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Are the Objective and Solutions of Dynamic User-Equilibrium Models Always Consistent?*

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Abstract

Traffic assignment models are an important component in analyzing the relationship between demand and supply in the transportation network for design, planning, and control purposes. The static traffic assignment model has been used in practice for several decades. With the latest development in the area of Advanced Traffic Management Systems (ATMS) and Advanced Traveler Information Systems (ATIS), there is an increasing demand for dynamic traffic assignment models to serve as a basis for studying various issues in these areas.

Existing dynamic user-equilibrium traffic assignment (DUETA) models are mostly expanded from the static user-equilibrium traffic assignment model by introducing the time dimension along with a group of additional constraints. Whereas the equivalency between the solution to the traffic assignment model and the user-equilibrium condition as defined by Wardrop is well established in the static case, the same may not be true for the dynamic case. This paper examines the general form of DUETA models as proposed in previous research and shows that, if queuing behavior is represented in the model at a minimal level, the solution to conventional DUETA models with an objective function of the form $\sum_t \sum_i \int_0^{x_i(t)} f(\omega) d\omega$ may not necessarily converge to or approximate the Wardropian user-equilibrium condition in the dynamic sense as defined by many researchers.

Keywords: Dynamic traffic assignment, Traffic flow, User-equilibrium, Mathematical Programming, Optimal Control.

Executive Summary

One of the major applications of dynamic traffic assignment models is to project the dynamic traffic flow pattern spatially over a network based on given sets of O-D demands. This is especially important to the ATMIS (Advanced Traffic Management and Information System) development in which control and information dissemination strategies are developed based on the dynamic evolution of traffic.

Conventionally, dynamic traffic assignment models were often formulated either from simulation or analytical approaches. The focus of this paper is on the property of the analytical model, in particular, the user-equilibrium dynamic traffic assignment (DUETA) model. It was often claimed that the analytical DUETA model has a desirable mathematical property that the solution obtained would reach or approximate the user-equilibrium state. We show in this paper, under queuing conditions, the solution to existing dynamic user-equilibrium traffic assinment models, either formulated as a mathematical programming problem or as an optimal control problem, may not necessarily converge to or approximate the Wardropian user-equilibrium condition in the dynamic sense as defined by many researchers.

The identified mathematical inconsistency illustrated in this paper indicates that the resulting traffic patterns on a network obtained from most existing analytical DUETA models could be drastically different from the ones in the user-equilibrium state, inconsistent with the modeling objective.

1 Introduction

In many transportation planning models, user-equilibrium is commonly used as a condition to define route choice or traffic assignment behavior. This condition has behavioral appeals. The criteria for attaining the user-equilibrium state is to assign traffic flow among the routes in a way that minimizes the cost for each traveler. Without control on route choice, as is the real-world situation, this condition can indeed make a strong contention.

In the traditional way of modeling traffic, also commonly known as static traffic assignment in which the time dimension of traffic flow is not considered, Wardrop (1952) defined the user-equilibrium condition as: no traveler can reduce his/her journey time by unilaterally changing to a new route. This static assignment implies that traffic flow is at a steady state in the network, and that the dynamics of traffic flow such as queue dissipation and formation are not captured. Its user-equilibrium state is usually obtained by solving a mathematical programming problem with the following objective function:

$$Z = \min \sum_i \int_0^{x_i} f(\omega) d\omega \tag{1}$$

where f is a travel cost function and x_i is the traffic flow on link i. (See for example, Sheffi (1985).) By minimizing this objective function subject to a group of flow conservation constraints, one can obtain x for every link in the network. The equivalency between the solution of this mathematical programming problem and the user-equilibrium condition is well established by Beckmann et al. (1956).

