Degenerate Competing Three-Particle Systems *

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Abstract

We study systems of three interacting particles, in which drifts and variances are assigned by rank. These systems are "degenerate": the variances corresponding to one or two ranks can vanish, so the corresponding ranked motions become ballistic rather than diffusive. Depending on which ranks are allowed to "go ballistic", the systems exhibit markedly different behavior which we study in some detail. Also studied are stability properties for the resulting planar process of gaps between successive ranks.

Keywords and Phrases: Competing particle systems; local times; reflected planar Brownian motion; triple collisions; structure of filtrations.

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1 Introduction

In recent years, systems of interacting particles that assign local characteristics to individual particles by rank, rather than by index ("name"), have received quite a bit of attention under the rubric of "competing particle systems". A crucial common feature of all these studies is non-degeneracy: particles of all ranks are assigned some local variance that is strictly positive.

We study here, and to the best of our knowledge for the first time, systems of such competing particles that are allowed to degenerate, meaning that the variances corresponding to some ranks can vanish. This kind of degeneracy calls for an entirely new theory to handle the resulting systems; we initiate such a theory in the context of systems consisting of three particles. Even with this simplification, the range of behavior these systems can exhibit is quite rich. We illustrate just how rich, by studying in detail the construction and properties of three such systems – in Sections 2–6, 7 and 8, respectively.

A salient feature emerging from the analysis, is that two purely ballistic ranked motions can never "pinch" a diffusive motion running between and reflected off from them (Proposition 4.1); whereas two diffusive ranked motions *can* pinch a purely ballistic one running in their midst and reflected off from

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them, resulting in a massive collision that involves all three particles (§7.2). Using simple excursion-theoretic ideas, we show in Proposition 7.1 how such a system can extricate itself from a triple collision; but also that the solution to the system of stochastic equations that describe its motion, which is demonstrably strong up until the first time a triple collision occurs, *ceases to be strong after that time*. This last question on the structure of filtrations had been open for several years.

The analysis of the three-particle systems is deeply connected to that of planar, semimartingale reflecting Brownian motions (SRBM) and their local times on the faces of the nonnegative orthant. Under appropriate conditions amounting to the so-called "skew-symmetry" for SRBM, the planar process of gaps between ranked particles has an invariant distribution. We exhibit this joint distribution explicitly in one instance (§8.3), and offer a conjecture for it in another (Remark 6.4). We show that the former is the product (8.25) of its exponential marginals, while the latter is determined by the distribution of the sum of its marginals as in (6.18) and is not of product form (Remark 6.2).

2 Diffusion in the Middle, with Ballistic Hedges

Given real numbers $\delta_1, \delta_2, \delta_3$ and $x_1 > x_2 > x_3$, we would like to start by constructing a filtered probability space $(\Omega, \mathfrak{F}, \mathbb{P}), \mathbb{F} = \{\mathfrak{F}(t)\}_{0 \le t < \infty}$ and on it three independent, adapted Brownian motions $B_1(\cdot), B_2(\cdot), B_3(\cdot)$, as well as the three continuous, adapted processes $X_1(\cdot), X_2(\cdot), X_3(\cdot)$, so that

$$X_i(\cdot) \,=\, x_i + \sum_{k=1}^{3} \delta_k \int_0^{\cdot} \mathbf{1}_{\{X_i(t)=R_k^X(t)\}}\, \mathrm{d}t + \int_0^{\cdot} \mathbf{1}_{\{X_i(t)=R_2^X(t)\}}\, \mathrm{d}B_i(t)\,, \quad i = 1,2,3\,, \tag{2.1}$$

$$\int_0^{\infty} \mathbf{1}_{\{R_k^X(t)=R_\ell^X(t)\}}\, \mathrm{d}t \,=\, 0\,, \quad \forall\ k < \ell \tag{2.2}$$

and

$$\big\{ t \in (0,\infty) \,:\, R_1^X(t) = R_3^X(t) \big\} \,=\, \emptyset \tag{2.3}$$

hold with probability one. Here we denote by

$$\max_{j=1,2,3} X_j(t) =: R_1^X(t) \ge R_2^X(t) \ge R_3^X(t) := \min_{j=1,2,3} X_j(t)\,, \qquad t \in [0,\infty) \tag{2.4}$$

the reverse order statistics, and adopt the convention of resolving ties always in favor of the lowest index i; for instance, we set

$$R_1^X(t) = X_1(t)\,,\ \ R_2^X(t) = X_3(t)\,,\ \ R_3^X(t) = X_2(t) \qquad \text{on}\ \ \big\{X_1(t) = X_3(t) > X_2(t)\big\}\,.$$

The dynamics of (2.1) mandate ballistic motions for the leader and laggard particles with drifts δ_1 and δ_3, respectively, which act here as "outer hedges"; and a diffusive (Brownian) motion with drift δ_2, for the particle in the middle. The condition (2.2) posits that collisions of particles are *non-sticky,* in the sense that the set of all collision times has zero LEBESGUE measure; while the condition (2.3) proscribes triple collisions altogether.

As a canonical example of this situation, it is useful to consider the symmetric case

$$\delta_3 = -\delta_1 = g > 0 = \delta_2\,; \tag{2.5}$$

that is, ballistic motion with negative drift $-g$ for the leading particle, ballistic motion with positive drift g for the laggard particle, and purely diffusive (Brownian) motion for the particle in the middle. In this case, the system of equations (2.1) takes the appealing, symmetric form

$$X_i(\cdot) = x_i + g \int_0^{\cdot} \Big(\mathbf{1}_{\{X_i(t)=R_3^X(t)\}} - \mathbf{1}_{\{X_i(t)=R_1^X(t)\}} \Big) \mathrm{d}t + \int_0^{\cdot} \mathbf{1}_{\{X_i(t)=R_2^X(t)\}}\, \mathrm{d}B_i(t)\,. \tag{2.6}$$

This system was first introduced and studied in the technical report FERNHOLZ (2010). Figure 1, which offers a nice illustration of the path behavior for this system, is taken from that report.

A very salient feature of the dynamics in (2.1) is that its dispersion structure is both degenerate and discontinuous. It should come then as no surprise, that the analysis of the system (2.1)-(2.3) might not be entirely trivial; in particular, it is not covered by the results in either STROOCK & VARADHAN (1979) or BASS & PARDOUX (1987). The question then, is whether a three-dimensional diffusion with the dynamics (2.1) and the properties (2.2), (2.3) exists; and if so, whether its distribution is uniquely determined, and whether one can compute its transition probabilities and characterize its ergodic behavior.

3 Analysis

Suppose that such a solution to the system of equation (2.1) subject to the conditions of (2.2), (2.3) as postulated in the previous section, has been constructed. Under these conditions, its reverse order statistics are given then as

$$R_1^X(t) \;=\; x_1 + \delta_1\, t + \frac{1}{2}\, \Lambda^{(1,2)}(t) \tag{3.1}$$

$$R_2^X(t) \;=\; x_2 + \delta_2\, t + W(t) - \frac{1}{2}\, \Lambda^{(1,2)}(t) + \frac{1}{2}\, \Lambda^{(2,3)}(t) \tag{3.2}$$

$$R_3^X(t) \;=\; x_3 + \delta_3\, t - \frac{1}{2}\, \Lambda^{(2,3)}(t) \tag{3.3}$$

for $0 \le t < \infty$, on the strength of the results in BANNER & GHOMRASNI (2008). Here the process

$$W(\cdot) := \sum_{i=1}^{3} \int_0^{\cdot} \mathbf{1}_{\{X_i(t)=R_2^X(t)\}}\, \mathrm{d}B_i(t) = \sum_{i=1}^{3} \left(X_i(\cdot) - x_i - \sum_{k=1}^{3} \delta_k \int_0^{\cdot} \mathbf{1}_{\{X_i(t)=R_k^X(t)\}} \mathrm{d}t \right) \tag{3.4}$$

is standard Brownian motion by the P. LÉVY theorem, and we denote by

$$\Lambda^{(k,\ell)}(t) \;\equiv\; L^{R_k^X - R_\ell^X}(t)\,, \qquad k < \ell \tag{3.5}$$

the local time accumulated at the origin by the continuous, nonnegative semimartingale $R_k^X(\cdot) - R_\ell^X(\cdot)$ over the time interval $[0, t]$. Here and in what follows, we use the convention

$$L^{\Xi}(\cdot) \equiv L^{\Xi}(\cdot\,;0) \;:=\; \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^{\cdot} \mathbf{1}_{\{\Xi(t)<\varepsilon\}}\, \mathrm{d}\langle M \rangle(t) \;=\; \int_0^{\cdot} \mathbf{1}_{\{\Xi(t)=0\}}\, \mathrm{d}\Xi(t) \;=\; \int_0^{\cdot} \mathbf{1}_{\{\Xi(t)=0\}}\, \mathrm{d}C(t) \tag{3.6}$$

for the "right" local time at the origin of a continuous, nonnegative semimartingale $\Xi(\cdot) = \Xi(0) + M(\cdot) + C(\cdot)$, with $M(\cdot)$ a continuous local martingale and $C(\cdot)$ a process of finite first variation on compact intervals of the real line. The local time process $L^{\Xi}(\cdot)$ is continuous, adapted and nondecreasing, flat off the set $\{t \ge 0 : \Xi(t) = 0\}$.

We set now

$$G(\cdot) \;:=\; R_1^X(\cdot) - R_2^X(\cdot)\,, \qquad H(\cdot) \;:=\; R_2^X(\cdot) - R_3^X(\cdot) \tag{3.7}$$

for the sizes of the gaps between the leader and the middle particle, and between the middle particle and the laggard, respectively, and obtain from (3.1)–(3.5) the semimartingale representations

$$G(t) \;=\; x_1 - x_2 - \big(\delta_2 - \delta_1\big)\, t - W(t) - \frac{1}{2}\, L^H(t) + L^G(t)\,, \qquad 0 \le t < \infty \tag{3.8}$$

$$H(t) \;=\; x_2 - x_3 - \big(\delta_3 - \delta_2\big)\, t + W(t) - \frac{1}{2}\, L^G(t) + L^H(t)\,, \qquad 0 \le t < \infty \tag{3.9}$$

where we recall $L^G(\cdot) \equiv \Lambda^{(1,2)}(\cdot)$, $L^H(\cdot) \equiv \Lambda^{(2,3)}(\cdot)$ from (3.7), (3.5). We introduce also the continuous semimartingales

$$U(t) = x_1 - x_2 - \big(\delta_2 - \delta_1\big)\, t - W(t) - \frac{1}{2}\, L^H(t)\,, \quad V(t) = x_2 - x_3 - \big(\delta_3 - \delta_2\big)\, t + W(t) - \frac{1}{2}\, L^G(t)\,,$$

and note

$$G(\cdot) \,=\, U(\cdot) + L^G(\cdot) \,\geq\, 0\,, \qquad \int_0^\infty \mathbf{1}_{\{G(t)>0\}}\, \mathrm{d}L^G(t) \,=\, 0 \tag{3.10}$$

$$H(\cdot) \,=\, V(\cdot) + L^H(\cdot)) \,\geq\, 0\,, \qquad \int_0^\infty \mathbf{1}_{\{H(t)>0\}}\, \mathrm{d}L^H(t) \,=\, 0\,. \tag{3.11}$$

In other words, the "gaps" $G(\cdot)$, $H(\cdot)$ are the SKOROKHOD reflections of the semimartingales $U(\cdot)$ and $V(\cdot)$, respectively. The theory of the SKOROKHOD reflection problem (e.g., Lemma 3.6.14 in KARATZAS & SHREVE (1991)) provides now the relationships

$$L^G(t) \,=\, \max_{0 \leq s \leq t} \big(-U(s)\big)^+ \,=\, \max_{0 \leq s \leq t} \Big(-(x_1 - x_2) + \big(\delta_2 - \delta_1\big)\, s + W(s) + \frac{1}{2}\, L^H(s)\Big)^+, \tag{3.12}$$

$$L^H(t) \,=\, \max_{0 \leq s \leq t} \big(-V(s)\big)^+ \,=\, \max_{0 \leq s \leq t} \Big(-(x_2 - x_3) + \big(\delta_3 - \delta_2\big)\, s - W(s) + \frac{1}{2}\, L^G(s)\Big)^+ \tag{3.13}$$

between the two local time processes $L^G(\cdot) \equiv \Lambda^{(1,2)}(\cdot)$ and $L^H(\cdot) \equiv \Lambda^{(2,3)}(\cdot)$, once the scalar Brownian motion $W(\cdot)$ has been specified.

