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Tests on Competitive Structures and Theories on Firm's Reputation Maintenance

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Tests on Competitive Structures and

Theories on Firm's Reputation Maintenance

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by

Yuta Yasui

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ABSTRACT OF THE DISSERTATION

Tests on Competitive Structures and
Theories on Firm's Reputation Maintenance

by

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Doctor of Philosophy in Economics
University of California, Los Angeles, 2021
Professor Ichiro Obara, Chair

This dissertation theoretically analyzes three topics in industrial organization. The first chapter introduces tests for price competition among multi-product firms. The tests are based on the firm's revealed preference (revealed profit function). I employ a demand structure introduced by Nocke and Schutz (2018), the discrete/continuous choice model, which nests the multinomial logit demand and CES demand functions. Any price and quantity data can be rationalized by price competition under a discrete/continuous choice model and increasing marginal costs. Adding more assumptions on the demand function, such as logit, CES, or the co-evolving and log-concave property produces some falsifiable restrictions.

The second and third chapters analyze seller's reputation maintenance behavior in dynamic models. Chapter 2 theoretically analyzes fake reviews on a platform market using models where a seller creates fake reviews through incentivized transactions, and its sales depend on its rating based on a review history. The platform can control the incentive for fake reviews by changing the parameters of the rating system, such as weights placed on old and new reviews and its filtering policy. At equilibrium, the number of fake reviews increases as quality increases but decreases as reputation improves. Since fake reviews have a positive relationship with a productâs underlying quality, rational consumers find a rating more informative when fake reviews exist, while credulous consumers suffer from a bias caused by boosted reputation. A stringent filtering policy can decrease the expected amount

of fake reviews and the bias of credulous consumers, but at the same time, it can decrease the informativeness of a rating system for rational consumers. In terms of the weight placed on the review history, rational consumers benefit from higher weights on past reviews than from optimal weights without fake reviews.

In Chapter 3, sellers build their reputation through their investment in product safety. I show that the platform's liability for the third-party products distorts the sellers' incentive to invest in product safety.

The dissertation of Yuta Yasui is approved.

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2021

To Shiho, who have supported and cheered me up all the time.

To my parents, who have supported me from afar.

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CHAPTER 1

Revealed Preference Tests for Price Competition in Multi-product Differentiated Markets

Assumptions on competitive structure are often crucial for marginal cost estimation and counterfactual predictions. This paper introduces tests for price competition among multi-product firms. The tests are based on the firm's revealed preference (revealed profit function). In contrast to other approaches based on estimated demand functions such as conduct parameter estimation, the proposed tests do not require any instrumental variables even though the models can accommodate structural error terms. In this paper, I employ a demand structure introduced by Nocke and Schutz (2018), the discrete/continuous choice model, which nests the multinomial logit demand and CES demand functions. Any price and quantity data can be rationalized by price competition under a discrete/continuous choice model and increasing marginal costs. Adding more assumptions on the demand function, such as logit, CES, or the *co-evolving* and *log-concave* property produces some falsifiable restrictions.

Keywords: revealed preference, multi-product, conduct, discrete/continuous

1.1 Introduction

In the industrial organization literature, we often assume specific competitive structures such as price competition or quantity competition. In many cases, competitive structure assumptions are crucial for empirical research. For instance, we often back out marginal costs from first-order conditions based on estimated demand functions and competitive structures. Results of counterfactual analysis, which often provide the main policy implication in research with structural models, also depend on the imposed competitive structures. Even though we can obtain parameter estimates in a structural model that fits to the data best, the structural model itself could possibly not fit the data. In other words, the data might not be rationalized by the model under any possible parameters. Furthermore, the data might not be rationalized by any realizations of structural error terms. This happens because some data points might be out of the support of the structural model. In this paper, I provide a systematic method to detect such inconsistency between the data and price competition among single- or multi-product firms under a certain class of demand functions.

Regarding consumer behavior, Afriat (1967) shows that finite data satisfy GARP if and only if they are rationalized by utility maximization given a finite set of price vectors. In other words, if the data violate GARP, then it cannot be explained by any (locally non-satiated) utility functions. Brown and Matzkin (1996) extend this idea to the general equilibrium framework. Carvajal et al. (2013) apply the idea to Cournot competition, and show that Cournot rationalizability can be checked by the existence of parameters that satisfy some inequalities. Carvajal et al. (2014) introduced a few variants of Carvajal et al. (2013): a test for multi-market-contact Cournot competition and a test for price competition in a differentiated market. However, they only focus on price competition where each firm produces a single product. Price competition with multi-product firms are often examined in the empirical industrial organization literature (e.g., Berry et al. (1995), Goldberg (1995)). One of the main difficulties to extend Carvajal et al.'s (2014) test to competition among multi-product firms arises from the substitution effects among the same firm's products (or cannibalization effects). We can circumvent such a difficulty by employing an important class of demand structure, the discrete/continuous choice model introduced by Nocke and Schutz (2018), which nests the multinomial logit demand function and constant elasticity of substitution (CES) demand function as special cases.

In order to test the competitive structure, we can also estimate the *conduct parameter*. Bresnahan (1982) shows that we can identify the conduct parameter in an industry from a rotation of the demand function over time (see Bresnahan (1989) for its estimation and applications). Researchers have recently estimated how much firms internalize other firms' profits, a measure that is closely related to the conduct parameter (e.g., Miller and Weinberg

(2017) and Sullivan (2016)). Alternatively, if we have data the on cost structure, we could compare marginal costs backed out from the model with the actual cost data since different competitive structures yield different first order conditions and in turn, different estimates of marginal costs (e.g., Wolfram (1999)). The revealed preference test examined in this paper provides an alternative approach with advantages and disadvantages. The first advantage is that the revealed preference test in this paper does not require any IVs while both conduct parameter estimation and the Wolfram (1999) approach require appropriate IVs. The reason we do not need IVs is that we do not estimate parameters to test the model, but check restrictions that is valid for any parameter values. This is analogous to Afriat (1967)'s theorem, which characterizes a set of data restrictions satisfied as long as consumers maximize their own utility, regardless of the underlying utility function' specification. Second, we only need the market-level price and quantity data, no other product characteristics, to implement the test. Due to this parsimonious data requirement, this test could be used as a pretest/sanity check before detailed estimation.

A disadvantage of the test is that the it is a joint test of competitive structure and demand/cost functions. Therefore, rejection of the model might imply other types of competition under a discrete/continuous demand structure, price competition under other demand functions, or other types of competition under other demand functions. Second, even though the discrete/continuous choice model is general enough to include the logit and CES demand function as special cases, it still has the independence-to-irrelevant-alternative (IIA) property. Therefore, the main theorem in this paper does not hold for the random coefficient logit model (e.g., Berry et al. (1995)). Another issue is that cost functions are assumed to be invariant over time in the tested model, even though the constant marginal costs are not implied since cost functions are assumed to be convex though time-invariant. Therefore, the test should be implemented for short panel data where the cost structure is not supposed to change during the time range. In practice, if a researcher has a long panel, then the data can be divided into many short panels, the test can be implemented for each short data segment, and the rejection ratio can be reported (e.g., Carvajal et al. (2014)). Incorporating the observed cost shifter alleviates this issue, as shown in Section 3.

In terms of the power of tests, any data would satisfy the rationalizability condition of the price competition under the general discrete/continuous demand function. This result is inconsistent with the findings of Carvajal et al. (2013) and Carvajal et al. (2014). The key difference is that they consider demand changes due to a common shock for different firms.[1] Naturally, we can obtain falsifiable restrictions by imposing a similar additional restriction that is compatible with the discrete/continuous demand function. We can also obtain falsifiable restrictions by restricting the underlying demand function to a sub-class of discrete/continuous demand structure, which still nests both logit and CES as its special cases. This also implies that price competition under a logit or CES demand function is falsifiable.

In general, we can check the set of restrictions by evaluating a loss function similar to those for moment inequality estimations. In principle, therefore, it shares some computational issues with moment inequality estimations. However, we can characterize the set of restrictions as a set of linear constraints over parameters by focusing on the logit demand function and considering a slightly modified data requirement, which must always be satisfied when researchers estimate logit demand functions. Then, we can implement the test through standard algorithms for linear constraints.

The remaining part of the paper is organized as follows. I introduce the main model and its special cases in Section 2. I first exemplify a revealed preference test under a logit demand function and then formalize and generalize the result. In Section 3, I discuss some extensions of the test with (i) additional demand restrictions discussed in earlier research, (ii) observed cost shifters, and (iii) the possibility of collusive conduct among firms. I discuss the algorithms for the tests in Section 4, and summarize in Section 5.

[1] Carvajal et al. (2013) consider Cournot competition in the homogenous goods market, where demand shock is common to all firms. Carvajal et al. (2014) consider multi-market contact Cournot competition and differentiated price competition. With differentiated price competition, they introduce additional restrictions that capture the idea of a common demand shock.

1.2 The model

In this paper, I consider a standard competition framework for a differentiated market, where each firm produces different products. The products can be similar but not completely the same. The demand functions are assumed to change over time, potentially because of a change in consumer tastes or product characteristics. The characteristics might or might not be observed by the econometrician. I denote $\mathcal{J} = \{1, 2, ..., J\}$ as a set of products and $Q_{j,t} : R_+^J \to R_+$ as a demand function of product $j \in \mathcal{J}$ at time $t \in \{1, ..., T\}$. The demand function is assumed to be in a class of *discrete/continuous* model, which is explained later. Firm $f \in \{1, ..., F\}$ produces a set of products $\mathcal{J}_f \subset \mathcal{J}$ s.t. $\mathcal{J}_f \cap \mathcal{J}_g = \phi$ for $f \neq g$ and denote $J_f = |\mathcal{J}_f|$. The cost function of product $j \in \mathcal{J}$, $C_j : R_+ \to R$, is assumed to be convex and twice continuously differentiable. In this section, I focus on time-invariant cost functions, which serve the similar purpose as time-invariant preference in Afriat (1967).[2] Then, the profit function for firm f at time t is written as $\pi_{f,t}(p) = \sum_{j \in \mathcal{J}_f} \{Q_{j,t}(p) p_j - C_j(Q_{j,t}(p))\}$.

In the following, I mainly utilize the first-order conditions of profit functions and cost convexity to derive testable data restrictions that should be satisfied regardless of the parameter values and structural error terms. Using the profit function defined above, the first-order condition w.r.t. p_j is written as

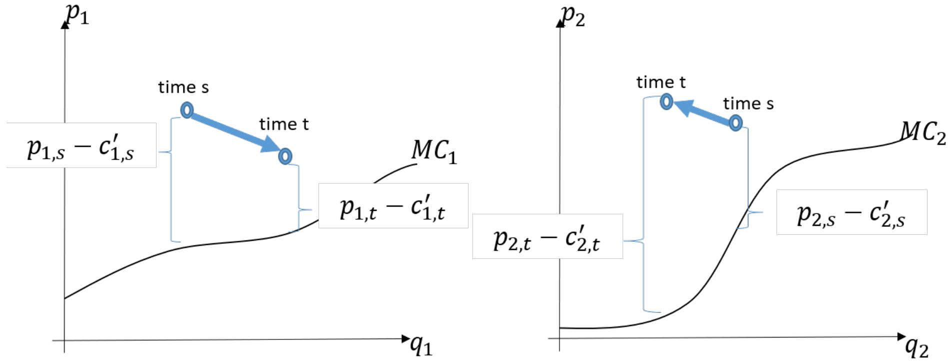
$$0 = Q_{j,t}(p) + \sum_{k \in J_f} \{p_k - C_k'(Q_{k,t}(p))\} \frac{\partial Q_{k,t}(p)}{\partial p_j}.$$

1.2.1 Example: Logit Demand Function

Before proceeding to the main result with a general specification, I demonstrate that some data cannot be explained by price competition with the logit demand function, which is a special case of the discrete/continuous model. By using a logit demand function, $Q_{j,t}(p) = M_t \frac{\exp(v_{jt} - \alpha p_{jt})}{1 + \sum_k \exp(v_{kt} - \alpha p_{kt})}$ for some M_t, $\alpha \in R_+$ and $(v_{j,t})_{j \in \mathcal{J}} \in R^J$, the first-order condition is

[2] In Section 3, I discuss an extension with time-variant and observed cost shifters.

Figure 1.1: Example: Logit Demand Function



rewritten as follows:[3]

$$0 = Q_{j,t}(p) - \left\{p_j - C_j'\left(Q_{j,t}(p)\right)\right\} \alpha Q_{j,t}(p) + \sum_{k \in J_f} \left\{p_k - C_k'\left(Q_{k,t}(p)\right)\right\} \frac{\alpha}{M_t} Q_{k,t}(p) Q_{j,t}(p)$$

By rearranging it, we obtain the following equation:

$$p_j - C_j'\left(Q_{j,t}(p)\right) = \frac{1}{\alpha} + \frac{1}{M_t} \sum_{k \in J_f} \left\{p_k - C_k'\left(Q_{k,t}(p)\right)\right\} Q_{k,t}(p).$$

In the above equation, the RHS applies in common to the goods produced by the same firm. Therefore,

$$p_j - C_j'\left(Q_{j,t}(p)\right) = p_k - C_k'\left(Q_{k,t}(p)\right) \text{ for any } j,\, k \in \mathcal{J}_f \tag{1.1}$$

must hold at the equilibrium. In this paper, I call this property the "common mark-up property." [4] The model is rejected by the common mark-up property together with the increasing marginal cost assumption, given the following data : $(\bar{p}_{j,\tau}, \bar{q}_{j,\tau})_{j=1,2,\, \tau=s,t}$ s.t. $\{1,2\} \subset \mathcal{J}_f$, $\bar{p}_{1,s} > \bar{p}_{1,t}$, $\bar{p}_{2,s} < \bar{p}_{2,t}$, $\bar{q}_{1,s} < \bar{q}_{1,t}$, and $\bar{q}_{2,s} > \bar{q}_{2,t}$. That is, the price and quantity of good 1 and those of good 2 moves in the opposite direction (see Fig. 1.1). Suppose that the data

[3] The following argument holds more generally with the time-variant α_t, instead of the time-invariant α. I use the time-invariant α simply because it is used more commonly in the literature.

[4] More generally, Nocke and Schutz (2018) call it the "common $\iota-$markup property" under the discrete/continuous choice model.

satisfy eq.(1.1) at time s. If the marginal costs are (weakly) increasing in own quantity, then $\bar{p}_{1,t} - C_1'(\bar{q}_{1,t}) < \bar{p}_{1,s} - C_1'(\bar{q}_{1,s}) = \bar{p}_{2,s} - C_2'(\bar{q}_{2,s}) < \bar{p}_{2,t} - C_2'(\bar{q}_{2,t})$. Therefore, eq.(1.1) cannot be satisfied at time t. Thus, these data, $(\bar{p}_{j,\tau}, \bar{q}_{j,\tau})_{j=1,2,\tau=s,t}$, cannot be explained by (a repetition of static) price competition under logit demand functions. This means these data cannot be explained by any set of parameters, α, m_t, and $v_{j,t}$ and non-parametric cost functions, $C_j(\cdot)$. This highlights a few important features of these restrictions on the data.

IVs not needed With the logit specification, one can understand an underlying mechanism of the revealed preference test better by comparing with an alternative procedure to check the competitive structure. A parameter of the demand function, α, is often estimated from aggregated data (with the use of IVs to deal with unobserved heterogeneity potentially correlated with prices), and δ's are backed out from the first-order condition. Potentially, we could check whether the obtained δ's are reasonable. In the revealed preference test, alternatively, a similar procedure is used for any possible $\alpha > 0$, instead of an estimated α pinned down by IVs. Therefore, we do not need IVs with unobserved heterogeneity that is potentially correlated with price or some other variables.

Interpretation of rejection Note that rejection or acceptance is not probabilistic even if the model has (only) the structural error term in the logit demand function. When we estimate logit demand functions from aggregate data, $v_{j,t}$ is decomposed as $v_{j,t} = x_{j,t}'\beta + \xi_{j,t}$ where $x_{j,t}$ is a vector of product j's observed characteristics, $\xi_{j,t}$ represents the unobserved characteristics, and β is a vector of parameters. In the logit demand estimation, $\xi_{j,t}$ is treated as a structural error term. However, eq.(1.1) should be satisfied regardless of what values of $\xi_{j,t}$ are realized as long as firms compete on price under logit demand functions (recall that I did not impose any assumptions on $v_{j,t}$). This is because each firm (but not to the econometrician) is assumed to know what $(\xi_{j,t})_{j\in\mathcal{J}}$ is realized, as is often assumed in the empirical IO literature. Therefore, any rejection of the model cannot be attributed to a peculiar realization of structural error terms.

(No) data restrictions by each assumption In this article, revealed preference tests are joint tests of demand and cost specifications and the competitive structure. However, it is worth noting that each of them by itself cannot be rejected by any data, $\{\bar{p}, \bar{q}\}$, but those can only be rejected together. Assuming only a logit demand function, for any data $(p_{j,t}, q_{j,t})_{j=1,2}$ at each t, we can back out the corresponding $(v_{j,t})_{j\in\mathcal{J}}$ by an inversion of the market share function as in Berry (1994). Thus, logit demand can fit the data since any changes in data over time can be captured by changes in $(v_{j,t})_{j\in\mathcal{J}}$ over time. As to the assumption of price competition and cost functions, any data can be rationalized by price competition under a more general demand function and convex time-invariant cost function, as explained in Section 2.2. This emphasizes that each assumption in this article is not trivially restrictive, especially when we have only price and quantity data.

In the following part, I provide a set of inequalities as a systematic method to detect data inconsistent with price competition, and show that such conditions are actually sufficient for rationalization by price competition. Instead of the logit demand function, I employ a class of demand functions by Nocke and Schutz (2018) that nests the logit demand function and CES demand function.

1.2.2 Discrete/Continuous Demand Function

In the following, I employ the discrete/continuous demand function introduced by Nocke and Schutz (2018), where the demand function for product j is written as

$$Q_j(p) = m\frac{-h_j'(p_j)}{h_0 + \sum_{k\in I} h_k(p_k)}$$

, where $h_j(\cdot)$ is decreasing and log-convex for every j, and m is a positive constant. An important example of this demand function is the logit model $h_j(p_j) = \exp(v_j - \alpha p_j)$ and $m = M/\alpha$, where $v_j \in R$ is the value of good j, $\alpha > 0$ is the coefficient for prices, $M > 0$

is the size of the market, and h_0 is the exponentiated value of the outside option.[5] Another important example is the CES model $h_j(p_j) = a_j p_j^{1-\sigma}$ and $m = \frac{I}{\sigma-1}$, where I is the income level of the consumer and σ is the elasticity of substitution.

In this paper, I utilize the fact that we can express the partial derivatives of the discrete/continuous demand function in a simple form:

$$\begin{aligned} \frac{\partial Q_{k,t}(p)}{\partial p_j} &= m \frac{-h'_{k,t}(p_k)}{\left(h_{0,t} + \sum_{k \in I} h_{k,t}(p_k)\right)^2} \left(-h'_{j,t}(p_j)\right). \\ &= m^{-1} Q_{k,t}(p) \cdot Q_{j,t}(p) \quad \forall k \neq j \end{aligned}$$

and

$$\begin{aligned} \frac{\partial Q_{j,t}(p)}{\partial p_j} &= m_t \frac{-h''_{j,t}(p_j)}{h_{0,t} + \sum_{k \in I} h_{k,t}(p_k)} + m_t \left(\frac{-h'_{j,t}(p_k)}{h_{0,t} + \sum_{k \in I} h_{k,t}(p_k)}\right)^2 \\ &= -Q_{j,t}(p) \frac{h''_{j,t}(p_k)}{-h'_{j,t}(p_k)} + m_t^{-1} \left(Q_{j,t}(p)\right)^2 \\ &= -m_t^{-1} Q_{j,t}(p) \left\{ m_t \frac{h''_{j,t}(p_k)}{-h'_{j,t}(p_k)} - Q_{j,t}(p) \right\}. \end{aligned}$$

It is worth noting that $m_t \frac{h''_{j,t}(p_k)}{-h'_{j,t}(p_k)} - Q_{j,t}(p)$ is positive because of the log concavity of $h_j(\cdot)$.[6]

[5] As discussed by Nocke and Schutz (2018), a discrete/continuous choice model with an outside option can be normalized to a discrete/continuous choice model without an outside option, $\tilde{Q}_j(p) = m \frac{-\tilde{h}'_j(p_j)}{\sum_{k \in I} \tilde{h}_k(p_k)}$, by letting $\tilde{h}_j(p_j) = \frac{1}{J} h_0 + h_j(p_j)$. In this paper, h_0 is explicitly denoted just to intuitively explain the results in the later parts.

[6] The log-concavity implies $\frac{h''_j(p)}{-h'_j(p)} > \frac{-h'_j(p)}{h_j(p)} \left(> \frac{-h'_j(p)}{h_0 + \sum h_k(p)} \right)$.

With the above expression, the FOC w.r.t. p_j is written as follows:

$$\begin{aligned} 0 &= 1+\sum_{k\in J_f}\left\{p_k - C_k'\left(Q_{k,t}\left(p\right)\right)\right\}\frac{\partial Q_{k,t}\left(p\right)}{\partial p_j}\frac{1}{Q_{j,t}\left(p\right)} \\ &= 1-m_t^{-1}\left\{p_j - C_j'\left(Q_{j,t}\left(p\right)\right)\right\}\left\{m_t\frac{h_{j,t}''\left(p_k\right)}{-h_{j,t}'\left(p_k\right)} - Q_{j,t}\left(p\right)\right\} \\ &\quad +m_t^{-1}\sum_{k\in J_f, k\neq j}\left\{p_k - C_k'\left(Q_{k,t}\left(p\right)\right)\right\}Q_{k,t}\left(p\right) \\ &= m_t - \left\{p_j - C_j'\left(Q_{j,t}\left(p\right)\right)\right\}m_t\frac{h_{j,t}''\left(p_k\right)}{-h_{j,t}'\left(p_k\right)} + \sum_{k\in J_f}\left\{p_k - C_k'\left(Q_{k,t}\left(p\right)\right)\right\}Q_{k,t}\left(p\right)\end{aligned}$$

Therefore, if the data $\{\bar{p}, \bar{q}\}$ are generated by price competition with (unknown) discrete/continuous demand function, there exists α_{jt}, δ_{jt}, which corresponds to $\frac{h_{j,t}''(\bar{p}_j)}{-h_{j,t}'(\bar{p}_j)}$ and $C_j'\left(\bar{q}_{j,t}\right)$, respectively, such that

$$0 = m_t - \left\{\bar{p}_j - \delta_{j,t}\right\}m_t\alpha_{j,t} + \sum_{k\in J_f}\left\{\bar{p}_k - \delta_{j,t}\right\}\bar{q}_{k,t}. \tag{1.2}$$

On the other hand, since $\delta_{j,t}$ corresponds to $C_j'\left(\bar{q}_{j,t}\right)$ and $C_j'\left(\cdot\right)$ is assumed to be increasing, $\delta_{j,t}$ must be greater than $\delta_{j,s}$ $(s\neq t)$ if $\bar{q}_{j,t}$ is greater than $\bar{q}_{j,s}$. This is summarized as an inequality:

$$0 \leq \left(\delta_{j,s} - \delta_{j,t}\right)\left(\bar{q}_{j,s} - \bar{q}_{j,t}\right). \tag{1.3}$$

Combining eq.(1.2) and eq.(1.3), we obtain a set of necessary conditions for the data to be rationalized by the model. Furthermore, the conditions also turn out to be sufficient for rationalization. They are summarized in the following theorem.

Theorem 1. *(Discrete/Continuous): The set of observations $\{\bar{p}, \bar{q}\}$ is Bertrand-rationalizable under a convex cost function and a discrete/continuous demand function if and only if there exist real numbers $\alpha_{j,t}$, $\delta_{j,t}$, and m_t for any $t\in\mathcal{T}$ and $j\in\mathcal{J}$ such that the following hold:*

1. $\alpha_{j,t} > 0$, $\delta_{j,t} > 0$, $m_t > 0$;

2. $0 = m_t - \left\{\bar{p}_{j,t} - \delta_{j,t}\right\}m_t\alpha_{j,t} + \sum_{k\in J_f}\left\{\bar{p}_{k,t} - \delta_{k,t}\right\}\bar{q}_{k,t}$; *and*

3. $0 \leq (\delta_{j,t'} - \delta_{j,t})(\bar{q}_{j,t'} - \bar{q}_{j,t})$.

