weaker than those derived in prior works.

Moreover, our main result (Theorem 7) does not assume almost-sure fulfilment of our connectivity criteria (see Assumption IV and Remark 7). Consequently, our main result significantly generalizes some of the known results on social learning.

2. ***Implications for Distributed Learning:*** Since the learning rule (4.1) is an exponentially fast algorithm for distributed learning [114, 122], our main result significantly extends the practicality of the results of [105, 110, 123, 124].

3. ***A Sufficient Condition for Consensus:*** Theorem 6 shows how uniform strong connectivity ensures that all the agents' beliefs converge to a consensus almost surely even when the true state is not identifiable.

4. ***Results on Related Learning Scenarios:*** Section 4.5 provides sufficient conditions for almost-sure asymptotic learning in certain variants of the original model such as learning via diffusion-adaptation and inertial non-Bayesian learning.

5. ***Methodological Contribution:*** The proofs of Theorems 7 and 6 illustrate the effectiveness of the less-known theoretical techniques of Class $\mathcal{P}^*$ and absolute probability sequences. Although these tools are typically used to analyze linear dynamics, our work entails a novel application of the same to a non-linear system. Specifically, the proof of Theorem 7 is an example of how these methods can be used to analyse dynamics that approximate linear systems arbitrarily well in the limit as $t \to \infty$.

Out of the available non-Bayesian learning models, we choose the one proposed in [105] for our analysis because its update rule is an analog of DeGroot's learning rule [103] in a learning environment that enables the agents to acquire external information in the form of private signals [122], and experiments have repeatedly shown that variants of DeGroot's model predict real-world belief evolution better than models that are founded solely on Bayesian rationality [125–127]. Moreover, DeGroot's learning rule is the only rule that satisfies the psychological assumptions of imperfect recall, label-neutrality, monotonicity and separability [63].

Related works: We first describe the main differences between this chapter and our prior work [39]:

1. The main result (Theorem 1) of [39] applies only to deterministic time-varying networks, whereas the main result (Theorem 7) of this chapter applies to random time-varying networks. Hence, Theorem 7 of this chapter is more general than the main result of [39]. As we will show in Remark 7, the results of this chapter apply to certain random graph sequences that almost surely fall outside the class of deterministic graph sequences considered in [39].

2. In addition to the corollaries reported in [39], this chapter provides three corollaries of our main results that apply to random networks. These corollaries are central to the sections on learning amid link failures, inertial non-Bayesian learning, and learning via diffusion and adaptation (Section 4.5.1 – Section 4.5.3).

As for other related works, [128] and [129] make novel connectivity assumptions, but unlike our work, neither of them allows for arbitrarily long periods of poor network connectivity. The same can be said about [120] and [130], even though they consider random networks and impose connectivity criteria only in the expectation sense. Finally, we note that [131] and [132] come close to our work because the former proposes an algorithm that allows for aperiodically varying network connectivity while the latter makes no connectivity assumptions. However, the sensor network algorithms proposed in [131] and [132] require each agent to have an *actual* belief and a *local* belief, besides using minimum-belief rules to update the actual beliefs. By contrast, the learning rule we analyze is more likely to mimic social networks because it is simpler and closer to the empirically supported DeGroot learning rule. Moreover, unlike our analysis, neither [131] nor [132] accounts for randomness in the network structure.

We begin by defining the model in Section 5.2. In Section 4.3, we review Class $\mathcal{P}^*$, a special but broad class of matrix sequences that forms an important part of our assumptions.

Next, Section 5.3 establishes our main result. We then discuss the implications of this result in Section 4.5. We conclude with a brief summary and future directions in Section 5.6.

Notation: We denote the set of real numbers by $\mathbb{R}$, the set of positive integers by $\mathbb{N}$, and define $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For any $n \in \mathbb{N}$, we define $[n] := \{1, 2, \ldots, n\}$.

We denote the vector space of n-dimensional real-valued column vectors by $\mathbb{R}^n$. We use the superscript notation ^{T} to denote the transpose of a vector or a matrix. All the matrix and vector inequalities are entry-wise inequalities. Likewise, if $v \in \mathbb{R}^n$, then $|v| := [|v_1| \; |v_2| \; \ldots \; |v_n|]^T$, and if $v > 0$ additionally, then $\log(v) := [\log(v_1) \; \log(v_2) \; \ldots \; \log(v_n)]^T$. We use I to denote the identity matrix (of the known dimension) and $\mathbf{1}$ to denote the column vector (of the known dimension) that has all entries equal to 1. Similarly, $\mathbf{0}$ denotes the all-zeroes vector of the known dimension.

We say that a vector $v \in \mathbb{R}^n$ is *stochastic* if $v \geq 0$ and $v^T \mathbf{1} = 1$, and a matrix A is stochastic if A is non-negative and if each row of A sums to 1, i.e., if $A \geq 0$ and $A\mathbf{1} = \mathbf{1}$. A stochastic matrix A is *doubly stochastic* if each column of A sums to 1, i.e., if $A \geq 0$ and $A^T \mathbf{1} = A\mathbf{1} = \mathbf{1}$. A sequence of stochastic matrices is called a *stochastic chain*. If $\{A(t)\}_{t=0}^{\infty}$ is a stochastic chain, then for any two times $t_1, t_2 \in \mathbb{N}_0$ such that $t_1 \leq t_2$, we define $A(t_2 : t_1) := A(t_2 - 1)A(t_2 - 2) \cdots A(t_1)$, and let $A(t_1 : t_1) := I$. If $\{A(t)\}_{t=0}^{\infty}$ is a random stochastic chain (a sequence of random stochastic matrices), then it is called an *independent* chain if the matrices $\{A(t)\}_{t=0}^{\infty}$ are P-independent with respect to a given probability measure P.

4.2 Problem Formulation

4.2.1 The Non-Bayesian Learning Model

We begin by describing our non-Bayesian learning model which is simply the extension of the model proposed in [105] to random network topologies.

As in [105], we let Θ denote the (finite) set of possible states of the world and let $\theta^* \in \Theta$ denote the true state. We consider a social network of n agents that seek to learn the identity of the true state with the help of their private measurements as well as their neighbors' beliefs.

Beliefs and Observations

For each $i \in [n]$ and $t \in \mathbb{N}_0$, we let $\mu_{i,t}$ be the probability measure on $(\Theta, 2^{\Theta})$ such that $\mu_{i,t}(\theta) := \mu_{i,t}(\{\theta\})$ denotes the degree of belief of agent i in the state θ at time t. Also, for each $\theta \in \Theta$, we let $\mu_t(\theta) := [\mu_{1,t}(\theta)\ \mu_{2,t}(\theta)\ \ldots\ \mu_{n,t}(\theta)]^T \in [0,1]^n$.

As in [105], we assume that the signal space (the space of privately observed signals) of each agent is finite. We let S_i denote the signal space of agent i, define $S := S_1 \times S_2 \times \cdots \times S_n$, and let $\omega_t = (\omega_{1,t}, \ldots, \omega_{n,t}) \in S$ denote the vector of observed signals at time t. Further, we suppose that for each $t \in \mathbb{N}$, the vector ω_t is generated according to the conditional probability measure $l(\cdot|\theta)$ given that $\theta^* = \theta$, i.e., ω_t is distributed according to $l(\cdot|\theta)$ if θ is the true state.

We now repeat the assumptions made in [105]:

1. $\{\omega_t\}_{t\in\mathbb{N}}$ is an i.i.d. sequence.

2. For every $i \in [n]$ and $\theta \in \Theta$, agent i knows $l_i(\cdot|\theta)$, the i^{th} marginal of $l(\cdot|\theta)$ (i.e., $l_i(s|\theta)$ is the conditional probability that $\omega_{i,t} = s$ given that θ is the true state).

3. $l_i(s|\theta) > 0$ for all $s \in S_i$, $i \in [n]$ and $\theta \in \Theta$. We let $l_0 := \min_{\theta\in\Theta} \min_{i\in[n]} \min_{s_i\in S_i} l_i(s_i|\theta)$. Note that $l_0 > 0$.

In addition, it is possible that some agents do not have the ability to distinguish between certain states solely on the basis of their private measurements because these states induce the same conditional probability distributions on the agents' measurement signals. To describe such situations, we borrow the following definition from [105].

Definition 40 (Observational equivalence). *Two states $\theta_1, \theta_2 \in \Theta$ are said to be observationally equivalent from the point of view of agent i if $l_i(\cdot|\theta_1) = l_i(\cdot|\theta_2)$.*

For each $i \in [n]$, let $\Theta_i^* := \{\theta \in \Theta : l_i(\cdot|\theta) = l_i(\cdot|\theta^*)\}$ denote the set of states that are observationally equivalent to the true state from the viewpoint of agent i. Also, let $\Theta^* := \cap_{j\in[n]} \Theta_j^*$

be the set of states that are observationally equivalent to θ^* from every agent's viewpoint. Since we wish to identify the subsets of agents that can collectively distinguish between the true state and the false states, we define two related terms.

Definition 41 (**Observational self-sufficience**). *If $\mathcal{O} \subset [n]$ is a set of agents such that $\cap_{j \in \mathcal{O}} \Theta_j^* = \Theta^*$, then $\mathcal{O}$ is said to be an observationally self-sufficient set.*

Definition 42 (**Identifiability**). *If $\Theta^* = \{\theta^*\}$, then the true state θ^* is said to be identifiable.*

Network Structure and the Update Rule

Let $\{G(t)\}_{t \in \mathbb{N}_0}$ denote the random sequence of n-vertex directed graphs such that for each $t \in \mathbb{N}_0$, there is an arc from node $i \in [n]$ to node $j \in [n]$ in $G(t)$ if and only if agent i influences agent j at time t. Let $A(t) = (a_{ij}(t))$ be a stochastic weighted adjacency matrix of the random graph $G(t)$, and for each $i \in [n]$, let $\mathcal{N}_i(t) := \{j \in [n] \setminus \{i\} : a_{ij}(t) > 0\}$ denote the set of in-neighbors of agent i in $G(t)$. We assume that at the beginning of the learning process (i.e., at $t = 0$), agent i has $\mu_{i,0}(\theta) \in [0,1]$ as her prior belief in state $\theta \in \Theta$. At time $t+1$, she updates her belief in θ as follows:

$$\mu_{i,t+1}(\theta) = a_{ii}(t)\mathrm{BU}_{i,t+1}(\theta) + \sum_{j \in \mathcal{N}_i(t)} a_{ij}(t)\mu_{j,t}(\theta), \tag{4.1}$$

where "BU" stands for "Bayesian update" and

$$\mathrm{BU}_{i,t+1}(\theta) := \frac{l_i(\omega_{i,t+1}|\theta)\mu_{i,t}(\theta)}{\sum_{\theta' \in \Theta} l_i(\omega_{i,t+1}|\theta')\mu_{i,t}(\theta')}.$$

Finally, we let $(\Omega, \mathcal{B}, \mathbb{P}^*)$ be the probability space such that $\{\omega_t\}_{t=1}^{\infty}$ and $\{A(t)\}_{t=0}^{\infty}$ are measurable w.r.t. $\mathcal{B}$, and $\mathbb{P}^*$ is a probability measure such that:

$$\mathbb{P}^*(\omega_1 = s_1, \omega_2 = s_2, \ldots, \omega_r = s_r) = \prod_{t=1}^{r} l(s_t|\theta^*)$$

for all $s_1, \dots, s_r \in S$ and all $r \in \mathbb{N} \cup \{\infty\}$. As in [105], we let $\mathbb{E}^*$ denote the expectation operator associated with $\mathbb{P}^*$.

4.2.2 Forecasts and Convergence to the Truth

At any time step t, agent i can use her current set of beliefs to estimate the probability that she will observe the signals $s_1, s_2, \dots, s_k \in S_i$ over the next k time steps. This is referred to as the *k-step-ahead forecast* of agent i at time t and denoted by $m_{i,t}^{(k)}(s_1, \dots, s_k)$. We thus have:

$$m_{i,t}^{(k)}(s_1, \dots, s_k) := \sum_{\theta \in \Theta} \prod_{r=1}^{k} l_i(s_r|\theta)\mu_{i,t}(\theta).$$

We use the following notions of convergence to the truth.

Definition 43 (Eventual Correctness [105]**).** *The k-step ahead forecasts of agent i are said to be eventually correct on a path $(A(0), \omega_1, A(1), \omega_2, \dots)$ if, along that path,*

$$m_{i,t}^{(k)}(s_1, s_2, \dots, s_k) \to \prod_{j=1}^{k} l_i(s_j|\theta^*) \quad as \quad t \to \infty.$$

Definition 44 (Weak Merging to the Truth [105]**).** *We say that the beliefs of agent i weakly merge to the truth on some path if, along that path, her k-step-ahead forecasts are eventually correct for all $k \in \mathbb{N}$.*

Definition 45 (Asymptotic Learning [105]**).** *Agent $i \in [n]$ asymptotically learns the truth on a path $(A(0), \omega_1, A(1), \omega_2, \dots)$ if, along that path, $\mu_{i,t}(\theta^*) \to 1$ (and hence $\mu_{i,t}(\theta) \to 0$ for all $\theta \in \Theta \setminus \{\theta^*\}$) as $t \to \infty$.*

Note that, if the belief of agent i weakly merges to the truth, it only means that agent i is able to estimate the probability distributions of her future signals/observations with arbitrary accuracy as time goes to infinity. On the other hand, if agent i asymptotically learns the truth, it means that, in the limit as time goes to infinity, agent i rules out all the false states and correctly figures out that the true state is θ^*. In fact, it can be shown that asymptotic learning implies weak merging to the truth, even though the latter does not imply the former [105].

4.3 Revisiting Class $\mathcal{P}^*$: A Special Class of Stochastic Chains

Our next goal is to deviate from the standard strong connectivity assumptions for social learning [105, 110, 123, 124]. We first explain the challenges involved in this endeavor. To begin, we express (4.1) as follows (Equation (4) in [105]):

$$\begin{aligned}&\mu_{t+1}(\theta) - A(t)\mu_t(\theta)\\ &= \text{diag}\left(\dots, a_{ii}(t)\left[\frac{l_i(\omega_{i,t+1}|\theta)}{m_{i,t}(\omega_{i,t+1})} - 1\right], \dots\right)\mu_t(\theta),\end{aligned} \tag{4.2}$$

where $m_{i,t}(s) := m_{i,t}^{(1)}(s)$ for all $s \in S_i$. Now, suppose $\theta = \theta^*$. Then, an extrapolation of the known results on non-Bayesian learning suggests the right-hand-side of (4.2) decays to 0 almost surely as $t \to \infty$. This means that for large values of t (say $t \geq T_0$ for some $T_0 \in \mathbb{N}$), the dynamics (4.2) for $\theta = \theta^*$ can be approximated as $\mu_{t+1}(\theta^*) \approx A(t)\mu_t(\theta^*)$. Hence, we expect the limiting value of $\mu_t(\theta^*)$ to be closely related to $\lim_{t\to\infty} A(t : T_0)$, whenever the latter limit exists. However, without standard connectivity assumptions, it is challenging to gauge the existence of limits of backward matrix products.

To overcome this difficulty, we use the notion of Class $\mathcal{P}^*$ introduced in [62]. This notion is based on Kolmogorov's ingenious concept of absolute probability sequences, which we now define.

Definition 46 (**Absolute Probability Sequence** [62])**.** *Let $\{A(t)\}_{t=0}^{\infty}$ be either a deterministic stochastic chain or a random process of independently distributed stochastic matrices. A deterministic sequence of stochastic vectors $\{\pi(t)\}_{t=0}^{\infty}$ is said to be an absolute probability* *sequence for* $\{A(t)\}_{t=0}^{\infty}$ *if*

$$\pi^T(t+1)\mathbb{E}[A(t)] = \pi^T(t) \quad \textit{for all } t \geq 0.$$

Note that every deterministic stochastic chain admits an absolute probability sequence [38]. Hence, every random sequence of independently distributed stochastic matrices also admits an absolute probability sequence.

Of interest to us is a special class of random stochastic chains that are associated with absolute probability sequences satisfying a certain condition. This class is defined below.

Definition 47 (**Class** $\mathcal{P}^*$ [62])**.** *We let (Class-)*$\mathcal{P}^*$ *be the set of random stochastic chains that admit an absolute probability sequence* $\{\pi(t)\}_{t=0}^{\infty}$ *such that* $\pi(t) \geq p^*\mathbf{1}$ *for some scalar* $p^* > 0$ *and all* $t \in \mathbb{N}_0$.

Remarkably, in scenarios involving a linear aggregation of beliefs, if $\{\pi(t)\}_{t=0}^{\infty}$ is an absolute probability sequence for $\{A(t)\}_{t=0}^{\infty}$, then $\pi_i(t)$ denotes the *Kolmogorov centrality* or *social power* of agent i at time t, which quantifies how influential the i-th agent is relative to other agents at time t [62, 63]. In view of Definition 47, this means that, if a stochastic chain belongs to Class $\mathcal{P}^*$, then the expected chain describes a sequence of influence graphs in which the social power of every agent exceeds a fixed threshold $p^* > 0$ at all times. Let us now now look at a concrete example.

Example 1. *Suppose* $A(t) = A_e$ *for all even* $t \in \mathbb{N}_0$*, and* $A(t) = A_o$ *for all odd* $t \in \mathbb{N}_0$*, where* A_e *and* A_o *are the matrices defined below:*

$$A_e := \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad A_o := \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix}.$$

Then one may verify that the alternating sequence $[\frac{2}{3}\ \frac{1}{3}]^T, [\frac{1}{3}\ \frac{2}{3}]^T, [\frac{2}{3}\ \frac{1}{3}]^T, \ldots$ *is an absolute probability sequence for the chain* $\{A(t)\}_{t=0}^{\infty}$*. Hence,* $\{A(t)\}_{t=0}^{\infty} \in \mathcal{P}^*$*.*

Let us now add a zero-mean independent noise sequence $\{W(t)\}_{t=0}^{\infty}$ to the original chain, where for all even $t \in \mathbb{N}_0$, the matrix $W(t)$ is the all-zeros matrix (and hence a degenerate random matrix), and for all odd $t \in \mathbb{N}_0$, the matrix $W(t)$ is uniformly distributed on $\{W_0, -W_0\}$, with W_0 given by

$$W_0 := \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{pmatrix}.$$

Then by Theorem 5.1 in [62], the random stochastic chain $\{A(t)+W(t)\}_{t=0}^{\infty}$ belongs to Class $\mathcal{P}^*$ because the expected chain $\{\mathbb{E}[A(t)+W(t)]\}_{t=0}^{\infty} = \{A(t)\}_{t=0}^{\infty}$ belongs to Class $\mathcal{P}^*$.

Remark 7. *Interestingly, Example 1 illustrates that a random stochastic chain may belong to Class $\mathcal{P}^*$ even though almost every realization of the chain lies outside Class $\mathcal{P}^*$. To elaborate, consider the setup of Example 1, and let $\tilde{A}(t) := A(t)+W(t)$. Observe that $A_o - W_0 = I$, which means that for any $B \in \mathbb{N}$ and $t_1 \in \mathbb{N}_0$, the probability that $A(t)+W(t) = I$ for all odd $t \in \{t_1, \ldots, t_1+2B-1\}$ is $\left(\frac{1}{2}\right)^B > 0$. Since $\{W(t)\}_{t=0}^{\infty}$ are independent, it follows that for $\mathbb{P}^*$-almost every realization $\{\hat{A}(t)\}_{t=0}^{\infty}$ of $\{\tilde{A}(t)\}_{t=0}^{\infty}$, there exists a time $\tau \in \mathbb{N}_0$ such that $\hat{A}(\tau+2B:\tau) = A_e \cdot I \cdot A_e \cdot I \cdots A_e \cdot I = A_e^B$. Therefore, if $\{\pi_R(t)\}_{t=0}^{\infty}$ is an absolute probability sequence for the deterministic chain $\{\hat{A}(t)\}_{t=0}^{\infty}$, we can use induction along with Definition 46 to show that $\pi_R^T(\tau+2B)\hat{A}(\tau+2B:\tau)$ equals $\pi_R^T(\tau)$. Thus,*

$$\pi_R^T(\tau) = \pi_R^T(\tau+2B)A_e^B \leq \mathbf{1}^T A_e^B.$$

Since the second entry of $\mathbf{1}^T A_e^B$ evaluates to $\frac{1}{2^B}$, and since B is arbitrary, it follows that there is no lower bound $p^* > 0$ on the second entry of $\pi_R(\tau)$. Hence, $\{\hat{A}(t)\}_{t=0}^{\infty} \notin \mathcal{P}^*$, implying that $\mathbb{P}^*$-almost no realization of $\{\tilde{A}(t)\}$ belongs to Class $\mathcal{P}^*$.

We now turn to a noteworthy subclass of Class $\mathcal{P}^*$: the class of *uniformly strongly connected* chains (Lemma 5.8, [62]). Below is the definition of this subclass (reproduced from [62]).

Definition 48 (**Uniform Strong Connectivity**). *A deterministic stochastic chain* $\{A(t)\}_{t=0}^{\infty}$ *is said to be uniformly strongly connected if:*

1. *there exists a* $\delta > 0$ *such that for all* $i, j \in [n]$ *and all* $t \in \mathbb{N}_0$*, either* $a_{ij}(t) \geq \delta$ *or* $a_{ij}(t) = 0$,

2. $a_{ii}(t) > 0$ *for all* $i \in [n]$ *and all* $t \in \mathbb{N}_0$*, and*

3. *there exists a constant* $B \in \mathbb{N}$ *such that the sequence of directed graphs* $\{G(t)\}_{t=0}^{\infty}$*, defined by* $G(t) = ([n], E(t))$ *where* $E(t) := \{(i,j) \in [n]^2 : a_{ji}(t) > 0\}$*, has the property that the graph:*

$$\mathcal{G}(k) := \left([n], \bigcup_{q=kB}^{(k+1)B-1} E(q)\right)$$

is strongly connected for every $k \in \mathbb{N}_0$.

Due to the last requirement above, uniformly strongly connected chains are also called B-strongly connected chains or simply B-connected chains. Essentially, a B-connected chain describes a time-varying network that may or may not be connected at every time instant but is guaranteed to be connected over bounded time intervals that occur periodically.

Besides uniformly strongly connected chains, we are interested in another subclass of Class $\mathcal{P}^*$: the set of independent *balanced* chains with *feedback property* (Theorem 4.7, [62]).

Definition 49 (**Balanced chains**). *A stochastic chain* $\{A(t)\}_{t=0}^{\infty}$ *is said to be balanced if there exists an* $\alpha \in (0, \infty)$ *such that:*

$$\sum_{i\in C}\sum_{j\in[n]\setminus C} a_{ij}(t) \geq \alpha \sum_{i\in[n]\setminus C}\sum_{j\in C} a_{ij}(t) \tag{4.3}$$

for all sets $C \subset [n]$ *and all* $t \in \mathbb{N}_0$.

Definition 50 (**Feedback property**). *Let* $\{A(t)\}_{t=0}^{\infty}$ *be a random stochastic chain, and let* $\mathcal{F}_t := \sigma(A(0), \ldots, A(t-1))$ *for all* $t \in \mathbb{N}$*. Then* $\{A(t)\}_{t=0}^{\infty}$ *is said to have feedback property if* *there exists a* $\delta > 0$ *such that*

$$\mathbb{E}[a_{ii}(t)a_{ij}(t)|\mathcal{F}_t] \geq \delta\mathbb{E}[a_{ij}(t)|\mathcal{F}_t] \quad \textit{a.s.}$$

for all $t \in \mathbb{N}_0$ *and all distinct* $i, j \in [n]$.

Intuitively, a balanced chain is a stochastic chain in which the total influence of any subset of agents $C \subset [n]$ on the complement set $\bar{C} := [n] \setminus C$ is neither negligible nor tremendous when compared to the total influence of $\bar{C}$ on C. As for the feedback property, we relate its definition to the *strong feedback property*, which has a clear interpretation.

Definition 51 (**Strong feedback property**). *We say that a random stochastic chain* $\{A(t)\}_{t=0}^{\infty}$ *has the strong feedback property with feedback coefficient* δ *if there exists a* $\delta > 0$ *such that* $a_{ii}(t) \geq \delta$ *a.s. for all* $i \in [n]$ *and all* $t \in \mathbb{N}_0$.

Intuitively, a chain with the strong feedback property describes a network in which all the agents' self-confidences are always above a certain threshold.

To see how the strong feedback property is related to the (regular) feedback property, note that by Lemma 4.2 of [62], if $\{A(t)\}_{t=0}^{\infty}$ has feedback property, then the expected chain, $\{\mathbb{E}[A(t)]\}_{t=0}^{\infty}$ has the strong feedback property. Thus, a balanced independent chain with feedback property describes a network in which complementary sets of agents influence each other to comparable extents, and every agent's mean self-confidence is always above a certain threshold.

Remark 8. *It may appear that every stochastic chain belonging to Class* $\mathcal{P}^*$ *is either uniformly strongly connected or balanced with feedback property, but this is not true. Indeed, one such chain is described in Example 1, wherein we have* $A(t) + W(t) = A_e$ *for even* $t \in \mathbb{N}_0$*, which implies that* (4.3) *is violated at even times. Hence,* $\{A(t) + W(t)\}_{t=0}^{\infty}$ *is not a balanced chain. As for uniform strong connectivity, recall from Remark 7 that* $\mathbb{P}^*$*-almost every realization of* $\{A(t) + W(t)\}_{t=0}^{\infty}$ *lies outside Class* $\mathcal{P}^*$*. Since Class* $\mathcal{P}^*$ *is a superset of the class of uniformly* *strongly connected chains (Lemma 5.8, [62]), it follows that* $\{A(t)+W(t)\}_{t=0}^{\infty}$ *is almost surely not uniformly strongly connected.*

Remark 9. *[133] provides examples of subclasses of Class* $\mathcal{P}^*$ *chains that are not uniformly strongly connected, such as the class of doubly stochastic chains. For instance, let* $\mathcal{D} \subset \mathbb{R}^{n\times n}$ *be any finite collection of doubly stochastic matrices such that* $I \in \mathcal{D}$*, and let* $\{A(t)\}_{t=0}^{\infty}$ *be a sequence of i.i.d. random matrices, each of which is uniformly distributed on* $\mathcal{D}$*. Then* $\{A(t)\}_{t=0}^{\infty}$*, being a doubly stochastic chain, belongs to Class* $\mathcal{P}^*$ *(see [133]). Now, for any* $B \in \mathbb{N}$ *and* $t_1 \in \mathbb{N}_0$*, the probability that* $A(t) = I$ *for all* $t \in \{t_1, \ldots, t_1+B-1\}$ *equals* $\frac{1}{|\mathcal{D}|^B} > 0$*. In light of the independence of* $\{A(t)\}_{t=0}^{\infty}$*, this implies that there almost surely exists a time interval* $\mathcal{T}$ *of length* B *such that* $A(t) = I$ *for all* $t \in \mathcal{T}$*, implying that there is no connectivity in the network during the interval* $\mathcal{T}$*. As the interval duration* B *is arbitrary, this means that the chain* $\{A(t)\}_{t=0}^{\infty}$ *is almost surely not uniformly strongly connected.*

4.4 The Main Result and its Derivation

We first introduce a network connectivity concept called γ-*epoch*, which plays a key role in our main result.

Definition 52 (**γ-epoch**). *For a given* $\gamma > 0$ *and* $t_s, t_f \in \mathbb{N}$ *satisfying* $t_s < t_f$*, the time interval* $[t_s, t_f]$ *is a* γ*-epoch if, for each* $i \in [n]$*, there exists an observationally self-sufficient set of agents,* $\mathcal{O}_i \subset [n]$*, and a set of time instants* $\mathcal{T}_i \subset \{t_s+1, \ldots, t_f\}$ *such that for every* $j \in \mathcal{O}_i$*, there exists a* $t \in \mathcal{T}_i$ *satisfying* $a_{jj}(t) \geq \gamma$ *and* $(A(t:t_s))_{ji} \geq \gamma$*. Moreover, if* $[t_s, t_f]$ *is a* γ*-epoch, then* $t_f - t_s$ *is the epoch duration.*

As an example, if $n \geq 9$ and if the sets $\{2,5,9\}$ and $\{7,9\}$ are observationally self-sufficient, then Fig. 4.1 illustrates the influences of agents 1 and n in the γ-epoch $[0,5]$.

Intuitively, γ-epochs are time intervals over which every agent strongly influences an observationally self-sufficient set of agents whose self-confidences are guaranteed to be above a certain threshold at the concerned time instants.

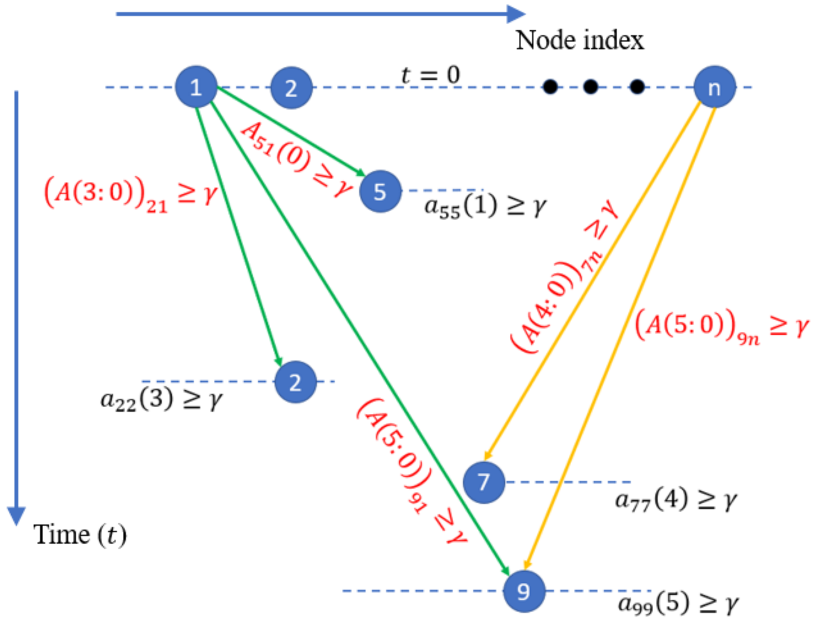

Figure 4.1. Example of a γ-epoch (from the viewpoint of nodes 1 and n)

We now list the assumptions underlying our main result.

I **(Recurring γ-epochs).** There exist constants $\gamma > 0$ and $B \in \mathbb{N}$, and an increasing sequence $\{t_k\}_{k=1}^{\infty} \subset \mathbb{N}$ such that $t_{2k} - t_{2k-1} \leq B$ for all $k \in \mathbb{N}$, and

$$\sum_{k=1}^{\infty} \mathbb{P}^* \left([t_{2k-1}, t_{2k}] \text{ is a } \gamma\text{-epoch}\right) = \infty.$$

This means that the probability of occurrence of a γ-epoch of bounded duration does not vanish too fast with time. Note, however, that $t_{2k+1} - t_{2k}$ (the time elapsed between two consecutive candidate γ-epochs) may be unbounded.

II **(Existence of a positive prior).** There exists an agent $j_0 \in [n]$ such that $\mu_{j_0,0}(\theta^*) > 0$, i.e., the true state is not ruled out entirely by every agent. We assume w.l.o.g. that $j_0 = 1$.

III **(Initial connectivity with the agent with the positive prior).** There *a.s.* exists a random time $T < \infty$ such that[1] $\mathbb{E}^* \left[\log \left(A(T:0)\right)_{i1}\right] > -\infty$ for all $i \in [n]$.

[1]In general, if $Q := \{j \in [n] : \mu_{j,0}(\theta^*) > 0\}$, then we only need $\mathbb{E}^*[\max_{j \in Q} \log((A(T:0))_{ij})] > -\infty$.

IV **(Class $\mathcal{P}^*$).** $\{A(t)\}_{t=0}^{\infty} \in \mathcal{P}^*$, i.e., the sequence of weighted adjacency matrices of the network belongs to Class $\mathcal{P}^*$ w.r.t. the probability measure $\mathbb{P}^*$.

V **(Independent chain).** $\{A_t\}_{t=0}^{\infty}$ is a $\mathbb{P}^*$-independent chain.

VI **(Independence of observations and network structure).** The sequences $\{\omega_t\}_{t=1}^{\infty}$ and $\{A_t\}_{t=0}^{\infty}$ are $\mathbb{P}^*$-independent of each other.

We are now ready to state our main results.

Theorem 5. *Suppose that the sequence $\{A(t)\}_{t=0}^{\infty}$ and the agents' initial beliefs satisfy Assumptions II - VI. Then:*

(i) If $\{A(t)\}_{t=0}^{\infty}$ either has the strong feedback property or satisfies Assumption I, then every agent's beliefs weakly merge to the truth $\mathbb{P}^*$-a.s. (i.e., $\mathbb{P}^*$-almost surely).

(ii) If Assumption I holds and θ^* is identifiable, then all the agents asymptotically learn the truth $\mathbb{P}^*$-a.s.

Theorem 7 applies to stochastic chains belonging to Class $\mathcal{P}^*$, and hence to scenarios in which the social power (Kolmogorov centrality) of each agent exceeds a fixed positive threshold at all times in the expectation sense (see Section 4.3). While Part (i) identifies the recurrence of γ-epochs as a sufficient connectivity condition for the agents' forecasts to be eventually correct, Part (ii) asserts that, if γ-epochs are recurrent and if the agents' observation methods enable them to collectively distinguish the true state from all other states, then they will learn the true state asymptotically almost surely.

Note that the sufficient conditions provided in Theorem 7 do not include uniform strong connectivity. However, it turns out that uniform strong connectivity as a connectivity criterion is sufficient not only for almost-sure weak merging to the truth but also for ensuring that all the agents asymptotically agree with each other almost surely, even when the true state is not identifiable. We state this result formally below.

Theorem 6. *Suppose Assumption II holds, and suppose* $\{A(t)\}_{t=0}^{\infty}$ *is deterministic and uniformly strongly connected. Then, all the agents' beliefs converge to a consensus* $\mathbb{P}^*$*-a.s., i.e., for each* $\theta \in \Theta$*, there exists a random variable* $C_\theta \in [0,1]$ *such that* $\lim_{t\to\infty} \mu_t(\theta) = C_\theta \mathbf{1}$ *a.s.*

Before proving Theorems 7 and 6, we look at the effectiveness of the concepts of Section 4.3 in analyzing the social learning dynamics studied in this chapter. We begin by noting the following implication of Assumption IV: there exists a deterministic sequence of stochastic vectors $\{\pi(t)\}_{t=0}^{\infty}$ and a constant $p^* > 0$ such that $\{\pi(t)\}_{t=0}^{\infty}$ is an absolute probability sequence for $\{A(t)\}_{t=0}^{\infty}$, and $\pi(t) \geq p^* \mathbf{1}$ for all $t \in \mathbb{N}_0$.

Using Absolute Probability Sequences and the Notion of Class $\mathcal{P}^*$ to Analyze Social Learning

1) Linear Approximation of the Update Rule: Consider the update rule (4.2) with $\theta = \theta^*$. Note that the only non-linear term in this equation is

$$u(t) := \text{diag}\left(\ldots, a_{ii}(t)\left[\frac{l_i(\omega_{i,t+1}|\theta^*)}{m_{i,t}(\omega_{i,t+1})} - 1\right], \ldots\right)\mu_t(\theta^*).$$

So, in case $\lim_{t\to\infty} u(t) = \mathbf{0}$, then the resulting dynamics for large t would be $\mu_t(\theta^*) \approx A(t)\mu_t(\theta^*)$, which is approximately linear and hence easier to analyze. This motivates us to use the following trick: we could take the dot product of each side of (4.2) with a non-vanishing positive vector $q(t)$, and then try to show that $q^T(t)u(t) \to 0$ as $t \to \infty$. Also, since $\{A(t)\} \in \mathcal{P}^*$, we could simply choose $\{q(t)\}_{t=0}^{\infty} = \{\pi(t)\}_{t=0}^{\infty}$ as our sequence of non-vanishing positive vectors.

Before using this trick, we need to take suitable conditional expectations on both sides of (4.2) so as to remove all the randomness from $A(t)$ and $a_{ii}(t)$. To this end, we define $\mathcal{B}_t := \sigma(\omega_1, \ldots, \omega_t, A(0), \ldots, A(t))$ for each $t \in \mathbb{N}$, and obtain the following from (4.2):

$$\mathbb{E}^*[\mu_{t+1}(\theta^*) \mid \mathcal{B}_t] - A(t)\mu_t(\theta^*) = \mathbb{E}^*[u(t) \mid \mathcal{B}_t],$$

where we used that $\mu_t(\theta^*)$ is measurable w.r.t. $\mathcal{B}_t$.

We now use the said trick as follows: we left-multiply both the sides of the above equation by the non-random vector $\pi^T(t+1)$ and obtain:

$$\pi^T(t+1)\mathbb{E}^*[u(t) \mid \mathcal{B}_t] = \pi^T(t+1)\mathbb{E}^*[\mu_{t+1}(\theta^*)|\mathcal{B}_t] - \pi^T(t+1)A(t)\mu_t(\theta^*).$$

Here, we use the definition of absolute probability sequences (Definition 46): we replace $\pi^T(t+1)$ with $\pi^T(t+2)\mathbb{E}^*[A(t+1)]$ in the first term on the right-hand-side. Consequently,

$$\begin{aligned}\pi^T(t+1)\mathbb{E}^*[u(t) \mid \mathcal{B}_t] &= \pi^T(t+2)\mathbb{E}^*[A(t+1)]\mathbb{E}^*[\mu_{t+1}(\theta^*) \mid \mathcal{B}_t] - \pi^T(t+1)A(t)\mu_t(\theta^*) \\ &\overset{(a)}{=} \mathbb{E}^*[\pi^T(t+2)A(t+1)\mu_{t+1}(\theta^*) \mid \mathcal{B}_t] - \pi^T(t+1)A(t)\mu_t(\theta^*), \end{aligned} \tag{4.4}$$

where (a) follows from Assumptions V and VI (for more details see Lemma 27). Now, to prove that $\lim_{t\to\infty} u(t) = \mathbf{0}$, we could begin by showing that the left-hand-side of (4.4) (i.e., $\pi^T(t+1)\mathbb{E}^*[u(t) \mid \mathcal{B}_t]$) goes to 0 as $t \to \infty$. As it turns out, this latter condition is already met: according to Lemma 21 (in the appendix), the right-hand side of (4.4) vanishes as $t \to \infty$. As a result,

$$\lim_{t\to\infty} \pi^T(t+1)\mathbb{E}^*[u(t) \mid \mathcal{B}_t] = 0 \quad \textit{a.s.}$$

Equivalently,

$$\sum_{i=1}^n \pi_i(t+1)a_{ii}(t)\mathbb{E}^*\left[\frac{l_i(\omega_{i,t+1}|\theta^*)}{m_{i,t}(\omega_{i,t+1})} - 1 \middle| \mathcal{B}_t\right]\mu_{i,t}(\theta^*) \longrightarrow 0 \quad \text{almost surely as} \quad t \to \infty,$$

where we have used that $a_{ii}(t)$ and $\mu_t(\theta^*)$ are measurable w.r.t. $\mathcal{B}_t$. To remove the summation from the above limit, we use the lower bound in Lemma 24 to argue that every summand in the above expression is non-negative. Thus, for each $i \in [n]$,

$$\lim_{t\to\infty} \pi_i(t+1)a_{ii}(t)\mathbb{E}^*\left[\frac{l_i(\omega_{i,t+1}|\theta^*)}{m_{i,t}(\omega_{i,t+1})} - 1 \middle| \mathcal{B}_t\right]\mu_{i,t}(\theta^*) = 0$$

a.s. More compactly, $\lim_{t\to\infty}\pi_i(t+1)\mathbb{E}^*[u_i(t)\mid\mathcal{B}_t]=0$ *a.s.* Here, Class $\mathcal{P}^*$ plays an important role: since $\pi(t+1)\geq p^*\mathbf{1}$, the multiplicand $\pi_i(t+1)$ can be omitted:

$$\lim_{t\to\infty}\mathbb{E}^*\left[u_i(t)\mid\mathcal{B}_t\right]=0 \quad \textit{a.s.} \tag{4.5}$$

We have thus shown that $\lim_{t\to\infty}\mathbb{E}^*[u(t)\mid\mathcal{B}_t]=\mathbf{0}$ *a.s.* With the help of some algebraic manipulation, we can now show that $\lim_{t\to\infty}u(t)=\mathbf{0}$ *a.s.* (see Lemma 25 for further details).

2) Analysis of 1-Step-Ahead Forecasts: Interestingly, the result $\lim_{t\to\infty}u(t)=\mathbf{0}$ *a.s.* can be strengthened further to comment on 1-step-ahead forecasts, as we now show.

Recall that $\pi(t)\geq p^*\mathbf{1}$ for all $t\in\mathbb{N}_0$. Since $\log(\mu_t(\theta^*))\leq\mathbf{0}$, this means that the following hold almost surely:

$$\begin{aligned}p^*\liminf_{t\to\infty}\sum_{i=1}^n\log(\mu_{i,t}(\theta^*))&=\liminf_{t\to\infty}p^*\mathbf{1}^T\log(\mu_t(\theta^*))\\&\geq\liminf_{t\to\infty}\pi^T\log(\mu_t(\theta^*))>-\infty,\end{aligned}$$

where the last step follows from Lemma 21. This is possible only if $\liminf_{t\to\infty}\log(\mu_{i,t}(\theta^*))>-\infty$ *a.s.* for each $i\in[n]$, which implies that $\liminf_{t\to\infty}\mu_{i,t}(\theta^*)>0$ *a.s.*, that is, there *a.s.* exist random variables $\delta>0$ and $T'\in\mathbb{N}$ such that $\mu_{i,t}(\theta^*)\geq\delta$ *a.s.* for all $t\geq T'$. Since $\lim_{t\to\infty}u_i(t)=0$ *a.s.*, it follows that $\lim_{t\to\infty}\frac{u_i(t)}{\mu_{i,t}(\theta^*)}=0$ *a.s.*, that is,

$$\lim_{t\to\infty}a_{ii}(t)\left(\frac{l_i(\omega_{i,t+1}|\theta^*)}{m_{i,t}(\omega_{i,t+1})}-1\right)=0 \quad \textit{a.s.}$$

On multiplying the above limit by $-m_{i,t}(\omega_{i,t+1})$, we observe that the following holds *a.s.* $\lim_{t\to\infty}a_{ii}(t)\left(m_{i,t}(\omega_{i,t+1})-l_i(\omega_{i,t+1}|\theta^*)\right)=0$. We now perform some simplification (see

Lemma 28) to show that

$$\lim_{t\to\infty} a_{ii}(t)\left(m_{i,t}(s) - l_i(s|\theta^*)\right) = 0 \quad \textit{a.s.} \text{ for all } s \in S_i. \tag{4.6}$$

Therefore, if there exists a sequence of times $\{t_k\}_{k=1}^{\infty}$ with $t_k \uparrow \infty$ such that the i-th agent's self-confidence $a_{ii}(t)$ exceeds a fixed threshold $\gamma > 0$ at times $\{t_k\}_{k=1}^{\infty}$, then (4.6) implies that its 1-step-ahead forecasts sampled at $\{t_k\}_{k=1}^{\infty}$ converge to the true forecasts, i.e., $\lim_{k\to\infty} m_{i,t_k}(s) = l_i(s|\theta^*)$ *a.s.*

The following lemma generalizes (4.6) to h-step-ahead forecasts. Its proof is based on induction and elementary properties of conditional expectation.

Lemma 20. *For all* $h \in \mathbb{N}$, $s_1, s_2, \ldots, s_h \in S_i$ *and* $i \in [n]$,

$$\lim_{t\to\infty} a_{ii}(t)\left(m_{i,t}^{(h)}(s_1, s_2, \ldots, s_h) - \prod_{r=1}^{h} l_i(s_r|\theta^*)\right) = 0 \quad \textit{a.s.}$$

Proof. We prove this lemma by induction. Observe that since $m_{i,t}(s) \leq 1$ for all $s \in S_i$ and $i \in [n]$, on multiplication by $m_{i,t}(s)$, Lemma 25 implies that Lemma 20 holds for $h = 1$. Now, suppose Lemma 20 holds for some $h \in \mathbb{N}$, and subtract both the sides of (4.2) from $\mathbb{E}^*[\mu_{t+1}(\theta)|\mathcal{B}_t]$ in order to obtain

$$\begin{aligned}
&\mathbb{E}^*[\mu_{t+1}(\theta)|\mathcal{B}_t] - \mu_{t+1}(\theta) \\
&= \mathbb{E}^*[\mu_{t+1}(\theta)|\mathcal{B}_t] - A(t)\mu_t(\theta) - \text{diag}\left(\ldots, a_{ii}(t)\left[\frac{l_i(\omega_{i,t+1}|\theta)}{m_{i,t}(\omega_{i,t+1})} - 1\right], \ldots\right)\mu_t(\theta).
\end{aligned}$$

Rearranging the above and using Lemma 26 results in

$$\mathbb{E}^*[\mu_{t+1}(\theta)|\mathcal{B}_t] - \mu_{t+1}(\theta) + \text{diag}\left(\ldots, a_{ii}(t)\left[\frac{l_i(\omega_{i,t+1}|\theta)}{m_{i,t}(\omega_{i,t+1})} - 1\right], \ldots\right)\mu_t(\theta) \xrightarrow{t\to\infty} 0$$

a.s. For $i \in [n]$, we now pick the i-th entry of the above vector limit, multiply both its sides by

$a_{ii}(t)\prod_{r=1}^{h} l_i(s_r|\theta)$ (where $s_1, s_2, \dots, s_r$ are chosen arbitrarily from S_i), and then sum over all $\theta \in \Theta$. As a result, the following quantity approaches 0 almost surely as $t \to \infty$

$$\sum_{\theta\in\Theta} a_{ii}(t)\left(\prod_{r=1}^{h} l_i(s_r|\theta)\right)\left(\mathbb{E}^*[\mu_{i,t+1}(\theta)|\mathcal{B}_t] - \mu_{i,t+1}(\theta)\right) + \sum_{\theta\in\Theta}\left(\prod_{r=1}^{h} l_i(s_r|\theta)\right) a_{ii}^2(t)\left[\frac{l_i(\omega_{i,t+1}|\theta)}{m_{i,t}(\omega_{i,t+1})} - 1\right]\mu_{i,t}(\theta) \tag{4.7}$$

On the other hand, the following holds almost surely

$$\begin{aligned}
&\sum_{\theta\in\Theta} a_{ii}(t)\left(\prod_{r=1}^{h} l_i(s_r|\theta)\right)\left(\mathbb{E}^*[\mu_{i,t+1}(\theta)|\mathcal{B}_t] - \mu_{i,t+1}(\theta)\right)\\
&= a_{ii}(t)\mathbb{E}^*\left[\sum_{\theta\in\Theta}\prod_{r=1}^{h} l_i(s_r|\theta)\mu_{i,t+1}(\theta)\ \Big|\ \mathcal{B}_t\right] - a_{ii}(t)\sum_{\theta\in\Theta}\prod_{r=1}^{h} l_i(s_r|\theta)\mu_{i,t+1}(\theta)\\
&= a_{ii}(t)\mathbb{E}^*\left[m_{i,t+1}^{(h)}(s_1,\dots,s_r)|\mathcal{B}_t\right] - a_{ii}(t)m_{i,t+1}^{(h)}(s_1,\dots,s_r)\\
&= \mathbb{E}^*\left[a_{ii}(t)\left(m_{i,t+1}^{(h)}(s_1,\dots,s_r) - \prod_{r=1}^{h} l_i(s_r|\theta)\right)\ \Big|\ \mathcal{B}_t\right]\\
&\quad - a_{ii}(t)\left(m_{i,t+1}^{(h)}(s_1,\dots,s_r) - \prod_{r=1}^{h} l_i(s_r|\theta)\right) \xrightarrow{t\to\infty} 0
\end{aligned} \tag{4.8}$$

where the last step follows from our inductive hypothesis and the Dominated Convergence Theorem for conditional expectations. Combining (4.7) and (4.8) now yields:

$$\sum_{\theta\in\Theta}\left(\prod_{r=1}^{h} l_i(s_r|\theta)\right) a_{ii}^2(t)\left[\frac{l_i(\omega_{i,t+1}|\theta)}{m_{i,t}(\omega_{i,t+1})} - 1\right]\mu_{i,t}(\theta) \to 0$$

a.s. as $t \to \infty$, which implies:

$$\frac{a_{ii}^2(t)}{m_{i,t}(\omega_{i,t+1})} m_{i,t}^{(h+1)}(\omega_{i,t+1}, s_1, \dots, s_h) - a_{ii}^2(t) m_{i,t}^{(h)}(s_1,\dots,s_h) \to 0 \quad \textit{a.s. } \text{as} \quad t\to\infty.$$

By the inductive hypothesis and the fact that $|a_{ii}(t)|$ and $|m_{i,t}(\omega_{i,t+1})|$ are bounded, the above means:

$$a_{ii}^2(t)m_{i,t}^{(h+1)}(\omega_{i,t+1},s_1,\ldots,s_h)-a_{ii}^2(t)m_{i,t}(\omega_{i,t+1})\prod_{r=1}^{h}l_i(s_r|\theta^*)\overset{t\to\infty}{\longrightarrow}0\quad \textit{a.s.}$$

Once again, the fact that $|m_{i,t}(\omega_{i,t+1})|$ is bounded along with Lemma 25 implies that the limit $a_{ii}(t)m_{i,t}(\omega_{i,t+1})-a_{ii}(t)l_i(\omega_{i,t+1}|\theta^*)\to 0$ holds *a.s.* and hence that

$$a_{ii}^2(t)m_{i,t}^{(h+1)}(\omega_{i,t+1},s_1,\ldots,s_h)-a_{ii}^2(t)l_i(\omega_{i,t+1}|\theta^*)\prod_{r=1}^{h}l_i(s_r|\theta^*)\overset{t\to\infty}{\longrightarrow}0\quad \textit{a.s.}$$

By the Dominated Convergence Theorem for Conditional Expectations, we have

$$a_{ii}^2(t)\mathbb{E}^*\left[m_{i,t}^{(h+1)}(\omega_{i,t+1},s_1,\ldots,s_h)-l_i(\omega_{i,t+1}|\theta^*)\prod_{r=1}^{h}l_i(s_r|\theta^*)|\mathcal{B}_t\right]\overset{t\to\infty}{\longrightarrow}0\quad \textit{a.s.},$$

which implies that

$$a_{ii}^2(t)\sum_{s_{h+1}\in S_i}l_i(s_{h+1}|\theta^*)\left(m_{i,t}^{(h+1)}(s_{h+1},s_1,\ldots,s_h)-\prod_{r=1}^{h+1}l_i(s_r|\theta^*)\right)\overset{t\to\infty}{\longrightarrow}0\quad \textit{a.s.}$$

Since $l_i(s_{h+1}|\theta^*)>0$ for all $s_{h+1}\in S_i$, it follows that

$$a_{ii}^2(t)\left(m_{i,t}^{(h+1)}(s_{h+1},s_1,\ldots,s_h)-\prod_{r=1}^{h+1}l_i(s_r|\theta^*)\right)\overset{t\to\infty}{\longrightarrow}0\quad \textit{a.s.}$$

for all $s_1,s_2,\ldots,s_{h+1}\in S_i$. We thus have

$$\begin{aligned}&\left[a_{ii}(t)\left(m_{i,t}^{(h+1)}(s_1,s_2,\ldots,s_{h+1})-\prod_{r=1}^{h+1}l_i\left(s_r\mid\theta^*\right)\right)\right]^2\\&=a_{ii}^2(t)\left(m_{i,t}^{(h+1)}(s_1,s_2,\ldots,s_{h+1})-\prod_{r=1}^{h+1}l_i\left(s_r\mid\theta^*\right)\right)\\&\cdot\left(m_{i,t}^{(h+1)}(s_1,s_2,\ldots,s_{h+1})-\prod_{r=1}^{h+1}l_i\left(s_r\mid\theta^*\right)\right),\end{aligned}$$

which decays to 0 almost surely as $t \to \infty$, because $\left| m_{i,t}^{(h+1)}(s_1, s_2, \ldots, s_{h+1}) - \prod_{r=1}^{h+1} l_i(s_r \mid \theta^*) \right|$ is bounded. This proves the lemma for $h+1$ and hence for all $h \in \mathbb{N}$. □

3) Asymptotic Behavior of the Agents' Beliefs: As it turns out, Lemma 21, which we used above to analyze the asymptotic behavior of $u(t)$, is a useful result based on the idea of absolute probability sequences. We prove this lemma below.

Lemma 21. *Let $\theta \in \Theta^*$. Then the following limits exist and are finite: $\mathbb{P}^*$-a.s: $\lim_{t\to\infty} \pi^T(t)\mu_t(\theta)$, $\lim_{t\to\infty} \pi^T(t+1)A(t)\mu_t(\theta)$ and $\lim_{t\to\infty} \pi^T(t)\log\mu_t(\theta^*)$. As a result, we can assert that $\mathbb{E}^*[\pi^T(t+2)A(t+1)\mu_{t+1}(\theta^*) \mid \mathcal{B}_t] - \pi^T(t+1)A(t)\mu_t(\theta^*)$ approaches 0 a.s. as $t \to \infty$.*

Proof. Let $\mathcal{B}_t' := \sigma(A(0), \ldots, A(t-1), \omega_1, \ldots, \omega_t)$ for all $t \in \mathbb{N}$. Taking the conditional expectation $\mathbb{E}[\cdot|\mathcal{B}_t']$ on both sides of (4.2) yields

$$
\begin{aligned}
&\mathbb{E}^*[\mu_{t+1}(\theta) \mid \mathcal{B}_t'] - \mathbb{E}^*[A(t) \mid \mathcal{B}_t']\mu_t(\theta) \\
&= \text{diag}\left(\ldots, \mathbb{E}^*\left[a_{ii}(t)\left(\frac{l_i(\omega_{i,t+1} \mid \theta)}{m_{i,t}(\omega_{i,t+1})} - 1\right)\middle|\mathcal{B}_t'\right], \ldots\right)\mu_t(\theta^*),
\end{aligned} \tag{4.9}
$$

where we used that $\mu_t(\theta)$ is measurable w.r.t. $\mathcal{B}_t'$. Now, observe that $\mathcal{B}_t' \subset \mathcal{B}_t$, which implies that

$$
\begin{aligned}
\mathbb{E}^*\left[a_{ii}(t)\left(\frac{l_i(\omega_{i,t+1} \mid \theta)}{m_{i,t}(\omega_{i,t+1})} - 1\right) \middle| \mathcal{B}_t'\right] &= \mathbb{E}^*\left[\mathbb{E}^*\left[a_{ii}(t)\left(\frac{l_i(\omega_{i,t+1} \mid \theta)}{m_{i,t}(\omega_{i,t+1})} - 1\right) \middle| \mathcal{B}_t\right] \middle| \mathcal{B}_t'\right] \\
&= \mathbb{E}^*\left[a_{ii}(t) \cdot \mathbb{E}^*\left[\frac{l_i(\omega_{i,t+1}|\theta)}{m_{i,t}(\omega_{i,t+1})} - 1 \middle| \mathcal{B}_t\right] \middle| \mathcal{B}_t'\right] \\
&\geq 0,
\end{aligned}
$$

where the inequality follows from the lower bound in Lemma 24. Hence, (4.9) implies that $\mathbb{E}^*[\mu_{t+1}(\theta) \mid \mathcal{B}_t'] \geq \mathbb{E}^*[A(t) \mid \mathcal{B}_t']\mu_t(\theta)$. Since $\mathbb{E}^*[A(t) \mid \mathcal{B}_t'] = \mathbb{E}^*[A(t)]$ by Assumptions V and VI, it follows that

$$\mathbb{E}^*[\mu_{t+1}(\theta) \mid \mathcal{B}'_t] \geq \mathbb{E}^*[A(t)]\mu_t(\theta) \tag{4.10}$$

a.s. for all $t \in \mathbb{N}$. Left-multiplying both the sides of (4.10) by $\pi^T(t+1)$ results in the following almost surely

$$\begin{aligned}\pi^T(t+1)\mathbb{E}^*[\mu_{t+1}(\theta)|\mathcal{B}'_t] &\geq \pi^T(t+1)\mathbb{E}^*[A(t)]\mu_t(\theta)\\ &= \pi^T(t)\mu_t(\theta),\end{aligned} \tag{4.11}$$

where the last step follows from the definition of absolute probability sequences (Definition 46). Since $\{\pi(t)\}_{t=0}^{\infty}$ is a deterministic sequence, it follows from (4.11) that

$$\mathbb{E}^*[\pi^T(t+1)\mu_{t+1}(\theta) \mid \mathcal{B}'_t] \geq \pi^T(t)\mu_t(\theta) \quad \textit{a.s.}$$

We have thus shown that $\{\pi^T(t)\mu_t(\theta)\}_{t=1}^{\infty}$ is a submartingale w.r.t. the filtration $\{\mathcal{B}'_t\}_{t=1}^{\infty}$. Since it is also a bounded non-negative sequence (because $\mathbf{0} \leq \pi(t), \mu_t(\theta) \leq \mathbf{1}$), it follows that $\{\pi^T(t)\mu_t(\theta)\}_{t=1}^{\infty}$ is a bounded non-negative submartingale. Hence, $\lim_{t\to\infty} \pi^T(t)\mu_t(\theta)$ exists and is finite $\mathbb{P}^*$-*a.s.*

The almost-sure existence of $\lim_{t\to\infty} \pi^T(t+1)A(t)\mu_t(\theta)$ and $\lim_{t\to\infty} \pi^T(t)\log\mu_t(\theta^*)$ can be proved using similar submartingale arguments, as we show below.