- Finally, we note that the equations of (3.8)-(3.9) can be cast in the form

$$\begin{pmatrix} G(t) \\ H(t) \end{pmatrix} =:\, \mathfrak{G}(t) \,=\, \mathfrak{g} + \mathfrak{Z}(t) + \mathcal{R}\,\mathfrak{L}(t)\,, \qquad 0 \leq t < \infty\,, \tag{3.14}$$

of HARRISON & REIMAN (1981), where

$$\mathcal{R} \,=\, \mathcal{I} - \mathcal{Q}\,, \qquad \mathcal{Q} \,:=\, \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}, \qquad \mathfrak{g} = \mathfrak{G}(0)\,, \qquad \mathfrak{L}(t) \,=\, \begin{pmatrix} L^G(t) \\ L^H(t) \end{pmatrix},$$

and

$$\mathfrak{Z}(t) \,=\, \begin{pmatrix} (\delta_1 - \delta_2)t - W(t) \\ (\delta_2 - \delta_3)t + W(t) \end{pmatrix}, \qquad 0 \leq t < \infty\,. \tag{3.15}$$

One reflects, in other words, off the faces of the nonnegative quadrant, the degenerate, two-dimensional Brownian motion $\mathfrak{Z}(\cdot)$ with drift vector and covariance matrix given respectively by

$$\boldsymbol{m} \,=\, \big(\delta_1 - \delta_2\,,\, \delta_2 - \delta_3\big)'\,, \qquad \mathcal{C} \,:=\, \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \tag{3.16}$$

The directions of reflection are the row vectors of the *reflection matrix* $\mathcal{R}$, and the matrix $\mathcal{Q} = \mathcal{I} - \mathcal{R}$ has spectral radius strictly less than 1, in agreement with the HARRISON & REIMAN (1981) theory.

4 Synthesis

We start with given real numbers $\delta_1,\, \delta_2,\, \delta_3$, and $x_1 > x_2 > x_3$, and construct a filtered probability space $(\Omega, \mathfrak{F}, \mathbb{P}),\, \mathbb{F} = \big\{\mathfrak{F}(t)\big\}_{0 \leq t < \infty}$ rich enough to support a scalar, standard Brownian motion $W(\cdot)$. In fact, we select the filtration $\mathbb{F}$ to be $\mathbb{F}^W = \big\{\mathfrak{F}^W(t)\big\}_{0 \leq t < \infty}$, the smallest right-continuous filtration to which the scalar Brownian motion $W(\cdot)$ is adapted.

Informed by the analysis of the previous section we consider, by analogy with (3.12)-(3.13), the system of equations

$$A(t) \,=\, \max_{0\le s\le t} \Big(-(x_1-x_2)+\big(\delta_2-\delta_1\big)\,s+W(s)+\frac{1}{2}\,\Gamma(s)\Big)^+, \qquad 0\le t<\infty \tag{4.1}$$

$$\Gamma(t) \,=\, \max_{0\le s\le t} \Big(-(x_2-x_3)+\big(\delta_3-\delta_2\big)\,s-W(s)+\frac{1}{2}\,A(s)\Big)^+, \qquad 0\le t<\infty \tag{4.2}$$

for two continuous, nondecreasing and adapted processes $A(\cdot)$ and $\Gamma(\cdot)$ with $A(0)=\Gamma(0)=0$. This system of equations is of the type studied in HARRISON & REIMAN (1981). From Theorem 1 of that paper, we know that the system (4.1), (4.2) possesses a unique, $\mathbb{F}^W-$adapted solution.

Once the solution $\big(A(\cdot),\Gamma(\cdot)\big)$ to this system has been constructed, we define the processes

$$U(t):=x_1-x_2-\big(\delta_2-\delta_1\big)\,t-W(t)-\frac{1}{2}\,\Gamma(t)\,,\quad V(t):=x_2-x_3-\big(\delta_3-\delta_2\big)\,t+W(t)-\frac{1}{2}\,A(t) \tag{4.3}$$

and then "fold" them to obtain their SKOROKHOD reflections; that is, the nonnegative continuous semimartingales

$$G(t)\,:=\,U(t)+\max_{0\le s\le t}\big(-U(s)\big)^+\,=\,x_1-x_2-\big(\delta_2-\delta_1\big)\,t-W(t)-\frac{1}{2}\,\Gamma(t)+A(t)\,\ge\,0 \tag{4.4}$$

$$H(t)\,:=\,V(t)+\max_{0\le s\le t}\big(-V(s)\big)^+\,=\,x_2-x_3-\big(\delta_3-\delta_2\big)\,t+W(t)-\frac{1}{2}\,A(t)+\Gamma(t)\,\ge\,0 \tag{4.5}$$

for $t\in[0,\infty)$, in accordance with (3.10)–(3.13). From the theory of the SKOROKHOD reflection problem once again, we deduce the a.s. properties

$$\int_0^\infty \mathbf{1}_{\{G(t)>0\}}\,\mathrm{d}A(t)\,=\,0\,,\qquad \int_0^\infty \mathbf{1}_{\{H(t)>0\}}\,\mathrm{d}\Gamma(t)\,=\,0\,; \tag{4.6}$$

and from the theory of semimartingale local time (KARATZAS & SHREVE (1991), Exercise 3.7.10) we obtain

$$\int_0^\infty \mathbf{1}_{\{G(t)=0\}}\,\mathrm{d}t=\int_0^\infty \mathbf{1}_{\{G(t)=0\}}\,\mathrm{d}\langle G\rangle(t)=0\,,\quad \int_0^\infty \mathbf{1}_{\{H(t)=0\}}\,\mathrm{d}t=\int_0^\infty \mathbf{1}_{\{H(t)=0\}}\,\mathrm{d}\langle H\rangle(t)=0\,. \tag{4.7}$$

4.1 Constructing the Ranks

We introduce now, by analogy with (3.1)-(3.3), the processes

$$R_1(t)\,:=\,x_1+\delta_1\,t+\frac{1}{2}\,A(t)\,,\qquad R_3(t)\,:=\,x_3+\delta_3\,t-\frac{1}{2}\,\Gamma(t)\,, \tag{4.8}$$

$$R_2(t)\,:=\,x_2+\delta_2\,t+W(t)-\frac{1}{2}\,A(t)+\frac{1}{2}\,\Gamma(t) \tag{4.9}$$

for $0\le t<\infty$ and note the relations $R_1(\cdot)-R_2(\cdot)=G(\cdot)\ge 0$, $R_2(\cdot)-R_3(\cdot)=H(\cdot)\ge 0$ in conjunction with (4.4) and (4.5). In other words, we have the a.s. comparisons, or "rankings", $R_1(\cdot)\ge R_2(\cdot)\ge R_3(\cdot)$. It is clear from the discussion following (4.1), (4.2), that these processes are adapted to the filtration generated by the driving Brownian motion $W(\cdot)$, whence the inclusion $\mathbb{F}^{(R_1,R_2,R_3)}\subseteq\mathbb{F}^W$.

Let us show that these rankings never collapse. To put things a bit colloquially: *"Two ballistic motions cannot squeeze a diffusive (Brownian) motion"*. We are indebted to Drs. Robert FERNHOLZ (cf. FERNHOLZ (2010)) and Johannes RUF for the argument that follows.

Proposition 4.1. *With probability one, we have:* $R_1(\cdot) - R_3(\cdot) = G(\cdot) + H(\cdot) > 0$.

Proof: We shall show that there cannot possibly exist numbers $T \in (0,\infty)$ and $r \in \mathbb{R}$, such that $R_1(T) = R_2(T) = R_3(T) = r$.

We argue by contradiction: If such a configuration were possible for some $\omega \in \Omega$ and some $T = T(\omega) \in (0,\infty)$, $r = r(\omega) \in \mathbb{R}$, we would have

$$r - \delta_3(T-t) \;\le\; R_3(t,\omega) \;\le\; R_2(t,\omega) \;\le\; R_1(t,\omega) \;\le\; r - \delta_1(T-t)\,, \qquad 0 \le t < T\,.$$

This is already impossible if $\delta_1 > \delta_3$, so let us assume $\delta_1 \le \delta_3$ and try to arrive at a contradiction in this case as well. The above quadruple inequality implies, a fortiori,

$$r - \delta_3(T-t) \;\le\; \overline{R}(t,\omega) := \frac{1}{3}\big(R_3(t,\omega) + R_2(t,\omega) + R_1(t,\omega)\big) \;\le\; r - \delta_1(T-t)\,, \qquad 0 \le t < T\,.$$

But we have $r - \overline{R}(t,\omega) = \overline{R}(T,\omega) - \overline{R}(t,\omega) = \overline{\delta}(T-t) + \big(W(T,\omega) - W(t,\omega)\big)/3$, where $\overline{\delta} := (\delta_1 + \delta_2 + \delta_3)/3$, and back into the above inequality this gives

$$3\left(\delta_1 - \overline{\delta}\right) \;\le\; \frac{W(T,\omega) - W(t,\omega)}{T-t} \;\le\; 3\left(\delta_3 - \overline{\delta}\right), \qquad 0 \le t < T\,.$$

However, from the PAYLEY-WIENER-ZYGMUND theorem for the Brownian motion $W(\cdot)$ (KARATZAS & SHREVE (1991), Theorem 2.9.18, p. 110), this double inequality would force ω into a $\mathbb{P}-$null set. $\square$

4.2 Identifying the Increasing Processes $A(\cdot)$, $\Gamma(\cdot)$ as Local Times

We claim that, in addition to (4.6) and (4.7), the properties

$$\int_0^\infty \mathbf{1}_{\{H(t)=0\}}\, \mathrm{d}A(t) \;=\; 0\,, \qquad \int_0^\infty \mathbf{1}_{\{G(t)=0\}}\, \mathrm{d}\Gamma(t) \;=\; 0 \tag{4.10}$$

are also valid a.s. Indeed, we know from (4.6) that $A(\cdot)$ is flat off the set $\{t \ge 0 : G(t) = 0\}$, so we have $\int_0^\infty \mathbf{1}_{\{H(t)=0\}}\, \mathrm{d}A(t) = \int_0^\infty \mathbf{1}_{\{H(t)=G(t)=0\}}\, \mathrm{d}A(t)$; but this last expression is a.s. equal to zero because, as we have shown, $\{t \ge 0 : G(t) = H(t) = 0\} = \emptyset$ holds mod. $\mathbb{P}$. This proves the first equality in (4.10); the second is argued similarly.

But now, the local time at the origin of the continuous, nonnegative semimartingale $G(\cdot)$ is given as

$$L^G(\cdot) = \int_0^\cdot \mathbf{1}_{\{G(t)=0\}}\, \mathrm{d}G(t) = \int_0^\cdot \mathbf{1}_{\{G(t)=0\}}\, \mathrm{d}A(t)$$
$$- \int_0^\cdot \mathbf{1}_{\{G(t)=0\}}\, \frac{\mathrm{d}\Gamma(t)}{2} + \big(\delta_1 - \delta_2\big) \int_0^\cdot \mathbf{1}_{\{G(t)=0\}}\, \mathrm{d}t$$

on the strength of (3.6) and (4.4). From (4.7) and (4.10) the last two integrals vanish, so (4.6) leads to

$$L^G(\cdot) \;=\; \int_0^\cdot \mathbf{1}_{\{G(t)=0\}}\, \mathrm{d}A(t) \;=\; A(\cdot)\,; \qquad \text{and} \qquad L^H(\cdot) \;=\; \Gamma(\cdot) \tag{4.11}$$

is shown similarly. In the light of Proposition 4.1, these local times satisfy the rather interesting property

$$\frac{1}{2}\Big(L^G(t) + L^H(t)\Big) \;>\; x_3 - x_1 + \big(\delta_3 - \delta_1\big)\, t\,, \qquad 0 < t < \infty\,. \tag{4.12}$$

Remark 4.1 (The Structure of Filtrations). We have identified the component processes of the pair $(A(\cdot), \Gamma(\cdot))$, solution of the system (4.1)-(4.2), as the local times at the origin of the continuous semimartingales $R_1(\cdot) - R_2(\cdot) = G(\cdot) \ge 0$, $R_2(\cdot) - R_3(\cdot)H(\cdot) \ge 0$. In particular, this implies $\mathbb{F}^{A,\Gamma} \subseteq \mathbb{F}^{(R_1,R_2,R_3)}$; and back in (4.9), it gives $\mathbb{F}^W \subseteq \mathbb{F}^{(R_1,R_2,R_3)}$.

But we have already established the reverse of this inclusion, and so we conclude that the process of ranks generates exactly the same filtration as its driving Brownian motion: $\mathbb{F}^{(R_1,R_2,R_3)} = \mathbb{F}^W$.

4.3 Constructing the "Names" (Individual Motions)

Once the "ranks" $R_1(\cdot) \geq R_2(\cdot) \geq R_3(\cdot)$ have been constructed in section 4.1 on the filtered probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, $\mathbb{F} = \{\mathfrak{F}(t)\}_{0 \leq t < \infty}$, with $\mathbb{F}$ selected as the smallest right-continuous filtration $\mathbb{F}^W = \{\mathfrak{F}^W(t)\}_{0 \leq t < \infty}$ to which the scalar Brownian motion $W(\cdot)$ is adapted, we can construct as in the proof of Theorem 5 in KARATZAS ET AL. (2016) the "names" that correspond to these ranks – that is, processes $X_1(\cdot), X_2(\cdot), X_3(\cdot)$, as well as a three-dimensional Brownian motion $(B_1(\cdot), B_2(\cdot), B_3(\cdot))$ defined on this same space and such that the equation (2.1) is satisfied and $R_k^X(\cdot) \equiv R_k(\cdot)$, $k = 1, 2, 3$.

It is also clear from our construction that the conditions (2.2) and (2.3) are also satisfied: the first thanks to the properties of (4.7), the second because of Proposition 4.1.