The first set of conditions comes from the underlying specifications of the demand and cost functions: $\alpha_{j,t} > 0$ from the assumption that h_j is decreasing and log-convex, $\delta_{j,t} > 0$ from the increasing cost functions, and $m_t > 0$ from the assumption that the quantity of each good is non-negative. The proof of sufficiency consists of two steps. First, given $\{\alpha_{j,t}\}$ and $\{\delta_{j.t}\}$ which satisfy the conditions, I construct demand functions $\left\{\bar{Q}_{j,t}(\cdot)\right\}$ and cost functions $\left\{\bar{C}_j(\cdot)\right\}$ to satisfy $\frac{\bar{h}''_{j,t}(\bar{p}_j)}{-\bar{h}'_{j,t}(\bar{p}_j)} = \alpha_{j,t}$ and $\bar{C}'_j(\bar{q}_{j,t}) = \delta_{j.t}$. [7]Then, the data $\{\bar{p}, \bar{q}\}$ satisfy the first-order conditions under the reconstructed demand and cost functions. In the second step, I show that the first-order conditions are a sufficient condition for profit maximization given the other firms' prices and the reconstructed demand cost functions. This result is not trivial since the profit function does not satisfy quasi-concavity. In this paper, sufficiency is proved by the unique solution of the first-order conditions. The uniqueness comes from the unique "common ι-markup" and a mapping from ι-markup to price vectors as in Nocke and Schutz (2018). See the appendix for the full proof.[8]

1.2.3 Tests for more restrictive models

For more restrictive specifications, such as logit or CES demand functions, we can easily derive the necessary condition for data to be rationalized by the models, by simply adding restrictions to the second condition in the above tests. Sufficiency of the restriction, however, is less trivial. In the proof of sufficiency in Theorem 1, I reconstruct the demand function as $\left\{\bar{Q}_{j,t}(\cdot)\right\}$, which still nests the logit demand function, but not CES. Such a reconstruction is sufficient for Theorem 1 since the re-constructed demand function $\left\{\bar{Q}_{j,t}(\cdot)\right\}$ is in the class of demand functions we are interested in. To test a model with the logit demand, we can apply

[7]In the proof of sufficiency, the constructed demand functions $\left\{\bar{Q}_{j,t}(\cdot)\right\}$ can be different from the actual demand functions $\{Q_{j,t}(\cdot)\}$. For instance, in the proof of Theorem 1, even if the data $\{\bar{p}, \bar{q}\}$ are actually generated from CES demand, which is a special case of the discrete/continuous choice model, the reconstructed $\left\{\bar{Q}_{j,t}(\cdot)\right\}$ can be a non-CES demand as long as it is another special case of the discrete/continuous choice model. This point is further discussed in the next subsection.

[8]Notably, the second condition is generally not linear because of the interaction of $\delta_{j,t}$ and $\alpha_{j,t}$, which contrasts with the finding of Carvajal et al. (2013, 2014). Thus, we cannot use algorithms for linear programming. One way to implement the above test is to consider an algorithm similar to moment inequalities.

a similar reconstruction of $\left\{\bar{Q}_{j,\,t}\left(\cdot\right)\right\}$ and the remaining is proved analogously. On the other hand, such a re-construction is no longer valid for a model with CES demand. As of today, I have not found a reconstructed CES demand function that provides the suffiiciency of FOC under arbitrary convex cost functions[9]. Assuming a constant marginal cost in addition to CES demand, we can regain the sufficiency by applying the proof of Nocke and Schutz (2018) for a unique solution of FOC.

Those conditions mentioned above are articulated in the following propositions (the specifications and results are summarized in Table 1.1 in the appendix).

Proposition 1. *(Logit) The set of observations $\{\bar{p},\, \bar{q}\}$ is Bertrand-rationalizable under convex cost functions and logit demand functions* ***if and only if*** *there exist real numbers α_t, $\delta_{j,t}$, and m_t for all $t \in \mathcal{T}$ and $j \in \mathcal{J}$, such that the following hold:*

1. $\alpha > 0,\ \delta_{j,t} > 0,\ m_t > 0$;
2. $0 = m_t - \{\bar{p}_{j,t} - \delta_{j,t}\}\, m_t\alpha_t + \sum_{k\in J_f}\{\bar{p}_{k,t} - \delta_{j,t}\}\,\bar{q}_{k,t}$; *and*
3. $0 \le (\delta_{j,t'} - \delta_{j,t})\,(\bar{q}_{j,t'} - \bar{q}_{j,t})$.

In the above statement, α_t is allowed to vary over time for the sake of generality. We can easily replace α_t the with time-invariant α. Such a simplified version is proved analogously to Proposition 1.

Corollary 1. *(CES): If the set of observations $\{\bar{p},\, \bar{q}\}$ is Bertrand-rationalizable under convex cost functions and CES demand functions, then there exist real numbers σ_t, $\delta_{j,t}$, and m_t for all $t \in \mathcal{T}$ and $j \in \mathcal{J}$, such that the following hold:*

1. $\sigma_t > 1,\ \delta_{j,t} > 0,\ m_t > 0$;
2. $0 = m_t - \frac{\bar{p}_{j,t} - \delta_{j,t}}{\bar{p}_{j,t}} m_t\sigma_t + \sum_{k\in J_f}\{\bar{p}_{k,t} - \delta_{j,t}\}\,\bar{q}_{k,t}$; *and*
3. $0 \le (\delta_{j,t'} - \delta_{j,t})\,(\bar{q}_{j,t'} - \bar{q}_{j,t})$.

[9] The corresponding matrix (similar to the negative semi-definite matrix in the proof of Theorem 1) would no longer be symmetric in a CES case. Therefore, the discussion on negative semi-definiteness does not apply.

As explained earlier, this necessary condition is derived as a special case of (the necessity part of) Theorem 1. Notably, the necessary condition alone works well to reject the model, but it would be harder to interpret when the model is not rejected; i.e., the data may or may not be rationalized by the model. Assuming constant (and time-invariant) marginal costs, we can easily prove the sufficiency since we can apply the proof in Nocke and Schutz (2018) for a unique solution of FOC.

Proposition 2. *(CES and constant MC): The set of observations* $\{\bar{p}, \bar{q}\}$ *is Bertrand rationalizable under linear cost functions and CES demand functions if and only if there exist real numbers* σ_t*,* δ_j*, and* m_t *for all* $t \in \mathcal{T}$ *and* $j \in \mathcal{J}$*, such that the following hold:*

1. $\sigma_t > 1$, $\delta_j > 0$, $m_t > 0$;
2. $0 = m_t - \frac{\bar{p}_{j,t} - \delta_j}{\bar{p}_{j,t}} m_t \sigma_t + \sum_{k \in J_f} \{\bar{p}_{k,t} - \delta_j\} \bar{q}_{k,t}$

Note that δ_j does not have a time index since constant and time-invariant marginal costs are imposed, instead of increasing marginal costs.

1.2.4 Falsifiability

Theorem 1 characterizes the necessary and sufficient condition for the data to be rationalized by the model of price competition under discrete/continuous demand functions and time-invariant convex cost functions. Meanwhile, readers might wonder how restrictive the conditions are. It turns out that the restriction in Theorem 1 is so loose that any data can be rationalized by the model with the general discrete/continuous demand functions. This can be surprising considering that even the general discrete/continuous choice model satisfies the IIA property. Any changes in price and quantity along a fixed discrete/continuous demand function must satisfy the IIA property, while demand functions themselves are allowed to change over time with the model in Theorem 1. To clearly understand how any data satisfy the restrictions, consider the following. For any given $\delta_{j,t}$ and m_t, the remaining parameter $\alpha_{j,t}$, which characterizes the demand functions, can be determined only through the first-order condition w.r.t. $p_{j,t}$, independently of the first-order condition w.r.t. $p_{k,s}$, where $k \neq j$ or $s \neq t$ (in contrast, under the logit demand, α_t is common to all goods so that data restrictions on different products are intertwined each other). Thus, for any data, we can find corresponding $\alpha_{j,t}$, $\delta_{j,t}$, m_t, and then, the sufficiency implies that any data are rationalized by the model.

Corollary 2. *Any data, $\{\bar{p}, \bar{q}\}$, are Bertrand-rationalizable under convex cost functions and discrete/continuous demand functions.*

Even though price competition under the general discrete/continuous choice model is not falsifiable, a model can be falsifiable under a more restrictive demand model such as the logit demand function as shown in the example. This naturally raises a question: How general is this falsifiability? In the following, I show that a subclass of discrete/continuous demand functions that nests both the logit and CES demand is falsifiable. Consider a discrete/continuous demand function generated by $h_{j,t}(\cdot)$ such that $\frac{h''_{j,t}(p_j)}{-h'_{j,t}(p_j)} = \frac{1}{a_t p_{j,t} + b_t}$ for some $a_t \geq 0$ and $b_t \geq 0$. Now, we can express the logit and CES demand function by setting $a_t = 0$ and setting $b_t = 0$, respectively. I call such a demand function as a discrete/continuous demand function with HARA h since h is characterized as analogous to a hyperbolic absolute risk averse vNM utility function, which nests CARA and CRRA as special cases. First, I introduce a modified version of the necessary condition for data to be rationalized by price competition under the modified specification. Similar to Corollary 1, the statement is only about the necessity of the data restriction.

Corollary 3. *If the set of observations $\{\bar{p}, \bar{q}\}$ is Bertrand-rationalizable under convex cost functions and discrete/continuous demand functions with HARA h, then there exist real numbers a_t, b_t, $\delta_{j,t}$, and m_t for all $t \in \mathcal{T}$ and $j \in \mathcal{J}$ such that the following hold:*

1. $a_t > 0$, $b_t > 0$ $\delta_{j,t} > 0$, $m_t > 0$;
2. $0 = m_t - \{\bar{p}_{j,t} - \delta_{j,t}\} m_t \frac{1}{a_t p_{j,t} + b_t} + \sum_{k \in J_f} \{\bar{p}_{k,t} - \delta_{j,t}\} \bar{q}_{k,t}$; *and*
3. $0 \leq (\delta_{j,t'} - \delta_{j,t})(\bar{q}_{j,t'} - \bar{q}_{j,t})$.

Note that the modified model is falsifiable, i.e., the model can be rejected under some data. By the second condition in the set of restriction, the data must satisfy $\frac{\bar{p}_{j,t} - \delta_{j,t}}{a_t \bar{p}_{j,t} + b_t} =$

$\frac{\bar{p}_{k,t}-\delta_{k,t}}{a_t\bar{p}_{k,t}+b_t}$ for all j, $k \in \mathcal{J}_f$ and all $t \in \mathcal{T}$. Consider a case where one firm produces products 1, 2, and 3 and generates the data $\{\bar{p}, \bar{q}\}$ such that $\bar{p}_{j,t} = \bar{p}_{j,s} \equiv \bar{p}_j$ and $\bar{q}_{j,t} = \bar{q}_{j,s} \equiv \bar{q}_j$ for $j = 1, 2$ and for some $t, s \in \mathcal{T}$, and $\bar{p}_{3,t} < \bar{p}_{3,s}$ and $\bar{q}_{3,t} > \bar{q}_{3,s}$.

Then, the above equality is rewritten as follows:

$$\frac{\bar{p}_1 - \delta_1}{\bar{p}_1 + b_t/a_t} = \frac{\bar{p}_2 - \delta_2}{\bar{p}_2 + b_t/a_t} = \frac{\bar{p}_{3,t} - \delta_{3,t}}{\bar{p}_{3,t} + b_t/a_t} \tag{1.4}$$

and

$$\frac{\bar{p}_1 - \delta_1}{\bar{p}_1 + b_s/a_s} = \frac{\bar{p}_2 - \delta_2}{\bar{p}_2 + b_s/a_s} = \frac{\bar{p}_{3,s} - \delta_{3,s}}{\bar{p}_{3,s} + b_s/a_s}. \tag{1.5}$$

Note that the equalities for goods 1 and 2 in eqs. (1.4) and (1.5) implies that $b_t/a_t = b_s/a_s \equiv b/a$. Therefore, all the terms in eqs. (1.4) and (1.5) must be the same. Thus, for good 3, $\frac{\bar{p}_{3,t}-\delta_{3,t}}{\bar{p}_{3,t}+b/a} = \frac{\bar{p}_{3,s}-\delta_{3,s}}{\bar{p}_{3,s}+b/a}$ must hold. This contradicts $\bar{p}_{3,t} < \bar{p}_{3,s}$ and $\bar{q}_{3,t} > \bar{q}_{3,s}$.

1.3 Extensions

This section introduces the following extensions of the revealed preference tests: (i) additional assumptions on demand function introduced by Carvajal et al. (2014), (ii) observable cost shifters as discussed in Carvajal et al. (2014), and (iii) collusive price competition, which can also work as an alternative hypothesis for the above tests.

1.3.1 Additional restrictions on demand

Even though the above results provide some testable restrictions, the general model is not falsifiable. We can obtain a stricter constraint by combining the demand assumption introduced by Carvajal et al. (2014). It is straightforward that the additional assumption in the model provides additional constraints in data. However, it is less obvious that the discrete/continuous demand function and the additional assumptions have a non-empty intersection.

In order to define the additional restrictions, I introduce some notations first. I denote

$\epsilon_{jt}(p) : R_+^J \to R$ as the relative decrease in the demand of good j at time t in response to an infinitesimal increase in its price. That is, given the demand function Q_{jt} for good j at time t ,

$$\epsilon_{jt}(p) = -\frac{\partial Q_{j,t}(p_j, p_{-j})}{\partial p_j} \frac{1}{Q_{jt}(p)}$$

Therefore, the own price elasticity is expressed as $p_j \epsilon_{jt}(p)$.

We then define the following properties of the demand functions.

Definition: A system of demand functions satisfies the *co-evolving* property if, for any s and $t \in \mathcal{T}$, either

$$\epsilon_{js}(p) \geq \epsilon_{jt}(p) \text{ for all } p \in R_+^J \text{ and all } j \in \mathcal{J}, \text{ or} \quad (1.6)$$

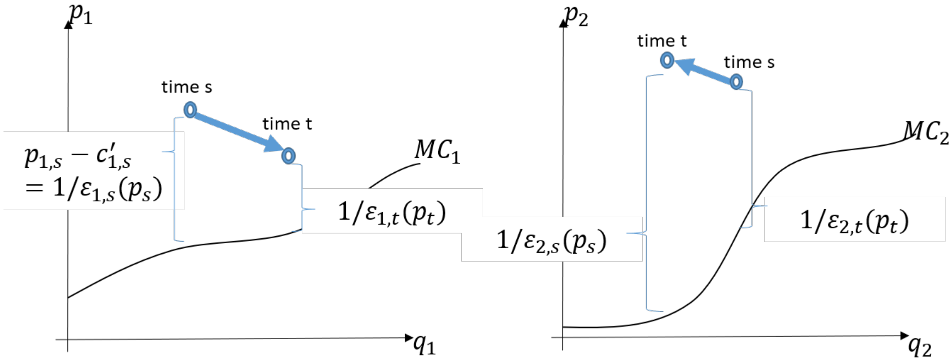
$$\epsilon_{js}(p) \leq \epsilon_{jt}(p) \text{ for all } p \in R_+^J \text{ and all } j \in \mathcal{J} \quad (1.7)$$

The co-evolving demand property captures the idea of *common demand shock* in Carvajal et al. (2013), which is a key component to obtain non-trivial data restrictions in their work. As seen in the above equations, if the relative slope of demand is higher for firm j in a market t, then so are the relative slopes for other firms $k \neq j$. In other words, we can construct a well-defined order of demands over $\mathcal{T}$, which is common to all firms according to the relative slopes.

The power of the co-evolving property is emphasized by two products produced by different firms. Consider the same prices and quantities as the previous example, but the two goods are produced by different firms: $(p_{j,\tau}, q_{j,\tau})_{j=1,2,\tau=s,t}$ s.t. $\mathcal{J}_1 = \{1\}$, $\mathcal{J}_2 = \{2\}$, $p_{1,s} > p_{1,t}$, $p_{2,s} < p_{2,t}$, $q_{1,s} < q_{1,t}$, and $q_{2,s} > q_{2,t}$ (see Fig. 1.2).[10] Since the two goods are produced by different firms, eq.(1.1) is no longer satisfied. However, the co-evolving property gives us an alternative restriction even under the general discrete/continuous model (instead of the logit demand function). If they are single-product firms, the first-order condition is re-written as follows: $p_j - C_j'(Q_{j,t}(p)) = 1/\epsilon_{j,t}(\bar{p}_{j,t})$. Since the marginal costs are increasing, we can obtain an inequality of the profit margins: $1/\epsilon_{1,s}(\bar{p}_s) = \bar{p}_{1,s} - C_1'(Q_{1,s}(\bar{p}_s)) >$

[10] The same logic is applied to multi-product firms just by ignoring other products.

Figure 1.2: Example: Rejection by Co-evolving Property



$\bar{p}_{1,t} - C_1'(Q_{1,t}(\bar{p}_t)) = 1/\epsilon_{1,t}(\bar{p}_t)$ for firm 1. Similarly, we also have $1/\epsilon_{2,s}(\bar{p}_s) < 1/\epsilon_{2,t}(\bar{p}_t)$. Therefore, the data imply $\epsilon_{1,s}(\bar{p}_s) < \epsilon_{1,t}(\bar{p}_t)$ and $\epsilon_{2,s}(\bar{p}_s) > \epsilon_{2,t}(\bar{p}_t)$. Assuming that $\epsilon_{j,s}(\cdot)$ is non-decreasing in the own price and decreasing in the other's price, we have $\epsilon_{1,s}(p) < \epsilon_{1,t}(p)$ but $\epsilon_{1,s}(p) > \epsilon_{1,t}(p)$, which contradicts the co-evolving property. In the following, I describe $\epsilon_{j,t}(\cdot)$ that is non-decreasing in the own price as *log-concave,* following Carvajal et al. (2014).[11]

Before stating the proposition, I demonstrate that the discrete/continuous model and co-evolving property have a non-empty intersection. In the multinomial logit demand, the co-evolving property is satisfied when v_{jt} and v_{kt} move almost in parallel over time. Since the logit demand function (with common M_t over t) requires that $\epsilon_{jt}(p) = \alpha - \frac{\alpha}{M} Q_{j,t}(p)$, $\epsilon_{jt}(p) \geq \epsilon_{js}(p)$ holds if and only if $Q_{j,t}(p) \geq Q_{j,s}(p)$ holds. The co-evolving property under the logit demand function requires that $Q_{j,t}(p) \geq Q_{j,s}(p)$ if and only if $Q_{k,t}(p) \geq Q_{k,s}(p)$. It can be satisfied when the change in the relative value of the outside option dominates the change in demand functions. Thus, the two properties have a non-empty intersection. Log-concavity (of $Q_{j,t}(p)$) is also satisfied if $\frac{h_j''(p_j)}{-h_j'(p_j)}$ is non-decreasing in p_j. In the following proposition, I combine the discrete/continuous choice model and the co-evolving property to derive a set of necessary conditions for the data to be rationalized by price competition.

Proposition 3. *The set of observations $\{\bar{p}, \bar{q}\}$ is Bertrand rationalizable under convex cost functions and discrete/continuous demand functions with log-concavity and co-evolving prop-*

[11] Carvajal et al. (2014) impose another condition, *substitutes condition*, that $\epsilon_{j,t}(\cdot)$ is decreasing in others' prices. Under discrete/continuous demand, however, this condition is always satisfied.

erty only if there is a permutation of $\mathcal{T}$, denoted by the function $\sigma : \mathcal{T} \to \mathcal{T}$, and real numbers $\alpha_{j,t}$, $\delta_{j,t}$, and m_t for all s, $t \in \mathcal{T}$ and $j \in \mathcal{J}$ such that the following hold:

1. $\alpha_{j,t} > 0$, $\delta_{j,t} > 0$, $m_t > 0$;
2. $0 = m_t - \{\bar{p}_j - \delta_{j,t}\} m_t \alpha_{j,t} + \sum_{k \in J_f} \{\bar{p}_k - \delta_{j,t}\} \bar{q}_{k,t}$;
3. $0 \leq (\delta_{j,t'} - \delta_{j,t}) (\bar{q}_{j,t'} - \bar{q}_{j,t})$; *and*
4. *if* $\bar{p}_{j,t} \geq \bar{p}_{j,s}$, $\bar{p}_{-j,t} \leq \bar{p}_{-j,s}$ *and* $\sigma(t) < \sigma(s)$, *then* $\alpha_{j,t} - m_t^{-1} \bar{q}_{j,t} \leq \alpha_{j,s} - m_s^{-1} \bar{q}_{j,s}$

See the appendix for the proof.

The last condition arises from the co-evolving property and log-concavity, which characterize the common order of $\epsilon_{jt}(p)$ over time. Under the discrete/continuous demand model, $\epsilon_{jt}(\bar{p}_t) = \frac{h''_{j,t}(\bar{p}_j, t)}{-h'_{j,t}(\bar{p}_j, t)} - m_t^{-1} Q_{j,t}(\bar{p}_t) = \alpha_{j,t} - m_t^{-1} \bar{q}_{j,t}$. The permutation, σ, is constructed to provide a common order for $\epsilon_{jt}(p)$ (if one exists). Notably, the co-evolving property is defined by comparing $\epsilon_{jt}(p)$ and $\epsilon_{js}(p)$ for all p, but we only observe values corresponding to $\epsilon_{jt}(\bar{p}_t)$ and $\epsilon_{js}(\bar{p}_s)$, where $\bar{p}_t$ and $\bar{p}_s$ can take different values. To deal with this subtlety, we add the inequalities "$\bar{p}_{j,t} \geq \bar{p}_{j,s}, \bar{p}_{-j,t} \leq \bar{p}_{-j,s}$" to the last condition. For this proposition, I prove only the necessity of the conditions. For the proof of sufficiency, I need to reconstruct the demand functions satisfying both a discrete/continuous structure and the co-evolving property from any parameters satisfying conditions 1-4.

1.3.2 Observed cost shock

One of the important assumptions in the above tests is the time-invariant cost function. In reality, however, the cost functions shifts over time because of change in input prices, for instance. We can accommodate such a shift to the revealed preference tests if cost shifters are observed.

Now, assume the following cost functions: $C_j(q_j, w_j)$ where $\frac{\partial C_j(q_j, w_j)}{\partial q_j}$ is increasing both in q_j and w_j. Assume also that we observe the cost shifter w in addition to price and quantity. Denote the observed price, quantity, and cost shifter as follows: $\{\bar{p}, \bar{q}, \bar{w}\}$ where

$\bar{x} = (\bar{x}_1', ..., \bar{x}_T')'$ and $\bar{x}_t = (\bar{x}_{1,t}, ..., \bar{x}_{J,t})'$ for $x = p, q, w$. Then, the restriction is modified as follows.

Remark 1.1. The set of observations $\{\bar{p}, \bar{q}, \bar{w}\}$ is Bertrand-rationalizable under marginal cost functions increasing in own quantity and a cost shifter and discrete/continuous demand functions only if there exist real numbers $\alpha_{j,t}$, $\delta_{j,t}$, and m_t for any $t \in \mathcal{T}$ and $j \in \mathcal{J}$ such that the following hold:

1. $\alpha_{j,t} > 0$, $\delta_{j,t} > 0$, $m_t > 0$;
2. $0 = m_t - \{\bar{p}_{j,t} - \delta_{j,t}\} m_t \alpha_{j,t} + \sum_{k \in J_f} \{\bar{p}_{k,t} - \delta_{k,t}\} \bar{q}_{k,t}$; and
3. $\delta_{j,t'} \geq \delta_{j,t}$ whenever $(\bar{q}_{j,t'}, \bar{w}_{j,t'}) > (\bar{q}_{j,t}, \bar{w}_{j,t})$.

In the above claim, I use a partial order for the third condition since, if $\bar{q}_{j,t'} > \bar{q}_{j,t}$ and $\bar{w}_{j,t'} < \bar{w}_{j,t}$, then we cannot tell when the marginal cost is higher. Tests for price competition under logit, CES, and HARA h, can be also derived analogously.