In the past decade or so, there has been a great deal of effort in extending this static assignment condition to a dynamic one, so that traffic flow dynamics which are inevitable in real-world situations can be captured. Dynamic traffic assignment deals with the time-varying demand at the origin and the time-varying flows on the links over the network. Though the existing dynamic user-equilibrium traffic assignment (DUETA) models vary in details (for a recent review, see Romph, 1994), the basic models are generally expanded from the static user-equilibrium traffic assignment model by introducing the time dimension along with a group of additional constraints. Through an optimal control or a mathematical programming approach, most of these new DUETA models adopted the following objective function by extending the static model's:

$$Z = \min \sum_t \sum_i \int_0^{x_i(t)} f(\omega) d\omega \tag{2}$$

where $x_i(t)$ is defined as the number of vehicles on link i at time t. Most of these formulations, albeit with slight variations, contended that they provided a temporal generalization of Wardrop's user-equilibrium condition. This temporal generalization requires that, when the user-equilibrium condition is achieved, at any time t, no vehicle can find a better route than the one assigned. It remains unclear, however, whether a simple extension of the objective function as expressed in **(2)** can adequately characterize dynamic traffic assignment as required by the above temporal generalization. Therefore, the purpose of this paper is to examine whether this augmented objective function (i.e., (2)) can truly capture the impact of queuing characteristics or dynamic congestion in its assignment, as is set out to be one of the ultimate goals of DUETA models. Our results seem to indicate that the new objective function may not necessarily converge to or approximate the user-equilibrium condition as defined earlier.

In this paper, we will develop this insight by constructing a simple counter-scenario, as explained in Section **2.** Section **3** will provide and discuss the assignment solution which is consistent with the dynamic user-equilibrium condition. In Section **4,** we will derive the assignment solution resulted from the augmented objective function. By contrasting the solutions obtained in Sections **3** and **4,** we will establish the potential problems introduced by the new objective function.

2 The Counter-Scenario

All traffic assignment models have a set of constraints to maintain flow propagation and conservation at a node or a link, and nonnegativity of variables. This set of constraints is fairly standard and applicable to any traffic assignment models, regardless of the objective of the assignment. If one is interested in deriving the traffic assignment pattern that minimizes overall system travel time, the same set of constraints can also be used. Routes for deriving the dynamic user-equilibrium condition are determined solely in the course of minimizing the objective function while satisfying the constraint set. With this understanding, the example shown in this section only emphasizes the form and values of the objective function.

To make the discussion below as general as possible, we avoid making reference to any specific form of the link travel cost function (sometimes also known as the link performance function) but assume the function to be non-negative, increasing, continuously differentiable in x, as widely adopted in the literature. The link travel cost is also assumed to be decreasing with increasing link capacity, as consistent with common sense. In the following, we use cumulative curves of vehicles to describe traffic departure and arrival patterns and to derive the user-equilibrium state.