In the case of $\lim_{t\to\infty} \pi^T(t+1)A(t)\mu_t(\theta)$, we derive an inequality similar to (4.10): we take conditional expectations on both sides of (4.2) and then use the lower bound in Lemma 24 to establish that

$$\mathbb{E}^*[\mu_{t+1}(\theta) \mid \mathcal{B}_t] \geq A(t)\mu_t(\theta) \quad \textit{a.s.} \tag{4.12}$$

Next, we observe that

$$\begin{aligned}
\pi^T(t+1)A(t)\mu_t(\theta) &\overset{(a)}{\leq} \pi^T(t+1)\mathbb{E}^*[\mu_{t+1}(\theta)|\mathcal{B}_t] \\
&= \pi^T(t+2)\mathbb{E}^*[A(t+1)]\mathbb{E}^*[\mu_{t+1}(\theta)|\mathcal{B}_t] \\
&\overset{(b)}{=} \pi^T(t+2)\mathbb{E}^*[A(t+1)|\mathcal{B}_t]\mathbb{E}^*[\mu_{t+1}(\theta)|\mathcal{B}_t] \\
&\overset{(c)}{=} \pi^T(t+2)\mathbb{E}^*[A(t+1)\mu_{t+1}(\theta)|\mathcal{B}_t] \\
&= \mathbb{E}^*[\pi^T(t+2)A(t+1)\mu_{t+1}(\theta)|\mathcal{B}_t] \quad \textit{a.s.},
\end{aligned} \tag{4.13}$$

where (a) follows from (4.12), and (b) and (c) each follow from Assumptions V and VI. Thus, $\{\pi^T(t+1)A(t)\mu_t(\theta)\}_{t=1}^{\infty}$ is a submartingale. It is also a bounded sequence. Hence, $\lim_{t\to\infty}\pi^T(t+1)A(t)\mu_t(\theta)$ exists and is finite *a.s.* Next, we use an argument similar to the proof of Lemma 2 in [105]: by taking the entry-wise logarithm of both sides of (4.1), using the concavity of the $\log(\cdot)$ function and then by using Jensen's inequality, we arrive at:

$$\log\mu_{i,t+1}(\theta^*) \geq a_{ii}(t)\log\mu_{i,t}(\theta^*) + a_{ii}(t)\log\left(\frac{l_i(\omega_{i,t+1}|\theta^*)}{m_{i,t}(\omega_{i,t+1})}\right) + \sum_{j\in\mathcal{N}_i(t)} a_{ij}(t)\log\mu_{j,t}(\theta^*). \tag{4.14}$$

Note that by Lemma 23 and Assumptions II and III, we have the following almost surely for all $i\in[n]$:

$$\mu_{i,T}(\theta^*) \geq (A(T:0))_{i1}(l_0/n)^T n\mu_{1,0}(\theta^*) > 0.$$

Therefore, (4.14) is well defined for all $t\geq T$ and $i\in[n]$. Next, for each $i\in[n]$, we have:

$$\mathbb{E}^*\left[\log\frac{l_i(\omega_{i,t+1}|\theta^*)}{m_{i,t}(\omega_{i,t+1})} \,\Big|\, \mathcal{B}'_t\right] = \sum_{s\in S_i} l_i(s|\theta^*)\log\left(\frac{l_i(s|\theta^*)}{m_{i,t}(s)}\right) = D(l_i(\cdot|\theta^*)\,\|\,m_{i,t}(\cdot)) \geq 0, \tag{4.15}$$

where $D(p\,\|\,q)$ denotes the relative entropy between two probability distributions p and q, and is always non-negative [134]. Taking conditional expectations on both the sides of (4.14) and then using (4.15) yields:

$$\mathbb{E}^*[\log \mu_{i,t+1}(\theta^*) \mid \mathcal{B}'_t] \geq \mathbb{E}^*[a_{ii}(t)] \log \mu_{i,t}(\theta^*) + \sum_{j \in \mathcal{N}_i(t)} \mathbb{E}^*[a_{ij}(t)] \log \mu_{j,t}(\theta^*) \tag{4.16}$$

a.s. for all $i \in [n]$ and t sufficiently large, which can also be expressed as $\mathbb{E}^*[\log \mu_{t+1}(\theta^*) \mid \mathcal{B}'_t] \geq \mathbb{E}^*[A(t)] \log \mu_t(\theta^*)$ *a.s.* Therefore, $\mathbb{E}^*[A(t)] \log \mu_t(\theta^*) \leq \mathbb{E}^*[\log \mu_{t+1}(\theta^*) \mid \mathcal{B}'_t]$ *a.s.*, and we have:

$$\begin{aligned}\pi^T(t) \log \mu_t(\theta^*) &= \pi^T(t+1)\mathbb{E}^*[A(t)] \log \mu_t(\theta^*) \\ &\leq \pi^T(t+1)\mathbb{E}^*[\log \mu_{t+1}(\theta^*) \mid \mathcal{B}'_t] \\ &= \mathbb{E}^*[\pi^T(t+1) \log \mu_{t+1}(\theta^*) \mid \mathcal{B}'_t] \end{aligned} \tag{4.17}$$

a.s. Thus, $\{\pi^T(t) \log \mu_t(\theta^*)\}_{t=0}^{\infty}$ is a submartingale. Now, recall that the following holds almost surely:

$$\mu_{i,T}(\theta^*) \geq (A(T:0))_{i1} \left(\frac{l_0}{n}\right)^T n\mu_{1,0}(\theta^*) > 0,$$

which, along with (4.17), implies that $\{\pi^T(t) \log \mu_t(\theta^*)\}_{t=T}^{\infty}$ is an integrable process. Since $\pi^T(t) \log \mu_t(\theta^*) < 0$ *a.s.*, it follows that the submartingale is also $L^1(\mathbb{P}^*)$-bounded. Hence, $\lim_{t\to\infty} \pi^T(t) \log \mu_t(\theta^*)$ exists and is finite almost surely.

Having shown that $\lim_{t\to\infty} \pi^T(t+1)A(t)\mu_t(\theta^*)$ exists *a.s.*, we use the Dominated Convergence Theorem for Conditional Expectations (Theorem 5.5.9 in [135]) to prove the last assertion of the lemma. We do this as follows: we note that we almost surely have $\lim_{t\to\infty} \left(\pi^T(t+2)A(t+1)\mu_{t+1}(\theta^*) - \pi^T(t+1)A(t)\mu_t(\theta^*)\right) = 0$. Therefore,

$$\begin{aligned}&\mathbb{E}^*[\pi^T(t+2)A(t+1)\mu_{t+1}(\theta^*) \mid \mathcal{B}_t] - \pi^T(t+1)A(t)\mu_t(\theta^*) \\ &= \mathbb{E}^*[\pi^T(t+2)A(t+1)\mu_{t+1}(\theta^*) - \pi^T(t+1)A(t)\mu_t(\theta^*) \mid \mathcal{B}_t] \to 0 \quad \textit{a.s.} \text{ as } \quad t \to \infty,\end{aligned}$$

where the second step follows from the Dominated Convergence Theorem for Conditional Expectations. □

We now use the above observations to prove Theorems 7 and 6.

Proof of Theorem 7

We prove each assertion of the theorem one by one.

Proof of (i): If $\{A(t)\}_{t=0}^{\infty}$ has the strong feedback property, then by Lemma 20, for all $s \in S_i$, $h \in \mathbb{N}$ and $i \in [n]$,

$$m_{i,t}^{(h)}(s_1, s_2, \ldots, s_h) - \prod_{r=1}^{h} l_i(s_r|\theta^*) \overset{t\to\infty}{\longrightarrow} 0 \quad \textit{a.s.}$$

which proves (i).

So, let us now ignore the strong feedback property and suppose that Assumption I holds. Let D_k denote the event that $[t_{2k-1}, t_{2k}]$ is a γ-epoch. Since $\{A(t)\}_{t=0}^{\infty}$ are independent, and since $\sum_{k=1}^{\infty} \Pr(D_k) = \infty$, we know from the Second Borel-Cantelli Lemma that $\Pr(D_k$ infinitely often$) = 1$ *a.s.* In other words, infinitely many γ-epochs occur *a.s.* So, for each $k \in \mathbb{N}$, suppose the k-th γ-epoch is the random time interval $[T_{2k-1}, T_{2k}]$. Then by the definition of γ-epoch, for each $k \in \mathbb{N}$ and $i \in [n]$, there almost surely exist $r_{i,k} \in [n]$, an observationally self-sufficient set, $\{\sigma_{i,k}(1), \ldots, \sigma_{i,k}(r_{i,k})\} \subset [n]$, and times $\{\tau_{i,k}(1), \ldots, \tau_{i,k}(r_{i,k})\} \subset \{T_{2k-1}, \ldots, T_{2k}\}$ such that

$$\min\left(a_{\sigma_{i,k}(q)\sigma_{i,k}(q)}(\tau_{i,k}(q)), (A(\tau_{i,k}(q) : T_{2k-1}))_{\sigma_{i,k}(q)\,i}\right) \geq \gamma$$

a.s. for all $q \in [r_i(k)]$. Since n is finite, there exist constants $r_1, r_2, \ldots, r_n \in [n]$ and a constant set of tuples $\{(\sigma_i(1), \ldots, \sigma_i(r_i))\}_{i\in[n]}$ such that $r_{i,k} = r_i$ and $(\sigma_{i,k}(1), \ldots, \sigma_{i,k}(r_{i,k})) = (\sigma_i(1), \ldots, \sigma_i(r_i))$ hold for all $i \in [n]$ and infinitely many $k \in \mathbb{N}$. Thus, we may assume that the same equalities hold for all $i \in [n]$ and all $k \in \mathbb{N}$ (by passing to an appropriate subsequence of

$\{T_k\}_{k=1}^{\infty}$, if necessary). Hence, by Lemma 20 and the fact that $a_{\sigma_i(q)\,\sigma_i(q)} \geq \gamma$, we have

$$m^{(h)}_{\sigma_i(q),\,\tau_{i,k}(q)}(s_1, \dots, s_h) \to \prod_{p=1}^{h} l_{\sigma_i(q)}(s_p|\theta^*)$$

a.s. for all $s \in [r]$ as $k \to \infty$, which means that the forecasts of each agent in $\{\sigma_i(q) : q \in [r_i]\}$ are asymptotically accurate along a sequence of times. Now, making accurate forecasts is possible only if agent $\sigma_i(q)$ rules out every state that induces on $S_{\sigma_i(q)}$ (the agent's signal space) a conditional probability distribution other than $l_{\sigma_i(q)}(\cdot|\theta^*)$. Such states are contained in $\Theta \setminus \Theta^*_{\sigma_i(q)}$. Thus, for every state $\theta \notin \Theta^*_{\sigma_i(q)}$, we have $\mu_{\sigma_i(q),\,\tau_{i,k}(q)}(\theta) \to 0$ *a.s.* as $k \to \infty$ (alternatively, we may repeat the arguments used in the proof of Proposition 3 of [105] to prove that $\mu_{\sigma_i(q),\,\tau_{i,k}(q)}(\theta) \to 0$ *a.s.* as $k \to \infty$).

On the other hand, since the influence of agent i on agent $\sigma_i(q)$ over the time interval $[T_{2k-1}, \tau_{i,k}(q)]$ exceeds γ, it follows from Lemma 23 that $\mu_{\sigma_i(q),\,\tau_{i,k}(q)}(\theta)$ is lower bounded by a multiple of $\mu_{i,T_{2k-1}}(\theta)$. To elaborate, Lemma 23 implies that for all $\theta \in \Theta \setminus \Theta^*_{\sigma_i(q)}$:

$$\begin{aligned}\mu_{i,T_{2k-1}}(\theta) &\leq \frac{\mu_{\sigma_i(q),\tau_{i,k}(q)}(\theta)}{(A(\tau_{i,k}(q) : T_{2k-1}))_{\sigma_i(q)i}} \cdot \left(\frac{n}{l_0}\right)^{\tau_{i,k}(q)-T_{2k-1}} \\ &\leq \frac{\mu_{\sigma_i(q),\tau_{i,k}(q)}(\theta)}{\gamma} \left(\frac{n}{l_0}\right)^{B}.\end{aligned}$$

Considering the limit $\mu_{\sigma_i(q),\,\tau_{i,k}(q)}(\theta) \to 0$, this is possible only if $\lim_{k\to\infty} \mu_{i,T_{2k-1}}(\theta) = 0$ *a.s.* for all $\theta \in \Theta \setminus \Theta^*_{\sigma_i(q)}$ and $q \in [r_i]$, i.e., $\lim_{k\to\infty} \mu_{i,T_{2k-1}}(\theta) = 0$ *a.s.* for all $\theta \in \cup_{q\in[r_i]} \left(\Theta \setminus \Theta^*_{\sigma_i(q)}\right)$. Since $\{\sigma_i(q) : q \in [r_i]\}$ is an observationally self-sufficient set, we are able to deduce that $\cup_{q\in[r_i]} \left(\Theta \setminus \Theta^*_{\sigma_i(q)}\right) = \Theta \setminus \Theta^*$ and hence that $\lim_{k\to\infty} \mu_{i,T_{2k-1}}(\theta) = 0$ *a.s.* for all $\theta \in \Theta \setminus \Theta^*$. Since $i \in [n]$ is arbitrary, this further implies that $\lim_{k\to\infty} \mu_{T_{2k-1}}(\theta) = \mathbf{0}$ for all $\theta \notin \Theta^*$. Hence, $\lim_{k\to\infty} \sum_{\theta\in\Theta^*} \mu_{T_{2k-1}}(\theta) = \mathbf{1}$ *a.s.*

To convert the above subsequence limit to a limit of the sequence $\{\sum_{\theta\in\Theta^*} \mu_t(\theta)\}_{t=0}^{\infty}$, we first show the existence of $\lim_{t\to\infty} \pi^T(t) \sum_{\theta\in\Theta^*} \mu_t(\theta)$ and use it to prove that

$$\lim_{t\to\infty} \sum_{\theta\in\Theta^*} \mu_t(\theta) = \mathbf{1} \quad \textit{a.s.} \tag{4.18}$$

This is done as follows. First, we note that $\lim_{t\to\infty} \pi^T(t) \sum_{\theta\in\Theta^*} \mu_t(\theta)$ exists *a.s.* because

$$\lim_{t\to\infty} \pi^T(t) \sum_{\theta\in\Theta^*} \mu_t(\theta) = \sum_{\theta\in\Theta^*} \lim_{t\to\infty} \pi^T(t) \mu_t(\theta),$$

which is a sum of limits that exist *a.s.* by virtue of Lemma 21. On the other hand, since $\lim_{k\to\infty} \sum_{\theta\in\Theta^*} \mu_{T_{2k-1}}(\theta) = \mathbf{1}$ *a.s.*, we have

$$\lim_{k\to\infty} \pi^T(T_{2k-1}) \sum_{\theta\in\Theta^*} \mu_{T_{2k-1}}(\theta) = \lim_{k\to\infty} \pi^T(T_{2k-1})\mathbf{1} = 1$$

a.s. because $\{\pi(t)\}_{t=1}^{\infty}$ are stochastic vectors. Hence, $\lim_{t\to\infty} \pi^T(t) \sum_{\theta\in\Theta^*} \mu_t(\theta) = 1$ *a.s.*, because the limit of a sequence is equal to the limit of each of its subsequences whenever the former exists.

We now prove that $\liminf_{t\to\infty} \sum_{\theta\in\Theta^*} \mu_t(\theta) = \mathbf{1}$ *a.s.* Suppose this is false, i.e., suppose there exists an $i \in [n]$ such that $\liminf_{t\to\infty} \sum_{\theta\in\Theta^*} \mu_{i,t}(\theta) < 1$. Then there exist $\varepsilon > 0$ and a sequence, $\{\varphi_k\}_{k=1}^{\infty} \subset \mathbb{N}$ such that $\sum_{\theta\in\Theta^*} \mu_{i,\varphi_k}(\theta) \leq 1-\varepsilon$ for all $k \in \mathbb{N}$. Since there also exists a $p^* > 0$ such that $\pi(t) \geq p^*\mathbf{1}$ *a.s.* for all $t \in \mathbb{N}$, we have for all $k \in \mathbb{N}$:

$$\begin{aligned}
\pi^T(\varphi_k) \sum_{\theta\in\Theta^*} \mu_{\varphi_k}(\theta^*) &\leq \pi_i(\varphi_k)(1-\varepsilon) + \sum_{j\in[n]\setminus\{i\}} \pi_j(\varphi_k) \cdot 1 \\
&= \sum_{j=1}^{n} \pi_j(\varphi_k) - \varepsilon \pi_i(\varphi_k) \\
&\leq 1 - \varepsilon p^* \\
&< 1,
\end{aligned}$$

which contradicts the conclusion of the previous paragraph. Hence, $\liminf_{t\to\infty}\sum_{\theta\in\Theta^*}\mu_t(\theta) = \mathbf{1}$ indeed holds *a.s.*, which means that

$$\lim_{t\to\infty}\sum_{\theta\in\Theta^*}\mu_t(\theta) = \mathbf{1} \quad \textit{a.s.} \tag{4.19}$$

In view of the definition of Θ^* (see Section 5.2), (4.18) means that the beliefs of agent i asymptotically concentrate only on those states that generate the i.i.d. signals $\{\omega_{i,t}\}_{t=1}^{\infty}$ according to the true probability distribution $l_i(\cdot|\theta^*)$. That is, agent i asymptotically rules out all those states that generate signals according to distributions that differ from the one associated with the true state. Since agent i knows that each of the remaining states generates $\{\omega_{i,t}\}_{t=1}^{\infty}$ according to $l_i(\cdot|\theta^*)$, this implies that agent i estimates the true distributions of her forthcoming signals with arbitrary accuracy as $t \to \infty$, i.e., her beliefs weakly merge to the truth. This claim is proved formally below.

For any $i \in [n]$ and $k \in \mathbb{N}$, we have

$$\begin{aligned}
m_{i,t}^{(k)}(s_1,\dots,s_k) &= \sum_{\theta\in\Theta}\prod_{r=1}^{k} l_i(s_r|\theta)\mu_{i,t}(\theta) \\
&\overset{(a)}{=} \sum_{\theta\in\Theta\setminus\Theta^*}\prod_{r=1}^{k} l_i(s_r|\theta)\mu_{i,t}(\theta) + \left(\prod_{r=1}^{k} l_i(s_r|\theta^*)\right)\sum_{\theta\in\Theta^*}\mu_{i,t}(\theta) \\
&\overset{t\to\infty}{\longrightarrow} \sum_{\theta\in\Theta\setminus\Theta^*}\prod_{r=1}^{k} l_i(s_r|\theta)\cdot 0 + \left(\prod_{r=1}^{k} l_i(s_r|\theta^*)\right)\cdot 1 \\
&= \prod_{r=1}^{k} l_i(s_r|\theta^*) \quad \mathbb{P}^*\text{-}\textit{a.s.}
\end{aligned}$$

where (a) follows from Definition 40 and the definition of Θ^*. Thus, every agent's beliefs weakly merge to the truth $\mathbb{P}^*$-*a.s*.

Proof of (ii): Next, we note that if Assumption I holds and θ^* is identifiable, then:

$$\lim_{t\to\infty}\mu_t(\theta^*) = \lim_{t\to\infty}\sum_{\theta\in\{\theta^*\}}\mu_t(\theta) = \lim_{t\to\infty}\sum_{\theta\in\Theta^*}\mu_t(\theta) = \mathbf{1}$$

a.s., where the last step follows from (4.18). This proves (ii).

Proof of Theorem 6

To begin, suppose $\{A(t)\}_{t=0}^{\infty}$ is a deterministic uniformly strongly connected chain, and let B denote the constant satisfying Condition 3 in Definition 48. Then one can easily verify that Assumptions I and III hold (see the proof of Lemma 22 for a detailed verification). Moreover, $\{A(t)\}_{t=0}^{\infty}\in\mathcal{P}^*$ by Lemma 5.8 of [62]. Thus, Assumptions I - VI hold (the last two of them hold trivially), implying that Equation (4.18) holds, which proves that $c_\theta = 0$ for all $\theta\in\Theta\setminus\Theta^*$. So, we restrict our subsequent analysis to the states belonging to Θ^*, and we let θ denote a generic state in Θ^*.

Since we aim to show that all the agents converge to a consensus, we first show that their beliefs attain *synchronization* as time goes to ∞ (i.e., $\lim_{t\to\infty}(\mu_{i,t}(\theta)-\mu_{j,t}(\theta)) = 0$ *a.s.* for all $i,j\in[n]$), and then show that the agents' beliefs converge to a steady state almost surely as time goes to ∞.

Synchronization

To achieve synchronization asymptotically in time, the quantity $|\max_{i\in[n]}\mu_{i,t}(\theta) - \min_{j\in[n]}\mu_{j,t}(\theta)|$, which is the difference between the network's maximum and minimum beliefs in the state θ, must approach 0 as $t\to\infty$. Since this requirement is similar to asymptotic stability, and since the update rule (4.2) involves only one non-linear term, we are motivated to identify a Lyapunov function associated with linear dynamics on uniformly strongly connected networks. One such function is the *quadratic comparison function* $V_\pi:\mathbb{R}^n\times\mathbb{N}_0\to\mathbb{R}$, defined as follows in [62]:

$$V_\pi(x,k) := \sum_{i=1}^{n} \pi_i(k)(x_i - \pi^T(k)x)^2.$$

Remarkably, the function $V_\pi(\cdot,k)$ is comparable in magnitude with the difference function $d(x) := |\max_{i\in[n]} x_i - \min_{j\in[n]} x_j|$. To be specific, Lemma 29 shows that for each $k \in \mathbb{N}_0$,

$$(p^*/2)^{\frac{1}{2}} d(x) \leq \sqrt{V_\pi(x,k)} \leq d(x). \tag{4.20}$$

As a result, just like V_π, the difference function $d(\cdot)$ behaves like a Lyapunov function for linear dynamics on a time-varying network described by $\{A(t)\}_{t=0}^{\infty}$. To elaborate, V_π being a Lyapunov function means that, for the linear dynamics $x(k+1) = A(k)x(k)$ with $x(0) \in \mathbb{R}^n$ as the initial condition, there exists a constant $\kappa \in (0,1)$ such that

$$V_\pi(x((q+1)B),(q+1)B) \leq (1-\kappa)^q V_\pi(x(0),0)$$

for all $q \in \mathbb{N}_0$ (see Equation (5.18) in [62]). This inequality can be combined with (4.20) to obtain a similar inequality for the function $d(\cdot)$ as follows: in the light of (4.20), the inequality above implies the following for all $q \in \mathbb{N}_0$:

$$d(x(q+1)B) \leq \sqrt{\frac{2(1-\kappa)^q}{p^*}} d(x(0)).$$

Now, note that there exists a $q_0 \in \mathbb{N}_0$ that is large enough for $\sqrt{\frac{2(1-\kappa)^{q_0}}{p^*}} < 1$ to hold. We then have

$$d(x(T_0)) \leq \alpha d(x(0)),$$

where $T_0 := (q_0+1)B$ and $\alpha := \sqrt{\frac{2(1-\kappa)^{q_0}}{p^*}} < 1$. More explicitly, we have $d(A(T_0:0)x_0) \leq \alpha d(x_0)$ for all initial conditions $x_0 \in \mathbb{R}^n$. Now, given any $r \in \mathbb{N}$, by the definition of uniform strong connectivity the truncated chain $\{A(t)\}_{t=rB}^{\infty}$ is also B-strongly connected. Therefore, the above inequality can be generalized to:

$$d(A(T_0+rB:rB)x_0) \leq \alpha d(x_0). \tag{4.21}$$

By using some algebra involving the row-stochasticity of the chain $\{A(t)\}_{t=0}^{\infty}$, Lemma 30 transforms (4.21) into the following, where $t_1, t_2 \in \mathbb{N}_0$ and $t_1 < t_2$:

$$d(A(t_2:t_1)x_0) \leq \alpha^{\frac{t_2-t_1}{T_0}-2} d(x_0). \tag{4.22}$$

For the linear dynamics $x(k+1) = A(k)x(k)$, (4.22) implies that $d(x(k)) \to 0$ as $k \to \infty$. Since we need a similar result for the non-linear dynamics (4.2), we first recast (4.2) into an equation involving backward matrix products (such as $A(t_2:t_1)$), and then use (4.22) to obtain the desired limit. The first step yields the following, which is straightforward to prove by induction [124]

$$\mu_{t+1}(\theta) = A(t+1:0)\mu_0(\theta) + \sum_{k=0}^{t} A(t+1:k+1)\rho_k(\theta), \tag{4.23}$$

where $\rho_k(\theta)$ is the vector with entries:

$$\rho_{i,k}(\theta) := a_{ii}(k)\left(\frac{l_i(\omega_{i,k+1}|\theta)}{m_{i,k}(\omega_{i,k+1})} - 1\right)\mu_{i,k}(\theta).$$

We now apply $d(\cdot)$ to both sides of (4.23) so that we can make effective use of (4.22). We do this below.

$$\begin{aligned} d(\mu_{t+1}(\theta)) &\overset{(a)}{\leq} d(A(t+1:0)\mu_0(\theta)) + \sum_{k=0}^{t} d(A(t+1:k+1)\rho_k(\theta)) \\ &\overset{(b)}{\leq} \alpha^{\frac{t+1}{T_0}-2} d(\mu_0(\theta)) + \sum_{k=0}^{t} \alpha^{\frac{t-k}{T_0}-2} d(\rho_k(\theta)). \end{aligned} \tag{4.24}$$

In the above chain of inequalities, (b) follows from (4.22), and (a) follows from the fact that

$d(x+y) \leq d(x)+d(y)$ for all $x,y \in \mathbb{R}^n$.

We will now show that $\lim_{t\to\infty} d(\mu_{t+1}(\theta)) = 0$ *a.s.* Observe that the first term on the right hand side of (4.24) vanishes as $t \to \infty$. To show that the second term also vanishes, we use some arguments of [124] below.

Note that Theorem 7 (i) implies that for all $i \in [n]$ and $\theta \in \Theta^*$:

$$l_i(\omega_{i,t+1}|\theta) - m_{i,t}(\omega_{i,t}) = l_i(\omega_{i,t+1}|\theta^*) - m_{i,t}(\omega_{i,t}) \to 0$$

a.s. as $t \to \infty$. It now follows from the definition of $\rho_k(\theta)$ that $\lim_{k\to\infty} \rho_k(\theta) = \mathbf{0}$ *a.s.* for all $\theta \in \Theta^*$. Thus, $\lim_{k\to\infty} d(\rho_k(\theta)) = 0$ *a.s.* for all $\theta \in \Theta^*$.

Next, note that $\sum_{k=0}^{t} \alpha^{\frac{t-k}{T_0}-2} \leq \alpha^{-2} \cdot \frac{1}{1-\alpha^{1/T_0}} < \infty$. Since $\lim_{k\to\infty} d(\rho_k(\theta)) = 0$ *a.s.*, we have $\lim_{t\to\infty} \sum_{k=0}^{t} \alpha^{\frac{t-k}{T_0}-2} d(\rho_k(\theta)) = 0$ *a.s.* by Toeplitz Lemma. Thus, (4.24) now implies that $\lim_{t\to\infty} d(\mu_{t+1}(\theta)) = 0$ *a.s.* for all $\theta \in \Theta^*$, i.e., synchronization is attained as $t \to \infty$.

Convergence to a Steady State

We now show that $\lim_{t\to\infty} \mu_{i,t}(\theta)$ exists *a.s.* for each $i \in [n]$ because $\lim_{t\to\infty} \pi^T(t)\mu_t(\theta)$ exists *a.s.* by Lemma 21. Formally, we have the following almost surely:

$$\begin{aligned}
\lim_{t\to\infty} \mu_{i,t}(\theta) &= \lim_{t\to\infty} \left(\mu_{i,t}(\theta) \sum_{j=1}^{n} \pi_j(t) \right) \\
&= \lim_{t\to\infty} \sum_{j=1}^{n} \pi_j(t) \left(\mu_{j,t}(\theta) + (\mu_{i,t}(\theta) - \mu_{j,t}(\theta)) \right) \\
&\overset{(a)}{=} \lim_{t\to\infty} \sum_{j=1}^{n} \pi_j(t) \mu_{j,t}(\theta) \\
&= \lim_{t\to\infty} \pi^T(t)\mu_t(\theta),
\end{aligned}$$

which exists almost surely. Here (a) holds because $\lim_{t\to\infty}(\mu_{i,t}(\theta) - \mu_{j,t}(\theta)) = 0$ *a.s.* as a result of asymptotic synchronization.

We have thus shown that $\lim_{t\to\infty} \mu_t(\theta)$ exists *a.s.* for all $\theta \in \Theta^*$, and we have also shown that $\lim_{t\to\infty} |\mu_{i,t}(\theta) - \mu_{j,t}(\theta)| = 0$ *a.s.* for all $i, j \in [n]$ and $\theta \in \Theta^*$. It follows that for each $\theta \in \Theta^*$, $\lim_{t\to\infty} \mu_t(\theta) = C_\theta \mathbf{1}$ *a.s.* for some scalar random variable $C_\theta = C_\theta(A(0), \omega_1, A(1), \omega_2, \ldots)$. This concludes the proof of the theorem.

4.5 Applications

We now establish a few useful implications of Theorem 7, some of which are either known results or their extensions.

4.5.1 Learning in the Presence of Link Failures

In the context of learning on random graphs, the following question arises naturally: is it possible for a network of agents to learn the true state of the world when the underlying influence graph is affected by random communication link failures? For simplicity, let us assume that there exists a constant stochastic matrix A such that $a_{ij}(t)$, which denotes the degree of influence of agent j on agent i at time t, equals 0 if the link (j, i) has failed and A_{ij} otherwise. Then, if the link failures are independent across time, the following result answers the question raised.

Corollary 5. *Let $([n], E)$ be a strongly connected directed graph whose weighted adjacency matrix $A = (A_{ij})$ satisfies $A_{ii} > 0$ for all $i \in [n]$. Consider a system of n agents satisfying the following criteria:*

1. *Assumption II holds.*
2. *The influence graph at any time $t \in \mathbb{N}$ is given by $G(t) = ([n], E - F(t))$, where $F(t) \subset E$ denotes the set of failed links at time t, and $\{F(t)\}_{t=0}^{\infty}$ are independently distributed random sets.*
3. *The sequences $\{\omega_t\}_{t=1}^{\infty}$ and $\{F(t)\}_{t=0}^{\infty}$ are independent.*
4. *At any time-step, any link $e \in E$ fails with a constant probability $\rho \in (0, 1)$. However, the failure of e may or may not be independent of the failure of other links.*

5. *The probability that $G(t)$ is connected at time t is at least $\sigma > 0$ for all $t \in \mathbb{N}_0$.*

Then, under the update rule (4.1)*, all the agents learn the truth asymptotically a.s.*

Proof. Since $\{F(t)\}_{t=0}^{\infty}$ are independent across time and also independent of the observation sequence, it follows that the chain $\{A(t)\}_{t=0}^{\infty}$ satisfies Assumptions V and VI.

Next, we observe that for any $t \in \mathbb{N}_0$, we have $\mathbb{E}^*[A(t)] = (1-\rho)A$ and hence, $\{\mathbb{E}^*[A(t)]\}$ is a static chain of irreducible matrices because A, being the weighted adjacency matrix of a strongly connected graph, is irreducible. Also, $\min_{i\in[n]} A_{ii} > 0$ implies that $\{\mathbb{E}^*[A(t)]\}$ has the strong feedback property. It now follows from Theorem 4.7 and Lemma 5.7 of [62] that $\{\mathbb{E}^*[A(t)]\}$ belongs to Class $\mathcal{P}^*$. As a result, Assumption IV holds.

We now prove that Assumption I holds. To this end, observe that $\{(G(nt), G(nt+1), G(nt+n-1))\}_{t=0}^{\infty}$ is a sequence of independent random tuples. Therefore, if we let L_r denote the event that all the graphs in the r^{th} tuple of the above sequence are strongly connected, then $\{L_r\}_{r=0}^{\infty}$ is a sequence of independent events. Note that $P(L_r) \geq \sigma^n$ and hence, $\sum_{r=0}^{\infty} P(L_r) = \infty$. Thus, by the Second Borel-Cantelli Lemma, infinitely many L_r occur *a.s.* Now, it can be verified that if L_r occurs, then at least one sub-interval of $[(r-1)n, rn]$ is a γ-epoch for some positive γ that does not depend on r. Thus, infinitely many γ-epochs occur *a.s.*

Finally, the preceding arguments also imply that there almost surely exists a time $T < \infty$ such that exactly 1 of the events $\{L_r\}_{r=0}^{\infty}$ has occurred until time T. With the help of the strong feedback property of $\{A(t)\}_{t=0}^{\infty}$ (which holds because $A_{ii} > 0$), it can be proven that $\log(A(T:0))_{i1} > -\infty$ *a.s.* for all $i \in [n]$. Thus, Assumption III holds.

We have shown that all of the Assumptions I - VI hold. Since θ^* is identifiable, it follows from Theorem 7 that all the agents learn the truth asymptotically *a.s.* □

4.5.2 Inertial Non-Bayesian Learning

In real-world social networks, it is possible that some individuals affected by psychological inertia cling to their prior beliefs in such a way that they do not incorporate their own observations in a fully Bayesian manner. This idea is closely related to the notion of prejudiced agents that motivated the popular Friedkin-Johnsen model in [136]. To describe the belief updates of such inertial individuals, we modify the update rule (4.1) by replacing the Bayesian update term $\text{BU}_{i,t+1}(\theta)$ with a convex combination of $\text{BU}_{i,t+1}(\theta)$ and the i^{th} agent's previous belief $\mu_{i,t}(\theta)$, i.e.,

$$\begin{aligned}\mu_{i,t+1}(\theta) &= a_{ii}(t)(\lambda_i(t)\mu_{i,t}(\theta) + (1-\lambda_i(t))\text{BU}_{i,t+1}(\theta)) \\ &\quad + \sum_{j\in\mathcal{N}_i(t)} a_{ij}(t)\mu_{j,t}(\theta), \end{aligned} \tag{4.25}$$

where $\lambda_i(t) \in [0,1]$ denotes the degree of inertia of agent i at time t. As it turns out, Theorem 7 implies that even if all the agents are inertial, they will still learn the truth asymptotically *a.s.* provided the inertias are all bounded away from 1.

Corollary 6. *Consider a network of n inertial agents whose beliefs evolve according to* (4.25). *Suppose that for each $i \in [n]$, the sequence $\{\lambda_i(t)\}_{t=0}^{\infty}$ is deterministic. Further, suppose $\lambda_{\max} := \sup_{t\in\mathbb{N}_0} \max_{i\in[n]} \lambda_i(t) < 1$ and that Assumptions II - VI hold. Then Assertions (i) and (ii) of Theorem 7 are true.*

Proof. In order to use Theorem 7 effectively, we first create a hypothetical copy of each of the n inertial agents and insert all the copies into the given inertial network in such a way that the augmented network (of $2n$ agents) has its belief evolution described by the original update rule (4.1). To this end, let $[2n]$ index the agents in the augmented network so that for each $i \in [n]$, the i^{th} real agent is still indexed by i whereas its copy is indexed by $i+n$. This means that for every $i \in [n]$, we let the beliefs, the signal structures and the observations of agent $i+n$ equal those of agent i at all times, i.e., let $\mu_{i+n,t}(\theta) := \mu_{i,t}(\theta)$, $S_{i+n} := S_i$, $l_{i+n}(\cdot|\theta) := l_i(\cdot|\theta)$ and

$\omega_{i+n,t} := \omega_{i,t}$ for all $\theta \in \Theta$ and all $t \in \mathbb{N}_0$. As a result, (4.25) can now be expressed as:

$$\mu_{i,t+1}(\theta)$$
$$= b_i(t)\mathrm{BU}_{i,t+1}(\theta) + w_i(t)\mu_{i+n,t}(\theta) + \sum_{j\in\mathcal{N}_i(t)} \frac{1}{2}a_{ij}(t)\mu_{j,t}(\theta) + \sum_{j\in\mathcal{N}_i(t)} \frac{1}{2}a_{ij}(t)\mu_{j+n,t}(\theta) \tag{4.26}$$

for all $i \in [n]$, where $b_i(t) := (1-\lambda_i(t))a_{ii}(t)$, and $w_i(t) := \lambda_i(t)a_{ii}(t)$ so that $a_{ii}(t) = b_i(t) + w_i(t)$. Now, let $b(t) \in \mathbb{R}^n$ and $w(t) \in \mathbb{R}^n$ be the vectors whose i^{th} entries are $b_i(t)$ and $w_i(t)$, respectively. Further, let $\hat{A}(t) \in \mathbb{R}^{n\times n}$ and $\tilde{A}(t) \in \mathbb{R}^{2n\times 2n}$ be the matrices defined by:

$$\hat{a}_{ij}(t) := (\hat{A}(t))_{ij} = \begin{cases} a_{ij}(t), & \text{if } i \neq j \\ 0, & \text{if } i = j \end{cases}$$

and

$$\tilde{A}(t) = \begin{pmatrix} \hat{A}(t)/2 + \mathrm{diag}(b(t)) & \hat{A}(t)/2 + \mathrm{diag}(w(t)) \\ \hat{A}(t)/2 + \mathrm{diag}(w(t)) & \hat{A}(t)/2 + \mathrm{diag}(b(t)) \end{pmatrix}.$$

Then, with the help of (4.26), one can verify that the evolution of beliefs in the augmented network is captured by:

$$\mu_{i,t+1}(\theta) = \tilde{a}_{ii}(t)\mathrm{BU}_{i,t+1}(\theta) + \sum_{j\in[2n]\setminus\{i\}} \tilde{a}_{ij}(t)\mu_{j,t}(\theta), \tag{4.27}$$

where $\tilde{a}_{ij}(t)$ is the (i,j)-th entry of $\tilde{A}(t)$.

We now show that the augmented network satisfies Assumptions II - VI with $\{\tilde{A}(t)\}_{t=0}^{\infty}$ being the associated sequence of weighted adjacency matrices.

It can be immediately seen that Assumption II holds for the augmented network because it holds for the original network.

Regarding Assumption III, we observe that $b_i(t) \geq (1-\lambda_{\max})a_{ii}(t)$ and also that $\tilde{a}_{ij}(t) \geq$

$\frac{1}{2}\hat{a}_{ij}(t) = \frac{1}{2}a_{ij}(t)$ for all distinct $i,j \in [n]$ and all $t \in \mathbb{N}_0$. Therefore,

$$\tilde{a}_{ij}(t) \geq \lambda_0 a_{ij}(t) \tag{4.28}$$

for all $i,j \in [n]$ and all $t \in \mathbb{N}_0$, where $\lambda_0 := \min\left\{1-\lambda_{\max}, \frac{1}{2}\right\}$. Note that $\lambda_0 > 0$ because $\lambda_{\max} < 1$. Since Assumption III holds for the original network, it follows that

$$(\tilde{A}(T:0))_{i1} \geq \lambda_0^T (A(T:0))_{i1} > 0 \quad \textit{a.s.}$$

for all $i \in [n]$. By using the fact that $\tilde{a}_{(n+i)\,(n+j)}(t) = \tilde{a}_{(n+i)\,j}(t) = \tilde{a}_{ij}(t)$ for all distinct $i,j \in [n]$, we can similarly show that $(\tilde{A}(T:0))_{(n+i)\,1} > 0$ *a.s.* for all $i \in [n]$.

As for Assumption IV, let $\{\pi(t)\}_{t=0}^{\infty}$ be an absolute probability process for $\{A(t)\}_{t=0}^{\infty}$ such that $\pi(t) \geq p^*\mathbf{1}$ for some scalar $p^* > 0$ (such a scalar exists because $\{A(t)\}_{t=0}^{\infty}$ satisfies Assumption IV). Now, let $\{\tilde{\pi}(t)\}_{t=0}^{\infty}$ be a sequence of vectors in $\mathbb{R}^{2n}$ defined by $\tilde{\pi}_{i+n}(t) = \tilde{\pi}_i(t) = \pi_i(t)/2$ for all $i \in [n]$ and all $t \in \mathbb{N}_0$. We then have $\tilde{\pi}(t) \geq \mathbf{0}$ and $\sum_{i=1}^{2n}\tilde{\pi}_i(t) = 1$. Moreover, for all $i \in [n]$:

$$\begin{aligned}
\left(\tilde{\pi}^T(t+1)\tilde{A}(t)\right)_i &= \sum_{j=1}^{2n} \tilde{a}_{ji}(t)\tilde{\pi}_j(t+1) \\
&= \sum_{j\in[n]\setminus\{i\}} \left(\tilde{a}_{ji}(t) + \tilde{a}_{n+j\,i}(t)\right)\frac{\pi_j(t)}{2} + \left(\tilde{a}_{ii}(t) + \tilde{a}_{n+i\,i}(t)\right)\frac{\pi_i(t+1)}{2} \\
&= \sum_{j\in[n]\setminus\{i\}} a_{ji}(t)\frac{\pi_j(t)}{2} + \left(b_i(t) + w_i(t)\right)\frac{\pi_j(t)}{2} \\
&= \frac{1}{2}\sum_{i=1}^{n} a_{ji}(t)\pi_j(t+1) \\
&= \frac{1}{2}\left(\pi^T(t+1)A(t)\right)_i,
\end{aligned}$$

and hence,

$$\mathbb{E}^*\left[\left(\tilde{\pi}^T(t+1)\tilde{A}(t)\right)_i \mid \mathcal{B}_t\right] = \frac{1}{2}\mathbb{E}^*\left[\left(\pi^T(t+1)A(t)\right)_i |\mathcal{B}_t\right] = \frac{1}{2}\pi_i(t) = \tilde{\pi}_i(t) \tag{4.29}$$

for all $i \in [n]$. We can similarly prove (4.29) for all $i \in \{n+1,\ldots,2n\}$. This shows that $\{\tilde{\pi}(t)\}_{t=0}^{\infty}$ is an absolute probability process for $\{\tilde{A}(t)\}_{t=0}^{\infty}$. Since $\tilde{\pi}^T(t) = \frac{1}{2}[\pi^T(t)\ \pi^T(t)]$ implies that $\tilde{\pi}(t) \geq \frac{p^*}{2}\mathbf{1}_{2n}$ for all $t \in \mathbb{N}_0$, it follows that $\{\tilde{A}(t)\}_{t=0}^{\infty} \in \mathcal{P}^*$, i.e., the augmented network satisfies Assumption IV.

Note that the augmented network also satisfies Assumptions V and VI because $\tilde{A}(t)$ is uniquely determined by $A(t)$ for every $t \in \mathbb{N}_0$ (under the assumption that $\{\lambda_i(t)\}_{t=0}^{\infty}$ is a deterministic sequence for each $i \in [n]$).

To complete the proof, we need to show that if $\{A(t)\}_{t=0}^{\infty}$ has feedback property (or satisfies Assumption I), then $\{\tilde{A}(t)\}_{t=0}^{\infty}$ also has feedback property (or satisfies Assumption I). Since the following holds for all $i \in [n]$:

$$\tilde{a}_{i+n\, i+n}(t) = \tilde{a}_{ii}(t) = (1-\lambda_i(t))a_{ii}(t) \geq (1-\lambda_{\max})a_{ii}(t), \tag{4.30}$$

it follows that $\{\tilde{A}(t)\}_{t=0}^{\infty}$ has feedback property if $\{\tilde{A}(t)\}_{t=0}^{\infty}$ has feedback property. Now, suppose the original chain, $\{A(t)\}_{t=0}^{\infty}$ satisfies Assumption I. Recall that $\tilde{a}_{(n+i)\,(n+j)}(t) = \tilde{a}_{(n+i)\,j}(t) = \tilde{a}_{ij}(t)$ for all distinct $i,j \in [n]$. In the light of this, (4.28) and (4.30) now imply that $\tilde{a}_{i+n\, j+n}(t) \geq \lambda_0 a_{ij}(t)$ for all $i,j \in [n]$ and all $t \in \mathbb{N}_0$. It follows that

$$\min\{(\tilde{A}(t_2:t_1))_{ij}, (\tilde{A}(t_2:t_1))_{n+i\, n+j}\} \geq \lambda_0^{t_2-t_1}(A(t_2:t_1))_{ij} \tag{4.31}$$

for all $i,j \in [n]$ and all $t_1,t_2 \in \mathbb{N}_0$ such that $t_1 \leq t_2$. Moreover, if $\mathcal{O}$ is an observationally self-sufficient set for the original network, then both $\mathcal{O}$ and $n+\mathcal{O}$ are observationally self-sufficient sets for the augmented network. Therefore, by (4.31), if $[t_1,t_2]$ is a γ-epoch of duration at most B for $\{A(t)\}_{t=0}^{\infty}$ then $[t_1,t_2]$ is a $\lambda_0^B\gamma$-epoch for $\{\tilde{A}(t)\}_{t=0}^{\infty}$. Assumption I thus holds for

$\{\tilde{A}(t)\}_{t=0}^{\infty}$.

An application of Theorem 7 to the augmented network now implies that the first two assertions of this theorem also hold for the original network. □

Remark 10. *Interestingly, Corollaries 5 and 6 imply that non-Bayesian learning (both inertial and non-inertial) occur almost surely on a sequence of independent Erdos-Renyi random graphs, provided the edge probabilities of these graphs are uniformly bounded away from 0 and 1 (i.e., if* $\rho(t)$ *is the edge probability of* $G(t)$*, then there should exist constants* $0 < \delta < \eta < 1$ *such that* $\delta \leq \rho(t) \leq \eta$ *for all* $t \in \mathbb{N}_0$*.) This is worth noting because a sequence of Erdos-Renyi networks is a.s. not uniformly strongly connected, which can be proved by using arguments similar to those used in Remarks 8 and 9.*

4.5.3 Learning via Diffusion and Adaptation

Let us extend our discussion to another variant of the original update rule (4.1). As per this variant, known as learning via *diffusion* and *adaptation* [110], every agent combines the Bayesian updates of her own beliefs with the most recent Bayesian updates of her neighbor's beliefs (rather than combining the Bayesian updates of her own beliefs with her neighbors' previous beliefs). As one might guess, this modification results in faster convergence to the truth in the case of static networks, as shown empirically in [110].

For a network of n agents, the time-varying analog of the update rule proposed in [110] can be stated as:

$$\mu_{i,t+1}(\theta) = \sum_{j=1}^{n} a_{ij}(t)\text{BU}_{j,t+1}(\theta) \tag{4.32}$$

for all $i \in [n]$, $t \in \mathbb{N}_0$ and $\theta \in \Theta$. On the basis of (4.32), we now generalize the theoretical results of [110] and establish that diffusion-adaptation almost surely leads to asymptotic learning even when the network is time-varying or random, provided it satisfies the assumptions stated earlier.

Corollary 7. *Consider a network* $\mathcal{H}$ *described by the rule* (4.32)*, and suppose that the sequence* $\{A(t)\}_{t=0}^{\infty}$ *and the agents' initial beliefs satisfy Assumptions II - VI. Then Assertions (i) and (ii)* *of Theorem 7 hold.*

Proof. Similar to the proof of Corollary 6, in order to use Theorem 7 appropriately, we construct a hypothetical network $\widetilde{\mathcal{H}}$ of $2n$ agents, and for each $i \in [n]$, we let the signal spaces and the associated conditional distributions of the i^{th} and the $(n+i)^{\text{th}}$ agents of $\widetilde{\mathcal{H}}$ be given by:

$$\tilde{S}_i = \tilde{S}_{n+i} = S_i,\ \tilde{l}_i(\cdot|\theta) = \tilde{l}_{n+i}(\cdot|\theta) = l_i(\cdot|\theta) \text{ for all } \theta \in \Theta, \tag{4.33}$$

respectively. Likewise, we let the prior beliefs of the agents of $\widetilde{\mathcal{H}}$ be given by $\tilde{\mu}_{i,0} = \tilde{\mu}_{n+i,0} = \mu_{i,0}$ for all $i \in [n]$. However, we let the observations of the hypothetical agents be given by $\tilde{\omega}_{i,2t} = \tilde{\omega}_{n+i,2t} := \omega_{i,t}$ and $\tilde{\omega}_{i,2t+1} = \tilde{\omega}_{n+i,2t+1} := \omega_{i,t}$ for all $t \in \mathbb{N}_0$. In addition, we let $W(t) := \text{diag}(a_{11}(t), \ldots, a_{nn}(t))$ and $\hat{A}(t) := A(t) - W(t)$ for all $t \in \mathbb{N}$ so that $\hat{a}_{ii}(t) = 0$ for all $i \in [n]$. Furthermore, let the update rule for the network $\widetilde{\mathcal{H}}$ be described by:

$$\tilde{\mu}_{i,t+1}(\theta) = \tilde{a}_{ii}(t)\widetilde{\text{BU}}_{i,t+1}(\theta) + \sum_{j \in [2n] \setminus \{i\}} \tilde{a}_{ij}(t)\tilde{\mu}_{j,t}(\theta), \tag{4.34}$$

where $\tilde{a}_{ij}(t)$ is the $(i,j)^{\text{th}}$ entry of the matrix $\tilde{A}(t)$ defined by

$$\tilde{A}(2t) = \begin{pmatrix} \frac{1}{2}\hat{A}(t-1) & \frac{1}{2}\hat{A}(t-1) + W(t-1) \\ \frac{1}{2}\hat{A}(t-1) + W(t-1) & \frac{1}{2}\hat{A}(t-1) \end{pmatrix}$$

and $\tilde{A}(2t+1) = I_{2n}$ for all $t \in \mathbb{N}_0$, and

$$\widetilde{\text{BU}}_{i,t+1}(\theta) := \frac{\tilde{l}_i(\tilde{\omega}_{i,t+1}|\theta)\tilde{\mu}_{i,t}(\{\theta\})}{\sum_{\theta' \in \Theta} \tilde{l}_i(\tilde{\omega}_{i,t+1}|\theta')\tilde{\mu}_{i,t}(\{\theta'\})}.$$

One can now verify that for all $t \in \mathbb{N}_0$:

$$\text{BU}_{i,t}(\theta) = \tilde{\mu}_{i,2t}(\theta) = \tilde{\mu}_{n+i,2t}(\theta) = \widetilde{\text{BU}}_{i,2t}(\theta)$$

and

$$\mu_{i,t}(\theta) = \tilde{\mu}_{i,2t+1}(\theta).$$

Hence, it suffices to prove that the first two assumptions of Theorem 7 apply to the hypothetical network $\widetilde{\mathcal{H}}$.

To this end, we begin by showing that if Assumption I holds for the original chain $\{A(t)\}_{t=0}^{\infty}$, then it also holds for the chain $\{\tilde{A}(t)\}_{t=0}^{\infty}$. First, observe that

$$\begin{pmatrix} I_n & I_n \end{pmatrix} \tilde{A}(2t+2) = \begin{pmatrix} A(t) & A(t) \end{pmatrix}.$$

Since $A(2t+3) = I_{2n}$, this implies:

$$\begin{aligned} &\begin{pmatrix} I_n & I_n \end{pmatrix} \tilde{A}(2t+5:2t+2) \\ &= \begin{pmatrix} A(t+1) & A(t+1) \end{pmatrix} \tilde{A}(2t+2) \\ &= \begin{pmatrix} A(t+1)A(t) & A(t+1)A(t) \end{pmatrix} \\ &= \begin{pmatrix} A(t+2:t) & A(t+2:t) \end{pmatrix}. \end{aligned}$$

By induction, this can be generalized to:

$$\begin{aligned} &\begin{pmatrix} I_n & I_n \end{pmatrix} \tilde{A}(2(t+k)+1:2t+2) \\ &= \begin{pmatrix} A(t+k:t) & A(t+k:t) \end{pmatrix}. \end{aligned} \tag{4.35}$$

for all $k \in \mathbb{N}$ and all $t \in \mathbb{N}_0$. On the other hand, due to block multiplication, for any $i, j \in [n]$, the $(i,j)^{\text{th}}$ entry of the left-hand-side of (4.35) equals $(\tilde{A}(2(t+k)+1:2t+2))_{ij} + (\tilde{A}(2(t+k)+1:2t+2))_{n+ij}$. Hence, (4.35) implies:

$$\max\left\{(\tilde{A}(2(t+k)+1:2t+2))_{ij}, (\tilde{A}(2(t+k)+1:2t+2))_{n+ij}\right\} \geq \frac{1}{2}(A(t+k:t))_{ij}. \tag{4.36}$$

Together with the fact that $\tilde{A}(2\tau+1) = I$ for all $\tau \in \mathbb{N}_0$, the inequality above implies the following: given that $i \in [n]$, $t_s, t_f \in \mathbb{N}_0$, and $k \in \mathbb{N}$, if there exist $\gamma > 0$, $C \subset [n]$ and $\mathcal{T} \subset \{t_s+1, \ldots, t_f\}$ such that for every $j \in C$, there exists a $t \in \mathcal{T}$ satisfying $a_{jj}(t) \geq \gamma$ and $(A(t:t_s))_{ji} \geq \gamma$, then there exists a set $\tilde{C} \subset [2n]$ such that $\{i \mod n : i \in \tilde{C}\} = C$ and for every $j \in \tilde{C}$, there exists a $t \in \mathcal{T}$ satisfying

$$\tilde{a}_{jj}(2t+1) \geq \gamma/2 \text{ and } (\tilde{A}(2t+1 : 2t_s+2))_{ji} \geq \gamma/2.$$

Next, we observe that if $\mathcal{O} \subset [n]$ is an observationally self-sufficient set for the original network $\mathcal{H}$, then (4.33) implies that any set $\tilde{\mathcal{O}} \subset [2n]$ that satisfies $\{i \mod n : i \in \tilde{\mathcal{O}}\} = \mathcal{O}$ is an observationally self-sufficient set for $\tilde{\mathcal{H}}$. In the light of the previous paragraph, this implies that if $[t_s, t_f]$ is a γ-epoch for $\mathcal{H}$, then $[2t_s+2, 2t_f+1]$ is a $\frac{\gamma}{2}$-epoch for $\widetilde{\mathcal{H}}$. Thus, if Assumption I holds for $\mathcal{H}$, then it also holds for $\widetilde{\mathcal{H}}$.

Now, since Assumption II holds for $\mathcal{H}$, it immediately follows that Assumption II also holds for $\widetilde{\mathcal{H}}$.

Next, on the basis of the block symmetry of $\tilde{A}(2t)$, we claim that the following analog of (4.36) holds for all $i, j \in [n]$:

$$\max\left\{(\tilde{A}(2(t+k)+1 : 2t+2))_{ij}, (\tilde{A}(2(t+k)+1 : 2t+2))_{in+j}\right\} \geq \frac{1}{2}(A(t+k:t))_{ij}.$$

This implies that for any $\tau \in \mathbb{N}$:

$$\max\left\{(\tilde{A}(2\tau+1 : 2))_{ij}, (\tilde{A}(2\tau+1 : 2))_{in+j}\right\} \geq \frac{1}{2}(A(\tau:0))_{ij}.$$

On the basis of this, it can be verified that Assumption III holds for $\widetilde{\mathcal{H}}$ whenever it holds for $\mathcal{H}$.

As for Assumption IV, it can be verified that if $\{\pi(t)\}_{t=0}^{\infty}$ is an absolute probability process for $\{A(t)\}_{t=0}^{\infty}$, then the sequence $\{\tilde{\pi}(t)\}_{t=0}^{\infty}$, defined by

$$\tilde{\pi}^T(2t) = \tilde{\pi}^T(2t-1) = \frac{1}{2}[\pi^T(t-1) \quad \pi^T(t-1)]$$

for all $t \in \mathbb{N}$ and $\tilde{\pi}^T(0) = \frac{1}{2}[\pi^T(0) \quad \pi^T(0)]$ is an absolute probability process for $\{\tilde{A}(t)\}_{t=0}^{\infty}$. Since $\{A(t)\}_{t=0}^{\infty} \in \mathcal{P}^*$, it follows that $\{\tilde{A}(t)\}_{t=0}^{\infty}$ also satisfies Assumption IV.