1.3.3 Collusive price competition

This section discusses a revealed preference test of collusive price competition. Each firm is assumed to choose their own price while (partially) internalizing the effect on the other firms as in Miller and Weinberg (2017) and Sullivan (2016). More specifically, firm f maximizes an objective function

$$\pi_f(p) = \sum_{f' \in \mathcal{F}} \left[\phi_{f,f'} \sum_{k \in J_{f'}} \{p_k - C_k'(Q_{k,t}(p))\} Q_{k,t}(p) \right]$$

given the others' prices at any time t, where $\phi_{f,f'} \in [0, 1]$ is firm f's weight on firm f''s profit. FOC w.r.t. p_j is written as follows:

$$0 = Q_{j,t}(p) + \sum_{f' \in \mathcal{F}} \left[\phi_{f,f'} \sum_{k \in J_{f'}} \{p_k - C_k'(Q_{k,t}(p))\} \frac{\partial Q_{k,t}(p)}{\partial p_j} \right].$$

This first-order condition is simplified by using the derivatives of discrete/continuous demand functions and dividing both sides by the market share of product j. Then, the first-order condition gives the following data restriction:

$$0 = m_t - \{\bar{p}_{j,t} - \delta_{j,t}\} m_t \alpha_{j,t} + \sum_{f' \in \mathcal{F}} \phi_{f,f'} \sum_{k \in J_f} \{\bar{p}_{k,t} - \delta_{k,t}\} \bar{q}_{k,t}.$$

Then, we state a modified version of the revealed preference test as follows:

Remark 1.2. The set of observations $\{\bar{p}, \bar{q}\}$ rationalizes a collusive price competition under convex cost functions and logit demand functions *only if* there exist real numbers α_t, $\delta_{j,t}$, and m_t for all $t \in \mathcal{T}$ and $j \in \mathcal{J}$ such that the following hold:

1. $\alpha_t > 0$, $\delta_{j,t} > 0$, $m_t > 0$;
2. $0 = m_t - \{\bar{p}_{j,t} - \delta_{j,t}\} m_t \alpha_t + \sum_{f' \in \mathcal{F}} \phi_{f,f'} \sum_{k \in J_f} \{\bar{p}_{k,t} - \delta_{k,t}\} \bar{q}_{k,t}$; and
3. $0 \leq (\delta_{j,t'} - \delta_{j,t}) (\bar{q}_{j,t'} - \bar{q}_{j,t})$.

In the above test, the demand is assumed to be a logit demand funcion since any data rationalize the model with the general discrete/continuous model, and the additional parameter $\phi_{f,f'}$ would further loosen the restriction. Under a logit demand function, on the other hand, the common markup property still holds, so the data shown in Subsection 2.1 do not rationalize the collusive price competition.

We can also a perfect collusion model as a special case of the above specification, where $\phi_{f,f'} = 1$ for all f and f'. Furthermore, it turns out to be the mathematically same model as the baseline model but with a different definition for a firm, therefore, we obtain the same result as in Section 2 with a different definition for the firm. To be more specific, I write an immediate corollary of Theorem 1 (characterization of the test) and Corollary 1 (unfalsifiability) for perfect collusion under a general discrete/continuous demand model.

Corollary 4. *The set of observations $\{\bar{p}, \bar{q}\}$ is collusion-rationalizable under convex cost functions and discrete/continuous demand functions if and only if there exist real numbers $\alpha_{j,t}$, $\delta_{j,t}$, and m_t for any $t \in \mathcal{T}$ and $j \in \mathcal{J}$ such that the following hold:*

1. $\alpha_{j,t} > 0$, $\delta_{j,t} > 0$, $m_t > 0$;

2. $0 = m_t - \{\bar{p}_{j,t} - \delta_{j,t}\} m_t \alpha_{j,t} + \sum_{f' \in \mathcal{F}} \sum_{k \in J_f} \{\bar{p}_{k,t} - \delta_{k,t}\} \bar{q}_{k,t}$; *and*

3. $0 \leq (\delta_{j,t'} - \delta_{j,t})(\bar{q}_{j,t'} - \bar{q}_{j,t})$.

Furthermore, any data, $\{\bar{p}, \bar{q}\}$, is collusion-rationalizable under convex cost functions and discrete/continuous demand functions.

By restricting a class of demand functions to the logit, we can also obtain a falsifiable model, which is summarized as an immediate corollary of Proposition 1.

Corollary 5. *(Logit) The set of observations $\{\bar{p}, \bar{q}\}$ is collusion-rationalizable under convex cost functions and logit demand functions if and only if there exist real numbers α_t, $\delta_{j,t}$, and m_t for all $t \in \mathcal{T}$ and $j \in \mathcal{J}$ such that the following hold:*

1. $\alpha > 0$, $\delta_{j,t} > 0$, $m_t > 0$;

2. $0 = m_t - \{\bar{p}_{j,t} - \delta_{j,t}\} m_t \alpha_t + \sum_{k \in J_f} \sum_{k \in J_f} \{\bar{p}_{k,t} - \delta_{j,t}\} \bar{q}_{k,t}$; *and*

3. $0 \leq (\delta_{j,t'} - \delta_{j,t})(\bar{q}_{j,t'} - \bar{q}_{j,t})$.

1.4 Implementation

The existence of parameters satisfying the inequality constraints can be checked by minimizing a loss function over a set of parameters, given the data observed, and checking whether the minimized value is close to zero. For instance, for an inequality $g(\theta; \bar{p}, \bar{q}) \geq 0$, we can construct a loss function $(\min\{0, g(\theta; \bar{p}, \bar{q})\})^2$. Similarly, for a vector of equality constraints $\boldsymbol{h}(\theta; \bar{p}, \bar{q}) = \mathbf{0}$ and a vector of inequality constraints $\boldsymbol{g}(\theta; \bar{p}, \bar{q}) \geq \mathbf{0}$, we can construct a loss function

$$\boldsymbol{h}(\theta; \bar{p}, \bar{q})^T \boldsymbol{h}(\theta; \bar{p}, \bar{q}) + \tilde{\boldsymbol{g}}(\theta; \bar{p}, \bar{q})^T \tilde{\boldsymbol{g}}(\theta; \bar{p}, \bar{q}),$$

where $\tilde{\boldsymbol{g}}(\theta; \bar{p}, \bar{q}) = [\min\{0, g_i(\theta; \bar{p}, \bar{q})\}]_i$. In general, this minimization faces computational issues similar to those of estimation with moment inequalities.[12] With logit demand and a slightly modified data requirement, the inequalities are written as linear constraints on parameters so that we can check the existence of parameters using off-the-shelf tools for linear constraints. In the following, I assume that the market size $\{\bar{M}_t\}_t$, prices $\{\bar{p}_{j,t}\}$, and quantities $\{\bar{q}_{j,t}\}$ are observable, as is always the case when the logit demand function can be estimated. The market shares of products at each time $\{\bar{s}_{j,t}\}$ are also observable since $\bar{s}_{j,t} = \frac{\bar{q}_{j,t}}{\bar{M}_t}$. Then, considering that $m = \frac{M}{\alpha}$ under logit demand, and with the replacement of $\frac{1}{\alpha} = \tilde{\alpha}$, the data restrictions for price competition under the logit demand function are characterized by a set of linear constraints on parameters $\tilde{\alpha}_{j,t}$, and $\delta_{j,t}$.

Corollary 6. *(Logit) The set of observations $\{\bar{p}, \bar{q}, \bar{M}\}$ is Bertrand-rationalizable under convex cost functions and logit demand functions **if and only if** there exists real numbers $\tilde{\alpha}_t$, $\delta_{j,t}$, and m_t for all $t \in \mathcal{T}$ and $j \in \mathcal{J}$ such that the following hold:*

1. $\tilde{\alpha}_t > 0$, $\delta_{j,t} > 0$;
2. $0 = \bar{M}_t\tilde{\alpha}_t - \{\bar{p}_{j,t} - \delta_{j,t}\} + \sum_{k \in J_f} \{\bar{p}_{k,t} - \delta_{j,t}\} \bar{q}_{k,t}$; *and*
3. $0 \leq (\delta_{j,t'} - \delta_{j,t})(\bar{q}_{j,t'} - \bar{q}_{j,t})$.

Proof. The proof is an immediate corollary of Proposition 1. □

Thus, we can use standard algorithms for linear constraints to check the constraint.

1.5 Summary

In this paper, I modify a Bertrand assumption test introduced by Carvajal et al. (2014) to allow it to be implemented for multi-product firms. To deal with difficulties caused by cannibalization effects, I employ the discrete/continuous demand function introduced

[12]The loss function tends to have a basin at the bottom with kinks around it. Therefore, the standard optimization algorithms do not work well.

by Nocke and Schutz (2018), which includes the multinomial logit demand function and the CES demand function as special cases. In the main theorem, I provide the necessary and sufficient condition for data to be rationalized by Bertrand competition among multi-product firms under the discrete/continuous model. The test is implementable without any IVs, and rejection by it deterministically implies misspecification of the model rather than a peculiar realization of structural error terms. Under the general discrete/continuous model, any data would satisfy the necessary and sufficient condition to be rationalized by the price competition, while some data is not rationalized by the price competition under more restrictive demand specifications such as the logit demand function, CES demand function, or a discrete/continuous demand function with HARA h. I also discuss additional restrictions on the demand function discussed in the previous research; a test with observed cost shifters; a test for collusive price competition; a simple implementation for the logit demand function.

APPENDIX

Proof of Theorem 1. For sufficiency, we only need to construct cost and demand functions for each firm whose profit function is maximized at $\bar{p}_{j,t}$, $\bar{q}_{j,t}$.

First, consider a reconstruction of demand function. If the data satisfy the restriction defined in Theorem 1, we should be able to find $\alpha_{j,t}$ which corresponds to $\frac{h''_{j,t}(\bar{p}_{j,t})}{-h'_{j,t}(\bar{p}_{j,t})}$ for each observation, where $h_{j,t} : R_+ \to R$ represents the true data-generating process. For reconstruction of demand functions, I consider $\bar{h}_{j,t} : R_+ \to R$ s.t. $\frac{\bar{h}''_{j,t}(p_j)}{-\bar{h}'_{j,t}(p_j)} = \alpha_{j,t}$ for all $p_j \in R_+$. This is analogous to the construction of utility function in Afriat (1967), where the gradient of the utility function is assumed to be locally constant. Since the constant $\frac{\bar{h}''_{j,t}(p_j)}{\bar{h}'_{j,t}(p_j)}$ implies that $\bar{h}_{j,t}(p_j)$ can be represented as CARA function with risk averseness $\alpha_{j,t}$, $\bar{h}_{j,t}(p_j) = \exp\{v_{jt} - \alpha_{jt}p_j\}$ for some v_{jt}. Then, we can construct a demand function, $\bar{Q}_{j,t}(p) = m_t \frac{-\bar{h}'_{j,t}(p_j)}{H_0 + \sum_k \bar{h}_{k,t}(p_j)} = m_t \frac{\alpha_{jt}\exp\{v_{jt} - \alpha_{jt}p_j\}}{H_0 + \sum_k \exp\{v_{kt} - \alpha_{kt}p_j\}}$. Here, I denote the reconstructed demand function as $\bar{Q}_{j,t}(p)$ in order to distinguish it from the demand function in the true data-generating process, $Q_{j,t}(p)$. Now, v_{jt} can be chosen to satisfy a system of K equations, $m_t \frac{\alpha_{jt}\exp\{v_{jt} - \alpha_{jt}\bar{p}_{jt}\}}{H_0 + \sum_k \exp\{v_{kt} - \alpha_{kt}\bar{p}_{jt}\}} = \bar{q}_{jt}$ for all j, as with an inversion of the share function in logit specifications, discussed in Berry (1994).

Since (δ_{jt}, q_{jt}) satisfies the co-monotone property, we can use monotone cubic interpolation to reconstruct the increasing and continuously differentiable $\bar{C}'(\cdot)$. Then, we can reconstruct $\bar{C}(q) = \int_0^q \bar{C}'(x)\, dx$, which is convex and twice continuously differentiable. [13]

The final step consists in proving that $(\bar{p}, \bar{q})$ is an equilibrium under reconstructed demand and cost functions. Since the reconstructed profit function is continuously differentiable, first-order conditions must be satisfied at the optimal price. Therefore, we only need to show that a solution of the first-order conditions for each firm is unique given the other firms' strategies. To do so, I use the common ι-markup property examined in Nocke and Schutz (2018). The following part is closely related to the proofs in Nocke and Schutz (2018)

[13] Carvajal et al. (2013, 2014) reconstruct the cost function as an upper envelope of linear cost functions, whose slope is determined by $\delta_{j,t}$'s. Instead, in this paper, I use cubic interpolation for differentiability, which is necessary for the inversion of ι-markup.

Table 1.1: Summary of results

	Demand	Cost	Extention	Neccesity	Sufficiency	Falsifiability
Theorem 1	DC	concave	-	Y	Y	N
Proposition 1	Logit	concave	-	Y	Y	Y
Corollary 1	CES	concave	-	Y	N	Y
Proposition 2	CES	Linear	-	Y	Y	Y
Corollary 3	DC with HARA h	concave	-	Y	N	Y
Proposition 3	*co-evolving* DC	concave	-	Y	N	Y
Remark 1	—	concave	cost shifter	Y	N	—
Remark 2	Logit	concave	potential collusion	Y	N	Y
Corollary 4	DC	concave	full collusion	Y	Y	N
Proposition 4	Logit	concave	full collusion	Y	Y	Y

(especially Lemma F), despite a few differences. First, we do not need to prove the existence of an equilibrium since we already have data as an equilibrium candidate. Therefore, we just need to show that those data is an equilibrium. Second, we consider a more general cost specification than Nocke and Schutz (2018). It complicates the inversion from μ^f to price vectors since marginal cost is not a constant, but a function of product quantity. Third, the reconstructed demand function is a special case of the demand function in Nocke and Schutz (2018). Therefore, we can circumvent the difficulty arising from a general cost function by specifying the shape of the demand function.

In the following, I omit the subscript for time t considering that the static NE is repeated and the following logic is applied for each t. Then, I denote the reconstructed demand function as $\bar{Q}_j(p) = m\frac{-\bar{h}'_j(p_j)}{H_0+\sum_k \bar{h}_k(p_j)} = m\frac{\alpha_j \exp\{v_j-\alpha_j p_j\}}{H_0+\sum_k \exp\{v_k-\alpha_k p_j\}}$ and $\bar{h}'_j(p_k) = -\alpha_j \exp\{v_j - \alpha_j p_j\}$, $\bar{h}''_j(p_k) = \alpha_j^2 \exp\{v_j - \alpha_j p_j\}$, and $\frac{\bar{h}''_j(p_k)}{-\bar{h}'_j(p_k)} = \alpha_j$. Since we now consider a maximization problem of a specific firm given the other firm's strategy, let us denote $\bar{h}_0 + \sum_{k \notin J_f} \bar{h}_k(p_k) = H_0$ and $J_f = \{1, ..., n\}$ without loss of generality. By the FOC, we have the following for any j

$$\left\{p_j - \bar{C}'_j\left(\bar{Q}_j(\mathbf{p})\right)\right\}\frac{\bar{h}''_j(p_j)}{-\bar{h}'_j(p_j)} = 1 + m^{-1}\sum_{k\in J_f}\left\{p_k - \bar{C}'_k\left(\bar{Q}_k(\mathbf{p})\right)\right\}\bar{Q}_k(\mathbf{p}) \tag{1.8}$$

Since the RHS is same for any $j \in J_f$, the solution of a system of equations defined by (1.8) for any $j \in J_f$ satisfies

$$\nu_j(\mathbf{p}) \equiv \left\{p_j - \bar{C}'_j\left(\bar{Q}_j(\boldsymbol{p})\right)\right\}\alpha_j = \mu^f$$

for any $j \in J_f$. Let $\nu(\mathbf{p}) = [\nu_1(\mathbf{p}), ..., \nu_n(\mathbf{p})]'$. Then, $\mathbf{p} = \nu^{-1}\left(\mathbf{1}\mu^f\right) \equiv r\left(\mu^f\right) \equiv \left[r_1\left(\mu^f\right), ..., r_n\left(\mu^f\right)\right]'$ at the solution of (1.8). Then, we can rewrite the condition (1.8) as

$$
\begin{aligned}
\mu^f &= 1+m^{-1}\sum_{k\in J_f}\left\{r_k\left(\mu^f\right)-\bar{C}'_k\left(r\left(\mu^f\right)\right)\right\}\bar{Q}_k\left(r\left(\mu^f\right)\right) \\
&= 1+m^{-1}\sum_{k\in J_f}\underbrace{\left\{r_k\left(\mu^f\right)-\bar{C}'_k\left(r\left(\mu^f\right)\right)\right\}\alpha_k}_{\mu^f}\frac{1}{\alpha_k}\bar{Q}_k\left(r\left(\mu^f\right)\right) \\
&= 1+m^{-1}\mu^f\sum_{k\in J_f}\frac{1}{\alpha_k}\bar{Q}_k\left(r\left(\mu^f\right)\right) \\
\Leftrightarrow 0 &= 1+\mu^f\left\{m^{-1}\sum_{k\in J_f}\frac{1}{\alpha_k}\bar{Q}_k\left(r\left(\mu^f\right)\right)-1\right\}\equiv\psi\left(\mu^f\right)
\end{aligned}
$$

Then, the uniqueness of the solution of the first-order condition is proved by the strict monotonicity of $\psi\left(\mu^f\right)$. Again, the existence of a solution can be omitted since the data satisfy the first-order condition by the construction of $\left(\bar{Q}_j\left(\cdot\right),\bar{C}_j\left(\cdot\right)\right)_{j\in J_f}$. By taking a derivative w.r.t. μ^f

$$
\begin{aligned}
\psi'\left(\mu^f\right) &= \underbrace{\sum_{k\in J_f}\frac{\exp\left\{v_k-\alpha_k r_k\left(\mu^f\right)\right\}}{H_0+\sum_l\exp\left\{v_l-\alpha_l r_l\left(\mu^f\right)\right\}}-1}_{<0} \\
&\quad \underbrace{+\mu^f m^{-1}\sum_{k\in J_f}\frac{1}{\alpha_k}\underbrace{\frac{\partial\bar{Q}_k\left(\mathbf{p}\right)}{\partial\mathbf{p}}|_{\mathbf{p}=r\left(\mu^f\right)}}_{1\times n}\underbrace{r'\left(\mu^f\right)}_{n\times 1}}_{\equiv A}
\end{aligned}
$$

It is enough to show that $A\leq 0$.

$$
\begin{aligned}
A &= \mu^f m^{-1}\underbrace{\left[\frac{1}{\alpha_1},\ldots,\frac{1}{\alpha_n}\right]}_{1\times n}\underbrace{\frac{\partial\bar{Q}\left(\mathbf{p}\right)}{\partial\mathbf{p}'}|_{\mathbf{p}=r\left(\mu^f\right)}}_{n\times n}\underbrace{\frac{\partial\nu^{-1}\left(\mathbf{m}\right)}{\partial\mathbf{m}'}|_{\mathbf{m}=\mathbf{1}\mu^f}}_{n\times n}\underbrace{\mathbf{1}}_{n\times 1} \\
&= \mu^f m^{-1}\underbrace{\mathbf{1}'}_{1\times n}\underbrace{\Lambda^{-1}\underbrace{\frac{\partial\bar{Q}\left(\mathbf{p}\right)}{\partial\mathbf{p}'}|_{\mathbf{p}=r\left(\mu^f\right)}}_{n\times n}\underbrace{\frac{\partial\nu^{-1}\left(\mathbf{m}\right)}{\partial\mathbf{m}'}|_{\mathbf{m}=\mathbf{1}\mu^f}}_{n\times n}}_{\equiv B}\underbrace{\mathbf{1}}_{n\times 1}
\end{aligned}
$$

where $\Lambda = \begin{bmatrix} \alpha_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha_n \end{bmatrix}$. Since $\mu^f > 0$ and $m > 0$, we only need to show that B is negative semi-definite.

To examine the derivatives of $\nu^{-1}(\mathbf{m})$, we first consider the derivative of ν. Recall that $\nu_j(\mathbf{p}) \equiv \left\{p_j - C_j'\left(\bar{Q}_j(p)\right)\right\}\alpha_j$. Then, the partial derivatives are

$$\begin{aligned}
\frac{\partial \nu_k(\mathbf{p})}{\partial p_k} &= \alpha_k \left(1 - c''\left(\bar{Q}_k(\mathbf{p})\right)\frac{\partial \bar{Q}_k(\mathbf{p})}{\partial p_k}\right) \\
\frac{\partial \nu_k(\mathbf{p})}{\partial p_j} &= -\alpha_k c''\left(\bar{Q}_k(\mathbf{p})\right)\frac{\partial \bar{Q}_k(\mathbf{p})}{\partial p_j}
\end{aligned}$$

Then,

$$\begin{aligned}
\frac{\partial \nu(\mathbf{p})}{\partial \mathbf{p}'} &= \Lambda \left\{ I - \begin{bmatrix} c''\left(\bar{Q}_1(\mathbf{p})\right)\frac{\partial \bar{Q}_1(\mathbf{p})}{\partial p_1} & c''\left(Q_1(\mathbf{p})\right)\frac{\partial Q_1(\mathbf{p})}{\partial p_n} \\ & \\ c''\left(Q_n(\mathbf{p})\right)\frac{\partial Q_n(\mathbf{p})}{\partial p_1} & c''\left(Q_n(\mathbf{p})\right)\frac{\partial Q_n(\mathbf{p})}{\partial p_n} \end{bmatrix} \right\} \\
&= \Lambda \left\{ I - \Gamma(\mathbf{p})\frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'} \right\}
\end{aligned}$$

where $\Gamma(\mathbf{p}) = \begin{bmatrix} c''(Q_1(\mathbf{p})) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & c''(Q_n(\mathbf{p})) \end{bmatrix}$. Then,

$$\begin{aligned}
B &= \Lambda^{-1}\frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'}|_{\mathbf{p}=\bar{\mathbf{p}}}\frac{\partial r(\mathbf{m})}{\partial \mathbf{m}'}|_{\mathbf{m}=\mathbf{1}\mu^f} \\
&= \Lambda^{-1}\frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'}|_{\mathbf{p}=\bar{\mathbf{p}}}\left[\frac{\partial \nu(\mathbf{p})}{\partial \mathbf{p}'}|_{\mathbf{p}=\bar{\mathbf{p}}}\right]^{-1} \\
&= \Lambda^{-1}\frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'}|_{\mathbf{p}=\bar{\mathbf{p}}}\left[\Lambda\left\{I-\Gamma(\mathbf{p})\frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'}\right\}\right]^{-1} \\
&= \Lambda^{-1}\left(\left(\frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'}|_{\mathbf{p}=\bar{\mathbf{p}}}\right)^{-1}\right)^{-1}\left\{I-\Gamma(\mathbf{p})\frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'}\right\}^{-1}\Lambda^{-1} \\
&= \Lambda^{-1}\left(\left\{I-\Gamma(\mathbf{p})\frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'}\right\}\left(\frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'}|_{\mathbf{p}=\bar{\mathbf{p}}}\right)^{-1}\right)^{-1}\Lambda^{-1} \\
&= \Lambda^{-1}\left(\left(\frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'}|_{\mathbf{p}=\bar{\mathbf{p}}}\right)^{-1}-\Gamma(\mathbf{p})\right)^{-1}\Lambda^{-1} \\
&= -\Lambda^{-1}\left(\Gamma(\mathbf{p})-\left(\frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'}|_{\mathbf{p}=\bar{\mathbf{p}}}\right)^{-1}\right)^{-1}\Lambda^{-1}
\end{aligned}$$

Now, B is negative definite as long as $\frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'}$ is negative semi definite;

$$\frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'} = m^{-1}\left\{\begin{bmatrix} Q_1(p)Q_1(p) & \cdots & Q_1(p)Q_n(p) \\ \vdots & \ddots & \vdots \\ Q_1(p)Q_n(p) & \cdots & Q_n(p)Q_n(p) \end{bmatrix} - m\begin{bmatrix} \alpha_1 Q_1(p) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha_n Q_n(p) \end{bmatrix}\right\}$$

Then,

$$
\begin{aligned}
x'\frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'}x &= m^{-1}\left\{x'Q(p)Q(p)'x - mx'\begin{bmatrix} \alpha_1 Q_1(p) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha_n Q_n(p)\end{bmatrix}x\right\} \\
&= m^{-1}\left\{\left(\sum_i x_i Q_i\right)^2 - m\sum_i x_i^2 \alpha_i Q_i\right\} \\
&= m^{-1}\left(\sum_i x_i Q_i\right)^2 - m^{-1}\sum_i x_i^2 Q_i^2 + m^{-1}\sum_i x_i^2 Q_i^2 - \sum_i x_i^2\alpha_i Q_i \\
&= -m^{-1}\underbrace{\left\{\sum_i x_i^2 Q_i^2 - \left(\sum_i x_i Q_i\right)^2\right\}}_{>0} - m^{-1}\left\{\sum_i x_i^2 Q_i \underbrace{(m\alpha_i - Q_i)}_{>0}\right\} < 0
\end{aligned}
$$

Then, $-\frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'}$ is positive definite, so as $\left(-\frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'}\right)^{-1}$. Therefore, $\Gamma(\mathbf{p}) - \left(\frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'}\right)^{-1}$ is positive definite since $\Gamma(\mathbf{p})$ is a diagonal matrix with positive components. Therefore,

$$
\begin{aligned}
x'Bx &= -x'\Lambda^{-1}\left(\Gamma(\mathbf{p}) - \left(\frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'}\right)^{-1}\right)^{-1}\Lambda^{-1}x \\
&= -\left(\left(\Lambda^{-1}\right)'x\right)'\left(\Gamma(\mathbf{p}) - \left(\frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'}\right)^{-1}\right)^{-1}\Lambda^{-1}x \\
&= -\left(\Lambda^{-1}x\right)'\left(\Gamma(\mathbf{p}) - \left(\frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'}\right)^{-1}\right)^{-1}\Lambda^{-1}x \\
&< 0
\end{aligned}
$$

Therefore, B is negative definite, so that $\psi'\left(\mu^f\right) < 0$.