• Alternatively, the construction of a pathwise unique, strong solution for the system (2.1) can be carried out along the lines of Proposition 8 in ICHIBA ET AL. (2013). We start at time $\tau_0 \equiv 0$ and follow the paths of the top particle and of the pair consisting of the bottom two particles *separately*, until the top particle collides with the leader of the bottom pair (at time ϱ_0). Then we follow the paths of the bottom particle and of the pair consisting of the top two particles *separately*, until the bottom particle collides with the laggard of the top pair (at time τ_1). We repeat the procedure until the first time we see a triple collision, obtain two interlaced sequences of stopping times $\{\tau_k\}_{k \in \mathbb{N}_0}$ and $\{\varrho_k\}_{k \in \mathbb{N}_0}$ with

$$0 = \tau_0 \leq \varrho_0 \leq \tau_1 \leq \varrho_1 \leq \cdots \leq \tau_k \leq \varrho_k \leq \cdots \ , \tag{4.13}$$

and denote by

$$\mathcal{S} := \inf \big\{ t \in (0, \infty) : X_1(t) = X_2(t) = X_3(t) \big\} = \lim_{k \to \infty} \tau_k = \lim_{k \to \infty} \varrho_k \tag{4.14}$$

the first time a triple collision occurs. During each interval of the form $[\tau_k, \varrho_k)$ or $[\varrho_k, \tau_{k+1})$, a pathwise unique, strong solution of the corresponding two-particle system is constructed as in Theorem 4.1 in FERNHOLZ ET. AL. (2013b).

We end up in this manner with a three-dimensional Brownian motion $(B_1(\cdot), B_2(\cdot), B_3(\cdot))$, and with three processes $X_1(\cdot), X_2(\cdot), X_3(\cdot)$ that satisfy the system of (2.1) as well as the requirement (2.2), once again thanks to results in FERNHOLZ ET AL. (2013b). For this system, the ranked processes $R_1^X(\cdot) \geq R_2^X(\cdot) \geq R_3^X(\cdot)$ as in (2.4), satisfy the equations we studied in Sections 2, 3 and generate the same filtration $\mathbb{F}^W = \{\mathfrak{F}^W(t)\}_{0 \leq t < \infty}$ as the scalar Brownian motion $W(\cdot)$ above (Remark 4.1). We have seen in Proposition 4.1 that for such a system there are no triple collisions, to wit, $\mathcal{S} = \infty$; thus the condition (2.3) is satisfied as well, and all the inequalities in (4.13) are strict.

• We have proved the following result.

Theorem 4.2. *The system of equations* (2.1) *admits a pathwise unique strong solution, satisfying the conditions of* (2.2) *and* (2.3).

Figure 1, reproduced here from FERNHOLZ (2011), illustrates the trajectories of these three scalar random motions $X_1(\cdot)$, $X_2(\cdot)$, $X_3(\cdot)$ in the case of the canonical example (2.5). It is very clear from this picture and from the construction in subsections 3.1, 3.2 that the middle particle $R_2(\cdot)$ undergoes Brownian motion $W(\cdot)$ with reflection at the upper and lower boundaries, respectively $R_1(\cdot)$ and $R_3(\cdot)$, of a time-dependent domain.

In contrast to the situation of the "double SKOROKHOD map" studied by KRUK et al. (2007), where the upper and lower reflecting boundaries are given constants, these boundaries $R_1(\cdot)$ and $R_3(\cdot)$ here are functions of time with finite first variation on compact intervals. They are "sculpted" by the Brownian motion $W(\cdot)$ itself, via the local times $L^G(\cdot) \equiv A(\cdot)$ and $L^H(\cdot) \equiv \Gamma(\cdot)$, in the manner of the system (4.1), (4.2). The upper (respectively, lower) boundary decreases (resp., increases) by linear segments at a 45^o angle, and increases (resp., decreases) by a singularly continuous "devil's staircase", governed by the local time $L^G(\cdot) \equiv A(\cdot)$ (resp., $L^H(\cdot) \equiv \Gamma(\cdot)$).

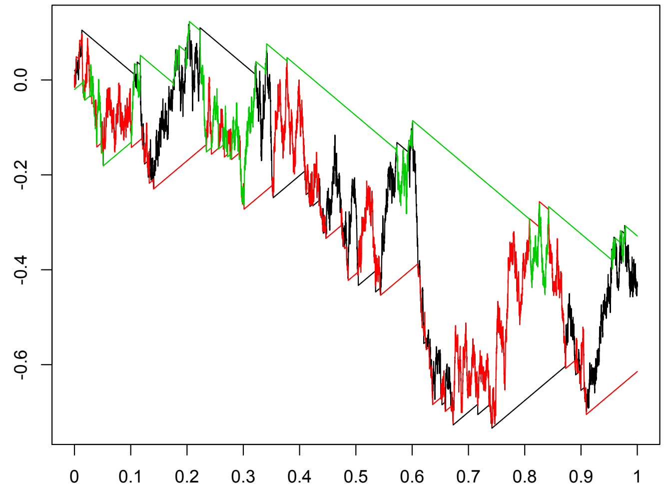

Figure 1: Simulated processes for the system in (2.6) with $g = 1$: Black $= X_1(\cdot)$, Red $= X_2(\cdot)$, Green $= X_3(\cdot)$. The 3-D process $(X_1(\cdot), X_2(\cdot), X_3(\cdot))$ carries the same information content as a scalar Brownian Motion. We are indebted to Dr. E.R. FERNHOLZ for this picture, which illustrates the "ballistic nature of the outer hedges" and the diffusive motion in the middle.

5 Positive Recurrence and Ergodicity

We present now a criterion for the process $(G(\cdot), H(\cdot))$ in (3.8)-(3.9) to reach an arbitrary open neighborhood of the origin in finite expected time.

We carry out this analysis along the lines of HOBSON & ROGERS (1993). The system studied by these authors is a *non-degenerate* reflected Brownian motion $(\boldsymbol{X}_\cdot, \boldsymbol{Y}_\cdot)$ in the first orthant driven by a planar Brownian motion $(\boldsymbol{B}_\cdot, \boldsymbol{W}_\cdot)$, namely

$$\boldsymbol{X}_t \,=\, \boldsymbol{x} + \boldsymbol{B}_t + \boldsymbol{\mu}\, t + \boldsymbol{L}_t^{\boldsymbol{X}} + \boldsymbol{\alpha} \boldsymbol{L}_t^{\boldsymbol{Y}}\,, \quad \boldsymbol{Y}_t \,=\, \boldsymbol{y} + \boldsymbol{W}_t + \boldsymbol{\nu}\, t + \boldsymbol{\beta} \boldsymbol{L}_t^{\boldsymbol{X}} + \boldsymbol{L}_t^{\boldsymbol{Y}}\,, \quad 0 \le t < \infty\,. \tag{5.1}$$

Here $(\boldsymbol{x}, \boldsymbol{y})$ is the initial state in the nonnegative quadrant, and $\boldsymbol{\mu}$, $\boldsymbol{\nu}$, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ are real constants. A necessary and sufficient condition for the positive recurrence of $(\boldsymbol{X}_\cdot, \boldsymbol{Y}_\cdot)$ in (5.1) is

$$\boldsymbol{\mu} + \boldsymbol{\alpha} \boldsymbol{\nu}^- < 0\,, \quad \boldsymbol{\nu} + \boldsymbol{\beta} \boldsymbol{\mu}^- < 0 \tag{5.2}$$

(Proposition 2.3 of HOBSON & ROGERS (1993)); and $x^- = \max(-x, 0)$ is the negative part of $x \in \mathbb{R}$.

By contrast, our system (3.8)-(3.9) is driven by the single Brownian motion $W(\cdot)$, thus *degenerate,* and has the form

$$\boldsymbol{X}_t \,=\, \boldsymbol{x} - \boldsymbol{W}_t + \boldsymbol{\mu} t + \boldsymbol{L}_t^{\boldsymbol{X}} + \boldsymbol{\alpha} \boldsymbol{L}_t^{\boldsymbol{Y}}\,, \quad \boldsymbol{Y}_t \,=\, \boldsymbol{y} + \boldsymbol{W}_t + \boldsymbol{\nu} t + \boldsymbol{\beta} \boldsymbol{L}_t^{\boldsymbol{X}} + \boldsymbol{L}_t^{\boldsymbol{Y}}\,, \quad 0 \le t < \infty \tag{5.3}$$

that one obtains by replacing formally the planar Brownian motion $(\boldsymbol{B}_\cdot, \boldsymbol{W}_\cdot)$ in (5.1) by $(-\boldsymbol{W}_\cdot, \boldsymbol{W}_\cdot)$. The system (3.8)-(3.9) can be cast in the form (5.3), if we replace formally the triple $(\boldsymbol{X}_\cdot, \boldsymbol{Y}_\cdot, \boldsymbol{W}_\cdot)$ by the triple $(G(\cdot), H(\cdot), W(\cdot))$ and substitute $\boldsymbol{\mu} = -(\delta_2 - \delta_1)$, $\boldsymbol{\nu} = -(\delta_3 - \delta_2)$, $\boldsymbol{\alpha} = \boldsymbol{\beta} = -1\,/\,2$.

We have the following result.

Proposition 5.1. *If the conditions*

$$2(\delta_3-\delta_2)+\big(\delta_1-\delta_2\big)^- >0\,,\qquad 2(\delta_2-\delta_1)+\big(\delta_2-\delta_3\big)^- >0\,, \tag{5.4}$$

hold, then the process $\big(G(\cdot),H(\cdot)\big)$ *in (3.8)-(3.9) is positive recurrent, has a unique invariant probability measure* $\boldsymbol{\pi}$ *with* $\boldsymbol{\pi}((0,\infty)^2)=1$*, and converges to this measure in distribution as* $t\to\infty$*.*

Let us note that the condition (5.4) is a simple recasting of (5.2). It is satisfied in the special case of (2.5); and more generally, when the strict ordering

$$\delta_1\,<\,\delta_2\,<\,\delta_3 \tag{5.5}$$

holds. This last condition is strictly stronger than (5.4); for example, the choices $\delta_1=1\,/\,3$, $\delta_2=0$, $\delta_3=1$ satisfy (5.4) but not (5.5). Let us also note that (5.4) implies $\delta_1<\delta_3$, as well as at least one of $\delta_2>\delta_1$, $\delta_3>\delta_2$; that is, (5.4) excludes the possibility $\delta_3\le\delta_2\le\delta_1$. These claims are discussed in detail in subsection 5.1.1.

Under the condition (5.4), the sum of local times $L^G(\cdot)+L^H(\cdot)$ dominates a straight line with positive slope, on account of (4.12).

Proof: Let us define inductively two sequences of stopping times $\tau:=\tau_1=\inf\{s\ge0:G(s)=0\}$, $\sigma:=\sigma_1=\inf\{s\ge\tau:H(s)=0\}$, $\tau_n:=\inf\{s\ge\sigma_{n-1}:G(s)=0\}$, $\sigma_n:=\inf\{s\ge\tau_n:H(s)=0\}$ for $n=2,3,\ldots$. Also let us define $\mathbf{T}_0:=\inf\{s\ge0:G(s)H(s)=0\}$ and

$$\mathbf{T}_\dagger:=\inf\{s\ge0:G(s)\le x_0,H(s)=0\}\,,\quad \mathbf{T}_r:=\inf\{s\ge0:(G(s),H(s))\in\mathrm{B}_0(r)\}\,,$$

where $\mathrm{B}_0(r)$ is the ball of radius $r>0$ centered at the origin. Most of the arguments in HOBSON & ROGERS (1993) carry over smoothly to the degenerate system (3.8)-(3.9). In fact, we can replace $B(\cdot)$ by $-W(\cdot)$ in the proof of Propositions 2.1-2.2 of HOBSON & ROGERS (1993), and deduce that, under (5.4), there exists a large enough $x_0>0$ such that for $x_1-x_2\ge x_0$ we have

$$\mathbb{E}^{(x_1-x_2,0)}[G(\sigma_1)]\le(x_1-x_2)/2\,,\quad \mathbb{E}^{(x_1-x_2,0)}[\sigma_1]\le 2C(x_1-x_2)\,,$$

where C is some positive constant. Moreover, again replacing $B(\cdot)$ by $-W(\cdot)$ in the first part of the proof of Proposition 2.3 of HOBSON & ROGERS (1993), we deduce

$$\mathbb{E}[\mathbf{T}_\dagger]\,\le\,C\left(1+\sqrt{(x_1-x_2)^2+(x_2-x_3)^2}\,\right)\,.$$

We conjecture that there exists a constant $\boldsymbol{\delta}>0$ such that a uniform estimate

$$\inf_{0<y\le x_0}\mathbb{P}^{(y,0)}\Big(\mathbf{T}_\varepsilon\le\mathbf{T}_{2x_0}\wedge1\Big)\ge\boldsymbol{\delta}>0 \tag{5.6}$$

holds; once (5.6) has been established, positive recurrence under the condition (5.4) will follow.