Finally, observe that $\{\tilde{A}(t)\}_{t=0}^{\infty}$ satisfies Assumptions V and VI because the original chain $\{A(t)\}_{t=0}^{\infty}$ satisfies them. In sum, Assumptions II - VI are all satisfied by $\widetilde{\mathcal{H}}$. As a result, Assertions (i) and (ii) of Theorem 7 hold for $\widetilde{\mathcal{H}}$. Hence, the same assertions apply to the original network as well. □

Remark 11. *The proof of Corollary 7 enables us to infer the following: it is possible for a network of agents following the original update rule* (4.1) *to learn the truth asymptotically almost surely despite certain agents not taking any new measurements at some of the time steps (which effectively means that their self-confidences are set to zero at those time steps). This could happen, for instance, when some of the agents intermittently lose contact with their external sources of information and therefore depend solely on their neighbors for updating their beliefs at the corresponding time instants. As a simple example, consider a chain* $\{A(t)\}_{t=0}^{\infty} \in \mathcal{P}^* \cap \mathbb{R}^{n\times n}$*, an increasing sequence* $\{\tau_k\}_{k=0}^{\infty} \in \mathbb{N}_0$ *with* $\tau_0 := 0$*, and a chain of permutation matrices,* $\{P(k)\}_{k=1}^{\infty} \subset \mathbb{R}^{n\times n}$ *such that* $P(k) \neq I_n$ *for any* $k \in \mathbb{N}$*. Then the chain,*

$$\begin{aligned}&A(0),\ldots,A(\tau_1-1),P^T(1)A(\tau_1),P(1),A(\tau_1+1),\ldots\\&\ldots,A(\tau_2-1),P^T(2)A(\tau_2),P(2),A(\tau_2+1),\ldots\end{aligned}$$

can be shown to belong to Class $\mathcal{P}^*$ *even though* $P_{ii}(k) = 0$ *for some* $i \in [n]$ *and infinitely many* $k \in \mathbb{N}$*. If, in addition,* $\{A(t)\}_{t=0}^{\infty}$ *satisfies Assumption I and* $\{\tau_k\}_{k=0}^{\infty}$ *have been chosen such that* $\tau_{k-1} < t_{2k-1} < t_{2k} < \tau_k$ *for each* $k \in \mathbb{N}$*, then it can be shown that even the modified chain satisfies Assumption I. In this case the assertions of Theorem 7 apply to the modified* *chain. Moreover, the modified chain violates Condition 2 of Definition 48, and hence, it is not a uniformly strongly connected chain. The upshot is that intermittent negligence of external information combined with the violation of standard connectivity criteria does not preclude almost-sure asymptotic learning.*

4.5.4 Learning on Deterministic Time-Varying Networks

We now provide some corollaries of Theorem 7 that apply to deterministic time-varying networks. We will need the following lemma in order to prove the corollaries.

Lemma 22. *Let $\{[A(t)]\}_{t=0}^{\infty}$ be deterministic and uniformly strongly connected. Then Assumptions I, III and IV hold.*

Proof. Let δ, B, $\{G(t)\}_{t=0}^{\infty}$ and $\{\mathcal{G}(k)\}_{k=0}^{\infty}$ be as defined in Definition 48. Consider Assumption I. By Definition 48, for any two nodes $i, j \in [n]$ and any time interval of the form $[kB, (k+1)B-1]$ where $k \in \mathbb{N}_0$, there exists a directed path from i to j in $\mathcal{G}(k)$, i.e., there exist an integer $q \in [B]$, nodes $s_1, s_2, \ldots, s_{q-1} \in [n]$ and times $\tau_1, \ldots, \tau_q \in \{kB, \ldots, (k+1)B-1\}$ such that

$$a_{j\,s_{q-1}}(\tau_q), a_{s_{q-1},s_{q-2}}(\tau_{q-1}), \ldots, a_{s_1 i}(\tau_1) > 0.$$

Observe that by Definition 48, each of the above quantities is lower bounded by δ. Also, $a_{rr}(t) > 0$ and hence, $a_{rr}(t) \geq \delta$ for all $r \in [n]$ and $t \in [kB, (k+1)B-1]$. Hence, for all $r \in [n]$ and $t_1, t_2 \in \{kB, \ldots, (k+1)B\}$ satisfying $t_1 \leq t_2$:

$$(A(t_2 : t_1))_{rr} \geq \prod_{t=t_1}^{t_2-1} a_{rr}(t) \geq \delta^{t_2-t_1} \geq \delta^B. \tag{4.37}$$

It follows that:

$$
\begin{aligned}
&(A(\tau_q+1:kB))_{ji} \\
&\geq a_{j,s_{q-1}}(\tau_q)(A(\tau_q:\tau_{q-1}+1))_{s_{q-1}s_{q-1}} \cdot a_{s_{q-1}s_{q-2}}(\tau_{q-1})(A(\tau_{q-1}:\tau_{q-2}+1))_{s_{q-2}s_{q-2}}\cdots \\
&\quad \cdots a_{s_1 i}(\tau_1)(A(\tau_1:kB))_{ii} \\
&\geq (\delta\cdot\delta^B)^q \\
&\geq \delta^{q(B+1)} \\
&\geq \delta^{B(B+1)}.
\end{aligned} \tag{4.38}
$$

Thus, setting $\gamma=\delta^{B(B+1)}$ ensures $(A(\tau:kB))_{ji}\geq\gamma$ as well as $a_{jj}(\tau)\geq\gamma$ for some $\tau\in\{kB+1,\ldots,(k+1)B\}$. Since $i,j\in[n]$ and $k\in\mathbb{N}_0$ were arbitrary, and since $[n]$ is observationally self-sufficient, it follows that $[kB,(k+1)B]$ is a γ-epoch for every $k\in\mathbb{N}_0$. Thus, by setting $t_{2k-1}=2kB$ and $t_{2k}=(2k+1)B$, we observe that the sequence $\{t_k\}_{k=1}^{\infty}$ satsifies the requirements of Assumption I.

As for Assumption III, (4.38) implies the existence of $\tau_1,\tau_2,\ldots,\tau_n\in[B]$ such that $(A(\tau_i:0))_{i1}>0$ for every $i\in[n]$. Then $(A(B:0))_{i1}\geq(A(B:\tau_i))_{ii}(A(\tau_i:0))_{i1}$ and the latter is positive since $(A(B:\tau_i))_{ii}>0$ by (4.37). Thus, Assumption III holds with $T=B$.

Finally, Assumption IV holds by Lemma 5.8 of [62]. □

An immediate consequence of Lemma 22 and Theorem 7 is the following result.

Corollary 8. *Suppose Assumption II holds and that $\{A(t)\}_{t=0}^{\infty}$ is a deterministic B-connected chain. Then all the agents' beliefs weakly merge to the truth a.s. Also, all the agents' beliefs converge to a consensus a.s. If, in addition, θ^* is identifiable, then the agents asymptotically learn θ^* a.s.*

Note that Corollary 8 is a generalization of the main result (Theorem 2) of [123] which imposes on $\{A(t)\}_{t=0}^{\infty}$ the additional restriction of double stochasticity.

Besides uniformly strongly connected chains, Theorem 7 also applies to balanced chains with strong feedback property, since these chains too satisfy Assumption IV.

Corollary 9. *Suppose Assumptions II and III hold, and that $\{A(t)\}_{t=0}^{\infty}$ is a balanced chain with strong feedback property. Then the assertions of Theorems 7 and 6 apply.*

Essentially, Corollary 9 states that if every agent's self-confidence is always above a minimum threshold and if the total influence of any subset S of agents on the complement set $\bar{S} = [n] \setminus S$ is always comparable to the total reverse influence (i.e., the total influence of $\bar{S}$ on S), then asymptotic learning takes place *a.s.* under mild additional assumptions.

It is worth noting that the following established result (Theorem 3.2, [124]) is a consequence of Corollaries 8 and 9.

Corollary 10 (Main result of [124]). *Suppose $\{A(t)\}_{t=0}^{\infty}$ is a deterministic stochastic chain such that $A(t) = \eta(t)A + (1-\eta(t))I$, where $\eta(t) \in (0,1]$ is a time-varying parameter and $A = (A_{ij})$ is a fixed stochastic matrix. Further, suppose that the network is strongly connected at all times, that there exists[2] a $\gamma > 0$ such that $A_{ii} \geq \gamma$ for all $i \in [n]$ (resulting in $a_{ii}(t) > 0$ for all $i \in [n]$ and $t \in \mathbb{N}_0$), and that $\mu_{j_0,0}(\theta^*) > 0$ for some $j_0 \in [n]$. Then the 1-step-ahead forecasts of all the agents are eventually correct a.s. Additionally, suppose $\sigma := \inf_{t \in \mathbb{N}_0} \eta(t) > 0$. Then all the agents converge to a consensus a.s. If, in addition, θ^* is identifiable, then all the agents asymptotically learn the truth a.s.*

Proof. Let $\delta := \min_{i,j \in [n]} \{A_{ij} : A_{ij} > 0\}$, let $C \subset [n]$ be an arbitrary index set and let $\bar{C} := [n] \setminus C$. Observe that since the network is always strongly connected, $A(t)$ is an irreducible matrix for every $t \in \mathbb{N}_0$. It follows that A is also irreducible. Therefore, there exist indices $p \in C$ and $q \in \bar{C}$ such that $A_{pq} > 0$. Hence, $A_{pq} \geq \delta$. Thus, for any $t \in \mathbb{N}$:

$$\sum_{i \in C} \sum_{j \in \bar{C}} a_{ij}(t) = \sum_{i \in C} \sum_{j \in \bar{C}} \eta(t) A_{ij} \geq \eta(t) A_{pq} \geq \eta(t)\delta.$$

[2]This assumption is stated only implicitly in [124]. It appears on page 588 (in the proof of Lemma 3.3 of the paper).

On the other hand, we also have:

$$\sum_{i\in\bar{C}}\sum_{j\in C} a_{ij}(t) = \eta(t)\sum_{i\in\bar{C}}\sum_{j\in C} A_{ij} \leq \eta(t)\sum_{i\in\bar{C}}\sum_{j\in C} 1 \leq n^2\eta(t).$$

Hence, $\sum_{i\in C}\sum_{j\in\bar{C}} a_{ij}(t) \geq \frac{\delta}{n^2}\sum_{i\in\bar{C}}\sum_{j\in C} a_{ij}(t)$ for all $t\in\mathbb{N}$. Moreover,

$$a_{ii}(t) = 1-\eta(t)(1-A_{ii}) \geq 1-1(1-\gamma) = \gamma > 0$$

for all $i\in[n]$ and $t\in\mathbb{N}$. Hence, $\{A(t)\}_{t=0}^{\infty}$ is a balanced chain with feedback property. In addition, we are given that Assumption II holds. Furthermore, feedback property and the strong connectivity assumption imply that Assumption III holds with $T = n-1$. Then by Corollary 9, all the agents' beliefs weakly merge to the truth. Thus, every agent's 1-step-ahead forecasts are eventually correct *a.s.*

Next, suppose $\inf_{t\in\mathbb{N}_0}\eta(t) > 0$, i.e., $\eta(t)\geq\sigma>0$ for all $t\in\mathbb{N}_0$. Then for all distinct $i,j\in[n]$, either $a_{ij}(t)\geq\sigma\delta$ or $a_{ij}(t)=0$. Along with the feedback property of $\{A(t)\}_{t=0}^{\infty}$ and the strong connectivity assumption, this implies that $\{A(t)\}_{t=0}^{\infty}$ is B-connected with $B=1$. We now invoke Corollary 8 to complete the proof. □

Finally, we note through the following example that uniform strong connectivity is not necessary for almost-sure asymptotic learning on time-varying networks.

Example 2. *Let $n=6$, let $\{2,3\}$ and $\{5,6\}$ be observationally self-sufficient sets, and suppose $\mu_{1,0}(\theta^*)>0$. Let $\{A(t)\}_{t=0}^{\infty}$ be defined by $A(0)=\frac{1}{6}\mathbf{1}\mathbf{1}^T$ and*

$$A(t) = \begin{cases} A_e & \text{if } t = 2^{2k} \text{ for some } k\in\mathbb{N}_0, \\ A_o & \text{if } t = 2^{2k+1} \text{ for some } k\in\mathbb{N}_0, \\ I & \text{otherwise}, \end{cases}$$

where

$$A_e := \begin{pmatrix} 1/3 & 1/3 & 1/3 & 0 & 0 & 0 \\ 1/8 & 1/2 & 3/8 & 0 & 0 & 0 \\ 1/4 & 1/2 & 1/4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1/8 & 3/8 & 1/2 \\ 0 & 0 & 0 & 1/2 & 1/4 & 1/4 \end{pmatrix}$$

and

$$A_o := \begin{pmatrix} 1/3 & 0 & 0 & 0 & 1/3 & 1/3 \\ 0 & 3/8 & 3/8 & 1/4 & 0 & 0 \\ 0 & 1/6 & 1/2 & 1/3 & 0 & 0 \\ 0 & 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 1/4 & 1/4 \\ 1/2 & 0 & 0 & 0 & 3/8 & 1/8 \end{pmatrix}.$$

Then it can be verified that $\{A(t)\}_{t=0}^{\infty}$ is a balanced chain with strong feedback property. Also, our choice of $A(0)$ ensures that Assumption III holds with $T = 1$. Moreover, we can verify that Assumption I holds with $t_{2k-1} = 2^k$ and $t_{2k} = 2^k + 1$ for all $k \in \mathbb{N}$. Therefore, by Corollary 9, all the agents asymptotically learn the truth a.s. This happens even though $\{A(t)\}_{t=0}^{\infty}$ is not B-connected for any finite B (which can be verified by noting that $\lim_{k\to\infty}(2^{2k+1} - 2^{2k}) = \infty$).

Remark 12. *Note that by Definition 49, balanced chains embody a certain symmetry in the influence relationships between the agents. Hence, the above example shows that asymptotic learning can be achieved even when some network connectivity is traded for influence symmetry.*

4.6 Conclusions and Future Directions

We extended the well-known model of non-Bayesian social learning [105] to study social learning over random directed graphs satisfying connectivity criteria that are weaker than uniform strong connectivity. We showed that if the sequence of weighted adjacency matrices associated to the network belongs to Class $\mathcal{P}^*$, implying that no agent's social power ever falls below a fixed threshold in the average case, then the occurrence of infinitely many γ-epochs (periods of sufficient connectivity) ensures almost-sure asymptotic learning. We then showed that our main result, besides generalizing a few known results, has interesting implications for related learning scenarios such as inertial learning or learning in the presence of link failures. We also showed that our main result subsumes time-varying networks described by balanced chains, thereby suggesting that influence symmetry aids in social learning. In addition, we showed how uniform strong connectivity guarantees that all the agents' beliefs almost surely converge to a consensus even when the true state is not identifiable. This means that, although periodicity in network connectivity is not necessary for social learning, it yields long-term social agreement, which may be desirable in certain situations.

In addition to the above results, we conjecture that our techniques can be useful to tackle the following problems.

1. ***Log-linear Learning:*** In the context of distributed learning in sensor networks, it is well-known that under standard connectivity criteria, log-linear learning rules (in which the agents linearly aggregate the *logarithms* of their beliefs instead of the beliefs themselves) also achieve almost-sure asymptotic learning but exhibit greater convergence rates than the learning rule that we have analyzed [6, 106]. We therefore believe that one can obtain a result similar to Theorem 7 by applying our Class $\mathcal{P}^*$ techniques to analyse log-linear learning rules.

2. ***Learning on Dependent Random Digraphs:*** As there exists a definition of Class $\mathcal{P}^*$ for dependent random chains [62], one may be able to extend the results of this chapter to comment on learning on dependent random graphs. Regardless of the potential challenges involved in this endeavor, our intuition suggests that recurring γ-epochs (which ensure a satisfactory level of communication and belief circulation in the network) in combination with the Class $\mathcal{P}^*$ requirement (which ensures that every agent is influential enough to

make a non-vanishing difference to others' beliefs over time) should suffice to achieve almost-sure asymptotic learning.

In future, we would like to derive a set of connectivity criteria that are both necessary and sufficient for asymptotic non-Bayesian learning on random graphs. Yet another open problem is to study asymptotic and non-asymptotic rates of learning in terms of the number of γ-epochs occurred.

Appendix: Relevant Lemmas

The lemma below provides a lower bound on the agents' future beliefs in terms of their current beliefs.

Lemma 23. *Given $t, B \in \mathbb{N}$ and $\Delta \in [B]$, the following holds for all $i, j \in [n]$ and $\theta \in \Theta$:*

$$\mu_{j,t+\Delta}(\theta) \geq (A(t+\Delta : t))_{ji} \left(\frac{l_0}{n}\right)^B n\mu_{i,t}(\theta). \tag{4.39}$$

Proof. We first prove the following by induction:

$$\mu_{j,t+\Delta}(\theta) \geq (A(t+\Delta : t))_{ji} \left(\frac{l_0}{n}\right)^\Delta n\mu_{i,t}(\theta). \tag{4.40}$$

Pick any two agents $i, j \in [n]$, and note that for every $\theta \in \Theta$, the update rule (4.1) implies that

$$\mu_{j,t+1}(\theta) \geq a_{jj}(t)\frac{l_j(\omega_{j,t+1}|\theta)}{m_{j,t}(\omega_{j,t+1})}\mu_{j,t}(\theta) \geq a_{jj}(t)l_0\mu_{j,t}(\theta),$$

whereas the same rule implies that $\mu_{j,t+1}(\theta) \geq a_{ji}(t)\mu_{i,t}(\theta)$ if $i \neq j$. As a result, we have $\mu_{j,t+1}(\theta) \geq a_{ji}(t)l_0\mu_{i,t}(\theta)$, which proves (4.40) for $\Delta = 1$. Now, suppose (4.40) holds with

$\Delta = m$ for some $m \in \mathbb{N}$. Then:

$$\begin{aligned} \mu_{j,t+m+1}(\theta) &\overset{(a)}{\geq} a_{jp}(t+m)\frac{l_0}{n} \cdot n\mu_{p,t+m}(\theta) \\ &\overset{(b)}{\geq} a_{jp}(t+m)(A(t+m:t))_{pi}\left(\frac{l_0}{n}\right)^{m+1} n^2\mu_{i,t}(\theta). \end{aligned} \tag{4.41}$$

for all $p \in [n]$, where (a) is obtained by applying (4.40) with $\Delta = 1$ and with t replaced by $t+m$, and (b) follows from the inductive hypothesis. Now, since

$$(A(t+m+1:t))_{ji} = \sum_{q=1}^{n} a_{jq}(t+m)(A(t+m:t))_{qi},$$

it follows that there exists a $p \in [n]$ satisfying

$$a_{jp}(t+m)A(t+m:t)_{pi} \geq A((t+m+1:t))_{ji}/n.$$

Combining this inequality with (4.41) proves (4.40) for $\Delta = m+1$ and hence for all $\Delta \in \mathbb{N}$. Suppose now that $\Delta \in [B]$. Then (4.40) immediately yields the following:

$$\mu_{j,t+\Delta}(\theta) \geq (A(t+\Delta:t))_{ji}\left(\frac{l_0}{n}\right)^{\Delta} n\mu_{i,t}(\theta) \geq (A(t+\Delta:t))_{ji}\left(\frac{l_0}{n}\right)^{B} n\mu_{i,t}(\theta),$$

where the second inequality holds because $\frac{l_0}{n} \leq l_0 \leq 1$ by definition. This completes the proof. □

Lemma 24. *There exists a constant $K_0 < \infty$ such that*

$$0 \leq \mathbb{E}^*\left[\frac{l_i(\omega_{i,t+1}|\theta)}{m_{i,t}(\omega_{i,t+1})} - 1 \,\middle|\, \mathcal{B}_t\right] \leq K_0$$

$\mathbb{P}^*$*-a.s. for all $\theta \in \Theta_i^*, i \in [n]$ and $t \in \mathbb{N}_0$. Moreover, the second inequality above holds for all $\theta \in \Theta$.*

Proof. By an argument similar to the one used in [105], since the function $\mathbb{R}_+ \ni x \to 1/x \in \mathbb{R}_+$ is strictly convex, by Jensen's inequality, we have the following almost surely for every $i \in [n]$ and $\theta \in \Theta_i^*$:

$$\mathbb{E}^*\left[\frac{l_i(\omega_{i,t+1}|\theta)}{m_{i,t}(\omega_{i,t+1})} \,\middle|\, \mathcal{B}_t\right] > \left(\mathbb{E}^*\left[\frac{m_{i,t}(\omega_{i,t+1})}{l_i(\omega_{i,t+1}|\theta)} \,\middle|\, \mathcal{B}_t\right]\right)^{-1}. \tag{4.42}$$

Also, (4.2) implies that $\mu_t(\theta)$ is completely determined by $\omega_1, \ldots, \omega_t, A(0), \ldots, A(t-1)$ and hence, it is measurable with respect to $\mathcal{B}_t$. Therefore, the following holds *a.s*:

$$\begin{aligned}
\mathbb{E}^*\left[\frac{m_{i,t}(\omega_{i,t+1})}{l_i(\omega_{i,t+1}|\theta)} \,\middle|\, \mathcal{B}_t\right] &= \mathbb{E}^*\left[\frac{\sum_{\theta'\in\Theta} l_i(\omega_{i,t+1}|\theta')\mu_{i,t}(\theta')}{l_i(\omega_{i,t+1}|\theta)} \,\middle|\, \mathcal{B}_t\right] \\
&= \sum_{\theta'\in\Theta} \mathbb{E}^*\left[\frac{l_i(\omega_{i,t+1}|\theta')}{l_i(\omega_{i,t+1}|\theta)} \,\middle|\, \mathcal{B}_t\right] \mu_{i,t}(\theta') \\
&\overset{(a)}{=} \sum_{\theta'\in\Theta} \mathbb{E}^*\left[\frac{l_i(\omega_{i,t+1}|\theta')}{l_i(\omega_{i,t+1}|\theta^*)} \,\middle|\, \sigma(\omega_1, \ldots, \omega_t)\right] \mu_{i,t}(\theta') \\
&\overset{(b)}{=} \sum_{\theta'\in\Theta} \sum_{s\in S_i} l_i(s|\theta')\mu_{i,t}(\theta') \\
&= 1,
\end{aligned}$$

where we have used the implication of observational equivalence and Assumption VI in (a), and the fact that $\{\omega_{i,t}\}_{t=0}^{\infty}$ are i.i.d. $\sim l_i(\cdot|\theta^*)$ in (b). Thus, (4.42) now implies the lower bound in Lemma 24.

As for the upper bound, since $l_0 > 0$, we also have:

$$\frac{l_i(\omega_{i,t+1}|\theta)}{m_{i,t}(\omega_{i,t+1})} \leq \frac{1}{m_{i,t}(\omega_{i,t+1})} = \frac{1}{\sum_{\theta\in\Theta} l_i(\omega_{i,t+1}|\theta)\mu_{i,t}(\theta)} \overset{(a)}{\leq} \frac{1}{l_0} < \infty,$$

where (a) follows from the fact that $\sum_{\theta\in\Theta} \mu_{i,t}(\theta) = 1$. This shows that $\mathbb{E}^*\left[\frac{l_i(\omega_{i,t+1}|\theta)}{m_{i,t}(\omega_{i,t+1})} - 1 \,\middle|\, \mathcal{B}_t\right] \leq \frac{1}{l_0} - 1$ *a.s.* for all $\theta \in \Theta$. Setting $K_0 = \frac{1}{l_0} - 1$ now completes the proof. □

The next lemma is one of the key steps in showing that the agents' beliefs weakly merge to the truth almost surely.

Lemma 25. *For all $i \in [n]$, we have*

$$u_i(t) := a_{ii}(t)\left(\frac{l_i(\omega_{i,t+1}|\theta^*)}{m_{i,t}(\omega_{i,t+1})} - 1\right)\mu_{i,t}(\theta^*) \to 0 \quad \textit{a.s. as} \quad t \to \infty.$$

Proof. Let $i \in [n]$ be a generic index. Similar to an argument used in [105], we observe that (4.5) implies the following:

$$\begin{aligned}
&a_{ii}(t)\mathbb{E}^*\left[\frac{l_i(\omega_{i,t+1}|\theta^*)}{m_{i,t}(\omega_{i,t+1})} - 1 \,\middle|\, \mathcal{B}_t\right]\mu_{i,t}(\theta^*) \\
&= a_{ii}(t)\mu_{i,t}(\theta^*)\sum_{s\in S_i} l_i(s|\theta^*)\left(\frac{l_i(s|\theta^*)}{m_{i,t}(s)} - 1\right) \\
&\overset{(a)}{=} a_{ii}(t)\mu_{i,t}(\theta^*)\sum_{s\in S_i}\left(l_i(s|\theta^*)\frac{l_i(s_i|\theta^*) - m_{i,t}(s)}{m_{i,t}(s)}\right) + a_{ii}(t)\mu_{i,t}(\theta^*)\sum_{s\in S_i}(m_{i,t}(s) - l_i(s|\theta^*)) \\
&= \sum_{s\in S_i} a_{ii}(t)\mu_{i,t}(\theta^*)\frac{(l_i(s|\theta^*) - m_{i,t}(s))^2}{m_{i,t}(s)} \overset{t\to\infty}{\longrightarrow} 0 \quad \textit{a.s.}
\end{aligned}$$

where (a) holds because $\sum_{s\in S_i} m_{i,t}(s) = \sum_{s\in S_i} l_i(s|\theta^*) = 1$ since both $l_i(\cdot|\theta^*)$ and $m_{i,t}(\cdot)$ are probability distributions on S_i. Since every summand in the last summation above is non-negative, it follows that for all $i \in [n]$:

$$a_{ii}(t)\mu_{i,t}(\theta^*)\frac{(l_i(s|\theta^*) - m_{i,t}(s))^2}{m_{i,t}(s)} \to 0 \quad \text{for all } s \in S_i$$

a.s. as $t \to \infty$. Therefore, for every $s \in S_i$ and $i \in [n]$,

$$\begin{aligned}
&\limsup_{t\to\infty}\left[a_{ii}(t)\left(\frac{l_i(s|\theta^*)}{m_{i,t}(s)} - 1\right)\mu_{i,t}(\theta^*)\right]^2 \\
&= \limsup_{t\to\infty}\left[a_{ii}(t)\mu_{i,t}(\theta^*)\frac{[l_i(s|\theta^*) - m_{i,t}(s)]^2}{m_{i,t}(s)} \cdot \frac{a_{ii}(t)\mu_{i,t}(\theta^*)}{m_{i,t}(s)}\right] \\
&\le \limsup_{t\to\infty}\left[a_{ii}(t)\mu_{i,t}(\theta^*)\frac{[l_i(s|\theta^*) - m_{i,t}(s)]^2}{m_{i,t}(s)} \cdot \frac{1}{l_0}\right] = 0 \quad \textit{a.s.},
\end{aligned}$$

which proves that

$$\lim_{t\to\infty}\left|a_{ii}(t)\left(\frac{l_i(s|\theta^*)}{m_{i,t}(s)}-1\right)\mu_{i,t}(\theta^*)\right|=0 \quad \textit{a.s.}$$

Since S_i is a finite set, this implies that

$$\lim_{t\to\infty}\max_{s\in S_i}\left|a_{ii}(t)\left(\frac{l_i(s|\theta^*)}{m_{i,t}(s)}-1\right)\mu_{i,t}(\theta^*)\right|=0 \quad \textit{a.s.},$$

which proves the lemma, because $\omega_{i,t+1}\in S_i$ for all $t\in\mathbb{N}_0$. □

We are now equipped to prove the following result which is similar to Lemma 3 of [105].

Lemma 26. *For all* $\theta\in\Theta$:

$$\mathbb{E}^*[\mu_{t+1}(\theta)|\mathcal{B}_t]-A(t)\mu_t(\theta)\to 0 \quad \textit{a.s. as} \quad t\to\infty.$$

Proof. We first note that $\sum_{s\in S_i} l_i(s|\theta)=\sum_{s\in S_i} l_i(s|\theta^*)=1$ implies that for all $\theta\in\Theta$:

$$\sum_{s\in S_i} l_i(s|\theta^*)\left(\frac{l_i(s|\theta)}{m_{i,t}(s)}-1\right)=\sum_{s\in S_i} l_i(s|\theta)\left(\frac{l_i(s|\theta^*)}{m_{i,t}(s)}-1\right).$$

Hence, for any $i\in[n]$ and $\theta\in\Theta$:

$$\begin{aligned} a_{ii}(t)\mathbb{E}^*\left[\frac{l_i(\omega_{i,t+1}|\theta)}{m_{i,t}(\omega_{i,t+1})}-1\,\middle|\,\mathcal{B}_t\right] &= a_{ii}(t)\sum_{s\in S_i} l_i(s|\theta^*)\left(\frac{l_i(s|\theta)}{m_{i,t}(s)}-1\right)\\ &= a_{ii}(t)\sum_{s\in S_i} l_i(s|\theta)\left(\frac{l_i(s|\theta^*)}{m_{i,t}(s)}-1\right)\\ &= \sum_{s\in S_i} l_i(s|\theta)a_{ii}(t)\left(\frac{l_i(s|\theta^*)}{m_{i,t}(s)}-1\right)\overset{t\to\infty}{\longrightarrow} 0 \quad \textit{a.s.}, \end{aligned} \tag{4.43}$$

where the last step follows from Lemma 25. Consequently, taking conditional expectations on both sides of (4.2) yields

$$\mathbb{E}^*[\mu_{t+1}(\theta)|\mathcal{B}_t] - A(t)\mu_t(\theta) = \text{diag}\left(\ldots, a_{ii}(t)\mathbb{E}^*\left[\frac{l_i(\omega_{i,t+1}|\theta)}{m_{i,t}(\omega_{i,t+1})} - 1 \,\middle|\, \mathcal{B}_t\right], \ldots\right)\mu_t(\theta) \overset{t\to\infty}{\longrightarrow} 0$$

almost surely, thus proving Lemma 26. □

Lemma 27.

$$\mathbb{E}^*[\pi^T(t+2)A(t+1)\mu_{t+1}(\theta^*) \mid \mathcal{B}_t] = \pi^T(t+2)\mathbb{E}^*[A(t+1)] \cdot \mathbb{E}^*[\mu_{t+1}(\theta^*) \mid \mathcal{B}_t]$$

Proof. We first prove that the following holds almost surely:

$$\mathbb{E}^*[A(t+1)\mu_{t+1}(\theta^*) \mid \mathcal{B}_t] = \mathbb{E}^*[A(t+1)]\mathbb{E}^*[\mu_{t+1}(\theta^*) \mid \mathcal{B}_t]. \tag{4.44}$$

To this end, observe from the update rule (4.1) that the belief vector $\mu_{t+1}(\theta^*)$ is determined fully by $\omega_1, \ldots, \omega_t, \omega_{t+1}$ and $A(0), \ldots, A(t)$. That is, there exists a deterministic vector function ψ such that

$$\mu_{t+1}(\theta^*) = \psi(\omega_1, \ldots, \omega_t, A(0), \ldots, A(t), \omega_{t+1}),$$

Consider now a realization w_0 of the tuple $(\omega_1, \ldots, \omega_t)$ and a realization A_0 of the tuple $(A(0), \ldots, A(t))$. Also, recall that $\omega_{t+1} \in S = \prod_{i=1}^n S_i$, and let $\phi : S \to [0, \infty)$ be the function defined by $\phi(s) := \psi(\mathrm{w}_0, \mathrm{A}_0, s)$. Then,

$$
\begin{aligned}
&\mathbb{E}^*\left[A(t+1)\mu_{t+1}(\theta^*) \mid \mathcal{B}_t\right]\Big|_{(\omega_1,\ldots,\omega_t,A(0),\ldots,A(t))=(\mathrm{w}_0,\mathrm{A}_0)} \\
&= \mathbb{E}^*\left[A(t+1)\mu_{t+1}(\theta^*) \mid \omega_1,\ldots,\omega_t,A(0),\ldots,A(t)\right]\Big|_{(\omega_1,\ldots,\omega_t,A(0),\ldots,A(t))=(\mathrm{w}_0,\mathrm{A}_0)} \\
&= \mathbb{E}^*\left[A(t+1)\mu_{t+1}(\theta^*) \mid (\omega_1,\ldots,\omega_t,A(0),\ldots,A(t)) = (\mathrm{w}_0,\mathrm{A}_0)\right] \\
&= \mathbb{E}^*\left[A(t+1)\psi(\omega_1,\ldots,\omega_t,A(0),\ldots,A(t),\omega_{t+1}) \mid (\omega_1,\ldots,\omega_t,A(0),\ldots,A(t)) = (\mathrm{w}_0,\mathrm{A}_0)\right] \\
&= \mathbb{E}^*\left[A(t+1)\psi(\mathrm{w}_0,\mathrm{A}_0,\omega_{t+1}) \mid (\omega_1,\ldots,\omega_t,A(0),\ldots,A(t)) = (\mathrm{w}_0,\mathrm{A}_0)\right] \\
&= \mathbb{E}^*[A(t+1)\phi(\omega_{t+1}) \mid (\omega_1,\ldots,\omega_t,A(0),\ldots,A(t)) = (\mathrm{w}_0,\mathrm{A}_0)] \\
&\overset{(a)}{=} \mathbb{E}^*[A(t+1)\phi(\omega_{t+1})] \\
&\overset{(b)}{=} \mathbb{E}^*[A(t+1)]\mathbb{E}^*[\phi(\omega_{t+1})] \\
&\overset{(c)}{=} \mathbb{E}^*[A(t+1)]\mathbb{E}^*[\phi(\omega_{t+1}) \mid (\omega_1,\ldots,\omega_t,A(0),\ldots,A(t)) = (\mathrm{w}_0,\mathrm{A}_0)] \\
&= \mathbb{E}^*[A(t+1)]\mathbb{E}^*[\psi(\mathrm{w}_0,\mathrm{A}_0,\omega_{t+1}) \mid (\omega_1,\ldots,\omega_t,A(0),\ldots,A(t)) = (\mathrm{w}_0,\mathrm{A}_0)] \\
&= \mathbb{E}^*[A(t+1)]\mathbb{E}^*[\psi(\omega_1,\ldots,\omega_t,A(0),\ldots,A(t),\omega_{t+1}) \mid (\omega_1,\ldots,\omega_t,A(0),\ldots,A(t)) = (\mathrm{w}_0,\mathrm{A}_0)] \\
&= \mathbb{E}^*[A(t+1)] \\
&\quad \cdot \mathbb{E}^*[\psi(\omega_1,\ldots,\omega_t,A(0),\ldots,A(t),\omega_{t+1}) \mid \omega_1,\ldots,\omega_t,A(0),\ldots,A(t)]\Big|_{(\boldsymbol{\omega}_{1:t},A(0),\ldots,A(t))=(\mathrm{w}_0,\mathrm{A}_0)} \\
&= \mathbb{E}^*[A(t+1)]\mathbb{E}^*[\psi(\omega_1,\ldots,\omega_t,A(0),\ldots,A(t),\omega_{t+1}) \mid \mathcal{B}_t]\Big|_{(\omega_1,\ldots,\omega_t,A(0),\ldots,A(t))=(\mathrm{w}_0,\mathrm{A}_0)} \\
&= \mathbb{E}^*[A(t+1)]\mathbb{E}^*\left[\mu_{t+1}(\theta^*) \mid \mathcal{B}_t\right]\Big|_{(\omega_1,\ldots,\omega_t,A(0),\ldots,A(t))=(\mathrm{w}_0,\mathrm{A}_0)}
\end{aligned}
$$

where (a) follows from Assumptions V and VI, (b) follows from Assumption VI, and (c) follows from Assumption VI and the assumption that $\{\omega_t\}_{t=1}^{\infty}$ are i.i.d. Since $(\mathrm{w}_0, \mathrm{A}_0)$ is arbitrary, the above chain of equalities holds for $\mathbb{P}^*$-almost every realization $(\mathrm{w}_0, \mathrm{A}_0)$ of $(\omega_1, \ldots, \omega_t, A(0), \ldots, A(t))$, and hence, (4.44) holds almost surely.

As a result, we have

$$\mathbb{E}^*\left[\pi^T(t+2)A(t+1)\mu_t(\theta^*) \mid \mathcal{B}_t\right] \overset{(a)}{=} \pi^T(t+2)\mathbb{E}^*\left[A(t+1)\mu_t(\theta^*) \mid \mathcal{B}_t\right]$$
$$\overset{(b)}{=} \pi^T(t+2)\mathbb{E}^*\left[A(t+1)\right]\mathbb{E}^*\left[\mu_t(\theta^*) \mid \mathcal{B}_t\right],$$

where (a) holds because $\pi(t+1)$ is a non-random vector, and (b) holds because of (4.44). This completes the proof. □

Lemma 28. *Let $i \in [n]$. Given that $\lim_{t\to\infty} a_{ii}(t)(m_{i,t}(\omega_{i,t+1}) - l_i(\omega_{i,t+1}|\theta^*)) = 0$ a.s., we have*

$$\lim_{t\to\infty} a_{ii}(t)(m_{i,t}(s) - l_i(s|\theta^*)) = 0 \quad \textit{a.s.} \quad \textit{for all } s \in S_i.$$

Proof. We first note that $\lim_{t\to\infty} |a_{ii}(t)(m_{i,t}(\omega_{i,t+1}) - l_i(\omega_{i,t+1}|\theta^*))| = 0$ *a.s.* So, by the Dominated Convergence Theorem for Conditional Expectations (Theorem 5.5.9 in [135]), we have

$$\lim_{t\to\infty} \mathbb{E}^*\left[|a_{ii}(t)(m_{i,t}(\omega_{i,t+1}) - l_i(\omega_{i,t+1}|\theta^*))| \mid \mathcal{B}_t\right] = 0 \quad \textit{a.s.} \tag{4.45}$$

Now, since $\omega_{i,t+1}$ is independent of $\{\omega_1, \ldots, \omega_t, A(0), \ldots, A(t)\}$ because of Assumption VI and the i.i.d. property of the observation vectors, we have

$$\mathbb{P}^*(\omega_{i,t+1} = s \mid \omega_1, \ldots, \omega_t, A(0), \ldots, A(t)) = \mathbb{P}^*(\omega_{i,t+1} = s) = l_i(s|\theta^*).$$

Also, the mapping $m_{i,t}(\cdot)$ is determined fully by $\omega_1, \ldots, \omega_t$ and $A(0), \ldots, A(t)$ (i.e., $m_{i,t}(s)$ is $\mathcal{B}_t$-measurable for all $s \in S_i$). Therefore, (4.45) is equivalent to the following:

$$\lim_{t\to\infty} \sum_{s\in S_i} l_i(s|\theta^*)|a_{ii}(t)(m_{i,t}(s) - l_i(s|\theta^*))| = 0 \quad \textit{a.s.}$$

Now, since $l_i(s|\theta^*) > 0$ for all $s \in S_i$, every summand in the above summation is non-negative, which implies that

$$\lim_{t\to\infty} l_i(s|\theta^*)|a_{ii}(t)(m_{i,t}(s)-l_i(s|\theta^*))| = 0 \quad \textit{a.s.} \text{ for all } s \in S_i.$$

Finally, since $l_i(s|\theta^*) > 0$ is independent of t, we can delete $l_i(s|\theta^*)$ from the above limit. This completes the proof. □

Lemma 29. *Let the function* $d : \mathbb{R}^n \to \mathbb{R}$ *be defined by* $d(x) := \max_{i\in[n]} x_i - \min_{j\in[n]} x_j$*, and let the function* $V_\pi : \mathbb{R}^n \times \mathbb{N}_0 \to \mathbb{R}$ *be defined by* $V_\pi(x,k) := \sum_{i=1}^n \pi_i(k)(x_i - \pi^T(k)x)^2$ *as in [62]. Then*

$$(p^*/2)^{\frac{1}{2}} d(x) \leq \sqrt{V_\pi(x,k)} \leq d(x)$$

for all $x \in \mathbb{R}^n$ *and* $k \in \mathbb{N}_0$*, where* $p^* > 0$ *is a constant such that* $\pi(k) \geq p^*\mathbf{1}$ *for all* $k \in \mathbb{N}_0$*.*

Proof. For any $x \in \mathbb{R}^n$, let us define $x_{\max} := \max_{i\in[n]} x_i$ and $x_{\min} := \min_{i\in[n]} x_i$. Then for any $k \in \mathbb{N}_0$:

$$\begin{aligned} V_\pi(x,k) &\geq p^* \sum_{i=1}^n (x_i - \pi^T(k)x)^2 \\ &\geq p^*(x_{\max} - \pi^T(k)x)^2 + p^*(\pi^T(k)x - x_{\min})^2 \\ &\geq \frac{p^*}{2}(x_{\max} - x_{\min})^2, \end{aligned} \tag{4.46}$$

which follows from the fact that $a^2 + b^2 \geq \frac{(a+b)^2}{2}$. Also, since $x_{\min} \leq x_i, \pi^T(k)x \leq x_{\max}$, we have:

$$V_\pi(x,k) \leq \sum_{i=1}^n \pi_i(k)(x_{\max} - x_{\min})^2 = (x_{\max} - x_{\min})^2. \tag{4.47}$$

As a result, (4.47) and (4.46) together imply that

$$(p^*/2)^{\frac{1}{2}} d(x) \leq \sqrt{V_\pi(x,k)} \leq d(x). \tag{4.48}$$

$\square$

Lemma 30. *Let $q_0 \in \mathbb{N}_0$, and suppose that $\{A(t)\}_{t=0}^{\infty}$ is a B-connected chain satisfying $d(A(T_0 + rB : rB)x) \leq \alpha d(x)$ for all $x \in \mathbb{R}_n$, $r \in \mathbb{N}_0$ and $T_0 := (q_0+1)B$. Then the following holds for all $x \in \mathbb{R}_n$:*

$$d(A(t_2 : t_1)x) \leq \alpha^{\frac{t_2 - t_1}{T_0} - 2} d(x). \tag{4.49}$$

Proof. We are given that $d(A(T_0 + rB : rB)x) \leq \alpha d(x)$. In particular, when $r = u(q_0+1)$ for some $u \in \mathbb{N}_0$, we have $rB = uT_0$, and hence:

$$d(A((u+1)T_0 : uT_0)x) \leq \alpha d(x)$$

for all $x \in \mathbb{R}^n$ and $u \in \mathbb{N}_0$. By induction, we can show that

$$d(A((u+k)T_0 : uT_0)x) \leq \alpha^k d(x)$$

for all $x \in \mathbb{R}^n$ and $u, k \in \mathbb{N}_0$. Furthermore, since $\{A(t)\}_{t=0}^{\infty}$ is a stochastic chain, we have $d(A(k_2 : k_1)x) \leq d(x)$ for all $k_1, k_2 \in \mathbb{N}_0$ such that $k_1 \leq k_2$. It follows that for any $v, w \in [T_0]$, $k \in \mathbb{N}_0$ and $x \in \mathbb{R}^n$:

$$\begin{aligned}
&d(A(v + (u+k)T_0 : uT_0 - w)x) \\
&= d(A(v + (u+k)T_0) : (u+k)T_0) \cdot A((u+k)T_0 : uT_0) \cdot A(uT_0 : uT_0 - w)x) \\
&\leq d(A((u+k)T_0 : uT_0) \cdot A(uT_0 : uT_0 - w)x) \\
&\leq \alpha^k d(A(uT_0 : uT_0 - w)x) \leq \alpha^k d(x).
\end{aligned}$$

Now, if $v + w \geq T_0$, it is possible that $k < 0$ and yet $v + (u+k)T_0 \geq uT_0 - w$. However, since

$\alpha < 1$, in case $k < 0$, we have:

$$d(A(v+(u+k)T_0 : uT_0 - w)x) \leq d(x) \leq \alpha^k d(x),$$

which shows that

$$d(A(v+(u+k)T_0 : uT_0 - w)x) \leq \alpha^k d(x) \tag{4.50}$$

holds whenever $v+(u+k)T_0 \geq uT_0 - w$. Now, for any $t_1, t_2 \in \mathbb{N}_0$ such that $t_1 \leq t_2$, on setting $u = \lceil t_1/T_0 \rceil$, $k = \lfloor t_2/T_0 \rfloor - \lceil t_1/T_0 \rceil$, $v = t_2 - \lfloor t_2/T_0 \rfloor T_0$ and $w = \lceil t_1/T_0 \rceil T_0 - t_1$, we observe that $t_2 = v+(u+k)T_0$ and $t_1 = uT_0 - w$ with $v, w \in [T_0]$. Since $k \geq \frac{t_2 - t_1}{T_0} - 2$, we can express (4.50) compactly as:

$$d(A(t_2 : t_1)x) \leq \alpha^{\frac{t_2 - t_1}{T_0} - 2} d(x), \tag{4.51}$$

which completes the proof. □

Chapter 4, in full, is a reprint of the material as it appears in Rohit Parasnis, Massimo Franceschetti, and Behrouz Touri, "Non-Bayesian Social Learning on Random Digraphs with Aperiodically Varying Network Connectivity", in *IEEE Transactions on Control of Network Systems*, in press (2022). The dissertation author was the primary investigator and author of this paper.

Chapter 5

Usefulness of the Age-Structured SIR Dynamics in Modelling COVID-19

5.1 Introduction

The global COVID-19 death toll has crossed 6 million [137], and it is no surprise that researchers all over the world have been forecasting the evolution of this pandemic to propose control policies aimed at minimizing its medical and economic impacts [138–143]. Their efforts have typically relied on classical epidemiological models or their variants (for an overview see [144] and the references therein). One such classical epidemic model is the Susceptible-Infected-Recovered (SIR) model. Proposed in [145], the SIR model is a compartmental model in which every individual belongs to one of three possible states at any given time instant: the *susceptible* state, the *infected* state, and the *recovered* state. The continuous-time SIR dynamics models the time-evolution of the fraction of individuals in any of these states using a set of ordinary differential equations (ODEs) parameterized by two quantities: the infection rate (the rate at which a given infected individual infects a given susceptible individual) and the recovery rate of infected individuals.

Even though the continuous-time SIR model is a deterministic model, it models an inherently random phenomenon in a large (but discrete) population. To bridge between the deterministic continuous-time SIR model and the underlying random processes over a finite population, researchers have shown that the associated (continuous-time) ODEs are the mean-field limits of continuous-time Markovian epidemic processes over a finite population [146, 147]. Similar results have been obtained for variants of the original model, such as for the SIR dynamics on a configuration model network [148, 149]. These results theoretically justify the SIR model ODEs.

Classical SIR models, however, (continuous and discrete-time) are homogeneous – the same infection and recovery rates apply to the whole social network despite differences in the individuals' age, gender, race, immunity level, and pre-existing medical conditions. For COVID-19, this assumption is inconsistent with studies showing that the contact rates between individuals and the recovery rates of infected individuals depend on factors such as age and location [150–153]. In addition, [154] argues that homogeneous models can introduce significant biases in forecasting the epidemic, including overestimation of the number of infections required to achieve herd immunity, overestimation of the strictness of optimal control policies, overestimation of the impact of policy relaxations, and incorrect estimation of the time of onset of the pandemic.

We therefore need to shift our focus to variants of the classical SIR model with heterogeneous contact rates. Examples include the multi-risk SIR model [141] and the age-stratified SIR models considered in [140, 155, 156], in which the population is partitioned into multiple groups and the rates of infection and recovery vary across groups. See [154] for a survey of these papers.

However, the models considered in the above works have two main shortcomings. On the one hand, barring exceptions such as [157], they are typically not validated using real data. On the other hand, they do not have a strong theoretical foundation because the dynamical processes studied in these works have not been established as the mean field limits of stochastic epidemic processes evolving on time-varying random graphs. We emphasize that even the convergence results obtained for homogeneous SIR models [146–149] make the unrealistic assumption that the network of physical contacts (in-person interactions) existing in the population is time-invariant.

As such, we cannot justify the use of these models in designing optimal control policies aimed at minimizing the impact of any epidemic. We therefore address the aforementioned shortcomings using the *age-structured* SIR model, a multi-group SIR model that partitions the population of a given region into different age groups and assigns different infection rates and recovery rates to the age groups. We note that, although we adopt the term *age-structured* in this chapter, our analysis also applies to populations partitioned on the basis of differences in geographical location, sex, immunity level, etc. Moreover, among existing heterogeneous models [154], the age-structured SIR model is the simplest and hence more mathematically and computationally tractable than other models.

The contributions of this chapter are as follows:

1. ***Modeling**:* We extend our previously proposed stochastic epidemic model [158] to a more general model that incorporates (a) a random and time-varying network of physical contacts (in-person interactions between pairs of individuals) that are updated asynchronously and at random times, (b) random transmissions of disease-causing pathogens from infected individuals to their susceptible neighbors, and (c) recoveries of infected individuals that occur at random times. We analyze the resulting dynamics and show that under certain independence assumptions, the expected trajectories of the fractions of susceptible/infected/recovered individuals in any age group converge in mean-square to the solutions of the age-structured SIR ODEs as the population size goes to ∞.

2. ***Convergence Rate Analysis**:* We derive a lower bound on the effective infection rate for a given pair of age groups in the stochastic model. This bound, as we show, is approximately linear in the reciprocal of the network update rate, which leads to the infection rate converging to its limit (specified by the ODEs) as fast as the reciprocal of the network update rate vanishes.

3. ***Validation**:* We validate our age-structured model empirically by estimating the parameters of our model using a Japanese COVID-19 dataset and, subsequently, by generating the

age-wise numbers of infected individuals as functions of time. In this process, we leverage the crucial fact that the ODEs defining our model are linear in the model parameters (transmission and recovery rates), which enables us to use a least-squares method for the system identification.

4. ***A Method to Detect Changes in Social Behavior:*** We design a simple algorithm that can be used to detect changes in social behavior throughout the duration of the pandemic. Given the age-wise daily infection counts, the algorithm estimates the dates around which the inter-age-group contact rates change significantly.

5. ***Insights into Epidemic Spreading:*** We interpret the results of our phase detection algorithm to identify the least and the most infectious age groups and the least and the most vulnerable age groups. Additionally, we analyze the data for the entire period from March 2020 to April 2021 to explain how certain social events influenced the propagation of COVID-19 in the prefecture of Tokyo.

The structure of this chapter is as follows: We introduce the age-structured SIR model and our stochastic epidemic model in Section 5.2. We establish the age-structured SIR ODEs as the mean-field limits of our stochastic model in Section 5.3. We also discuss the limitations of (converse result for) our model in Section 5.3. Next, we describe the empirical validation of our model (in the context of the COVID-19 outbreak in Tokyo) in Section 5.5. We conclude with a brief summary and future directions in Section 5.6.

Related Works: [159] proposes a heterogeneous epidemic model with time-varying parameters to show that heterogeneous susceptibility to infection results in a temporary weakening of the COVID-19 pandemic but not in herd immunity. The model is validated using the death tolls (and not the case numbers) reported for New York and Chicago for a period of about 80 days. [155] uses the age-structured SEIQRD model to predict the number of deaths with a reasonable accuracy, but unlike our work, it does not use the proposed model to generate the number of new cases as a function of time. [160] uses heterogeneous variants of the SEIR model to study the impact of the lockdown policy implemented in France, but it does not validate these models empirically. [152] reports contact rate matrices for the population of the UK based on the self-reported data of 36,000 volunteers. However, the study ignores the time-varying nature of these contact rates, which we capture in our phase detection algorithm (Section 5.5). Another study that uses time-invariant model parameters is [161], which proposes the age-structured SEIRA model and uses it to simulate the number of new infections in different social groups of Chile.

[162] uses a heterogeneous SEIRD model to predict the effects of various relaxation policies on infection counts in certain regions of Italy. The model therein is empirically validated only using the data obtained during the first 60 days of the pandemic. In [156], the authors propose an age-structured SIRD model and calibrate it with the data obtained from [138]. Unlike this chapter, however, [156] divides the population into only two age groups, and does not compare the model-generated values of the number of infections with the official case counts. Two other studies that use two-age-group SIR models are [163] and [164]. While [163] argues that in Florida, old and socially inert adults have been possibly infected by the young, [164] argues that age-group-targeted policies are more effective than uniform policies in reducing the economic impact of COVID-19. [165] proposes a heterogeneous SIR model with feedback and forecasts the economic and medical impacts of various policies aimed at controlling the pandemic in Chile. Unlike our study, however, [165] ignores the time-varying nature of contact rates. [157] proposes the SEIR-HC-SEC-AGE model, a heterogeneous SEIR model that sub-divides each age-group further into risk sectors with different vulnerabilities to the SARS-CoV-2 virus. The model therein, which is calibrated to predict the effects of different lockdown policies in certain regions of Italy, simulates the time-evolution of the observed death toll with a high accuracy. By contrast, we pick a much simpler heterogeneous model and examine whether it fits the observed case numbers well. [140] and [141] use an age-structured SIR model to show that control policies that target different age groups differently perform better than uniform policies. However, these results assume that inter-age-group contact rates are the same for all pairs of age groups, an

assumption that is inconsistent with our empirical results (Section 5.5). Hence, deriving optimal policies in the framework of the age-structured SIR model under more general assumptions is an important open problem.

Notation: We let $\mathbb{N}$ denote the set of natural numbers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We let $[l] := \{1, 2, \ldots, \ell\}$ for $\ell \in \mathbb{N}$. We denote the set of real and positive real numbers by $\mathbb{R}$ and $\mathbb{R}_+$, respectively. For $x \in \mathbb{R}$, we let $x_+ := \max\{x, 0\}$ denote the positive part of x.

The symbols t and k are used as a continuous-time and discrete-time indices, respectively. We use the notation $z(t)$ for functions $z : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}$ and $z[k]$ for functions $z : \mathbb{N} \to \mathbb{R}$. We occasionally omit the time index (t) when the value of t is clear from the context.

We use the Bachmann-Landau asymptotic notation $O(f(n))$ for a given function $f : \mathbb{N} \to \mathbb{R}$ in the context of $n \to \infty$. We use $o(\Delta t)$ in the context of $\Delta t \to 0$. In addition, for a given function $g : [0, \infty) \to \mathbb{R}$, we use the notation $g' = g'(t)$ to denote $\frac{dg}{dt}$, the first derivative of g with respect to time.

For a set $\mathcal{S}$, we let $|\mathcal{S}|$ denote the cardinality of $\mathcal{S}$. In this chapter, all random events and random variables are with respect to a probability space $(\Omega, \mathcal{F}, \text{Pr})$, where Ω is the sample space, $\mathcal{F}$ is the set of events, and $\text{Pr}(\cdot)$ is the probability measure on this space. We denote random variables and random events using capital letters, and for a random event C, we define 1_C to be the indicator random variable associated with C, i.e, $1_C : \Omega \to \mathbb{R}$ is the random variable with $1_C(\omega) = 1$ if $\omega \in C$ and $1_C(\omega) = 0$, otherwise. For an event $C \in \mathcal{F}$, $\bar{C}$ represents the complement of C. For a random variable X, $\mathbb{E}[X]$ denotes the expected value of X and $\mathbb{E}[X \mid C]$ denotes the conditional expectation of X given the event C. For random variables X and Y and a random event C, we define

$$\mathbb{E}[X \mid Y, C] = \frac{\mathbb{E}[X 1_C \mid Y]}{\mathbb{E}[1_C \mid Y]}.$$

Therefore, for an event $F \in \mathcal{F}$

$$\Pr(F \mid Y, C) = \mathbb{E}[1_F \mid Y, C] = \frac{\mathbb{E}[1_{F\cap C} \mid Y]}{\mathbb{E}[1_C \mid Y]}.$$

We denote tuples of length $r > 1$ using bold-face letters and random tuples using bold-face capital letters. For a tuple $\mathbf{x}$ of length $r \in \mathbb{N}$ and an index $\ell \in [r]$, we let $x_\ell = (\mathbf{x})_\ell$ denote the ℓ-th entry of $\mathbf{x}$.

For $n \in \mathbb{N}$ and $E \subset [n] \times [n]$, we use $G = ([n], E)$ to denote the directed graph (digraph) with vertex set $[n]$ and edge set E. Finally, for a graph $G = ([n], E)$, given two distinct nodes $a, b \in [n]$, we let $\langle a, b\rangle := (a-1)(n-1) + b - \chi_{b-a}$, where $\chi_\alpha = 1$ if $\alpha > 0$ and $\chi_\alpha = 0$, otherwise. Note that $\langle \cdot, \cdot \rangle$ maps the edges between (distinct) nodes of the graph to the numbers $1, \dots, n^2 - n$ in lexicographic order.

5.2 Problem Formulation

We now introduce two epidemic models, of which the first describes a deterministic dynamical system and the second describes a stochastic process on a finite population. One of the main objectives of this work is to relate these models, which is achieved in Section 5.3.