Proof of Proposition 3:

We need to derive the last condition in this proposition.

By the co-evolving property, we can find a permutation such that $\sigma(t) < \sigma(s)$ implies $\epsilon_{j,t}(p) \le \epsilon_{j,s}(p)$ for all $j \in J$ and for all p. If $\bar{p}_{i,t} \ge \bar{p}_{i,s}$, $\bar{p}_{-i,t} \le \bar{p}_{-i,s}$ and $\sigma(t) < \sigma(s)$, then $\alpha_{j,t} - m_t^{-1}\bar{q}_{j,t} = \epsilon_{j,t}(\bar{p}_{jt}, \bar{p}_{-jt}) \leq \epsilon_{j,t}(\bar{p}_{js}, \bar{p}_{-jt}) \leq \epsilon_{j,s}(\bar{p}_{js}, \bar{p}_{-js}) \leq \epsilon_{j,s}(\bar{p}_{js}, \bar{p}_{-js}) = \alpha_{j,s} - m_s^{-1}\bar{q}_{j,s}$. Thus, $\alpha_{j,t} - m_t^{-1}\bar{q}_{j,t} \leq \alpha_{j,s} - m_s^{-1}\bar{q}_{j,s}$

CHAPTER 2

Controlling Fake Reviews

This paper theoretically analyzes fake reviews on a platform market using models where a seller creates fake reviews through incentivized transactions, and its sales depend on its rating based on a review history. The platform can control the incentive for fake reviews by changing the parameters of the rating system, such as weights placed on old and new reviews and its filtering policy. At equilibrium, the number of fake reviews increases as quality increases but decreases as reputation improves. Since fake reviews have a positive relationship with a product's underlying quality, rational consumers find a rating more informative when fake reviews exist, while credulous consumers suffer from a bias caused by boosted reputation. A stringent filtering policy can decrease the expected amount of fake reviews and the bias of credulous consumers, but at the same time, it can decrease the informativeness of a rating system for rational consumers. In terms of the weight placed on the review history, rational consumers benefit from higher weights on past reviews than from optimal weights without fake reviews.

2.1 Introduction

Online platform markets are growing worldwide, such that both businesses and their customers increasingly rely on reviews on the platforms.[1] At the same time, incentives for sellers to make fake reviews are also growing. Washington Post (Dwoskin and Timberg,

[1]Hollenbeck (2018); Hollenbeck et al. (2019) show that ratings work as a substitute of other form of advertisement or brand names, and this pattern is getting stronger over time in the hotel industry. Reimers and Waldfogel (2020) exhibit that the existence of star ratings has 15 times as the impact on consumer surplus as the professional reviews on New York Times. For the institutional details and data analysis on platforms and ratings, see also Belleflamme and Peitz (2018)

Figure 2.1: An example of a refund offer

Person Red, who is suspected as a seller on Amazon, posts pictures of its products and offers full refunds of the products after reviews of them. About an hour after of the post, Person Blue, who is suspected as a fake reviewer, shows an interest on the products and refunds.

2018) reports that based on fake review detection algorithms, 50.7% of reviews for Bluetooth headphones, 58.2% for Bluetooth speakers, 55.6% for weight loss pills, and 67.0% for testosterone boosters on Amazon are suspicious. How do sellers make fake reviews? The sellers can post information of their products with refund offers, which are typically finalized via PayPal after purchases and positive reviews on Amazon. (See Fig. 2.1 for an example of such an offer.)[2] These reviews correspond to verified purchases and are reflected to the star rating (until they are detected by Amazon).[3] He et al. (2020) connect such refund offers on Facebook with product listings on Amazon and show a positive correlation between refund offers on Facebook and a product's performance on Amazon such as its ratings, sales ranking, and the number of reviews. Regulators have been concerned about fake reviews, and their attitude toward fake reviews is becoming stringent. For instance, in 2019, the Federal Trade Commission (FTC) filed the first case against paid fake reviews by CureEncapsulations on Amazon. Online platforms have restricted fake reviews in their own ways, but regulators put increasingly high pressure on online platforms to maintain a stricter attitude against fake reviews. [4]

[2]For more details on evasive practice by incentivized reviewers and agents who contact buyers to incentivize them to write reviews, see Oak (2021).

[3]Offers of such fake reviews from fake reviewers have been found on eBay.

[4]For instance, in 2019, the Competition and Markets Authority (CMA) in U.K. launched work programme "has written to Facebook and eBay this week urging them to conduct an urgent review of their sites to prevent

However, the impact of fake reviews on consumers on a platform is not clear. First, consumers might not be fooled by fake reviews if they know that there are fake reviews. In the standard work of Holmström (1999), the market can correctly anticipate the behavior of long-lived players and debias the signal. Furthermore, customers might be able to elicit additional information from fake reviews. If only high-quality sellers make fake reviews to boost their initial reputation, the boosted rating can be an even better signal of good quality. Such a behavior might be possible if low quality is revealed via word of mouth, and only a high-quality seller can reap benefits from future sales, as suggested by Nelson (1970,1974) in the context of advertising. [5]

In this study, we examine a theoretical model in which sales are determined by the seller's reputation level and the seller chooses the amount of positive fake reviews at each instance. Consumers perceive a seller's reputation based on the potentially boosted ratings displayed on the platform. The platform can control how strictly it filters fake reviews and how much the rating reflects the information of past feedback (i.e., how fast the rating evolves). A key assumption in this study is that it becomes harder for a seller to make fake reviews as its reputation improves because of the higher reimbursement necessary to incentivize reviewers due to the higher price.[6] This brings more fake reviews from the seller with low reputation. This also generates the dependence of fake reviews on the seller's quality-type. Because high-quality sellers benefit more from their high reputation, high-quality sellers generate more fake reviews at equilibrium. Because of this positive relationship between the number of fake reviews and quality, consumers sometimes benefit from lenient policies on fake reviews. In the literature on signaling promotion, the complementarity between quality

fake and misleading online reviews from being bought and sold". In responses, both Facebook and eBay have immediately deleted posts identified by CMA, and updated their policy to explicitly prohibit offers of fake reviews. In 2020, May, CMA has launched new investigation into online websites on how they currently detect fake or misleading reviews.

[5] Ananthakrishnan et al. (2020) analyze the display of fake reviews from a different perspective and show that the consumers form more trust on the platform if it shows the fake reviews with flags indicating them as fake reviews, rather than deleting them from the platform.

[6] We can see the interaction between fake reviews and reputation more commonly. For instance, fake reviews might be crowded out if the seller receives many organic feedback due to large demand caused by high reputation. Then, the effective fake review would be costly for such a seller.

and reputation is understudied because, in most research, promotion is done only once at the beginning of a game. In this study, the complementarity comes from the future cost-saving effect rather than an increase in revenue.

The opposite dependence of fake reviews on a reputation about quality and on the underlying true quality also provides some cautions on empirical analysis on signaling promotion. That is, reputation-based indices, such as customer rating, can be a bad proxy for a product's underlying quality. Researchers can estimate opposite results if they use customer rating as a proxy for quality. Furthermore, even if the true quality is measured, it is important to control for the reputation level when estimating the relationship between promotion and the underlying quality. Fig. 2.2 exemplifies the possibility of an omitted variable issue; that is, the promotion level and the true quality of a product can be negatively correlated without being conditioned upon a firm's reputation level, even though quality and promotion have a positive relationship, *ceteris paribus*.

The negative relationship between fake reviews and a firm's reputation also increases the speed at which the rating changes. That is, in the presence of fake reviews, when the rating goes down (up), it more quickly goes up (down) than when the rating system has no fake reviews. This distorts the informativeness of the rating system. How fast the rating changes relates to the relative weight of new information in the rating system. The greater is the weight of new information (and the lower the weight of old information), the faster is the transition of the rating. Thus, the equilibrium effect that makes the transition faster has the effect of distorting upward the weight of the new information (and downward the weight of the old information). Therefore, given the existence of fake reviews, the platform needs to make some adjustments. The platform should set a lower weight for new information (and higher weight for old information) compared with a rating system that has no fake reviews.

The discussion above is based on the assumption of rational consumers who know the seller's strategy. However, the regulator's concern is not necessarily on sophisticated consumers but more on naive consumers, who are vulnerable to fake reviews.[7] In this study,

[7]For instance, Federal Trade Commission (FTC)'s mission is "[p]rotecting consumers and competition by preventing anticompetitive, deceptive, and unfair business practices through ...".

we also incorporate such consumers and show how much they become biased as a result of fake reviews by the sellers. Even though in general the relationship between the bias and the censorship policy is not monotonous, stringent censorship generally reduces the naive consumer's bias under a reasonable range of parameters.

Thus, the regulator might face a trade-off between the precision of the information for rational consumers and the bias that credulous consumers suffer from. This study provides a framework for analyzing such a trade-off.

The remainder of this paper is organized as follows. In Section 2, we review related literature. In Section 3, we analyze a model with rational buyers. In Section 4, we introduce credulous consumers. Section 5 concludes. Most of the proofs are deferred to the Appendix.

2.2 Literature Review

This paper mainly contributes to two streams of literature: rating design and signaling through promotion. The literature on rating design can be divided into two strands: (i) how to reveal the known quality level or estimated quality index (i.e., whether to reveal full information or add noise/coarsen the information) and (ii) how to generate the index of an unknown quality based on the multiple sources of information on a player's performance.

The first strand is often framed in the context of certification, such as the works of Lizzeri (1999), Ostrovsky and Schwarz (2009), Boleslavsky and Cotton (2015), Harbaugh and Rasmusen (2018), Hopenhayn and Saeedi (2019), Hui et al. (2018). Some models are made tractable by the representation with posterior distribution in the line of Bayesian persuasion proliferated by Rayo and Segal (2010) and Kamenica and Gentzkow (2011). Saeedi and Shourideh (2020) extend the framework wherein the quality is endogenously chosen by the seller rather than the exogenous variable.

This paper relates to another strand of literature, as it analyzes how to aggregate the players' actions into a single index. In a one-shot model, Ball (2019) analyzes the optimal

(https://www.ftc.gov/about-ftc)

way to aggregate the various sources of potentially manipulated signals. In a dynamic setting based on Holmström's (1999) signal jamming/career concern model, **?** show that the effort level of a long-lived player is maximized by a rating that is linear to past observations. Vellodi (2020) analyzes the impact of rating on the entry/exit behavior of a firm and derives an optimal rating that prevents high-quality sellers from exiting from the market due to a reputation trap of failing to accumulate good reputation because of initial bad luck. Bonatti and Cisternas (2019) examine a long-lived consumer's Ratchet effect. The consumers try to hide its willingness to pay to avoid the personalized pricing by short-lived monopolist, so that the consumption does not perfectly reflect their willingness to pay. Similarly to **?** and Bonatti and Cisternas (2019), this study examines the relationship between a signal-jamming structure and a linear rating system. In contrast to **?**, the equilibrium strategy is dependent on the hidden quality and reputation, such that the seller's strategy changes the informativeness of the rating on the equilibrium path, as in Bonatti and Cisternas (2019).[8] In contrast to Bonatti and Cisternas (2019), where the effect of the manipulation is endogenously determined via the short-lived player's belief, in this study, the platform controls for the effectiveness of the manipulation so that we can analyze the impact of censorship by the platform. In addition, this study departs from the literature by analyzing the impact of manipulation on naive/credulous consumers, which is often the concern of regulators.

This paper also contributes to the literature on promotion and signaling. Nelson (1970, 1974) argues that even if the promotion does not have any intrinsic information, "burning money" itself can be a signal of good quality because such a signal pays off only for high-quality firms through repeated purchases in the future. This idea is formalized later by Kihlstrom and Riordan (1984), Milgrom and Roberts (1986) and many others as separating equilibria in signaling models. Using a one-shot signal-jamming framework instead of a signaling model, Mayzlin (2006) shows a negative relationship between promotion through fake

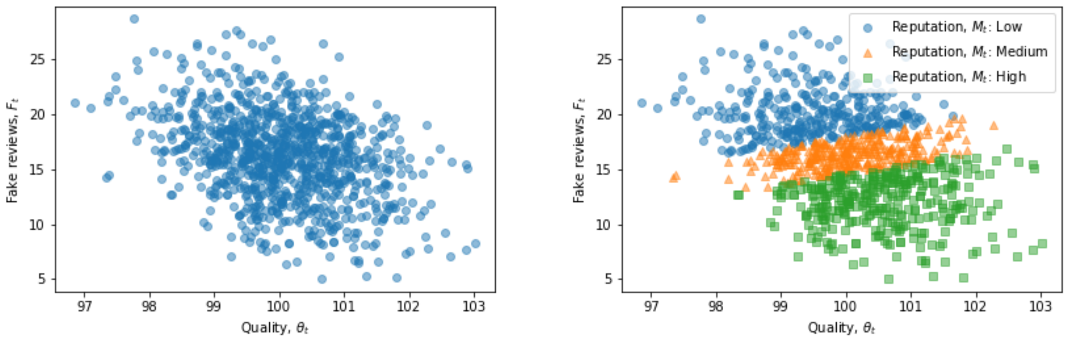
[8]Another contrast to **?** is that they start from a general information structure so that they can represent any reputation by changing the information structure. Then, they can focus on the resulted process of reputation level in a similar way that researchers focus on the resulted outcome by the revelation principle in the context of the mechanism design. On the other hand, this paper and Bonatti and Cisternas (2019) use more specific information structure, so that we should examine how the consumers interpret the resulted rating.

reviews and quality, and Dellarocas (2006) generalizes conditions for the positive/negative correlation in a one-shot signal-jamming model. Bar-isaac and Deb (2014) examine the effects of vertically/horizontally heterogeneous preferences, and Grunewald and Kräkel (2017) examine the effect of competition between firms. Most studies on the signaling role of promotion are based on models with one-shot promotion, except for Horstmann and MacDonald (1994), where the experience of the product is an imperfect signal of the quality, and the signaling via advertising is done only after establishing a reputation so that it is hard for low-quality sellers to mimic high-quality sellers' behavior.[9]. In this study, I examine a dynamic signal-jamming model, where reputational concern is the driving force for the positive correlation between quality and promotion. It also generates non-degenerate dynamics consistent with an observation by Luca and Zervas (2016) that strategic manipulation increases after a drop in reputation.

The dependence of fake reviews on reputation also provides some implications for the empirical literature on signaling promotion. The literature has had weak support regarding the correlation between quality and promotion. For instance, Kwoka (1984) observes that optometrists with more advertisements provide less thorough eye examination, and Horstmann and Moorthy (2003) observe that advertising is hump-shaped in terms of quality among restaurants in New York. Recently, Sahni and Nair (2019) implement a quasi-experiment to isolate the intrinsic information and signaling effect of burning money and show that the consumer positively responds to the burning of money. They point out that it is difficult to show the relationship between quality and promotion level because it is difficult to obtain a reliable measure of quality. This paper emphasizes this point. A reputation-based index, such as customer rating, can be a bad proxy for the underlying quality. The reputation level and the underlying quality level have opposite impacts on the promotion level in equilibrium. Furthermore, even if the true quality is measured, it is important to control for the reputation level. As shown in Fig. 2.2 , the level of promotion and the true quality can

[9]Aside from the context of the rating system or the signaling promotion, Grugov and Troya-Martinez (2019) examine the biasing behavior of the seller in a model a. la. Holmström (1999) incorporating a detection rule and credulous consumers, and show that the biasing behavior increases as the authority requires stricter rule and the share of credulous consumer increases.

Figure 2.2: A simulated distribution of quality levels and the amount of fake reviews



The left panel show that the the amount of the fake reviews is negatively correlated with the quality level, unconditional on the level of reputation. On the other hand, the right panel shows that the amount of the fake reviews is increasing in the quality level, conditional on the reputation level.

be negatively correlated without being conditioned upon the reputation level, even though quality and promotion have a positive relationship, *ceteris paribus*.

2.3 Rating Design for Rational Consumers

In this study, we examine both models with rational consumers and naive consumers. In this section, we first introduce a baseline model with a mass of rational consumers. The consumers rationally expect that a long-lived seller makes fake reviews following a linear strategy. However, they cannot induce the seller's exact action at time t because the quality is still hidden, even though the strategy and the current reputation are known to the consumers.

Then, in the next section, we introduce a market with naive consumers who do not expect any fake reviews while the seller makes fake reviews, such that the reputation is biased upward. In each model, we examine the impact of the platform's filtering/censoring policy on reviews, the weights of new and old reviews, and the precision of genuine reviews.

2.3.1 Model

The model is in a continuous time and infinite horizon, $t \in [0, \infty)$. At each instance t, a long-lived seller sells q units of its product, whose quality is denoted as θ_t, and makes F_t units of fake reviews. A sufficiently large mass, n, of consumers forms a demand function such that the price $p_t = E[\theta_t|Y_t] \equiv M_t$ clears the market, where Y_t is the rating of the product at time t.[10] The price being a representation of the reputation of the hidden quality is the standard assumption in the literature on reputation. The quality θ_t governs consumers' willingness to pay for the product, so the price is high when the expected quality of the product is high. A more specific underlying model, that can incorporate naive consumers is suggested in the Appendix.

The quality, θ_t, and rating, Y_t, change over time. The quality, θ_t, follows an exogenous mean-reverting process:

$$d\theta_t = \kappa\left(-\theta_t + \mu\right)dt + \sigma_\theta dZ_t^\theta \tag{2.1}$$

while the rating, Y_t, is characterized by the following differential equation:

$$dY_t = -\phi Y_t + d\xi_t \tag{2.2}$$

where $d\xi_t$ is defined as:

$$d\xi_t = aF_t dt + bq\theta_t dt + \sqrt{bq}\sigma_\xi dZ_t^\xi \tag{2.3}$$

where $\left(Z_t^\theta, Z_t^\xi\right)$ is a standard Brownian motion; a is the effectiveness of the fake review; b is the feedback rate from customers; μ is the mean of θ_t in the stationary distribution, and σ_θ and σ_ξ govern the standard deviations of the disturbance. The exogenous mean-reverting process of θ_t is understood as resulting from the competition over quality among sellers. The relative quality of a firm's product might decrease due to the rise of other sellers with even higher quality. The firm's product's relative quality might increase when a competitor increases its product's price. The transition of the rating, Y_t, is interpreted in a discrete time analogue that the future rating, Y_{t+dt}, is a weighted sum of the new reviews, $d\xi_t$, and the previous reviews, Y_t, with weights of 1 and $1 - \phi dt$, respectively. After filtering suspicious reviews, the new reviews consist of two components: "organic" reviews and the remaining fake reviews. The second and third terms of Eq. (2.3) correspond to organic reviews. Higher

[10] Saeedi (2019) showed that the reputation is the major determinant of the price on eBay market.

quality tends to generate high reviews, and the information becomes precise when there is feedback from many transactions (i.e., high q) or a high response rate (i.e., high b). The disturbance, $\sigma_\xi dZ_t^\xi$, is caused by the heterogeneity of the criteria among customers.[11] The first term is the effect of the fake reviews. The seller tries to boost the average review through fake reviews, but some of them are detected by the platform, and the remaining reviews enter as $aF_t dt$. Thus, a small a implies stringent censorship. As in **?**, Vellodi (2020), and Bonatti and Cisternas (2019), the rating, Y_t, does not exactly capture 5-star rating on Amazon, Yelp, or some other online platform. The level of Y_t is dependent on the mean of θ_t and other parameters. By this specification of the rating, we can rely on the normality to simplify the analysis.

The seller's instantaneous payoff is defined as:

$$\pi_t = (1 - \tau)\, p_t\, (q + F_t) - p_t \cdot F_t - \frac{c}{2} F_t^2$$

where τ denotes the transaction fees imposed by the platform. The first term is the total revenue from all transactions, including those corresponding to fake reviews, and the second term is the reimbursement cost to the fake reviewers. The last term expresses that generating more fake reviews is harder. The seller might find it challenging to search for incentivized reviewers through communities such as Facebook. Some fake review services may charge a higher price for fake reviews. Furthermore, increasing the number of fake reviews come with a higher risk of being detected by the platform. The cost of production is abstracted out from the model. [12] The long-lived seller maximizes its discounted present value by choosing $(F_t)_{t\geq 0}$.

[11] In this paper, the mechanism behind the customer feedback is abstracted and assumed that the fixed portion of consumers keep reviewing. For detailed analysis on the customer feedback, see Chevalier et al. (2018) and the literature cited in it. They analyze the relationship with managerial responses to reviews.

[12] Whether the high quality seller or low quality seller face high costs of production is arguable by itself. If high quality come from the seller's high productivity, the high quality seller can produce with lower costs. If the low quality is by the seller's choice rather than the difference in the production technology among sellers, the low quality product would be associated with low production cost. The different specifications on the production costs can cause different pattern in fake reviews, but those extensions are deferred to the future research.

The instantaneous profit becomes easier to compare with the previous research when it is rewritten as follows:

$$\pi_t = (1-\tau)\, M_t \cdot q - \tau M_t \cdot F_t - \frac{c}{2} F_t^2. \tag{2.4}$$

Without the second term in eq. (2.4), the model becomes effectively a special case of **?**, which is based on Holmström's (1999) signal-jamming model and uses a general information structure as a rating. However, due to the existence of this term, the marginal cost of the manipulation depends on the current reputation level. Therefore, the equilibrium manipulation level depends on the current rating in contrast to **?**, where the equilibrium action turns out to be state-independent. Instead of relying on the time- and state-invariant action, we apply the idea of Bonatti and Cisternas (2019) to focus on a linear strategy, and a Gaussian stationary distribution of $(\theta_t,\, Y_t)$. Then, the Hamilton-Jacobi-Bellman equation gives a simple quadratic value function, which is solved by the guess-and-verify method. It is verified that as τ approaches zero, the equilibrium strategy becomes invariant to θ_t, Y_t, (and t).

The interaction between the current reputation and the current action is considered as the driving force of the non-degenerate Markov equilibrium strategy. In this study, this interaction between reputation and manipulation is derived from the reimbursement to fake reviewers; however, such an interaction can be more commonly observed in the context of fake reviews. For instance, if the reputation is high, then a large demand can crowd-out fake reviews, such that the effective fake reviews can be more costly given the high reputation. In the Appendix, an alternative model with such an interpretation is discussed. A model with a changing quantity that is isomorphic to the main model is discussed in Appendix 2.C.

Definition of the Equilibrium As mentioned above, we focus on a linear Markov strategy equilibria, where a linear Markov strategy maximizes the seller's discounted present value among any admissible strategies.

A linear strategy (in θ_t and Y_t) is defined as:

$$F_t = \hat{\alpha}\theta_t + \hat{\beta} Y_t + \hat{\gamma}$$

Note that θ_t does not directly appear in the instantaneous payoff function, but it appears in the transition of the payoff relevant state variable, Y_t. Thus, the seller is potentially sensitive to the level of θ_t. Now the equilibrium is defined as follows:

Definition 1. A linear Markov strategy $F = (F_t)_{t\geq 0}$ s.t. $F_t = \hat{\alpha}\theta_t + \hat{\beta}Y_t + \hat{\gamma}$ is a stationary Gaussian linear Markov equilibrium if

1. $F = \arg\max_{\left(\tilde{F}_t\right)_{t\geq 0}} E_0\left[\int_0^\infty e^{-tr}\pi_t\right]$ where $\left(\tilde{F}_t\right)_{t\geq 0}$ is admissible,
2. $M_t = E\left[\theta_t|Y_t\right]$, and
3. $(\theta_t, Y_t)_{t\geq 0}$ induced by F is a stationary Gaussian.