Instead of showing (5.6), we shall show under the condition (5.4) that for every $0<\varepsilon<x_0$, there exists a positive constant $\boldsymbol{\delta}>0$ such that

$$\inf_{\varepsilon<y\le x_0}\mathbb{P}^{(y,0)}\big(\widetilde{\mathbf{T}}_\varepsilon\le\widetilde{\mathbf{T}}_{2x_0}\wedge\mathbf{t}_0(y)\big)\;\ge\;\boldsymbol{\delta}>0\,, \tag{5.7}$$

where $\mathbf{t}_0(y):=(y-(5/6)\varepsilon)\,/\,(\delta_3-\delta_1-(1/2)(\delta_3-\delta_2)^+)>0$, $\varepsilon<y\le x_0$ and

$$\widetilde{\mathbf{T}}_r:=\inf\{s\ge0:G(s)+H(s)=r\}\,,\quad r\ge0\,.$$

In fact, we evaluate the smaller probability $\inf_{\varepsilon<y\le x_0}\mathbb{P}^{(y,0)}(\widetilde{\mathbf{T}}_\varepsilon \le \widetilde{\mathbf{T}}_{2x_0}\wedge \mathbf{t}_0(y)\,,\widetilde{\mathbf{T}}_\varepsilon<\tau)$, where we recall $\tau := \inf\{s>0 : G(s)=0\}$, that is, the probability that the process $(G(\cdot),H(\cdot))$, starting from the point $G(0)=y\in(0,x_0]$, $H(0)=0$ in the axis, reaches the neighborhood of the origin, before going away from the origin and before attaining the other axis. The process $(G(\cdot),H(\cdot))$ does not accumulate any local time $A(\cdot)$ before time τ:

$$0<G(t)\;=\;y-(\delta_2-\delta_1)t-W(t)-(1/2)\Gamma(t)\,,\quad 0\le H(t)\;=\;-(\delta_3-\delta_2)t+W(t)+\Gamma(t)\,,$$

and consequently $G(t)+H(t)\;=\;y-(\delta_3-\delta_1)t+(1/2)\Gamma(t)$, for $0\le t\le\tau$. From the SKOHOKHOD construction, we obtain the upper bound for the local time $\Gamma(\cdot)$:

$$\Gamma(t)\;=\;\max_{0\le s\le t}\big(-W(s)+(\delta_3-\delta_2)s\big)^+\le\max_{0\le s\le t}(-W(s))^+ +(\delta_3-\delta_2)^+t\,,\quad 0\le t\le\tau\,.$$

Thus we obtain

$$G(t)\ge y-\Big(\delta_2-\delta_1+\frac{1}{2}(\delta_3-\delta_2)^+\Big)t-W(t)-\frac{1}{2}\max_{0\le s\le t}\big(-W(s)\big)^+\,, \tag{5.8}$$

$$G(t)+H(t)\le y-\Big(\delta_3-\delta_1-\frac{1}{2}(\delta_3-\delta_2)^+\Big)t+\frac{1}{2}\max_{0\le s\le t}\big(-W(s)\big)^+\,;\quad 0\le t\le\tau\,. \tag{5.9}$$

Now let us consider the event

$$A(y):=\Big\{\omega\in\Omega:\max_{0\le s\le \boldsymbol{t}_0(y)}|W(s,\omega)|\le\varepsilon\,/\,3\Big\}\,,\qquad \varepsilon<y\le x_0\,.$$

Since $\delta_3-\delta_1-(1/2)(\delta_3-\delta_2)^+\le\delta_2-\delta_1+(1/2)(\delta_3-\delta_2)^+$, for every $\omega\in A(y)$ we obtain

$$\min_{0\le t\le \boldsymbol{t}_0(y)}\Big[y-\Big(\delta_2-\delta_1+\frac{1}{2}(\delta_3-\delta_2)^+\Big)t-W(t,\omega)-\frac{1}{2}\max_{0\le s\le t}\big(-W(s,\omega)\big)^+\Big]\ge\frac{\varepsilon}{3}>0\,,$$

and hence, combining with (5.8), we obtain $A(y)\subset\{\mathbf{t}_0(y)<\tau\}$. Moreover, for every $\omega\in A(y)$ we have

$$\min_{0\le t\le\mathbf{t}_0(y)}\big(G(t,\omega)+H(t,\omega)\big)\le\min_{0\le t\le\mathbf{t}_0(y)}\Big[y-\Big(\delta_3-\delta_1-\frac{1}{2}(\delta_3-\delta_2)^+\Big)t+\frac{1}{2}\max_{0\le s\le t}\big(-W(s,\omega)\big)^+\Big]\le\varepsilon\,,$$

thus also

$$\max_{0\le t\le\mathbf{t}_0(y)}\big(G(t,\omega)+H(t,\omega)\big)\le x_0+\varepsilon<2x_0\,.$$

Thus $A(y)\subset\{\widetilde{\mathbf{T}}_\varepsilon\le\widetilde{\mathbf{T}}_{2x_0}\wedge\mathbf{t}_0(y),\widetilde{\mathbf{T}}_\varepsilon<\tau\}$. Therefore, by the reflection principle for Brownian motion,

$$\inf_{\varepsilon<y\le x_0}\mathbb{P}^{(y,0)}(\widetilde{\mathbf{T}}_\varepsilon\le\widetilde{\mathbf{T}}_{2x_0}\wedge\mathbf{t}_0(y))\ge\inf_{\varepsilon<y\le x_0}\mathbb{P}^{(y,0)}(\widetilde{\mathbf{T}}_\varepsilon\le\widetilde{\mathbf{T}}_{2x_0}\wedge\mathbf{t}_0(y),\widetilde{\mathbf{T}}_\varepsilon<\tau)$$

$$\ge\inf_{\varepsilon<y\le x_0}\mathbb{P}^{(y,0)}(A(y))\ge 1-\Big(\frac{\mathbf{t}_0(x_0)}{2\pi}\Big)^{1/2}\cdot\frac{4}{\varepsilon/3}\cdot\exp\Big(-\frac{(\varepsilon/3)^2}{2\mathbf{t}_0(x_0)}\Big)$$

(cf. Problem 2.8.2 of KARATZAS & SHREVE (1991)). Letting its right-hand side be $\boldsymbol{\delta}>0$, we obtain (5.7). Appealing to the second half of the proof of Proposition 2.3 in HOBSON & ROGERS (1993), page 393, we conclude that the system (3.8)-(3.9) is neighborhood positive recurrent under (5.4).

For the remaining claims of the Proposition, let us recall the equations of (3.8)-(3.9) written in the HARRISON & REIMAN (1981) form (3.14)-(3.16), and note that the process $\mathfrak{Z}(\cdot)$ of (3.15) has independent, stationary increments with $\mathfrak{Z}(t) = 0$ and $\mathbb{E}\big|\mathfrak{Z}(1)\big| < \infty$. Now, as is relatively easy to verify (and shown in subsection 5.1.1), the conditions of (5.4) imply that the components of the vector

$$-\mathcal{R}^{-1}\,\mathbb{E}\big(\mathfrak{Z}(1)\big) \;=\; \frac{2}{3}\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}\begin{pmatrix}\delta_2-\delta_1\\ \delta_3-\delta_2\end{pmatrix} \;=\; \frac{2}{3}\begin{pmatrix}\delta_2+\delta_3-2\delta_1\\ 2\delta_3-\delta_1-\delta_2\end{pmatrix} \;=:\; \begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix} \;=\; \boldsymbol{\lambda} \tag{5.10}$$

are both strictly positive; cf. (5.16) below. Then Corollary 2.1 in KELLA & RAMASUBRAMANIAN (2012) (cf. KHAS'MINSKII (1960), KELLA & WHITT (1996), Theorem 3.4 in KONSTANTOPOULOS ET AL. (2004), SARANTSEV (2016, 2017)) implies that $\mathfrak{G}(\cdot) = \big(G(\cdot), H(\cdot)\big)$ is positive recurrent, has a unique invariant probability measure $\boldsymbol{\pi}$, and converges to this measure in distribution as $t \to \infty$. In addition, for any bounded, measurable function $f \,:\, [0,\infty)^2 \to \mathbb{R}$ we have the strong law of large numbers

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f\big(G(t),H(t)\big)\,\mathrm{d}t \;=\; \int_{[0,\infty)^2} f(g,h)\,\boldsymbol{\pi}(\mathrm{d}g,\mathrm{d}h)\,, \qquad \text{a.s.}$$

The claim $\boldsymbol{\pi}\big((0,\infty)^2\big) = 1$ follows now from (4.7).

5.1 Strong Laws of Large Numbers for Local Times

We shall suppose in this section that the drifts in the system (2.1) satisfy the conditions of (5.4).

Proposition 5.2. *Under the conditions of (5.4), the local times accumulated at the origin by the "gap" processes $G(\cdot)$ and $H(\cdot)$ satisfy the strong laws of large numbers*

$$\lim_{t\to\infty}\frac{L^G(t)}{t} \;=\; \lambda_1\,, \qquad\qquad \lim_{t\to\infty}\frac{L^H(t)}{t} \;=\; \lambda_2 \tag{5.11}$$

almost surely, in the notation of (5.10).

Proof: As we just argued, under the condition (5.4) the two-dimensional process $(G(\cdot), H(\cdot))$ of gaps has a unique invariant probability measure $\boldsymbol{\pi}$ on $\mathcal{B}\big((0,\infty)^2\big)$, to which it converges in distribution. This implies, *a fortiori*, that

$$\lim_{t\to\infty}\frac{G(t)}{t} \;=\; 0 \qquad \text{and} \qquad \lim_{t\to\infty}\frac{H(t)}{t} \;=\; 0 \tag{5.12}$$

hold in distribution, thus also in probability. Back into (4.4), (4.5) and in conjunction with the law of large numbers for the Brownian motion $W(\cdot)$, these observations give that

$$\lim_{t\to\infty}\frac{2\,L^G(t)-L^H(t)}{2\,t} \;=\; \delta_2-\delta_1\,, \qquad \lim_{t\to\infty}\frac{2\,L^H(t)-L^G(t)}{2\,t} \;=\; \delta_3-\delta_2 \tag{5.13}$$

hold in probability, and thus the same is true of (5.11).

There exist then sequences $\{t_k\}_{k\in\mathbb{N}} \subset (0,\infty)$ and $\{\tau_k\}_{k\in\mathbb{N}} \subset (0,\infty)$ which increase strictly to infinity, and along which we have

$$\lim_{k\to\infty}\frac{L^G(\tau_k)}{\tau_k} \;=\; \frac{2}{3}\big(\delta_2+\delta_3-2\,\delta_1\big)\,, \qquad \lim_{k\to\infty}\frac{L^H(t_k)}{t_k} \;=\; \frac{2}{3}\big(2\,\delta_3-\delta_1-\delta_2\big)$$

almost surely. However, Theorem II.2 in AZÉMA ET AL. (1967) implies that the limits $\lim_{t\to\infty}\big(L^G(t)/t\big)$ and $\lim_{t\to\infty}\big(L^H(t)/t\big)$ do exist almost surely (see also HARRISON & WILLIAMS (1987), sections 7 and 8). It follows from these considerations, that the limiting relations of (5.11) are valid not just in probability, but also almost surely; the same is true then for those of (5.12). $\square$

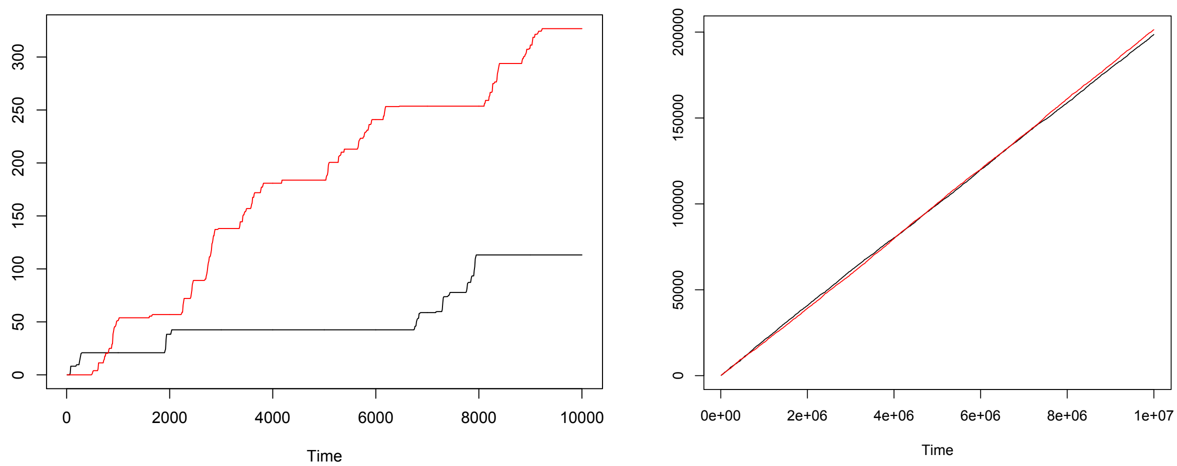

Figure 2: Simulated local times $L^G(\cdot)$ (black) and $L^H(\cdot)$ (red) for a short time (left panel) and for a long time (right panel) with $\delta_1 = 0.01$, $\delta_2 = 0.02$, $\delta_3 = 0.03$, thus $\lambda_1 = \lambda_2 = 0.02$. The long-term growth rates converge in the manner of (5.11), as the time-horizon increases; whereas over short time horizons, the CANTOR-function-like nature of local time becomes quite evident.