5.2.1 The Age-Structured SIR Model

Consider a population of individuals spanning m age groups[1]. Suppose a part of this population contracts a communicable disease at time $t = 0$. Let $s_i(t), \beta_i(t)$, and $r_i(t)$ denote, respectively, the fractions of susceptible, infected, and recovered individuals in the i-th age group at (a continuous) time $t \geq 0$, so that $s_i(t) + \beta_i(t) + r_i(t)$ equals the fraction of individuals in the i-th age group for all $t \geq 0$. As the disease spreads across the population, susceptible individuals

[1]As mentioned before, throughout this chapter, we could generalize the discussions involving *age groups* to subpopulations distinguished by geographical locations, pre-existing health conditions, sex, etc.

get infected, and infected individuals recover in accordance with the system of ODEs given by

$$\begin{aligned}\dot{s}_i(t) &= -s_i(t)\sum_{j=1}^{m} A_{ij}\beta_j(t),\\ \dot{\beta}_i(t) &= s_i(t)\sum_{j=1}^{m} A_{ij}\beta_j(t) - \gamma_i\beta_i(t),\\ \dot{r}_i(t) &= \gamma_i\beta(t),\end{aligned} \tag{5.1}$$

where for each $i, j \in [m]$, the constant A_{ij} represents the rate of infection transmission from an individual in age group j to an individual in age group i, and γ_i denotes the recovery rate of an infected individual in age group i. Hereafter, we refer to A_{ij} as the *contact rate of age group j with age group i*. Note that the third equation in (5.1) can be obtained from the first two equations simply by using the fact that $\dot{s}_i(t) + \dot{\beta}_i(t) + \dot{r}_i(t) = 0$ for all $t \geq 0$. Also, if $m = 1$, the above model reduces to the classical (homogeneous and continuous-time) SIR model.

5.2.2 A Stochastic Epidemic Model

Let us now define a continuous-time Markov chain that describes an age-structured process of epidemic spreading occurring over a finite (atomic) population composed of individuals that are connected through a random, time-varying network $G(t)$.

Age Groups

Let $n \in \mathbb{N}$ denote the total population size, and let $[n]$ be the vertex set of the time-varying graph $G(t)$, so that the vertex set indexes all the individuals/nodes in the network. We assume that $[n]$ is partitioned into m age groups $\{\mathcal{A}_i\}_{i=1}^{m}$ and that $|\mathcal{A}_i|$ (the number of individuals in the i-th age group) scales linearly with n for all $i \in [m]$. In the following, $i, j \in [m]$ are generic age group indices.

State Space

The state space of our random process is the space $\mathbb{S} = \{-1,0,1\}^n \times \{0,1\}^{2n(n-1)}$. The network state is a tuple $\mathbf{x} = (x_1, x_2, \ldots, x_{2n^2-n}) \in \mathbb{S}$, where

(i) $\{x_\ell\}_{\ell \in [n]}$ denotes the disease states of the nodes in the network, i.e., for $\ell \in [n]$, we set $x_\ell = 0$, 1, or -1 accordingly as node ℓ is susceptible, infected, or recovered, respectively.

(ii) For $\ell \in \{n+1, n+2, \ldots, n^2\}$, we let x_ℓ denote the *edge state* of the ℓ-th pair in the lexicographic order of pairs of distinct nodes given below.

$$(1,2), \ldots, (1,n), (2,1), \ldots, (2,n), \ldots, (n,1), \ldots, (n,n-1)$$

In other words, for any node pair $(a,b) \in [n] \times [n]$ such that $a \neq b$, we set $x_{\langle a,b \rangle} = 1$ if there is a directed edge from b to a in the network G, and $x_{\langle a,b \rangle} = 0$, otherwise. For notational convenience, we let $1_{(a,b)}(\mathbf{x}) := x_{\langle a,b \rangle}$.

(iii) For $\ell \in \{n^2+1, \ldots, 2n^2-n\}$, we let x_ℓ be a binary variable whose value flips (becomes $1 - x_\ell$) whenever the $(\ell - n^2)$-th edge state gets updated (re-initialized). However, the direction of this flip (whether x_ℓ changes from 0 to 1 or from 1 to 0) carries no significance.

State Attributes

For all $\mathbf{x} \in \mathbb{S}$, we let $\mathcal{S}_i(\mathbf{x}) := \{a \in \mathcal{A}_i : x_a = 0\}$, $\mathcal{I}_i(\mathbf{x}) := \{a \in \mathcal{A}_i : x_a = 1\}$, and $\mathcal{R}_i(\mathbf{x}) := \{a \in \mathcal{A}_i : x_a = -1\}$ denote, respectively, the set of susceptible individuals, the set of infected individuals, and the set of recovered individuals in $\mathcal{A}_i$ given that the network state is $\mathbf{x}$. We let $\mathcal{S}(\mathbf{x}) := \cup_{i=1}^m \mathcal{S}_i(\mathbf{x})$ and $\mathcal{I}(\mathbf{x}) := \cup_{i=1}^m \mathcal{I}_i(\mathbf{x})$. Additionally, for every node $a \in [n]$, we let $E_j^{(a)}(\mathbf{x}) := \sum_{c \in \mathcal{I}_j(\mathbf{x})} 1_{(a,c)}(\mathbf{x})$ be the number of arcs from $\mathcal{I}_j(\mathbf{x})$ to a.

The Markov Process

Let $\mathbf{X}(t) \in \mathbb{S}$ denote the state of the network at any time $t \geq 0$. Then we assume that $\{\mathbf{X}(t) : t \geq 0\}$ is a right-continuous time-homogeneous Markov process in which every transition from a state $\mathbf{x} \in \mathbb{S}$ to a state $\mathbf{y} \in \mathbb{S} \setminus \{\mathbf{x}\}$ belongs to one of the following categories:

1. *Infection transition:* This occurs when a node $a \in \mathcal{S}_i(\mathbf{x})$ gets infected by a node in $\cup_{k=1}^m \mathcal{I}_k(\mathbf{x})$, while the disease states of all other nodes and the edge states of all the node pairs remain the same. In other words, $x_a = 0$, $y_a = 1$, and $x_\ell = y_\ell$ for all $\ell \neq a$. Denoting the state-independent rate of pathogen transmission from a node in $\mathcal{I}_k(\mathbf{x})$ to an adjacent node in $\mathcal{S}_i(\mathbf{x})$ by B_{ik}, we note that the rate of infection transmission from any node $c \in \mathcal{I}_k(\mathbf{x})$ to a is $B_{ik} 1_{(a,c)}(\mathbf{x})$. Hence, the total rate at which a receives pathogens from $\mathcal{I}_k$ is $\sum_{c \in \mathcal{I}_k(\mathbf{x})} B_{ik} 1_{(a,c)}(\mathbf{x}) = B_{ik} E_k^{(a)}(\mathbf{x})$, assuming that different edges transmit the infection independently of each other during vanishingly small time intervals. As a result, the effective rate at which a gets infected is $\sum_{k=1}^m B_{ik} E_k^{(a)}(\mathbf{x})$. We denote the successor state $\mathbf{y}$ of $\mathbf{x}$, where the node a turns from susceptible to infected, by $\mathbf{x}_{\uparrow a}$.

2. *Recovery transition:* This occurs when a node $a \in \mathcal{I}_i(\mathbf{x})$ recovers, i.e., $x_a = 1$, $y_a = -1$, and $x_\ell = y_\ell$ for all $\ell \neq a$. We let γ_i denote the rate at which an infected node in $\mathcal{A}_i$ (such as a) recovers. For such a transition, we denote $\mathbf{y} = \mathbf{x}_{\downarrow a}$.

3. *Edge update transition:* This occurs when $x_{\langle a,b \rangle}$, the edge state of a node pair $(a,b) \in \mathcal{A}_i \times \mathcal{A}_j$, is updated or re-initialized, i.e., $y_{n^2+\langle a,b \rangle} = 1 - x_{n^2+\langle a,b \rangle}$, and $y_\ell = x_\ell$ for all $\ell \notin \{\langle a,b \rangle, n^2 + \langle a,b \rangle\}$. We let λ denote the *edge update rate* or the rate at which an edge state is updated. In addition, given that the edge state of (a,b) is updated at time $t \geq 0$, the probability that $1_{(a,b)}(t) = 1$ (i.e., the edge (a,b) exists after the re-initialization) equals $\frac{\rho_{ij}}{n}$, where $\rho_{ij} > 0$ is constant in time. Therefore, if $y_{\langle a,b \rangle} = 1$ (meaning that (a,b) exists as an arc in G in the network state $\mathbf{y}$), then the rate of transition from $\mathbf{x}$ to $\mathbf{y}$ equals $\lambda \frac{\rho_{ij}}{n}$, whereas if $y_{\langle a,b \rangle} = 0$, then the rate of transition from $\mathbf{x}$ to $\mathbf{y}$ equals $\lambda \left(1 - \frac{\rho_{ij}}{n}\right)$. In the former case, we write $\mathbf{y} = \mathbf{x}_{\uparrow(a,b)}$, while in the latter case, we write $\mathbf{y} = \mathbf{x}_{\downarrow(a,b)}$. Note that the rate of transition from $\mathbf{x}$ to $\mathbf{x}_{\downarrow(a,b)}$ or $\mathbf{x}_{\uparrow(a,b)}$ does not depend on $\mathbf{x}$.

   The edge update transition of (a,b) can be described informally as follows. Throughout the evolution of the pandemic, a and b decide whether or not to meet each other at a constant rate $\lambda > 0$, i.e., their decision times $\{T_\ell^{(a,b)}\}_{\ell=1}^\infty$ form a Poisson process with rate λ. Each time they make such a decision, they decide to interact with probability $\frac{\rho_{ij}}{n}$, and they decide not to interact with probability $1-\frac{\rho_{ij}}{n}$, independently of their past decisions. The probability of interaction is assumed to scale inversely with n so that the mean degree of every node is constant with respect to n.

To summarize, the rate of transition from any state $\mathbf{x} \in \mathbb{S}$ to any state $\mathbf{y} \in \mathbb{S} \setminus \{\mathbf{x}\}$ is given by $\mathbf{Q}$, the infinitesimal generator of the Markov chain $\{\mathbf{X}(t) : t \geq 0\}$, where for $\mathbf{x} \neq \mathbf{y}$

$$\mathbf{Q}(\mathbf{x},\mathbf{y}) := \begin{cases} \sum_{k=1}^m B_{ik} E_k^{(a)}(\mathbf{x}) & \text{if } \mathbf{y} = \mathbf{x}_{\uparrow a} \text{ for some } a \in \mathcal{S}_i(\mathbf{x}), i \in [m] \\ \gamma_i & \text{if } \mathbf{y} = \mathbf{x}_{\downarrow a} \text{ for some } a \in \mathcal{I}_i(\mathbf{x}), i \in [m] \\ \lambda \frac{\rho_{ij}}{n} & \text{if } \mathbf{y} = \mathbf{x}_{\uparrow (a,b)} \text{ for some } (a,b) \in \mathcal{A}_i \times \mathcal{A}_j, i,j \in [m] \\ \lambda \left(1 - \frac{\rho_{ij}}{n}\right) & \text{if } \mathbf{y} = \mathbf{x}_{\downarrow (a,b),} \text{ for some } (a,b) \in \mathcal{A}_i \times \mathcal{A}_j, i,j \in [m] \\ 0 & \text{otherwise} \end{cases},$$

and $\mathbf{Q}(\mathbf{x},\mathbf{x}) := -\sum_{\mathbf{y} \in \mathbb{S} \setminus \{\mathbf{x}\}} \mathbf{Q}(\mathbf{x},\mathbf{y})$. In addition, we say that $\mathbf{y}$ *succeeds* $\mathbf{x}$ *potentially* iff $\mathbf{Q}(\mathbf{x},\mathbf{y}) > 0$.

5.3 Main Result

To provide a rigorous mean-field derivation of the dynamics (5.1), we now consider a *sequence* of social networks with increasing population sizes such that each network obeys the theoretical framework described in Section 5.2. Given a network from this sequence with population size $n \in \mathbb{N}$, we let $\mathcal{S}_j^{(n)}(t) := \mathcal{S}_j(\mathbf{X}(t))$, $\mathcal{I}_j^{(n)}(t) := \mathcal{I}_j(\mathbf{X}(t))$, and $\mathcal{R}_j^{(n)}(t) := \mathcal{R}_j(\mathbf{X}(t))$ denote the (random) sets of infected, susceptible, and infected individuals in the j-th age group, respectively, and we let $s_j^{(n)}(t) := \frac{1}{n}|\mathcal{S}_j^{(n)}(t)|$, $\beta_j^{(n)}(t) := \frac{1}{n}|\mathcal{I}_j^{(n)}(t)|$ and $r_j^{(n)}(t) := \frac{1}{n}|\mathcal{R}_j^{(n)}(t)|$ denote the fractions of susceptible, infected, and recovered individuals in the j-th age group, respectively. As for the absolute numbers, we let $S_j^{(n)}(t) := |\mathcal{S}_j^{(n)}(t)|$, $I_j^{(n)}(t) := |\mathcal{I}_j^{(n)}(t)|$, and

$R_j^{(n)}(t) := |\mathcal{R}_j^{(n)}(t)|$. Additionally, we let $E^{(n)}(t)$ denote the edge set of the network at time t, and we drop the superscript $^{(n)}$ when the context makes our reference to the n-th network clear.

Another quantity that varies with n is $\lambda^{(n)}$, the edge update rate. To obtain the desired mean-field limit in Theorem 7, we assume that $\lambda^{(n)} \to \infty$ as $n \to \infty$. To interpret this assumption, consider any pair of individuals $(a,b) \in \mathcal{A}_i \times \mathcal{A}_j$ that are in contact with each other at time $t \geq 0$ during the epidemic. Since the edge state of (a,b) is updated to 0 (the state of non-existence) at a time-invariant rate of $\lambda^{(n)}\left(1 - \frac{\rho_{ij}}{n}\right)$, the assumption implies that the mean interaction time of b with a, which is $\frac{1}{\lambda^{(n)}} + O\left(\frac{1}{n\lambda^{(n)}}\right)$, vanishes as the population size increases. This is a possible real-world scenario, because as n increases, the population density of the given geographical region increases, which could result in overcrowding and rapidly changing interaction patterns in the network. This may be especially true in the case of public places such as supermarkets and subway stations at a time when the society is already aware of an evolving epidemic. Another implication of $\lim_{n\to\infty} \lambda^{(n)} = \infty$ is that the rate at which a given infected node contacts and transmits pathogens to a given susceptible node vanishes as the population size goes to ∞ (see Remark 13 for an explanation). This implication is weaker than the often-assumed condition that the rate of pathogen transmission is proportional to the reciprocal of the population size [166, 167].

We are now ready to state our main result. Its proof is based on the theory of continuous-time Markov chains and an analysis of how the disease propagation process is affected by random updates occurring in the network structure at random times (which results in Propositions 12 and 13) in addition to the proof techniques used in [166]. The proofs of all these results are available in the appendix.

Theorem 7. *Suppose that* $\lim_{n\to\infty} \lambda^{(n)} = \infty$ *and that for every* $i \in [m]$*, there exist* $s_{i,0}, \beta_{i,0} \in [0,1]$ *such that* $\lim_{n\to\infty} s_i^{(n)}(0) = s_{i,0}$ *and* $\lim_{n\to\infty} \beta_i^{(n)}(0) = \beta_{i,0}$*. Then for each* $i \in [m]$*,*

$$\lim_{n\to\infty} \mathbb{E}\left[\left\|\left(s_i^{(n)}(t), \beta_i^{(n)}(t)\right) - (y_i(t), w_i(t))\right\|_2^2\right] = 0.$$

on any finite time interval $[0,T_0]$*, where* $(y_i(t),w_i(t))$ *is the solution to the ODE system given by the first two equations in* (5.1)*, i.e.,* $(y_i(t),w_i(t))$ *satisfies*

(I) $\dot{y}_i = -y_i \sum_{j=1}^{m} A_{ij} w_j, \quad y_i(0) = s_{i,0},$

(II) $\dot{w}_i = y_i \sum_{j=1}^{m} A_{ij} w_j - \gamma_i w_i, \quad w_i(0) = \beta_{i,0},$

and $A \in \mathbb{R}^{m\times m}$ *is defined by* $A_{ij} := \rho_{ij} B_{ij}$*.*

Theorem 7 relies on the following proposition.

Proposition 12. *For each* $i,j \in [m]$*, let*

$$\chi_{ij} = \chi_{ij}(t,\mathcal{S},\mathcal{I}) := \mathbb{E}[1_{(a,b)}(t) \mid \mathcal{S}(t),\mathcal{I}(t)] = \Pr((a,b) \in E(t) \mid \mathcal{S}(t),\mathcal{I}(t))$$

be the random variable that denotes the conditional probability that a pair of nodes $(a,b) \in \mathcal{S}_i(t) \times \mathcal{I}_j(t)$ *are in physical contact at time* t *given the state of the network at time* t*. Then the following equations hold for all* $t \geq 0$*:*

(i) $\mathbb{E}[s_i]' = -\sum_{j=1}^{m} B_{ij}\mathbb{E}[n\chi_{ij}s_i\beta_j],$

(ii) $\mathbb{E}[\beta_i]' = \sum_{j=1}^{m} B_{ij}\mathbb{E}[n\chi_{ij}s_i\beta_j] - \gamma_i\mathbb{E}[\beta_i],$

(iii) $\mathbb{E}[s_i^2]' = -\sum_{j=1}^{m}\left(2B_{ij}\mathbb{E}[n\chi_{ij}s_i^2\beta_j] - B_{ij}\mathbb{E}[n\chi_{ij}s_i\beta_j]/n\right),$

(iv) $\mathbb{E}[\beta_i^2]' = \sum_{j=1}^{m} B_{ij}(2\mathbb{E}[n\chi_{ij}s_i\beta_j\beta_i] + \mathbb{E}[n\chi_{ij}s_i\beta_j]/n) - \gamma_i\left(2\mathbb{E}[\beta_i^2] - \mathbb{E}[\beta_i]/n\right).$

Proof. We derive the equations one by one.

Proof of (i).

Suppose $i,j \in [m]$ and $t \geq 0$ are given. Let $W_i(t) := \min\{\ell : \ell \in \mathcal{S}_i(t)\}$ and $X_j(t) := \min\{\ell : \ell \in \mathcal{I}_j(t)\}$. Then, as per our model of disease propagation, for any $\Delta t > 0$, we have

$$\begin{aligned}\Pr(\cup_{\tau\in[t,t+\Delta t)}\{X_j(t) \overset{\tau}{\rightsquigarrow} W_i(t)\}\\ \mid (W_i(t),X_j(t)) \in \cap_{\tau\in[t,t+\Delta t)}E(\tau),\mathcal{S},\mathcal{I})\\ = B_{ij}\Delta t + o(\Delta t),\end{aligned} \tag{5.2}$$

wherein we have used the Bachmann-Landau asymptotic notation $o(x)$, which means that

$$\lim_{x\to 0}\frac{o(x)}{x} = 0.$$

Before we proceed, we define the random variable $\chi_{ij}(t,\mathcal{S},\mathcal{I})$ to be the following conditional probability:

$$\chi_{ij}(t,\mathcal{S},\mathcal{I}) = \Pr((W_i(t),X_j(t)) \in E(t) \mid \mathcal{S}(t),\mathcal{I}(t)).$$

Observe that $\chi_{ij}(t,\mathcal{S},\mathcal{I})$ is a conditional probability that there exists an edge at time t between the susceptible node of $\mathcal{A}_i$ with the smallest index and the infected node of $\mathcal{A}_j$ with the smallest index. However, recall from Section 5.2 that the probabilities of disease transmission and edge existence are independent of how the nodes are labelled *within* their respective age groups. Hence, $\chi_{ij}(t,\mathcal{S},\mathcal{I})$ is simply the conditional probability that *any* two nodes $a \in \mathcal{S}_i(t)$ and $b \in \mathcal{I}_j(t)$ are in contact with each other at time t, given that $\mathcal{S}(t)$ and $\mathcal{I}(t)$ are known.

Now, suppose Δt is small enough so that $[t,t+\Delta t) \subset [kT,(k+1)T)$ for some $k \in \mathbb{N}_0$.

It then follows that $E(\tau)$ is constant for $\tau \in [t, t+\Delta t)$ and hence,

$$\begin{aligned}
&\Pr((W_i(t), X_j(t)) \in \cap_{\tau \in [t,t+\Delta t)} E(\tau) \mid \mathcal{S}(t), \mathcal{I}(t)) \\
&= \Pr((W_i(t), X_j(t)) \in E(t) \mid \mathcal{S}(t), \mathcal{I}(t)) = \chi_{ij}. \qquad (5.3)
\end{aligned}$$

Another implication of $[t, t+\Delta t) \subset [kT, (k+1)T)$ is that

$$\begin{aligned}
\cup_{\tau \in [t,t+\Delta t)} \{X_j(t) \overset{\tau}{\rightsquigarrow} W_i(t)\} &\subset \cup_{\tau \in [t,t+\Delta t)]} \{(W_i(t), X_j(t)) \in E(\tau)\} \\
&= \{(W_i(t), X_j(t)) \in \cap_{\tau \in [t,t+\Delta t)} E(\tau)\}.
\end{aligned}$$

Therefore, on combining (5.2) and (5.3) we obtain

$$\Pr(\cup_{\tau \in [t,t+\Delta t)} \{X_j(t) \overset{\tau}{\rightsquigarrow} W_i(t)\} \mid \mathcal{S}(t), \mathcal{I}(t)) = B_{ij} \chi_{ij} \Delta t + o(\Delta t).$$

By the label symmetry arguments discussed above, the above implies that for any node pair $(a, b) \in \mathcal{S}_i(t) \times \mathcal{I}_j(t)$,

$$\Pr(\cup_{\tau \in [t,t+\Delta t)} \{b \overset{\tau}{\rightsquigarrow} a\} \mid \mathcal{S}(t), \mathcal{I}(t)) = B_{ij} \chi_{ij} \Delta t + o(\Delta t). \qquad (5.4)$$

Next, we evaluate the probability of multiple transmissions occurring during $[t, t+\Delta t)$ as shown below.

$$\begin{aligned}
&\Pr\left(\cup_{\tau_1,\tau_2 \in [t,t+\Delta t)} (\{b_1 \overset{\tau_1}{\rightsquigarrow} a_1\} \cap \{b_2 \overset{\tau_2}{\rightsquigarrow} a_2\}) \mid \mathcal{S}, \mathcal{I}\right) \\
&= \Pr\left(\cup_{\tau_1,\tau_2 \in [t,t+\Delta t)} (\{b_1 \overset{\tau_1}{\rightsquigarrow} a_1\} \cap \{b_2 \overset{\tau_2}{\rightsquigarrow} a_2\}) \mid (a_1, b_1), (a_2, b_2) \in \cap_{\tau \in [t,t+\Delta t)} E(\tau), \mathcal{S}, \mathcal{I}\right) \\
&\quad \times \Pr((a_1, b_1), (a_2, b_2) \in \cap_{\tau \in [t,t+\Delta t)} E(\tau) \mid \mathcal{S}, \mathcal{I}) \\
&\le \Pr\left(\cup_{\tau_1,\tau_2 \in [t,t+\Delta t)} (\{b_1 \overset{\tau_1}{\rightsquigarrow} a_1\} \cap \{b_2 \overset{\tau_2}{\rightsquigarrow} a_2\}) \mid (a_1, b_1), (a_2, b_2) \in \cap_{\tau \in [t,t+\Delta t)} E(\tau), \mathcal{S}, \mathcal{I}\right).
\end{aligned}$$

In continuation of the above sequence of inequalities, we have

$$\begin{aligned}
&\Pr\left(\cup_{\tau_1,\tau_2\in[t,t+\Delta t)}\left(\{b_1 \overset{\tau_1}{\rightsquigarrow} a_1\}\cap\{b_2 \overset{\tau_2}{\rightsquigarrow} a_2\}\right) \mid (a_1,b_1),(a_2,b_2)\in\cap_{\tau\in[t,t+\Delta t)}E(\tau),\mathcal{S},\mathcal{I}\right)\\
&\overset{(a)}{=} \Pr(\cup_{\tau\in[t,t+\Delta t)}\{b_1 \overset{\tau}{\rightsquigarrow} a_1\} \mid (a_1,b_1)\in\cap_{\tau\in[t,t+\Delta t)}E(\tau),\mathcal{S},\mathcal{I})\\
&\quad\times\Pr(\cup_{\tau\in[t,t+\Delta t)}\{b_2 \overset{\tau}{\rightsquigarrow} a_2\} \mid (a_2,b_2)\in\cap_{\tau\in[t,t+\Delta t)}E(\tau),\mathcal{S},\mathcal{I})\\
&=(B_{ij}\Delta t+o(\Delta t))^2\\
&=o(\Delta t),
\end{aligned}$$

where (a) follows from Assumption V. We now use (5.4) along with the inclusion-exclusion principle to obtain the following:

$$\begin{aligned}
&\Pr(\cap_{\tau\in[t,t+\Delta t)}\{\mathcal{I}_j \overset{\tau}{\rightsquigarrow} \mathcal{S}_i\} \mid \mathcal{S},\mathcal{I})\\
&=\sum_{(a,b)\in\mathcal{S}_i\times\mathcal{I}_j}\Pr(\cap_{\tau\in[t,t+\Delta t)}\{b \overset{\tau}{\rightsquigarrow} a\} \mid \mathcal{S},\mathcal{I})\\
&\quad-\sum_{(a_1,b_1),}\ \sum_{(a_2,b_2)\in\mathcal{S}_i\times\mathcal{I}_j}\Pr\Big(\cup_{\tau\in[t,t+\Delta t)}\big(\{b_1 \overset{\tau}{\rightsquigarrow} a_1\}\\
&\qquad\qquad\cap\{b_2 \overset{\tau}{\rightsquigarrow} a_2\}\big) \mid \mathcal{S},\mathcal{I}\Big)+\cdots\\
&\overset{(a)}{=}\sum_{(a,b)\in\mathcal{S}_i\times\mathcal{I}_j}(B_{ij}\chi(t,\mathcal{S},\mathcal{I})\Delta t+o(\Delta t))\\
&\quad-\sum_{(a_1,b_1),}\ \sum_{(a_2,b_2)\in\mathcal{S}_i\times\mathcal{I}_j}(B_{ij}\chi(t,\mathcal{S},\mathcal{I})\Delta t+o(\Delta t))^2+\cdots\\
&=S_iI_j\left(B_{ij}\chi(t,\mathcal{S},\mathcal{I})\Delta t+o(\Delta t)\right)-o(\Delta t)+o(\Delta t)-\cdots\\
&=B_{ij}\chi_{ij}S_iI_j\Delta t+o(\Delta t),
\end{aligned}\tag{5.5}$$

where (a) follows from the assumption that different edges transmit independently of each other during vanishingly small time intervals. By using a similar argument based on the inclusion-exclusion principle, we can invoke the same assumption to show that

$$
\begin{aligned}
&\Pr(\mathcal{S}_i \text{ is infected during } [t,t+\Delta t) \mid \mathcal{S},\mathcal{I}) \\
&= \Pr\left(\cup_{j=1}^{m} \cup_{\tau\in[t,t+\Delta t)} \{\mathcal{I}_j \overset{\tau}{\rightsquigarrow} \mathcal{S}_i\} \mid \mathcal{S},\mathcal{I}\right) \\
&= \sum_{j=1}^{m} \Pr\left(\cup_{\tau\in[t,t+\Delta t)}\{\mathcal{I}_j \overset{\tau}{\rightsquigarrow} \mathcal{S}_i\} \mid \mathcal{S},\mathcal{I}\right) + o(\Delta t) \\
&= \sum_{j=1}^{m} B_{ij}\chi_{ij} S_i I_j \Delta t + o(\Delta t).
\end{aligned} \tag{5.6}
$$

Now, we need to use the above expressions to compute the expected decrease in the number of susceptible individuals over a small time interval. To this end, we first let $\Delta_t S_i := S_i(t+\Delta t) - S_i(t)$ and observe that it is unlikely for more than one susceptible individual to be infected during a small time interval:

$$
\Pr\left(\cap_{l=1}^{d}\left(\cup_{\tau\in[t,t+\Delta t)}\{b_\ell \overset{\tau}{\rightsquigarrow} a_\ell\}\right) \mid \mathcal{S},\mathcal{I}\right) = \prod_{l=1}^{d}\left(O(\Delta t) + o(\Delta t)\right) = o(\Delta t) \quad \text{for all} \quad d \geq 2. \tag{5.7}
$$

Thus,

$$
\Pr(\mathcal{S}_i \text{ is infected during } [t,t+\Delta t) \mid \mathcal{S},\mathcal{I}) - \Pr(\Delta_t S_i = -1 \mid \mathcal{S},\mathcal{I}) = o(\Delta t).
$$

Consequently,

$$
\begin{aligned}
\mathbb{E}[S_i(t+\Delta t) - S_i(t) \mid \mathcal{S},\mathcal{I}] &= (-1)\cdot \Pr(\Delta_t S_i = -1 \mid \mathcal{S},\mathcal{I}) + \sum_{j=2}^{S_i(t)} (-j)\cdot \Pr(\Delta_t S_i = -j \mid \mathcal{S},\mathcal{I}) \\
&= -\left(\sum_{j=1}^{m} B_{ij}\chi(t,\mathcal{S},\mathcal{I}) S_i I_j \Delta t + o(\Delta t)\right) + \sum_{j=2}^{S_i(t)} (-j)\cdot o(\Delta t).
\end{aligned}
$$

Since $S_i(t) \le n < \infty$, we have

$$\begin{aligned}
\mathbb{E}[S_i(t+\Delta t) - S_i(t) \mid \mathcal{S},\mathcal{I}] &\overset{(a)}{=} -\left(\sum_{j=1}^{m} B_{ij}\chi(t,\mathcal{S},\mathcal{I}) S_i I_j \Delta t + o(\Delta t)\right) + o(\Delta t) \\
&= -\sum_{j=1}^{m} B_{ij}\chi_{ij} S_i I_j \Delta t + o(\Delta t),
\end{aligned} \tag{5.8}$$

Taking expectations on both sides of (5.8) and dividing by Δt results in the following:

$$\mathbb{E}\left[\frac{S_i(t+\Delta t) - S_i(t)}{\Delta t}\right] = -\sum_{j=1}^{m} B_{ij}\mathbb{E}[\chi_{ij} S_i I_j] + \frac{o(\Delta t)}{\Delta t}. \tag{5.9}$$

Thus,

$$\begin{aligned}
(\mathbb{E}[S_i(t)])' &= \lim_{\Delta t \to 0} \frac{\mathbb{E}[S_i(t+\Delta t)] - \mathbb{E}[S_i(t)]}{\Delta t} \\
&= -\frac{1}{n}\sum_{j=1}^{m} B_{ij}\mathbb{E}[n\chi_{ij} S_i I_j].
\end{aligned}$$

Dividing both sides of the above by n yields (i).

Proof of (ii).

For any time $t \in [0,\infty)$ and any node $a \in \cup_{j=1}^{m} \mathcal{I}_j(t)$, let D_a denote the event that a recovers during the time interval $[t, t+\Delta t)$. Then

$$\begin{aligned}
&\Pr(\cup_{i \in \mathcal{I}_i} D_i \mid \mathcal{S}(t),\mathcal{I}(t)) \\
&= \sum_{a \in \mathcal{I}_i} \Pr(D_a \mid \mathcal{S},\mathcal{I}) - \sum_{a_1 \in \mathcal{I}_i}\sum_{a_2 \in \mathcal{I}_i} \Pr(D_{a_1} \cap D_{a_2} \mid \mathcal{S},\mathcal{I}) + \dots \\
&\overset{(a)}{=} \sum_{a \in \mathcal{I}_i} (\gamma_i \Delta t + o(\Delta t)) - \sum_{a_1 \in \mathcal{I}_i}\sum_{a_2 \in \mathcal{I}_i} \Pr(D_{a_1} \mid \mathcal{S},\mathcal{I}) \cdot \Pr(D_{a_2} \mid \mathcal{S},\mathcal{I}) + \dots \\
&= \gamma_i I_i \Delta t + o(\Delta t) - \sum_{a_1 \in \mathcal{I}_i}\sum_{a_2 \in \mathcal{I}_i} (\gamma_i \Delta t + o(\Delta t))^2 + \dots \\
&= \gamma_i I_i \Delta t + o(\Delta t),
\end{aligned} \tag{5.10}$$

where (a) holds because of the assumption that different nodes recover independently of each other during vanishingly small time intervals. Consequently,

$$
\begin{aligned}
&\gamma_i I_i \Delta t + o(\Delta t) \\
&= \Pr(\cup_{a \in I_i} D_a \mid \mathcal{S}, \mathcal{I}) \\
&\geq \Pr(\mathcal{I}_i(t+\Delta t) \subsetneq \mathcal{I}_i(t) \mid \mathcal{S}, \mathcal{I}) \\
&= \Pr\left((\cup_{a \in \mathcal{I}_i} D_a) \cap \{\Delta_t S_i = 0\} \mid \mathcal{S}, \mathcal{I}\right) \\
&\overset{(\dagger)}{\geq} \Pr\left(\cup_{a \in \mathcal{I}_i} D_a \mid \mathcal{S}, \mathcal{I}\right) \Pr\left(\Delta_t S_i = 0 \mid \mathcal{S}, \mathcal{I}\right) \\
&= (\gamma_i I_i \Delta t + o(\Delta t)) \left(1 + o(\Delta t) - \sum_{j=1}^{m} B_{ij} \chi_{ij} S_i I_j \Delta t\right) \\
&= \gamma_i I_i \Delta t + o(\Delta t),
\end{aligned}
$$

which shows that $\Pr(I_i(t+\Delta t) < I_i(t) \mid \mathcal{S}, \mathcal{I}) = \gamma_i I_i \Delta t + o(\Delta t)$. Similarly, we can show that

$$
\begin{aligned}
&\Pr(I_i(t+\Delta t) > I_i(t) \mid \mathcal{S}, \mathcal{I}) \\
&= \Pr(S_i(t+\Delta t) < S_i(t) \mid \mathcal{S}, \mathcal{I}) + o(\Delta t) \\
&= \sum_{j=1}^{m} B_{ij} \chi_{ij} S_i I_j \Delta t + o(\Delta t).
\end{aligned}
$$

Moreover, by repeating some of the arguments used in deriving (5.35), we can show that $\Pr(I_i(t+\Delta t) - I_i(t) \leq -2 \mid \mathcal{S}, \mathcal{I}) = o(\Delta t)$. Therefore,

$$
\begin{aligned}
&\mathbb{E}[\Delta_t I_i(t) \mid \mathcal{S}, \mathcal{I}] \\
&= +1 \cdot \Pr(I_i(t+\Delta t) > I_i(t) \mid \mathcal{S}, \mathcal{I}) \\
&\quad - 1 \cdot \Pr(\mathcal{I}_i(t+\Delta t) < I_i(t) \mid \mathcal{S}, \mathcal{I}) \\
&= \frac{1}{n} \sum_{i=1}^{m} B_{ij} (n \chi_{ij}) S_i I_j \Delta t - \gamma_i I_i \Delta t + o(\Delta t).
\end{aligned}
\tag{5.11}
$$

The rest of the derivation parallels that of (i); take expectations on both the sides of (5.11), divide by Δt, let $\Delta t \to 0$ and divide both sides by n so as to obtain (ii).

Proof of (iii).

Observe that when $\Delta_t S_i = -1$, we have $S_i^2(t+\Delta t) - S_i^2(t) = 1 - 2S_i(t)$. Therefore,

$$\begin{aligned}
&\mathbb{E}[S_i^2(t+\Delta t) - S_i^2(t) \mid \mathcal{S},\mathcal{I}] \\
&= (1-2S_i(t)) \cdot \Pr(\Delta_t S_i = -1 \mid \mathcal{S},\mathcal{I}) + o(\Delta t) \\
&= \frac{1}{n} B_{ij} n \chi_{ij} \left(\sum_{j=1}^{m} S_i I_j \Delta t - \frac{2}{n} \sum_{j=1}^{m} S_i^2 I_j \Delta t \right) + o(\Delta t).
\end{aligned}$$

Taking expectations on both sides, dividing by Δt, letting $\Delta t \to 0$, and dividing both sides by n^2 yields (iii).

Proof of (iv).

Observe that if $\Delta_t I_i(t) = -1$, we have $I_i^2(t+\Delta t) - I_i^2(t) = 1 - 2I_i(t)$, and if $\Delta_t I_i(t) = 1$, we have $I_i^2(t+\Delta t) - I_i^2(t) = 1 + 2I_i(t)$.

Thus,

$$\begin{aligned}
&\mathbb{E}[I_i^2(t+\Delta t) - I_i^2(t) \mid \mathcal{S},\mathcal{I}] \\
&= (1-2I_i(t)) \cdot \Pr(I_i(t+\Delta t) < I_i(t) \mid \mathcal{S},\mathcal{I}) \\
&+ (1+2I_i(t)) \cdot \Pr(I_i(t+\Delta t) > I_i(t) \mid \mathcal{S},\mathcal{I}).
\end{aligned} \tag{5.12}$$

On substituting the probabilities above with the expressions derived earlier, taking expectations on both sides, dividing by $n^2 \Delta t$ and letting $\Delta t \to 0$, we obtain (iv). □

We point out that if $\chi_{ij} = \frac{\rho_{ij}}{n}$ then Equations (i) and (ii) have the same coefficients as (I) and (II). It is then natural to ask: how does the conditional edge probability χ_{ij} compare to the unconditional edge probability $\frac{\rho_{ij}}{n}$? The following proposition provides an answer. As we show in Remark 13, our answer helps characterize the rate at which the infection transmission rates converge to their respective limits, an analysis missing from other works such as [166] and [167].

Proposition 13. *For all $t \geq 0$, $n \in \mathbb{N}$ and $i, j \in [m]$,*

$$\frac{\rho_{ij}}{n}\left(1 - \frac{B_{ij}}{\lambda^{(n)}}\left(1 - e^{-\lambda^{(n)} t}\right)\right) \leq \chi_{ij} \leq \frac{\rho_{ij}}{n}.$$

Remark 13. *Given $(\mathcal{S}(t), \mathcal{I}(t))$, note that the conditional probability that a given infected node in $\mathcal{A}_j$ infects a given susceptible node in $\mathcal{A}_i$ during a time interval $[t, t+\Delta t)$ is $B_{ij}\chi_{ij}(t, \mathcal{S}, \mathcal{I})\Delta t + o(\Delta t)$. In light of Proposition 13, this means that the associated conditional infection rate $B_{ij}\chi_{ij}(t, \mathcal{S}, \mathcal{I})$ belongs to the interval*

$$\left[\frac{1}{n}A_{ij}\left(1 - \frac{B_{ij}}{\lambda^{(n)}}(1 - e^{-\lambda^{(n)} t})\right), \frac{1}{n}A_{ij}\right].$$

On taking expectations, we realize that the same applies to the associated unconditional infection rate as well. Hence, the total rate of infection transmission from all of $\mathcal{A}_j$ to any given node of $\mathcal{A}_i$ is at least $I_j^{(n)}(t) \times \frac{1}{n}A_{ij}\left(1 - \frac{B_{ij}}{\lambda^{(n)}}(1 - e^{-\lambda^{(n)} t})\right) = A_{ij}\beta_j^{(n)}(t)\left(1 - \frac{B_{ij}}{\lambda^{(n)}}(1 - e^{-\lambda^{(n)} t})\right)$ and at most $A_{ij}\beta_j^{(n)}(t)$. Since we assume $\lim_{n\to\infty}\lambda^{(n)} = \infty$, this further implies that the concerned rate is approximately $A_{ij}\beta_j^{(n)}(t)$ for large n, thereby giving us an interpretation of the 'contact rate' A_{ij} as a normalized infection rate. That is, in the limit as $n \to \infty$, the matrix A quantifies the infection transmission rates between any two age groups relative to the level of infectedness (fraction of infected persons) of the transmitting age group. Moreover, Proposition 13 also implies that the difference between the age-wise infection transmission rates and their respective mean-field limits (which exist as per Theorem 7) is $O\left(\frac{1}{\lambda^{(n)}}\right)$.

5.4 A Converse Result

The purpose of this section is to argue that the age-structured SIR dynamics does not model an epidemic well if the infection rates B_{ij} are high enough to be comparable to the edge update rate of the network.

Theorem 8. *Suppose* $\lambda_\infty := \lim_{n\to\infty} \lambda^{(n)} < \infty$ *and that for every* $p \in [m]$*, there exist* $s_{p,0}, \beta_{p,0} \in [0,1]$ *such that* $s_p^{(n)}(0) \to s_{p,0}$ *and* $\beta_i^{(n)}(0) \to \beta_{p,0}$ *as* $n \to \infty$*. In addition, let* $\{(y_q(t), w_q(t))\}$ *be the solutions of the ODEs (I) and (II). Then, there exists no interval* $[t_1, t_2] \subset [0, \infty)$ *for which* $\min_{p,q\in[m]} \min_{t\in[t_1,t_2]} y_p(t) w_q(t) > 0$ *and on which the pairs* $\left\{\left(s_q^{(n)}(t), \beta_q^{(n)}(t)\right)\right\}_{q=1}^m$ *uniformly converge in probability to the corresponding pairs in* $\{(y_q(t), w_q(t))\}_{q=1}^m$*. More precisely, for every interval* $[t_1, t_2] \subset \mathbb{R}$ *such that* $y_p(t) > 0$ *and* $w_p(t) > 0$ *for all* $p \in [m]$ *and* $t \in [t_1, t_2]$*, there exists a* $q \in [m]$ *and an* $\varepsilon_q > 0$ *such that*

$$\liminf_{n\to\infty} \sup_{t\in[t_1,t_2]} \Pr\left(\left\|\left(s_q^{(n)}(t), \beta_q^{(n)}(t)\right) - (y_q(t), w_q(t))\right\|_2 > \varepsilon_q\right) > 0.$$

Proof. Suppose, on the contrary, that there exists a time interval $[t_1, t_2] \subset [0, \infty)$ such that $y_p(t) > 0$ and $w_p(t) > 0$ for all $p \in [m]$ and $t \in [t_1, t_2]$, and the following holds for all $q \in [m]$ and all $\varepsilon_q > 0$:

$$\liminf_{n\to\infty} \sup_{t\in[t_1,t_2]} \Pr\left(\left\|\left(s_q^{(n)}(t), \beta_q^{(n)}(t)\right) - (y_q(t), w_q(t))\right\|_2 > \varepsilon_q\right) = 0,$$

i.e., for a fixed $\varepsilon > 0$, there exists a sequence $\{\pi(n)\}_{n=1}^\infty \subset \mathbb{N}$ such that

$$\lim_{n\to\infty} \sup_{t\in[t_1,t_2]} \Pr\left(\left\|\left(s_q^{(\pi(n))}(t), \beta_q^{(\pi(n))}(t)\right) - (y_q(t), w_q(t))\right\|_2 > \varepsilon\right) = 0.$$

We then arrive at a contradiction, as shown below.

We first choose an $\eta > 2\left(1 + \frac{A_{\max}}{\lambda_\infty}\right)$ (where $A_{\max} := \max\{A_{pq} : p, q \in [m]\}$) and a $t \in (t_1, t_2)$. By our hypothesis and norm equivalence, for $\kappa_0 := \frac{1}{\eta\lambda_\infty}$ and for every $\delta > 0$, there exists an $N_{\varepsilon,\delta} \in \mathbb{N}$ such that

$$\Pr\left(\left\|\left(s_q^{(\pi(n))}(\tau), \beta_q^{(\pi(n))}(\tau)\right) - (y_q(\tau), w_q(\tau))\right\|_1 \le \varepsilon\right) \ge 1 - \delta$$

for all $n \geq N_{\varepsilon,\delta}$ and all $\tau \in [t-\kappa_0, t]$.

We now define $\alpha_0 := \min_{p,q\in[m]} \min_{t\in[t_1,t_2]} y_p(t) w_q(t)$ and we let a and b be any two nodes such that $(a,b) \in \mathcal{S}_i(t) \times \mathcal{I}_j(t)$ for arbitrary $i,j \in [m]$. Additionally, we let $K := t - \inf\{\tau \geq 0 : b \in \mathcal{I}(\tau)\}$ denote the (random) time elapsed between the time at which b gets infected and time t. We then have

$$\begin{aligned}
\Pr(K \leq \kappa_0 \mid \mathcal{S}(t), \mathcal{I}(t)) &= \Pr(b \text{ is infected during } [t-\kappa_0, t] \mid \mathcal{S}(t), \mathcal{I}(t)) \\
&\overset{(a)}{\leq} 1 - e^{-A_{\max}\kappa_0} \\
&\overset{(b)}{\leq} A_{\max}\kappa_0 \\
&= \frac{A_{\max}}{\eta\lambda_\infty}, \qquad (5.13)
\end{aligned}$$

where (b) holds because $1 - e^{-u} \leq u$ for all $u \geq 0$, and (a) can be explained as follows: given $(\mathcal{S}(\tau), \mathcal{I}(\tau))$ and an infected node $c \in \mathcal{I}_q(\tau)$ for some time $\tau \in [t-\kappa_0, t)$, and given that $b \in \mathcal{S}_j(\tau)$, we know from Proposition 13 that the conditional probability of the edge (b,c) existing in the network at time τ is at most $\frac{\rho_{jq}}{\pi(n)}$. Also, as per the definition of our stochastic epidemic model, given that $(b,c) \in E(\tau)$ and given $(\mathcal{S}(\tau), \mathcal{I}(\tau))$ (and hence, also that $(b,c) \in \mathcal{S}_j(\tau) \times \mathcal{I}_q(\tau)$), the conditional rate of infection transmission from c to b at time τ is B_{jq}. Hence, given $(\mathcal{S}(\tau), \mathcal{I}(\tau))$ (and hence, that $(b,c) \in \mathcal{S}_j(\tau) \times \mathcal{I}_q(\tau)$), the conditional rate of infection transmission from c to b is at most $B_{jq}\frac{\rho_{jq}}{\pi(n)} = \frac{A_{jq}}{\pi(n)}$. Under our modelling assumption that distinct edges transmit the infection independently of each other during vanishingly small time intervals, this means that, conditional on $\mathcal{S}(\tau)$ and $\mathcal{I}(\tau)$, the conditional total rate at which b receives infection is at most

$$\sum_{q\in[m]} \sum_{c\in\mathcal{I}_q(\tau)} \frac{A_{jq}}{\pi(n)} = \sum_{q\in[m]} |\mathcal{I}_q(\tau)| \frac{A_{jq}}{\pi(n)} = \sum_{q\in[m]} \beta_q^{(\pi(n))}(\tau) A_{jq} \leq A_{\max} \sum_{q\in[m]} \beta_q^{(\pi(n))}(\tau) \leq A_{\max}.$$

Note that this upper bound is time-invariant and does not depend on $\mathcal{S}(\tau)$ or $\mathcal{I}(\tau)$ for any time τ. It thus follows that, conditional on $(\mathcal{S}(t), \mathcal{I}(t))$, the rate at which b gets infected is at most $A_{\max}$

throughout the interval $[t-\kappa_0, t)$ and hence, the probability that b does not get infected during an interval of length κ_0 is at least $e^{-A_{\max}\kappa_0}$. This implies (a).

We now infer from (5.13) that

$$\Pr(K \geq \kappa_0 \mid \mathcal{S}(t), \mathcal{I}(t)) \geq 1 - \frac{A_{\max}}{\eta\lambda_\infty}. \tag{5.14}$$

Next, we lower-bound $\Pr(T \geq \kappa_0 \mid \mathcal{S}(t), \mathcal{I}(t))$. To this end, note from Proposition 13 that $\Pr((a,b) \in E(t) \mid \mathcal{S}(t), \mathcal{I}(t)) \leq \frac{\rho_{ij}}{\pi(n)}$. As a result, we have

$$\begin{aligned}
&|\Pr(T \geq \kappa_0 \mid \mathcal{S}(t), \mathcal{I}(t)) - \Pr(T \geq \kappa_0 \mid (a,b) \notin E(t), \mathcal{S}(t), \mathcal{I}(t))| \\
&= |\Pr(T \geq \kappa_0 \mid (a,b) \notin E(t), \mathcal{S}(t), \mathcal{I}(t))(1 - \Pr((a,b) \in E(t) \mid \mathcal{S}(t), \mathcal{I}(t))) \\
&\quad + \Pr(T \geq \kappa_0 \mid (a,b) \in E(t), \mathcal{S}(t), \mathcal{I}(t)) \cdot \Pr((a,b) \in E(t) \mid \mathcal{S}(t), \mathcal{I}(t)) \\
&\quad - \Pr(T \geq \kappa_0 \mid (a,b) \notin E(t), \mathcal{S}(t), \mathcal{I}(t))| \\
&\leq \frac{\rho_{ij}}{\pi(n)},
\end{aligned}$$

which also means that

$$|\Pr(T < \kappa_0 \mid (\mathcal{S}(t), \mathcal{I}(t))) - \Pr(T < \kappa_0 \mid (a,b) \notin E(t), (\mathcal{S}(t), \mathcal{I}(t)))| \leq \frac{\rho_{ij}}{\pi(n)}. \tag{5.15}$$

Moreover, for any realization $(\mathcal{S}_0, \mathcal{I}_0)$ of $(\mathcal{S}(t), \mathcal{I}(t))$, Remark 17 asserts that

$$\Pr(T \leq \kappa_0 \mid K = \kappa, (\mathcal{S}(t), \mathcal{I}(t)) = (\mathcal{S}_0, \mathcal{I}_0), (a,b) \notin E(t)) \leq 1 - e^{-\lambda\kappa_0}$$

for all $0 \leq \kappa \leq t$. Since the right-hand-side above is independent of both κ and $(\mathcal{S}_0, \mathcal{I}_0)$, it follows that

$$\Pr(T \leq \kappa_0 \mid (\mathcal{S}(t), \mathcal{I}(t)), (a,b) \notin E(t)) \leq 1 - e^{-\lambda\kappa_0} \leq \lambda\kappa_0. \tag{5.16}$$

Therefore, as a consequence of (5.14), (5.15), (5.16), and the union bound, we have

$$\begin{aligned}\Pr(T \geq \kappa_0, K \geq \kappa_0 \mid \mathcal{S}(t),\mathcal{I}(t)) &= 1 - \Pr(\{T < \kappa_0\} \cup \{K < \kappa_0\} \mid \mathcal{S}(t),\mathcal{I}(t)) \\ &\geq 1 - \Pr(T < \kappa_0 \mid \mathcal{S}(t),\mathcal{I}(t)) - \Pr(K < \kappa_0 \mid \mathcal{S}(t),\mathcal{I}(t)) \\ &\geq 1 - \frac{A_{\max}}{\eta\lambda_\infty} - \frac{\lambda^{(\pi(n))}}{\eta\lambda_\infty} - \frac{\rho_{ij}}{\pi(n)}. \end{aligned} \tag{5.17}$$

This further yields,

$$\begin{aligned}&\chi_{ij}(t,\mathcal{S},\mathcal{I}) \\ &= \Pr((a,b) \in E(t) \mid \mathcal{S}(t),\mathcal{I}(t)) \\ &= \Pr((a,b) \in E(t) \mid T \geq \kappa_0, K \geq \kappa_0, \mathcal{S}(t),\mathcal{I}(t)) \cdot \Pr(T \geq \kappa_0, K \geq \kappa_0 \mid \mathcal{S}(t),\mathcal{I}(t)) \\ &\quad + \Pr((a,b) \in E(t) \mid \{T < \kappa_0\} \cup \{K < \kappa_0\}, \mathcal{S}(t),\mathcal{I}(t)) \cdot \Pr(\{T < \kappa_0\} \cup \{K < \kappa_0\} \mid \mathcal{S},\mathcal{I}) \\ &\leq \frac{\rho_{ij}}{\pi(n)} e^{-B_{ij}\kappa_0} \left(1 - \frac{A_{\max}}{\eta\lambda_\infty} - \frac{\lambda^{(\pi(n))}}{\eta\lambda_\infty} - \frac{\rho_{ij}}{\pi(n)}\right) + \left(\frac{A_{\max}}{\eta\lambda_\infty} + \frac{\lambda^{(\pi(n))}}{\eta\lambda_\infty} + \frac{\rho_{ij}}{\pi(n)}\right) \frac{\rho_{ij}}{\pi(n)} \\ &= \frac{\rho_{ij}}{\pi(n)} e^{-\frac{B_{ij}}{\eta\lambda_\infty}} \left(1 - \frac{A_{\max}}{\eta\lambda_\infty} - \frac{\lambda^{(\pi(n))}}{\eta\lambda_\infty} - \frac{\rho_{ij}}{\pi(n)}\right) + \left(\frac{A_{\max}}{\eta\lambda_\infty} + \frac{\lambda^{(\pi(n))}}{\eta\lambda_\infty} + \frac{\rho_{ij}}{\pi(n)}\right) \frac{\rho_{ij}}{\pi(n)} \end{aligned} \tag{5.18}$$

where the inequality is a consequence of (5.17) and Remark 16. Recall that $\lim_{n\to\infty} \lambda^{(n)} = \lambda_\infty$, which means that the right hand side can be made smaller than $(1+\varepsilon)\frac{\rho_{ij}}{\pi(n)} e^{-\frac{B_{ij}}{\eta\lambda_\infty}}$ by choosing n large enough. Moreover (5.18) holds for all $t \in [t_1, t_2]$.

Proposition 12 now implies that for all $t \in [t_1, t_2]$ and large enough n,

$$\mathbb{E}[s_i]' = -\sum_{j=1}^{m} B_{ij}\mathbb{E}[n\chi_{ij}s_i\beta_j] > -\sum_{j=1}^{m} A_{ij}(1+\varepsilon)e^{-\frac{B_{ij}}{\eta\lambda_\infty}}\mathbb{E}[s_i\beta_j], \tag{5.19}$$

Now, observe that for any $t \in [t_1, t_2]$, we have

$$
\begin{aligned}
&\mathbb{E}[s_i(t)\beta_j(t)] \\
&\leq 1 \cdot \Pr\left(\left\|\left(s_q^{(n)}(\tau), \beta_q^{(n)}(\tau)\right) - (y_q(\tau), w_q(\tau))\right\|_1 > \varepsilon\right) \\
&\quad + (w_i(t)+\varepsilon)(y_j(t)+\varepsilon) \cdot \Pr\left(\left\|\left(s_q^{(n)}(\tau), \beta_q^{(n)}(\tau)\right) - (y_q(\tau), w_q(\tau))\right\|_1 \leq \varepsilon\right) \\
&\leq \delta + y_i(t)w_j(t) + 2\varepsilon + \varepsilon^2. \qquad (5.20)
\end{aligned}
$$

Therefore, assuming that δ and ε are small enough to satisfy $\delta + 2\varepsilon + \varepsilon^2 < \alpha_0 \left(\frac{e^{\frac{B_{ij}}{\eta\lambda_\infty}}}{1+\varepsilon} - 1\right)$, (5.19) implies the existence of a constant $\varepsilon' > 0$ such that :

$$
\mathbb{E}[s_i]' = -\sum_{j=1}^{m} B_{ij}\mathbb{E}[n\chi_{ij}s_i\beta_j] \geq -\sum_{j=1}^{m} A_{ij}y_i(t)w_j(t) + \varepsilon' = y_i'(t) + \varepsilon'.
$$

Since this holds for all $t \in [t_1, t_2]$, we have

$$
\mathbb{E}[s_i^{(\pi(n))}(t_2)] - y_i(t_2) \geq \mathbb{E}[s_i^{(\pi(n))}(t_1)] - y_i(t_1) + (t_2 - t_1)\varepsilon'
$$

for all sufficiently large n. Here, we observe that $\{s_i^{(n)}(t_1), \beta_j^{(n)}(t_1) : i, j \in [m], n \in \mathbb{N}\}$ are bounded by the constant function 1, which is integrable with respect to probability measures. Therefore, $\{s_i^{(n)}(t_1) : i, j \in [m], n \in \mathbb{N}\}$ are uniformly integrable. Since they converge in probability to $\{y_i(t_1) : i, j \in [m]\}$ (by hypothesis), it follows by Vitali's Convergence Theorem that they also converge in L^1-norm. Thus, $\mathbb{E}[s_i^{(n)}(t_1)] - y_i(t_1) \to 0$ as $n \to \infty$, thereby implying that

$$
\liminf_{n\to\infty} \left(\mathbb{E}[s_i^{(\pi(n))}(t_2)] - y_i(t_2)\right) \geq (t_2 - t_1)\varepsilon'. \qquad (5.21)
$$

On the other hand, Vitali's Convergence Theorem and our hypothesis also imply that $\mathbb{E}[s_i^{(n)}(t_2)] \to y_i(t_2)$ as $n \to \infty$, which contradicts (5.21). Hence, our hypothesis that the fractions $\{s_q^{(n)}(t), \beta_q^{(n)}(t)\}_{q \in [m]}$ converge in probability to the solutions of (I) and (II) uniformly on the interval $[t_1, t_2]$ is false. This completes the proof. □

Before interpreting Theorem 8, we first note that the result only applies to the time intervals on which $\{y_i(t)w_j(t) : i, j \in [m]\}$ are positive throughout the interval. Although this condition appears stringent, it is mild from the viewpoint of epidemic spreading in the real world. This is because, in practice we are only interested in time periods during which every age group has infected cases (which ensures that $\beta_j(t) > 0$ for all $j \in [m]$), and most epidemics leave behind uninfected individuals (thereby ensuring that $s_i(t) > 0$ for all $i \in [m]$). Therefore, Theorem 8 applies to all time intervals of practical interest.