We do not know that $(\theta_t, Y_t)_{t\geq 0}$ is stationary or Gaussian *ex ante* because Y_t is endogenously determined by F_t. However, given a linear strategy, the condition for $(\theta_t, Y_t)_{t\geq 0}$ to be a stationary Gaussian is simply characterized by an inequality—similar to Bonatti and Cisternas (2019)—by Eqs. (2.2) and (2.3), and the definition of the linear strategy,

$$\begin{aligned} dY_t &= -\phi Y_t dt + aF_t dt + bq\theta_t dt + \sqrt{bq}\sigma_\xi dZ_t^\xi \\ &= -\left(\phi - a\hat{\beta}\right)Y_t dt + \left(a\hat{\alpha} + bq\right)\theta_t dt + a\delta\mu dt + \sqrt{bq}\sigma_\xi dZ_t^\xi \end{aligned} \tag{2.5}$$

Thus, an inequality, $\phi - a\hat{\beta} > 0$, must hold for $(\theta_t, Y_t)_{t\geq 0}$ to have a stationary distribution (otherwise, the process of Y_t diverges). When (θ_t, Y_t) is a stationary Gaussian, by the projection theorem on the Gaussian distribution,

$$M_t \equiv E\left[\theta_t|Y_t\right] = E\left[\theta_t\right] + \frac{Cov\left(\theta_t, Y_t\right)}{Var\left(Y_t\right)}\left[Y_t - E\left[Y_t\right]\right] \tag{2.6}$$

Furthermore, if it is stationary, all expectations in Eq.(2.6) are constants. By letting $\lambda \equiv \frac{Cov(\theta_t, Y_t)}{Var(Y_t)}$ and $\nu \equiv E\left[Y_t\right]$ (and $\mu = E\left[\theta_t\right]$ by construction), Eq.(2.6) is written as $M_t = \mu + \lambda[Y_t - \nu]$. In the following part of this section, we use M_t instead of Y_t as a state variable for the sake of expositional simplicity. Then, the linear strategy is redefined as

$$F_t = \alpha\theta_t + \beta M_t + \delta\mu$$

The stationary condition is summarized as follows:

Lemma 1. *(Stationarity and the characterization of the long-run moments) Suppose* $F_t = \alpha\theta_t + \beta M_t + \delta\mu$ *where* $M_t \equiv E[\theta_t|Y_t]$ *for all* $t \geq 0$. *Then, a process* $(\theta_t,\, Y_t)_{t\geq 0}$ *is a stationary Gaussian if and only if*

i. $M_t = \mu + \lambda[Y_t - \nu]$ *for all* t

ii. $a\lambda\beta - \phi < 0$, *and*

iii. $(\theta_0,\, Y_0)' \sim \mathcal{N}\left([\mu,\, \nu]',\, \Gamma\right)$ *is independent of* $\left(Z_t^\theta,\, Z_t^\xi\right)_{t\geq 0}$ *where* Γ *is the variance-covariance matrix in the stationary distribution.*

The third condition is required so that the game starts from a stationary distribution. Now, the HJB equation is simply written by using Ito's lemma:

$$\begin{aligned} rV(\theta,\, M) = &\sup_{F\in\mathbb{R}} (1-\tau)\, M\cdot q - \tau M\cdot F - \frac{c}{2}F^2 \\ &- \kappa(\theta-\mu)\, V_\theta \\ &+ \left\{a\lambda F + bq\lambda\theta - \phi\left[M - \bar{\theta} + \lambda\bar{Y}\right]\right\} V_M \\ &+ \frac{\sigma_\theta^2}{2} V_{\theta\theta} \\ &+ \frac{bq\lambda^2\sigma_\xi^2}{2} V_{MM} \end{aligned} \tag{2.7}$$

By guessing the quadratic form of the value function, $V = v_0 + v_1\theta + v_2 M + v_3\theta^2 + v_4 M^2 + v_5\theta M$, and the linear strategy, we can verify the existence and uniqueness of the value function and the linear strategy via the matching coefficient.

2.3.2 Equilibrium Characterization

The equilibrium strategy is characterized by guessing the quadratic value function and the linear strategy and by matching coefficients α, β, δ, $(v_k)_{k=0}^{5}$ of the first-order conditions, envelop conditions, and the stationarity condition characterized in Lemma 1. In the proof, the characterizing conditions are summarized into one equation $h(L) = 0$ with an aggregator $L \equiv a\lambda\beta$, and then all the equilibrium coefficients are derived as a function of L. Aggregator L is interpreted as an equilibrium effect on the speed of the rating transition or the equilibrium effect on the relative weight of new information. When L is positive, the rating transition effectively speeds up because the low rating is soon boosted back to the average rating by fake reviews.

By analyzing the existence and uniqueness of the aggregator L and examining the corresponding equilibrium coefficients, we obtain the following theorem:

Theorem 2 (Existence and uniqueness)**.** *There is always a stationary linear Markov equilibrium. For any equilibrium,* $\alpha > 0$, $\beta \in \left(-\frac{\tau}{c}, 0\right)$, $\lambda > 0$ *and* $L > 0$ *hold. Furthermore, if* $h'(L) < 0$ *holds, then such an equilibrium is unique, and the equilibrium coefficients* α, β, *and* δ *are differentiable in the parameters.*

$h'(L) < 0$ *holds for any* $L > 0$ *if* $6\kappa\phi + 4r^2 + 2\kappa r + 17r\phi + 19\phi^2 > \kappa^2$.

Note that $6\kappa\phi + 4r^2 + 2\kappa r + 17r\phi + 19\phi^2 > \kappa^2$ is a loose and reasonable condition. ϕ is the transition speed of the rating, and κ is the transition speed of the quality. The required inequality is reasonable as long as the rating system is meant to help estimate the current quality. For instance, even if the true quality does not drift much (i.e., $\kappa \simeq 0$), the rating should drift toward the underlying true quality (i.e., $\phi > 0$).

Intuition of the Equilibrium Strategy In Theorem 2, it is shown that high-quality types make more fake reviews ($\alpha > 0$), conditional on its reputation level. and high-reputation type makes fewer fake reviews ($\beta < 0$) conditional on the quality type. Given the logic of Nelson (1970; 1074), $\alpha > 0$ (and $\beta < 0$) might look intuitive, but this model adds different reasons than the previous research.

I start from the negative β. From the first-order condition, the optimal strategy is expressed as

$$F_t = -\frac{\tau}{c}M + a\lambda \underbrace{\{v_2 + 2M_t v_4 + \theta v_5\}}_{=V_M}$$

Then, $\beta = -\frac{\tau}{c} + \frac{2a\lambda}{c}v_4$. Furthermore, the envelope condition gives an expression for v_4 so that it is rewritten as $\beta = -\frac{\tau}{c} - \frac{\tau}{c}\frac{a\beta\lambda}{(-a\beta\lambda+r+2\phi)}$. The first term comes from the interaction of the reputation level and the fake reviews in the cost term, $\tau M_t F_t$. If the reputation is high, then the marginal cost of the fake review is high. Therefore, the seller will make fewer fake reviews given a higher reputation. The second term corresponds to the fake review's marginal benefit in the future. Given the equilibrium strategy, $v_4 = -\frac{\beta\tau}{2(-a\beta\lambda+r+2\phi)}$ is positive, meaning that the marginal benefit in the future increases with the reputation. This is because the future self will reduce the amount of fake reviews after observing the boosted reputation due to today's fake reviews. Furthermore, this effect increases with M_t because the future reputation M_{t+dt} tends to be high given a high M_t, so the interaction term

$$\tau M_{t+dt}F_{t+dt} = \alpha\tau M_{t+dt}\theta_{t+dt} + \tau\beta M_{t+dt}^2 + \delta\mu\tau M_{t+dt} \tag{2.8}$$

decreases quadratically given a negative β. It turns out that the first term dominates the second term; thus β remains negative.

The intuition of positive α comes from the complementarity between the quality, θ, and the reputation, M, in the seller's value function. With high quality θ_t today, the reputation in the future tends to be higher than the case with low quality today, given the same level of reputation M_t today. Furthermore, as previously stated, the future benefit from the reputation boost is higher given a higher reputation in the future. Thus, high quality results in a high incentive for fake reviews. Mathematically, the equilibrium coefficient α is characterized as

$$\alpha = a\lambda v_5 = \frac{a\lambda}{\kappa + r + \phi}\{2\left(a\alpha + bq\right)\lambda v_4 - \alpha\tau\} \tag{2.9}$$

The first equality reveals that the sign of α comes from the complementarity of θ and M in the value function. In the last expression, $\left(a\alpha + bq\right)\lambda$ indicate that the high θ_t results in a high M_{t+dt}. It is multiplied with positive v_4, which represents an increasing marginal value with respect to M_{t+dt}. This is the driving force of the positive α. The remaining term of Eq. (2.9), $-\alpha\tau$, states that such an incentive is attenuated because the quality in the near future θ_{t+dt} tends to be high given high θ_t; thus, today's fake reviews increase the cost in the future via the first term of Eq. (2.8).

In summary, the driving force of $\beta < 0$ is the incentive to reduce $\tau M_t F_t$ today given a high M_t. α is positive because of the complementarity of θ_t and M_t through cost savings. Readers might wonder why an increase in revenue (like Nelson, 1970, 1974) does not appear in the above argument. If θ_t is high, the boosted revenue would stay high for a long time; but in this model, such a product would eventually achieve a high reputation through organic feedback even without fake reviews. Therefore, the *marginal future revenue* $\frac{dp_s}{dF_t}$ $(s \geq t)$ is independent of θ_t. It is worth noting that the same intuition applies even in a variant of the model with a fixed price p and time-varying quantity q_t discussed in the Appendix.

2.3.2.1 Properties of the equilibrium

Before examining the normative properties of the equilibrium, we check some positive properties of the equilibrium.

First, the expected amount of fake reviews is increasing in a. This is simply because the marginal benefit of fake reviews in the future would increase if the platform loosens the censorship policy. The model does not guarantee a positive amount of fake reviews in general, but it is also shown that the expected amount of fake reviews is positive under some parameters.

Proposition 4. *$E[F_t]$ increases with L and L increases with a. Furthermore, $E[F_t] \geq 0$ holds for sufficiently large a.*

Thus, the model can represent a reasonable situation under some parameters where fake reviews have non-trivial effect (i.e., a is significantly high). There still remains a small probability that F_t becomes negative due to the normal distribution, but the model can approximate a reasonable distribution of the fake reviews, under which the negative revenue is rarely observed, as shown in Fig. 2.2.

The precision of "organic" feedback from normal customers also monotonically changes the expected amount of fake reviews. When the organic feedback from customers varies a lot, it is hard for the seller to manipulate the reputation because a boosted rating is attributed to a large variation in the feedback.

Proposition 5. *$E\left[F_t\right]$ is decreasing in $\left(\frac{\sigma_\xi}{\sigma_\theta}\right)$.*

Even though a stringent policy decreases the expected amount of fake reviews, as shown in Proposition 4, it does not imply that the seller's strategy gets closer to the no-fake strategy of $\{\alpha,\ \beta,\ \delta\} = \{0,,\ 0,\ 0\}$. Moreover, the stringent policy might have unintentional effects of increasing the absolute value of the equilibrium coefficients.

Proposition 6. *$|\alpha|$ increases in $\frac{\tau}{c}$ and decreases in $\frac{\sigma_\xi}{\sigma_\theta}$. $|\beta|$ decreases in a and increases in $\left(\frac{\sigma_\xi}{\sigma_\theta}\right)$.*

Under a stringent policy (small a), the marginal benefit of fake review decreases because fake reviews are reflected less in the rating; but at the same time, the dependence of the marginal benefit on the current reputation also decreases. Mathematically, the second term of $\beta = -\frac{\tau}{c} + \frac{\tau}{c}\frac{-a\beta\lambda}{(-a\beta\lambda+r+2\phi)}$ decreases while the marginal cost still depends on the current reputation regardless of the censoring policy. Therefore, $|\beta|$ increases owing to the less countervailing effect.

In the proof of the proposition, the intensity of dynamic consideration is also captured by an aggregator $L = -a\lambda\beta$, which is the equilibrium effect on the reputation transition speed. L becomes smaller when the dynamic incentive becomes smaller; thus, α, which only comes from the future marginal benefit, becomes smaller, and $|\beta|$, to which the future marginal benefit only works as a counteracting effect, becomes greater because the present cost reduction incentive prevails. L is shown to be increasing in $\frac{a\tau}{c}$ and decreasing in $\frac{\sigma_\xi}{\sigma_\theta}$.

Lemma 2. *L at the equilibrium increases in $\frac{a\tau}{c}$ and decreases in $\frac{\sigma_\xi}{\sigma_\theta}$. Furthermore, $L \to 0$ as $\frac{a\tau}{c} \to 0$ and $L \to \infty$ as $\frac{a\tau}{c} \to \infty$.*

This concludes Proposition 6. α does not necessarily increase in a because α is a function in a and L, so the change in a affects directly and indirectly via L, and the net impact is not clear. $|\beta|$ does not necessarily decrease in $\frac{\tau}{c}$ for an analogous reason even though a limit of $\tau \to 0$ is known.

Proposition 6 implies less signaling (smaller α) and more distortion in the effective transition speed of the rating (greater $|\beta|$) when the aggregator on the strategic effect L is small. This suggests less information from the rating system when the strategic effect L is small. In the following section, we formally examine this effect.

Some limits of the equilibrium strategy are worth noting before jumping into a normative analysis. Since the negative β comes from the interaction term in the cost of the fake reviews, whose coefficient is τ, β approaches zero as τ approaches zero. At the same time, α also approaches zero because the complementarity of θ and M is caused by future cost savings via negative β. In this limit, the fake reviews become constant as in Holmström (1999). This is summarized in the following proposition.

Proposition 7. $|\alpha|$, $|\beta| \to 0$ *as* $\tau \to 0$.

2.3.3 Optimal Rating System for Rational Consumers

In this study, we focus on the informativeness of the rating system as a normative criterion for two reasons. First is from the viewpoint of consumer protection: as the rating system gets more informative about the quality of a product, the price is likely to be close to the underlying quality. Thus, it becomes less likely that consumers would face huge regret from the purchase of the product. Second is from the viewpoint of the platform: the informativeness of the rating is crucial to attracting consumers in the long run. If consumers find it uninformative, they, as well as the sellers, can move to other platforms, given less consumers in the market. Thus, the informativeness of the rating would be the first priority when the platform controls it.

Since rational customers can form an unbiased estimate from any current rating, $M_t = E[\theta_t|Y_t]$, the informativeness of the rating is defined by the variance of the customer's estimate of quality. Owing to the normality assumption, this is rewritten as $Var\left(\theta_t|Y_t\right) = Var\left(\theta_t\right)\left(1-\rho^2\right)$, where ρ^2 is the correlation between θ_t and Y_t. Therefore, we use ρ^2 as the criterion for the informativeness of the rating.

Given an equilibrium strategy, the stochastic differential equations—Eqs. (2.1) and (2.5)—give us ρ^2 as a function of the parameters and the equilibrium strategy. Therefore, the change of a parameter directly affects ρ^2 and indirectly affects it via a change of the equilibrium strategy. Fortunately, by representing the equilibrium coefficients α and β as functions of the equilibrium aggregator $L = a\beta\lambda$, all the direct and indirect effects of the censorship (a) are expressed as an effect through L. Comparative statics about other parameters, such as ϕ and σ_ξ/σ_θ, can also be examined by the indirect effect through L and the direct effect.

Lemma 3. *At the equilibrium, ρ^2 is expressed as a function:*

$$\rho^2\left(L;\,\phi,\,\kappa,\,\sigma_\xi,\,\sigma_\theta,\,r,\,bq\right) = \frac{(\phi+L)}{(\kappa+\phi+L)}\frac{\left(A\left(L;\,\phi,\,\kappa,\,r,\,bq\right)+1\right)^2}{\left(\left(A\left(L;\,\phi,\,\kappa,\,r,\,bq\right)+1\right)^2+\kappa(\sigma_\xi/\sigma_\theta)^2(\kappa+\phi L)\right)}$$

on which a, c, τ have effects only through L.

$A\left(L;\,\phi,\,\kappa,\,r,\,bq\right)$ summarizes all the direct and indirect effects of a on the informativeness as a function of L.

2.3.3.1 Filtering/Censoring Reviews

First, we analyze the impact of a filtering/censoring policy, a. Do fake reviews damage the informativeness of the rating system compared with the case without fake reviews? Does filtering or censoring the reviews (i.e., decrease in a) increase the rating's informativeness?

As a benchmark, we derive informativeness *without* fake reviews. By construction, we can do this by letting $\alpha = \beta = \delta = 0$.[13] The same informativeness is also replicated by setting $L = 0$ in $\rho^2\left(L;\,\phi,\,\kappa,\,\sigma_\xi,\,\sigma_\theta\right)$ to make it easier to compare with the informativeness

[13] Actually, δ does not enter in the formula for the informativeness, so $\delta = 0$ does not matter in terms of the informativeness.

at the equilibrium.

Lemma 4. *$\rho^2(0; \phi, \kappa, \sigma_\xi, \sigma_\theta)$ coincides with ρ^2 under the no-fake strategy.*

Note that $L = 0$ does not necessarily mean $\alpha = \beta = \delta = 0$. For instance, L approaches 0 as a approaches 0; but at the same time, β converges to some negative value. The lemma says that even under such a situation, informativeness is the same as that without fake reviews. Lemma 2, which is about the relationship between L and parameters, and Lemma 4 together lead us to the following proposition:

Proposition 8. *The informativeness of the rating system in equilibrium converges to that of the "no-fake" strategy as $\frac{a\tau}{c} \to 0$.*

Thus, even though the equilibrium strategy at the limit of $\frac{a\tau}{c}$ is not necessarily the no-fake strategy, the informativeness converges to that of the no-fake strategy.

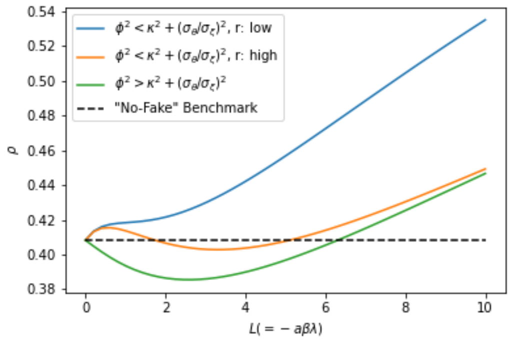
By analyzing the behavior of $\rho^2(L; \phi, \kappa, \sigma_\xi, \sigma_\theta)$ with respect to L, we can conclude that the informativeness can be even higher under some parameters where a positive amount of the fake reviews is expected. In other words, stringent censorship can decrease the informativeness of the rating system.

Proposition 9. *The equilibrium strategy is more informative than no-fake strategy under a set of parameters such that*

1. *$\frac{a\tau}{c}$ is sufficiently large, or*
2. *$\frac{a\tau}{c}$ is sufficiently small and $\phi^2 < \kappa^2 + \frac{\sigma_\theta^2}{\sigma_\xi^2}$*

Fig. 2.3 shows the behavior of ρ^2 with respect to L. The first part of the proposition comes from the fact that ρ^2 converges to 1 as L approaches infinity. Since L is increasing $\frac{a\tau}{c}$ from zero to infinity, the equilibrium informativeness surpasses that of the no-fake benchmark at some point as $\frac{a\tau}{c}$ increases. The second part is derived from the behavior of ρ^2 around $L = 0$. The derivative of ρ^2 with respect to L is determined by the relative size of ϕ^2 and

Figure 2.3: Change of the informativeness in the aggregator L



The graph indicates that the informativeness is (i) increasing in L if ϕ and r are relatively low, (ii) increasing in L around zero, then decreasing, and then increasing if ϕ is relatively low but r is relatively high, and (iii) decreasing in L around zero and then increasing in L if ϕ is relatively high. It also indicates the rating becomes more informative than the no-fake benchmark as L gets large.

$(\kappa^2 + \sigma_\theta^2/\sigma_\xi^2)$: If $\phi^2 < \kappa^2 + \frac{\sigma_\theta^2}{\sigma_\xi^2}$, then ρ^2 decreases in L; thus, decreases in $\frac{a\tau}{c}$. [14]

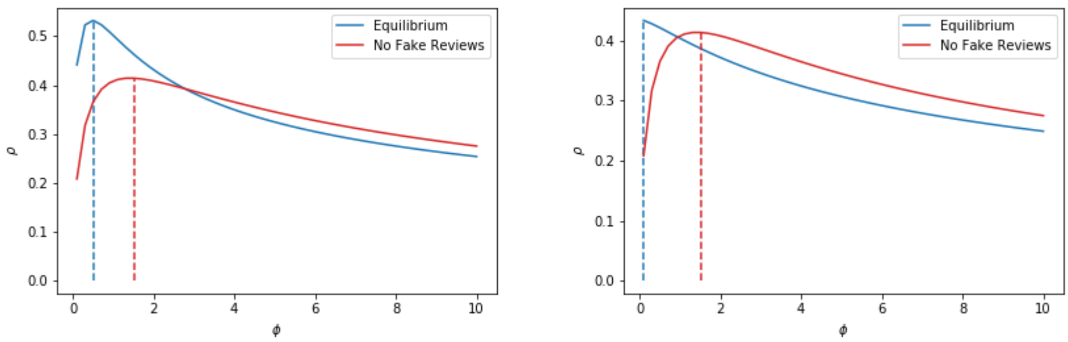
The intuition of this proposition consists of two parts: (i) As mentioned in Subsection 2.3.2.1, the sensitivity of fake reviews to θ_t decreases as the strategic effect L decreases. Thus, the strict censorship policy, which reduces the equilibrium aggregator L, decreases the signaling effect of the fake reviews. (ii) Meanwhile, $L > 0$ increases the effective transition speed of reputation to $\phi + L$. It can be good or bad in terms of informativeness, depending on the original transition speed, ϕ. More specifically, the threshold of $\sqrt{\kappa^2 + \sigma_\theta^2/\sigma_\xi^2} \equiv \phi^0$ is the informativeness-maximizing ϕ, given no fake reviews. Therefore, if ϕ is smaller than ϕ^0, the faster transition improves informativeness. It turns out that the first effect dominates in the case of a large L and the second effect dominates in the case of L close to zero.

2.3.3.2 Weights on New/Previous Reviews

Next, we analyze the optimal weights of the new and old reviews. Again, the informativeness without fake reviews is expressed by $\rho^2(0;\, \phi,\, \kappa,\, \sigma_\xi,\, \sigma_\theta)$. Therefore, the optimal weight at this benchmark is simply characterized by $\frac{\partial}{\partial \phi}\rho^2(0;\, \phi,\, \kappa,\, \sigma_\xi,\, \sigma_\theta) = 0$. Let ϕ^0 be the solu-

[14]Note that $E[F_t]$ is increasing in L and positive for large L (by Proposition 4). Thus, the high informativeness is not due to negative fake reviews, but associated with the positive amount of fake reviews.

Figure 2.4: Change of the informativeness in ϕ



The left panel shows change of the informativeness in ϕ when r is relatively low, while the right panel shows that of a relatively high r. The informativeness is maximized at a lower ϕ under the equilibrium than the maximizer under the no-fake benchmark.

tion to this equation. Meanwhile, at equilibrium, ϕ changes the equilibrium aggregator L. Thus, the optimal weight at equilibrium is characterized by $\frac{d\rho^2}{d\phi} = \frac{\partial}{\partial \phi}\rho^2 \left(L;\, \phi,\, \kappa,\, \sigma_\xi,\, \sigma_\theta\right) + \frac{\partial}{\partial L}\rho^2 \left(L;\, \phi,\, \kappa,\, \sigma_\xi,\, \sigma_\theta\right) \frac{dL}{d\phi} = 0$. Let the solution of this equation be ϕ^*. Now, we have the following proposition. [15]

Proposition 10. $\frac{d\rho^2}{d\phi} < 0$ *at* $\phi = \phi^0$. *Furthermore, if* r *is sufficiently small, then*

$$\rho^2 \left(L\left(\phi^*\right);\, \phi^*,\, \kappa,\, \sigma_\xi,\, \sigma_\theta\right) > \rho^2 \left(0;\, \phi^0,\, \kappa,\, \sigma_\xi,\, \sigma_\theta\right).$$

The first part of the proposition states that the platform should reduce the speed of transition ϕ, given the existence the fake reviews. Intuitively, this is explained as follows. At equilibrium, the transition of the rating score Y_t is $\phi + L$ where L is non-negative. Therefore, to cancel the strategic impact on the transition speed, the platform should decrease ϕ, compared with the no-fake benchmark ϕ^0. Again, the transition speed is interpreted as the relative weight of the new information. At the equilibrium, the number of fake reviews decreases in the current rating; thus, the fake reviews cancel the past performance. In other words, the new information is effectively weighted more than the platform intends. Thus,

[15] ϕ corresponding to disaggregated information, ϕ^d, is an alternative benchmark as in Bonatti and Cisternas (2019). In this model, we obtain a mixed result for the comparison of ϕ^* and ϕ^d. See the appendix for more details.

the platform can increase the informativeness by adjusting it downward.

The second part of the proposition is even more striking. If the seller is sufficiently concerned about the future, the platform can achieve higher informativeness than the no-fake review benchmark by adjusting the speed of updating the rating. The implication is similar to Proposition 5, but is slightly different from it. The right panel of Fig. 3 illustrates that informativeness at equilibrium is greater than that without fake reviews under some parameters (e.g., $\phi = 0.9$), as shown in Proposition 5, but it can still be lower than the maximum informativeness without fake reviews (maximized around $\phi = 1.6$). The second part of Proposition 6 states that even when we compare the maximum informativeness of the rating with and without fake reviews, the one with fake reviews will be higher if the seller cares enough about the future as shown in the left panel of Fig. 3.