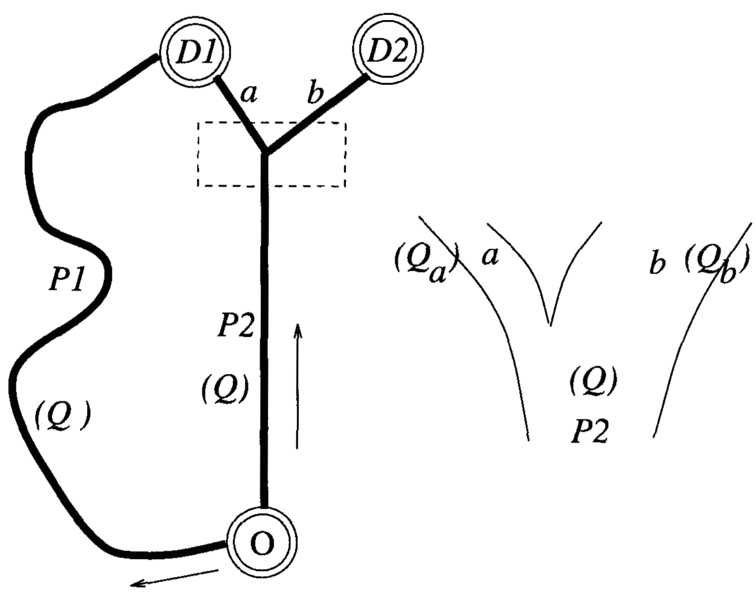


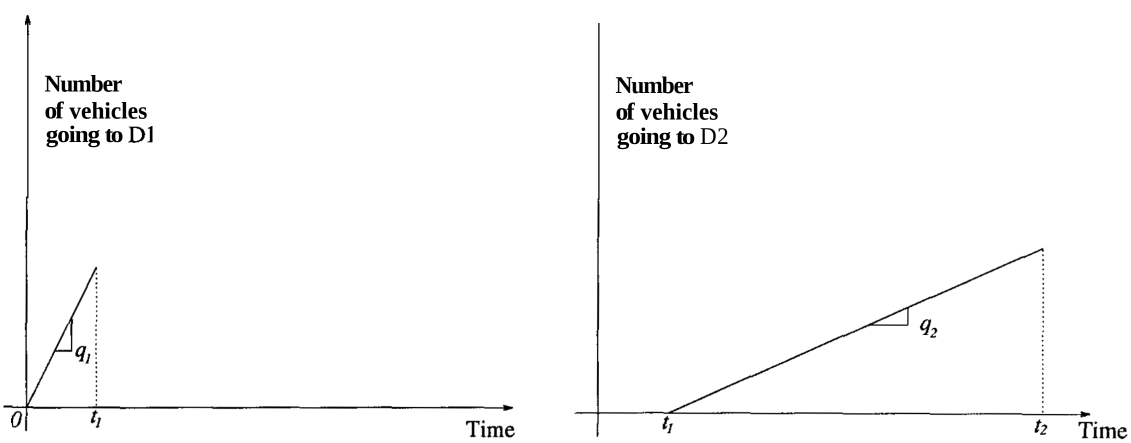
Figure 1: Network topology of the example

2.1 Network Topology

Consider a network with a single origin (O) and two destinations ($D1$ and 02). The roadway geometry is given in Figure 1. There are two paths ($P1$ and $P2a$) from O to $D1$ and a single path ($P2b$) from O to 02. $P1$ is longer than $P2a$ and homogeneous everywhere. $P2a$ has a bottleneck in the diverge branch leading to $D1$ as illustrated in the same Figure. The capacities (given in the parenthesis in the Figure) for the two main roads, $P1$ and $P2$, are both Q; the capacities for the two diverge branches a and b are Q_a and Q_b respectively with $Q_a \ll Q_b < Q$.

2.2 Traffic Demands

Two traffic streams, going from origin **O** to each of the two destinations $D1$ and **02**, are generated sequentially in two non-overlapping time intervals, $[0, t_1]$ and $[t_1, t_2]$. Their respective cumulative departure curves from the origin are shown in Figure **2** (a) and (b). Traffic stream 1 to $D1$ departs at the origin at a constant rate of q_1 in the time interval $[0, t_1]$. Traffic stream **2** to **0 2** departs at the origin at a constant rate of q_2 only in the time interval $[t_1, t_2]$. We assume that $Q_a \ll q_1 = Q$ and $q_2 = Q_b$. The total demand during the entire time period is thus $q_1 t_1 + q_2(t_2 - t_1)$.



(a) Travel demand to $D1$ (b) Travel demand to **0 2**

Figure **2:** Travel demands to $D1$ and **0 2**

3 The User-Equilibrium Solution Consistent with Its Definition

By considering the travel time experienced by the travelers, we can derive the traffic flow pattern under the user-equilibrium condition. Due to the nature of the demand pattern and the network topology, the traffic assignment procedure can be derived empirically in two steps.