Restricting our focus to such intervals, Theorem 8 asserts that, if the edge update rate does not go to ∞ with the population size, then there exists a positive lower bound on the probability of the age-wise infected and susceptible fractions differing significantly from the corresponding solutions of the age-structured SIR ODEs at one or more points of time in the considered time interval. At this point, we remark that for large populations, the edge update rate λ is approximately the reciprocal of the mean duration of every interaction in the network. This means that the greater the value of λ, the faster will be the changes that occur in the social interaction patterns of the network. Therefore, in conjunction with Theorem 7, Theorem 8 enables us to draw the following inference: the age-structured SIR model can be expected to approximate a real-world epidemic spreading in a large population accurately if and only if the social interaction patterns of the network change rapidly with time. This is more likely to be the case in crowded public places such as supermarkets and airports.

There is another way to interpret Theorems 7 and 8. Note that we have assumed that the sequence of edge states realized during the timeline of the epidemic are independent for every pair of nodes in the network. Therefore, for greater values of λ, the network structure becomes more unrecognizable from its past realizations. Thus, the age-structured SIR model can be expected to approximate epidemic spreading well if and only if the network is highly memoryless, i.e., if and only if the network continually "forgets" its past interaction patterns throughout the timeline of the epidemic under study.

Remark 14. *Observe from the proof of Theorem 8 that the difference between* $\mathbb{E}[s_i]'$*, the first derivative of the expected fraction of infected nodes in* $\mathcal{A}_i$*, and* y_i'*, the first derivative of the corresponding ODE solution* $y_i(t)$*, is small only if* $e^{-\frac{B_{ij}}{\eta\lambda_\infty}}$ *is close to 1, which happens when* $\lambda_\infty \gg B_{ij}$*. Moreover, this observation is consistent with Remark 13, according to which the total infection rate from* $[n] = \cup_{j=1}^m \mathcal{A}_j$ *to any given susceptible node in* $\mathcal{A}_i$ *is close to* $\sum_{j=1}^m A_{ij}\beta_j(t)$ *(and hence, in close agreement with the ODEs* (5.1)*) when* $\frac{B_{ij}}{\lambda^{(n)}} \ll 1$*. Along with Theorems 7 and 8, this means that the age-structured SIR model is likely to approximate real-world epidemic spreading well if and only if the infection transmission rates are negligible when compared to the social mixing rate* λ*.*

Intuitively, when $\frac{B_{ij}}{\lambda} \ll 1$, the time scales (the mean duration of time) over which the concerned disease spreads from any age group to any other age group are orders of magnitude greater than the time scale over which the network is updated. As a result, the independence of the sequences of edge state updates ensures that most of the possible realizations of the network structure are attained over the time scale of infection transmission. Equivalently, from the viewpoint of the pathogens causing the disease, the effective network structure (the network topology averaged over any of the age-wise infection timescales) is close to being a complete graph. Hence, by extrapolating the existing results on mean-field limits of epidemic processes on complete graphs (such as [166]) to heterogeneous epidemic models, we can assert that the age-structured SIR ODEs are able to approximate the epidemic propagation with a high accuracy.

On the other hand, if the infection rates B_{ij} are too high (and hence, comparable to the social mixing rate λ, which is always finite in reality), the pathogens perceive a randomly generated network even on the time scale of infection transmission. Since this random network is sparse (because we assume the expected node degrees to be constant, which results in the edge probability scaling inversely with the population size), it follows that the number of transmissions occurring in any given time period is likely to be smaller than in the case of a complete graph. Thus, the age-structued SIR ODEs overestimate the rate of growth of age-wise infected fractions. This is further confirmed by the sign of the inequality in (5.21).

5.5 Empirical Validation

We now validate the age-structured SIR model in the context of the COVID-19 pandemic in Japan as follows: we first estimate the model parameters using the data provided by the Government of Japan, and we then compare the trajectories generated by the model with the reference data.

5.5.1 Dataset

We use a dataset provided by the Government of Japan at [168]. This dataset partitions the population of the prefecture of Tokyo into $m = 5$ age groups: 0 - 19, 20 - 39, 40 - 59, 60 - 79, and 80+ years old individuals. For each age group $i \in [m]$ and each day k in the year-long timeline $\Gamma = \{\text{March 10, 2020}, \ldots, \text{April 9, 2021}\}$, the dataset lists the total number of people infected in the age group until date k. We denote this number by $I_i^T[k]$.

5.5.2 Preprocessing

Due to several factors, such as lack of reporting/testing on the weekends, the raw data has missing information and is contaminated with noise. Therefore, using a moving average filter with a window size of 15 days, we de-noise the raw data to obtain the estimated total number of infected individuals by day k in age group i, denoted by $I_i^T[k]$. We then estimate from the smoothed data the number of susceptible, infected, and recovered individuals in age group $i \in [m]$ on day k, denoted by $S_i[k]$, $I_i[k]$, and $R_i[k]$, respectively. We do this as follows: for any age group $i \in [m]$ and day $k \in \Gamma$, we have $I_i^T[k] = I_i[k] + R_i[k]$, because the cumulative number of infections $I_i^T[k]$ includes both active COVID-19 cases and closed cases (cases of individuals who were infected in the past but recovered/succumbed by day k). Therefore, to estimate $I_i[k]$ and

$R_i[k]$ from $I_i^T[k]$, we assume that every infected individual takes exactly $T_R = 14$ days to recover. This assumption is consistent with WHO's criteria for discharging patients from isolation (i.e., discontinuing transmission-based precautions) [169] after a period involving the first 10 days from the onset of symptoms and 3 additional symptom-free days (if the patient is originally symptomatic) or after 10 days from being tested positive for SARS-CoV-2 (if the patient is asymptomatic). After the required period, the patients were not required to re-test. Under such an assumption on the recovery time, we have $R_i[k] = I_i^T[k-T_R]$ and $I_i[k] = I_i^T[k] - I_i^T[k-T_R]$. Next, we obtain $S_i[k]$ by subtracting $I_i^T[k]$ from the total population of $\mathcal{A}_i$, which is obtained from the age distribution and the total population of Tokyo.

We must mention that in the subsequent analysis, all infected individuals are considered infectious, i.e., they can potentially transmit the SARS-CoV-2 virus to their susceptible contacts. This assumption, on which the classical SIR model and all its variants are based, is consistent with the CDC's understanding of the first wave of SARS-CoV-2 infection, which claims that every infected individual remains infectious for up to about 10 days from the onset of symptoms, though the exact duration of the period of infectiousness remains uncertain [170].

5.5.3 Parameter Estimation Algorithm

Before estimating the parameters of our model, we discretize the ODEs (5.1) with a step size of 1 day and obtain the following:

$$\begin{aligned} s_i[k+1] - s_i[k] &= -s_i[k] \sum_{j=1}^{m} A_{ij}\beta_j[k], \\ \beta_i[k+1] - \beta_i[k] &= s_i[k] \sum_{j=1}^{m} A_{ij}\beta_j[k] - \gamma_i\beta_i[k], \\ r_i[k+1] - r_i[k] &= \gamma_i\beta_i[k], \end{aligned} \tag{5.22}$$

A key observation here is that these equations are linear in the model parameters. Therefore, given the sets of fractions $\{s_i[k] : i \in [m], k \in \Gamma\}$, $\{\beta_i[k] : i \in [m], k \in \Gamma\}$, and

$\{r_i[k] : i \in [m], k \in \Gamma\}$ (which we obtain by implementing the data processing steps described above) for all $i \in [m]$, we can express (5.22) in the form of a matrix equation $Cx = d$, where the column vector $x \in \mathbb{R}^{m^2+m}$ is a stack of the parameters $\{A_{ij} : 1 \leq i, j \leq m\}$ and $\{\gamma_i : 1 \leq i \leq m\}$, the column vector d is a stack of the increments $\{s_i[k+1] - s_i[k] : i \in [m], k \in \Gamma\}$, $\{\beta_i[k+1] - \beta_i[k] : i \in [m], k \in \Gamma\}$, and $\{r_i[k+1] - r_i[k] : i \in [m], k \in \Gamma\}$, and C is a matrix of coefficients. Thus, solving the least-squares problem (5.23) gives us the best estimates of the model parameters $\{A_{ij} : i, j \in [m]\} \cup \{\gamma_i : i \in [m]\}$ in the mean-square sense.

$$\hat{x} = \underset{x \geq 0}{\operatorname{argmin}} \|Cx - d\|_2. \tag{5.23}$$

However, the values of the contact rates A_{ij} change as and when the patterns of social interaction in the network change during the course of the pandemic. For this reason, we assume that the pandemic timeline splits up into multiple phases, say $\Gamma_1, \ldots, \Gamma_s$, with the contact rates varying across phases, and we perform the required optimization separately for each phase. At the same time, we do not expect the contact rates to make quantum leaps (or falls) from one phase to the next. Therefore, for every $\ell \geq 2$, in the objective function corresponding to Phase ℓ we introduce a regularization term that penalizes any deviation of the optimization variables from the model parameters estimated for the previous phase (Phase $\ell - 1$). Adding this term also ensures that our parameter estimation algorithm does not overfit the data associated with any one phase. Our optimization problem for Phase ℓ thus becomes

$$\hat{x}^{(\ell)} = \underset{x \geq 0}{\operatorname{argmin}} \left(\|Cx - d\|_2 + \lambda \|x^{(\ell)} - x^{(\ell-1)}\|_2 \right), \tag{5.24}$$

where $x^{(\ell)}$ is the parameter vector estimated for Phase ℓ.

We now summarize this parameter estimation algorithm for Phase $\ell \in [s]$.

Algorithm 1. Parameter Estimation Algorithm for Phase ℓ

Input: $(s_i[k], \beta_i[k], r_i[k])$ for all $i \in [m]$ and $k \in \Gamma_\ell$

Output: $\hat{x}$

```
function GET_PARAMETERS((s_i[k], β_i[k], r_i[k]))
    for each day, each age group do
        Stack the difference equations (5.22) vertically
    Obtain the matrix equation Cx = d
    Solve Least Squares Problem (5.24)
    return x̂
```

5.5.4 Phase Detection Algorithm

We now provide an algorithm that divides the timeline of the pandemic into multiple phases in such a way that the beginning of each new phase indicates a significant change in one or more of the contact rates $\{A_{ij} : i, j \in [m]\}$.

Given the pandemic timeline $\{p_0, \ldots, p_s\}$ (where p_0 denotes March 10, 2020 and p_s denotes April 9, 2021), our phase detection algorithm outputs $s-1$ phase boundaries $p_1 \leq p_2 \leq \cdots \leq p_{s-1}$ that divide $[p_0, p_s)$ into s phases, namely $\Gamma_1 = [p_0, p_1), \Gamma_2 = [p_1, p_2), \ldots, \Gamma_s = [p_{s-1}, p_s)$. Central to the algorithm are the following optimization problems:

Problem (a): Unconstrained Optimization

$$\underset{x \geq 0}{\text{minimize}} \quad \|C_{[p,p+w)} x - d_{[p,p+w)}\|_2. \tag{5.25}$$

Problem (b): Constrained Optimization

$$\begin{aligned} \underset{x \geq 0}{\text{minimize}} \quad & \|C_{[p,p+w)} x - d_{[p,p+w)}\|_2, \\ \text{subject to} \quad & \|x - \bar{x}_{[p,p+w)}\|_2 \leq \varepsilon \|\bar{x}_{[p-\Delta p, p-\Delta p+w)}\|_2. \end{aligned} \tag{5.26}$$

In these problems, $p \in \Gamma$ denotes the *start date* (chosen recursively as described in Algorithm 2), $w \in \mathbb{N}$ is the *optimization window*, $\Delta p < w$ is the algorithm step size, $[p, p+w)$ denotes a w-day period from day p, $C_{[p,p+w)}$ and $d_{[p,p+w)}$ are obtained from $\{(s_i[k], \beta_i[k], r_i[k]) : i \in [m], k \in \{p, \ldots, p+w\}\}$ by using the procedure described in Section 5.5.3, and $\bar{x} := \arg\min_{x \geq 0} \|C_{[p,p+w)}x - d_{[p,p+w)}\|_2$ is the parameter vector estimated by Problem (a). We set $w = 30$ (days), and the quantities Δp and ε are pre-determined algorithm parameters whose choice is discussed in the next subsection.

Observe that both Problem and Problem (b) result in the minimization of the mean-square error (5.27), where $\{(\hat{s}_i[k], \hat{\beta}_i[k], \hat{r}_i[k]) : i \in [m], k \in \{p, \ldots, p+w\}\}$ are the model-generated values (estimates) of the susceptible, infected, and recovered fractions $\{(s_i[k], \beta_i[k], r_i[k]) : i \in [m], k \in \{p, \ldots, p+w\}\}$. Also note that Problem (a) performs this minimization while ignoring all the previously estimated model parameters, whereas Problem (b) performs the same minimization while constraining x to remain close to the parameter vector estimated for the period $[p - \Delta p, p - \Delta p + w)$. However, if the contact rates do not change significantly around day p, then the additional constraint imposed in Problem (b) should be satisfied automatically (without imposition) in Problem (a), which should in turn result in the same mean-square error for both the problems.

$$\mathcal{E} = \frac{1}{3m(w+1)} \sum_{i \in [m]} \sum_{k \in \{p, \ldots, p+w\}} \left((s_i[k] - \hat{s}_i[k])^2 + (\beta_i[k] - \hat{\beta}_i[k])^2 + (r_i[k] - \hat{r}_i[k])^2 \right). \tag{5.27}$$

Therefore, after solving Problems (a) and (b), our phase detection algorithm compares $\mathcal{E}_{(a)p}$ (the mean-square error for Problem (a)) with $\mathcal{E}_{(b)p}$ (the mean-square error for Problem (b)) as follows: using (5.27), the algorithm first computes $\mathcal{E}_{(a)p}$ and $\mathcal{E}_{(b)p}$. It then compares

$\frac{|\mathcal{E}_{(b)p} - \mathcal{E}_{(a)p}|}{\mathcal{E}_{(a)p}}$ with δ, a positive threshold whose choice is discussed in the next subsection. If

$$\frac{|\mathcal{E}_{(b)p} - \mathcal{E}_{(a)p}|}{\mathcal{E}_{(a)p}} > \delta, \tag{5.28}$$

then p is identified as a phase boundary. Otherwise, the algorithm increments the value of p by Δp, checks whether the interval $[p, p+w)$ is part of the timeline Γ, and repeats the entire procedure described above.

Finally, the algorithm merges every short phase (length ≤ 20 days) with its predecessor by deleting the appropriate phase boundary(s). There are two reasons for this step. First, the contact rates are believed to change not instantly but with a transition period of positive duration. Second, since the data used is noisy, to avoid overfitting the data it is necessary for the number of data points per phase (given by $2m$ times the number of days per phase) to significantly exceed $m^2 + m$, the number of model parameters to be estimated per phase.

We now provide the pseudocode for the entire algorithm. Observe that Problems (a) and (b) are both convex optimization problems. This enables us to use the Embedded Conic Solver (ECOS) [171] of CVXPY [172, 173] to implement our algorithm.

Algorithm 2. Phase Detection Algorithm

Input: $(s_i[k], \beta_i[k], r_i[k])$ for all $i \in [m]$ and $k \in \Gamma$

Output: Set of phase boundaries $\mathcal{B}$

function DETECT_PHASES($(s_i[k], \beta_i[k], r_i[k])$)
    Initialize set of phase boundaries $\mathcal{B} \leftarrow \phi$
    Initialize start date $p \leftarrow \Delta p$
    while $p \in \Gamma$ **do**
        Solve Problem (a) for window $[p, p+w)$
        Solve Problem (b) for window $[p, p+w)$
        if condition (5.28) holds **then**
            $\mathcal{B} \leftarrow \mathcal{B} \cup \{p\}$
        $p \leftarrow p + \Delta p$
    Initialize $p_{\text{start}} \leftarrow 0$
    Initialize $\mathbf{b} \leftarrow$ list($\mathcal{B}$)
    Sort $\mathbf{b}$ in ascending order
    for $p \in \mathbf{b}$ **do**
        if $p - p_{\text{start}} \leq 20$ **then**
            $\mathcal{B} \leftarrow \mathcal{B} \setminus \{p\}$
        else
            $p_{\text{start}} = p$
    return $\mathcal{B}$

5.5.5 Selection of Algorithm Parameters

We now explain our parameter choices for the algorithms described above.

Phase Detection Algorithm

As mentioned earlier, for Algorithm 2, we set $\Delta p = 5$ days and the optimization window $w = 30$ days. This ensures that the optimization window is large enough for the number of model parameters to be significantly smaller than the number of data points used to estimate these parameters in Problems (a) and (b). In addition, we set $\delta = 3$, and $\varepsilon = 10^{-4}$ for the following reasons:

1. $\varepsilon = 10^{-4}$: If both $[p, p+w)$ and $[p-\Delta p, p-\Delta p+w)$ are sub-intervals of the same phase, then the same set of contact rates (and hence the same parameter vector x) should apply to the network during both the time intervals.

2. $\delta = 3$: If day p marks the beginning of a new phase (i.e., a new set of contact rates), we expect the least-squares error (5.27) to increase significantly upon the imposition of the constraint introduced in (5.26).

Parameter Estimation Algorithm

We set $\lambda = 10^{-5}$ in (5.24). This small but non-zero value is consistent with our belief that around every phase boundary, contact rates change gradually but significantly during a transition period involving the phase boundary.

5.5.6 Results

We now present the results of implementing both the algorithms on our chosen dataset.

Phase Detection

Algorithm 2 detects the following phases.

Table 5.1. Phases Detected by Algorithm 2 [1, 2]

Phase	From	To	Corresponding Events
1	Mar 10 2020	Mar 28 2020	Closure of Schools
2	Mar 28 2020	April 23 2020	Issuance of State of Emergency
3	April 23 2020	May 20 2020	
4	May 20 2020	Jun 22 2020	
5	Jun 22 2020	Jul 24 2020	Summer Vacation
6	Jul 24 2020	Aug 25 2020	Obon, Summer Vacation
7	Aug 25 2020	Sep 23 2020	Summer Vacation
8	Sep 23 2020	Oct 20 2020	"Go to Travel" Campaign
			Relaxation of Immigration Policy
9	Oct 20 2020	Nov 14 2020	"Go to Eat" Campaign
			"Go to Travel" Campaign
10	Nov 14 2020	Dec 19 2020	
11	Dec 19 2020	Jan 12 2021	Issuance of State of Emergency
			Winter Vacation
12	Jan 12 2021	Feb 07 2021	
13	Feb 07 2021	Apr 09 2021	

Although some of the detected phases can be accounted for by identifying changes in governmental policies and major social events, many of them seem to result from changes in social interaction patterns that cannot be explained using public information sources (such as news websites). However, this is consistent with out intuition that social behavior is inherently dynamic – it displays significant changes even in the absence of government diktats and important calendar events. Moreover, except for the first phase, the length of every phase is at least 25 days, which points to the likely scenario that it takes at least 3 to 4 weeks for the contact rates to change significantly. This could be true because social behavior is often unorganized. In particular, the interaction patterns of any one individual are often not in synchronization with those of others.

Another noteworthy inference to be drawn from Table 5.1 and Figure 5.1 is that policy changes initiated by governments have a delayed effect at times. For example, the "Go to Travel" and the "Go to Eat" campaigns, launched between mid-September and mid-November (Phases 8 and 9), seem to have caused a spike in daily case counts in the subsequent phases (Phases 10 and 11). Likewise, the State of Emergency issued in Phase 11 seems to have come to fruition in Phase 12 and its effects appear to have remained until the last phase (Phase 13).

Parameter Estimation and Its Implications

Figure 5.1 below plots the original and the model-generated fractions of infected individuals in each age group as functions of time.

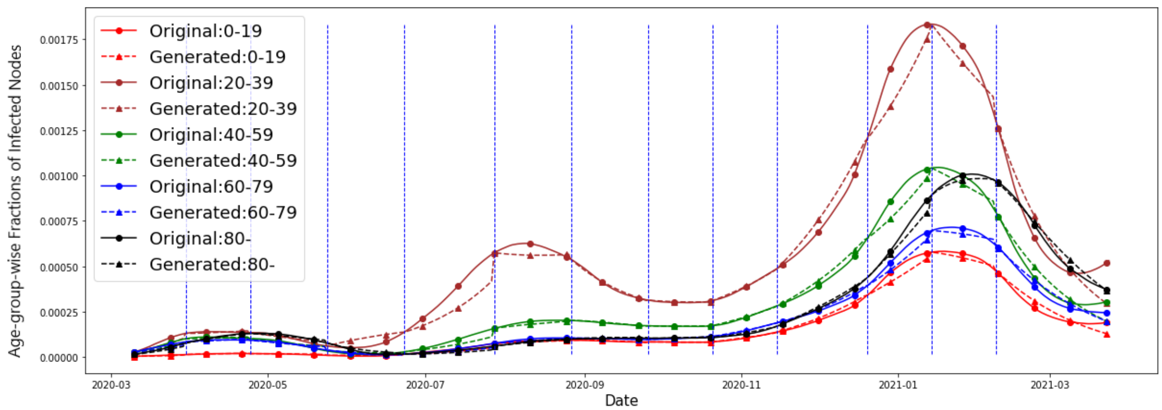

Figure 5.1. Age-wise Daily Fractions of Infected Individuals in Tokyo, Japan: Original and Generated Trajectories

Figure 5.2 plots the estimated contact rates and labels the 10 most significant ones among the 25 rates.

As seen in Figure 5.1, three COVID-19 surges or "waves" occur during the considered timeline. For each wave, we explain below the corresponding contact rate variations and their implications with the help of the mobility data of Tokyo (Figure 5.3) collected by Google [174].

The First Wave (March 2020 - June 2020, Phases 1 - 3)

This wave corresponds to a rapid surge in daily cases across the world followed by various governmental measures such as issuance of national emergencies, tightening of immigration policies, home quarantines, and school closures. In Japan, the national emergency consisted of various measures such as restrictions on service times in restaurants and bars, enforcement of

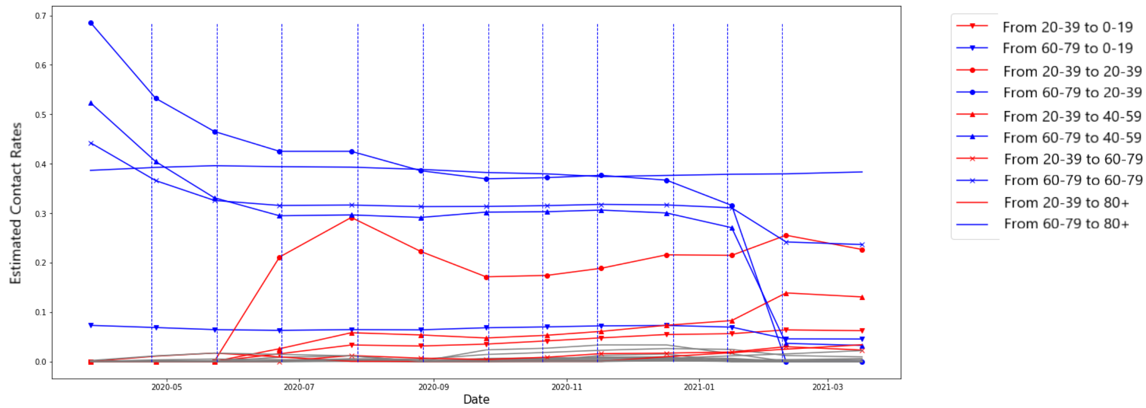

Figure 5.2. Estimated Contact Rate Between Groups

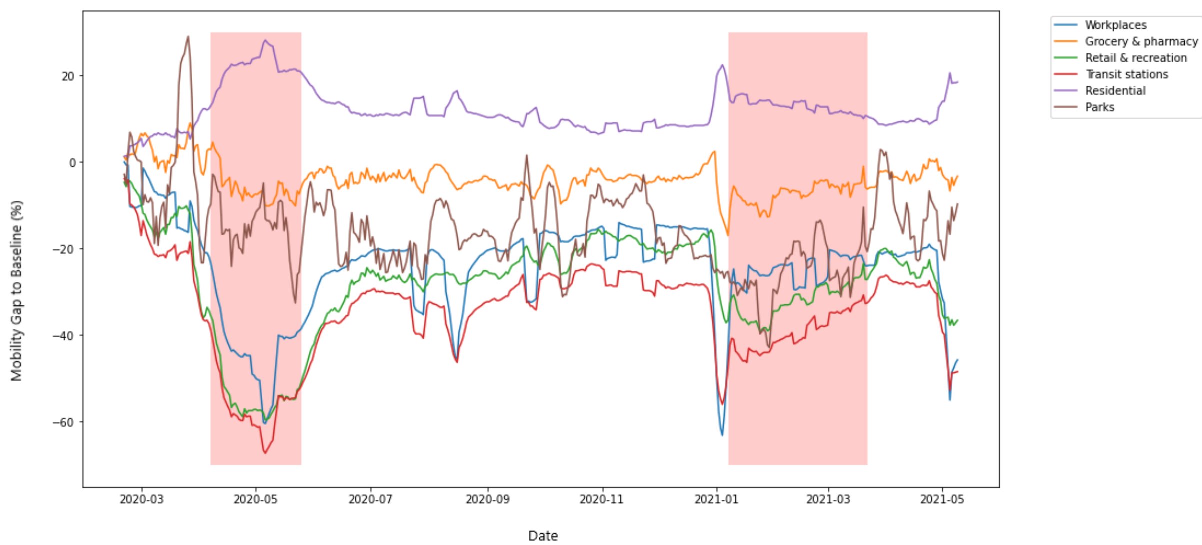

Figure 5.3. Mobility for Each Type of Place by Google
The period in which state of emergency is issued is highlighted in red.

work from home, and a limit on the number of people attending public events. As a result of these measures, the mobility of workplaces, retail and recreation, and transit stations dropped dramatically in April 2020 and remained low for over a month (Figure 5.3).

This drop is reflected in our simulation results (Figure 5.2), which show that the three greatest contact rates decreased steadily from April to June. However, Figure 5.2 also shows that contact rates from the age group 60-79 to most other age groups (shown in blue) remained remarkably high throughout the timeline Γ. This may be because most people admitted to nursing homes are aged above 60 and frequently come in contact with the relatively younger care-taking staff. More strikingly, the contact rate from age group 60-79 to age group 80+ is consistently high. This could be because there is a significant number of married couples with members from both these age groups (thereby resulting in a high value of ρ_{ij} for $(i,j) = (m,m-1)$) and because the age group 80+ has the lowest immunity levels, which leads to a large effective B_{ij} (infection rate) for $(i,j) = (m,m-1)$.

The Second Wave (July 2020 - September 2020, Phases 4 - 7)

The most intriguing aspect of the second wave is that the wave subsided without any significant governmental interventions (such as the issuance of a nationwide emergency). To explain this phenomenon, some researchers point out that (i) the rate of PCR testing increased in July and thus more infections were detected in the first few weeks of the second wave, and (ii) people's mobility decreased in August during the Japanese summer vacation period called "Obon" [2]. As we can infer from Figure 5.3, this decrease in mobility occurs primarily at workplaces and transit stations [2].

Besides Figure 5.3, our simulation results provide some insight into the second wave. Figure 5.1 shows that the contact rates from age group 60-79 to other age groups do not show any increase during the first few weeks of the wave. However, the intra-group contact rate of the age group 20-39 increases rapidly before this period and drops significantly in August, corresponding to a decrease in daily cases. This strongly suggests that the social activities of those aged between 20-39 played a key role in the second wave. Meanwhile, contact rates from the age group 60-79 decreased after the first wave, possibly because of an increase in the proportion of quarantined individuals among the elderly, which in turn could have resulted from an increased public awareness of older age groups' higher susceptibility to the virus.

The Third Wave (October 2020 - January 2021, Phases 8 - 11)

This wave was the most severe of the three because in October, a policy promoting domestic travel (the "Go to Travel" campaign) was implemented in the Tokyo prefecture and eating out was promoted as well (as part of the “Go to Eat” campaign). In addition, Japan started relaxing its immigration policy in October. [2] points out that the “major factors for this rise include the government’s implementation of further policies to encourage certain activities, relaxed immigration restrictions, and people not reducing their level of activity”. This observation is supported by Figure 5.3, which shows that there is no drop in mobility in any category during the third wave. As a result, daily infection counts dropped only after the second state of emergency was issued by the government on January 7, 2021.

In agreement with these observations are our simulation results (Figure 5.1), which show that the age group 60-79 remained the most infectious throughout the third wave, and that the contact rates from the age group 20-39 gradually increased in the early weeks of the wave. This was followed by a remarkable decrease in the intra-group contact rate of the age group 60-79 from mid-January onwards.

Comparing the Age Groups on the Basis of Infectiousness as well as Susceptibility

It is evident from Figure 5.2 that among all the five age groups, members of the youngest age group (0-19) are the least likely to contract COVID-19. This validates the current understanding of the scientific community that children and teenagers are more immune to the disease than adults. At the opposite extreme, the age groups 80+ and 20-39 appear to be the most vulnerable, possibly because members of the former group have the lowest immunity levels and the latter group exhibits the highest levels of mobility and social activity.

Besides throwing light on how the likelihood of receiving infection varies across age groups, Figure 5.2 also throws light on how the likelihood of transmitting the infection varies across age groups. From the figure, the two most infectious age groups are clearly 60-79 and 20-39. Surprisingly, the age group 80+ is found to be less infectious than the group 60-79, perhaps because of the lower social mobility of the former. The figure also shows that the age groups 0-19, 40-59 and 80+ are remarkably less infectious than the other two age groups. However, we need additional empirical evidence to validate these findings, and it would be interesting to see whether our inferences are echoed by future empirical studies.

5.6 Conclusion and Future Directions

We have analyzed the age-structured SIR model of epidemic spreading from both theoretical and empirical viewpoints. Starting from a stochastic epidemic model, we have shown that the ODEs defining the age-structured SIR model are the mean-field limits of a continuous-time Markov process evolving over a time-varying network that involves random, asynchronous interactions if and only if the social mixing rate grows unboundedly with the population size. We have also provided a lower-bound on the associated convergence rates in terms of the social mixing rate. As for empirical validation, we have proposed two algorithms: a least-square method to estimate the model parameters based on real data and a phase detection algorithm to detect changes in contact rates and hence also the most significant social behavioral changes that possibly occurred during the observed pandemic timeline. We have validated our model empirically by using it to approximate the trajectories of the numbers of susceptible, infected, and recovered individuals in the prefecture of Tokyo, Japan, over a period of more than 12 months. Our results show that for the purpose of forecasting the future of the COVID-19 pandemic and designing appropriate control policies, the age-structured SIR model is likely to be a strong contender among compartmental epidemic models.

Our analysis, however, has a few limitations. First, it is not clear whether the large number of phases detected by Algorithm 2 indicates rapidly changing social interaction patterns or simply that our model is unable to approximate the pandemic over timescales significantly longer than a month. Second, the outputs of our algorithms have a few surprising implications that are as yet unconfirmed by independent empirical studies. For example, the estimated contact rates indicate that the age group 60-79 is consistently more infectious than the age group 20-39, a finding that is inconsistent with the widely held belief that younger age groups are significantly more mobile than the older ones. Such apparent anomalies highlight the need for age-stratified mobility datasets that would enable further investigation into the dynamic interplay between social behavior and epidemic spreading.

Appendix

Our first aim is to prove Proposition 12, which is based on the Lemma 31. This lemma describes a known property of continuous-time Markov chains, but we prove it nevertheless. The proof is based on the concept of *jump times*, defined below.

Definition 53 (Jump Times)**.** *The* jump times *of the Markov chain* $\{\mathbf{X}(\tau) : \tau \geq 0\}$ *are the random times defined by* $J_0 := 0$ *and* $J_\ell := \inf\{\tau \geq 0 : \mathbf{X}(J_{\ell-1} + \tau) \neq \mathbf{X}(J_{\ell-1})\}$ *for all* $\ell \in \mathbb{N}$.

Note that jump times are simply the times at which the Markov chain jumps or transitions to a new state.

Lemma 31. *Let* $\mathbf{x} \in \mathbb{S}$ *and let* $[t, t + \Delta t) \subset [0, \infty)$*. Given that* $\mathbf{X}(t) = \mathbf{x}$*, the conditional probability that more than one state transitions occur during* $[t, t + \Delta t)$ *is* $o(\Delta t)$.

Proof. Let $\mathbf{y}, \mathbf{z} \in \mathbb{S}$ be any two states such that $\mathbf{y}$ and $\mathbf{z}$ potentially succeed $\mathbf{x}$ and $\mathbf{y}$, respectively. Also, let $\{\mathbf{X}(J_i)\}_{i=0}^{\infty}$ be the embedded jump chain of $\{\mathbf{X}(\tau)\}_{\tau \geq 0}$ (where $J_0 := 0$). Then, given that $\mathbf{X}(0) = \mathbf{x}$, $\mathbf{X}(J_1) = \mathbf{y}$, and $\mathbf{X}(J_2) = \mathbf{z}$, the holding times J_1 and $J_2 - J_1$ are conditionally independent exponential random variables with parameters $q_x := |\mathbf{Q}(\mathbf{x}, \mathbf{x})|$ and $q_y := |\mathbf{Q}(\mathbf{y}, \mathbf{y})|$, respectively. Therefore, given that the original Markov chain makes its first and second transitions from $\mathbf{x}$ to $\mathbf{y}$ and from $\mathbf{y}$ to $\mathbf{z}$ respectively, the conditional probability that both of these transitions occur during $[0, \Delta t)$ is given by $\Pr(J_2 < \Delta t \mid (\mathbf{X}(0), \mathbf{X}(J_1), \mathbf{X}(J_2)) = (\mathbf{x}, \mathbf{y}, \mathbf{z}))$, which is upper-bounded by $\Pr(J_2 - J_1 < \Delta t, J_1 < \Delta t \mid (\mathbf{X}(0), \mathbf{X}(J_1), \mathbf{X}(J_2)) = (\mathbf{x}, \mathbf{y}, \mathbf{z}))$, which equals

$$\begin{aligned}
&\Pr(J_2 - J_1 < \Delta t \mid (\mathbf{X}(0), \mathbf{X}(J_1), \mathbf{X}(J_2)) = (\mathbf{x}, \mathbf{y}, \mathbf{z})) \\
&\quad \cdot \Pr(J_1 < \Delta t \mid (\mathbf{X}(0), \mathbf{X}(J_1), \mathbf{X}(J_2)) = (\mathbf{x}, \mathbf{y}, \mathbf{z})) \\
&= (1 - e^{-q_y \Delta t})(1 - e^{-q_x \Delta t}) = o(\Delta t).
\end{aligned} \tag{5.29}$$

Therefore, $\Pr(J_2 < \Delta t \mid \mathbf{X}(0) = \mathbf{x})$, which is a quantity upper bounded by $\max_{\mathbf{y}, \mathbf{z} \in \mathbb{S}} \Pr(J_2 < \Delta t \mid (\mathbf{X}(0), \mathbf{X}(J_1), \mathbf{X}(J_t) = (\mathbf{x}, \mathbf{y}, \mathbf{z}))$, is $o(\Delta t)$. Hence, given that $\mathbf{X}(0) = \mathbf{x}$, the conditional probability that at least two state transitions occur during $[0, \Delta t)$ is $o(\Delta t)$. By time-homogeneity, this means the following: given that $\mathbf{X}(t) = \mathbf{x}$, the conditional probability that at least two state transitions occur during $[t, t + \Delta t)$ is $o(\Delta t)$. □

Proof of Proposition 12

Proof. We derive the equations one by one.

Proof of (i)

Consider any state $\mathbf{x} \in \mathbb{S}$. Then, by the definition of $\mathbf{Q}$, for any $i \in [n]$ and $a \in \mathcal{S}_i(\mathbf{x})$, we have

$$\Pr(\mathbf{X}(t + \Delta t) = \mathbf{x}_{\uparrow a} \mid \mathbf{X}(t) = \mathbf{x}) = \mathbf{Q}(\mathbf{x}, \mathbf{x}_{\uparrow a}) \Delta t + o(\Delta t) = \left(\sum_{k=1}^{m} B_{ik} E_k^{(a)}(\mathbf{x}) \right) \Delta t + o(\Delta t). \tag{5.30}$$

We now use (5.30) to evaluate the probability of the event $\{S_i(t + \Delta t) = S_i(t) - 1\}$. To this end, let $D_\ell(U, t, \Delta t))$ denote the event that exactly ℓ nodes in a given set $U \subset [n]$ recover during $[t, t + \Delta t)$ (i.e., there exist exactly ℓ indices $r_1, \ldots, r_\ell$ in U such that $X_{r_k}(t) = 1$ and $X_{r_k}(t + \Delta t) = -1$). Similarly, let $\mathscr{I}_\ell(U, t, \Delta t)$ denote the event that exactly ℓ nodes in U get infected during $[t, t+\Delta t)$. Then,

$$\begin{aligned}
&\Pr(S_i(t+\Delta t) - S_i(t) = -1 \mid \mathbf{X}(t) = \mathbf{x}) \\
&= \Pr(\mathscr{I}_1(\mathcal{A}_i, t, \Delta t) \mid \mathbf{X}(t) = \mathbf{x}) \\
&\overset{(a)}{=} \Pr(D_0([n], t, \Delta t) \cap \mathscr{I}_0([n] \setminus \mathcal{A}_i, t, \Delta t) \cap \mathscr{I}_1(\mathcal{A}_i, t, \Delta t) \mid \mathbf{X}(t) = \mathbf{x}) + o(\Delta t) \\
&= \Pr\left(\cup_{a \in \mathcal{S}_i(\mathbf{x})} \{\mathbf{X}(t+\Delta t) = \mathbf{x}_{\uparrow a}\} \mid \mathbf{X}(t) = \mathbf{x}\right) + o(\Delta t) \\
&= \sum_{a \in \mathcal{S}_i(\mathbf{x})} \Pr(\mathbf{X}(t+\Delta t) = \mathbf{x}_{\uparrow a} \mid \mathbf{X}(t) = \mathbf{x}) + o(\Delta t) \\
&\overset{(b)}{=} \left(\sum_{a \in \mathcal{S}_i(\mathbf{x})} \sum_{j=1}^{m} B_{ij} E_j^{(a)}(\mathbf{x}) \right) \Delta t + o(\Delta t) \\
&= \left(\sum_{j=1}^{m} \sum_{a \in \mathcal{S}_i(\mathbf{x})} \sum_{b \in \mathcal{I}_j(\mathbf{x})} B_{ij} 1_{(a,b)}(\mathbf{x}) \right) \Delta t + o(\Delta t). \qquad (5.31)
\end{aligned}$$

where (a) is a straightforward consequence of Lemma 31, and (b) follows from (5.30). Since this holds for all $\mathbf{x} \in \mathbb{S}$, we have

$$\Pr(S_i(t+\Delta t) - S_i(t) = -1 \mid \mathbf{X}(t)) = \left(\sum_{j=1}^{m} \sum_{a \in \mathcal{S}_i(t)} \sum_{b \in \mathcal{I}_j(t)} B_{ij} 1_{(a,b)}(t) \right) \Delta t + o(\Delta t), \qquad (5.32)$$

where $\mathcal{S}_i(t)$, $\mathcal{I}_i(t)$, and $1_{(a,b)}(t)$ stand for $\mathcal{S}_i(\mathbf{X}(t))$, $\mathcal{I}_i(\mathbf{X}(t))$, and $1_{(a,b)}(\mathbf{X}(t))$, respectively. Since $\mathcal{S}(t)$ and $\mathcal{I}(t)$ are determined by $\mathbf{X}(t)$, we may express (5.32) as

$$\Pr(S_i(t+\Delta t) - S_i(t) = -1 \mid \mathcal{S}(t), \mathcal{I}(t), \mathbf{X}(t)) = \left(\sum_{j=1}^{m} \sum_{a \in \mathcal{S}_i(t)} \sum_{b \in \mathcal{I}_j(t)} B_{ij} 1_{(a,b)}(t) \right) \Delta t + o(\Delta t). \qquad (5.33)$$

As a result, we have

$$\Pr(S_i(t+\Delta t)-S_i(t)=-1\mid \mathcal{S}(t),\mathcal{I}(t))=\left(\sum_{j=1}^{m}\sum_{a\in\mathcal{S}_i(t)}\sum_{b\in\mathcal{I}_j(t)}B_{ij}\mathbb{E}[1_{(a,b)}(t)\mid \mathcal{S}(t),\mathcal{I}(t)]\right)\Delta t + o(\Delta t). \tag{5.34}$$

At this point we note that

$$\mathbb{E}[1_{(a,b)}(t)\mid \mathcal{S}(t),\mathcal{I}(t)]=\Pr((a,b)\in E(t)\mid \mathcal{S}(t),\mathcal{I}(t))=\chi_{ij}(t,\mathcal{S},\mathcal{I}).$$

We thus have the following for $\Delta_t S_i := S_i(t+\Delta t)-S_i(t)$:

$$\begin{aligned}\mathbb{E}[\Delta_t S_i\mid \mathcal{S}(t),\mathcal{I}(t)] &= -\Pr(\Delta_t S_i=-1\mid \mathcal{S}(t),\mathcal{I}(t))-\sum_{\ell=2}^{n}\ell\cdot\Pr(\Delta_t S_i=-\ell)\\ &\overset{(a)}{=} -\Pr(\Delta_t S_i=-1\mid \mathcal{S}(t),\mathcal{I}(t))+o(\Delta t)\\ &= -\left(\sum_{j=1}^{m}\sum_{a\in\mathcal{S}_i(t)}\sum_{b\in\mathcal{I}_j(t)}B_{ij}\chi_{ij}(t)\right)\Delta t+o(\Delta t)\\ &= -\left(\sum_{j=1}^{m}B_{ij}\chi_{ij}(t)S_i(t)I_j(t)\right)\Delta t+o(\Delta t),\end{aligned} \tag{5.35}$$

where (a) follows from Lemma 31. Taking expectations on both sides of (5.35) and dividing the resulting relation by Δt now yields

$$\mathbb{E}\left[\frac{S_i(t+\Delta t)-S_i(t)}{\Delta t}\right]=-\sum_{j=1}^{m}B_{ij}\mathbb{E}[n\chi_{ij}(t)s_i(t)I_j(t)]+\frac{o(\Delta t)}{\Delta t}, \tag{5.36}$$

where we used that $S_i(t)=ns_i(t)$. On letting $\Delta t\to 0$ and then dividing both the sides of (5.36) by n, we obtain (i).

Proof of (ii)

Observe that for any $\mathbf{x} \in \mathbb{S}$, we have

$$\begin{aligned}
&\Pr(I_i(t+\Delta t) - I_i(t) = 1 \mid \mathbf{X}(t) = \mathbf{x}) \\
&= \Pr(|\mathcal{I}_i(\mathbf{X}(t+\Delta t))| - |\mathcal{I}_i(\mathbf{X}(t))| = 1 \mid \mathbf{X}(t) = \mathbf{x}) \\
&= \Pr(\cup_{\ell=0}^{n}(D_\ell(\mathcal{A}_i, t, \Delta t) \cap \mathscr{I}_{\ell+1}(\mathcal{A}_i, t, \Delta t)) \mid \mathbf{X}(t) = \mathbf{x}) \\
&\overset{(a)}{=} \Pr(D_0(\mathcal{A}_i, t, \Delta t) \cap \mathscr{I}_1(\mathcal{A}_i, t, \Delta t) \mid \mathbf{X}(t) = \mathbf{x}) + o(\Delta t) \\
&\overset{(b)}{=} \Pr(D_0([n], t, \Delta t) \cap \mathscr{I}_0([n] \setminus \mathcal{A}_i, t, \Delta t) \cap \mathscr{I}_1(\mathcal{A}_i, t, \Delta t) \mid \mathbf{X}(t) = \mathbf{x}) + o(\Delta t) \\
&= \Pr\left(\cup_{c \in \mathcal{S}_i(\mathbf{x})}\{\mathbf{X}(t+\Delta t) = \mathbf{x}_{\uparrow c}\} \mid \mathbf{X}(t) = \mathbf{x}\right) + o(\Delta t) \\
&= \sum_{c \in \mathcal{S}_i(\mathbf{x})} \Pr(\mathbf{X}(t+\Delta t) = \mathbf{x}_{\uparrow c} \mid \mathbf{X}(t) = \mathbf{x}) + o(\Delta t) \\
&\overset{(c)}{=} \left(\sum_{c \in \mathcal{S}_i(\mathbf{x})} \sum_{j=1}^{m} B_{ij} E_j^{(c)}(\mathbf{x})\right) \Delta t + o(\Delta t), \qquad (5.37)
\end{aligned}$$

where (a) and (b) follow from Lemma 31 and (c) follows from (5.30).

On the other hand,

$$\begin{aligned}
&\Pr(I_i(t+\Delta t) - I_i(t) = -1 \mid \mathbf{X}(t) = \mathbf{x}) \\
&= \Pr(\cup_{\ell=0}^{n}(D_{\ell+1}(\mathcal{A}_i, t, \Delta t) \cap \mathscr{I}_{\ell}(\mathcal{A}_i, t, \Delta t)) \mid \mathbf{X}(t) = \mathbf{x}) \\
&\overset{(a)}{=} \Pr(D_1(\mathcal{A}_i, t, \Delta t) \cap \mathscr{I}_0(\mathcal{A}_i, t, \Delta t) \mid \mathbf{X}(t) = \mathbf{x}) + o(\Delta t) \\
&\overset{(b)}{=} \Pr(D_1(\mathcal{A}_i, t, \Delta t) \cap D_0([n] \setminus \mathcal{A}_i, t, \Delta t) \cap \mathscr{I}_0([n], t, \Delta t) \mid \mathbf{X}(t) = \mathbf{x}) + o(\Delta t) \\
&= \sum_{c \in \mathcal{I}_i(\mathbf{x})} \Pr(\mathbf{X}(t+\Delta t) = \mathbf{x}_{\downarrow c} \mid \mathbf{X}(t) = \mathbf{x}) + o(\Delta t) \\
&= \sum_{c \in \mathcal{I}_i(\mathbf{x})} (\mathbf{Q}(\mathbf{x}, \mathbf{x}_{\downarrow c}) \Delta t + o(\Delta t)) + o(\Delta t) \\
&= \sum_{c \in \mathcal{I}_i(\mathbf{x})} \gamma_i \Delta t + o(\Delta t) \\
&= \gamma_i |I_i(\mathbf{x})| \Delta t + o(\Delta t). \qquad (5.38)
\end{aligned}$$

As a result of (5.37), (5.38), and Lemma 31, we have

$$\mathbb{E}[I_i(t+\Delta t) - I_i(t) \mid \mathbf{X}(t)] = \left(\sum_{c \in \mathcal{S}_i(\mathbf{x})} \sum_{j=1}^{m} B_{ij} E_j^{(c)}(\mathbf{X}(t)) - \gamma_i |\mathcal{I}_i(\mathbf{X}(t))| \right) \Delta t + o(\Delta t).$$

We can repeat the arguments used in the proof of (i) to prove that

$$\mathbb{E}[I_i(t+\Delta t) - I_i(t) \mid \mathcal{S}(t), \mathcal{I}(t)] = \left(\sum_{j=1}^{m} B_{ij} \chi_{ij}(t) S_i(t) I_j(t) - \gamma_i I_i(t) \right) \Delta t + o(\Delta t),$$

which implies that

$$\mathbb{E}\left[\frac{I_i(t+\Delta t) - I_i(t)}{\Delta t} \right] = \sum_{j=1}^{m} B_{ij} \mathbb{E}[n \chi_{ij}(t) s_i(t) I_j(t)] - \gamma_i \mathbb{E}[I_i(t)] + \frac{o(\Delta t)}{\Delta t}.$$

On dividing both sides by n and then letting $\Delta t \to 0$, we obtain (ii).

Proof of (iii)

Observe that when $\Delta_t S_i = -1$, we have $S_i^2(t+\Delta t) - S_i^2(t) = 1 - 2S_i(t)$. Therefore,

$$\begin{aligned}
\mathbb{E}[S_i^2(t+\Delta t) - S_i^2(t) \mid \mathcal{S}(t), \mathcal{I}(t)] &= (1 - 2S_i(t)) \cdot \Pr(\Delta_t S_i = -1 \mid \mathcal{S}(t), \mathcal{I}(t)) + o(\Delta t) \\
&= \left(\sum_{j=1}^{m} B_{ij} \chi_{ij} S_i I_j \Delta t - 2 \sum_{j=1}^{m} B_{ij} \chi_{ij} S_i^2 I_j \Delta t \right) + o(\Delta t).
\end{aligned}$$

Taking expectations on both sides, dividing by Δt, letting $\Delta t \to 0$, and dividing both sides by n^2 yields (iii).

Proof of (iv)

Observe that if $\Delta_t I_i(t) = -1$, we have $I_i^2(t+\Delta t) - I_i^2(t) = 1 - 2I_i(t)$, and if $\Delta_t I_i(t) = 1$, we have $I_i^2(t+\Delta t) - I_i^2(t) = 1 + 2I_i(t)$.

Thus,

$$
\begin{aligned}
&\mathbb{E}[I_i^2(t+\Delta t) - I_i^2(t) \mid \mathcal{S}(t), \mathcal{I}(t)] \\
&= (1-2I_i(t)) \cdot \Pr(\Delta_t I_i = -1 \mid \mathcal{S}(t), \mathcal{I}(t)) + (1+2I_i(t)) \cdot \Pr(\Delta_t I_i = 1 \mid \mathcal{S}(t), \mathcal{I}(t)) + o(\Delta t)
\end{aligned}
$$

On substituting the probabilities above with the expressions derived earlier, taking expectations on both sides, dividing by $n^2 \Delta t$ and letting $\Delta t \to 0$, we obtain (iv). □

Lemma 32. *Let $a \in \mathcal{A}_i$ and $b \in \mathcal{A}_j$ be any two nodes, let $t \in [0, \infty)$ be any time, and let $T \in [0, t]$ be the random variable such that $t - T$ is the time at which $1_{(a,b)}$ is updated for the last time during the interval $[0, t]$. Then the random variables T and $1_{(a,b)}(t)$ are independent.*

Proof. For $\tau \in [0, t]$, let N_τ denote the number of times $1_{(a,b)}$ is updated in the open interval $(t-\tau, t)$, and let U_τ denote the zero-probability event that $1_{(a,b)}$ is updated at time $t - \tau$. Note that $\mathbf{Q}(\mathbf{x}, \mathbf{x}_{\uparrow(a,b)}) + \mathbf{Q}(\mathbf{x}, \mathbf{x}_{\downarrow(a,b)}) = \lambda$ for all $\mathbf{x} \in \mathbb{S}$, which means that the rate at which $1_{(a,b)}$ is updated is time-invariant and independent of the network state. This means that the sequence of times at which $1_{(a,b)}$ is updated is a Poisson process, which further means that the updates of $1_{(a,b)}$ occurring in disjoint time intervals are independent. It follows that N_τ is a Poisson random variable (with mean $\lambda\tau$) that is independent of U_τ and $1_{(a,b)}(t-\tau)$. As a result,

$$
\begin{aligned}
\Pr((a,b) \in E(t) \mid T = \tau) &\overset{(a)}{=} \Pr((a,b) \in E(t-\tau) \mid U_\tau, N_\tau = 0) \\
&\overset{(b)}{=} \frac{\Pr(N_\tau = 0 \mid (a,b) \in E(t-\tau), U_\tau)}{\Pr(N_\tau = 0 \mid U_\tau)} \cdot \Pr((a,b) \in E(t-\tau) \mid U_\tau) \\
&= \frac{\Pr(N_\tau = 0 \mid 1_{(a,b)}(t-\tau) = 1, U_\tau)}{\Pr(N_\tau = 0 \mid U_\tau)} \cdot \Pr((a,b) \in E(t-\tau) \mid U_\tau) \\
&= \frac{\Pr(N_\tau = 0)}{\Pr(N_\tau = 0)} \cdot \Pr((a,b) \in E(t-\tau) \mid U_\tau) \\
&\overset{(c)}{=} \frac{\rho_{ij}}{n},
\end{aligned}
$$

where (a) follows from the definition of T, (b) follows from Bayes' rule, and (c) follows from the model definition (Section 5.2). Thus, $\Pr(1_{(a,b)}(t) = 1 \mid T = \tau)$ and $\Pr(1_{(a,b)}(t) = 0 \mid T = \tau)$ do not depend on τ, which means that T and $1_{(a,b)}(t)$ are independent. $\square$

The proof of Proposition 13 is based on the concepts of transition sequences and agnostic transition sequences, which we define below.

Definition 54 (Transition Sequence)**.** *Consider any time $t \geq 0$, integer $r \in \mathbb{N}_0$, tuples denoted by $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(r)} \in \mathbb{S}$, and times $0 \leq t_1 < t_2 < \cdots < t_r \leq t$. Let $F = \{\mathbf{x}^{(0)} \xrightarrow{t_1} \mathbf{x}^{(1)} \xrightarrow{t_2} \cdots \xrightarrow{t_r} \mathbf{x}^{(r)} \xrightarrow{t} \mathbf{x}^{(r)}\}$ denote the event that the embedded jump chain $\{\mathbf{X}(J_\ell) : \ell \in \mathbb{N}_0\}$ satisfies $\mathbf{X}(J_\ell) = \mathbf{x}^{(\ell)}$ and $J_\ell = t_\ell$ for all $\ell \in [r]$, and $J_{r+1} > t$. Then F is said to be a transition sequence for the time interval $[0, t]$.*

Note that if F is a transition sequence for $[0, t]$, then for every tuple $\mathbf{x} \in \mathbb{S}$, we either have $F \subset \{\mathbf{X}(t) = \mathbf{x}\}$ or $F \subset \{\mathbf{X}(t) \neq \mathbf{x}\}$.

Definition 55 ((a,b)-Complement)**.** *Let $\mathbf{x} \in \mathbb{S}$. Then the (a,b)-complement of $\mathbf{x}$, denoted by $\mathbf{x}_{\overline{(a,b)}}$, is defined by*

$$(\mathbf{x}_{\overline{(a,b)}})_\ell = \begin{cases} x_\ell & \text{if } \ell \in [2n^2 - n] \setminus \{\langle a,b \rangle\}, \\ 1 - x_\ell & \text{if } \ell = \langle a,b \rangle. \end{cases}$$

Definition 56 ((a,b)-Agnostic Transition Sequence)**.** *Let $a, b \in [n]$, and let $F = \{\mathbf{x}^{(0)} \xrightarrow{t_1} \mathbf{x}^{(1)} \xrightarrow{t_2} \cdots \xrightarrow{t_r} \mathbf{x}^{(r)} \xrightarrow{t} \mathbf{x}^{(r)}\}$ be a transition sequence for a time interval $[0, t]$. Further, let $\Lambda_{(a,b)}(F)$ be defined by $\Lambda_{(a,b)}(F) :=$*

$$\begin{cases} \max\left\{\ell \in [r] : \mathbf{x}^{(\ell)} \in \{\mathbf{x}^{(\ell-1)}_{\uparrow(a,b)}, \mathbf{x}^{(\ell-1)}_{\downarrow(a,b)}\}\right\} & \text{if } \left\{\ell \in [r] : \mathbf{x}^{(\ell)} \in \{\mathbf{x}^{(\ell-1)}_{\uparrow(a,b)}, \mathbf{x}^{(\ell-1)}_{\downarrow(a,b)}\}\right\} \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Then the (a,b)-agnostic transition sequence for F is the event $F_{?(a,b)} = F \cup F_{\overline{(a,b)}}$, where $F_{\overline{(a,b)}}$, *defined by*

$$F_{\overline{(a,b)}} := \Big\{ \mathbf{x}^{(0)} \xrightarrow{t_1} \cdots \xrightarrow{t_{\Lambda_{(a,b)}(F)-1}} \mathbf{x}^{(\Lambda_{(a,b)}(F)-1)} \xrightarrow{t_{\Lambda_{(a,b)}(F)}} \mathbf{x}_{\overline{(a,b)}}^{(\Lambda_{(a,b)}(F))}$$
$$\xrightarrow{t_{\Lambda_{(a,b)}(F)+1}} \mathbf{x}_{\overline{(a,b)}}^{(\Lambda_{(a,b)}(F)+1)} \xrightarrow{t_{\Lambda_{(a,b)}(F)+2}} \cdots \xrightarrow{t_r} \mathbf{x}_{\overline{(a,b)}}^{(r)} \xrightarrow{t} \mathbf{x}_{\overline{(a,b)}}^{(r)} \Big\},$$

is called the (a,b)*-complement of* F*.*

Given that F occurs, $t_{\Lambda_{(a,b)}(F)}$ denotes the time at which the edge state of (a,b) is updated for the last time during the time interval $[0,t]$ (note that, if the edge state of (a,b) is not updated during $[0,t]$, then $t_{\Lambda_{(a,b)}(F)} = t_0 := 0$). Therefore, the only difference between F and $F_{\overline{(a,b)}}$ is that the last edge state of (a,b) to be realized during the interval $[0,t]$ is different for F and $F_{(a,b)}$. Stated differently, if $F \subset \{(a,b) \in E(t)\}$, then $F_{\overline{(a,b)}} \subset \{(a,b) \notin E(t)\}$, and vice-versa. As a result, the event $F_{?(a,b)}$ is (a,b)-agnostic in that the occurrence of this event does not provide any information about the edge state of (a,b) at time t.