We note that both limits in (5.11) are equal to $2\,g$ in the special case (2.5); and as a sanity check, we verify in subsection 5.1.1 that both these limits are strictly positive under the conditions of (5.4). Let us also note that the range of the particles' configuration

$$R_1^X(t) - R_3^X(t) \;=\; G(t) + H(t) \;=\; x_1 - x_3 - \big(\delta_3 - \delta_1\big)\, t + \frac{1}{2}\big(L^G(t) + L^H(t)\big)\,, \qquad 0 \le t < \infty$$

is a process of finite variation: it decreases linearly (with slope $\delta_3 - \delta_1$); increases in Cantor-function-like fashion (as the sums of local times), a behavior that can be gleaned very clearly from Figure 1; and satisfies $\lim_{t\to\infty}\big(\big(R_1^X(t) - R_3^X(t)\big)/t\big) = 0$ a.s., on the strength of (5.12).

The long-term average growth rates of local times $(L^G(\cdot), L^H(\cdot))$ in (5.11) are consistent with simulated local times based on the SKOROKHOD map in HARRISON & REIMAN (1981). The simulations, reported in Figure 2, demonstrate the long-term linear growth of these local times with the rates of (5.11).

Remark 5.1. Elementary stochastic calculus applied to the equations (4.4), (4.5) leads to the dynamics

$$\mathrm{d}\big(G^2(t) + G(t)H(t) + H^2(t)\big) \;=\; \Big[1 - \frac{3}{2}\big(\lambda_1 G(t) + \lambda_2 H(t)\big)\Big]\mathrm{d}t + \big(H(t) - G(t)\big)\,\mathrm{d}W(t). \tag{5.14}$$

As the planar process $(G(t), H(t))$ approaches the origin, the drift in this expression gets close to 1 and pushes the process away from the origin; on the other hand, when either of the components of the vector $(G(t), H(t))$ gets very large, there is a strong negative drift in the above expression (5.14), which tends to bring $(G(\cdot), H(\cdot))$ back toward the origin. This behavior is consistent with the existence of an invariant probability measure for the process $(G(\cdot), H(\cdot))$.

Straightforward computation shows that $V(g,h) = \exp\big\{\sqrt{g^2 + gh + h^2}\,\big\}$, $(g,h) \in [0,\infty)^2 \setminus (0,0)$ is a LYAPOUNOV function for the semimartingale reflecting Brownian motion $(G(\cdot), H(\cdot))$, leading to yet another derivation of the positive recurrence and stochastic stability (existence and uniqueness of an invariant distribution) for this process. We refer to DUPUIS & WILLIAMS (1994), BRAMSON, DAI

& HARRISON (2009) and BRAMSON (2011) for construction of a LYAPOUNOV function and fluid paths, leading to the positive recurrence of (non-degenerate) semimartingale reflecting Brownian motion. Here, Proposition 5.1 deals with the degenerate case.

5.1.1 Discussion of Condition (5.4), and a Sanity Check

We note that if $\delta_1 \geq \delta_2 \geq \delta_3$, the conditions of (5.4) cannot hold; this is because we have then

$$2(\delta_2 - \delta_1) + \big(\delta_2 - \delta_3\big)^- \,=\, 2(\delta_2 - \delta_1) \leq 0\,, \qquad 2(\delta_3 - \delta_2) + \big(\delta_1 - \delta_2\big)^- \,=\, 2(\delta_3 - \delta_2) \leq 0\,.$$

Thus, under (5.4), we have either $\delta_2 > \delta_1$ or $\delta_3 > \delta_2$. We shall consider the following three cases, the only ones that are compatible with (5.4):

$$\text{(i) } \delta_1 < \delta_2 < \delta_3\,, \qquad \text{(ii) } \delta_2 > \delta_1 \text{ and } \delta_2 \geq \delta_3\,, \qquad \text{(iii) } \delta_3 > \delta_2 \text{ and } \delta_1 \geq \delta_2\,.$$

It can be shown that, in all three cases, the conditions of (5.4) imply

$$\delta_3 > \delta_1 \tag{5.15}$$

as well as

$$2\delta_3 - \delta_1 - \delta_2 > 0\,, \qquad \delta_2 + \delta_3 - 2\delta_1 > 0\,. \tag{5.16}$$

Then the a.s. limits in (5.11) are positive.

Let us close this paragraph by observing that the inequalities of (5.16) imply both (5.15) and (5.4). Thus, the condition of (4.4) is equivalent to that in (5.16), and also equivalent to the component-wise inequality $\mathcal{R}^{-1}\mathbb{E}(\mathfrak{Z}(1)) < 0$ in (5.10).

6 Basic Adjoint Relation (BAR) and LAPLACE Transforms

Under the condition (5.4), can the invariant probability measure $\boldsymbol{\pi}$ of the two-dimensional process $(G(\cdot), H(\cdot))$ of gaps be computed explicitly? We do not know the answer to this question, but will try to make some progress on it in the present section, culminating with the conjecture of Remark 6.4 which pertains to what we call the "symmetric case" for this problem.

For every bounded continuous function $f : [0,\infty)^2 \to \mathbb{R}$ of class $\mathcal{C}^2_b\big((0,\infty)^2\big) \cap \mathcal{C}^1_b\big([0,\infty)^2 \setminus \{0\}\big)$, simple stochastic calculus gives

$$f(\mathfrak{G}(T)) - f(\mathfrak{G}(0)) \,=\, \int_0^T \nabla f(\mathfrak{G}(t)) \cdot \mathrm{d}\big(\mathfrak{Z}(t) + \mathcal{R}\mathfrak{L}(t)\big) + \int_0^T \frac{1}{2}\big(D^2_{gg} + D^2_{hh} - 2D^2_{gh}\big) f(\mathfrak{G}(t))\mathrm{d}t$$

where the processes $\big(\mathfrak{G}(\cdot), \mathfrak{Z}(\cdot), \mathfrak{L}(\cdot)\big)$ and the matrix $\mathcal{R}$ are defined in (3.14)-(3.16). Taking expectation on both sides, then integrating with respect to the invariant probability measure $\boldsymbol{\pi}$ for the vector process $\mathfrak{G}(\cdot) \,=\, \big(G(\cdot), H(\cdot)\big)$, we obtain by FUBINI's theorem

$$\begin{aligned} 0 \,=\, T \int_0^\infty \int_0^\infty \Big[\, \frac{1}{2}\big(D^2_{gg} + D^2_{hh} - 2D^2_{gh}\big) f(g,h) + \boldsymbol{m} \cdot \nabla f(g,h) \Big]\, \boldsymbol{\pi}(\mathrm{d}g, \mathrm{d}h) \\ + \frac{T}{2}\bigg(\int_0^\infty \Big(D_g - \frac{1}{2}D_h\Big) f(0,h)\, \boldsymbol{\nu}_1(\mathrm{d}h) + \int_0^\infty \Big(D_h - \frac{1}{2}D_g\Big) f(g,0)\, \boldsymbol{\nu}_2(\mathrm{d}g)\bigg) \end{aligned} \tag{6.1}$$

for $0 < T < \infty$, where $\boldsymbol{m} = \big(\delta_1 - \delta_2\,,\, \delta_2 - \delta_3\big)'$ is the drift vector of $\mathfrak{Z}(\cdot)$ in (3.15), (3.16).

We have denoted here by $\boldsymbol{\nu}_1$ (respectively, by $\boldsymbol{\nu}_2$) the σ-finite measure on the axis $\{(g,h)\in[0,\infty)^2 \,:\, g=0\}$ (respectively, on the axis $\{(g,h)\in[0,\infty)^2 \,:\, h=0\}$) induced by the vector $\mathfrak{L}(\cdot)=\big(L^G(\cdot),L^H(\cdot)\big)'$ of local times under the invariant probability measure $\boldsymbol{\pi}$; namely,

$$\mathbb{E}_{\boldsymbol{\pi}}\int_0^T f\big(\mathfrak{G}(t)\big)\,\mathrm{d}\mathfrak{L}(t) \,=\, \frac{T}{2}\left(\int_0^\infty f(0,h)\,\boldsymbol{\nu}_1(\mathrm{d}h)\,,\,\int_0^\infty f(g,0)\,\boldsymbol{\nu}_2(\mathrm{d}g)\right)', \tag{6.2}$$

or equivalently

$$\boldsymbol{\nu}_1(A) \,=\, \mathbb{E}_{\boldsymbol{\pi}}\int_0^2 \mathbf{1}_{\{H(t)\in A\}}\,\mathrm{d}L^G(t)\,, \qquad \boldsymbol{\nu}_2(A) \,=\, \mathbb{E}_{\boldsymbol{\pi}}\int_0^2 \mathbf{1}_{\{G(t)\in A\}}\,\mathrm{d}L^H(t) \tag{6.3}$$

for $A\in\mathcal{B}\big((0,\infty)\big)$; see AZÉMA ET AL. (1967). Dividing both sides of (6.1) by $T/2$, we obtain for the invariant probability measure $\boldsymbol{\pi}$ of $(G(\cdot),H(\cdot))$ in (3.8)-(3.9) the *Basic Adjoint Relation* (BAR)

$$\int_0^\infty\int_0^\infty \Big(\big(D^2_{gg}+D^2_{hh}-2D^2_{gh}\big)+2\big(\delta_1-\delta_2\big)D_g+2\big(\delta_2-\delta_3\big)D_h\Big)f(g,h)\,\boldsymbol{\pi}(\mathrm{d}g,\mathrm{d}h)+ \tag{6.4}$$

$$+\int_0^\infty\Big(D_g-\frac{1}{2}D_h\Big)f(0,h)\,\boldsymbol{\nu}_1(\mathrm{d}h)+\int_0^\infty\Big(D_h-\frac{1}{2}D_g\Big)f(g,0)\,\boldsymbol{\nu}_2(\mathrm{d}g)\,=\,0\,.$$

This Basic Adjoint Relationship (BAR) was introduced, and studied in detail, for non-degenerate reflected Brownian motions, by HARRISON & WILLIAMS (1987). A probability measure $\boldsymbol{\pi}$ on $\mathcal{B}\big((0,\infty)^2\big)$ is invariant for the vector process $\mathfrak{G}(\cdot)=(G(\cdot),H(\cdot))$ if it, together with two finite measures $\boldsymbol{\nu}_1$, $\boldsymbol{\nu}_2$ on $\mathcal{B}((0,\infty))$, satisfies the BAR (6.4); cf. DAI & KURTZ (2003).

6.1 LAPLACE Transforms and Ramifications

The Basic Adjoint Relation of (6.4) allows us to express the LAPLACE transform $\widehat{\boldsymbol{\pi}}$ of the invariant probability measure $\boldsymbol{\pi}$ in terms of the LAPLACE transforms $\widehat{\boldsymbol{\nu}}_1$, $\widehat{\boldsymbol{\nu}}_2$ of the measures $\boldsymbol{\nu}_1$, $\boldsymbol{\nu}_2$ in (6.2), (6.3) on the axes. Indeed, substituting $f(g,h)=\exp(-\alpha_1 g-\alpha_2 h)$ into (6.4) with $\alpha_1\ge 0$, $\alpha_2\ge 0$, we see that the LAPLACE transforms

$$\widehat{\boldsymbol{\pi}}(\boldsymbol{\alpha}) \,:=\, \widehat{\boldsymbol{\pi}}(\alpha_1,\alpha_2) \,=\, \mathbb{E}_{\boldsymbol{\pi}}\Big[e^{-\alpha_1 G(t)-\alpha_2 H(t)}\Big]\,, \qquad \widehat{\boldsymbol{\nu}}_i(\alpha_j) \,:=\, \int_0^\infty e^{-\alpha_j x}\boldsymbol{\nu}_i(\mathrm{d}x)$$

for $i\neq j\in\{1,2\}$ satisfy the equation

$$\Big[(\alpha_1-\alpha_2)^2+2(\delta_2-\delta_1)\alpha_1+2(\delta_3-\delta_2)\alpha_2\Big]\,\widehat{\boldsymbol{\pi}}(\alpha_1,\alpha_2) \,=$$

$$=\,\Big(\alpha_1-\frac{\alpha_2}{2}\Big)\,\widehat{\boldsymbol{\nu}}_1(\alpha_2)+\Big(\alpha_2-\frac{\alpha_1}{2}\Big)\,\widehat{\boldsymbol{\nu}}_2(\alpha_1)\,. \tag{6.5}$$

The following observations, in the form of bullets, are consequences of this last equation (6.5).

• For any pair $(\alpha_1,\alpha_2)\in[0,\infty)^2$ that satisfies $(\alpha_1-\alpha_2)^2+2(\delta_2-\delta_1)\alpha_1+2(\delta_3-\delta_2)\alpha_2\neq 0$, the equation (6.5) yields

$$\mathbb{E}_{\boldsymbol{\pi}}\Big[e^{-\alpha_1 G(t)-\alpha_2 H(t)}\Big] \,=\, \frac{(2\alpha_1-\alpha_2)\widehat{\boldsymbol{\nu}}_1(\alpha_2)+(2\alpha_2-\alpha_1)\widehat{\boldsymbol{\nu}}_2(\alpha_1)}{2(\alpha_1-\alpha_2)^2+4(\delta_2-\delta_1)\alpha_1+4(\delta_3-\delta_2)\alpha_2} \,=\, \widehat{\boldsymbol{\pi}}(\alpha_1,\alpha_2)\,. \tag{6.6}$$

In other words, the invariant distribution $\boldsymbol{\pi}$ on $(0,\infty)^2$ can be obtained from the measures $\boldsymbol{\nu}_1$, $\boldsymbol{\nu}_2$ of (6.2) on the two axes.