In the first step, we only consider traffic stream 1 since it is the only traffic in the network for the time interval $[0, t_1]$. We assume that the network topology is such that $P1$ is considerably longer than $P2a$. $P2a$ always yields shorter travel times for **all** the demands destined to $D1$ despite the existence of a bottleneck in $P2a$ and even if the free-flow condition prevails in $P1$. The underlying condition for this to happen and the travel delay to each vehicle in traffic stream 1 can be illustrated with a set of cumulative curves of vehicles as shown in Figure **3.** For the given cumulative demand (or departure)

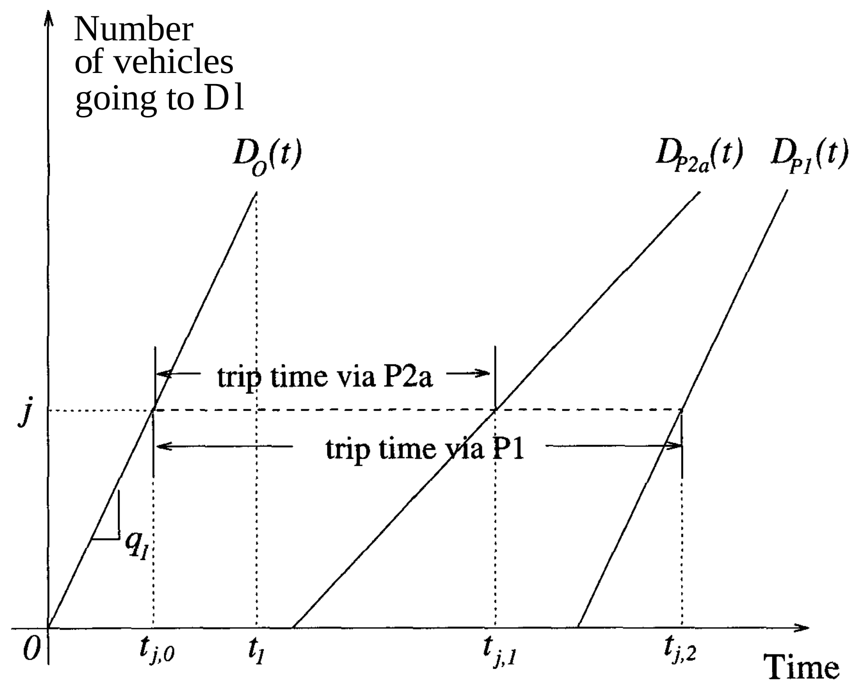


Figure **3:** The trip time from the origin to Dl via paths $P1$ and $P2a$

curve $D_O(t) = q_1 t$ for $t \in [0,t_1]$, the cumulative arrival curve D_{P1} $(D_{P2a}(t))$ is constructed by loading **all** demands for Dl on path $P1$ $(P2a)$. The vehicle trip time from O to $D1$ via path $P1$ $(P2a)$ can be obtained by the horizontal distance between curves $D_O(t)$ and $D_{P1}(t)$ $(D_{P2a}(t))$. **As** an example, the jth vehicle leaving **O** at $t_{j,0}$ will reach $D1$ at $t_{j,1}$ by taking $P2a$ and at $t_{j,2}$ by taking $P1$. The time saving for the vehicle is $D_{P1}^{-1}(j) - D_{P2a}^{-1}(j) = t_{j,2} - t_{j,1}$. **As** shown in the curves, $D_{P1}^{-1}(k) - D_{P2a}^{-1}(k) > 0$ for **all** k in traffic stream 1, suggesting that, for each vehicle in traffic stream 1, the congested travel condition on $P2a$ (represented by the low discharging rate for curve $D_{P2a}(t)$) is better than the free-flow condition on $P1$ in terms of travel times. The user-equilibrium state for traffic stream 1 is thus achieved when all the vehicles in this group are assigned to $P2a$. This is so because no one can shorten his/her travel time by switching to $P1$. Therefore, assigning all stream 1 vehicles to path $P2a$ indeed satisfies the dynamic user-equilibrium condition.