Note that if F is a transition sequence, then F, $F_{\overline{(a,b)}}$, and $F_{?(a,b)}$ are all zero-probability events. We now approximate these events with the help of suitable positive-probability events.

Definition 57 (δ-Approximation)**.** *Let* $F = \{\mathbf{x}^{(0)} \xrightarrow{t_1} \mathbf{x}^{(1)} \xrightarrow{t_2} \cdots \xrightarrow{t_r} \mathbf{x}^{(r)} \xrightarrow{t} \mathbf{x}^{(r)}\}$ *be a transition sequence. Then, for a given* $\delta > 0$*, the* δ*-approximation of* F *is the event* $F^\delta := \{\mathbf{X}(0) = \mathbf{x}^{(0)}, \ldots, \mathbf{X}(J_r) = \mathbf{x}^{(r)}, \Delta_1 J \in [\Delta_1 t, \Delta_1 t + \delta), \ldots, \Delta_r J \in [\Delta_r t, \Delta_r t + \delta), \Delta_{r+1} J > t - t_r\}$*, where* $\Delta_\ell J := J_\ell - J_{\ell-1}$*,* $\Delta_\ell t := t_\ell - t_{\ell-1}$*, and* $t_0 := 0$*. Also, the* δ*-approximation of* $F_{?(a,b)}$ *is the event* $F^\delta_{?(a,b)} := F^\delta \cup F^\delta_{\overline{(a,b)}}$ *(where* $F^\delta_{\overline{(a,b)}}$ *is the* δ*-approximation of* $F_{\overline{(a,b)}}$*).*

The following lemma evaluates the probability of occurrence of a δ-approximation event.

Lemma 33. *Let* $F = \{\mathbf{x}^{(0)} \xrightarrow{t_1} \mathbf{x}^{(1)} \xrightarrow{t_2} \cdots \xrightarrow{t_r} \mathbf{x}^{(r)} \xrightarrow{t} \mathbf{x}^{(r)}\}$ *be a transition sequence. Then for all sufficiently small* $\delta > 0$*, the ratio* $\frac{\Pr(F^\delta)}{\Pr(\mathbf{X}(0)=\mathbf{x}^{(0)})}$ *equals*

$$e^{-q_r(t-t_r)} \prod_{\ell=1}^{r} q_{\ell-1,\ell}\left(e^{-q_{\ell-1}(t_\ell - t_{\ell-1})}\delta + o(\delta)\right),$$

where $t_0 := 0$, $q_\ell := -\mathbf{Q}(\mathbf{x}^{(\ell)}, \mathbf{x}^{(\ell)})$, *and* $q_{\ell,\ell+1} := \mathbf{Q}(\mathbf{x}^{(\ell)}, \mathbf{x}^{(\ell+1)})$ *for each* $\ell \in [r]$.

Proof. Observe that

$$\begin{aligned}
\frac{\Pr(F^\delta)}{\Pr(\mathbf{X}(0) = \mathbf{x}^{(0)})} &= \Pr(\cap_{\ell\in[r]}\{\mathbf{X}(J_\ell) = \mathbf{x}^{(\ell)}, \Delta_\ell J \in [\Delta_\ell t, \Delta_\ell t + \delta)\} \mid \mathbf{X}(0) = \mathbf{x}^{(0)}) \\
&\quad \times \Pr(\Delta_{r+1} J > t - t_r \mid \cap_{\ell\in[r]}\{\mathbf{X}(J_\ell) = \mathbf{x}^{(\ell)}, \Delta_\ell J \in [\Delta_\ell t, \Delta_\ell t + \delta)\}) \\
&\overset{(a)}{=} \prod_{\ell=1}^{r} \Pr(\Delta_\ell J \in [\Delta_\ell t, \Delta_\ell t + \delta), \mathbf{X}(J_\ell) = \mathbf{x}^{(\ell)} \mid \mathbf{X}(J_{\ell-1}) = \mathbf{x}^{(\ell-1)}) \\
&\quad \times \Pr(\Delta_{r+1} J > t - t_r \mid \mathbf{X}(J_r) = \mathbf{x}^{(r)}) \\
&\overset{(b)}{=} \prod_{\ell=1}^{r} \Pr(\Delta_1 J \in [\Delta_\ell t, \Delta_\ell t + \delta), \mathbf{X}(J_1) = \mathbf{x}^{(\ell)} \mid \mathbf{X}(0) = \mathbf{x}^{(\ell-1)}) \\
&\quad \times \Pr(\Delta_1 J > t - t_r \mid \mathbf{X}(0) = \mathbf{x}^{(r)}) \\
&\overset{(c)}{=} \prod_{\ell=1}^{r} \Big(\Pr(\Delta_1 J \in [\Delta_\ell t, \Delta_\ell t + \delta) \mid \mathbf{X}(0) = \mathbf{x}^{(\ell-1)}) \\
&\quad \cdot \Pr(\mathbf{X}(J_1) = \mathbf{x}^{(\ell)} \mid \mathbf{X}(0) = x^{(\ell-1)}) \Big) \\
&\quad \times \Pr(\Delta_1 J > t - t_r \mid \mathbf{X}(0) = \mathbf{x}^{(r)}) \\
&\overset{(d)}{=} \prod_{\ell=1}^{r} \left(\left(q_{\ell-1} e^{-q_{\ell-1}\Delta_\ell t} \delta + o(\delta) \right) \frac{q_{\ell-1,\ell}}{q_{\ell-1}} \right) \times e^{-q_r(t-t_r)} \\
&= e^{-q_r(t-t_r)} \prod_{\ell=1}^{r} q_{\ell-1,\ell} \left(e^{-q_{\ell-1}(t_\ell - t_{\ell-1})} \delta + o(\delta) \right),
\end{aligned}$$

where (a) follows from the strong Markov property and the fact that jump times are stopping times, (b) follows from Proposition 3.2 of [14], (c) follows from the fact that $\mathbf{X}(J_1)$ and $\Delta_1 J$ (which equals J_1) are conditionally independent given $\mathbf{X}(0)$ (see Proposition 3.1 of [14]), and (d) follows from the following two facts:

1. $\Delta_1 J$ is conditionally exponentially distributed with mean $q_{\ell-1}^{-1}$ given that $\mathbf{X}(0) = \mathbf{x}^{(\ell-1)}$.

2. For the embedded jump chain, the probability of transitioning from $\mathbf{x} \in \mathbb{S}$ to $\mathbf{y} \in \mathbb{S}$ is $\frac{\mathbf{Q}(\mathbf{x},\mathbf{y})}{|\mathbf{Q}(\mathbf{x},\mathbf{x})|}$.

□

For the rest of the appendix, let $a \in \mathcal{A}_i$ and $b \in \mathcal{A}_j$ be any two nodes, let $t \in [0,\infty)$ be any time instant, let $T \in [0,t]$ be the random variable such that $t-T$ is the time at which $1_{(a,b)}$ is updated for the last time during the interval $[0,t)$, and let $K := t - \min\{t, \inf\{\tau : b \in \mathcal{I}(\tau)\}\}$, where $\inf\{\tau : b \in \mathcal{I}(\tau)\}$ is the time at which b gets infected.

Lemma 34. *The PDF of T has $[0,t]$ as its support and is given by*

$$f_T(\tau) = \lambda e^{-\lambda \tau} + e^{-\lambda t} \delta_D(\tau - t),$$

where $\delta_D(\cdot)$ is the Dirac-delta function.

Proof. The definition of T implies that the support of its PDF is $[0,t]$. To derive the required closed-form expression for this PDF, recall that $\mathbf{Q}(\mathbf{x}, \mathbf{x}_{\uparrow(a,b)}) + \mathbf{Q}(\mathbf{x}, \mathbf{x}_{\downarrow(a,b)}) = \lambda$ for all $\mathbf{x} \in \mathbb{S}$, which means that the edge state of (a,b) is updated at a constant rate of λ at all times. Therefore, for any $\tau \in [0,t)$, the quantity $\Pr(T > \tau)$ (the probability that $1_{(a,b)}$ is not updated during $[t-\tau, t]$) is given by $e^{-\lambda \tau}$. However, $\Pr(T > t) = 0$, implying that $\Pr(T = t) = \Pr(T \geq t) = \lim_{\tau \to t^-} \Pr(T > \tau) = e^{-\lambda t}$. Hence, the CDF of T is $F(\tau) = 1 - e^{-\lambda \tau}$ for all $\tau \in [0,t)$, and $F(t) = 1$. Taking the first derivative of this CDF now yields the required expression for f_T. □

To prove the next lemma, we need the notion of agnostic superstates, which is defined below.

Definition 58 ((a,b)-Agnostic Superstate)**.** *Given a node pair $(a,b) \in [n] \times [n]$, a collection of states $\mathbb{X} \subset \mathbb{S}$ is an (a,b)-*agnostic superstate *if $\mathbb{X}$ can be expressed as $\mathbb{X} = \left\{\mathbf{x}, \mathbf{x}_{\overline{(a,b)}}, \mathbf{y}, \mathbf{y}_{\overline{(a,b)}}\right\}$ for a pair of states $\mathbf{x}, \mathbf{y} \in \mathbb{S}$ satisfying $y_{n^2+\langle a,b \rangle} = 1 - x_{n^2+\langle a,b \rangle}$ and $x_\ell = y_\ell$ for all $\ell \in [2n^2 - n] \setminus \{n^2 + \langle a,b \rangle\}$.*

Note that an (a,b)-agnostic superstate specifies the disease states of all the nodes and the edge states of all the node pairs except (a,b).

Definition 59 (((a,b)-Agnostic Jump Times)**.** *Given* $(a,b) \in [n] \times [n]$*, the* (a,b)-agnostic jump times *of the chain* $\{\mathbf{X}(\tau) : \tau \geq 0\}$*, denoted by* $\{L_k\}_{k=0}^{\infty}$*, are defined by* $L_0 := 0$ *and* $L_k := \inf\{J_\ell : \ell \in \mathbb{N}, J_\ell > L_{k-1}, \mathbf{X}(J_\ell) \notin \{(\mathbf{X}(J_{\ell-1}))_{\uparrow(a,b)}, (\mathbf{X}(J_{\ell-1}))_{\downarrow(a,b)}\}\}$ *for all* $k \in \mathbb{N}$*.*

Note that $\{L_k\}_{k=0}^{\infty} \subset \{J_\ell\}_{\ell=0}^{\infty}$ and that the (a,b)-agnostic jump times of $\{\mathbf{X}(\tau)\}$ are the jump times of the chain at which the edge state of (a,b) is not updated.

Lemma 35. *K is independent of* $(T, 1_{(a,b)}(t))$*.*

Proof. Note that K is a function of $\tilde{K} := \inf\{\tau \geq 0 : b \in \mathcal{I}(\tau)\}$, the time at which b gets infected. Hence, it suffices to prove that $\tilde{K}$ is independent of $(T, 1_{(a,b)}(t))$.

Consider now any $\kappa \geq 0$ and note that $\{\tilde{K} \geq \kappa\} = \cup_{N=0}^{\infty} \left(\{\tilde{K} = L_N\} \cap \{L_N \geq \kappa\}\right)$. To examine the probability of $\{\tilde{K} = L_N\}$, we let $\mathcal{F}_N(\kappa)$ denote the set of all the events of the form $F = \{\mathbf{X}(0) \in \mathbb{X}^{(0)}, \mathbf{X}(L_1) \in \mathbb{X}^{(1)}, \ldots, \mathbf{X}(L_N) \in \mathbb{X}^{(N)}\}$ (where $\mathbb{X}^{(0)}, \ldots, \mathbb{X}^{(N)}$ are (a,b)-agnostic superstates satisfying $\mathbb{X}^{(k)} \neq \mathbb{X}^{(k-1)}$ for all $k \in [N]$) that satisfy $F \subset \{\tilde{K} = L_N\}$, and we observe that $\{\tilde{K} = L_N\} = \cup_{F \in \mathcal{F}_N(\kappa)} F$.

We now examine $\Pr(F)$ for an arbitrary $F = \{\mathbf{X}(0) \in \mathbb{X}^{(0)}, \mathbf{X}(L_1) \in \mathbb{X}^{(1)}, \ldots, \mathbf{X}(L_N) \in \mathbb{X}^{(N)}\} \in \mathcal{F}_N(\kappa)$. Pick any $k \in [N]$ and $\mathbf{x} \in \mathbb{X}^{(k-1)}$. Note that $F \subset \{\tilde{K} = L_N\}$ implies that $b \in \mathcal{S}(\mathbf{x})$. In view of our definition of $\mathbf{Q}$, this means that $\mathbf{Q}(\mathbf{x}, \mathbf{x}) = -\sum_{\mathbf{z} \in \mathbb{S} \setminus \{\mathbf{x}\}} \mathbf{Q}(\mathbf{x}, \mathbf{z})$, which possibly depends on the disease states $\{x_1, \ldots, x_n\}$ and on the edge states $\{1_{(c,d)}(\mathbf{x}) : (c,d) \in [n] \times [n] : d \in \mathcal{I}(\mathbf{x})\}$, does not depend on $1_{(a,b)}(\mathbf{x})$. We next observe that, by the definitions of (a,b)-agnostic states and jump times, none of the possible transitions from $\mathbb{X}^{(k-1)}$ to $\mathbb{X}^{(k)}$ involves an edge state update for (a,b), which means that the values of both $X_{\langle a,b \rangle} = 1_{(a,b)}(\mathbf{X})$ and $X_{n^2 + \langle a,b \rangle}$ are preserved in such transitions. Therefore, for every $\mathbf{x} \in \mathbb{X}^{(k-1)}$, there exists at most one state $\mathbf{y} \in \mathbb{X}^{(k)}$ that potentially succeeds $\mathbf{x}$. For such a state $\mathbf{y}$, the transition rate

$\mathbf{Q}(\mathbf{x},\mathbf{y})$, given by

$$\mathbf{Q}(\mathbf{x},\mathbf{y}) = \begin{cases} \sum_{q=1}^{m}\sum_{d\in\mathcal{I}_q(\mathbf{x})} B_{pq}1_{(c,d)}(\mathbf{x}) & \text{if } \exists\, c \in \mathcal{S}_p(\mathbf{x}) \text{ such that } \mathbf{y} = \mathbf{x}_{\uparrow c} \\ \gamma_p & \text{if } \exists\, c \in \mathcal{I}_p(\mathbf{x}) \text{ such that } \mathbf{y} = \mathbf{x}_{\downarrow c}, \\ \lambda\frac{\rho_{pq}}{n} & \text{if } \exists\, (c,d) \in \mathcal{A}_p \times \mathcal{A}_q \setminus \{(a,b)\} \text{ with } \mathbf{y} = \mathbf{x}_{\uparrow(c,d)}, \\ \lambda\left(1-\frac{\rho_{pq}}{n}\right) & \text{if } \exists\, (a,b) \in \mathcal{A}_p \times \mathcal{A}_q \setminus \{(a,b)\} \text{ with } \mathbf{y} = \mathbf{x}_{\downarrow(c,d)}, \end{cases}$$

does not depend on $x_{\langle a,b\rangle} = y_{\langle a,b\rangle}$ or on $x_{n^2+\langle a,b\rangle} = y_{n^2+\langle a,b\rangle}$, because $b \notin \mathcal{I}(\mathbf{x})$ implies that $(a,b) \notin \cup_{p=1}^{m} \cup_{c\in\mathcal{S}_p(\mathbf{x})} \cup_{q=1}^{m} \{(c,d) : d \in \mathcal{I}_q(\mathbf{x})\}$. It follows that the rate at which the Markov chain $\{\mathbf{X}(\tau)\}$ transitions from $\mathbf{x}$ to a state in $\mathbb{X}^{(k)}$, given by $\sum_{\mathbf{z}\in\mathbb{X}^{(k)}} \mathbf{Q}(\mathbf{x},\mathbf{z}) = \mathbf{Q}(\mathbf{x},\mathbf{y})$, is time-invariant and takes the same value for every $\mathbf{x} \in \mathbb{X}^{(k-1)}$. This means that, as long as the Markov chain $\{\mathbf{X}(\tau) : \tau \geq 0\}$ does not leave the (a,b)-agnostic superstate $\mathbb{X}^{(k-1)}$, the rate at which the chain transitions to $\mathbb{X}^{(k)}$ remains the same regardless of transitions within $\mathbb{X}^{(k-1)}$. We can express this formally as

$$\lim_{\Delta\tau\to 0} \frac{\Pr(\mathbf{X}(\tau+\Delta\tau) \in \mathbb{X}^{(k)} \mid \mathbf{X}(\tau) \in \mathbb{X}^{(k-1)}, \mathbf{X}(\tau) = \mathbf{x})}{\Delta\tau} = \mathbf{Q}(\mathbb{X}^{(k-1)}, \mathbb{X}^{(k)})$$

for all $\tau \geq 0$ and $\mathbf{x} \in \mathbb{X}^{(k-1)}$, where

$$\mathbf{Q}(\mathbb{X}^{(k-1)}, \mathbb{X}^{(k)}) := \lim_{\Delta\tau\to 0} \frac{\Pr(\mathbf{X}(\tau+\Delta\tau) \in \mathbb{X}^{(k)} \mid \mathbf{X}(\tau) \in \mathbb{X}^{(k-1)})}{\Delta\tau}$$

denotes the time-invariant rate of transitioning from $\mathbb{X}^{(k-1)}$ to $\mathbb{X}^{(k)}$.

By Markovity, this implies that[2]

$$\lim_{\Delta\tau\to 0} \frac{\Pr(\mathbf{X}(\tau+\Delta\tau) \in \mathbb{X}^{(k)} \mid \mathbf{X}(\tau) = \mathbf{x}, \{\mathbf{X}(\tau') : 0 \leq \tau' < \tau\})}{\Delta\tau} = \mathbf{Q}(\mathbb{X}^{(k-1)}, \mathbb{X}^{(k)}),$$

[2]In this chapter, conditioning an event H on $\{\mathbf{X}(\tau') : 0 \leq \tau' \leq \tau\}$ means conditioning H on every $\mathbf{X}(\tau')$ for $0 \leq \tau' \leq \tau$, i.e., conditioning H on the trajectory traced by the Markov chain during the interval $[0,\tau]$ and not just on the random set of tuples $\{\mathbf{X}(\tau') : 0 \leq \tau' \leq \tau\}$. Conditioning on the set $\{\mathbf{X}(\tau') : 0 \leq \tau' \leq \tau\}$ is not sufficient because sets, by definition, are unordered.

for all $\tau \geq 0$ and $\mathbf{x} \in \mathbb{X}^{(k-1)}$, which means that the conditional rate at which the chain leaves $\mathbb{X}^{(k-1)}$ time τ is independent of the history $\{\mathbf{X}(\tau') : \tau' \in [0, \tau]\}$. Since $\{1_{(a,b)}(\tau') : 0 \leq \tau' \leq \tau\}$ are determined by $\{\mathbf{X}(\tau') : 0 \leq \tau' \leq \tau\}$, it follows that

$$\lim_{\Delta\tau \to 0} \frac{\Pr(\mathbf{X}(\tau + \Delta\tau) \in \mathbb{X}^{(k)} \mid \mathbf{X}(\tau) = \mathbf{x}, \{1_{(a,b)}(\tau') : 0 \leq \tau' \leq \tau\})}{\Delta\tau} = \mathbf{Q}(\mathbb{X}^{(k-1)}, \mathbb{X}^{(k)})$$

for all $\tau \geq 0$ and $\mathbf{x} \in \mathbb{X}^{(k-1)}$. Now, let $T_{\uparrow}(\tau) := \inf\{\tau' \geq 0 : \mathbf{X}(\tau + \tau') = (\mathbf{X}((\tau + \tau')^{-}))_{\uparrow(a,b)}\}$ be the (random) time elapsed between time τ and the first of the updates of $1_{(a,b)}$ that occur after time τ and result in $1_{(a,b)} = 1$. Then, the following holds for all $\mathbf{x} \in \mathbb{X}^{(k-1)}$, $\tau \geq 0$, $\sigma > 0$ and sufficiently small $\Delta\tau > 0$:

$$\begin{aligned}
&\Pr(T_{\uparrow}(\tau) \geq \sigma \mid \mathbf{X}(\tau) = \mathbf{x}, \{1_{(a,b)}(\tau') : 0 \leq \tau' \leq \tau\}, \mathbf{X}(\tau + \Delta\tau) \in \mathbb{X}^{(k)}) \\
&= \Pr(T_{\uparrow}(\tau + \Delta\tau) \geq \sigma - \Delta\tau \\
&\qquad\qquad \mid \mathbf{X}(\tau) = \mathbf{x}, \{1_{(a,b)}(\tau') : 0 \leq \tau' \leq \tau\}, \mathbf{X}(\tau + \Delta\tau) \in \mathbb{X}^{(k)}, T_{\uparrow}(\tau) \geq \Delta\tau) \\
&\quad \cdot \Pr(T_{\uparrow}(\tau) \geq \Delta\tau \mid \mathbf{X}(\tau) = \mathbf{x}, \{1_{(a,b)}(\tau') : 0 \leq \tau' \leq \tau\}, \mathbf{X}(\tau + \Delta\tau) \in \mathbb{X}^{(k)}) \\
&= \Pr(T_{\uparrow}(\tau + \Delta\tau) \geq \sigma - \Delta\tau \mid \mathbf{X}(\tau) = \mathbf{x}, \{1_{(a,b)}(\tau') : 0 \leq \tau' \leq \tau\}, \mathbf{X}(\tau + \Delta\tau) \in \mathbb{X}^{(k)}, \\
&\qquad\qquad\qquad\qquad\qquad\qquad X_{n^2 + \langle a,b \rangle}(\tau') = X_{n^2 + \langle a,b \rangle}(\tau) \, \forall \tau' \in [\tau, \tau + \Delta\tau)) \\
&\quad \cdot \Pr(T_{\uparrow}(\tau) \geq \Delta\tau \mid \mathbf{X}(\tau) = \mathbf{x}, \{1_{(a,b)}(\tau') : 0 \leq \tau' \leq \tau\}, \mathbf{X}(\tau + \Delta\tau) \in \mathbb{X}^{(k)}) \\
&\overset{(a)}{=} e^{-\lambda \frac{\rho_{ij}}{n}(\sigma - \Delta\tau)} \cdot \Pr(T_{\uparrow}(\tau) \geq \Delta\tau \mid \mathbf{X}(\tau) = \mathbf{x}, \{1_{(a,b)}(\tau') : 0 \leq \tau' \leq \tau\}, \mathbf{X}(\tau + \Delta\tau) \in \mathbb{X}^{(k)}) \\
&\overset{\Delta\tau \to 0}{\longrightarrow} e^{-\lambda \frac{\rho_{ij}}{n}\sigma} \cdot 1 \\
&\overset{(b)}{=} \Pr(T_{\uparrow}(\tau) \geq \sigma \mid \mathbf{X}(\tau) = \mathbf{x}, \{1_{(a,b)}(\tau') : 0 \leq \tau' \leq \tau\}),
\end{aligned}$$

where (a) and (b) follow from Markovity and the fact that $\mathbf{Q}(\mathbf{z}, \mathbf{z}_{\uparrow(a,b)}) = \lambda \frac{\rho_{ij}}{n}$ for all $F \in \mathbb{S}$. Now, let $T_{\uparrow}^{(1)}(\tau) := T_{\uparrow}(\tau)$ and $T_{\uparrow}^{(\ell)}(\tau) := T_{\uparrow}(T_{\uparrow}^{(\ell-1)}(\tau))$ for all $\ell \in \mathbb{N}$. Then, since $\{T_{\uparrow}^{(\ell)}\}_{\ell=1}^{\infty}$ are stopping times, similar arguments can be used to show the following for all $\sigma_1, \sigma_2, \ldots, \sigma_\ell \geq 0$

and all $\ell \in \mathbb{N}$

$$\lim_{\Delta\tau\to 0} \Pr\Big(T_{\uparrow}^{(1)}(\tau) \geq \sigma_1, \ldots, T_{\uparrow}^{(\ell)}(\tau) \geq \sigma_\ell \\ \mid \mathbf{X}(\tau) = \mathbf{x}, \{1_{(a,b)}(\tau') : 0 \leq \tau' \leq \tau\}, \mathbf{X}(\tau+\Delta\tau) \in \mathbb{X}^{(k)}\Big) \\ = \Pr\left(T_{\uparrow}^{(1)}(\tau) \geq \sigma_1, \ldots, T_{\uparrow}^{(\ell)}(\tau) \geq \sigma_\ell \mid \mathbf{X}(\tau) = \mathbf{x}, \{1_{(a,b)}(\tau') : 0 \leq \tau' \leq \tau\}\right).$$

Similarly, if we let $T_{\downarrow}^{(1)} := \inf\{\tau' \geq 0 : \mathbf{X}(\tau+\tau') = (\mathbf{X}((\tau+\tau')^-))_{\downarrow(a,b)}\}$ and $T_{\downarrow}^{(\ell)}(\tau) := T_{\downarrow}(T_{\downarrow}^{(\ell-1)}(\tau))$ for all $\ell \in \mathbb{N}$, then we can show that for all $\sigma_{\uparrow 1}, \ldots, \sigma_{\uparrow \ell}, \sigma_{\downarrow_1}, \ldots, \sigma_{\downarrow \ell'} \geq 0$ and all $\ell, \ell' \in \mathbb{N}$,

$$\lim_{\Delta\tau\to 0} \Pr\Big(T_{\uparrow}^{(1)}(\tau) \geq \sigma_{\uparrow 1}, \ldots, T_{\uparrow}^{(\ell)}(\tau) \geq \sigma_{\uparrow \ell}, T_{\downarrow}^{(1)}(\tau) \geq \sigma_{\downarrow 1}, \ldots, T_{\downarrow}^{(\ell')}(\tau) \geq \sigma_{\downarrow \ell'} \\ \mid \mathbf{X}(\tau) = \mathbf{x}, \{1_{(a,b)}(\tau') : 0 \leq \tau' \leq \tau\}, \mathbf{X}(\tau+\Delta\tau) \in \mathbb{X}^{(k)}\Big) \\ = \Pr\Big(T_{\uparrow}^{(1)}(\tau) \geq \sigma_{\uparrow 1}, \ldots, T_{\uparrow}^{(\ell)}(\tau) \geq \sigma_{\uparrow \ell}, T_{\downarrow}^{(1)}(\tau) \geq \sigma_{\downarrow 1}, \ldots, T_{\downarrow}^{(\ell')}(\tau) \geq \sigma_{\downarrow \ell'} \\ \mid \mathbf{X}(\tau) = \mathbf{x}, \{1_{(a,b)}(\tau') : 0 \leq \tau' \leq \tau\}\Big).$$

As a result, we have the following for all $\sigma_{\uparrow 1}, \ldots, \sigma_{\uparrow \ell}, \sigma_{\downarrow_1}, \ldots, \sigma_{\downarrow \ell'} \geq 0$ and all $\ell, \ell' \in \mathbb{N}$:

$$\frac{\Pr\left(\mathbf{X}(\tau+\Delta\tau) \in \mathbb{X}^{(k)} \mid \mathbf{X}(\tau) = \mathbf{x}, \{1_{(a,b)}(\tau')\}, \{T_{\uparrow}^{(\xi)}(\tau) \geq \sigma_{\uparrow\xi}\}, \{T_{\downarrow}^{(\xi)}(\tau) \geq \sigma_{\downarrow\xi}\}\right)}{\Delta\tau} \\ \stackrel{(a)}{=} \frac{\Pr\left(\{T_{\uparrow}^{(\xi)}(\tau) \geq \sigma_{\uparrow\xi}\}, \{T_{\downarrow}^{(\xi)}(\tau) \geq \sigma_{\downarrow\xi}\} \mid \mathbf{X}(\tau) = \mathbf{x}, \{1_{(a,b)}(\tau')\}, \mathbf{X}(\tau+\Delta\tau) \in \mathbb{X}^{(k)}\right)}{\Pr\left(\{T_{\uparrow}^{(\xi)}(\tau) \geq \sigma_{\uparrow\xi}\}_{\xi=1}^{\ell}, \{T_{\downarrow}^{(\xi)}(\tau) \geq \sigma_{\downarrow\xi}\}_{\xi=1}^{\ell'} \mid \mathbf{X}(\tau) = \mathbf{x}, \{1_{(a,b)}(\tau')\}_{\tau' \in [0,\tau]}\right)} \\ \times \frac{\Pr\left(\mathbf{X}(\tau+\Delta\tau) \in \mathbb{X}^{(k)} \mid \mathbf{X}(\tau) = \mathbf{x}, \{1_{(a,b)}(\tau')\}_{\tau' \in [0,\tau]}\right)}{\Delta\tau} \\ \stackrel{\Delta\tau\to 0}{\longrightarrow} 1 \times \mathbf{Q}(\mathbb{X}^{(k-1)}, \mathbb{X}^{(k)}),$$

i.e.,

$$\lim_{\Delta\tau\to 0} \frac{\Pr\Big(\mathbf{X}(\tau+\Delta\tau)\in\mathbb{X}^{(k)} \mid \mathbf{X}(\tau)=\mathbf{x}, \{1_{(a,b)}(\tau')\}_{\tau'\in[0,\tau]}, \{T_\uparrow^{(\xi)}(\tau)\}_{\xi=1}^{\ell}, \{T_\downarrow^{(\xi)}(\tau)\}_{\xi=1}^{\ell'}\Big)}{\Delta\tau}$$
$$= \mathbf{Q}(\mathbb{X}^{(k-1)}, \mathbb{X}^{(k)})$$

for all $\ell, \ell' \in \mathbb{N}$. Now, observe that if we are given $\{1_{(a,b)}(\tau') : 0 \le \tau' \le \tau\}$, then $\{1_{(a,b)}(\tau') : \tau \le \tau' \le t\}$ are determined by a subset of the random variables $\{T_\uparrow^{(\ell)}\}_{\ell=1}^{\infty} \cup \{T_\downarrow^{(\ell)}\}_{\ell=1}^{\infty}$ and this subset is random but almost surely finite. Hence, the above limit implies the following for all $\mathbf{x} \in \mathbb{X}^{(k-1)}$ and $\tau \ge 0$:

$$\lim_{\Delta\tau\to 0} \frac{\Pr\Big(\mathbf{X}(\tau+\Delta\tau)\in\mathbb{X}^{(k)} \mid \mathbf{X}(\tau)=\mathbf{x}, \{1_{(a,b)}(\tau') : 0 \le \tau' \le t\}\Big)}{\Delta\tau} = \mathbf{Q}(\mathbb{X}^{(k-1)}, \mathbb{X}^{(k)}).$$

Moreover, since the above arguments remain valid if we replace $\mathbb{X}^{(k)}$ with an arbitrary (a,b)-agnostic superstate $\mathbb{Y} \ne \mathbb{X}^{(k-1)}$, we can generalize the above to

$$\lim_{\Delta\tau\to 0} \frac{\Pr\Big(\mathbf{X}(\tau+\Delta\tau)\in\mathbb{Y} \mid \mathbf{X}(\tau)=\mathbf{x}, \{1_{(a,b)}(\tau') : 0 \le \tau' \le t\}\Big)}{\Delta\tau} = \mathbf{Q}(\mathbb{X}^{(k-1)}, \mathbb{Y})$$

for all (a,b)-agnostic superstates $\mathbb{Y} \ne \mathbb{X}^{(k-1)}$. It follows that

$$\lim_{\Delta\tau\to 0} \frac{\Pr\Big(\mathbf{X}(\tau+\Delta\tau)\notin\mathbb{X}^{(k-1)} \mid \mathbf{X}(\tau)=\mathbf{x}, \{1_{(a,b)}(\tau') : 0 \le \tau' \le t\}\Big)}{\Delta\tau} = \sum_{\mathbb{Y}\ne\mathbb{X}^{(k-1)}} \mathbf{Q}(\mathbb{X}^{(k-1)}, \mathbb{Y}) \tag{5.39}$$

for all $\mathbf{x} \in \mathbb{X}^{(k-1)}$ and all $\tau \ge 0$. This means that, given $\{\mathbf{X}(L_{k-1}) = \mathbf{x}\}$ for some $\mathbf{x} \in \mathbb{X}^{(k-1)}$, the random quantity $L_k - L_{k-1}$, which is the duration of time spent by the Markov chain in $\mathbb{X}^{(k-1)}$, is conditionally exponentially distributed with rate $\sum_{\mathbb{Y}\ne\mathbb{X}^{(k-1)}} \mathbf{Q}(\mathbb{X}^{(k-1)}, \mathbb{Y})$ and it is conditionally independent of $\{1_{(a,b)}(\tau') : 0 \le \tau' \le t\}$. Besides, the above deductions also imply the following: given $\{1_{(a,b)}(\tau') : 0 \le \tau' \le t\}$ and given that the chain exits $\mathbb{X}^{(k-1)}$ from state $\mathbf{x}$

at time $\tau \geq 0$, the conditional probability that it enters $\mathbb{X}^{(k)}$ at time τ is

$$\begin{aligned}
&\Pr(\mathbf{X}(\tau) \in \mathbb{X}^{(k)} \mid \mathbf{X}(\tau^-) = \mathbf{x}, \{1_{(a,b)}(\tau') : 0 \leq \tau' \leq t\}, \mathbf{X}(\tau) \notin \mathbb{X}^{(k-1)}) \\
&= \lim_{\Delta\tau \to 0} \Pr(\mathbf{X}(\tau) \in \mathbb{X}^{(k)} \mid \mathbf{X}(\tau - \Delta\tau) = \mathbf{x}, \{1_{(a,b)}(\tau') : 0 \leq \tau' \leq t\}, \mathbf{X}(\tau) \notin \mathbb{X}^{(k-1)}) \\
&= \lim_{\Delta\tau \to 0} \frac{\Pr(\mathbf{X}(\tau) \in \mathbb{X}^{(k)} \mid \mathbf{X}(\tau - \Delta\tau) = \mathbf{x}, \{1_{(a,b)}(\tau') : 0 \leq \tau' \leq t\})}{\Pr(\mathbf{X}(\tau) \notin \mathbb{X}^{(k-1)} \mid \mathbf{X}(\tau - \Delta\tau) = \mathbf{x}, \{1_{(a,b)}(\tau') : 0 \leq \tau' \leq t\})} \\
&= \lim_{\Delta\tau \to 0} \frac{\Pr(\mathbf{X}(\tau) \in \mathbb{X}^{(k)} \mid \mathbf{X}(\tau - \Delta\tau) = \mathbf{x}, \{1_{(a,b)}(\tau') : 0 \leq \tau' \leq t\})}{\Delta\tau} \\
&\quad \times \lim_{\Delta\tau \to 0} \left(\frac{\Pr(\mathbf{X}(\tau) \notin \mathbb{X}^{(k-1)} \mid \mathbf{X}(\tau - \Delta\tau) = \mathbf{x}, \{1_{(a,b)}(\tau') : 0 \leq \tau' \leq t\})}{\Delta\tau} \right)^{-1} \\
&= \frac{\mathbf{Q}(\mathbb{X}^{(k-1)}, \mathbb{X}^{(k)})}{\sum_{\mathbb{Y} \neq \mathbb{X}^{(k-1)}} \mathbf{Q}(\mathbb{X}^{(k-1)}, \mathbb{Y})}.
\end{aligned}$$

By invoking Markovity in the preceding arguments, the above can be generalized to

$$\begin{aligned}
&\Pr(\mathbf{X}(\tau) \in \mathbb{X}^{(k)} \mid \mathbf{X}(\tau^-) = \mathbf{x}, \{\mathbf{X}(\tau')\}_{\tau' \in [0,\tau)}, \{1_{(a,b)}(\tau')\}_{\tau' \in [0,t]}, \mathbf{X}(\tau) \notin \mathbb{X}^{(k-1)}) \\
&= \frac{\mathbf{Q}(\mathbb{X}^{(k-1)}, \mathbb{X}^{(k)})}{\sum_{\mathbb{Y} \neq \mathbb{X}^{(k-1)}} \mathbf{Q}(\mathbb{X}^{(k-1)}, \mathbb{Y})},
\end{aligned} \tag{5.40}$$

which implies that

$$\begin{aligned}
\Pr(\mathbf{X}(\tau) \in \mathbb{X}^{(k)} \mid \mathbf{X}(\tau') = \mathbf{x} \forall \tau' \in [L_{k-1}, \tau), \{\mathbf{X}(\tau')\}\{1_{(a,b)}(\tau')\}_{\tau' \in [0,t]}, \mathbf{X}(\tau) \notin \mathbb{X}^{(k-1)}) \\
= \frac{\mathbf{Q}(\mathbb{X}^{(k-1)}, \mathbb{X}^{(k)})}{\sum_{\mathbb{Y} \neq \mathbb{X}^{(k-1)}} \mathbf{Q}(\mathbb{X}^{(k-1)}, \mathbb{Y})}.
\end{aligned}$$

Equivalently,

$$\begin{aligned}
\Pr(\mathbf{X}(L_k) \in \mathbb{X}^{(k)} \mid \mathbf{X}(L_{k-1}) = \mathbf{x}, \{\mathbf{X}(\tau')\}_{\tau' \in [0, L_{k-1})}, \{1_{(a,b)}(\tau')\}_{\tau' \in [0,t]}, L_k = \tau) \\
= \frac{\mathbf{Q}(\mathbb{X}^{(k-1)}, \mathbb{X}^{(k)})}{\sum_{\mathbb{Y} \neq \mathbb{X}^{(k-1)}} \mathbf{Q}(\mathbb{X}^{(k-1)}, \mathbb{Y})}.
\end{aligned}$$

for all $\tau > 0$ and $\mathbf{x} \in \mathbb{X}^{(k-1)}$. Hence,

$$\begin{aligned}
&\Pr(\mathbf{X}(L_k) \in \mathbb{X}^{(k)} \mid \mathbf{X}(L_{k-1}) \in \mathbb{X}^{(k-1)}, \{\mathbf{X}(\tau')\}_{\tau' \in [0, L_{k-1})}, \{1_{(a,b)}(\tau')\}_{\tau' \in [0,t]}, L_k) \\
&= \frac{\mathbf{Q}(\mathbb{X}^{(k-1)}, \mathbb{X}^{(k)})}{\sum_{\mathbb{Y} \neq \mathbb{X}^{(k-1)}} \mathbf{Q}(\mathbb{X}^{(k-1)}, \mathbb{Y})}.
\end{aligned} \tag{5.41}$$

Since the entire analysis above holds for all $k \in [N]$, we have the following for all indices $\sigma_1, \sigma_2, \ldots, \sigma_N \geq 0$.

$$\begin{aligned}
&\Pr\left(\left(\cap_{k=1}^N \{L_k - L_{k-1} \geq \sigma_k\}\right) \cap \left(\cap_{k=0}^N \{\mathbf{X}(L_k) \in \mathbb{X}^{(k)}\}\right) \mid \mathbf{X}(0) \in \mathbb{X}^{(0)}, \{1_{(a,b)}(\tau')\}\right) \\
&= \prod_{k=1}^N \Pr(\mathbf{X}(L_k) \in \mathbb{X}^{(k)}, L_k - L_{k-1} \geq \sigma_k \mid \{\mathbf{X}(L_\xi) \in \mathbb{X}^{(\xi)}, L_\xi - L_{\xi-1} \geq \sigma_\xi\}, \{1_{(a,b)}(\tau')\}) \\
&= \prod_{k=1}^N \Pr(L_k - L_{k-1} \geq \sigma_k \mid \{\mathbf{X}(L_\xi) \in \mathbb{X}^{(\xi)}, L_\xi - L_{\xi-1} \geq \sigma_\xi\}_{\xi=0}^{k-1}, \{1_{(a,b)}(\tau')\}_{\tau' \in [0,t]}) \\
&\quad \times \prod_{k=1}^N \Pr(\mathbf{X}(L_k) \in \mathbb{X}^{(k)} \mid L_k - L_{k-1} \geq \sigma_k, \{\mathbf{X}(L_\xi) \in \mathbb{X}^{(\xi)}, L_\xi - L_{\xi-1} \geq \sigma_\xi\}, \{1_{(a,b)}(\tau')\}) \\
&\overset{(a)}{=} \prod_{k=1}^N \Pr(L_k - L_{k-1} \geq \sigma_k \mid \mathbf{X}(L_{k-1}) \in \mathbb{X}^{(k-1)}, \{1_{(a,b)}(\tau')\}_{\tau' \in [0,t]}) \\
&\quad \times \prod_{k=1}^N \Pr(\mathbf{X}(L_k) \in \mathbb{X}^{(k)} \mid L_k - L_{k-1} \geq \sigma_k, \{\mathbf{X}(L_\xi) \in \mathbb{X}^{(\xi)}, L_\xi - L_{\xi-1} \geq \sigma_\xi\}, \{1_{(a,b)}(\tau')\}) \\
&\overset{(b)}{=} \prod_{k=1}^N \exp\left(-\sigma_k \sum_{\mathbb{Y} \neq \mathbb{X}^{(k-1)}} \mathbf{Q}(\mathbb{X}^{(k-1)}, \mathbb{Y})\right) \times \prod_{k=1}^N \frac{\mathbf{Q}(\mathbb{X}^{(k-1)}, \mathbb{X}^{(k)})}{\sum_{\mathbb{Y} \neq \mathbb{X}^{(k-1)}} \mathbf{Q}(\mathbb{X}^{(k-1)}, \mathbb{Y})},
\end{aligned} \tag{5.42}$$

where (a) is a consequence of the strong Markov property and the fact that $\{L_k\}_{k=1}^N$ are stopping times, and (b) follows from (5.39) and (5.41). Since $\sigma_1, \ldots, \sigma_N$ are arbitrary and since the above expression is independent of $\{1_{(a,b)}(\tau) : 0 \leq \tau \leq t\}$, we have shown that for the event

$$F \cap \{L_N \geq \kappa\} = \left\{\mathbf{X}(0) \in \mathbb{X}^{(0)}, \ldots, \mathbf{X}(L_N) \in \mathbb{X}^{(N)}, \sum_{k=1}^N (L_k - L_{k-1}) \geq \kappa\right\},$$

we have $\Pr(F \cap \{L_N \geq \kappa\} \mid \{1_{(a,b)}(\tau) : 0 \leq \tau \leq t\}) = \Pr(F \cap \{L_N \geq \kappa\})$. As a result,

$$\begin{aligned}
&\Pr(\{\tilde{K} = L_N\} \cap \{L_N \geq \kappa\} \mid \{1_{(a,b)}(\tau) : 0 \leq \tau \leq t\}) \\
&= \Pr\left(\cup_{F \in \mathcal{F}_N(\kappa)} (F \cap \{L_N \geq \kappa\}) \mid \{1_{(a,b)}(\tau) : 0 \leq \tau \leq t\}\right) \\
&\overset{(a)}{=} \sum_{F \in \mathcal{F}_N(\kappa)} \Pr\left(F \cap \{L_N \geq \kappa\} \mid \{1_{(a,b)}(\tau) : 0 \leq \tau \leq t\}\right) \\
&= \sum_{F \in \mathcal{F}_N(\kappa)} \Pr(F \cap \{L_N \geq \kappa\}) \\
&\overset{(b)}{=} \Pr\left(\cup_{F \in \mathcal{F}_N(\kappa)} (F \cap \{L_N \geq \kappa\})\right) \\
&= \Pr(\{\tilde{K} = L_N\} \cap \{L_N \geq \kappa\}),
\end{aligned}$$

where (a) and (b) hold because the definition of $\mathcal{F}_N(\kappa)$ implies that $\mathcal{F}_N(\kappa)$ is a collection of disjoint events. Since $\{\tilde{K} \geq \kappa\} = \cup_{N=0}^{\infty} \left(\{\tilde{K} = L_N\} \cap \{L_N \geq \kappa\}\right)$ and since $\{\tilde{K} = L_1\} \cap \{L_1 \geq \kappa\}, \{\tilde{K} = L_2\} \cap \{L_2 \geq \kappa\}, \ldots$ are disjoint events, it follows that $\Pr(\tilde{K} \geq \kappa \mid \{1_{(a,b)}(\tau) : 0 \leq \tau \leq t\}) = \Pr(\tilde{K} \geq \kappa)$. Moreover, since $\kappa \geq 0$ is arbitrary, this means that $\tilde{K}$ is independent of $\mid \{1_{(a,b)}(\tau) : 0 \leq \tau \leq t\}$. Finally, since K and $(T, 1_{(a,b)}(t))$ are functions of $\tilde{K}$ and $\{1_{(a,b)}(\tau) : 0 \leq \tau \leq t\}$, respectively, it follows that K and $(T, 1_{(a,b)}(t))$ are independent. □

Remark 15. *Observe that in the proof of Lemma 35,* (5.42) *implies that the event* $\{L_N \geq \kappa\} \cap \left(\cap_{k=0}^{N} \{\mathbf{X}(L_k) \in \mathbb{X}^{(k)}\}\right)$ *is independent of* $\{1_{(a,b)}(\tau) : 0 \leq \tau \leq t\}$ *(since* $L_N = \sum_{k=1}^{N} (L_k - L_{k-1})$ *and since the initial state* $\mathbf{X}(0)$ *is assumed to be non-random). Note that this is true for all the choices of* (a,b)*-agnostic superstates* $\{\mathbb{X}^{(k)}\}_{k=0}^{N}$ *that satisfy* $\cap_{k=0}^{N} \{\mathbf{X}(L_k) \in \mathbb{X}^{(k)}\} \subset \{\tilde{K} = L_N\}$ *and hence also for all* $\{\mathbb{X}^{(k)}\}_{k=0}^{N}$ *that satisfy* $\cap_{k=0}^{N} \{\mathbf{X}(L_k) \in \mathbb{X}^{(k)}\} \subset \{\tilde{K} = L_N\} \cap \{\mathbf{X}(\tilde{K}) \in \mathbb{Y}\}$*, where* $\mathbb{Y}$ *is an arbitrary* (a,b)*-agnostic superstate. Now, let us by* $\mathcal{X}$ *the set of all* $\{\mathbb{X}^{(k)}\}_{k=0}^{N}$ *satisfying* $\cap_{k=0}^{N} \{\mathbf{X}(L_k) \in \mathbb{X}^{(k)}\} \subset \{\tilde{K} = L_N\} \cap \{\mathbf{X}(\tilde{K}) \in \mathbb{Y}\}$*, we have*

$$\cup_{\{\mathbb{X}^{(k)}\}_{k=0}^{N} \in \mathcal{X}} \left(\cap_{k=0}^{N} \{\mathbf{X}(L_k) \in \mathbb{X}^{(k)}\}\right) = \{\tilde{K} = L_N\} \cap \{\mathbf{X}(\tilde{K}) \in \mathbb{Y}\}.$$

Then, by the preceding arguments we have

$$\begin{aligned}
&\Pr(\{\tilde{K}=L_N\}\cap\{\mathbf{X}(\tilde{K})\in\mathbb{Y}\}\cap\{L_N\geq\kappa\}\mid\{1_{(a,b)}(\tau):0\leq\tau\leq t\})\\
&=\Pr\left(\cup_{\{\mathbb{X}^{(k)}\}_{k=0}^N\in\mathcal{X}}\left(\cap_{k=0}^N\{\mathbf{X}(L_k)\in\mathbb{X}^{(k)}\}\right)\cap\{L_N\geq\kappa\}\mid\{1_{(a,b)}(\tau):0\leq\tau\leq t\}\right)\\
&=\sum_{\{\mathbb{X}^{(k)}\}_{k=0}^N\in\mathcal{X}}\Pr\left(\cap_{k=0}^N\{\mathbf{X}(L_k)\in\mathbb{X}^{(k)}\}\cap\{L_N\geq\kappa\}\mid\{1_{(a,b)}(\tau):0\leq\tau\leq t\}\right)\\
&=\sum_{\{\mathbb{X}^{(k)}\}_{k=0}^N\in\mathcal{X}}\Pr\left(\cap_{k=0}^N\{\mathbf{X}(L_k)\in\mathbb{X}^{(k)}\}\cap\{L_N\geq\kappa\}\right)\\
&=\Pr\left(\cup_{\{\mathbb{X}^{(k)}\}_{k=0}^N\in\mathcal{X}}\left(\cap_{k=0}^N\{\mathbf{X}(L_k)\in\mathbb{X}^{(k)}\}\cap\{L_N\geq\kappa\}\right)\right)\\
&=\Pr(\{\tilde{K}=L_N\}\cap\{\mathbf{X}(\tilde{K})\in\mathbb{Y}\}\cap\{L_N\geq\kappa\}),
\end{aligned}\tag{5.43}$$

which shows that $\{\tilde{K}=L_N\}\cap\{\mathbf{X}(\tilde{K})\in\mathbb{Y}\}\cap\{L_N\geq\kappa\}$ *is independent of* $\{1_{(a,b)}(\tau):0\leq\tau\leq t\}$. *Since* $\{\tilde{K}\geq\kappa\}\cap\{\mathbf{X}(\tilde{K})\in\mathbb{Y}\}=\cup_{N=0}^{\infty}\left(\{\tilde{K}=L_N\}\cap\{\mathbf{X}(\tilde{K})\in\mathbb{Y}\}\cap\{L_N\geq\kappa\}\right)$*, it follows that* $\{\tilde{K}\geq\kappa\}\cap\{\mathbf{X}(\tilde{K})\in\mathbb{Y}\}$ *is independent of* $\{1_{(a,b)}(\tau):0\leq\tau\leq t\}$. *As a consequence of this observation, the fact that* $\mathbb{Y}$ *is an arbitrary* (a,b)*-agnostic superstate and the fact that* κ *is an arbitrary non-negative number, we have that* $(\mathbb{X}(\tilde{K}),\tilde{K})$ *are independent of* $\{1_{(a,b)}(\tau):0\leq\tau\leq t\}$*, where* $\mathbb{X}(\tilde{K})$ *denotes the* (a,b)*-agnostic superstate of the chain at time* $\tilde{K}$.

In order to state the remaining lemmas, we need to introduce some additional notation. For two nodes $(a,b)\in[n]\times[n]$, we let $b\overset{t}{\rightsquigarrow}a$ denote the event that b transmits pathogens to a at time t. For a given time interval $[t,t+\Delta t)\subset[0,\infty)$, we let $\left\{b\overset{t,\Delta t}{\rightsquigarrow}a\right\}:=\cup_{\tau\in[t,t+\Delta t)}\{b\overset{\tau}{\rightsquigarrow}a\}$. The complement of this event is denoted by $\left\{b\overset{t,\Delta t}{\not\rightsquigarrow}a\right\}$. For two given node sets $A,B\subset[n]$, we use $\{B\overset{t}{\rightsquigarrow}A\}$ to denote the event that some node(s) of B infect(s) one or more nodes in A at time t.

We now provide a sequence of lemmas that we later use to prove Proposition 13.

Lemma 36. *Suppose* $a\in\mathcal{A}_i$, $b\in\mathcal{A}_j$, $y\in\{0,1\}$*, and* $t_1,t_2\in[0,\infty)$ *such that* $t_1<t_2$. *Given that* $b\in\mathcal{I}_j(t_1):=\mathcal{I}_j(\mathbf{X}(t_1))$ *and that* $1_{(a,b)}(\tau):=1_{(a,b)}(\mathbf{X}(\tau))=y$ *for all* $\tau\in[t_1,t_2)$*, the conditional*

probability that b neither recovers nor infects a during the interval $[t_1,t_2)$ is $e^{-(B_{ij}\delta_{1y}+\gamma_j)(t_2-t_1)}$, where δ_{ij} is the Kronecker delta.