2.3.3.3 The Precision of Genuine Reviews

Lastly, we examine the impact of the precision of organic feedback, $\frac{\sigma_\xi}{\sigma_\theta}$. As discussed in Subsection 2.3.2.1, increasing $\frac{\sigma_\xi}{\sigma_\theta}$ and decreasing a have similar effects on the equilibrium strategy. However, they differ in terms of the impact on informativeness. This is because a affects informativeness only through the equilibrium aggregator L, but $\frac{\sigma_\xi}{\sigma_\theta}$ affects informativeness directly as well. Intuitively, if the reviews consist of less precise feedback (i.e., higher $\frac{\sigma_\xi}{\sigma_\theta}$), the rating score, by definition, is less informative about quality. The indirect effect consists of two parts, like the comparative statics over a: (i) Higher $\frac{\sigma_\xi}{\sigma_\theta}$ implies a smaller strategic effect L, which implies less signaling effect. (ii) $L > 0$ effectively increases the rating transition to $\phi + L$. The following proposition shows that the direct effect and the first indirect effect dominate the second indirect effect for any parameter.

Proposition 11. *The informativeness at the equilibrium decreases in* $\frac{\sigma_\xi}{\sigma_\theta}$.

Thus, the precise organic feedback increases informativeness even though it comes with more fake reviews.

2.4 Rating Design for Naive Consumers

The model with rational consumers is a standard starting point for any economic model, but in the context of customer reviews, it is natural to consider the impact on naive consumers who do not expect any fake reviews. The regulator often tries to protect customers from fake reviews, with the assumption that the fake reviews can fool or at least confuse consumers. In this section, we assume that some consumers do not expect any fake reviews on the platform. They are modeled by assuming that the reputation (and the price) is characterized as $\widetilde{M_t} = \mu + \widetilde{\lambda} [Y_t - \widetilde{\nu}]$ where $\widetilde{\lambda}$ and $\widetilde{\nu}$ are characterized by the stochastic differential equations Eqs. (2.1) and (2.5), where $\alpha = \beta = \delta = a = 0$. Meanwhile, the long-lived seller faces the same problem as in the previous chapter, except for the definition p_t.

2.4.1 Model / Equilibrium Characterization

In this section, the price is assumed to be a convex combination of a rational reputation M and a naive reputation $\widetilde{M}$.

$$\begin{aligned} p &= \eta M + (1-\eta)\widetilde{M} \\ &= \eta \{\mu + \lambda [Y_t - \nu]\} + (1-\eta)\left\{\mu + \lambda^{naive}\left[Y_t - \nu^{naive}\right]\right\} \\ &= \mu - \left(\eta\lambda\nu + (1-\eta)\lambda^{naive}\nu^{naive}\right) + \left(\eta\lambda + (1-\eta)\lambda^{naive}\right) Y_t \end{aligned}$$

One interpretation is that each consumer can be partially rational. Their expectation about the quality of the product is somewhere in between the totally sophisticated expectation and the totally naive expectation. The rationality of each consumer is captured by η.

Another interpretation is that η is the ratio of rational consumers among all consumers. Then, the market price is set somewhere in between the rational expectation and the naive expectation. When the ratio of rational consumers increases, it converges to the rational expectation. The linear specification captures such a relationship in a simple manner. Furthermore, it can be rationalized as an equilibrium price given a specific utility function of buyers. Suppose that there are n consumers in the market and $\eta \cdot n$ of them are rational and the other $(1-\eta)\cdot n$ are naive. Consumer $i \in [0,\, n]$ feels $u_{t,i} = \theta_t + \epsilon_{t,i} - p_t$ if the consumer buys the product, and 0 otherwise, where $\epsilon_{t,i}$ is identically and independently distributed. Rational and naive consumers differ only in terms of how they form their expectation based on the same observation of the rating Y_t. Conditional on Y_t, a rational consumer purchases the product if and only if $M_t + \epsilon_i - p \geq 0$, while a naive consumer purchases it if and only if $\widetilde{M}_t + \epsilon_i - p \geq 0$. Therefore, the total demand function is expressed as

$$\eta \cdot n \cdot (1 - F(p - M)) + (1-\eta)\cdot n \cdot \left(1 - F\left(p - \widetilde{M}\right)\right)$$

where $F(p)$ is the c.d.f. of the random variable ϵ_i. By letting $n = 2q$ and assuming that ϵ_i is distributed uniformly and symmetrically around zero. We obtain $p = \eta M + (1-\eta)\widetilde{M}$ to clear the market.

In this section, we consider a linear strategy $F_t = \hat{\alpha}\theta_t + \hat{\beta} Y_t + \hat{\gamma}$ and the HJB equation with state variables θ and Y because Y keeps track of both M and $\widetilde{M}$ in a simple manner:

$$\begin{aligned} rV(\theta,\, Y) = {} & \sup_{F\in\mathbb{R}} (1-\tau)\, p\cdot q - \tau p \cdot F - \frac{c}{2}F^2 \\ & - \kappa(\theta - \mu) V_\theta \\ & + \{-\phi Y_t + aF_t dt + bq\theta_t\} V_Y \\ & + \frac{\sigma_\theta^2}{2} V_{\theta\theta} \\ & + \frac{b^2 q^2 \sigma_\xi^2}{2} V_{YY} \end{aligned} \tag{2.10}$$

The following theorem states that, even with credulous consumers, we have the existence and uniqueness given the same condition as the baseline model.

Theorem 3. *For any $\eta \in [0,1]$, a stationary linear Markov equilibrium always exists. For any equilibrium, $\alpha > 0$, $\beta \in \left(-\frac{\tau}{c},\, 0\right)$, $\lambda > 0$ and $L > 0$ hold. Furthermore, if $h'(L) < 0$ holds, then such an equilibrium is unique and the equilibrium coefficients α, β, and δ are* *differentiable in the parameters.*

$h'(L) < 0$ *holds for any* $L > 0$ *if* $6\kappa\phi + 4r^2 + 2\kappa r + 17r\phi + 19\phi^2 > \kappa^2$.

In addition, surprisingly, the existence of naive consumers reduces the seller's strategic behavior.

Proposition 12. *The equilibrium with naive consumers ($\eta \in [0,1)$) generates a smaller $|\alpha|$, a larger $|\beta|$,and a smaller $E[F_t]$ compared with the equilibrium without naive consumers ($\eta = 1$).*

This is because rational consumers are more sensitive to the change in ratings compared with naive consumers. Rational consumers know that the rating is boosted, but they also know that the rating is boosted more by a firm with a high quality product. Therefore, rational consumers attribute the boosted rating to high quality, and set a high price for such a boosted rating. Meanwhile, naive consumers are unaware of such a strategic correlation between quality and a rating. Therefore, with naive consumers, the price is less responsive to the boost of the ratings; thus, the seller faces a smaller *marginal* benefit of fake reviews, which leads fewer fake reviews in expectation.

Readers might wonder why the seller does not become more exploitative of naive consumers. This is simply because the fake review strategy against rational consumers generates more fake reviews for different reasons than exploiting consumers. If only a small number of naive consumers exist and observe the ratings, naive consumers would form even more biased estimates because the seller makes more fake reviews to send a good signal to rational consumers.

2.4.2 Optimal Rating System for Naive Consumers

Criteria: Bias in the Reputation. In this section, we evaluate the impact of fake reviews on naive consumers. To do so, we introduce a bias in the naive consumer's expectation caused by the boosted rating:

$$\begin{aligned} Bias &\equiv E\left[\widetilde{M}_t - \theta_t\right] \\ &= E\left[\mu - \theta_t + \widetilde{\lambda}\left[Y_t - \widetilde{\nu}\right]\right] \\ &= \widetilde{\lambda}\left[\nu - \widetilde{\nu}\right] \end{aligned}$$

where $\widetilde{\lambda}$ is the sensitivity of the reputation to the rating, and ν and $\widetilde{\nu}$ are the actual mean of the rating and the estimate of the mean of the rating by the naive consumers, respectively. The decomposition of the bias, as shown above, is intuitive: the positive bias is due to the boosted reputation. Because naive consumers do not expect any fake reviews, they interpret a high rating ($Y_t > \widetilde{\nu}$) as a result of high quality, even though it is actually the average level of the rating at equilibrium ($Y_t = \nu > \widetilde{\nu}$).

Therefore, as long as the seller makes a positive amount of fake reviews (in expectation) to boost the rating, naive consumers are positively biased. This intuition is verified in the following lemma.

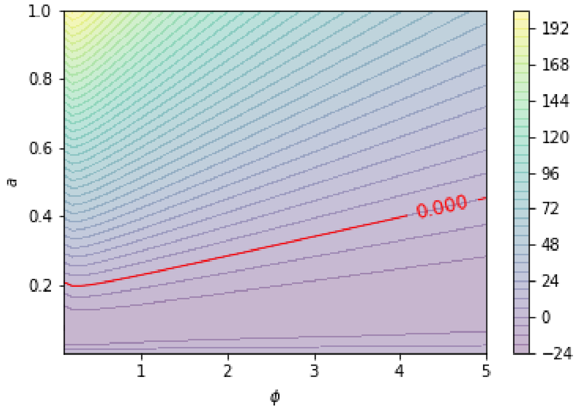
Lemma 5. *$Bias \geq 0$ if and only if $E\left[F_t\right] \geq 0$.*

2.4.2.1 Filtering/Censoring Reviews

In the following section, for the sake of tractability, I focus on the case of $\eta = 0$, where only naive consumers exist in the market. Numerical exercises for $\eta \in (0,\, 1)$ can be found in the Appendix.

First, we examine the impact of a filtering policy, for which regulators are arguably concerned the most. The following proposition provides a theoretical background of a stringent policy that procect the naive customers. Note that even though the statement seems pretty intuitive, it is not trivial because the model predicts a non-monotonouse relationship between censorship and bias in general. Fortunately, in a realistic range of parameters, where naive consumers suffer from a positive bias in their reputation, stringent censorship will reduce such a bias.

Figure 2.5: Impact of censorship intensity and the weights of reviews on naive consumer's bias.



Proposition 13. *Suppose Bias* ≥ 0*; then, Bias increases in a.*

Combined with Lemma 5, the condition for a stringent policy to work for naive consumers is translated as the condition of a measure observable by the platform.

Corollary 7. *Stringent censorship reduces the bias of naive consumers whenever the expected amount of fake reviews is positive.*

Thus, as long as a positive number of fake reviews are observed, the stringent policy is beneficial for naive consumers, even though it can reduce informativeness of rating for the rational consumers.

2.4.2.2 Weights on New/Previous Reviews

As shown in Fig. 2.5, the bias tends to be hump-shaped in ϕ. This is intuitive because fake reviews would be effective only when the rating is believed to be informative by the consumers so that the consumers react to the rating. Since the informativeness is hump shaped in ϕ, so is the bias caused by the fake reviews. This emphasizes that the trade-off between bias and informativeness can be an inherent feature of fake reviews.

Some readers might want an integrated criteria for bias and the informativeness. The mean squared error (MSE) is a natural candidate. It does not provide a clear-cut prediction, but a simulation of MSE is provided in the Appendix.

2.5 Conclusions

In this study, the effects of fake reviews on rational and credulous consumers are analyzed. The key assumption is that a high reputation results in a high cost of fake reviews. This is rationalized by the high reimbursement to reviewers or high demand for the product and the substantial, authentic feedback crowding-out the fake reviews.

At equilibrium, the amount of fake reviews increases (decreases) as product quality (firm reputation) increases (improves), which implies difficulties in the empirical analysis of signaling promotion. Stringent censorship reduces the expected amount of fake reviews, while decreasing the signaling effect and increasing the effective transition speed of the rating.

This leads to a normative result wherein the rating under a less strict filtering policy can be more informative than the rating under a strict policy or the rating with no fake reviews. In terms of the weights of new and old information in a rating system where fake reviews exist, the platform should reduce the weight of new information to maximize the informativeness of the rating, compared with a rating system that does not have fake reviews. Since fake reviews effectively attenuate the impact of old information and increase the relative weight of the new information, the platform should make the necessary adjustments.

The existence of credulous consumers decreases the expected amount of fake reviews since they are less responsive to the rating without being aware of the positive relationship between fake reviews and the quality. Moreover, they are vulnerable to fake reviews and pay more than the true quality in expectation. The model predicts that as long as a positive amount of the fake reviews is observed, the regulator or the platform can reduce such biased behaviors by enhancing censorship.

The results emphasize that regulators or platforms would face a trade-off between the degree of informativeness and the bias caused by fake reviews. As long as the rating is considered informative, the incentive to make fake reviews arises.

APPENDIX

2.A Proofs

Proof of Theorem 2. By $M_t = \mu + \lambda [Y_t - \nu] \Leftrightarrow \lambda Y_t = M_t - \mu + \lambda\nu$, and the linear strategy $F_t = \alpha\theta_t + \beta M_t + \delta\mu$, the increment of M_t is written as

$$\begin{aligned} dM_t &= d(\lambda Y_t) \\ &= (-\phi + a\lambda\beta) M_t dt \\ &+ (a\lambda\alpha + bq\lambda)\theta_t dt \\ &+ (\phi\mu - \phi\lambda\nu + a\lambda\delta\mu) dt \\ &+ bq\lambda\sigma_\xi dZ_t^\xi \end{aligned}$$

Now, we look for a quadratic value function

$$V = v_0 + v_1\theta + v_2 M + v_3\theta^2 + v_4 M^2 + v_5\theta M \tag{2.11}$$

satisfying the HJB equation:

$$\begin{aligned} rV(\theta, M) &= \sup_{F\in\mathbb{R}} (1-\tau) M\cdot q - \tau M \cdot F - \frac{c}{2}F^2 \\ &- \kappa(\theta - \mu) V_\theta \\ &+ \left\{a\lambda F + bq\lambda\theta - \phi\left[M - \bar{\theta} + \lambda\bar{Y}\right]\right\} V_M \\ &+ \frac{\sigma_\theta^2}{2} V_{\theta\theta} \\ &+ \frac{bq\lambda^2\sigma_\xi^2}{2} V_{MM} \end{aligned}$$

By the first-order condition,

$$
\begin{aligned}
0 &= -\tau M - cF + a\lambda V_M \\
\Leftrightarrow F &= -\frac{\tau}{c}M + \frac{a\lambda}{c}V_M \\
&= \frac{a\lambda}{c}v_5\theta + \left(2\frac{a\lambda}{c}v_4 - \frac{\tau}{c}\right) M + \frac{a\lambda}{c}v_2
\end{aligned}
$$

By matching coefficients with $F = \alpha\theta + \beta M + \delta\mu$,

$$
\begin{aligned}
\alpha &= \frac{a\lambda}{c}v_5 \\
\beta &= 2\frac{a\lambda}{c}v_4 - \frac{\tau}{c} \\
\delta\mu &= \frac{a\lambda}{c}v_2
\end{aligned}
$$

By solving them for v_k's,

$$\frac{c}{a\lambda}\alpha = v_5 \tag{2.12}$$

$$\frac{c}{2a\lambda}\left(\beta + \frac{\tau}{c}\right) = v_4 \tag{2.13}$$

$$\frac{\delta\mu c}{a\lambda} = v_2 \tag{2.14}$$

By the Envelop condition w.r.t. M,[16]

$$
\begin{aligned}
rV_M = &(1-\tau)\, q - \tau F \\
&- \kappa\,(\theta - \mu)\, V_{\theta M} \\
&- \phi V_M \\
&+ \{a\lambda F + bq\lambda\theta - \phi\,[M - \mu + \lambda\nu]\}\, V_{MM}
\end{aligned}
$$

By inserting the derivatives of eq.(2.11) and equating the coefficients of θ, M, and constants

[16]The envelop condition w.r.t. θ gives conditions characterizing v_1 and v_3, and one characterizing v_5, which coincides with the condition from the envelop condition w.r.t. M.

on LHS and RHS,

$$(r+\phi)\,v_5 = -\tau\alpha - \kappa v_5 + \{a\lambda\alpha + bq\lambda\}\,2v_4$$
$$2\,(r+\phi)\,v_4 = -\tau\beta + \{a\lambda\beta - \phi\}\,2v_4$$
$$(r+\phi)\,v_2 = (1-\tau)\,q - \tau\delta\bar{\theta} + \kappa\mu v_5 + \{a\lambda\delta\mu + \phi\mu - \phi\lambda\nu\}\,2v_4$$

Then, inserting eq.(2.12) to eq (2.14),

$$(r+\phi+\kappa)\,\frac{c}{a\lambda}\alpha = -\tau\alpha + \{a\lambda\alpha + bq\lambda\}\,2\frac{c}{2a\lambda}\left(\beta+\frac{\tau}{c}\right) \tag{2.15}$$
$$2\,(r+\phi)\,\frac{c}{2a\lambda}\left(\beta+\frac{\tau}{c}\right) = -\tau\beta + \{a\lambda\beta - \phi\}\,2\frac{c}{2a\lambda}\left(\beta+\frac{\tau}{c}\right) \tag{2.16}$$
$$(r+\phi)\,\frac{\delta\mu c}{a\lambda} = (1-\tau)\,q - \tau\delta\mu + \kappa\mu\frac{c}{a\lambda}\alpha + \{a\lambda\delta\mu + \phi\mu - \phi\lambda\nu\}\,2\frac{c}{2a\lambda}\left(\beta+\frac{\tau}{c}\right) \tag{2.17}$$

By combining with the consistency of λ: $\lambda = \frac{(a\alpha+bq)\sigma_\theta^2(\phi-a\beta\lambda)}{(\phi-a\beta\lambda+\kappa)\kappa bq\sigma_\xi^2+\sigma_\theta^2(a\alpha+bq)^2}$, we can characterize α, β, δ, λ. In the following, I do so by using an aggregator $L=-a\beta\lambda$ so that the stationarity condition is easier to verify. First, by replacing λ to $-\frac{L}{a\beta}$ in the above four equations,

$$0 = -\frac{bq(\beta c+\tau)}{a} + \alpha\tau - \alpha(\beta c+\tau) - \frac{\alpha\beta c\kappa}{L} - \frac{\alpha\beta c\phi}{L} - \frac{\alpha\beta cr}{L} \tag{2.18}$$
$$0 = \beta\tau - \beta(\beta c+\tau) - \frac{2\beta\phi(\beta c+\tau)}{L} - \frac{\beta r(\beta c+\tau)}{L} \tag{2.19}$$
$$0 = \frac{\nu\phi(\beta c+\tau)}{a} - \delta\mu(\beta c+\tau) \tag{2.20}$$
$$+\frac{\alpha\beta c\kappa\mu}{L} - \frac{\beta c\delta\mu\phi}{L} + \frac{\beta\mu\phi(\beta c+\tau)}{L} - \frac{\beta c\delta\mu r}{L} + \delta\mu\tau + q\tau - q \tag{2.21}$$
$$-\frac{L}{a\beta} = \frac{\sigma_\theta^2(L+\phi)(a\alpha+bq)}{\sigma_\theta^2(a\alpha+bq)^2 + \kappa bq\sigma_\xi^2(\kappa+L+\phi)} \tag{2.22}$$

By solving (2.19) for β, we get $\beta = -\frac{\tau}{c}\left(\frac{r+2\phi}{r+2\phi+L}\right) \equiv B(L)$. By inserting this into (2.18) and solving it for α, we get $\alpha = \frac{bq}{a}\frac{L^2}{(r+2\phi)(r+\phi+\kappa+L)} \equiv A(L)$. By plugging $\beta = B(L)$ and

$\alpha = A(L)$ into (2.22), we obtain an equation characterizing L:

$$-\frac{L}{aB(L)} = \frac{\sigma_\theta^2(L+\phi)(aA(L)+bq)}{\sigma_\theta^2(aA(L)+bq)^2+\kappa bq\sigma_\xi^2(\kappa+L+\phi)}$$

Rearranging it , we get

$$\begin{aligned} 1 &= \frac{\sigma_\theta^2(L+\phi)(aA(L)+bq)}{\sigma_\theta^2(aA(L)+bq)^2+\kappa bq\sigma_\xi^2(\kappa+L+\phi)}\frac{-aB(L)}{L} \\ &\equiv h(L) \end{aligned}$$

To evaluate $h(L)$, the sign of L is useful to characterize.

Lemma 6. *$\beta < 0$ and $L > 0$ under the linear stationary Gaussian equilibrium.*

Proof. By the stationarity, we must have $\phi + L > 0$. Then,

$$\begin{aligned} \beta &= -\frac{\tau}{c}\left(\frac{r+2\phi}{r+2\phi+L}\right) \\ &= -\frac{\tau}{c}\left(\frac{r+2\phi}{r+\phi+\phi+L}\right) \\ &< 0 \end{aligned}$$

Then, $\alpha = \frac{bq}{a}\frac{L^2}{(r+2\phi)(r+\phi+\kappa+L)} > 0$ and $\lambda = \frac{(a\alpha+bq)\sigma_\theta^2(\phi+L)}{(\phi+L+\kappa)b^2q^2\kappa\sigma_\xi^2+\sigma_\theta^2(a\alpha+bq)^2} > 0$. Now, we can conclude $-a\beta\lambda \equiv L > 0$. □

Now, it is shown that $\lim_{L\to 0} h(L) = \infty$ and $\lim_{L\to\infty} h(L) = 0$. Then, combined with the continuity of $h(L)$, there exist some L such that $h(L) = 1$. The uniqueness is proved by checking whether $h'(L) < 0$ holds. It is shown that

$$h'(L) = -h_1(L)\left\{h_2(L) + L^4\left(-\kappa^2+6\kappa\phi+4r^2+2\kappa r+17r\phi+19\phi^2\right)\right\}$$

where $h_1(L), h_2(L) > 0$ for all $L > 0$. Thus, $6\kappa\phi+4r^2+2\kappa r+17r\phi+19\phi^2 > \kappa^2$ is sufficient for $h'(L) < 0$ □

Proof of Lemma 2. By plugging $\alpha(L)$ and $\beta(L)$ in to h, it can be written as $h(L) = \frac{a\tau}{c}\frac{h_3}{L(L+r+2\phi)\left(h_4+\left(\sigma_\xi/\sigma_\theta\right)^2 h_5\right)}$ □

where $h_3 = (L+\phi)(r+2\phi)^2(\kappa+L+r+\phi)\left(L^2+L(r+2\phi)+(r+2\phi)(\kappa+r+\phi)\right)$, $h_4 = bq\left(L^2+L(r+2\phi)+(r+2\phi)(\kappa+r+\phi)\right)^2$, $h_5 = \kappa(r+2\phi)^2(\kappa+L+\phi)(\kappa+L+r+\phi)^2$. Note that h_3, h_4, h_5 are positive and independent of a and σ_ξ/σ_θ. Thus, h is increasing in $\frac{a\tau}{c}$ and decreasing in σ_ξ/σ_θ. Since $h'(L) < 0$ is shown in the proof of Theorem 1, the implicit function theorem tells that L is increasing in a and decreasing in σ_ξ/σ_θ. Furthermore, $h(L) \to \infty$ if L is bounded above and $\frac{a\tau}{c} \to \infty$. Thus, to satisfy the equilibrium condition: $1 = h(L)$, L goes infinite as $\frac{a\tau}{c}$ goes infinite. Similarly, $h(L) \to 0$ if L is bounded away from zero and $\frac{a\tau}{c} \to 0$. Thus, L goes infinite as $\frac{a\tau}{c}$ goes infinite to satisfy the equilibrium condition.

Proof of Proposition 1 and 2. Since $E[M_t] = E[E[\theta_t|Y_t]] = \mu$, we have

$$E[F_t] = E[\alpha\theta_t + \beta M_t + \delta\mu] = (\alpha+\beta+\delta)\mu.$$

By expressing α, β, δ as a function of the equilibrium aggregator L, it is written as

$$E[F_t] = \frac{cLq(1-\tau)(L+r+2\phi) - \mu\tau^2\left(r^2+3r\phi+2\phi^2\right)}{c\tau\left(L^2+L(r+2\phi)+r^2+3r\phi+2\phi^2\right)}$$

and the partial derivative with respect to L is

$$\frac{\partial E[F_t]}{\partial L} = \frac{\left(r^2+3r\phi+2\phi^2\right)(2L+r+2\phi)\left(cq(1-\tau)+\mu\tau^2\right)}{c\tau\left(L^2+L(r+2\phi)+r^2+3r\phi+2\phi^2\right)^2} > 0.$$

Since a, σ_ξ, and σ_θ affects $E[F_t]$ only through the aggregator L, we can show the effects of a and $\frac{\sigma_\xi}{\sigma_\theta}$ by analyzing the sign of $\frac{dL}{da}$ and $\frac{dL}{d(\sigma_\xi/\sigma_\theta)}$. By Lemma 2, we can conclude $E[F_t]$ increasing in a and decreasing in $\frac{\sigma_\xi}{\sigma_\theta}$.