In the second step, we consider the assignment of traffic stream 2 to $D2$ during the time interval $[t_1, t_2]$. Since $P2b$ is the only route choice, so all stream 2 vehicles should be assigned there. The resulting assignment from these two steps is indeed the user-equilibrium solution for both traffic streams since no vehicle can be better off by changing routes. It should be obvious that increasing the demand to $D2$, accomplished by extending t_2, will not alter the assignment solution. In summary, the dynamic user-equilibrium condition requires that $P1$ should not be used at all in the assignment.

During the entire second assignment period, $[t_1, t_2]$, one should note that the queue upstream of the junction, initially formed by the stream 1 vehicles to $D1$, shall persist. This is because the demand for destination $D2$ in the second period, q_2, equals the capacity at the diverge branch b, Q_b. There is no extra capacity to recover or shorten the existing queue. On the contrary, if there were no stream 1 vehicles on path $P2$, there would not be a queue for the stream 2 vehicles. In Figure 4, the shading region represents the total delay to traffic stream 2 induced by the queue. All stream 2 vehicles will experience the same delay, regardless of their departure time from the origin.

4 The User-Equilibrium Solution by Minimizing the Objective Function

In this section, we contrast the solution obtained in Section 3, which is consistent with the definition of user-equilibrium, with the solution obtained by optimizing the objective function of DUETA models, as discussed in Section 1. Here we argue that the objective function will derive solutions that assign traffic stream 1 vehicles onto $P1$, thus contradicting the solutions obtained in Section 3. We will show that the equilibrium solution of Section 3 is not a local minimum of (2) because a small feasible perturbation can be used to reduce (2).

To begin, let the objective function value of the dynamic user-equilibrium solution

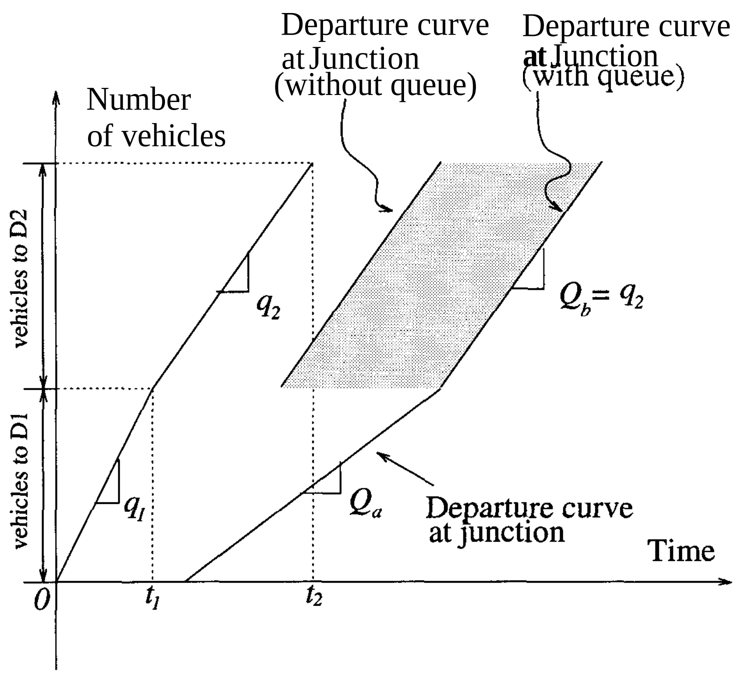


Figure 4: The delay to stream 2 vehicles incurred by traffic stream 1

obtained in Section 3 be expressed as:

$$\sum_t \sum_i \int_0^{x_i(t)} f(\omega) d\omega = C_1 + C_2.$$

where C_1 and C_2 are the respective objective function values associated with traffic streams 1 and 2.

Let's consider this alternate assignment scheme by reassigning the last m ($0 < m \leq q_1 t_1$) vehicles in traffic stream 1 to $P1$ and the rest to $P2a$ as before. The rerouting of the last m vehicles have no impact on stream 1 vehicles departed earlier. However, as a result of rerouting these m vehicles to the longer route $P1$, the objective function value associated with traffic stream 1 will be increased from C_1 to C_1'. We denote this increase, $C_1' - C_1$, by M.