Proof. Let $\mathbb{X} := \{\mathbf{x} \in \mathbb{S} : b \in \mathcal{I}_j(\mathbf{x}), 1_{(a,b)}(\mathbf{x}) = y\}$. Also, let $\Delta t > 0$. Since the rate of infection transmission from b to a at time t_1 is $B_{ij}1_{(a,b)}(\mathbf{X}(t_1))$, we have the following for all $\mathbf{x} \in \mathbb{X}$:

$$\Pr\left(b \overset{t_1,\Delta t}{\rightsquigarrow} a \,\middle|\, \mathbf{X}(t_1) = \mathbf{x}\right) = B_{ij}\delta_{1y}\Delta t + o(\Delta t).$$

On the other hand, denoting the event that b recovers during $[t_1, t_1 + \Delta t)$ by D_b, we have

$$\Pr(D_b \mid \mathbf{X}(t) = \mathbf{x}) = \gamma_j \Delta t + o(\Delta t).$$

Similarly, if we let $F_{(a,b)}$ denote the event that the edge state $1_{(a,b)}$ flips (i.e., changes from y to $1-y$) during $[t_1, t_1 + \Delta t)$, we have $\Pr(F_{(a,b)} \mid \mathbf{X}(t) = \mathbf{x}) = \lambda\left(y\left(1 - \frac{\rho_{ij}}{n}\right) + (1-y)\frac{\rho_{ij}}{n}\right)\Delta t + o(\Delta t)$. As a result, we have

$$\begin{aligned}
&\Pr\left(\left\{b \overset{t_1,\Delta t}{\not\rightsquigarrow} a\right\} \cap \bar{D}_b \cap \bar{F}_{(a,b)} \,\middle|\, \mathbf{X}(t_1) = \mathbf{x}\right)\\
&= 1 - \Pr\left(\left\{b \overset{t_1,\Delta t}{\rightsquigarrow} a\right\} \cup D_b \cup F_{(a,b)} \,\middle|\, \mathbf{X}(t_1) = \mathbf{x}\right)\\
&\overset{(a)}{=} 1 - \Pr\left(b \overset{t_1,\Delta t}{\rightsquigarrow} a \,\middle|\, \mathbf{X}(t_1) = \mathbf{x}\right) - \Pr(D_b \mid \mathbf{X}(t_1) = \mathbf{x}) - \Pr(F_{(a,b)} \mid \mathbf{X}(t_1) = \mathbf{x}\}) + o(\Delta t)\\
&= 1 - (B_{ij}\delta_{1y}\Delta t + o(\Delta t)) - (\gamma_j \Delta t + o(\Delta t)) - \left(\lambda\left(y\left(1 - \frac{\rho_{ij}}{n}\right) + (1-y)\frac{\rho_{ij}}{n}\right)\Delta t + o(\Delta t)\right)\\
&\quad + o(\Delta t)\\
&= 1 - \left(B_{ij}\delta_{1y} + \gamma_j + \lambda\left(y\left(1 - \frac{\rho_{ij}}{n}\right) + (1-y)\frac{\rho_{ij}}{n}\right)\right)\Delta t + o(\Delta t),
\end{aligned}$$

where (a) follows from Lemma 31 and the Inclusion-Exclusion principle. Since this holds for all

$\mathbf{x} \in \mathbb{X}$, the above implies that

$$\begin{aligned}
&\Pr\left(\left\{b \overset{t_1,\Delta t}{\not\rightsquigarrow} a\right\} \cap \bar{D}_b \cap \bar{F}_{(a,b)} \,\middle|\, \mathbf{X}(t_1) \in \mathbb{X}\right) \\
&= 1 - \left(B_{ij}\delta_{1y} + \gamma_j + \lambda\left(y\left(1 - \frac{\rho_{ij}}{n}\right) + (1-y)\frac{\rho_{ij}}{n}\right)\right)\Delta t + o(\Delta t).
\end{aligned}$$

Now, consider any $\ell \in \mathbb{N}_0$. By replacing t_1 with $t_1 + \ell\Delta t$ in the above relation, we obtain

$$\begin{aligned}
&\Pr\left(\left\{b \overset{t_1+\ell\Delta t,\Delta t}{\not\rightsquigarrow} a\right\} \cap \bar{D}_b^{(\ell)} \cap \bar{F}_{(a,b)}^{(\ell)} \,\middle|\, \mathbf{X}(t_1+\ell\Delta t) \in \mathbb{X}\right) \\
&= 1 - \left(B_{ij}\delta_{1y} + \gamma_j + \lambda\left(y\left(1 - \frac{\rho_{ij}}{n}\right) + (1-y)\frac{\rho_{ij}}{n}\right)\right)\Delta t + o(\Delta t),
\end{aligned}$$

where $D_b^{(\ell)}$ is the event that b recovers during $[t_1+\ell\Delta t, t_1+(\ell+1)\Delta t)$ and $F_{(a,b)}^{(\ell)}$ is the event that $1_{(a,b)}$ flips during $[t_1+\ell\Delta t, t_1+(\ell+1)\Delta t)$. Therefore, on setting $\Delta t = \frac{t_2-t_1}{N}$ for an arbitrary $N \in \mathbb{N}$, it follows that

$$\begin{aligned}
&\Pr\left(\{b \in \mathcal{I}(t)\} \cap \left\{b \overset{t_1,t_2-t_1}{\not\rightsquigarrow} a\right\} \cap \{1_{(a,b)}(\tau) = y \,\forall \tau \in [t_1,t_2)\} \,\middle|\, \mathbf{X}(t_1) = \mathbf{x}\right) \\
&= \prod_{\ell=1}^{N-1} \Pr\left(\left\{b \overset{t_1+\ell\Delta t,\Delta t}{\not\rightsquigarrow} a\right\} \cap \bar{D}_b^{(\ell)} \cap \bar{F}_{(a,b)}^{(\ell)} \right. \\
&\qquad\qquad \left.\middle|\, b \overset{t_1,\ell\Delta t}{\not\rightsquigarrow} a, \bar{D}_b, \bar{D}_b^{(1)}, \ldots, \bar{D}_b^{(\ell-1)}, \bar{F}_{(a,b)}, \ldots, \bar{F}_{(a,b)}^{(\ell-1)}, \mathbf{X}(t_1) = \mathbf{x}\right) \\
&\quad \times \Pr\left(\left\{b \overset{t_1,\Delta t}{\not\rightsquigarrow} a\right\} \cap \bar{D}_b \cap \bar{F}_{(a,b)} \,\middle|\, \mathbf{X}(t_1) = \mathbf{x}\right) \\
&= \prod_{\ell=1}^{N-1} \Pr\left(\left\{b \overset{t_1+\ell\Delta t,\Delta t}{\not\rightsquigarrow} a\right\} \cap \bar{D}_b^{(\ell)} \cap \bar{F}_{(a,b)}^{(\ell)}\right. \\
&\overset{(a)}{=} \prod_{\ell=1}^{N-1} \left(1 - \left(B_{ij}\delta_{1y} + \gamma_j + \lambda\left(y\left(1 - \frac{\rho_{ij}}{n}\right) + (1-y)\frac{\rho_{ij}}{n}\right)\right)\Delta t + o(\Delta t)\right) \\
&\quad \times \left(1 - \left(B_{ij}\delta_{1y} + \gamma_j + \lambda\left(y\left(1 - \frac{\rho_{ij}}{n}\right) + (1-y)\frac{\rho_{ij}}{n}\right)\right)\Delta t + o(\Delta t)\right), \qquad (5.44)
\end{aligned}$$

i.e.,

$$
\begin{aligned}
&\Pr\left(\{b\in\mathcal{I}(t)\}\cap\left\{b\overset{t_1,t_2-t_1}{\not\rightsquigarrow}a\right\}\cap\{1_{(a,b)}(\tau)=y\,\forall\tau\in[t_1,t_2)\}\;\middle|\;\mathbf{X}(t_1)=\mathbf{x}\right)\\
&\qquad\qquad\left|\;\mathbf{X}(t_1+\ell\Delta t)\in\mathbb{X},b\overset{t_1,\ell\Delta t}{\not\rightsquigarrow}a,\{\bar{D}_b^{(\sigma)}\}_{\sigma=0}^{\ell-1},\{\bar{F}_{(a,b)}^{(\sigma)}\}_{\sigma=0}^{\ell-1},\mathbf{X}(t_1)=\mathbf{x}\right)\\
&\quad\times\left(1-\left(B_{ij}\delta_{1y}+\gamma_j+\lambda\left(y\left(1-\frac{\rho_{ij}}{n}\right)+(1-y)\frac{\rho_{ij}}{n}\right)\right)\Delta t+o(\Delta t)\right)\\
&=\left(1-\left(B_{ij}\delta_{1y}+\gamma_j+\lambda\left(y\left(1-\frac{\rho_{ij}}{n}\right)+(1-y)\frac{\rho_{ij}}{n}\right)\right)\left(\frac{t_2-t_1}{N}\right)\right)^N+o\left(\frac{1}{N}\right),
\end{aligned}\tag{5.45}
$$

where (a) follows from the following observation: for any $\mathbf{y}\in\mathbb{X}$, Markovity implies that

$$
\begin{aligned}
&\Pr\left(\left\{b\overset{t_1+\ell\Delta t,\Delta t}{\not\rightsquigarrow}a\right\}\cap\bar{D}_b^{(\ell)}\cap\bar{F}_{(a,b)}^{(\ell)}\right.\\
&\qquad\left|\;\mathbf{X}(t_1+\ell\Delta t)=\mathbf{y},b\overset{t_1,\ell\Delta t}{\not\rightsquigarrow}a,\{\bar{D}_b^{(\sigma)}\}_{\sigma=0}^{\ell-1},\{\bar{F}_{(a,b)}^{(\sigma)}\}_{\sigma=0}^{\ell-1},\mathbf{X}(t_1)=\mathbf{x}\right)\\
&=\Pr\left(\left\{b\overset{t_1+\Delta t,\Delta t}{\not\rightsquigarrow}a\right\}\cap\bar{D}_b^{(1)}\cap\bar{F}_{(a,b)}^{(1)}\;\middle|\;\mathbf{X}(t_1+\Delta t)=\mathbf{y}\right)\\
&=1-\left(B_{ij}\delta_{1y}+\gamma_j+\lambda\left(y\left(1-\frac{\rho_{ij}}{n}\right)+(1-y)\frac{\rho_{ij}}{n}\right)\right)\Delta t+o(\Delta t),
\end{aligned}
$$

which further implies that

$$
\begin{aligned}
&\Pr\left(\left\{b\overset{t_1+\ell\Delta t,\Delta t}{\not\rightsquigarrow}a\right\}\cap\bar{D}_b^{(\ell)}\cap\bar{F}_{(a,b)}^{(\ell)}\right.\\
&\qquad\left|\;\mathbf{X}(t_1+\ell\Delta t)\in\mathbb{X},b\overset{t_1,\ell\Delta t}{\not\rightsquigarrow}a,\{\bar{D}_b^{(\sigma)}\}_{\sigma=0}^{\ell-1},\{\bar{F}_{(a,b)}^{(\sigma)}\}_{\sigma=0}^{\ell-1},\mathbf{X}(t_1)=\mathbf{x}\right)\\
&=1-\left(B_{ij}\delta_{1y}+\gamma_j+\lambda\left(y\left(1-\frac{\rho_{ij}}{n}\right)+(1-y)\frac{\rho_{ij}}{n}\right)\right)\Delta t+o(\Delta t).
\end{aligned}
$$

Now, since (5.44) holds for all $N \in \mathbb{N}$, it follows that

$$\Pr\left(\{b \in \mathcal{I}(t)\} \cap \left\{b \overset{t_1, t_2 - t_1}{\not\leadsto} a\right\} \cap \{1_{(a,b)}(\tau) = y \,\forall \tau \in [t_1, t_2)\} \,\middle|\, \mathbf{X}(t_1) = \mathbf{x}\right)$$
$$= \lim_{N \to \infty} \left(\left(1 - \left(B_{ij}\delta_{1y} + \gamma_j + \lambda y \left(1 - \frac{\rho_{ij}}{n}\right) + \lambda (1-y) \frac{\rho_{ij}}{n}\right) \left(\frac{t_2 - t_1}{N}\right)\right)^N + o\left(\frac{1}{N}\right)\right)$$
$$= e^{-\left(B_{ij}\delta_{1y} + \gamma_j + \lambda\left(y\left(1 - \frac{\rho_{ij}}{n}\right) + (1-y)\frac{\rho_{ij}}{n}\right)\right)(t_2 - t_1)}. \tag{5.46}$$

Similarly, we can show that

$$\Pr\left(1_{(a,b)}(\tau) = y \,\forall \tau \in [t_1, t_2) \,\middle|\, \mathbf{X}(t_1) = \mathbf{x}\right) = e^{-\lambda\left(y\left(1 - \frac{\rho_{ij}}{n}\right) + (1-y)\frac{\rho_{ij}}{n}\right)(t_2 - t_1)}. \tag{5.47}$$

As a result of (5.46) and (5.47),

$$\Pr\left(\{b \in \mathcal{I}(t)\} \cap \left\{b \overset{t_1, t_2 - t_1}{\not\leadsto} a\right\} \,\middle|\, \{1_{(a,b)}(\tau) = y \,\forall \tau \in [t_1, t_2)\}, \mathbf{X}(t_1) = \mathbf{x}\right)$$
$$= \frac{\Pr\left(\{b \in \mathcal{I}(t)\} \cap \left\{b \overset{t_1, t_2 - t_1}{\not\leadsto} a\right\} \cap \{1_{(a,b)}(\tau) = y \,\forall \tau \in [t_1, t_2)\} \,\middle|\, \mathbf{X}(t_1) = \mathbf{x}\right)}{\Pr\left(1_{(a,b)}(\tau) = y \,\forall \tau \in [t_1, t_2) \,\middle|\, \mathbf{X}(t_1) = \mathbf{x}\right)}$$
$$= e^{-(B_{ij}\delta_{1y} + \gamma_j)(t_2 - t_1)}.$$

Since the above holds for all $\mathbf{x} \in \mathbb{X}$, it follows that

$$\Pr\left(\{b \in \mathcal{I}(t)\} \cap \left\{b \overset{t_1, t_2 - t_1}{\not\leadsto} a\right\} \,\middle|\, \{1_{(a,b)}(\tau) = y \,\forall \tau \in [t_1, t_2)\}, \mathbf{X}(t_1) \in \mathbb{X}\right)$$
$$= e^{-(B_{ij}\delta_{1y} + \gamma_j)(t_2 - t_1)},$$

which proves the lemma. □

Lemma 37. *Let $T_{on} := \int_{t-K}^{t-T} 1_{(a,b)}(\sigma) d\sigma$ denote the total duration of time for which the edge (a,b) exists in the network during $[t-K, t-T]$. Then, for all $\kappa, \tau \in [0, t]$ and all $\tau_{on} \in [0, (\kappa - \tau)_+]$,* *we have*

$$\Pr\left(b \overset{0,t}{\not\rightsquigarrow} a \,\middle|\, (K,T,T_{on}) = (\kappa,\tau,\tau_{on}), b \in \mathcal{I}(t^-), (a,b) \notin E(t)\right) = e^{-B_{ij}\tau_{on}},$$

where we define $\mathcal{I}(\sigma^-) := \cup_{\varepsilon>0} \cap_{\tau' \in [\sigma-\varepsilon,\sigma)} \mathcal{I}(\tau')$ *for all* $\sigma \geq 0$*. In other words,* $c \in \mathcal{I}(\sigma^-)$ *iff there exists an* $\varepsilon > 0$ *such that* $c \in \mathcal{I}(\tau')$ *for all* $\tau' \in [\sigma-\varepsilon,\sigma)$*.*

Proof. We first show that $\left\{b \overset{t-\kappa,\kappa-\tau}{\not\rightsquigarrow} a\right\}$ is conditionally independent of $\{(a,b) \notin E(t)\}$ given $(K,T,T_{\text{on}}) = (\kappa,\tau,\tau_{\text{on}})$ and $b \in \mathcal{I}(t-\tau)$:

$$\begin{aligned}
&\Pr\left((a,b) \in E(t) \,\middle|\, (K,T,T_{\text{on}}) = (\kappa,\tau,\tau_{\text{on}}), b \in \mathcal{I}(t-\tau), b \overset{t-\kappa,\kappa-\tau}{\not\rightsquigarrow} a\right) \\
&\overset{(a)}{=} \Pr\left((a,b) \in E(t-\tau) \,\middle|\, (K,T,T_{\text{on}}) = (\kappa,\tau,\tau_{\text{on}}), b \in \mathcal{I}(t-\tau), b \overset{t-\kappa,\kappa-\tau}{\not\rightsquigarrow} a\right) \\
&\overset{(b)}{=} \frac{\rho_{ij}}{n} \\
&= \Pr\left((a,b) \in E(t-\tau) \mid T = \tau\right) \\
&\overset{(c)}{=} \Pr((a,b) \in E(t) \mid T = \tau),
\end{aligned} \tag{5.48}$$

where (a) and (c) hold because $1_{(a,b)}$ is not updated during the interval $[t-\tau,t)$, and (b) follows from the modeling assumption that the probability of the edge (a,b) existing in the network following an edge state update is $\frac{\rho_{ij}}{n}$ (independent of the past states $\{\mathbf{X}(\tau') : 0 \leq \tau' < t-\tau\}$), the fact that $\{b \in \mathcal{I}(t-\tau)\} = \{b \in \mathcal{I}((t-\tau)^-)\}$ almost surely, and from the observation that $t-\tau$ is an update time for $1_{(a,b)}$ given $T = \tau$.

In view of (5.48), the definitions of K, T, and T_{on} imply that

$$\begin{aligned}
&\Pr\left(b \overset{0,t}{\not\rightsquigarrow} a \,\middle|\, (K,T,T_{\text{on}}) = (\kappa,\tau,\tau_{\text{on}}), b \in \mathcal{I}(t^-), (a,b) \notin E(t)\right) \\
&= \Pr\left(b \overset{t-\kappa,\kappa-\tau}{\not\rightsquigarrow} a \,\middle|\, (K,T,T_{\text{on}}) = (\kappa,\tau,\tau_{\text{on}}), b \in \mathcal{I}(t^-), (a,b) \notin E(t)\right) \\
&= \Pr\left(b \overset{t-\kappa,\kappa-\tau}{\not\rightsquigarrow} a \,\middle|\, (K,T,T_{\text{on}}) = (\kappa,\tau,\tau_{\text{on}}), b \in \mathcal{I}(t^-)\right),
\end{aligned}$$

which means that

$$\begin{aligned}
&\Pr\left(b \overset{0,t}{\not\rightsquigarrow} a \,\middle|\, (K,T,T_{\text{on}}) = (\kappa,\tau,\tau_{\text{on}}), b \in \mathcal{I}(t^-), (a,b) \notin E(t)\right) \\
&= \Pr\left(b \overset{t-\kappa,\kappa-\tau}{\not\rightsquigarrow} a \,\middle|\, (K,T,T_{\text{on}}) = (\kappa,\tau,\tau_{\text{on}}), b \in \mathcal{I}(t-\tau), b \notin \cup_{\tau' \in (t-\tau,t)} \mathcal{R}(\tau')\right) \\
&= \Pr\left(b \overset{t-\kappa,\kappa-\tau}{\not\rightsquigarrow} a \,\middle|\, (K,T,T_{\text{on}}) = (\kappa,\tau,\tau_{\text{on}}), b \in \mathcal{I}(t-\tau)\right), \qquad (5.49)
\end{aligned}$$

where the last step holds because $\mathbf{Q}(\mathbf{x}, \mathbf{x}_{\downarrow b}) = \gamma_j$ for all $\mathbf{x} \in \mathbb{S}$ satisfying $x_b = 1$, which implies that, given $b \in \mathcal{I}(t-\tau)$ and any other conditioning event, node b recovers during $(t-\tau, t)$ at a constant rate of γ_j independently of all past edge states and past disease states (and therefore independently of past transmissions as well). Hence, $\left\{b \overset{t-\kappa,\kappa-\tau}{\not\rightsquigarrow} a\right\}$ and $\{b \notin \cup_{\tau' \in (t-\tau,t)} \mathcal{R}(\tau')\}$ are conditionally independent given $(K,T,T_{\text{on}}) = (\kappa,\tau,\tau_{\text{on}})$ and $b \in \mathcal{I}(t-\tau)$.

We now evaluate the right-hand side of (5.49) as follows. Let C denote the (random) number of times $1_{(a,b)}$ flips (changes) during $[t-K, t-T]$, and let the times of these changes be $T_1 < \cdots < T_C$. We assume that C is even (as the case of C being odd is handled similarly) and that $1_{(a,b)}(\tau') = 0$ for $\tau' \in [t-K, T_1]$ (the case $1_{(a,b)}(\tau') = 1$ for $\tau' \in [t-K, T_1]$ is handled similarly). Then, for a given $c \in \mathbb{N}$ and a collection of times $t_1, \ldots, t_c, \tau_{\text{on}}$, we have $\{C = c, T_1 = t_1, \ldots, T_c = t_c\} \subset \{T_{\text{on}} = \tau_{\text{on}}\}$ iff $\sum_{k=1}^{c/2} (t_{2k} - t_{2k-1}) = \tau_{\text{on}}$. Suppose this condition holds. Then, observe that

$$\begin{aligned}
&\Pr\left(b \overset{t-\kappa,\kappa-\tau}{\not\rightsquigarrow} a, b \in \mathcal{I}(t-\tau) \,\middle|\, (K,T,T_{\text{on}}) = (\kappa,\tau,\tau_{\text{on}}), C = c, (T_1,\ldots,T_c) = (t_1,\ldots,t_c)\right) \\
&= \Pr\Bigg(b \overset{t-\kappa,\kappa-\tau}{\not\rightsquigarrow} a, b \in \mathcal{I}(t-\tau) \\
&\qquad \Bigg| (K,T,T_{\text{on}}) = (\kappa,\tau,\tau_{\text{on}}), 1_{(a,b)}(\tau') = 1 \text{ iff } \tau' \in \cup_{k=1}^{c/2} [t_{2k-1}, t_{2k}]\Bigg),
\end{aligned}$$

which means that

$$
\begin{aligned}
&\Pr\left(b \overset{t-\kappa,\kappa-\tau}{\not\rightsquigarrow} a, b \in \mathcal{I}(t-\tau) \middle| (K,T,T_{\text{on}}) = (\kappa,\tau,\tau_{\text{on}}), C = c, (T_1,\ldots,T_c) = (t_1,\ldots,t_c)\right) \\
&\overset{(a)}{=} \prod_{k=1}^{c/2} \Pr\left(b \overset{t_{2k-1},t_{2k}-t_{2k-1}}{\not\rightsquigarrow} a, b \in \mathcal{I}(t_{2k}) \middle| 1_{(a,b)}(\tau') = 1 \,\forall \tau' \in [t_{2k-1}, t_{2k}], b \in \mathcal{I}(t_{2k-1})\right) \\
&\quad \times \prod_{k=1}^{c/2+1} \Pr\Bigg(b \overset{t_{2k-2},t_{2k-1}-t_{2k-2}}{\not\rightsquigarrow} a, b \in \mathcal{I}(t_{2k-1}) \\
&\qquad\qquad \Bigg| 1_{(a,b)}(\tau') = 0 \,\forall \tau' \in [t_{2k-2}, t_{2k-1}], b \in \mathcal{I}(t_{2k-1})\Bigg) \\
&\overset{(b)}{=} \prod_{k=1}^{c/2} e^{-(B_{ij}+\gamma_j)(t_{2k}-t_{2k-1})} \times \prod_{k=1}^{c/2+1} e^{-\gamma_j(t_{2k-1}-t_{2k-2})} \\
&= e^{-B_{ij}\sum_{k=1}^{c/2}(t_{2k}-t_{2k-1})} \cdot e^{-\gamma_j \sum_{k=1}^{c+1}(t_k - t_{k-1})} \\
&= e^{-B_{ij}\tau_{\text{on}}} e^{-\gamma_j(\kappa-\tau)},
\end{aligned}
\tag{5.50}
$$

where (b) follows from Lemma 36, and (a) follows from the following fact: since the definition of our epidemic model implies that the rate of pathogen transmission from b to a at any time instant t' depends only on $1_{(a,b)}(t')$ and the disease state of b at time t', transmission events corresponding to disjoint time intervals are conditionally independent if we are given $1_{(a,b)}$ and the disease state of b as functions of time.

On the other hand, we have

$$
\begin{aligned}
&\Pr(b \in \mathcal{I}(t-\tau) \mid (K,T,T_{\text{on}}) = (\kappa,\tau,\tau_{\text{on}}), C = c, (T_1,\ldots,T_c) = (t_1,\ldots,t_c)) \\
&= \Pr(b \notin \mathbb{R}(t-\tau) \mid (K,T,T_{\text{on}}) = (\kappa,\tau,\tau_{\text{on}}), C = c, (T_1,\ldots,T_c) = (t_1,\ldots,t_c)) \\
&= e^{-\gamma_j((t-\tau)-(t-\kappa))} \\
&= e^{-\gamma_j(\kappa-\tau)},
\end{aligned}
\tag{5.51}
$$

where the second equality holds because our model assumes that the rate of recovery of an infected node is time-invariant and independent of all the edge states and the disease states of other nodes (precisely, $\mathbf{Q}(\mathbf{x},\mathbf{x}_{\downarrow b}) = \gamma_j$ for all $\mathbf{x} \in \mathbb{S}$ such that $b \in \mathcal{I}(\mathbf{x})$).

As a result of (5.50) and (5.51), we have

$$\begin{aligned}
&\Pr\left(b \overset{t-\kappa,\kappa-\tau}{\not\rightsquigarrow} a \,\middle|\, (K,T,T_{\text{on}}) = (\kappa,\tau,\tau_{\text{on}}), b \in \mathcal{I}(t-\tau), (C,T_1,\ldots,T_C) = (c,t_1,\ldots,t_c)\right) \\
&= \frac{\Pr\left(b \overset{t-\kappa,\kappa-\tau}{\not\rightsquigarrow} a, b \in \mathcal{I}(t-\tau) \,\middle|\, (K,T,T_{\text{on}}) = (\kappa,\tau,\tau_{\text{on}}), (C,T_1,\ldots,T_C) = (c,t_1,\ldots,t_c)\right)}{\Pr\left(b \in \mathcal{I}(t-\tau) \,\middle|\, (K,T,T_{\text{on}}) = (\kappa,\tau,\tau_{\text{on}}), (C,T_1,\ldots,T_C) = (c,t_1,\ldots,t_c)\right)} \\
&= e^{-B_{ij}\tau_{\text{on}}}.
\end{aligned}$$

Since $(c,t_1,\ldots,t_c)$ was an arbitrary tuple satisfying $\{(C,T_1,\ldots,T_C) = (c,t_1,\ldots,t_c)\} \subset \{T_{\text{on}} = \tau_{\text{on}}\}$, it follows that

$$\Pr\left(b \overset{t-\kappa,\kappa-\tau}{\not\rightsquigarrow} a \,\middle|\, (K,T,T_{\text{on}}) = (\kappa,\tau,\tau_{\text{on}}), b \in \mathcal{I}(t-\tau)\right) = e^{-B_{ij}\tau_{\text{on}}}.$$

Invoking (5.49) now completes the proof. □

Observe that in the above proof, given that $K = \kappa$ and that $(a,b) \notin E(t)$, $(C,T_1,\ldots,T_C)$ uniquely determines $\{1_{(a,b)}(\tau) : t-K \le \tau \le t\}$. Therefore, as an implication of the above proof, we have

$$\Pr\left(b \overset{t-K,t}{\not\rightsquigarrow} a \mid K = \kappa, \{1_{(a,b)}(\tau) : t-K \le \tau \le t\}, b \in \mathcal{I}(t^-), (a,b) \notin E(t)\right) = e^{-B_{ij}T_{\text{on}}}.$$

The dependence on the random variable T_{on} holds because T_{on} is a function of $\{1_{(a,b)}(\tau) : t-K \le \tau \le t\}$. By invoking Markovity, this result can be extended to

$$\Pr\left(b \overset{t-K,t}{\not\rightsquigarrow} a \mid K = \kappa, \{1_{(a,b)}(\tau) : 0 \le \tau \le t\}, b \in \mathcal{I}(t^-), (a,b) \notin E(t)\right) = e^{-B_{ij}T_{\text{on}}},$$

which is equivalent to the following lemma.

Lemma 38. *Let* $T_{on} := \int_{t-K}^{t-T} 1_{(a,b)}(\sigma)d\sigma$ *denote the total duration of time for which the edge* (a,b) *exists in the network during* $[t-K, t-T]$*. Then, for all* $\kappa \in [0,t]$*, we have*

$$\Pr\left(b \overset{0,t}{\not\rightsquigarrow} a \,\middle|\, K = \kappa, \{1_{(a,b)}(\tau) : 0 \le \tau \le t\}, b \in \mathcal{I}(t^-), (a,b) \notin E(t)\right) = e^{-B_{ij}T_{on}}.$$

Lemma 39. *Recall from Lemma 38 that* $T_{on} = \int_{t-K}^{t-T} 1_{(a,b)}(\sigma)d\sigma$*. Then for all* $\kappa, \tau \in [0,t]$*, we have*

$$\begin{aligned}&\Pr\left((\mathcal{S}(t),\mathcal{I}(t)) = (\mathcal{S}_0,\mathcal{I}_0) \,\middle|\, (K,T,T_{on}) = (\kappa,\tau,\tau_{on}), b \overset{0,t}{\not\rightsquigarrow} a, (a,b) \notin E(t), b \in \mathcal{I}(t^-)\right)\\ &= \Pr\left((\mathcal{S}(t),\mathcal{I}(t)) = (\mathcal{S}_0,\mathcal{I}_0) \,\middle|\, K = \kappa, b \overset{0,t}{\not\rightsquigarrow} a, (a,b) \notin E(t), b \in \mathcal{I}(t^-)\right).\end{aligned}$$

Proof. We first examine the following conditional probability for an arbitrary (a,b)-agnostic superstate $\mathbb{Y}$:

$$\begin{aligned}\Pr\Bigg(&(\mathcal{S}(t),\mathcal{I}(t)) = (\mathcal{S}_0,\mathcal{I}_0), \mathbf{X}(\tilde{K}) \in \mathbb{Y}, K = \kappa, b \overset{0,t}{\not\rightsquigarrow} a, b \in \mathcal{I}(t^-)\\ &\Bigg| \{1_{(a,b)}(\tau) : 0 \le \tau \le t\}, (a,b) \notin E(t)\Bigg).\end{aligned}$$

To begin, note that the proof of Lemma 35, Remark 15, and the fact that K is a function of $\tilde{K}$ together imply that

$$f_{K|\{1_{(a,b)}(\tau):0\le\tau\le t\},(a,b)\notin E(t)}(\kappa) = f_K(\kappa)$$

and that

$$\Pr(\mathbf{X}(\tilde{K}) \in \mathbb{Y} \mid K = \kappa, \{1_{(a,b)}(\tau) : 0 \le \tau \le t\}, (a,b) \notin E(t)) = \Pr(\mathbf{X}(\tilde{K}) \in \mathbb{Y} \mid K = \kappa). \tag{5.52}$$

Next, for the event $\{b \in \mathcal{I}(t^-)\}$, we have

$$\begin{aligned}
&\Pr\Big(b \in \mathcal{I}(t^-) \mid K = \kappa, \mathbf{X}(\tilde{K}) \in \mathbb{Y}, \{1_{(a,b)}(\tau) : 0 \le \tau \le t\}, (a,b) \notin E(t)\Big) \\
&\overset{(a)}{=} e^{-\gamma_j (t-\kappa)} \\
&\overset{(b)}{=} \Pr\Big(b \in \mathcal{I}(t^-) \mid K = \kappa\Big) \qquad (5.53)
\end{aligned}$$

where (a) and (b) follow from our modelling assumption that $\mathbf{Q}(\mathbf{x}, \mathbf{x}_{\downarrow b}) = \gamma_j$ for all $\mathbf{x}$ satisfying $x_b = 1$, which means that the recovery time of b depends only on the time of infection of b and is conditionally independent of all other disease states and all the edge states. Similarly, we have

$$\begin{aligned}
&\Pr\left(b \overset{0,t}{\not\rightsquigarrow} a \mid \mathbf{X}(\tilde{K}) \in \mathbb{Y}, K = \kappa, \{1_{(a,b)}(\tau) : 0 \le \tau \le t\}, (a,b) \notin E(t), b \in \mathcal{I}(t^-)\right) \\
&\overset{(a)}{=} \Pr\left(b \overset{0,t}{\not\rightsquigarrow} a \mid K = \kappa, \{1_{(a,b)}(\tau) : 0 \le \tau \le t\}, (a,b) \notin E(t), b \in \mathcal{I}(t^-)\right) \\
&\overset{(b)}{=} e^{-B_{ij} T_{\text{on}}} \qquad (5.54)
\end{aligned}$$

where (a) follows from our modelling assumptions, which imply that the rate of infection transmission along an edge depends only on the edge state of the transmitting edge and the disease state of the transmitting node and is conditionally independent of other disease states and edge states (which are captured by the (a,b)-agnostic superstate of the chain) and (b) follows from Lemma 38. Note that T_{on} is a function of T and hence also of $\{1_{(a,b)}(\tau) : 0 \le \tau \le t\}$.

It remains for us to analyze

$$\begin{aligned}
\Pr\Bigg(&(\mathcal{S}(t), \mathcal{I}(t)) = (\mathcal{S}_0, \mathcal{I}_0) \\
&\Bigg| \mathbf{X}(\tilde{K}) \in \mathbb{Y}, K = \kappa, b \overset{0,t}{\not\rightsquigarrow} a, b \in \mathcal{I}(t^-), \{1_{(a,b)}(\tau) : 0 \le \tau \le t\}, (a,b) \notin E(t)\Bigg).
\end{aligned}$$

To do so, we first let L_N denote the time of the first (a,b)-agnostic jump to occur after b gets infected, i.e., $N := \inf\{\ell \in \mathbb{N} : L_\ell \ge \tilde{K}\}$, and we note the following: given the conditioning events and variables above (including the event that b does not infect a during $[0,t]$), the total conditional rate at which a receives pathogens at any time $\tau \leq L_N$ is

$$\sum_{q=1}^{m} \sum_{d \in \mathcal{I}_q(\mathbf{X}(\tilde{K}) \setminus \{b\}} B_{iq} 1_{(a,d)}(\mathbf{X}(\tilde{K})),$$

which is determined uniquely by $\mathbb{Y}$, the (a,b)-agnostic superstate of the chain at time $\tilde{K}$. Therefore, this rate is conditionally independent of $1_{(a,b)}(\tau)$ for any τ. Similarly, for all age groups $\ell \in [m]$, given the conditioning events and variables above, the conditional rate at which a node $d \in \mathcal{I}_\ell(\mathbf{X}(\tilde{K}))$ recovers, which equals γ_ℓ, and the total conditional rate at which a node $c \in \mathcal{A}_\ell \setminus \{a\}$ receives pathogens, which equals $\sum_{q=1}^{m} \sum_{d \in \mathcal{I}_q(\mathbf{X}(\tilde{K}))} B_{\ell q} 1_{(c,d)}(\mathbf{X}(\tilde{K}))$, are both conditionally independent of $1_{(a,b)}(\tau)$ given that $\mathbf{X}(\tilde{K}) \in \mathbb{Y}$. Therefore, by using arguments similar to those made in the proof of Lemma 35, we can show that $(\mathcal{S}(L_N), \mathcal{I}(L_N))$ is conditionally independent of $\{1_{(a,b)}(\tau) : 0 \leq \tau \leq t\}$ given the rest of the conditioning events and variables. Moreover, by repeating the above for subsequent (a,b)-agnostic jumps, we can generalize this conditional independence assertion to $(\mathcal{S}(t), \mathcal{I}(t))$, which means that

$$\begin{aligned}
&\Pr\Big((\mathcal{S}(t), \mathcal{I}(t)) = (\mathcal{S}_0, \mathcal{I}_0) \\
&\qquad \Big| \, \mathbf{X}(\tilde{K}) \in \mathbb{Y}, K = \kappa, b \overset{0,t}{\not\rightsquigarrow} a, b \in \mathcal{I}(t^-), \{1_{(a,b)}(\tau) : 0 \leq \tau \leq t\}, (a,b) \notin E(t)\Big) \\
&= \Pr\Big((\mathcal{S}(t), \mathcal{I}(t)) = (\mathcal{S}_0, \mathcal{I}_0) \mid \mathbf{X}(\tilde{K}) \in \mathbb{Y}, K = \kappa, b \overset{0,t}{\not\rightsquigarrow} a, b \in \mathcal{I}(t^-), (a,b) \notin E(t)\Big). \quad (5.55)
\end{aligned}$$

Combining (5.52), (5.53), (5.54) and (5.55) now yields

$$\Pr\Bigg((\mathcal{S}(t),\mathcal{I}(t)) = (\mathcal{S}_0,\mathcal{I}_0), \mathbf{X}(\tilde{K}) \in \mathbb{Y}, b \overset{0,t}{\not\rightsquigarrow} a, b \in \mathcal{I}(t^-) \\ \Bigg| K = \kappa, \{1_{(a,b)}(\tau) : 0 \le \tau \le t\}, (a,b) \notin E(t)\Bigg) \\ = \Pr\Bigg((\mathcal{S}(t),\mathcal{I}(t)) = (\mathcal{S}_0,\mathcal{I}_0) \Bigg| \mathbf{X}(\tilde{K}) \in \mathbb{Y}, K = \kappa, b \overset{0,t}{\not\rightsquigarrow} a, b \in \mathcal{I}(t^-), (a,b) \notin E(t)\Bigg) \\ \times e^{-B_{ij}T_{\text{on}}} \times \Pr(b \in \mathcal{I}(t^-) \mid K = \kappa) \times \Pr(\mathbf{X}(\tilde{K}) \in \mathbb{Y} \mid K = \kappa)$$

Summing both the sides of the above equation over the space of all (a,b)-agnostic superstates $\mathbb{Y}$ gives

$$\Pr\Bigg((\mathcal{S}(t),\mathcal{I}(t)) = (\mathcal{S}_0,\mathcal{I}_0), b \overset{0,t}{\not\rightsquigarrow} a, b \in \mathcal{I}(t^-) \mid K = \kappa, \{1_{(a,b)}(\tau) : 0 \le \tau \le t\}, (a,b) \notin E(t)\Bigg) \\ = e^{-B_{ij}T_{\text{on}}} \times \Pr(b \in \mathcal{I}(t^-) \mid K = \kappa) \\ \times \sum_{\mathbb{Y}} \Bigg(\Pr\Bigg((\mathcal{S}(t),\mathcal{I}(t)) = (\mathcal{S}_0,\mathcal{I}_0) \mid \mathbf{X}(\tilde{K}) \in \mathbb{Y}, K = \kappa, b \overset{0,t}{\not\rightsquigarrow} a, b \in \mathcal{I}(t^-), (a,b) \notin E(t)\Bigg) \\ \cdot \Pr(\mathbf{X}(\tilde{K}) \in \mathbb{Y} \mid K = \kappa)\Bigg). \tag{5.56}$$

Here, we recall from our earlier arguments that

$$e^{-B_{ij}T_{\text{on}}} \times \Pr(b \in \mathcal{I}(t^-) \mid K = \kappa) \\ = \Pr\Big(b \overset{0,t}{\not\rightsquigarrow} a \mid K = \kappa, \{1_{(a,b)}(\tau) : 0 \le \tau \le t\}, (a,b) \notin E(t), b \in \mathcal{I}(t^-)\Big) \\ \times \Pr\Big(b \in \mathcal{I}(t^-) \mid K = \kappa, \{1_{(a,b)}(\tau) : 0 \le \tau \le t\}, (a,b) \notin E(t)\Big) \\ = \Pr\Big(b \overset{0,t}{\not\rightsquigarrow} a, b \in \mathcal{I}(t^-) \mid \{1_{(a,b)}(\tau) : 0 \le \tau \le t\}, (a,b) \notin E(t)\Big).$$

In light of (5.56), this means that

$$\Pr\left((\mathcal{S}(t),\mathcal{I}(t)) = (\mathcal{S}_0,\mathcal{I}_0), b \overset{0,t}{\not\rightsquigarrow} a, b \in \mathcal{I}(t^-) \mid K = \kappa, \{1_{(a,b)}(\tau) : 0 \le \tau \le t\}, (a,b) \notin E(t)\right)$$
$$= \Pr\left(b \overset{0,t}{\not\rightsquigarrow} a, b \in \mathcal{I}(t^-) \mid \{1_{(a,b)}(\tau) : 0 \le \tau \le t\}, (a,b) \notin E(t)\right)$$
$$\times \sum_{\mathbb{Y}} \left(\Pr\left((\mathcal{S}(t),\mathcal{I}(t)) = (\mathcal{S}_0,\mathcal{I}_0) \mid \mathbf{X}(\tilde{K}) \in \mathbb{Y}, K = \kappa, b \overset{0,t}{\not\rightsquigarrow} a, b \in \mathcal{I}(t^-), (a,b) \notin E(t)\right) \cdot \Pr(\mathbf{X}(\tilde{K}) \in \mathbb{Y} \mid K = \kappa) \right).$$

Dividing both the sides of this equation by

$$\Pr\left(b \overset{0,t}{\not\rightsquigarrow} a, b \in \mathcal{I}(t^-) \mid \{1_{(a,b)}(\tau) : 0 \le \tau \le t\}, (a,b) \notin E(t)\right)$$

gives

$$\Pr\left((\mathcal{S}(t),\mathcal{I}(t)) = (\mathcal{S}_0,\mathcal{I}_0) \mid K = \kappa, b \overset{0,t}{\not\rightsquigarrow} a, b \in \mathcal{I}(t^-), \{1_{(a,b)}(\tau) : 0 \le \tau \le t\}, (a,b) \notin E(t)\right)$$
$$= \sum_{\mathbb{Y}} \left(\Pr\left((\mathcal{S}(t),\mathcal{I}(t)) = (\mathcal{S}_0,\mathcal{I}_0) \mid \mathbf{X}(\tilde{K}) \in \mathbb{Y}, K = \kappa, b \overset{0,t}{\not\rightsquigarrow} a, b \in \mathcal{I}(t^-), (a,b) \notin E(t)\right) \cdot \Pr(\mathbf{X}(\tilde{K}) \in \mathbb{Y} \mid K = \kappa) \right)$$
$$= \Pr\left((\mathcal{S}(t),\mathcal{I}(t)) = (\mathcal{S}_0,\mathcal{I}_0) \mid K = \kappa, b \overset{0,t}{\not\rightsquigarrow} a, b \in \mathcal{I}(t^-), (a,b) \notin E(t)\right),$$

where the last step holds because the summation is independent of $\{1_{(a,b)}(\tau) : 0 \le \tau \le t\}$ given that $(a,b) \notin E(t)$. We have thus shown the following: given $K = \kappa, b \overset{0,t}{\not\rightsquigarrow} a$, and $b \in \mathcal{I}(t^-)$, the event $\{(\mathcal{S}(t),\mathcal{I}(t)) = (\mathcal{S}_0,\mathcal{I}_0)\}$ is conditionally independent of $\{1_{(a,b)}(\tau) : 0 \le \tau \le t\}$. Since T and T_{on} are functions of $\{1_{(a,b)}(\tau) : 0 \le \tau \le t\}$, the assertion of the lemma follows. □

Proof of Proposition 13

Before we prove Proposition 13, we recall that for any transition sequence $F = \{\mathbf{x}^{(0)} \xrightarrow{t_1} \mathbf{x}^{(1)} \xrightarrow{t_2} \cdots \xrightarrow{t_r} \mathbf{x}^{(r)} \xrightarrow{t} \mathbf{x}^{(r)}\}$ on a time interval $[0,t]$, the index $\Lambda_{(a,b)}(F)$ indexes the transition in which $(a.b)$ is updated for the last time during $[0,t]$ given that F occurs. We now define another similar index below:

$$\Gamma(a,b)(F) = \begin{cases} \min\left\{\ell \in [r] : \mathbf{x}^{(\ell)} = \mathbf{x}^{(\ell-1)}_{\uparrow b}\right\} & \text{if } \left\{\ell \in [r] : \mathbf{x}^{(\ell)} = \mathbf{x}^{(\ell-1)}_{\uparrow b}\right\} \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Observe that $\Gamma_b(F)$ indexes the transition in which b gets infected given that F occurs.

Proof. Consider any realization $(\mathcal{S}_0, \mathcal{I}_0)$ of $(\mathcal{S}(t), \mathcal{I}(t))$, and let $\mathcal{F}$ be the set of all the transition sequences for $[0,t]$ that result in the occurrence of $\{(\mathcal{S}(t), \mathcal{I}(t)) = (\mathcal{S}_0, \mathcal{I}_0)\}$, so that $\{(\mathcal{S}(t), \mathcal{I}(t)) = (\mathcal{S}_0, \mathcal{I}_0)\} = \cup_{F \in \mathcal{F}} F$.

Consider now any pair of nodes $(a,b) \in \mathcal{A}_i \cap \mathcal{S}_0 \times \mathcal{A}_j \cap \mathcal{I}_0$ (so that we have $a \in \mathcal{S}_i(t)$ and $b \in \mathcal{I}_j(t)$ in the event that $(\mathcal{S}(t), \mathcal{I}(t)) = (\mathcal{S}_0, \mathcal{I}_0)$), and note that for any transition sequence $F \in \mathcal{F}$, we have $F_{\overline{(a,b)}} \in \mathcal{F}$, because both $F_{\overline{(a,b)}}$ and F involve the same node recoveries and disease transmissions (all of which occur along edges other than (a,b)). Therefore, $F_{?(a,b)} \subset \mathcal{F}$ for each $F \in \mathcal{F}$, and it follows that $\{(\mathcal{S}(t), \mathcal{I}(t)) = (\mathcal{S}_0, \mathcal{I}_0)\} = \cup_{F \in \mathcal{F}} F_{?(a,b)}$.

Hence, we can derive bounds on $\chi_{ij}(t)$ (defined to be $\Pr((a,b) \in E(t) \mid \mathcal{S}(t), \mathcal{I}(t))$) by bounding $\Pr((a,b) \in E(t) \mid (\mathcal{S}(t), \mathcal{I}(t)) = (\mathcal{S}_0, \mathcal{I}_0)) = \Pr((a,b) \in E(t) \mid \cup_{F \in \mathcal{F}} F_{?(a,b)})$. To this end, we pick $F \in \mathcal{F}$ and $\delta > 0$, and apply Bayes' rule to $\Pr((a,b) \in E(t) \mid F^{\delta}_{?(a,b)})$ as follows.

$$\begin{aligned} \Pr((a,b) \in E(t) \mid F^{\delta}_{?(a,b)}) &= \left(\frac{\Pr(F^{\delta}_{?(a,b)})}{\Pr(F^{\delta}_{?(a,b)} \mid (a,b) \in E(t)) \cdot \Pr((a,b) \in E(t))} \right)^{-1} \\ &= \left(1 + \frac{\Pr(F^{\delta}_{?(a,b)} \mid (a,b) \notin E(t))}{\Pr(F^{\delta}_{?(a,b)} \mid (a,b) \in E(t))} \cdot \frac{\Pr((a,b) \notin E(t))}{\Pr((a,b) \in E(t))} \right)^{-1} \end{aligned} \quad (5.57)$$

At this point, note that $\Pr((a,b) \in E(t)) = \frac{\rho_{ij}}{n}$, which is the probability that the edge (a,b) exists in the network after the last of the updates of $1_{(a,b)}$ to occur during $[0,t]$. Therefore,

$$\Pr((a,b) \in E(t) \mid F^{\delta}_{?(a,b)}) = \left(1 + \frac{\Pr(F^{\delta}_{?(a,b)} \mid (a,b) \notin E(t))}{\Pr(F^{\delta}_{?(a,b)} \mid (a,b) \in E(t))} \cdot \frac{1-\rho_{ij}/n}{\rho_{ij}/n}\right)^{-1}. \tag{5.58}$$

We now estimate $\frac{\Pr(F^{\delta}_{?(a,b)} \mid (a,b) \notin E(t))}{\Pr(F^{\delta}_{?(a,b)} \mid (a,b) \in E(t))}$. Note that if δ is small enough, either $F^{\delta} \subset \{(a,b) \in E(t)\}$ or $F^{\delta} \subset \{(a,b) \notin E(t)\}$. Assume w.l.o.g. that $F^{\delta} \subset \{(a,b) \notin E(t)\}$ (equivalently, $F^{\delta}_{\overline{(a,b)}} \subset \{(a,b) \in E(t)\}$), and observe that

$$\begin{aligned}\frac{\Pr(F^{\delta}_{?(a,b)} \mid (a,b) \notin E(t))}{\Pr(F^{\delta}_{?(a,b)} \mid (a,b) \in E(t))} &= \frac{\Pr(F^{\delta}_{?(a,b)} \cap \{(a,b) \notin E(t)\})}{\Pr(F^{\delta}_{?(a,b)} \cap \{(a,b) \in E(t)\})} \cdot \frac{\Pr(a,b) \in E(t)}{\Pr(a,b) \notin E(t)} \\ &= \frac{\Pr(F^{\delta})}{\Pr(F^{\delta}_{\overline{(a,b)}})} \left(\frac{\rho_{ij}/n}{1-\rho_{ij}/n}\right). \end{aligned} \tag{5.59}$$

Thus, the next step is to evaluate $\frac{\Pr(F^{\delta})}{\Pr(F^{\delta}_{\overline{(a,b)}})}$. To do so, suppose $F = \{\mathbf{x}^{(0)} \xrightarrow{t_1} \mathbf{x}^{(1)} \xrightarrow{t_2} \cdots \xrightarrow{t_r} \mathbf{x}^{(r)} \xrightarrow{t_{r+1}} x^{(r)}\}$ with $t_{r+1} := t$, $\Lambda_{(a,b)}(F) = \zeta \in \{0,1,\ldots,r\}$, $\Gamma_b(F) = \xi \in \{0,1,\ldots,r\}$, $t_{\Lambda_{(a,b)}(F)} = t_\zeta = t - \tau$ for some $\tau \in [0,t]$, and $t_{\Gamma_b(F)} = t_\xi = t - \kappa$ for some $\kappa \in [0,\tau]$. Then the assumption $F^{\delta} \subset \{(a,b) \notin E(t)\}$ implies that $\mathbf{x}^{(\zeta)} = \mathbf{x}^{(\zeta-1)}_{\downarrow(a,b)}$ and hence also that $\mathbf{x}^{(\zeta)}_{\overline{(a,b)}} = \mathbf{x}^{(\zeta-1)}_{\uparrow(a,b)}$. As a result,

$$F_{\overline{(a,b)}} = \left\{\mathbf{x}^{(0)} \xrightarrow{t_1} \cdots \xrightarrow{t_{\zeta-1}} \mathbf{x}^{(\zeta-1)} \xrightarrow{t-\tau} \mathbf{x}^{(\zeta-1)}_{\uparrow(a,b)} \xrightarrow{t_{\zeta+1}} \mathbf{x}^{(\zeta+1)}_{\overline{(a,b)}} \xrightarrow{t_{\zeta+2}} \cdots \xrightarrow{t_r} \mathbf{x}^{(r)}_{\overline{(a,b)}} \xrightarrow{t} \mathbf{x}^{(r)}_{\overline{(a,b)}}\right\}.$$

It now follows from Lemma 33 that

$$\frac{\Pr(F^{\delta})}{\Pr(F^{\delta}_{\overline{(a,b)}})} = \frac{e^{-q_r(t-t_r)} \prod_{\ell=\zeta}^{r} q_{\ell-1,\ell}(e^{-q_{\ell-1}(t_\ell - t_{\ell-1})}\delta + o(\delta))}{e^{-\bar{q}_r(t-t_r)} \prod_{\ell=\zeta}^{r} \bar{q}_{\ell-1,\ell}(e^{-\bar{q}_{\ell-1}(t_\ell - t_{\ell-1})}\delta + o(\delta))}, \tag{5.60}$$

where $q_{\ell-1,\ell} := \mathbf{Q}(\mathbf{x}^{(\ell-1)}, \mathbf{x}^{(\ell)})$ and $\bar{q}_{\ell-1,\ell} := \mathbf{Q}(\mathbf{x}^{(\ell-1)}_{\overline{(a,b)}}, \mathbf{x}^{(\ell)}_{\overline{(a,b)}})$ for all $\ell \in \{\zeta+1,\ldots,r\}$, $q_\ell := -\mathbf{Q}(\mathbf{x}^{(\ell)}, \mathbf{x}^{(\ell)})$ and $\bar{q}_\ell := -\mathbf{Q}(\mathbf{x}^{(\ell)}_{\overline{(a,b)}}, \mathbf{x}^{(\ell)}_{\overline{(a,b)}})$ for all $\ell \in \{\zeta,\ldots,r\}$, $q_{\zeta-1,\zeta} := \mathbf{Q}(\mathbf{x}^{(\zeta-1)}, \mathbf{x}^{(\zeta-1)}_{\downarrow(a,b)}) =$

$\lambda\left(1-\frac{\rho_{ij}}{n}\right)$, $\bar{q}_{\zeta-1,\zeta} := \mathbf{Q}(\mathbf{x}^{(\zeta-1)}, \mathbf{x}^{(\zeta-1)}_{\uparrow(a,b)}) = \lambda\frac{\rho_{ij}}{n}$, and $\bar{q}_{\zeta-1} = q_{\zeta-1} := -\mathbf{Q}(\mathbf{x}^{(\zeta-1)}, \mathbf{x}^{(\zeta-1)})$.

The above definitions imply that $\frac{q_{\zeta-1,\zeta}}{\bar{q}_{\zeta-1,\zeta}} = \frac{1-\rho_{ij}/n}{\rho_{ij}/n}$. To evaluate $\frac{q_{\ell-1,\ell}}{\bar{q}_{\ell-1,\ell}}$ for $\ell \in \{\zeta+1,\dots,r\}$, observe that by the definition of $\Lambda_{(a,b)}(F)$, we have $\mathbf{x}^{(\ell)} \notin \{\mathbf{x}^{(\ell-1)}_{\uparrow(a,b)}, \mathbf{x}^{(\ell-1)}_{\downarrow(a,b)}\}$ for $\ell > \zeta = \Lambda_{(a,b)}(F)$. Moreover, the facts $F \subset \{\mathcal{S}(t) = \mathcal{S}_0\}$ and $a \in \mathcal{S}_0$ together imply that $\mathbf{x}^{(\ell)} \neq \mathbf{x}^{(\ell-1)}_{\uparrow a}$ for all $\ell \in [r]$. Hence, $\mathbf{x}^{(\ell)} \notin \{\mathbf{x}^{(\ell-1)}_{\uparrow(a,b)}, \mathbf{x}^{(\ell-1)}_{\downarrow(a,b)}, \mathbf{x}^{(\ell-1)}_{\uparrow a}\}$ for all $\ell > \zeta$. It now follows from the definition of $\mathbf{Q}$ (Section 5.2) that $\mathbf{Q}(\mathbf{x}^{(\ell-1)}, \mathbf{x}^{(\ell)}) = \mathbf{Q}(\mathbf{x}^{(\ell-1)}_{\overline{(a,b)}}, \mathbf{x}^{(\ell)}_{\overline{(a,b)}})$ for all $\ell \in \{\zeta+1,\dots,r\}$. Thus, $\frac{\prod_{\ell=\zeta}^{r} q_{\ell-1,\ell}}{\prod_{\ell=\zeta}^{r} \bar{q}_{\ell-1,\ell}} = \frac{1-\rho_{ij}/n}{\rho_{ij}/n}$.