The reverse is also true: Setting $\alpha_1=2\alpha_2>0$ (resp., $\alpha_2=2\alpha_1>0$) in (6.6), we get respectively,

$$\widehat{\nu_1}(\alpha_2) \,=\, \frac{2}{3}\Big(\alpha_2+2\big(\delta_2+\delta_3\big)-4\delta_1\Big)\,\widehat{\pi}\big(2\alpha_2,\alpha_2\big)\,, \qquad \widehat{\nu_2}(\alpha_1) \,=\, \frac{2}{3}\Big(\alpha_1-2\big(\delta_2+\delta_1\big)+4\delta_3\Big)\,\widehat{\pi}\big(\alpha_1,2\alpha_1\big)\,.$$

• If $(\alpha_1, \alpha_2) \in [0,\infty)^2$ lies on the segment of the parabola

$$(\alpha_1 - \alpha_2)^2 + 2(\delta_2 - \delta_1)\alpha_1 + 2(\delta_3 - \delta_2)\alpha_2 \;=\; 0\,, \tag{6.7}$$

then (6.5) yields

$$(2\alpha_1 - \alpha_2)\,\widehat{\boldsymbol{\nu}}_1(\alpha_2) + (2\alpha_2 - \alpha_1)\,\widehat{\boldsymbol{\nu}}_2(\alpha_1) \;=\; 0\,.$$

Under the conditions of (5.4), the segment is non-empty provided $\delta_2 < \delta_1 < (\delta_2 + \delta_3)/2 < \delta_3$ or $\delta_1 < (\delta_1 + \delta_2)/2 < \delta_3 < \delta_2$.

On the other hand, under the condition (5.5), the segment on the parabola (6.7) degenerates to the origin $(\alpha_1, \alpha_2) = (0, 0)$ and thus (6.6) holds then for every $(\alpha_1, \alpha_2) \in [0,\infty)^2 \setminus \{(0,0)\}$.

• Dividing (6.5) by $\alpha_j > 0$, then letting $\alpha_j \uparrow \infty$, $j = 1,2$, we obtain

$$\lim_{\alpha_2\uparrow\infty} \alpha_2\, \widehat{\boldsymbol{\pi}}(\alpha_1,\alpha_2) \;=\; \widehat{\boldsymbol{\nu}}_2(\alpha_1)\,, \qquad \lim_{\alpha_1\uparrow\infty} \alpha_1\, \widehat{\boldsymbol{\pi}}(\alpha_1,\alpha_2) \;=\; \widehat{\boldsymbol{\nu}}_1(\alpha_2)\,, \qquad (\alpha_1,\alpha_2) \in [0,\infty)^2\,. \tag{6.8}$$

We deduce that the measures $\boldsymbol{\nu}_1$, $\boldsymbol{\nu}_2$ of (6.2) are appropriately normalized traces on the two axes, of the the invariant probability measure $\boldsymbol{\pi}$.

• Now, let us take $\alpha_1 = \alpha_2 = \alpha > 0$ in (6.5); we see that the LAPLACE transform of the invariant distribution for the sum $G(\cdot) + H(\cdot)$ of the gaps is expressed as

$$\mathbb{E}_{\boldsymbol{\pi}}\Big[e^{-\alpha(G(T)+H(T))}\Big] \;=\; \widehat{\boldsymbol{\pi}}(\alpha,\alpha) \;=\; \frac{\widehat{\boldsymbol{\nu}}_1(\alpha) + \widehat{\boldsymbol{\nu}}_2(\alpha)}{4(\delta_3 - \delta_1)} \;=\; \frac{\widehat{\boldsymbol{\nu}}_1(\alpha) + \widehat{\boldsymbol{\nu}}_2(\alpha)}{2(\lambda_1 + \lambda_2)}\;; \qquad \alpha > 0\,. \tag{6.9}$$

In conjunction with Proposition 4.1, this shows that the measure $\boldsymbol{\nu}_1 + \boldsymbol{\nu}_2$ is supported on the open half-line $(0,\infty)$.

Letting $\alpha \downarrow 0$ in the above equation gives the total mass of the two measures on the axes under the stationary distribution, namely

$$\big(\boldsymbol{\nu}_1 + \boldsymbol{\nu}_2\big)((0,\infty)) \;=\; \widehat{\boldsymbol{\nu}}_1(0) + \widehat{\boldsymbol{\nu}}_2(0) \;=\; 4(\delta_3 - \delta_1) > 0\,; \tag{6.10}$$

this is consistent with the strong laws of large numbers (5.11), because of the normalization (6.2) and

$$\lim_{T\to\infty} \frac{1}{T}\big(L^G(T) + L^H(T)\big) \;=\; \frac{1}{2}\,\big(\boldsymbol{\nu}_1 + \boldsymbol{\nu}_2\big)\big((0,\infty)\big) \;=\; 2(\delta_3 - \delta_1) \;=\; \lambda_1 + \lambda_2\,.$$

In particular, the two measures $\boldsymbol{\nu}_1$, $\boldsymbol{\nu}_2$ of (6.2) are both finite.

• Now let us take the limit in (6.6) as $\alpha_2 \downarrow 0$, to obtain

$$\mathbb{E}_{\boldsymbol{\pi}}\Big[e^{-\alpha_1 G(T)}\Big] \;=\; \widehat{\boldsymbol{\pi}}(\alpha_1, 0) \;=\; \frac{2\,\widehat{\boldsymbol{\nu}}_1(0) - \widehat{\boldsymbol{\nu}}_2(\alpha_1)}{2\alpha_1 + 4(\delta_2 - \delta_1)}\,;$$

next we let $\alpha_1 \downarrow 0$ and get

$$2\,\widehat{\boldsymbol{\nu}}_1(0) - \widehat{\boldsymbol{\nu}}_2(0) \;=\; 4(\delta_2 - \delta_1)\,.$$

In conjunction with (6.10), this gives the total mass of each of the two measures on the axes, namely

$$\boldsymbol{\nu}_1\big((0,\infty)\big) \;=\; \widehat{\boldsymbol{\nu}}_1(0) \;=\; \frac{4}{3}\big(\delta_2 + \delta_3 - 2\,\delta_1\big) \;=\; 2\,\lambda_1 \;=\; 2\,\lim_{T\to\infty} \frac{L^G(T)}{T}\,, \tag{6.11}$$

$$\boldsymbol{\nu}_2\big((0,\infty)\big) \;=\; \widehat{\boldsymbol{\nu}}_2(0) \;=\; \frac{4}{3}\big(2\,\delta_3 - \delta_1 - \delta_2\big) \;=\; 2\,\lambda_2 \;=\; 2\,\lim_{T\to\infty} \frac{L^H(T)}{T}\,, \tag{6.12}$$

in accordance with (6.2) and (5.11). This way we express the LAPLACE transform for the first marginal

$$\mathbb{E}_{\boldsymbol{\pi}}\Big[e^{-\alpha_1 G(T)}\Big] \;=\; \widehat{\boldsymbol{\pi}}(\alpha_1, 0) \;=\; \frac{4\,\lambda_1 - \widehat{\boldsymbol{\nu}}_2(\alpha_1)}{2\,\alpha_1 + 4(\delta_2 - \delta_1)}\,, \tag{6.13}$$

and likewise the LAPLACE transform for the second marginal

$$\mathbb{E}_{\boldsymbol{\pi}}\Big[e^{-\alpha_2 H(T)}\Big] \;=\; \widehat{\boldsymbol{\pi}}(0,\alpha_2) \;=\; \frac{4\,\lambda_2 - \widehat{\boldsymbol{\nu}}_1(\alpha_2)}{2\,\alpha_2 + 4(\delta_3-\delta_2)}\,, \tag{6.14}$$

in terms of the LAPLACE transforms of the traces $\boldsymbol{\nu}_2$ and $\boldsymbol{\nu}_1$, respectively.

Remark 6.1 (Absolute Continuity). It can be shown as in section 8 of HARRISON & WILLIAMS (1987) that the measures $\boldsymbol{\nu}_1(\cdot)$ and $\boldsymbol{\nu}_2(\cdot)$ are absolutely continuous with respect to LEBESGUE measure; in other words, that there exist probability density functions $\boldsymbol{\sigma}_1(\cdot)$ and $\boldsymbol{\sigma}_2(\cdot)$ on $(0,\infty)$, such that

$$\boldsymbol{\nu}_j(A) = 2\,\lambda_j \int_A \boldsymbol{\sigma}_j(z)\,\mathrm{d}z\,, \qquad A\in\mathcal{B}\big([0,\infty)\big)\,, \quad j=1,2\,. \tag{6.15}$$

It follows now from (6.9) that the invariant distribution of the sum of gaps $G(\cdot)+H(\cdot)$ is also absolutely continuous with respect to LEBESGUE measure, with probability density function $\mathbb{P}_{\boldsymbol{\pi}}\big(G(T)+H(T)\in \mathrm{d}z\big) = \boldsymbol{\sigma}(z)\,\mathrm{d}z$ given by

$$\boldsymbol{\sigma}(z) \;=\; \frac{\lambda_1}{\lambda_1+\lambda_2}\,\boldsymbol{\sigma}_1(z) + \frac{\lambda_2}{\lambda_1+\lambda_2}\,\boldsymbol{\sigma}_2(z)\,, \qquad z\in(0,\infty)\,. \tag{6.16}$$

Remark 6.2 (No Product Form). It is seen from (6.2), (6.11) and the definition of the local time that

$$\begin{aligned}(2\lambda_1)^{-1}\int_0^\infty e^{-\alpha_2 h}\,\boldsymbol{\nu}_1(\mathrm{d}h) \;&=\; \Big(\mathbb{E}_\pi\big[L^G(T)\big]\Big)^{-1}\cdot\mathbb{E}_\pi\Big[\int_0^T e^{-\alpha_1 G(t)-\alpha_2 H(t)}\mathrm{d}L^G(t)\Big]\\ &=\;\Big(\mathbb{E}_\pi\Big[\lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\int_0^T \mathbf{1}_{\{G(t)<\varepsilon\}}\mathrm{d}t\Big]\Big)^{-1}\cdot\mathbb{E}_\pi\Big[\lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\int_0^T e^{-\alpha_1 G(t)-\alpha_2 H(t)}\mathbf{1}_{\{G(t)<\varepsilon\}}\mathrm{d}t\Big]\\ &=\;\lim_{\varepsilon\downarrow 0}\big(\boldsymbol{\pi}(g<\varepsilon)\big)^{-1}\int_0^\infty\int_0^\infty e^{-\alpha_1 g-\alpha_2 h}\cdot\mathbf{1}_{\{g<\varepsilon\}}\,\boldsymbol{\pi}(\mathrm{d}g,\mathrm{d}h)\,,\quad (\alpha_1,\alpha_2)\in(0,\infty)^2\end{aligned}$$

holds for all $T\in(0,\infty)$. Hence, by the uniqueness of the LAPLACE transform and (6.15) we obtain

$$\mathbb{P}_{\boldsymbol{\pi}}\big(H(T)\in\mathrm{d}z\,\big|\,G(T)=0\big) \;=\; \boldsymbol{\sigma}_1(z)\,\mathrm{d}z\,;\quad \text{similarly}\quad \mathbb{P}_{\boldsymbol{\pi}}\big(G(T)\in\mathrm{d}z\,\big|\,H(T)=0\big) \;=\; \boldsymbol{\sigma}_2(z)\,\mathrm{d}z\,.$$

With this interpretation in mind, it becomes clear that *the joint distribution of the two gaps under the invariant probability measure cannot possibly be the product of their two marginal distributions.*

6.2 The General Symmetric Case

Let us consider now the general symmetric case

$$\delta_2-\delta_1 \;=\; \delta_3-\delta_2 \;=:\; g \;>\; 0\,; \tag{6.17}$$

the configuration of (2.5) is a special case of this situation, with $\delta_2=0$.