Since $E[F_t] > 0$ for sufficiently large L and $L \to \infty$ as $a \to \infty$, $E[F_t] > 0$ holds for sufficiently large a. □

Proof of Proposition 3. The equilibrium condition gives $\alpha = \frac{bq}{a}\frac{L^2}{(r+2\phi)(r+\phi+\kappa+L)}$ and $\beta = -\frac{\tau}{c}\left(\frac{r+2\phi}{r+2\phi+L}\right)$. Furthermore, it is shown that $\frac{\partial\alpha}{\partial L} > 0$ and $\frac{\partial\beta}{\partial L} > 0$. Then, Lemma 2 concludes the proposition. □

Proof of Lemma 3 and 4. An arbitrary strategy α, β, δ satisfying $\phi - a\beta\lambda$ (not necessarily the equilibrium strategy) generates a stationary distribution. Using the vairance-covariance matrix of the stationary distribution, the informativeness is written as

$$\rho^2 = \frac{(\phi - a\beta\lambda)(a\alpha + bq)^2}{(\kappa + \phi - a\beta\lambda)\left((a\alpha + bq)^2 + \kappa bq\left(\sigma_\xi/\sigma_\theta\right)^2(\kappa + \phi - a\beta\lambda)\right)}$$

Thus, the informativeness without fake reviews is

$$\rho^2 = \frac{\phi(bq)^2}{(\kappa + \phi)\left((bq)^2 + \kappa bq\left(\sigma_\xi/\sigma_\theta\right)^2(\kappa + \phi)\right)}$$

.On the other hand, at the equilibrium, $-a\beta\lambda$ can be replaced to L, and $a\alpha$ is written as a function in L: $a\alpha = bq\frac{L^2}{(r+2\phi)(r+\phi+\kappa+L)}$ such that $a\alpha = 0$ when $L = 0$. Note that a does not appear in the RHS, so the direct and indirect effects of a on $a \cdot \alpha$ are all captured by L. Now the equilibrium informativeness is written as:

$$\rho^2(L;\,\phi,\,\kappa,\,\sigma_\xi,\,\sigma_\theta) = \frac{(\phi + L)(a\alpha + bq)^2}{(\kappa + \phi + L)\left((a\alpha + bq)^2 + \kappa bq\left(\sigma_\xi/\sigma_\theta\right)^2(\kappa + \phi + L)\right)}.$$

Note that $\rho^2(0;\,\phi,\,\kappa,\,\sigma_\xi,\,\sigma_\theta) = \frac{\phi(bq)^2}{(\kappa+\phi)\left((bq)^2+\kappa bq\left(\sigma_\xi/\sigma_\theta\right)^2(\kappa+\phi)\right)}$ coincides with the informativeness without fake reviews. This concludes Lemma 4. □

Proof of Proposition 5. The first part is proved by the limit as $L \to \infty$:

$$
\begin{aligned}
&\lim_{L\to\infty} \rho^2 \left(L;\, \phi,\, \kappa,\, \sigma_\xi,\, \sigma_\theta\right) \\
=&\lim_{L\to\infty} \frac{(\phi + L)}{(\kappa + \phi + L)} \frac{(a\alpha + bq)^2}{\left((a\alpha + bq)^2 + \kappa bq \left(\sigma_\xi/\sigma_\theta\right)^2 (\kappa + \phi + L)\right)} \\
=&1
\end{aligned}
$$

The second part comes from the derivative of ρ^2 with respect to L around zero. □

Proof of Proposition 6. The optimal ϕ without fake reviews is characterized by

$$
\frac{\partial}{\partial \phi} \rho^2 \left(0;\, \phi,\, \kappa,\, \sigma_\xi,\, \sigma_\theta\right) = 0
$$

which yields $\phi^0 = \sqrt{bq \left(\sigma_\theta/\sigma_\xi\right)^2 + \kappa^2}$ as the optimal level. On the other hand, the effect of ϕ at the equilibrium is

$$
\begin{aligned}
\frac{d\rho^2}{d\phi} &= \frac{\partial}{\partial \phi} \rho^2 \left(L;\, \phi,\, \kappa,\, \sigma_\xi,\, \sigma_\theta\right) + \frac{\partial}{\partial L} \rho^2 \left(L;\, \phi,\, \kappa,\, \sigma_\xi,\, \sigma_\theta\right) \frac{dL}{d\phi} \\
&= \frac{\partial}{\partial \phi} \rho^2 \left(L;\, \phi,\, \kappa,\, \sigma_\xi,\, \sigma_\theta\right) - \frac{\partial}{\partial L} \rho^2 \left(L;\, \phi,\, \kappa,\, \sigma_\xi,\, \sigma_\theta\right) \frac{\partial h}{\partial \phi} / \frac{\partial h}{\partial L}
\end{aligned}
$$

By evaluating this at $\phi = \phi^0$, we obtain $\frac{d\rho^2}{d\phi}|_{\phi=\phi^0} < 0$.

The second part is proved by two inequalities:

$$
\rho^2 \left(0;\, \phi^0,\, \kappa,\, \sigma_\xi,\, \sigma_\theta\right) < \rho^2 \left(L\left(\phi^0\right);\, \phi^0,\, \kappa,\, \sigma_\xi,\, \sigma_\theta\right) \leq \rho^2 \left(L\left(\phi^*\right);\, \phi^*,\, \kappa,\, \sigma_\xi,\, \sigma_\theta\right).
$$

The first inequality is proved as follows. For any $L > 0$,

$$
\begin{aligned}
&\rho^2 \left(L;\, \phi^0,\, \kappa,\, \sigma_\xi,\, \sigma_\theta\right) - \rho^2 \left(0;\, \phi^0,\, \kappa,\, \sigma_\xi,\, \sigma_\theta\right) \\
=&r \cdot g_1 + g_2
\end{aligned}
$$

where g_1 is polynomial in r and L and $g_2 > 0$ is polynomial in L and does not depend on r. Since $L \to C$ for some $C > 0$ as $r \to 0$, $r \cdot g_1 + g_2$ converges to a positive number. Thus, for sufficiently small r, the first inequality holds. The second inequality holds by definition. □

Proof of Proposition 7. Similarly to Proposition 6, the total effect of σ_ξ/σ_θ is written as $\frac{d\rho^2}{d(\sigma_\xi/\sigma_\theta)} = \frac{\partial}{\partial(\sigma_\xi/\sigma_\theta)}\rho^2\left(L;\, \phi,\, \kappa,\, \sigma_\xi,\, \sigma_\theta\right) - \frac{\partial}{\partial L}\rho^2\left(L;\, \phi,\, \kappa,\, \sigma_\xi,\, \sigma_\theta\right)\frac{\partial h}{\partial(\sigma_\xi/\sigma_\theta)}/\frac{\partial h}{\partial L}$. It is shown that $\frac{d\rho^2}{d(\sigma_\xi/\sigma_\theta)} < 0$. □

Proof of Theorem 2. Now, we look for a quadratic value function

$$V = v_0 + v_1\theta + v_2 Y + v_3\theta^2 + v_4 Y^2 + v_5\theta Y \tag{2.23}$$

satisfying the HJB equation:

$$\begin{aligned} rV\left(\theta,\, Y\right) = \sup_{F\in\mathbb{R}} & \left(1-\tau\right)p\cdot q - \tau p\cdot F - \frac{c}{2}F^2 \\ & - \kappa\left(\theta-\mu\right)V_\theta \\ & + \left(aF + bq\theta - \phi Y\right)V_Y \\ & + \frac{\sigma_\theta^2}{2}V_{\theta\theta} \\ & + \frac{bq\sigma_\xi^2}{2}V_{YY} \\ \text{s.t. } & p = \mu - \left(\eta\lambda + (1-\eta)\,\tilde\lambda\right)Y + \left(\eta\lambda\nu + (1-\eta)\,\tilde\lambda\tilde\nu\right) \end{aligned}$$

The first order condition and gives

$$v_5 = \frac{\alpha c}{a} \tag{2.24}$$

$$v_4 = \frac{\beta c + \hat{\lambda}\tau}{2a} \tag{2.25}$$

$$v_2 = \frac{c\delta\mu + \mu\tau - \widehat{\lambda\nu}\tau}{a} \tag{2.26}$$

where $\hat{\lambda} = \left(\eta\lambda + (1-\eta)\,\tilde{\lambda}\right)$ and $\widehat{\lambda\nu} = \left(\eta\lambda\nu + (1-\eta)\,\tilde{\lambda}\tilde{\nu}\right)$, and the envelop condition gives

$$0 = \hat{\lambda}\alpha\tau - 2a\alpha v_4 - 2bqv_4 + rv_5 + \kappa v_5 + v_5\phi \tag{2.27}$$

$$0 = -2a\beta v_4 + \beta\hat{\lambda}\tau + 2rv_4 + 4v_4\phi \tag{2.28}$$

$$0 = -2a\delta\mu v_4 + \delta\mu\hat{\lambda}\tau + \hat{\lambda}q\tau - \hat{\lambda}q + rv_2 - \kappa\mu v_5 + v_2\phi \tag{2.29}$$

By inserting eq.(2.25) into (2.28) and solving it for $\hat{\lambda}$ and by letting $L = a\beta$, we obtain

$$\hat{\lambda} = \frac{cL(L + r + 2\phi)}{a\tau(r + 2\phi)} \equiv \hat{\lambda}\,(L)$$

On the other hand, the stochastic differential equation for $(\theta,\,Y)$ gives

$$\lambda = \frac{bq\sigma_\theta^2(L+\phi)\,(A\,(L)+1)}{\sigma_\theta^2\,(bqA\,(L)+bq)^2 + \kappa bq\sigma_\xi^2(\kappa + L + \phi)} \equiv \lambda\,(L)$$

$$\tilde{\lambda} = \frac{bq\sigma_\theta^2\phi}{\sigma_\theta^2\,(bq)^2 + \kappa bq\sigma_\xi^2(\kappa + \phi)} = \lambda\,(0)$$

Then, by rearranging

$$\hat{\lambda} = \left(\eta\lambda + (1-\eta)\,\tilde{\lambda}\right)$$

$$\Rightarrow 1 = \frac{\eta\lambda\,(0) + (1-\eta)\,\lambda\,(L)}{\hat{\lambda}\,(L)} \equiv h\,(L;\eta)$$

Note that $\lim_{L\to 0} h\,(L;\eta) = \infty$ and $\lim_{L\to\infty} h\,(L;\eta) = 0$. Then, $h_L\,(L;\eta) < 0$ holds for any $\eta \in [0,\,1]$ as long as $h_L\,(L;1) < 0$. □

Proof of Proposition 8. Since $\lambda(0) \leq \lambda(L)$ for any $L \geq 0$, we have $h(L;\eta) \leq h(L;1)$ for any $\eta \in [0,\, 1]$. Thus, the equilibrium L will be smaller given $\eta < 1$ than the equilibrium L given $\eta = 1$.

The expected amount of the fake reviews is

$$E[F_t] = \alpha\mu + \beta\nu + \delta\mu$$

By plugging the equilibrium conditions and taking derivative with respect to L, we can show $\frac{\partial}{\partial L}E[F_t] \geq 0$. □

Proof of Proposition 9. At the equilibrium, $\frac{\partial bias}{\partial L} \geq 0$ always holds and $\frac{\partial bias}{\partial a} \geq 0$ holds if $bias \geq 0$. □

2.B An interpretation of the pricing rule

this pricing rule as a result of competition among heterogeneous consumers, to which we can easily introduce a mixture of rational and naive consumers in the next section. Suppose that consumer $i \in [0,\, n]$ feels $u_{t,i} = \theta_t + \epsilon_{t,i} - p_t$ if the consumer buy the product, and 0 otherwise, where $\epsilon_{t,i}$ is identically and independently distributed. Then, given the rating shown on the platform, Y_t, the consumer will choose to purchase the product if and only if $E[\theta_t|Y_t] + \epsilon_{t,i} - p_t \geq 0$. Therefore, the demand function is expressed as $n \cdot (1 - F(p_t - M_t))$ where $F(\cdot)$ is a c.d.f. of the random variable $\epsilon_{t,i}$. By letting $n = 2q$ and assuming that $\epsilon_{t,i}$ is distributed symmetrically around zero. We obtain $p_t = M_t$ as the market clearing price.

2.C An Alternative Model with Changing q

The same results with the base line model can be generated with a slightly different specification of the model with the quantity level dependent on the reputation level.

Now, suppose that the seller sells q_t units of the product at a fixed price of p, and makes F_t units of fake reviews. The quality of the product is denoted as θ_t. A sufficiently large mass of consumers forms a belief on the quality $E[\theta_t|Y_t] \equiv M_t$ and the demand function based on that. Since the price is fixed, high reputation results in large quantity: $q_t = M_t$.

The quality θ_t evolves in the same way as the main model. The new information as

$$aF_t dt + bq_t \left(\theta_t dt + \sigma_\xi dZ_t^\xi\right) \tag{2.30}$$

The difference from the main model is that the quantity varies over time and the coefficient of dZ_t^ξ is now defined as $bq_t\sigma_\xi$ instead of $\sqrt{bq_t}\sigma_\xi$. In this specification, we can analyze the effect of the organic reviews crowding out the fake reviews, but not the effect of the large transaction generating intrinsically more precise information by the large sample.

The seller's instantaneous payoff is defined as:

$$\pi_t = (1-\tau)\,p\,(q_t + F_t) - p \cdot F_t - \frac{c}{2}\left(\frac{F_t}{q_t}\right)^2$$

where τ is transaction fees imposed by the platform. The specification of the quadratic cost is now different from the base line model: the seller needs to pay a large cost if the seller tries to increase the share of the fake reviews among the all the reviews. The revenue and the reimbursement cost is still the same as the baseline model.

$$\begin{aligned}\pi_t &= (1-\tau)\,pq_t - \tau p \cdot F_t - \frac{c}{2}\left(\frac{F_t}{q_t}\right)^2 \\ &= (1-\tau)\,pM_t - \tau p \cdot M_t\frac{F_t}{M_t} - \frac{c}{2}\left(\frac{F_t}{M_t}\right)^2\end{aligned}$$

By changing the choice variable of the seller from F_t to $\frac{F_t}{M_t}$, which is the combination of the original variable and a constant at time t, we can write the instantaneous profit isomorphic to one in the baseline model. To simplify the analysis, we assume that the platform use an average information at time t to update the ratings:

$$d\xi = \frac{a}{b}\frac{F_t}{M_t}dt + \theta_t dt + \sigma_\xi dZ_t^\xi \tag{2.31}$$

The model is then isomorphic to the baseline model, so generates the same results as those from the baseline model.

2.D Simulation Results

2.D.1 Mixture of the Rational and Naive Consumers

In the main part, the correlation of the rating with the underlying true quality for rational consumers, and the bias for the naive consumers are examined. There is a trade-off of the correlation and the bias. Then, natural questions are (i) how to integrate such indices into one objective function, and (ii) how it changes as the market's rationality changes from totally naive to totally rational. In this section, we suggest a mean squared error of the price since the price is considered as the whole market's prediction about the underlying quality. The minimization of the mean squared errors minimizes the customers' *ex post* regret on average, so increases the value-added of the platform, and attracts the customers in long-run.

2.D.1.1 Mean Squared Error

The mean squared errors of the price is defined and written with the equilibrium variables as follows:

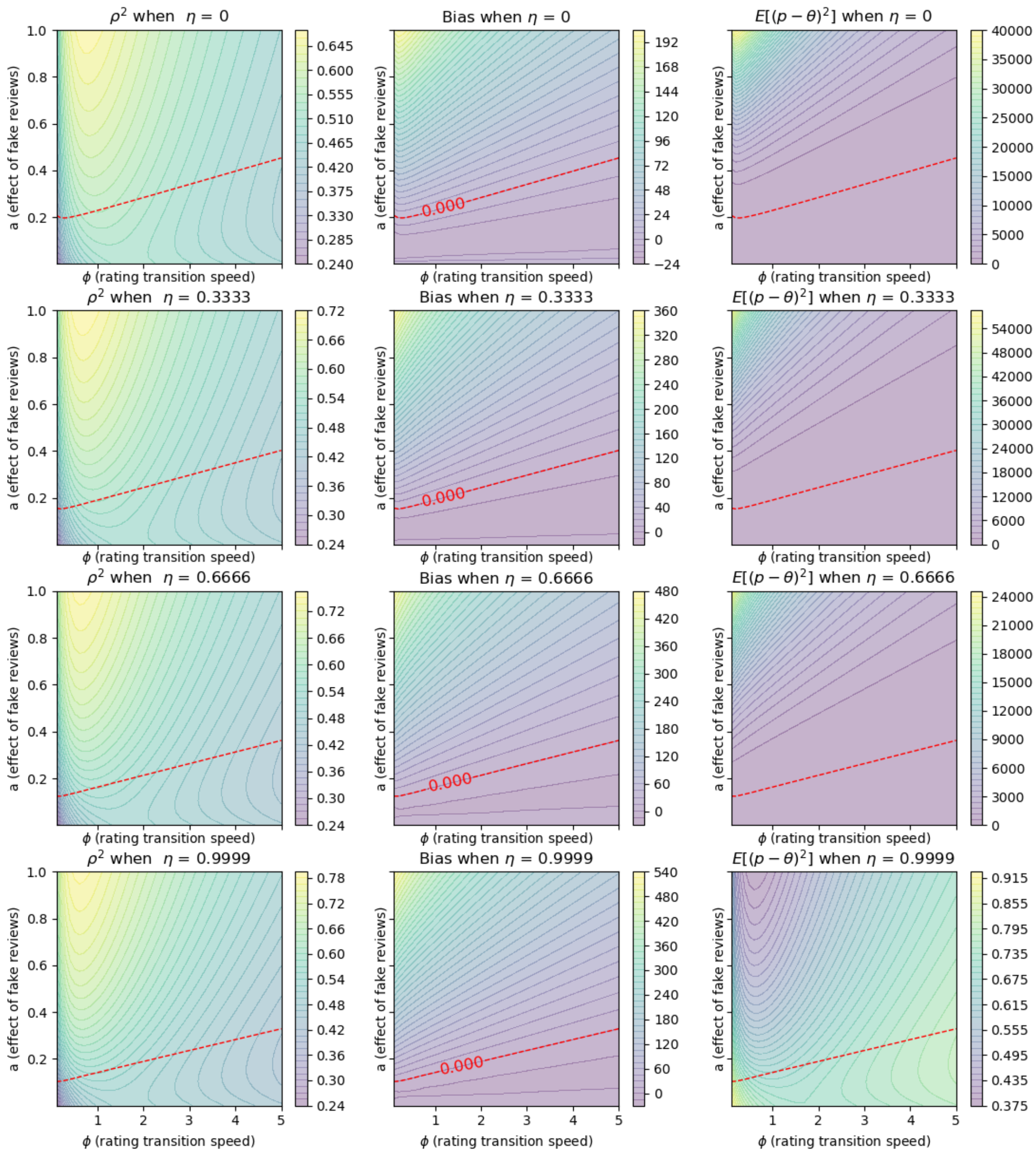
$$
\begin{aligned}
MSE_p &= E\left[(p_t-\theta_t)^2\right] \\
&= E\left[\left(\eta\left\{\mu+\lambda\left[Y_t-\nu\right]\right\}+(1-\eta)\left\{\mu+\widetilde{\lambda}\left[Y_t-\widetilde{\nu}\right]\right\}-\theta_t\right)^2\right] \\
&= Var\left(Y_t\right)\left\{\left(\eta\lambda+(1-\eta)\widetilde{\lambda}\right)^2-2\left(\eta\lambda+(1-\eta)\widetilde{\lambda}\right)\lambda\right\}+(1-\eta)^2 Bias^2+Var\left(\theta_t\right)
\end{aligned}
$$

Note that, when $\eta=1$, minimization of MSE is reduced to maximization of the correlation of the rating Y_t and θ_t:

$$
\begin{aligned}
MSE_p &= -\lambda^2 Var\left(Y_t\right)+Var\left(\theta_t\right) \\
&= Var\left(\theta_t\right)\left\{1-\frac{Cov\left(Y_t,\theta_t\right)^2}{Var\left(Y_t\right)^2}\frac{Var\left(Y_t\right)}{Var\left(\theta_t\right)}\right\} \\
&= Var\left(\theta_t\right)\left\{1-\rho^2\right\}
\end{aligned}
$$

For different levels of η, we calculate the correlation of Y and θ as a criteria for the rational consumers, the bias as a criteria for naive consumers, and the mean squared error as a criteria for the whole market. See fig.5 for the simulation results. The correlation of the rating with the underlying quality show the similar pattern regardless of the level of η, while it is scaled up as the rationality increases. So does the bias the naive consumers faces. This is consistent with Proposition 12. As the market becomes more rational, the consumers takes the signaling effect of the seller's fake reviews ($\alpha>0$), so the market becomes more sensitive to the rating. Then, the seller will have more incentive to make fake reviews, resulting in more bias for naive consumers. At the same time, the signaling effect ($\alpha>0$) is also enhanced by this increased manipulation by the seller. Therefore, the rating becomes more informative for rational consumers. Roughly speaking, the mean squared error integrates the correlation and the bias into one. As the ratio of the rational consumers increases, the correlation becomes more important. As the ratio of the naive consumers increases, the bias comes more important. Fig. 5 exhibits this. For $\eta = 0$, 0.3333, 0.6666, the MSE shows the similar pattern as the bias, while the MSE shows the similar pattern as the correlation for $\eta = 0.9999$. Given other parameters used in the simulation, the bias is the dominant force in MSE for most of η. This results depends on the parameter setting, so is ultimately an empirical question, but suggests that decreasing the bias is more important than increasing the informativeness for rational consumers.

Figure 2.6: Correlation, bias, and mean squared errors



From top to the bottom, the rationality of the market is increased from 0, 0.3333, 0.6666, to 0.9999. The left panels are contours of the correlation of the rating Y_t with θ_t based on rational expectations taking the seller's strategy into account. The middle panels show biases the naive consumers faces. The right panels show the mean squared errors of the market price as a whole market's prediction of the underlying quality. Red dashed lines border sets of parameters which predict realistic positive bias (positive number of positive fake reviews) at the equilibrium. Areas above red lines corresponds to the positive number of positive fake reviews.

CHAPTER 3

Note on Platform Liability for Third-Party Defective Products

3.1 Introduction

In August 2016, Angela Bolger, bought a replacement laptop battery listed on Amazon. The battery was sold by a third-party vendor named "E-Life," which is a fictitious name used by Lenoge Technology Ltd (Lenoge) in Hong Kong. A few months later, the battery exploded and Bolger suffered serious burns and was hospitalized for two weeks. Bolger filed a lawsuit against Amazon and several other companies allegedly involved in the design, manufacture, distribution, or sale of the battery. While Amazon "suppressed" listings of the battery (in other words, hid the product from search) and refunded a credit of the purchase price to Bolger, it claimed that it could not be held liable for defects in the battery since it was merely a provider of an online marketplace, but not a seller of the product. In the first round, Amazon's claim was accepted and summary judgment was granted. In August 2020, however, the California Court of Appeal reversed the lower court's decision and judged that Amazon was strictly liable for defective products by third-party vendors. [1]

This article theoretically analyzes effects of the platform liability for the third-party products on incentives of the third-party vendors to make efforts to produce safe products. For this purpose, I suppose that the platform behaves as a mere marketplace and does not screen vendors or products sold there before occurrence of a defect. Sellers still have incentives to make effort to build their own reputation. In this setting, the platform liability

[1]See Bolger v. Amazon.com, 53 Cal.App.5th 431 (Cal. Ct. App.2020)

for third-party products has an unintended effect to reduce incentive to safe products.

In the context of product liability of in a simple buyer-seller relationship, sellers can choose the product and monitor its manufacturing process of the product if strict liability for its own product is imposed. The monitoring and screening create incentives for manufacturers to produce safe products. However, the marketplace may not be able to choose products sold there or monitor the manufacturing process closely. This is considered an essential feature of the online marketplace because it allows small vendors of niche products to access the global market. However, if there is no (or less) screening or monitoring of the product, the incentive to provide safe products is not passed through the supply chain. Moreover, the platform's strict liability might diminish third-party sellers' incentives to build their reputation on the platform because consumers become less sensitive to the sellers' reputation. This study embodies this logic in a model based on Jovanovic's (2021) analysis of product recalls, where sellers' hidden efforts reduce the probability of a defect. However, in this study, sellers can avoid compensation at the occurrence of a defect in contrast to Jovanovic's (2021) analysis. An interpretation is that the sellers in a platform can be relatively small foreign venders, and the platform has to pay a fraction of the damage caused by the defect. In a setting where the platform does not enhance its screening or monitoring over third-party vendors, I show that the effort levels of the best equilibrium decrease in the extent of the platform's compensation for the damage. It would be important (theoretically and practically) to see how the platform changes its screening or monitoring policy over third-party vendors in response to the liability rule, but it is deferred to future extensions.