By shifting these m vehicles to $P1$, the size of the queue on $P2$ to be faced by the incoming traffic stream 2 vehicles will be reduced. That is, for some i's during the time interval $[t_1, t_2]$, $x_i(t)$ will be lower[1]. Since the link performance function, f, is non-negative

[1] This should be expected because, to traffic stream 2, the presence of queues acts like a dynamic

and increasing in $x_i(t)$, the objective function value associated with traffic stream 2 should be decreased from C_2 to C_2'. Instead of comparing the change in each individual $x_i(t)$, we consider an aggregated X_T that represents the total demand to 0 2 for the entire assignment period and let $C_2 - C_2' = g(X_T)$.

Recall from Section **3** that the queue at the bottleneck, initially formed by the stream 1 vehicles, does not dissipate with the incoming traffic flow rate of q_2. Reducing the bottleneck delay or shortening the queue length by shifting m stream 1 vehicles away to path $P1$ will benefit all stream 2 vehicles heading for 0 2, because each stream 2 vehicle has a shorter queue to wait through. Basically, the more stream 2 vehicles there are, the higher will be the reduction term $g(X_T)$. In other words, $g(X_T)$ will increase with X_T. In this scenario, while $C_1' - C_1$ is maintained at M, $C_2 - C_2'$ can be made to increase indefinitely by letting $t_2 \to +\infty$ and hence $X_T \to +\infty$, since X_T is equal to $q_2(t_2 - t_1)$. In summary, as long as t_2 is extended sufficiently long, one can find a demand pattern for which $C_2 - C_2' > M = C_1' - C_1$, that means the objective function value can be reduced by assigning m vehicles to path $P1$. The assignment solution obtained by minimizing this objective function therefore clearly contradicts the solution obtained in Section **3,** which requires no vehicle be assigned to path $P1$ regardless of t_2.

5 Discussions

The example given in the previous sections shows that the DUETA models with an objective function of the form '& $\sum_i \int_0^{x_i(t)} f(\omega)d\omega$ cannot always guarantee a traffic pattern consistent with the dynamic user-equilibrium condition. The problem arises on a network in which queues are formed in the junction area. As shown in our example, the impact on delay in one time interval would continue to later time intervals. Consequently, if the objective function is to minimize the delay function for the entire duration, then the minimization may violate the user-equilibrium condition in order to reduce the overall objective function value.

The dynamic bottleneck shown in the example resembles the gridlock phenomena discussed in Daganzo, 1995. Despite the hypothetical nature of the scenario, similar

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bottleneck, taking away some effective capacities. The cost of going through a "long" bottleneck should be higher than the cost of going through a "short" bottleneck.

bottleneck situations can be observed in many locations. For example, if we view the diverge branch to $D1$ to be the one that leads to a toll plaza, we would expect traffic back up past the junction during peak hours, effectively blocking the traffic to the other branch (e.g. similar phenomenon can be observed in the junction area of 1-80 and 580 in the San Francisco Bay Area).

This example exhibits several inherent distinctions between dynamic and static assignment. In the static case, bottlenecks are fixed, embedded in the roadway geometry. In the dynamic case, however, bottlenecks can result from vehicle interactions (e.g. competing traffic flow from ramps, temporary lane closure due to incidents, or queuing at diverge junctions such as the one shown in the example). Consequently, bottlenecks are dynamic, which arise and vanish in time, depending on the traffic condition especially on the formation and dissipation of queues.

Time dimension is one important issue that complicates dynamic traffic assignment. For static assignment, without modeling the time dimension or differentiating the departure and arrival time of traffic streams, all traffic streams coexist at the same "time". However, in dynamic user-equilibrium traffic assignment, the time sequence in which the events happen has to be observed. The impacts of earlier events on subsequent events have to be carefully introduced. Apparently, according to our example, the DUETA models extended from the static model with an additive and separable objective function do not seem to be able to handle this problem well.

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