We have now $\lambda_1=\lambda_2=2\,g=:\lambda$ in (5.10), as well as $\boldsymbol{\nu}_1(\cdot)\equiv\boldsymbol{\nu}_2(\cdot)=:\boldsymbol{\nu}(\cdot)$, $\boldsymbol{\sigma}_1(\cdot)\equiv\boldsymbol{\sigma}_2(\cdot)\equiv\boldsymbol{\sigma}(\cdot)$, and (6.9), (6.6) lead to $\widehat{\boldsymbol{\nu}}(\alpha)\;=\;2\,\lambda\,\widehat{\boldsymbol{\pi}}(\alpha,\alpha)$ and to the *functional equation*

$$\widehat{\boldsymbol{\pi}}(\alpha_1,\alpha_2) \;=\; \frac{\lambda}{(\alpha_1-\alpha_2)^2+\lambda(\alpha_1+\alpha_2)}\Big[(2\alpha_1-\alpha_2)\,\widehat{\boldsymbol{\pi}}\big(\alpha_2,\alpha_2\big)+(2\alpha_2-\alpha_1)\,\widehat{\boldsymbol{\pi}}\big(\alpha_1,\alpha_1\big)\Big] \tag{6.18}$$

for the LAPLACE transform of the joint distribution of the gaps. To wit, in the symmetric case of (6.17) and in steady state, *the joint distribution of the gaps is determined by the distribution of their sum* – or for that matter by the common marginal distribution of each of these gaps, as in this case

$$\widehat{\boldsymbol{\nu}}(\alpha) \;=\; 2\,\lambda\,\widehat{\boldsymbol{\pi}}(\alpha,\alpha)\,, \qquad \widehat{\boldsymbol{\pi}}(\alpha,0) \;=\; \widehat{\boldsymbol{\pi}}(0,\alpha) \;=\; \frac{\lambda}{\alpha+\lambda}\big[\,2-\widehat{\boldsymbol{\pi}}(\alpha,\alpha)\big]\,; \qquad \alpha\geq 0\,. \tag{6.19}$$

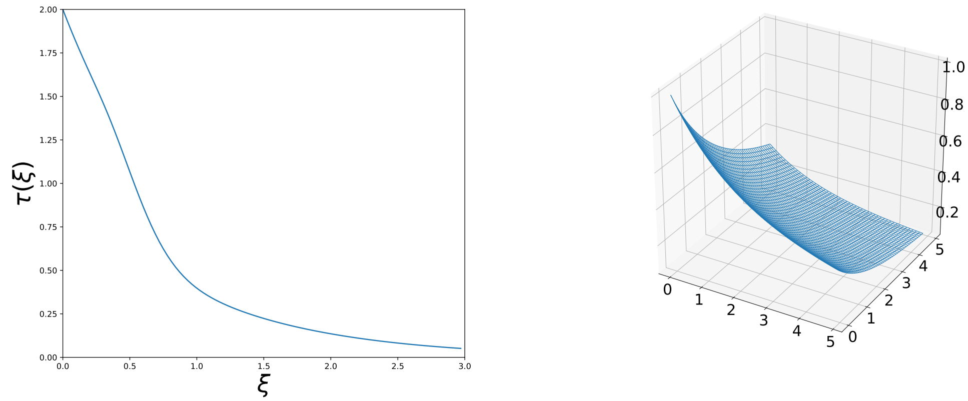

Figure 3: The marginal probability density function $\boldsymbol{\tau}(\cdot)$ in (6.25) (left) and the joint Laplace transform $\widehat{\boldsymbol{\pi}}(\alpha_1, \alpha_2)$ in (6.26) (right) under the conjecture on $\boldsymbol{\sigma}(\cdot)$ in (6.23).

This last equation suggests that, in the symmetric case of (6.17), the marginal invariant distributions of the gaps have common probability density function

$$\frac{\mathbb{P}_{\boldsymbol{\pi}}\big(G(t) \in \mathrm{d}\xi\big)}{\mathrm{d}\xi} = \frac{\mathbb{P}_{\boldsymbol{\pi}}\big(H(t) \in \mathrm{d}\xi\big)}{\mathrm{d}\xi} = \boldsymbol{\tau}(\xi) = \lambda \left[2\, e^{-\lambda\xi} - \int_0^{\xi} e^{-\lambda(\xi - z)}\, \boldsymbol{\sigma}(z)\, \mathrm{d}z \right] \tag{6.20}$$

for $\xi \in (0, \infty)$. In particular, the invariant distribution for the sum of the two gaps has finite moment-generating function, thus moments of all orders:

$$\int_0^{\infty} e^{\lambda z}\, \boldsymbol{\sigma}(z)\, \mathrm{d}z \leq 2\,. \tag{6.21}$$

Remark 6.3 (The Average Gaps in Steady-State). Always under the condition (6.17), suppose that the pair of processes $(G(\cdot), H(\cdot))$ runs under its stationary distribution $\boldsymbol{\pi}$. Then by taking the expectation in the expression (5.14) of Remark 5.1, one obtains $\mathbb{E}_{\boldsymbol{\pi}}[\,1 - 3gG(t) - 3gH(t)\,] = 0$; due to symmetry, this shows

$$\mathbb{E}_{\boldsymbol{\pi}}\big[G(t)\big] = \mathbb{E}_{\boldsymbol{\pi}}\big[H(t)\big] = \frac{1}{6\,g} = \frac{1}{3\,\lambda}\,. \tag{6.22}$$

The first-moment computation (6.22) *rules out exponential marginal distributions for the gaps* in this symmetric case (6.17). For if $\boldsymbol{\tau}(\xi) = \beta\, e^{-\beta\,\xi}$, $\xi \in (0, \infty)$ were valid for some constant $\beta > 0$, then the equation

$$\boldsymbol{\tau}(\xi)\, e^{\lambda\,\xi} = \lambda \left[2 - \int_0^{\xi} e^{\lambda\, z}\, \boldsymbol{\sigma}(z)\, \mathrm{d}z \right], \qquad 0 < \xi < \infty$$

from (6.20) would force $\beta = 2\,\lambda$ and $\boldsymbol{\tau}(\xi) = \boldsymbol{\sigma}(\xi) = \beta\, e^{-\beta\,\xi}$, thus $\mathbb{E}_{\boldsymbol{\pi}}\big[G(t)\big] = \mathbb{E}_{\boldsymbol{\pi}}\big[H(t)\big] = 1/\beta = 1/(2\lambda)$, contradicting (6.22).

Remark 6.4 (A Conjecture Involving the Gamma Distribution). Always in the symmetric case (6.17), we conjecture that under the stationary distribution $\boldsymbol{\pi}$, the density function $\boldsymbol{\sigma}(\cdot)$ for the sum of the gaps $G(\cdot) + H(\cdot)$ is the Gamma probability density with parameters $(\lambda \mathrm{u}, (2/3)\mathrm{u})$, i.e.,

$$\boldsymbol{\sigma}(\xi) = \frac{(\lambda \mathrm{u})^{2\mathrm{u}/3}}{\boldsymbol{\Gamma}(2\,\mathrm{u}/3)}\, \xi^{\,(2\mathrm{u}/3)-1}\, e^{-\lambda \mathrm{u} \xi}\,, \qquad 0 < \xi < \infty\,. \tag{6.23}$$

Here $\mathbf{\Gamma}(z) = \int_0^\infty x^{z-1} e^{-x} \mathrm{d}x$ is the Gamma function, and the positive constant u the unique solution of the transcendental equation $2u \log(u/(u-1)) \, = \, 3 \log 2$. Equivalently, u is given as

$$\mathrm{u} \, := \, \frac{3 \log 2}{3 \log 2 + 2 \, \mathbf{W}(-(3 \log 2)/(4\sqrt{2}))} \tag{6.24}$$

in terms of $\mathbf{W}(\cdot)$, the LAMBERT *W-function* or "product logarithm", with the property $z = \mathbf{W}(ze^z)$.

With this probability density function $\boldsymbol{\sigma}(\cdot)$ as in (6.23), the condition (6.21) is satisfied as equality:

$$\int_0^\infty e^{\lambda z} \boldsymbol{\sigma}(z) \mathrm{d}z \, = \, \Big(\frac{\mathrm{u}}{\mathrm{u}-1} \Big)^{2\mathrm{u}/3} \, = \, 2 \, .$$

It follows from (6.20) that the common marginal probability density for the gaps becomes then

$$\boldsymbol{\tau}(\xi) \, = \, \lambda e^{-\lambda \xi} \Big[\Big(\int_0^\infty - \int_0^\xi \Big) e^{\lambda z} \boldsymbol{\sigma}(z) \mathrm{d}z \Big] \, = \, \lambda e^{-\lambda \xi} \cdot \int_\xi^\infty e^{\lambda z} \boldsymbol{\sigma}(z) \mathrm{d}z \, , \qquad 0 < \xi < \infty \tag{6.25}$$

under the invariant distribution; that the LAPLACE transform in (6.18) for the joint distribution of the gaps takes the form

$$\widehat{\boldsymbol{\pi}}(\alpha_1, \alpha_2) \, = \, \frac{\lambda}{(\alpha_1 - \alpha_2)^2 + \lambda(\alpha_1 + \alpha_2)} \left[(2\alpha_1 - \alpha_2) \Big(\frac{\lambda \mathrm{u}}{\lambda \mathrm{u} + \alpha_2} \Big)^{\frac{2\mathrm{u}}{3}} + (2\alpha_2 - \alpha_1) \Big(\frac{\lambda \mathrm{u}}{\lambda \mathrm{u} + \alpha_1} \Big)^{\frac{2\mathrm{u}}{3}} \right] \tag{6.26}$$

for $(\alpha_1, \alpha_2) \in (0, \infty)^2$, as in Figure 3; and that the first-moment condition of (6.22) holds, namely,

$$\mathbb{E}_{\boldsymbol{\pi}}[G(t)] = \mathbb{E}_{\boldsymbol{\pi}}[H(t)] \, = \, \int_0^\infty \xi \cdot \lambda e^{-\lambda \xi} \Big[\int_\xi^\infty e^{\lambda z} \boldsymbol{\sigma}(z) \mathrm{d}z \Big] \mathrm{d}\xi \, = \, \frac{1}{\lambda} - \frac{2}{3\,\lambda} \, = \, \frac{1}{3\,\lambda} \, .$$

7 The "Obverse" of (2.1)–(2.3): Ballistic Middle Motion, Diffusive Hedges

We take up in this section the "obverse" of the three-particle system in (2.1)-(2.3), by which we mean replacing the equations in (2.1) by

$$X_i(\cdot) \, = \, x_i + \sum_{k=1}^3 \delta_k \int_0^\cdot \mathbf{1}_{\{X_i(t) = R_k^X(t)\}} \, \mathrm{d}t + \int_0^\cdot \Big(\mathbf{1}_{\{X_i(t) = R_1^X(t)\}} + \mathbf{1}_{\{X_i(t) = R_3^X(t)\}} \Big) \, \mathrm{d}B_i(t) \tag{7.1}$$

for $i = 1, 2, 3$, and replacing the conditions of (2.2), (2.3) by

$$\int_0^\infty \mathbf{1}_{\{R_k^X(t) = R_\ell^X(t)\}} \, \mathrm{d}t \, = \, 0 \, , \quad \forall \; k < \ell \, ; \qquad L^{R_1^X - R_3^X}(\cdot) \equiv 0 \tag{7.2}$$

in the notation of (2.4) and (3.6). The processes $B_1(\cdot), B_2(\cdot), B_3(\cdot)$, are again independent scalar Brownian motions. In words: it is now the leading and laggard particles that undergo diffusion, and the particle in the middle that undergoes purely ballistic motion. Once again, the dynamics of the system (7.1) involve dispersion functions that are both discontinuous and degenerate.

In contrast to Proposition 4.1, we shall see here that *"the two diffusive motions can squeeze the ballistic motion in the middle"*, and thus triple points will occur with probability one; yet the resulting *triple collisions are "soft"*, in that the local time $L^{R_1^X - R_3^X}(\cdot)$ associated with them is identically equal to zero, as postulated in the second requirement of (7.2). The first requirement there, mandates that all collisions are non-sticky.

7.1 Analysis

Let us assume that a weak solution to this system of (7.1), (7.2) has been constructed on an appropriate filtered probability space $(\Omega, \mathfrak{F}, \mathbb{P}), \mathbb{F} = \{\mathfrak{F}(t)\}_{0 \le t < \infty}$. Reasoning as before, we have the analogues

$$
\begin{aligned}
R_1^X(t) &= x_1 + \delta_1\, t + W_1(t) + \frac{1}{2}\,\Lambda^{(1,2)}(t)\,,\\
R_2^X(t) &= x_2 + \delta_2\, t - \frac{1}{2}\,\Lambda^{(1,2)}(t) + \frac{1}{2}\,\Lambda^{(2,3)}(t)\,,\\
R_3^X(t) &= x_3 + \delta_3\, t + W_3(t) - \frac{1}{2}\,\Lambda^{(2,3)}(t)\,; \quad t \ge 0
\end{aligned} \tag{7.3}
$$

of (3.1)-(3.3) in the notation of (3.5), and as in (3.4) with the processes

$$
W_k(\cdot) \,:=\, \sum_{i=1}^{3} \int_0^{\cdot} \mathbf{1}_{\{X_i(t) = R_k^X(t)\}}\, \mathrm{d}B_i(t)\,, \qquad k = 1, 3 \tag{7.4}
$$

which are independent Brownian motions by the P. LÉVY theorem. It is fairly clear that the center of gravity of this system evolves as Brownian motion with drift, since

$$
\sum_{i=1}^{3} X_i(t) \,=\, x + \delta\, t + \sqrt{2}\, V(t)\,, \qquad 0 \le t < \infty\,,
$$

for $x = x_1 + x_2 + x_3$, $\delta = \delta_1 + \delta_2 + \delta_3$, and $V = (W_1 + W_3)/\sqrt{2}$ a standard Brownian motion.