3.2 Related Literature

This study is related to several strands of the literature. Previous research on product liability mostly analyzes the necessity and validity of product liability in a seller-buyer relationship (e.g., Polinsky and Shavell (2010), Daughety and Reinganum (2013), Hua and Spier (2020)). In this context, a seller or manufacturer is liable and has to compensate for the damage caused by defects. In this study, a seller can escape after a defect is found, and the a platform has to compensate for part of the damage caused by the defect.

It is debatable whether online platforms are marketplaces or middlemen. Biglaiser (1993) argues that intermediaries have larger incentives to build reputations than sellers because they stay in the market for a longer time than sellers. Biglaiser and Li (2018) on the other hand, showed that the existence of a middleman, who can choose products, might exacerbate the moral hazard of a seller. The conclusion of this paper might look similar to that of Biglaiser and Li (2018). In contrast to previous research, however, a platform does not choose products sold in the marketplace, and only a buyer and a seller have some active roles in the model. The liability rule forces a platform to behave as an intermediary responsible for its products, but in this study, the platform cannot screen or monitor the seller. Even in this setting, the existence of a liability rule exacerbate a moral hazard problem of the seller.

With the growth of the platform economy, the liability of platforms in general is attracting attention. Buiten et al. (2020) and Lefouili and Madio (2021) provide a broad perspective on platforms' liability for content on platforms, including the liability on social media such as Facebook. Busch (2021) summarizes recent cases and regulatory strategies for platform liability on third-party products. This paper focuses on an online marketplace, such as Amazon, and points out that the platform's product liability for third-party products distorts the reputational incentive of third-party vendors.

Finally, this paper is related to the literature on firm dynamics, such as Jovanovic (1982) and Hopenhayn (1992), and recent studies on reputation dynamics, such as Atkeson et al. (2015), Board and Meyer-ter vehn (2020), and Jovanovic (2021). The most closely related paper is Jovanovic (2021), where sellers are strictly liable for their products, and their prices depend on reputation because their efforts change both the quality of their product and the probability of a defect. In this study, sellers can evade compensation for their product defects. Therefore, if the damage is not covered by the platform, a high probability of a defect leads to a low price. However, if the damage is covered by the platform, the price does not depend on the seller's effort level because consumers are not concerned about potential defects compensated by the platform. This generates an unintended effect of liability (i.e., disincentivizing sellers to invest in product safety).

3.3 Model

Time is a continuous and infinite horizon. At each moment t, each long-seller chooses its effort level x_t with quadratic costs $\frac{1}{2}x_t^2$ to reduce the probability of a defect in its product. A unit mass of consumers buy the products. Therefore, if there are no defects of its product produced at time t, the seller's instantaneous payoff is

$$\pi_t^s = (1-\tau)\, p_t - \frac{1}{2}x_t^2$$

where p_t is the price of product at time t. At each moment, a defect occurs with the hazard of $(\lambda - x_t)_+ \equiv \max\{0, \lambda - x_t\}$. This leads to the cumulative distribution function of the time elapsed without a defect, s:

$$Pr\,(s \leq t) = 1 - \exp\left(-\int_0^t (\lambda - x_s)_+ \, ds\right).$$

At the occurrence of the defect, the seller exits from the marketplace and re-enter with a new name.[2]

At each moment t, there is a mass one of consumers. Each buyer's utility from purchasing the product is $u_t = \begin{cases} \bar{u} - p_t & \text{if there is no defect} \\ \bar{u} - p_t - (1-\alpha)\, D & \text{if there is a defect} \end{cases}$, and 0 from not purchasing it, where D is the amount of damage caused by a defect, and α is the platform's compensation level. A probability that a defect occurs at this moment t is expressed by the hazard at time t, $(\lambda - x_t^*)_+$, where x_t^* is a buyer's belief in the seller's action at time t. Then, the expected utility from purchasing the product is $E\,[u_t] = \bar{u} - (1-\alpha)\, D\, (\lambda - x_t^*)_+ - p_t$. For the sake of tractability, I assume that each seller is a monopolist of its own product, and therefore set its price to $p_t = \bar{u} - (1-\alpha)\, D\, (\lambda - x_t^*)_+$.

[2]In this work, the goal is to identify the roles of platform liability for defects when buyers cannot get compensated from third-party vendors of defective products. Therefore, I focus on an extreme case in which all vendors evade compensation for defects.

Then, the seller's Hamilton-Jacobi-Bellman (HJB) equation is written as follows:

$$rv_t = \max_{x_t \leq \lambda} \left((1-\tau)\left(\bar{u} - (1-\alpha) D \left(\lambda - x_t^*\right)_+\right) - \frac{x^2}{2} - (\lambda - x_t)_+ (v_t - k) + \frac{dv_t}{dt} \right) \tag{3.1}$$

where k is the value of the seller when the seller exits the market.

At equilibrium, the seller solves the above HJB equation. The buyers form the rational belief: $x_t^* = x_t$, where x_t is the maximizer of the right-hand side of the above equation, and k is set to v_0 because the seller of the defective product re-enters the market with a different identity.

In this article, I analyze the impact of the platform liability level α on the efforts to reduce defects. The platform takes the liability rule as given and provides the marketplace to market participants. The platform's reaction to the liability rule is deferred to a future extension.

3.3.1 Equilibrium Characterization

Suppose that $x_t < \lambda$ holds for all t. Then, by a first-order condition of the right-hand side of eq.(3.1),

$$x_t = (v_t - k) \tag{3.2}$$

Plugging this into the HJB equation,

$$rv_t = (1-\tau)\left(\bar{u} - (1-\alpha) D \left(\lambda - x_t^*\right)\right) - \frac{x_t^2}{2} - (\lambda - x_t)\, x_t + \frac{dv_t}{dt}. \tag{3.3}$$

Furthermore, eq. (3.2) implies that

$$\begin{aligned}
\frac{dx_t}{dt} &= \frac{dv_t}{dt} \\
&= r\left(x_t + k\right) - (1-\tau)\,\bar{u} + (1-\tau)(1-\alpha) D (\lambda - x_t) + \frac{x_t^2}{2} + (\lambda - x_t)\, x_t && (3.4) \\
&= rk + (r + \lambda - (1-\tau)(1-\alpha) D)\, x_t - (1-\tau)\,\bar{u} + (1-\tau)(1-\alpha) D\lambda - \frac{x_t^2}{2} && (3.5)
\end{aligned}$$

where the second equality is derived from eq. (3.3). Then, I can find limit points, $\lim_{t\to\infty} x_t$, by setting the right-hand side of (3.5) to zero. This equation has two roots:

$$x^- = (r + \lambda - (1-\tau)(1-\alpha) D) - A$$
$$x^+ = (r + \lambda - (1-\tau)(1-\alpha) D) + A$$

where $A = \sqrt{(r+\lambda-(1-\tau)(1-\alpha)D)^2 + 2(rk - (1-\tau)\{\bar{u} - (1-\alpha) D\lambda\})}$

Given these results, (3.5) can be written as $\frac{1}{\rho(x-x^-)(x-x^+)}dx = dt$, where $\rho = -\frac{1}{2}$. This is further rewritten as

$$\frac{1}{\rho(x^+ - x^-)}\left(\frac{1}{(x_t - x^+)} - \frac{1}{(x_t - x^-)}\right) dx_t = dt$$

By integrating both sides,

$$\frac{1}{\rho(x^+ - x^-)}\left(\ln\left(x^+ - x_t\right) - \ln\left(x_t - x^-\right)\right) = t + C$$

for some constant $C \in \mathbb{R}$. Then, by rearranging this,

$$\ln\left(\frac{x^+ - x_t}{x_t - x^-}\right) = \rho\left(x^+ - x^-\right)(t + C)$$
$$x_t = \frac{x^+ + x^- \exp\{\rho(x^+ - x^-)(t+C)\}}{1 + \exp\{\rho(x^+ - x^-)(t+C)\}}$$

The constant C is determined by the first-order condition at $t = 0$:

$$v_0 - k = x_0 = \frac{x^+ + x^- \exp\{\rho(x^+ - x^-)(C)\}}{1 + \exp\{\rho(x^+ - x^-)(C)\}}$$
$$\Leftrightarrow \frac{x^+ - (v_0 - k)}{(v_0 - k) - x^-} = \exp\left\{\rho\left(x^+ - x^-\right)C\right\}$$

Therefore, we obtain the explicit characterization of the effort path:

$$x_t = \frac{x^+ + x^- \exp\{\rho(x^+ - x^-)t\}\frac{x^+ - (v_0 - k)}{(v_0 - k) - x^-}}{1 + \exp\{\rho(x^+ - x^-)t\}\frac{x^+ - (v_0 - k)}{(v_0 - k) - x^-}}.$$

Furthermore, when $k = v_0$, the equilibrium path becomes $x_t = \frac{x^+ + x^- \exp\left\{\rho\left(x^+ - x^-\right)t\right\} \frac{x^+}{-x^-}}{1+\exp\{\rho(x^+ - x^-)t\} \frac{x^+}{-x^-}}$, which implies $x_0 = 0$. At the same time, eq.(3.2) characterizes the seller's value function: $v_t = v_0 + x_t = v_0 + \frac{x^+ + x^- \exp\left\{\rho\left(x^+ - x^-\right)t\right\} \frac{x^+}{-x^-}}{1+\exp\{\rho(x^+ - x^-)t\} \frac{x^+}{-x^-}}$. Because x_t converges to x^+ as $t \to \infty$, v_t also converges to $v_0 + x^+$.

Since x^+ is a function of the seller's value after exiting from the market, k, the re-entry condition, $k = v_0$, implies that x^+ is a function of v_0. Therefore, a range of admissible x^+ is translated to a range of admissible v_0 (and a range of admissible $\lim_{t\to\infty} v_t$). In the following section, I characterize the range of admissible x^+.

Range of x^+ In this article, I suppose x^+ to be non-negative because an effort level x_t for any $t \geq 0$ is non-zero. Furthermore, it is proved that $x^+ \leq \lambda$ must hold at equilibrium.

Lemma 7. *$x^+ \leq \lambda$ at equilibrium*

Proof. Suppose the contrary, $x^+ > \lambda$ at equilibrium. Then, at some point in time $t = t_\lambda$, x_t reaches λ. Then, for $t \geq t_\lambda$, we have a zero probability of a defect, and the flow profit becomes $(1-\tau)\,\bar{u} - \frac{\lambda^2}{2}$. This leads to $v_t = \frac{1}{r}\left\{(1-\tau)\,\bar{u} - \frac{\lambda^2}{2}\right\}$ for $t \geq t_\lambda$. For $t \geq t_\lambda$, if $x_t = \lambda$ maximizes

$$(1-\tau)\left(\bar{u} - (1-\alpha)\,D\,(\lambda - x_t^*)_+\right) - \frac{x^2}{2} - (\lambda - x_t)_+\,(v_t - k) + \frac{dv_t}{dt},$$

then $v_t - k \geq \lambda \Leftrightarrow v_t \geq k + \lambda$ must hold. Therefore,

$$\frac{1}{r}\left\{(1-\tau)\,\bar{u} - \frac{\lambda^2}{2}\right\} \geq k + \lambda \tag{3.6}$$

must hold.

On the other hand, $x^+ > \lambda$ is equivalent to

$$\begin{aligned}(r + \lambda - (1-\tau)\,(1-\alpha)\,D) + A &> \lambda \\ \Leftrightarrow \lambda + \frac{\lambda^2}{2r} + k - \frac{(1-\tau)\,\bar{u}}{r} &> 0\end{aligned}$$

$$\Leftrightarrow \lambda + k > \frac{1}{r}\left\{(1-\tau)\,\bar{u} - \frac{\lambda^2}{2}\right\} \tag{3.7}$$

Therefore, eq.(3.6) and eq. (3.7) contradict each other. Thus, $x^+ \leq \lambda$ must hold at equilibrium. □

Highest-effort equilibrium. Now, the equilibrium with the highest effort is characterized by $x^+ = \lambda$. The corresponding v_0 is derived by solving the following equation:

$$(r + \lambda - (1-\tau)(1-\alpha)\,D) + \tilde{A} = \lambda$$

where $\tilde{A} = \sqrt{(r + \lambda - (1-\tau)(1-\alpha)\,D)^2 + 2\,(rv_0 - (1-\tau)\{\bar{u} - (1-\alpha)\,D\lambda\})}$. This leads to

$$v_0 = \frac{(1-\tau)\{\bar{u} - (1-\alpha)\,D\lambda\}}{r} + \frac{((1-\tau)(1-\alpha)\,D - r)^2 - ((1-\tau)(1-\alpha)\,D - r - \lambda)^2}{2r} \equiv v_0^{best}$$

Therefore, the best-effort equilibrium is characterized as follows: $v_0 = v_0^{best}$, $x_+ = \lambda$, and $x_t = x_t^{best} = x^- + \frac{\lambda - x^-}{1 + \frac{\lambda}{-x^-}\exp\left\{-\frac{1}{2}(\lambda - x^-)t\right\}}$. This characterizes the upper bound of the effort path $\{x_t\}$. This upper bound of the equilibria paths is still a function in α, the liability level, through x^-, which is a function of v_0 and α. To examine the effect of α on the upper bound, we set $v_0 = v_0^{best}(\alpha)$ and analyze the total impact of α on x^-:

$$\begin{aligned}
x^-\left(v_0^{best}(\alpha), \alpha\right) &= (r + \lambda - (1-\tau)(1-\alpha)\,D) \\
&\quad - \sqrt{(r + \lambda - (1-\tau)(1-\alpha)\,D)^2 + ((1-\tau)(1-\alpha)\,D - r)^2 - ((1-\tau)(1-\alpha)\,D - r - \lambda)^2} \\
&= (r + \lambda - (1-\tau)(1-\alpha)\,D) - ((1-\tau)(1-\alpha)\,D - r) \\
&= \lambda + 2r - 2\,(1-\tau)(1-\alpha)\,D
\end{aligned}$$

Thus, $x^-\left(v_0^{best}(\alpha), \alpha\right)$ is increasing in α. Since it is shown that x_t^{best} is decreasing in x^-, x_t^{best} is decreasing in α. This is summarized as the following proposition. As the platform liability level, α, increases, x_t^{best} grows slower over t and the level of x_t^{best} decreases for all $t > 0$. In other words, the upper bound of the equilibrium effort decreases as the platform liability for the third-party product is enhanced. Intuition of the proposition is explained as follows. High liability means high demand from consumers regardless of the effort level of the seller. Therefore, the seller has less incentive to reduce the probability of a defect. Therefore, the reputation (expected level of effort) builds relatively slower. This also implies that the level of effort decreases because x_0 is always zero.

Lowest-effort equilibrium The lowest-effort equilibrium is characterized by $x^+ = 0$. This leads to $x_t = 0$ and $v_t = v_0$ for all t. The corresponding v_0 is characterized by $x^+ (v_0) = 0$:

$$x^+ (v_0) = 0 \text{ (or, } v_t (v_0) = v_0 + x^+ (v_0) = v_0)$$

$$v_0^{worst} = \frac{(1-\tau) \{\bar{u} - (1-\alpha) D\lambda\}}{r}$$

Clearly, at the worst equilibrium, the platform liability level α does not change the effort level. A high α also reduces the expected damage caused to consumers, and this increases the seller's surplus by increasing the price.

3.4 Summary and Future Extensions

This study demonstrated that the upper bound of the sellers' effort levels decreases in the platform liability for the third-party product. On the contrary, the lower bound of the effort level is not affected by the liability rule because the third-party seller puts no effort in this case. Therefore, regulators should exercise caution about determining whether a third-party seller will respond to the liability rule.

In the current study, the platform did not take any active role. However, the liability rule intends to encourage platform monitoring. The platform's reactions to the liability rule will generate important extensions in the future.

Bibliography

Afriat, S. N., "The Construction of Utility Functions from Expenditure Data," International Economic Review, 1967, 8 (1), 67–77.

Ananthakrishnan, Uttara M., Beibei Li, and Michael D. Smith, "A Tangled Web: Should Online Review Portals Display Fraudulent Reviews?," Information Systems Research, jun 2020, Article in, 1–22.

Atkeson, Andrew, Christian Hellwig, and Guillermo Ordoñez, "Optimal regulation in the presence of reputation concerns," Quarterly Journal of Economics, 2015, 130 (1), 415–464.

Ball, Ian, "Scoring Strategic Agents," 2019, (November), 1–57.

Bar-isaac, Heski and Joyee Deb, "What is a Good Reputation? Career Concerns with Heterogeneous Audiences," International Journal of Industrial Organization, 2014, 34, 44–50.

Belleflamme, Paul and Martin Peitz, "Inside the Engine Room of Digital Platform: Reviews, Ratings, and Recommendations," in "Economic Analysis of the Digital Revolution" 2018, pp. 75–114.

Berry, Steven, James Levinsohn, and Ariel Pakes, "Automobile Prices in Market Equilibrium," Econometrica, 1995, 63 (4), 841–890.

Berry, Steven T., "Estimating Discrete-choice Models of Product Differentiation," The RAND Journal of Economics, 1994, 25 (2), 242–262.

Biglaiser, Gary, "Middlemen as Experts," RAND Journal of Economics, 1993, 24 (2), 212–223.

_ and Fei Li, "Middlemen: the Good, the Bad, and the Ugly," RAND Journal of Economics, 2018, 49 (1), 3–22.

Board, Simon and Moritz Meyer ter vehn, "A Reputational Theory of Firm Dynamics," 2020, pp. 1–35.

Boleslavsky, Raphael and Christopher Cotton, “Grading standards and education quality,” American Economic Journal: Microeconomics, 2015, 7 (2), 248–279.

Bonatti, Alessandro and Gonzalo Cisternas, “Consumer Scores and Price Discrimination,” Review of Economic Studies, 2019.

Bresnahan, T, “Empirical studies of industries with market power,” Handbook of Industrial Organization, 1989, II, 1012–1057.

Bresnahan, Timothy F., “The oligopoly solution concept is identified,” Economics Letters, 1982, 10 (1-2), 87–92.

Brown, Donald J. and Rosa L. Matzkin, “Testable Restrictions on the Equilibrium Manifold,” Econometrica, 1996, 64 (6), 1249–1262.

Buiten, Miriam C., Alexandre De Streel, and Martin Peitz, “Rethinking Liability Rules for Online Hosting Platforms,” International Journal of Law and Information Technology, 2020, 28 (2), 139–166.

Busch, Christoph, “Rethinking Product Liability Rules for Online Marketplaces: A Comparative Perspective,” Available at SSRN, 2021.

Carvajal, Andrés, Rahul Deb, James Fenske, and John K.-H. Quah, “Revealed Preference Tests of the Cournot Model,” Econometrica, 2013, 81 (506), 2351–2379.

_ , _ , _ , and _ , “A Nonparametric Analysis of Multi-product Oligopolies,” Economic Theory, 2014, 57 (2), 253–277.

Chevalier, Judith A., Yaniv Dover, and Dina Mayzlin, “Channels of Impact: User Reviews When Quality Is Dynamic and Managers Respond,” Marketing Science, 2018, 37 (5), 688–709.

Daughety, Andrew F. and Jennifer F. Reinganum, “Economic analysis of products liability: Theory,” Research Handbook on the Economics of Torts, 2013, (11), 69–96.

Dellarocas, Chrysanthos, "Strategic Manipulation of Internet Opinion Forums: Implications for Consumers and Firms," Management Science, 2006, 52 (10), 1577–1593.

Dwoskin, Elizabeth and Craig Timberg, "How merchants use Facebook to flood Amazon with fake reviews," 2018.

Goldberg, Pinelopi Koujianou, "Product Differentiation and Oligopoly in International Markets: The Case of the US Automobile Industry," Econometrica, 1995, 63 (4), 891–951.

Grugov, Mikhail and Marta Troya-Martinez, "Vague Lies and Lax Standards of Proof: On the Law and Economics," Journal of Economics Management Strategy, 2019, 28, 298–315.

Grunewald, Andreas and Matthias Kräkel, "Advertising as signal jamming," International Journal of Industrial Organization, 2017, 55, 91–113.

Harbaugh, Rick and Eric Rasmusen, "Coarse Grades: Informing the Public by Withholding Information," American Economic Journal: Microeconomics, 2018, 10 (1), 210–235.

He, Sherry, Brett Hollenbeck, and Davide Proserpio, "The Market for Fake Reviews," SSRN Electronic Journal, 2020, pp. 1–38.

Hollenbeck, Brett, "Online Reputation Mechanisms and the Decreasing Value of Chain Affiliation," Journal of Marketing Research, oct 2018, 55 (5), 636–654.

_ , Sridhar Moorthy, and Davide Proserpio, "Advertising Strategy in the Presence of Reviews: An Empirical Analysis," Marketing Science, 2019, 38 (5), 793–811.

Holmström, Bengt, "Managerial Incentive Problems: A Dynamic Perspective," Review of Economic Studies, 1999, 66 (1), 169–182.

Hopenhayn, Hugo A., "Entry, exit, and firm dynamics in long run equilibrium," Econometrica, 1992, 60 (5), 1127–1150.

Hopenhayn, Hugo and Maryam Saeedi, "Optimal Ratings and Market Outcomes," NBER Working Paper Series, 2019, pp. 1–39.

Horstmann, Ignatius J. and Glen M. MacDonald, “When is Advertisnig a Signal of Product Quality?,” Journal of Economics and Management Strategy, 1994, 3 (3), 561–584.

_ and Sridhar Moorthy, “Advertising Spending and Quality for Services: The Role of Capacity,” Quantitative Marketing and Economics, 2003, 1 (3), 337–365.

Hua, Xinyu and Kathryn E. Spier, “Product Safety, Contracts, and Liability,” RAND Journal of Economics, 2020, 51 (1), 233–259.

Hui, Xiang, Maryam Saeedi, Giancarlo Spagnolo, and Steve Tadelis, “Certification, Reputation and Entry: An Empirical Analysis,” 2018, pp. 1–59.

Jovanovic, Boyan, “Selection and the Evolution of Industry,” Econometrica, 1982.

_ , “Product Recalls and Firm Reputation,” 2021.

Kamenica, Emir and Matthew Gentzkow, “Bayesian Persuasion,” American Economic Review, 2011, 101 (6), 2590–2615.

Kihlstrom, Richard E. and Michael H. Riordan, “Advertising as a Signal,” Journal of Political Economy, 1984, 92 (3), 427–450.

Lefouili, Yassine and Leonardo Madio, “The Economics of Platform Liability,” 2021.

Lizzeri, Alessandro, “Information Revelation and Certification Intermediaries,” RAND Journal of Economics, 1999, 30 (2), 214–231.

Luca, Michael and Georgios Zervas, “Fake It Till You Make It: Reputation, Competition, and Yelp Review Fraud,” Management Science, 2016, 62 (12), 3412–3427.

Mayzlin, Dina, “Promotional Chat on the Internet,” Marketing Science, 2006, 25 (2), 155–163.

Milgrom, Paul and John Roberts, “Price and Advertising Signals of Product Quality,” Journal of Political Economy, 1986, 94 (4), 796–821.

Miller, Nathan H. and Matthew C. Weinberg, "Understanding the Price Effects of the MillerCoors Joint Venture," Econometrica, 2017, 85 (6), 1763–1791.

Nocke, Volker and Nicolas Schutz, "Multiproduct-Firm Oligopoly: An Aggregative Games Approach," Econometrica, 2018, 86 (2), 523–557.

Ostrovsky, Michael and Michael Schwarz, "Information disclosure and unraveling in matching markets," American Economic Journal: Microeconomics, 2009, 2 (2), 34–63.

Polinsky, A. Mitchell and Steven Shavell, "The uneasy case for product liability," Harvard Law Review, 2010, 123 (6), 1437–1492.

Rayo, Luis and Ilya Segal, "Optimal Information Disclosure," Journal of Political Economy, 2010, 118 (5), 949–987.

Reimers, Imke C. and Joel Waldfogel, "Digitization and Pre-Purchase Information: The Causal and Welfare Impacts of Reviews and Crowd Ratings," National Bureau of Economic Research, 2020.

Saeedi, Maryam, "Reputation and Adverse Selection: Theory and Evidence from eBay," RAND Journal of Economics, 2019, 50 (4), 822–853.

_ and Ali Shourideh, "Optimal Rating Design," 2020, pp. 1–47.

Sahni, Navdeep S and Harikesh S Nair, "Does Advertising Serve as a Signal? Evidence from a Field Experiment in Mobile Search," The Review of Economic Studies, 2019, (October 2019), 1529–1564.

Sullivan, Christopher J., "The Ice Cream Split: Empirically Distinguishing Price and Product Space Collusion," 2016, pp. 1–54.

Vellodi, Nikhil, "Ratings Design and Barriers to Entry," SSRN Electronic Journal, 2020, pp. 1–63.

Wolfram, Catherine D., "Measuring Duopoly Power in the British Electricity Spot Market," American Economic Review, 1999, 89 (4), 805–826.