To relate $\bar{q}_\ell$ to q_ℓ, note that $F \subset \{(a,b) \notin E(t)\}$ implies that $\mathbf{x}^{(\ell)}_{\langle a,b\rangle} = 0$ and hence also that $\left(\mathbf{x}^{(\ell)}_{\overline{(a,b)}}\right)_{\langle a,b\rangle} = 1$ for $\ell \geq \zeta$. Since $\mathbf{x}^{(\ell)}_u = \left(\mathbf{x}^{(\ell)}_{\overline{(a,b)}}\right)_u$ for all $u \neq \langle a,b\rangle$, we have

$$\begin{aligned}
\mathbf{Q}\left(\mathbf{x}^{(\ell)}_{\overline{(a,b)}}, \left(\mathbf{x}^{(\ell)}_{\overline{(a,b)}}\right)_{\uparrow a}\right) &= \sum_{k=1}^{m} B_{ik} E_k^{(a)}\left(\mathbf{x}^{(\ell)}_{\overline{(a,b)}}\right) \\
&= \sum_{k=1}^{m} \sum_{c \in \mathcal{I}_k\left(\mathbf{x}^{(\ell)}_{\overline{(a,b)}}\right)} B_{ik} 1_{(a,c)}\left(\mathbf{x}^{(\ell)}_{\overline{(a,b)}}\right) \\
&= \sum_{k=1}^{m} \sum_{c \in \mathcal{I}_k\left(\mathbf{x}^{(\ell)}\right)} B_{ik} 1_{(a,c)}\left(\mathbf{x}^{(\ell)}\right) + B_{ij} 1_{(a,b)}\left(\mathbf{x}^{(\ell)}_{\overline{(a,b)}}\right) \\
&= \mathbf{Q}\left(\mathbf{x}^{(\ell)}, \mathbf{x}^{(\ell)}_{\uparrow a}\right) + B_{ij}
\end{aligned}$$

for all $\ell \in \{\zeta, \dots, r\}$ such that $b \in \mathcal{I}_j(\mathbf{x}^{(\ell)})$, and

$$\mathbf{Q}\left(\mathbf{x}^{(\ell)}_{\overline{(a,b)}}, \left(\mathbf{x}^{(\ell)}_{\overline{(a,b)}}\right)_{\uparrow a}\right) = \mathbf{Q}\left(\mathbf{x}^{(\ell)}, \mathbf{x}^{(\ell)}_{\uparrow a}\right)$$

for all $\ell \in \{\zeta,\dots,r\}$ such that $b \notin \mathcal{I}_j(\mathbf{x}^{(\ell)})$. As a result,

$$\mathbf{Q}\left(\mathbf{x}^{(\ell)}_{\overline{(a,b)}}, \left(\mathbf{x}^{(\ell)}_{\overline{(a,b)}}\right)_{\uparrow a}\right) = \begin{cases} \mathbf{Q}\left(\mathbf{x}^{(\ell)}, \mathbf{x}^{(\ell)}_{\uparrow a}\right) & \text{if } \zeta \leq \ell < \xi \\ \mathbf{Q}\left(\mathbf{x}^{(\ell)}, \mathbf{x}^{(\ell)}_{\uparrow a}\right) + B_{ij} & \text{if } \max\{\zeta,\xi\} \leq \ell \leq r. \end{cases} \tag{5.61}$$

Moreover, using the definition of $\mathbf{Q}$ one can easily verify that regardless of whether the network is in state $\mathbf{x}^{(\ell)}$ or in state $\mathbf{x}^{(\ell)}_{\overline{(a,b)}}$, the rates of infection of nodes in $\mathcal{S}(\mathbf{x}^{(\ell)}) \setminus \{a\}$, the recovery rates of nodes in $\mathcal{I}(x^{(\ell)})$, and the edge update rate (which is λ) are the same. Given that $\mathbf{Q}(\mathbf{x},\mathbf{x}) = -\sum_{\mathbf{y}\in\mathbb{S}\setminus\{\mathbf{x}\}} \mathbf{Q}(\mathbf{x},\mathbf{y})$ for all $\mathbf{x} \in \mathbb{S}$, it now follows from (5.61) that

$$\bar{q}_\ell - q_\ell = \begin{cases} 0 & \text{if } \zeta \leq \ell < \xi \\ B_{ij} & \text{if } \max\{\zeta,\xi\} \leq \ell \leq r. \end{cases} \tag{5.62}$$

Combining the above observations with (5.60) yields

$$\begin{aligned}
\frac{\Pr(F^\delta)}{\Pr(F^\delta_{\overline{(a,b)}})} &= \left(\frac{1-\rho_{ij}/n}{\rho_{ij}/n}\right) e^{(\bar{q}_r - q_r)(t-t_r)} \prod_{\ell=\zeta-1}^{r-1} \left(e^{(\bar{q}_\ell - q_\ell)(t_{\ell+1}-t_\ell)}\delta + o(\delta)\right) \\
&\overset{(a)}{=} \left(\frac{1-\rho_{ij}/n}{\rho_{ij}/n}\right) \prod_{\ell=\zeta-1}^{r} \left(e^{(\bar{q}_\ell - q_\ell)(t_{\ell+1}-t_\ell)}\delta + o(\delta)\right) \\
&= \left(\frac{1-\rho_{ij}/n}{\rho_{ij}/n}\right) \prod_{\ell=\zeta-1}^{\max\{\zeta,\xi\}-1} (1+o(\delta)) \prod_{\ell=\max\{\zeta,\xi\}}^{r} \left(e^{B_{ij}(t_{\ell+1}-t_\ell)}\delta + o(\delta)\right) \\
&= \left(\frac{1-\rho_{ij}/n}{\rho_{ij}/n}\right) e^{B_{ij}\left(t - t_{\max\{\zeta,\xi\}}\right)} + o(\delta),
\end{aligned}$$

which means that

$$\frac{\Pr(F^\delta)}{\Pr(F^\delta_{\overline{(a,b)}})} = \left(\frac{1-\rho_{ij}/n}{\rho_{ij}/n}\right) e^{B_{ij}\min\{\tau,\kappa\}} + o(\delta). \tag{5.63}$$

We now use (5.57) and (5.59) along with (5.63) to show that

$$\begin{aligned}
\Pr((a,b) \in E(t) \mid F^\delta_{?(a,b)}) &= \left(1 + \left(\frac{1-\rho_{ij}/n}{\rho_{ij}/n}\right) e^{B_{ij}\min\{\tau,\kappa\}} + o(\delta)\right)^{-1} \\
&= \frac{\rho_{ij}}{n} \cdot \frac{1}{\frac{\rho_{ij}}{n}\cdot 1 + \left(1 - \frac{\rho_{ij}}{n}\right) e^{B_{ij}\min\{\tau,\kappa\}}} + o(\delta).
\end{aligned}$$

In the limit as $\delta \to 0$, this yields

$$\Pr((a,b) \in E(t) \mid F_{?(a,b)}) = \frac{\rho_{ij}}{n} \cdot \frac{1}{\frac{\rho_{ij}}{n} \cdot 1 + \left(1 - \frac{\rho_{ij}}{n}\right) e^{B_{ij} \min\{\tau,\kappa\}}} \tag{5.64}$$

Note that these bounds hold for all $F \in \mathcal{F}$ satisfying $t - t_{\Lambda_{(a,b)}(F)} = \tau$ and $t - t_{\Gamma_b(F)} = \kappa$. We now recall that T is the difference between t and the time at which $1_{(a,b)}$ is updated for the last time during $[0,t]$, so that we have $T = t - t_{\Lambda_{(a,b)}(F)}$ whenever $F_{?(a,b)}$ occurs. Likewise, $K = t - t_{\Gamma_b(F)}$ on $F_{?(a,b)}$. Therefore, (5.64) holds for all $F \in \mathcal{F}$ satisfying $F \subset \{T = \tau\} \cap \{K = \kappa\}$. As a result, we have

$$\Pr((a,b) \in E(t) \mid \cup_{F \in \mathcal{F}} F_{?(a,b)}, T = \tau, K = \kappa) = \frac{\rho_{ij}}{n} \cdot \frac{1}{\frac{\rho_{ij}}{n} \cdot 1 + \left(1 - \frac{\rho_{ij}}{n}\right) e^{B_{ij} \min\{\tau,\kappa\}}}.$$

Since $\cup_{F \in \mathcal{F}} F_{?(a,b)} = \{(\mathcal{S}(t),\mathcal{I}(t)) = (\mathcal{S}_0,\mathcal{I}_0)\}$ as argued earlier, it follows that

$$\begin{aligned} &\Pr((a,b) \in E(t) \mid (\mathcal{S}(t),\mathcal{I}(t)) = (\mathcal{S}_0,\mathcal{I}_0), T = \tau, K = \kappa) \\ &= \frac{\rho_{ij}}{n} \cdot \frac{1}{\frac{\rho_{ij}}{n} \cdot 1 + \left(1 - \frac{\rho_{ij}}{n}\right) e^{B_{ij} \min\{\tau,\kappa\}}}. \end{aligned} \tag{5.65}$$

Observe that $0 \leq \min\{\kappa,\tau\} \leq \tau$, which means that

$$\begin{aligned} \frac{\rho_{ij}}{n} &\geq \Pr((a,b) \in E(t) \mid (\mathcal{S}(t),\mathcal{I}(t)) = (\mathcal{S}_0,\mathcal{I}_0), T = \tau, K = \kappa) \\ &\geq \frac{\rho_{ij}}{n} \cdot \frac{1}{\frac{\rho_{ij}}{n} \cdot 1 + \left(1 - \frac{\rho_{ij}}{n}\right) e^{B_{ij}\tau}} \\ &\geq \frac{\rho_{ij}}{n} e^{-B_{ij}\tau}. \end{aligned} \tag{5.66}$$

Furthermore, since the above bounds do not depend on κ, we can remove the conditioning on K to obtain

$$\frac{\rho_{ij}}{n} \geq \Pr((a,b) \in E(t) \mid (\mathcal{S}(t),\mathcal{I}(t)) = (\mathcal{S}_0,\mathcal{I}_0), T = \tau) \geq \frac{\rho_{ij}}{n} e^{-B_{ij}\tau}. \tag{5.67}$$

Consequently,

$$\begin{aligned}\frac{\rho_{ij}}{n} &\geq \Pr((a,b) \in E(t) \mid (\mathcal{S}(t),\mathcal{I}(t)) = (\mathcal{S}_0,\mathcal{I}_0)) \\ &\geq \frac{\rho_{ij}}{n}\int_0^t e^{-B_{ij}\tau} f_{T|(\mathcal{S}(t),\mathcal{I}(t))=(\mathcal{S}_0,\mathcal{I}_0)}(\tau)d\tau \\ &\geq \frac{\rho_{ij}}{n}\left(1-\frac{B_{ij}}{\lambda}(1-e^{-\lambda t})\right),\end{aligned}$$

where the second inequality follows from Lemmas 41 and 42. Since $(\mathcal{S}_0,\mathcal{I}_0)$ is an arbitrary realization of $(\mathcal{S}(t),\mathcal{I}(t))$, the assertion of the proposition follows. □

Remark 16. *The proof of Proposition 13 enables us to make a stronger statement about the conditional probability of b being in contact with a at time t. Indeed, consider* (5.65) *and observe that it holds for all realizations $(\mathcal{S}_0,\mathcal{I}_0)$ of $(\mathcal{S}(t),\mathcal{I}(t))$ that satisfy $a \in \mathcal{S}_0 \cap \mathcal{A}_i$ and $b \in \mathcal{I}_0 \cap \mathcal{A}_j$. It follows that*

$$\Pr((a,b) \in E(t) \mid \mathcal{S}(t),\mathcal{I}(t),T=\tau,K=\kappa) = \frac{\rho_{ij}}{n}\cdot\frac{1}{\frac{\rho_{ij}}{n}\cdot 1+\left(1-\frac{\rho_{ij}}{n}\right)e^{B_{ij}\min\{\tau,\kappa\}}}$$

holds for all node pairs $(a,b) \in \mathcal{S}_i(t)\times\mathcal{I}_j(t)$. Equivalently, the following holds for all $(a,b) \in \mathcal{S}_i(t)\times\mathcal{I}_j(t)$:

$$\Pr((a,b) \in E(t) \mid \mathcal{S}(t),\mathcal{I}(t),T,K) = \frac{\rho_{ij}}{n}\cdot\frac{1}{\frac{\rho_{ij}}{n}\cdot 1+\left(1-\frac{\rho_{ij}}{n}\right)e^{B_{ij}\min\{T,K\}}}.$$

Lemma 40. *Let $\tau_1,\tau_2 \in [0,t]$. Then*

$$1 \leq \frac{\Pr\left(b \overset{0,t}{\not\rightsquigarrow} a \,\middle|\, b \in \mathcal{I}(t^-), K=\kappa, T=\tau_2, (a,b)\notin E(t)\right)}{\Pr\left(b \overset{0,t}{\not\rightsquigarrow} a \,\middle|\, b \in \mathcal{I}(t^-), K=\kappa, T=\tau_1, (a,b)\notin E(t)\right)} \leq e^{B_{ij}(\tau_2-\tau_1)}.$$

Proof. Consider $T_{\text{on}}^{(\tau_2)} := \left(\int_{t-\kappa}^{t-\tau_2} 1_{(a,b)}(\tau')d\tau'\right)_+$, which denotes the duration of time for which b

is in contact with a during the interval $[t-\kappa, t-\tau_2)$. Then, for any $\sigma \in [0, (\kappa-\tau_2)_+]$, we have

$$\begin{aligned}
&\Pr\left(b \overset{0,t}{\not\rightsquigarrow} a \,\middle|\, T_{\text{on}}^{(\tau_2)} = \sigma, b \in \mathcal{I}(t^-), K = \kappa, T = \tau_1, (a,b) \notin E(t)\right) \\
&= \Pr\left(b \overset{t-\kappa,\kappa-\tau_1}{\not\rightsquigarrow} a \,\middle|\, T_{\text{on}}^{(\tau_2)} = \sigma, b \in \mathcal{I}(t^-), K = \kappa, T = \tau_1, (a,b) \notin E(t)\right) \\
&= \Pr\left(b \overset{t-\kappa,\kappa-\tau_2}{\not\rightsquigarrow} a \,\middle|\, T_{\text{on}}^{(\tau_2)} = \sigma, b \in \mathcal{I}(t^-), K = \kappa, T = \tau_1, (a,b) \notin E(t)\right) \\
&\quad \times \Pr\left(b \overset{t-\tau_2,\tau_2-\tau_1}{\not\rightsquigarrow} a \,\middle|\, T_{\text{on}}^{(\tau_2)} = \sigma, b \in \mathcal{I}(t^-), K = \kappa, T = \tau_1, (a,b) \notin E(t)\right) \\
&\overset{(a)}{\geq} \Pr\left(b \overset{t-\kappa,\kappa-\tau_2}{\not\rightsquigarrow} a \,\middle|\, T_{\text{on}}^{(\tau_2)} = \sigma, b \in \mathcal{I}(t^-), K = \kappa, T = \tau_1, (a,b) \notin E(t)\right) e^{-B_{ij}(\tau_2-\tau_1)} \\
&\overset{(b)}{\geq} e^{-B_{ij}\sigma} \cdot e^{-B_{ij}(\tau_2-\tau_1)} \\
&\overset{(c)}{=} \Pr\left(b \overset{t-\kappa,\kappa-\tau_2}{\not\rightsquigarrow} a \,\middle|\, T_{\text{on}}^{(\tau_2)} = \sigma, b \in \mathcal{I}(t^-), K = \kappa, T = \tau_2, (a,b) \notin E(t)\right) e^{-B_{ij}(\tau_2-\tau_1)} \\
&= \Pr\left(b \overset{0,t}{\not\rightsquigarrow} a \,\middle|\, T_{\text{on}}^{(\tau_2)} = \sigma, b \in \mathcal{I}(t^-), K = \kappa, T = \tau_2, (a,b) \notin E(t)\right) e^{-B_{ij}(\tau_2-\tau_1)}, \qquad (5.68)
\end{aligned}$$

where (a) follows from the fact that $\int_{t-\tau_2}^{t-\tau_1} 1_{(a,b)}(\tau')d\tau' \leq \tau_2 - \tau_1$ and from the observation that

$$\begin{aligned}
&\Pr\left(b \overset{t-\tau_2,\tau_2-\tau_1}{\not\rightsquigarrow} a \,\middle|\, T_{\text{on}}^{(\tau_2)} = \sigma, b \in \mathcal{I}(t^-), K = \kappa, T = \tau_1, (a,b) \notin E(t), \int_{t-\tau_2}^{t-\tau_1} 1_{(a,b)}(\tau')d\tau'\right) \\
&= e^{-B_{ij}\left(\int_{t-\tau_2}^{t-\tau_1} 1_{(a,b)}(\tau')d\tau'\right)},
\end{aligned}$$

the proof of which parallels the proof of Lemma 37, and (b) and (c) follow from the fact that

$$\Pr\left(b \overset{t-\kappa,\kappa-\tau_2}{\not\rightsquigarrow} a \,\middle|\, T_{\text{on}}^{(\tau_2)} = \sigma, b \in \mathcal{I}(t^-), K = \kappa, T = \tau', (a,b) \notin E(t)\right) = e^{-B_{ij}\sigma}$$

holds for all $\tau' \in [0,t]$, the proof of which also parallels the proof of Lemma 37.

We now eliminate $T_{\text{on}}^{(\tau_2)}$ from (5.68). To do so, we first note the following: given that $t - T \geq t - \tau_2$, the random variable $T_{\text{on}}^{(\tau_2)}$ is by definition conditionally independent of $t - T$ (the time of the last edge update of (a,b) during $[t-\tau_2, t]$) because the edge update process for (a,b) is a Poisson process and hence, for any collection of disjoint time intervals, the times at which $1_{(a,b)}$ is updated during the intervals are independent of each other. Since $\{T=\tau_2\},\{T=\tau_1\}\subset\{t-T\geq t-\tau_2\}$ and since $\{t-T\geq t-\tau_2\}=\{T\leq\tau_2\}$, it follows that

$$\begin{aligned}
&\Pr\left(b \overset{0,t}{\not\rightsquigarrow} a \middle| b\in\mathcal{I}(t^-),K=\kappa,T=\tau_1,(a,b)\notin E(t)\right)\\
&=\int_0^{(\kappa-\tau_2)+}\Pr\left(b \overset{0,t}{\not\rightsquigarrow} a \middle| T_{\text{on}}^{(\tau_2)}=\sigma,b\in\mathcal{I}(t^-),K=\kappa,T=\tau_1,(a,b)\notin E(t)\right)\\
&\qquad\qquad\cdot f_{T_{\text{on}}^{(\tau_2)}|b\in\mathcal{I}(t^-),K=\kappa,T=\tau_1,(a,b)\notin E(t)}(\sigma)d\sigma\\
&=\int_0^{(\kappa-\tau_2)+}\Pr\left(b \overset{0,t}{\not\rightsquigarrow} a \middle| T_{\text{on}}^{(\tau_2)}=\sigma,b\in\mathcal{I}(t^-),K=\kappa,T=\tau_1,(a,b)\notin E(t)\right)\\
&\qquad\qquad\cdot f_{T_{\text{on}}^{(\tau_2)}|b\in\mathcal{I}(t^-),K=\kappa,T\leq\tau_2,(a,b)\notin E(t)}(\sigma)d\sigma,
\end{aligned} \tag{5.69}$$

and likewise,

$$\begin{aligned}
&\Pr\left(b \overset{0,t}{\not\rightsquigarrow} a \middle| b\in\mathcal{I}(t^-),K=\kappa,T=\tau_2,(a,b)\notin E(t)\right)\\
&=\int_0^{(\kappa-\tau_2)+}\Pr\left(b \overset{0,t}{\not\rightsquigarrow} a \middle| T_{\text{on}}^{(\tau_2)}=\sigma,b\in\mathcal{I}(t^-),K=\kappa,T=\tau_2,(a,b)\notin E(t)\right)\\
&\qquad\qquad\cdot f_{T_{\text{on}}^{(\tau_2)}|b\in\mathcal{I}(t^-),K=\kappa,T\leq\tau_2,(a,b)\notin E(t)}(\sigma)d\sigma.
\end{aligned}$$

Therefore, taking conditional expectations on both sides of (5.68) w.r.t. the probability density function $f_{T_{\text{on}}^{(\tau_2)}|b\in\mathcal{I}(t^-),K=\kappa,T\leq\tau_2,(a,b)\notin E(t)}$ yields

$$\begin{aligned}
&\Pr\left(b \overset{0,t}{\not\rightsquigarrow} a \middle| b\in\mathcal{I}(t^-),K=\kappa,T=\tau_1,(a,b)\notin E(t)\right)\\
&\geq\Pr\left(b \overset{0,t}{\not\rightsquigarrow} a \middle| b\in\mathcal{I}(t^-),K=\kappa,T=\tau_2,(a,b)\notin E(t)\right)e^{-B_{ij}(\tau_2-\tau_1)},
\end{aligned}$$

which proves the upper bound. For the lower bound, we again proceed as in (5.68), but reverse the inequality signs:

$$
\begin{aligned}
&\Pr\left(b \overset{0,t}{\not\rightsquigarrow} a \,\middle|\, T_{\text{on}}^{(\tau_2)} = \sigma, b \in \mathcal{I}(t^-), K = \kappa, T = \tau_1, (a,b) \notin E(t)\right) \\
&= \Pr\left(b \overset{t-\kappa,\kappa-\tau_2}{\not\rightsquigarrow} a \,\middle|\, T_{\text{on}}^{(\tau_2)} = \sigma, b \in \mathcal{I}(t^-), K = \kappa, T = \tau_1, (a,b) \notin E(t)\right) \\
&\quad \times \Pr\left(b \overset{t-\tau_2,\tau_2-\tau_1}{\not\rightsquigarrow} a \,\middle|\, T_{\text{on}}^{(\tau_2)} = \sigma, b \in \mathcal{I}(t^-), K = \kappa, T = \tau_1, (a,b) \notin E(t)\right) \\
&\leq \Pr\left(b \overset{t-\kappa,\kappa-\tau_2}{\not\rightsquigarrow} a \,\middle|\, T_{\text{on}}^{(\tau_2)} = \sigma, b \in \mathcal{I}(t^-), K = \kappa, T = \tau_1, (a,b) \notin E(t)\right) \\
&= e^{-B_{ij}\sigma} \\
&= \Pr\left(b \overset{t-\kappa,\kappa-\tau_2}{\not\rightsquigarrow} a \,\middle|\, T_{\text{on}}^{(\tau_2)} = \sigma, b \in \mathcal{I}(t^-), K = \kappa, T = \tau_2, (a,b) \notin E(t)\right) \\
&= \Pr\left(b \overset{0,t}{\not\rightsquigarrow} a \,\middle|\, T_{\text{on}}^{(\tau_2)} = \sigma, b \in \mathcal{I}(t^-), K = \kappa, T = \tau_2, (a,b) \notin E(t)\right)
\end{aligned}
$$

In light of (5.69) and (41), the above inequality implies the required lower bound. □

Lemma 41. *Let T denote the random time defined earlier. Then*

$$\int_0^t e^{-B_{ij}\tau} f_{T|(\mathcal{S}(t),\mathcal{I}(t))=(\mathcal{S}_0,\mathcal{I}_0),(a,b)\notin E(t)}(\tau)d\tau \geq 1 - \frac{B_{ij}}{\lambda}(1-e^{-\lambda t}).$$

Proof. We first use Bayes' rule to note that for any $\kappa \in [0,t]$,

$$
\begin{aligned}
&f_{T|K=\kappa,(\mathcal{S}(t),\mathcal{I}(t))=(\mathcal{S}_0,\mathcal{I}_0),(a,b)\notin E(t)}(\tau) \\
&= \Pr((\mathcal{S}(t),\mathcal{I}(t)) = (\mathcal{S}_0,\mathcal{I}_0) \mid T = \tau, K = \kappa, (a,b) \notin E(t)) \\
&\quad \cdot f_{K|T=\tau,(a,b)\notin E(t)}(\kappa) \cdot f_{T|(a,b)\notin E(t)}(\tau) \\
&\quad \cdot \left(\int_0^t \Pr((\mathcal{S}(t),\mathcal{I}(t)) = (\mathcal{S}_0,\mathcal{I}_0) \mid T = \tau', K = \kappa, (a,b) \notin E(t)) \right. \\
&\qquad \left. \cdot f_{K|T=\tau',(a,b)\notin E(t)}(\kappa) \cdot f_{T|(a,b)\notin E(t)}(\tau')d\tau' \right)^{-1}.
\end{aligned}
\tag{5.70}
$$

We consider each multiplicand one by one. First, we use Lemmas 32 and 35 to note that $f_{K|T=\tau,(a,b)\notin E(t)}(\kappa)\cdot f_{T|(a,b)\notin E(t)}(\tau) = f_K(\kappa)f_T(\tau)$. To deal effectively with the other multiplicands, we let $T_{\text{on}} := \left(\int_{t-K}^{t-T} 1_{(a,b)}(\tau')d\tau'\right)_+$ denote the total duration of time for which b is in contact with a during the time interval $[t-K, t-T]$, and we observe that

$$
\begin{aligned}
&\Pr((\mathcal{S}(t),\mathcal{I}(t)) = (\mathcal{S}_0,\mathcal{I}_0) \mid T=\tau, K=\kappa, (a,b)\notin E(t)) \\
&= \int_0^{(\kappa-\tau)_+} \Big(\Pr((\mathcal{S}(t),\mathcal{I}(t)) = (\mathcal{S}_0,\mathcal{I}_0) \mid T=\tau, K=\kappa, T_{\text{on}}=\tau_{\text{on}}, (a,b)\notin E(t)) \\
&\qquad\qquad \cdot f_{T_{\text{on}}|K=\kappa,T=\tau,(a,b)\notin E(t)}(\tau_{\text{on}})\Big) d\tau_{\text{on}} \\
&= \int_0^{(\kappa-\tau)_+} \Big(\Pr((\mathcal{S}(t),\mathcal{I}(t)) = (\mathcal{S}_0,\mathcal{I}_0) \mid T=\tau, K=\kappa, T_{\text{on}}=\tau_{\text{on}}, (a,b)\notin E(t), b \overset{0,t}{\not\rightsquigarrow} a, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad b\in\mathcal{I}(t^-)) \\
&\qquad\qquad \cdot \Pr(b \overset{0,t}{\not\rightsquigarrow} a \mid b\in\mathcal{I}(t^-), T=\tau, K=\kappa, T_{\text{on}}=\tau_{\text{on}}, (a,b)\notin E(t)) \\
&\qquad\qquad \cdot \Pr(b\in\mathcal{I}(t^-) \mid T=\tau, K=\kappa, T_{\text{on}}=\tau_{\text{on}}, (a,b)\notin E(t)) \\
&\qquad\qquad \cdot f_{T_{\text{on}}|K=\kappa,T=\tau,(a,b)\notin E(t)}(\tau_{\text{on}})\Big) d\tau_{\text{on}} \\
&\overset{(a)}{=} \int_0^{(\kappa-\tau)_+} \Big(\Pr((\mathcal{S}(t),\mathcal{I}(t)) = (\mathcal{S}_0,\mathcal{I}_0) \mid K=\kappa, (a,b)\notin E(t), b \overset{0,t}{\not\rightsquigarrow} a, b\in\mathcal{I}(t^-)) \\
&\qquad\qquad \cdot e^{-B_{ij}\tau_{\text{on}}} \cdot e^{-\gamma_j\kappa} \cdot f_{T_{\text{on}}|K=\kappa,T=\tau,(a,b)\notin E(t)}(\tau_{\text{on}})\Big) d\tau_{\text{on}},
\end{aligned}
$$

where (a) follows from Lemmas 37 and 39 and from the modelling assumption that b recovers at rate γ_j independently of any edge state. On substituting the above expression into (5.70), we obtain

$$
f_{T|K=\kappa,(\mathcal{S}(t),\mathcal{I}(t))=(\mathcal{S}_0,\mathcal{I}_0),(a,b)\notin E(t)}(\tau) = \frac{\left(\int_0^{(\kappa-\tau)_+} e^{-B_{ij}\tau_{\text{on}}}\psi_{\kappa,\tau}(\tau_{\text{on}})d\tau_{\text{on}}\right) f_T(\tau)}{\int_0^t \left(\int_0^{(\kappa-\tau')_+} e^{-B_{ij}\tau_{\text{on}}}\psi_{\kappa,\tau'}(\tau_{\text{on}})d\tau_{\text{on}}\right) f_T(\tau')d\tau'}, \tag{5.71}
$$

where $\psi_{\kappa,\tau}(\cdot) := f_{T_{\text{on}}|K=\kappa,T=\tau,(a,b)\notin E(t)}(\cdot)$. Now, Lemma 37 implies that

$$\begin{aligned}
&\int_0^{(\kappa-\tau)_+} e^{-B_{ij}\tau_{\text{on}}}\psi_{\kappa,\tau}(\tau_{\text{on}})d\tau_{\text{on}} \\
&= \int_0^{(\kappa-\tau)_+} \Bigg(\Pr\left(b \overset{0,t}{\not\rightsquigarrow} a \,\middle|\, T_{\text{on}} = \tau_{\text{on}}, b \in \mathcal{I}(t^-), K = \kappa, T = \tau, (a,b) \notin E(t)\right) \\
&\qquad\qquad \cdot f_{T_{\text{on}}|K=\kappa,T=\tau,(a,b)\notin E(t)}(\tau_{\text{on}})\Bigg) d\tau_{\text{on}} \\
&\overset{(a)}{=} \int_0^{(\kappa-\tau)_+} \Pr\left(b \overset{0,t}{\not\rightsquigarrow} a \,\middle|\, T_{\text{on}} = \tau_{\text{on}}, b \in \mathcal{I}(t^-), K = \kappa, T = \tau, (a,b) \notin E(t)\right) \\
&\qquad\qquad \cdot f_{T_{\text{on}}|b\in\mathcal{I}(t^-),K=\kappa,T=\tau,(a,b)\notin E(t)}(\tau_{\text{on}}) d\tau_{\text{on}} \\
&= \Pr\left(b \overset{0,t}{\not\rightsquigarrow} a \,\middle|\, b \in \mathcal{I}(t^-), K = \kappa, T = \tau, (a,b) \notin E(t)\right),
\end{aligned}$$

where (a) holds because the recovery time of b is conditionally independent of T_{on} given $K = \kappa$ (recall that b recovers at rate γ_j independently of any edge state (and hence independently of T_{on}), and $\{b \in \mathcal{I}(t^-)\}$ is precisely the event that the recovery time of b is at least K). Hence, (5.71) implies that for any $\tau_1, \tau_2 \in [0,t]$ satisfying $\tau_1 \leq \tau_2$, we have

$$\frac{g(\tau_2)}{g(\tau_1)} = \frac{\Pr\left(b \overset{0,t}{\not\rightsquigarrow} a \,\middle|\, b \in \mathcal{I}(t^-), K = \kappa, T = \tau_2, (a,b) \notin E(t)\right)}{\Pr\left(b \overset{0,t}{\not\rightsquigarrow} a \,\middle|\, b \in \mathcal{I}(t^-), K = \kappa, T = \tau_1, (a,b) \notin E(t)\right)} \cdot \frac{f_T(\tau_2)}{f_T(\tau_1)}, \tag{5.72}$$

where $g(\cdot) := f_{T|K=\kappa(\mathcal{S}(t),\mathcal{I}(t))=(\mathcal{S}_0,\mathcal{I}_0),(a,b)\notin E(t)}(\cdot)$.

As a consequence of (5.72) and Lemma 40, we have

$$\frac{g(\tau_2)}{g(\tau_1)} \leq e^{B_{ij}(\tau_2-\tau_1)} \frac{f_T(\tau_2)}{f_T(\tau_1)}.$$

Since $f_T(\tau) = \lambda e^{-\lambda\tau} + e^{-\lambda t}\delta(\tau - t)$ for $\tau \in [0,t]$ (see Lemma 34), we have the following for all

$0 \leq \tau_1 \leq \tau_2 < t$:

$$g(\tau_2) \leq e^{-(\lambda - B_{ij})(\tau_2 - \tau_1)} g(\tau_1), \tag{5.73}$$

and for all $0 \leq \tau < t$, we have

$$\tilde{g}(t) \leq \frac{e^{-(\lambda - B_{ij})(t-\tau)}}{\lambda} g(\tau), \tag{5.74}$$

where $\tilde{g}(t)$ scales $\delta(0)$ so that $g(t) = \tilde{g}(t)\delta(0)$. Since δ is the Dirac-delta function, (5.74) simply means that

$$\Pr(T = t \mid K = \kappa, (\mathcal{S}(t), \mathcal{I}(t)) = (\mathcal{S}_0, \mathcal{I}_0), (a,b) \notin E(t)) \leq \frac{e^{-(\lambda - B_{ij})(t-\tau)}}{\lambda} g(\tau).$$

Our next goal is to use (5.73) and (5.74) to show that

$$\int_0^t e^{-B_{ij}\tau} g(\tau) d\tau \geq \int_0^t e^{-B_{ij}\tau} \varphi(\tau) d\tau, \tag{5.75}$$

where φ is the probability density function defined by

$$\varphi(\tau) := (\lambda - B_{ij}) e^{-(\lambda - B_{ij})\tau} + e^{-(\lambda - B_{ij})t} \delta(\tau - t)$$

for all $\tau \in [0,t]$ and $\varphi(\tau) = 0$ for $\tau > t$. To this end, we compare g with φ under the following two cases.

Case 1: There exists a time $\tau_0 \in [0,t)$ such that $g(\tau_0) < \varphi(\tau_0)$. In this case, (5.73) implies that for all $\tau \in [\tau_0, t)$,

$$
\begin{aligned}
g(\tau) &\leq e^{-(\lambda - B_{ij})(\tau - \tau_0)} g(\tau_0) \\
&< e^{-(\lambda - B_{ij})(\tau - \tau_0)} (\lambda - B_{ij}) e^{-(\lambda - B_{ij})(\tau_0)} \\
&= \varphi(\tau),
\end{aligned}
$$

which means that the set $\{\tau \in [0,t) : g(\tau) < \varphi(\tau)\}$ is either $[\tau^*, t)$ or (τ^*, t), where $\tau^* := \inf\{\tau : g(\tau) < \varphi(\tau)\}$. Also, by the definition of τ^*, we have $g(\tau) \geq \varphi(\tau)$ for all $\tau \in [0, \tau^*)$. Next, to compare g and φ at $\tau = t$, we use (5.74) to note that

$$
\begin{aligned}
g(t) &\leq \frac{e^{-(\lambda - B_{ij})(t - \tau_0)}}{\lambda} g(\tau_0) \delta(0) \\
&\leq \frac{e^{-(\lambda - B_{ij})(t - \tau_0)}}{\lambda} (\lambda - B_{ij}) e^{-(\lambda - B_{ij})\tau_0} \delta(0) \\
&= \left(1 - \frac{B_{ij}}{\lambda}\right) e^{-(\lambda - B_{ij})t} \delta(0) \\
&\leq \varphi(t). \qquad (5.76)
\end{aligned}
$$

Thus, $g(\tau) - \varphi(\tau) \geq 0$ for all $\tau \in [0, \tau^*)$ and $g(\tau) - \varphi(\tau) \leq 0$ for all $\tau \in (\tau^*, t]$. Now, since g and φ are both PDFs, we must have $\int_0^\infty (g(\tau) - \varphi(\tau)) d\tau = 0$ or equivalently,

$$
\int_0^{\tau^*} (g(\tau) - \varphi(\tau)) d\tau = \int_{\tau^*}^{\infty} (\varphi(\tau) - g(\tau)) d\tau.
$$

Since both the integrands above are non-negative, we have

$$
\begin{aligned}
&\int_0^{\tau^*} e^{-B_{ij}\tau} (g(\tau) - \varphi(\tau)) d\tau \\
&\geq e^{-B_{ij}\tau^*} \int_0^{\tau^*} (g(\tau) - \varphi(\tau)) d\tau \\
&= e^{-B_{ij}\tau^*} \int_{\tau^*}^{\infty} (\varphi(\tau) - g(\tau)) d\tau \\
&\geq \int_{\tau^*}^{\infty} e^{-B_{ij}\tau} (\varphi(\tau) - g(\tau)) d\tau.
\end{aligned}
$$

Adding $\int_{\tau^*}^{\infty} e^{-B_{ij}\tau} g(\tau) d\tau + \int_0^{\tau^*} e^{-B_{ij}\tau} \varphi(\tau) d\tau$ to both sides now yields (5.75).

Case 2: $g(\tau) \geq \varphi(\tau)$ for all $\tau \in [0,t)$. In this case, we can simply set $\tau^* = t$ and repeat the arguments following (5.76) in Case 1 to show that (5.75) holds.

Next, we use the definition of φ to evaluate $\int_0^t e^{-B_{ij}\tau} \varphi(\tau) d\tau$, and we then restate (5.75) as follows:

$$\int_0^t e^{-B_{ij}\tau} f_{T|K=\kappa,(\mathcal{S}(t),\mathcal{I}(t))=(\mathcal{S}_0,\mathcal{I}_0),(a,b)\notin E(t)}(\tau) d\tau \geq 1 - \frac{B_{ij}}{\lambda}(1-e^{-\lambda t}). \tag{5.77}$$

Since this holds for all $\kappa \in [0,t)$, the assertion of the lemma follows. □

Using arguments very similar to the proof above, we can prove the following result.

Lemma 42. *Let T denote the random time defined earlier. Then*

$$\int_0^t e^{-B_{ij}\tau} f_{T|(\mathcal{S}(t),\mathcal{I}(t))=(\mathcal{S}_0,\mathcal{I}_0),(a,b)\in E(t)}(\tau) d\tau \geq 1 - \frac{B_{ij}}{\lambda}(1-e^{-\lambda t}).$$

Remark 17. *In the proof of Lemma 41, if, instead of using* (5.72) *along with the upper bound in Lemma 40, we had used* (5.72) *along with the lower bound in Lemma 40, we would have obtained*

$$\frac{g(\tau_2)}{g(\tau_1)} \geq \frac{f_T(\tau_2)}{f_T(\tau_1)} = e^{-\lambda(\tau_2-\tau_1)}.$$

In addition, if we had subsequently replaced t with a generic $\tau \in [0,t)$ and the weighting function $[0,\infty) \ni \tau \to e^{-B_{ij}\tau} \in (0,\infty)$ by the constant function 1, and if we had defined φ by $\varphi(\tau) := \lambda e^{-\lambda\tau} + e^{-\lambda t}\delta(\tau - t)$, then using the same arguments but with reversed inequality signs, we would have been able to prove that

$$\int_0^{\tau} 1 \cdot g(\tau') d\tau' \leq \int_0^{\tau} 1 \cdot \varphi(\tau') d\tau'.$$

Since the integral on the left-hand-side is $\Pr(T \leq \tau \mid K = \kappa, (\mathcal{S}(t),\mathcal{I}(t)) = (\mathcal{S}_0,\mathcal{I}_0), (a,b) \notin E(t))$

and since the right-hand-side evaluates to $1-e^{-\lambda\tau}$*, we conclude that*

$$\Pr(T \leq \tau \mid K=\kappa, (\mathcal{S}(t),\mathcal{I}(t)) = (\mathcal{S}_0,\mathcal{I}_0), (a,b) \notin E(t)) \leq 1-e^{-\lambda\tau}$$

for all $\tau \in [0,t)$ *and all* $\kappa \in [0,t]$.

Some Auxiliary Lemmas

In addition to the above results, the proof of Theorem 7 relies on the following lemmas, which we reproduce from [166].

Lemma 43. *For random variables* Y *and* Z*, we have* $\mathit{Var}[Y+Z] \leq 2(\mathit{Var}[Y]+\mathit{Var}[Z])$.

Lemma 44. *For a random variable* $Y \in [0,1]$*, we have* $\mathit{Var}[Y^2] \leq 4\mathit{Var}[Y]$.

Lemma 45. *For random variables* Y *and* Z *in* $[0,1]$*,*

$$|\mathrm{E}[YZ]-\mathrm{E}[Y]\mathrm{E}[Z]| \leq (\mathrm{Var}[Y]+\mathrm{Var}[Z])/2$$

$$\left|\mathrm{E}\left[Y^2Z\right]-\mathrm{E}[Y]^2\mathrm{E}[Z]\right| \leq 2(\mathrm{Var}[Y]+\mathrm{Var}[Z]).$$

The following result is a straightforward consequence of the above lemmas.

Corollary 11. *For non-negative random variables* Y *and* Z *satisfying* $0 \leq Y+Z \leq 1$*, we have*

$$\mathit{Var}[YZ] \leq 8(\mathit{Var}[Y]+\mathit{Var}[Z]).$$

Proof. Note that $4\mathrm{Var}[YZ] = \mathrm{Var}[2YZ] = \mathrm{Var}[(Y+Z)^2+(-1)(Y^2+Z^2)]$. Hence,

$$\begin{aligned}
4\mathrm{Var}[YZ] &\overset{(a)}{\leq} 2(\mathrm{Var}[(Y+Z)^2]+(-1)^2\mathrm{Var}[Y^2+Z^2]) \\
&\overset{(b)}{\leq} 2(4\mathrm{Var}[Y+Z]+2(\mathrm{Var}[Y^2]+\mathrm{Var}[Z^2])) \\
&\overset{(c)}{\leq} 2(4\mathrm{Var}[Y+Z]+2(4\mathrm{Var}[Y]+4\mathrm{Var}[Z])),
\end{aligned}$$

where (a) follows from Lemma 43, (b) from both Lemma 43 and Lemma 44, and (c) from Lemma 44 alone. Thus,

$$\text{Var}[YZ] \leq 2(\text{Var}[Y+Z] + 2(\text{Var}[Y] + \text{Var}[Z])) \leq 8(\text{Var}[Y] + \text{Var}[Z]),$$

where the last inequality follows from Lemma 43. □

Proof of Theorem 7

Proof. The proof is based on Proposition 13 and it follows the approach used in [166]. We first modify Equations (i) - (iv) (Proposition 12) by expressing the expectations of cross-terms such as $\mathbb{E}[s_i \beta_j]$ in terms of expectations of individual terms such as $\mathbb{E}[s_i^2]$ and $\mathbb{E}[\beta_j]$. To begin, we apply Lemma 45 to $\mathbb{E}[s_i(t)\beta_j(t)]$ and obtain

$$|\mathbb{E}[s_i(t)\beta_j(t)] - \mathbb{E}[s_i(t)]\mathbb{E}[\beta_j(t)]| \leq \frac{1}{2}(\text{Var}[s_i(t)] + \text{Var}[\beta_j(t)]).$$

Therefore, there exists a function $h_{i,j,1,n} : [0,\infty) \to [-1,1]$ such that

$$\mathbb{E}[s_i(t)\beta_j(t)] = \mathbb{E}[s_i(t)]\mathbb{E}[\beta_j(t)] + \frac{h_{i,j,1,n}(t)}{2}(\text{Var}[s_i(t)] + \text{Var}[\beta_j(t)]).$$

Similarly, we can use Lemma 45 to show that there exists a function $h_{i,j,2,n} : [0,\infty) \to [-1,1]$ such that

$$\mathbb{E}[s_i^2(t)\beta_j(t)] = \mathbb{E}[s_i(t)]^2\mathbb{E}[\beta_j(t)] + 2h_{i,j,2,n}(t)(\text{Var}[s_i(t)] + \text{Var}[\beta_j(t)]).$$

Next, we use Corollary 11 to express $\mathbb{E}[s_i(t)\beta_j(t)\beta_i(t)]$ as

$$\mathbb{E}[s_i(t)\beta_j(t)\beta_i(t)] = \mathbb{E}[s_i(t)\beta_j(t)]\mathbb{E}[\beta_i(t)] + \frac{h_{i,j,5,n}(t)}{2}(\text{Var}[s_i(t)\beta_j(t)] + \text{Var}[\beta_i(t)]),$$

which means that

$$\mathbb{E}[s_i(t)\beta_j(t)\beta_i(t)] = \Big(\mathbb{E}[s_i(t)]\mathbb{E}[\beta_j(t)] + \frac{h_{i,j,1,n}(t)}{2}(\text{Var}[s_i(t)] + \text{Var}[\beta_j(t)])\Big)\mathbb{E}[\beta_i(t)]$$
$$+ \frac{h_{i,j,5,n}(t)}{2}(h_{i,j,6,n}(t)(8\text{Var}[s_i(t)] + 8\text{Var}[\beta_j(t)]) + \text{Var}[\beta_i(t)]),$$

where $h_{i,j,5,n}(t) \in [-1,1]$ and $h_{i,j,6,n}(t) \in [0,1]$.

We thus obtain the following relations:

(I) $\mathbb{E}[s_i\beta_j] = \mathbb{E}[s_i]\mathbb{E}[\beta_j] + \frac{h_{i,j,1,n}}{2}(\text{Var}[s_i] + \text{Var}[\beta_j])$,

(II) $\mathbb{E}[s_i^2\beta_j] = \mathbb{E}[s_i]^2\mathbb{E}[\beta_j] + 2h_{i,j,2,n}(\text{Var}[s_i] + \text{Var}[\beta_j])$,

(III) $\mathbb{E}[s_i\beta_j\beta_i] = \Big(\mathbb{E}[s_i]\mathbb{E}[\beta_j] + \frac{h_{i,j,1,n}}{2}(\text{Var}[s_i] + \text{Var}[\beta_j])\Big)$
$+\frac{h_{i,j,5,n}}{2}(h_{i,j,6,n}(8\text{Var}[s_i] + 8\text{Var}[\beta_j]) + \text{Var}[\beta_i])$.

To handle terms of the form $B_{ij}\mathbb{E}[n \cdot \chi_{ij}(t,\mathcal{S},\mathcal{I}) \cdot U]$ where U is some random variable, we use Proposition 13 to obtain

$$A_{ij}\left(1 - \frac{B_{ij}}{\lambda^{(n)}}(1 - e^{-\lambda^{(n)}t})\right)\mathbb{E}[U] \leq B_{ij}\mathbb{E}[n\chi_{ij}(t,\mathcal{S},\mathcal{I})U] \leq A_{ij}\mathbb{E}[U].$$

As a result, if $\Pr(|U| \leq 1) = 1$, then there exists a function $h_{i,j,U,n} : [0,\infty) \to [0, B_{ij}A_{ij}]$ such that

$$B_{ij}\mathbb{E}[n\chi_{ij}(t,\mathcal{S},\mathcal{I})U] = A_{ij}\mathbb{E}[U] - \frac{h_{i,j,U,n}(t)}{\lambda^{(n)}}. \tag{5.78}$$

By making the above substitutions in (i) - (iv) and by using the identity $\text{Var}[Y]' = \mathbb{E}[Y^2]' - 2\mathbb{E}[Y]\mathbb{E}[Y]'$, we obtain the following differential equations:

(I) $\mathbb{E}[s_i]' = \sum_{j=1}^m \frac{h_{i,j,7,n}}{\lambda^{(n)}} - \sum_{j=1}^m A_{ij}\left(\mathbb{E}[s_i]\mathbb{E}[\beta_j] + \frac{h_{i,j,1,n}}{2}(\text{Var}[s_i] + \text{Var}[\beta_j])\right)$,

(II) $\mathbb{E}[\beta_i]' = \sum_{j=1}^m A_{ij}\left(\mathbb{E}[s_i]\mathbb{E}[\beta_j] + \frac{h_{i,j,1,n}}{2}(\text{Var}[s_i] + \text{Var}[\beta_j])\right) - \sum_{j=1}^m \frac{h_{i,j,7,n}}{\lambda^{(n)}} - \gamma_i\mathbb{E}[\beta_i]$,

(III)

$$
\begin{aligned}
\text{Var}[s_i]' = & -2\sum_{j=1}^{m} A_{ij}\Big(\mathbb{E}[s_i]^2\mathbb{E}[\beta_j] + 2h_{i,j,2,n}(\text{Var}[s_i] + \text{Var}[\beta_j])\Big) \\
& + \sum_{j=1}^{m} A_{ij}\left(\mathbb{E}[s_i]\mathbb{E}[\beta_j] + \frac{h_{i,j,1,n}}{2}(\text{Var}[s_i] + \text{Var}[\beta_j])\right)\left(2\mathbb{E}[s_i] + \frac{1}{n}\right) \\
& + \sum_{j=1}^{m}\left(\frac{2h_{i,j,8,n}}{\lambda^{(n)}} - \frac{h_{i,j,7,n}}{n\lambda^{(n)}} - 2\mathbb{E}[s_i]\frac{h_{i,j,7,n}}{\lambda^{(n)}}\right),
\end{aligned}
$$

(IV)

$$
\begin{aligned}
\text{Var}[\beta_i]' = 2\sum_{j=1}^{m} A_{ij}\Bigg(& \mathbb{E}[s_i]\mathbb{E}[\beta_j]\mathbb{E}[\beta_i] + \frac{h_{i,j,1,n}}{2}(\text{Var}[s_i] + \text{Var}[\beta_j])\mathbb{E}[\beta_i] \\
& + \frac{h_{i,j,5,n}}{2}\left(h_{i,j,6,n}(8\text{Var}[s_i] + 8\text{Var}[\beta_j]) + \text{Var}[\beta_i]\right)\Bigg) \\
+ \sum_{j=1}^{m} A_{ij}\left(\frac{1}{n} - 2\mathbb{E}[\beta_i]\right) & \left(\mathbb{E}[s_i]\mathbb{E}[\beta_j]\frac{h_{i,j,1,n}}{2}(\text{Var}[s_i] + \text{Var}[\beta_j])\right) - 2\gamma_i\text{Var}[\beta_i] \\
+ \gamma_i\frac{\mathbb{E}[\beta_i]}{n} \quad & - \sum_{j=1}^{m}\left(\frac{2h_{i,j,9,n}}{\lambda^{(n)}} + \frac{h_{i,j,7,n}}{n\lambda^{(n)}}\right),
\end{aligned}
$$

where $h_{i,j,7,n}$, $h_{i,j,8,n}$, and $h_{i,j,9,n}$ are functions from $[0,\infty)$ to $[0, B_{ij}A_{ij}]$ and are defined on the basis of (5.78).

The above equations constitute a proper system of differential equations with the same variables $\{\mathbb{E}[s_i]\}_{i=1}^m$, $\{\text{Var}[s_i]\}_{i=1}^m$, $\{\mathbb{E}[\beta_i]\}_{i=1}^m$, and $\{\text{Var}[\beta_i]\}_{i=1}^m$ appearing on both the sides. To express these equations compactly, we define $z^{(n)} \in [0,1]^{4m}$ as the vector whose entries are given by $z_{i,1}^{(n)} := z_{4(i-1)+1}^{(n)} := \mathbb{E}[s_i^{(n)}]$, $z_{i,2}^{(n)} := z_{4(i-1)+2}^{(n)} := \mathbb{E}[\beta_i^{(n)}]$, $z_{i,3}^{(n)} := z_{4(i-1)+3}^{(n)} := \text{Var}[s_i^{(n)}]$, and $z_{i,4}^{(n)} := z_{4i}^{(n)} := \text{Var}[\beta_i^{(n)}]$. Then $z^{(n)}(t)$ is a solution to the initial value problem $(z^{(n)})' = g_n(t, z^{(n)}; 1/n, 1/\lambda^{(n)})$ and $z(0) = z_0^{(n)}$, where

(I) $g_{i,n}^{(1)}(t,z;\varepsilon_1,\varepsilon_2) := -\sum_{j=1}^m A_{ij}\left(z_{i,1}z_{j,2} + \frac{h_{i,j,1,n}}{2}(z_{i,3} + z_{j,4})\right) + \varepsilon_2\sum_{j=1}^m h_{i,j,7,n}$,

(II) $g_{i,n}^{(2)}(t,z;\varepsilon_1,\varepsilon_2) := \sum_{j=1}^m A_{ij}\left(z_{i,1}z_{j,2} + \frac{h_{i,j,1,n}}{2}(z_{i,3} + z_{j,4})\right) - \gamma_i z_{i,2} - \varepsilon_2\sum_{j=1}^m h_{i,j,7,n}$,

(III)

$$g_{i,n}^{(3)}(t,z;\varepsilon_1,\varepsilon_2) := -2\sum_{j=1}^{m} A_{ij}((z_{i,1})^2 z_{j,2} + 2h_{i,j,2,n}(z_{i,3}+z_{j,4}))$$
$$+\sum_{j=1}^{m} A_{ij}\left(z_{i,1}z_{j,2} + \frac{h_{i,j,1,n}}{2}(z_{i,3}+z_{j,4})\right)(2z_{i,1}+\varepsilon_1)$$
$$+\sum_{j=1}^{m}(2h_{i,j,8,n}\varepsilon_2 - h_{i,j,7,n}\varepsilon_1\varepsilon_2 - 2\mathbb{E}[s_1]h_{i,j,7,n}\varepsilon_2),$$

(IV)

$$g_{i,n}^{(4)}(t,z;\varepsilon_1,\varepsilon_2) = 2\sum_{j=1}^{m} A_{ij}\Bigg(z_{i,1}z_{j,2}z_{i,2} + \frac{h_{i,j,1,n}}{2}(z_{i,3}+z_{j,4})z_{i,2}$$
$$+\frac{h_{i,j,5,n}}{2}\left(h_{i,j,6,n}(8z_{i,3}+8z_{j,4})+z_{i,4}\right)\Bigg)$$
$$+\sum_{j=1}^{m} A_{ij}(\varepsilon_1 - 2z_{i,2})\left(z_{i,1}z_{j,2} + \frac{h_{i,j,1,n}}{2}(z_{i,3}+z_{j,4})\right)$$
$$-2\gamma_i z_{i,4} + \varepsilon_1\gamma_i z_{i,2} - \sum_{j=1}^{m}(2h_{i,j,9,n}\varepsilon_2 + h_{i,j,7,n}\varepsilon_1\varepsilon_2),$$

(V) $z_0^{(n)} = (s_1^{(n)}(0), \beta_1^{(n)}(0), 0, 0, s_2^{(n)}(0), \beta_2^{(n)}(0), 0, 0, \ldots, s_m^{(n)}(0), \beta_m^{(n)}(0), 0, 0)$.

Observe that irrespective of the value of n, the solution $(\bar{z}_{i,1}(t), \bar{z}_{i,2}(t), \bar{z}_{i,3}(t), \bar{z}_{i,4}(t)) := (y_i(t), w_i(t), 0, 0)$ solves the initial value problem $z' = g_n(t,z;0,0)$ and $z(0) = z_0$, where

$$z_0 := (s_{1,0}, \beta_{1,0}, 0, 0, s_{2,0}, \beta_{2,0}, 0, 0, \ldots, s_{m,0}, \beta_{m,0}, 0, 0).$$

Next, we need to bound $\left\|z^{(n)}(t) - \bar{z}(t)\right\|$ (where $\bar{z}(t) \in [0,1]^{4m}$ is the unique vector satisfying $\bar{z}_{4(i-1)+\ell}(t) = \bar{z}_{i,\ell}(t)$ for all $i \in [m]$ and $\ell \in [4]$). For this purpose, we will need the following lemma, which we borrow from [166].

Lemma 46. *Consider the initial value problems $x' = f_1(t,x)$, $x(0) = x_1$ and $x' = f_2(t,x)$, $x(0) = x_2$ with solutions $\varphi_1(t)$ and $\varphi_2(t)$ respectively. If f_1 is Lipschitz in x with constant L and*

$\|f_1(t,x) - f_2(t,x)\| \leq M$, *then* $\|\varphi_1(t) - \varphi_2(t)\| \leq (\|x_1 - x_2\| + M/L)\, e^{Lt} - M/L$.

Now, note that the domain of z for $g_n(t,z;\varepsilon_1,\varepsilon_2)$ can be chosen to be bounded because $0 \leq \mathbb{E}[s_i], \mathbb{E}[\beta_i] \leq 1$ and $\text{Var}[s_i] \leq \mathbb{E}[s_i^2] \leq 1$. Similarly, $\text{Var}[\beta_i] \leq 1$. Also, we let $\varepsilon_1, \varepsilon_2 \in (0,1)$ and define $\varepsilon := \max\{\varepsilon_1, \varepsilon_2\}$. Since $g_n(t,z;0,0)$ is a polynomial in z, it is Lipschitz-continuous with some Lipschitz constant $L \in (0,\infty)$. In addition, we use the bounds on z and the functions $\{h_{i,j,\ell,n} : 1 \leq \ell \leq 9\}$ as follows:

$$\begin{aligned}
\|g_n(t,z;\varepsilon_1,\varepsilon_2) - g_n(t,z;0,0)\| &\leq 2\sum_{i=1}^{m}\sum_{j=1}^{m} A_{ij}\varepsilon \left| z_{i,1}z_{j,2} + \frac{h_{i,j,1,n}}{2}(z_{i,3} + z_{j,4}) \right| + \sum_{i=1}^{m}\gamma_i\varepsilon \\
&\quad + 10\sum_{i=1}^{m}\sum_{j=1}^{m} A_{ij}B_{ij}\varepsilon \\
&\leq \left(\sum_{i=1}^{m}\sum_{j=1}^{m} A_{ij}(4 + 10B_{ij}) + \sum_{i=1}^{m}\gamma_i \right)\varepsilon,
\end{aligned}$$

i.e.,

$$\|g_n(t,z;\varepsilon_1,\varepsilon_2) - g_n(t,z;0,0)\| \leq M(\varepsilon),$$

where $M(\varepsilon) := \left(\sum_{i=1}^{m}\sum_{j=1}^{m} A_{ij}(4 + 10B_{ij}) + \sum_{i=1}^{m}\gamma_i \right)\varepsilon$.

We now apply Lemma 46 after setting

$$f_1(t,x) = g_n(t,x;0,0), \quad f_2(t,x) = g_n(t,x;1/n, 1/\lambda^{(n)}), \quad x_1 = z_0, \quad x_2 = z_0^{(n)}.$$

Also, we let $\varphi_1 = \bar{z}$ and $\varphi_2 = z^{(n)}$. Then we have

$$\left\| z^{(n)}(t) - \bar{z}(t) \right\| \leq \left(\left\| z_0 - z_0^{(n)} \right\| + \frac{M\alpha_n}{L} \right) e^{Lt} - \frac{M\alpha_n}{L},$$

where $\alpha_n := \max\left\{\frac{1}{n}, \frac{1}{\lambda^{(n)}}\right\}$. Thus, for all $t \leq T$,

$$\left\|z^{(n)}(t) - \bar{z}(t)\right\| \leq \left(\left\|z_0 - z_0^{(n)}\right\| + \frac{M\alpha_n}{L}\right)e^{LT} - \frac{M\alpha_n}{L}. \tag{5.79}$$

Since $\lim_{\varepsilon\to 0} M(\varepsilon) = 0$, $\lim_{n\to\infty} z_0^{(n)} = z_0$, and $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\max\left\{\frac{1}{n}, \frac{1}{\lambda^{(n)}}\right\} = 0$, the right hand side of (5.79) goes to zero as $n \to \infty$. Hence we have the uniform convergence $z^{(n)} \to \bar{z}$ over any finite time interval $[0,T]$. The last step is to show that $z^{(n)} \to \bar{z}$ implies L^2-convergence, i.e., $\mathbb{E}[\|(s_i^{(n)} - y_i, \beta_i^{(n)} - w_i)\|_2] \to 0$ as $n \to \infty$. To this end, we have

$$\begin{aligned}
\mathbb{E}[\|(s_i^{(n)} - y_i, \beta_i^{(n)} - w_i)\|_2^2] &= \mathbb{E}[(s_i^{(n)} - y_i)^2] + \mathbb{E}[(\beta_i^{(n)} - w_i)^2] \\
&= (\mathbb{E}[s_i^{(n)}] - y_i])^2 + (\mathbb{E}[\beta_i^{(n)}] - w_i)^2 + \text{Var}[s_i^{(n)}] + \text{Var}[\beta_i^{(n)}] \\
&\leq |\mathbb{E}[s_i^{(n)}] - y_i]| + |\mathbb{E}[\beta_i^{(n)}] - w_i]| + \text{Var}[s_i^{(n)}] + \text{Var}[\beta_i^{(n)}] \\
&= |z_{i,1}^{(n)} - \bar{z}_{i,1}| + |z_{i,2}^{(n)} - \bar{z}_{i,2}| + |z_{i,3}^{(n)} - \bar{z}_{i,3}| + |z_{i,4}^{(n)} - \bar{z}_{i,4}|,
\end{aligned}$$

where we used that $\bar{z}_{i,3} = \bar{z}_{i,4} = 0$, and the inequality holds because $y_i, \mathbb{E}[s_i^{(n)}], w_i, \mathbb{E}[\beta_i^{(n)}] \in [0,1]$. Thus, the uniform convergence of $z^{(n)}$ to $\bar{z}$ over $[0,T]$ proves that $\mathbb{E}[\|(s_i^{(n)} - y_i, \beta_i^{(n)} - w_i)\|_2] \to 0$ as $n \to \infty$.

□

Chapter 5, in full, is a reprint of the material as it appears in Rohit Parasnis, Ryosuke Kato, Amol Sakhale, Massimo Franceschetti, and Behrouz Touri, "Usefulness of the Age-Structured SIR Dynamics for Modelling COVID-19", *arXiv preprint arXiv:2203.05111* (2022). The dissertation author was the primary investigator and author of this article.

Chapter 5, in full, is currently being prepared for submission for publication as Rohit Parasnis, Ryosuke Kato, Amol Sakhale, Massimo Franceschetti, and Behrouz Touri, "Usefulness of the Age-Structured SIR Dynamics for Modelling COVID-19" (the publication venue is to be determined). The dissertation author was the primary investigator and author of this article.

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