Then the gaps $G(\cdot) := R_1^X(\cdot) - R_2^X(\cdot)$ and $H(\cdot) := R_2^X(\cdot) - R_3^X(\cdot)$ are given as

$$
G(t) \,=\, U(t) + L^G(t)\,, \qquad H(t) \,=\, V(t) + L^H(t)\,, \qquad 0 \le t < \infty
$$

in the manner of (3.8), (3.9), where again $L^G(\cdot) \equiv \Lambda^{(1,2)}(\cdot)$, $L^H(\cdot) \equiv \Lambda^{(2,3)}(\cdot)$, and now

$$
U(t) := x_1 - x_2 - \big(\delta_2 - \delta_1\big)\, t + W_1(t) - \frac{1}{2}\, L^H(t)\,, \qquad V(t) := x_2 - x_3 - \big(\delta_3 - \delta_2\big)\, t - W_3(t) - \frac{1}{2}\, L^G(t)\,.
$$

The theory of the SKOROKHOD reflection problem gives the analogues

$$
L^G(t) \,=\, \max_{0 \le s \le t} \big(- U(s) \big)^+ \,=\, \max_{0 \le s \le t} \Big(- (x_1 - x_2) + \big(\delta_2 - \delta_1\big)\, s - W_1(s) + \frac{1}{2}\, L^H(s) \Big)^+ \tag{7.5}
$$

$$
L^H(t) \,=\, \max_{0 \le s \le t} \big(- V(s) \big)^+ \,=\, \max_{0 \le s \le t} \Big(- (x_2 - x_3) + \big(\delta_3 - \delta_2\big)\, s + W_3(s) + \frac{1}{2}\, L^G(s) \Big)^+ \tag{7.6}
$$

of the equations (3.12), (3.13), a system of equations linking the two local time processes $L^G(\cdot)$, $L^H(\cdot)$.

7.2 Synthesis

We start again with the given real numbers $\delta_1, \delta_2, \delta_3$ and $x_1 > x_2 > x_3$, and construct a filtered probability space $(\Omega, \widetilde{\mathfrak{F}}, \mathbb{P}), \widetilde{\mathbb{F}} = \big\{\widetilde{\mathfrak{F}}(t)\big\}_{0 \le t < \infty}$ and on it three independent, standard Brownian motions $W_i(\cdot)$, $i = 1, 2, 3$. We consider the analogue

$$
A(t) \,=\, \max_{0 \le s \le t} \Big(- (x_1 - x_2) + \big(\delta_2 - \delta_1\big)\, s - W_1(s) + \frac{1}{2}\, \Gamma(s) \Big)^+, \quad 0 \le t < \infty \tag{7.7}
$$

$$
\Gamma(t) \,=\, \max_{0 \le s \le t} \Big(- (x_2 - x_3) + \big(\delta_3 - \delta_2\big)\, s + W_3(s) + \frac{1}{2}\, A(s) \Big)^+, \quad 0 \le t < \infty \tag{7.8}
$$

of the system of equations (7.5) and (7.6) for two continuous, nondecreasing and adapted processes $A(\cdot)$ and $\Gamma(\cdot)$ with $A(0) = \Gamma(0) = 0$. Once again, the theory of HARRISON & REIMAN (1981) guarantees the existence of a unique continuous solution $\big(A(\cdot), \Gamma(\cdot)\big)$ for the system (7.7), (7.8), adapted to the filtration $\mathbb{F}^{(W_1, W_3)}$ generated by the 2-D Brownian motion $(W_1(\cdot), W_3(\cdot))$:

$$\mathfrak{F}^{(A,\Gamma)}(t) \subseteq \mathfrak{F}^{(W_1,W_3)}(t), \qquad 0 \le t < \infty. \tag{7.9}$$

With the processes $A(\cdot)$, $\Gamma(\cdot)$ thus in place, we consider the continuous semimartingales

$$U(t) := x_1 - x_2 - \big(\delta_2 - \delta_1\big)\, t + W_1(t) - \frac{1}{2}\,\Gamma(t), \quad V(t) := x_2 - x_3 - \big(\delta_3 - \delta_2\big)\, t - W_3(t) - \frac{1}{2}\, A(t)$$

and then "fold" them to obtain their SKOROKHOD reflections

$$G(t) := U(t) + \max_{0 \le s \le t} \big(- U(s)\big)^+ \tag{7.10}$$

$$= x_1 - x_2 - \big(\delta_2 - \delta_1\big)\, t + W_1(t) - \frac{1}{2}\,\Gamma(t) + A(t) \ge 0$$

$$H(t) := V(t) + \max_{0 \le s \le t} \big(- V(s)\big)^+ \tag{7.11}$$

$$= x_2 - x_3 - \big(\delta_3 - \delta_2\big)\, t - W_3(t) - \frac{1}{2}\, A(t) + \Gamma(t) \ge 0$$

for $t \in [0, \infty)$. This system of equations (7.10), (7.11) can be cast in the HARRISON-REIMAN form

$$\begin{pmatrix} G(t) \\ H(t) \end{pmatrix} = \begin{pmatrix} G(0) \\ H(0) \end{pmatrix} + \mathfrak{Z}(t) + \mathcal{R}\,\mathfrak{L}(t), \qquad 0 \le t < \infty$$

of (3.14), now with covariance matrix

$$\mathcal{C} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \text{reflection matrix} \qquad \mathcal{R} = \mathcal{I} - \mathcal{Q}, \quad \mathcal{Q} := \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix},$$

and

$$\mathfrak{L}(t) = \begin{pmatrix} L^G(t) \\ L^H(t) \end{pmatrix}, \qquad \mathfrak{Z}(t) = \begin{pmatrix} (\delta_1 - \delta_2)t + W_1(t) \\ (\delta_2 - \delta_3)t - W_3(t) \end{pmatrix}, \qquad 0 \le t < \infty.$$

Once again, we obtain easily the analogues

$$\int_0^\infty \mathbf{1}_{\{G(t)>0\}}\, \mathrm{d}A(t) = 0, \qquad \int_0^\infty \mathbf{1}_{\{H(t)>0\}}\, \mathrm{d}\Gamma(t) = 0 \tag{7.12}$$

and

$$\int_0^\infty \mathbf{1}_{\{G(t)=0\}}\, \mathrm{d}t = 0, \qquad \int_0^\infty \mathbf{1}_{\{H(t)=0\}}\, \mathrm{d}t = 0 \tag{7.13}$$

of the properties in (4.6), (4.7) using, respectively, the theories of the SKOROKHOD reflection problem and of semimartingale local time.

We claim that we also have the analogues

$$\int_0^\infty \mathbf{1}_{\{H(t)=0\}}\, \mathrm{d}A(t) = 0, \qquad \int_0^\infty \mathbf{1}_{\{G(t)=0\}}\, \mathrm{d}\Gamma(t) = 0 \tag{7.14}$$

of the properties in (4.10), though now for a different reason.

Let us elaborate: the system of (7.10), (7.11) characterizes a two-dimensional Brownian motion $(G(\cdot),\, H(\cdot))$ with drift $(\delta_1 - \delta_2,\, \delta_2 - \delta_3)$ and reflection along the faces of the nonnegative quadrant. According to the theory of VARADHAN & WILLIAMS (1985) (see also ICHIBA & KARATZAS (2010)), this process will hit the corner of the quadrant, with probability one: $\mathbb{P}(G(t) = H(t) = 0,\ \text{ for some } t > 0) = 1$. Yet we have, again with probability one,

$$\int_0^\infty \mathbf{1}_{\{G(t)=0\}}\, \mathrm{d}\Gamma(t) \;=\; \int_0^\infty \mathbf{1}_{\{G(t)=H(t)=0\}}\, \mathrm{d}\Gamma(t) \;=\; 0 \tag{7.15}$$

and

$$\int_0^\infty \mathbf{1}_{\{H(t)=0\}}\, \mathrm{d}A(t) \;=\; \int_0^\infty \mathbf{1}_{\{G(t)=H(t)=0\}}\, \mathrm{d}A(t) \;=\; 0\,, \tag{7.16}$$

where the first equalities come from those in (7.12) and the second equalities from Theorem 1 in REIMAN & WILLIAMS (1988). The claims in (7.14) are thus established.

Armed with the properties (7.12)-(7.14), we obtain here again the identifications $L^G(\cdot) \equiv A(\cdot)$, $L^H(\cdot) \equiv \Gamma(\cdot)$ of the processes $A(\cdot),\, \Gamma(\cdot)$ in (7.7), (7.8) as local times. Details are omitted, being very similar to what was done before.

- *Construction of the Ranked Motions:* We introduce now, by analogy with (4.8)-(4.9), the processes

$$\begin{aligned} R_1(t) \;&:= x_1 + \delta_1\, t + W_1(t) + \frac{1}{2}\, A(t) \\ R_2(t) \;&:= x_2 + \delta_2\, t - \frac{1}{2}\, A(t) + \frac{1}{2}\, \Gamma(t) \\ R_3(t) \;&:= x_3 + \delta_3\, t + W_3(t) - \frac{1}{2}\, \Gamma(t) \end{aligned} \tag{7.17}$$

for $0 \le t < \infty$, and note again the relations $R_1(\cdot) - R_2(\cdot) = G(\cdot) \ge 0$, $R_2(\cdot) - R_3(\cdot) = H(\cdot) \ge 0$ and the comparisons $R_1(\cdot) \ge R_2(\cdot) \ge R_3(\cdot)$. The range

$$R_1(t) - R_3(t) = G(t) + H(t) = x_1 - x_3 + \big(\delta_1 - \delta_3\big)\, t + W_1(t) - W_3(t) + \frac{1}{2}\Big(A(t) + \Gamma(t)\Big)\,, \quad 0 \le t < \infty$$

is a nonnegative semimartingale with $\langle R_1 - R_3 \rangle(t) = 2\, t$ and local time at the origin

$$L^{R_1 - R_3}(\cdot) \;=\; \int_0^{\cdot} \mathbf{1}_{\{G(t)+H(t)=0\}} \Big[\, \big(\delta_1 - \delta_3\big)\, \mathrm{d}t + \frac{1}{2}\Big(\mathrm{d}A(t) + \mathrm{d}\Gamma(t)\Big)\, \Big] \;=\; 0 \tag{7.18}$$

by virtue of (3.6) and (7.13), (7.14). This is in accordance with the second property posited in (7.2).

Whereas, we argued already that the first time

$$\mathcal{S} \;:=\; \inf\big\{ t \ge 0 : R_1(t) = R_3(t) \big\} \tag{7.19}$$

of a triple collision, is a.e. finite: $\mathbb{P}(\mathcal{S} < \infty) = 1$.

Remark 7.1 (On the Structure of Filtrations). It follows from (7.17), (7.9) that the so-constructed triple $(R_1(\cdot), R_2(\cdot), R_3(\cdot))$ is adapted to the filtration $\mathbb{F}^{(W_1, W_3)}$ generated by the 2-D Brownian motion $(W_1(\cdot), W_3(\cdot))$:

$$\mathfrak{F}^{(R_1, R_2, R_3)}(t) \;\subseteq\; \mathfrak{F}^{(W_1, W_3)}(t)\,, \qquad 0 \le t < \infty\,. \tag{7.20}$$

On the other hand, the identifications

$$A(\cdot) = L^G(\cdot) = L^{R_1 - R_2}(\cdot)\,, \qquad \Gamma(\cdot) = L^H(\cdot) = L^{R_2 - R_3}(\cdot)\,,$$

show that $(A(\cdot),\, \Gamma(\cdot))$ is adapted to the filtration $\mathbb{F}^{(R_1,R_2,R_3)}$ generated by the triple $(R_1(\cdot), R_2(\cdot), R_3(\cdot))$; on account of (7.17), it follows that the same is true of the 2-D Brownian motion $(W_1(\cdot), W_3(\cdot))$. In other words, the reverse inclusion of (7.20) is also valid, and we conclude that the triple $(R_1(\cdot), R_2(\cdot), R_3(\cdot))$ and the pair $(W_1(\cdot), W_3(\cdot))$ generate exactly the same filtration:

$$\mathfrak{F}^{(R_1,R_2,R_3)}(t) \,=\, \mathfrak{F}^{(W_1,W_3)}(t)\,, \qquad 0 \le t < \infty\,. \tag{7.21}$$

• *Construction of the Individual Motions Up Until the First Triple Collision:* The same methodologies as those deployed already in subsection 4.3, show here as well how to construct a *strong* solution to the system (7.1) subject to the requirements of (7.2), up until the first time $\mathcal{S}$ of (7.19) that a triple collision occurs. The difference now, of course, is that this happens in the present context with probability one in finite time, i.e., $\mathbb{P}(\mathcal{S} < \infty) = 1$, so we need to find another way to construct a solution *beyond* this time.

• *Construction of the Individual Motions After a Triple Collision:* In order to construct the processes that satisfy (7.1) after the first triple collision time $\mathcal{S}$, we consider the excursions of the rank-gap process $(G(\cdot), H(\cdot))$ and unfold them, by permuting the names of the individual components.

More precisely, for the semimartingales $G(\cdot)$ and $H(\cdot)$ let us define the first passage time

$$\sigma_0 \,:=\, \inf \big\{ t > 0 : G(t) \wedge H(t) \,=\, 0 \big\}\,,$$

the zero sets

$$\mathfrak{Z}^G := \{ t \ge 0 : G(t) \,=\, 0 \}\,, \qquad \mathfrak{Z}^H := \{ t \ge 0 : H(t) \,=\, 0 \}\,,$$

and the corresponding countably-many excursion intervals $\{\mathcal{C}^G_\ell, \ell \in \mathbb{N}\}$, $\{\mathcal{C}^H_m, m \in \mathbb{N}\}$ in a measurable manner, i.e.,

$$\mathbb{R}_+ \setminus \mathfrak{Z}^G \,=\, \bigcup_{\ell \in \mathbb{N}} \mathcal{C}^G_\ell\,, \qquad \mathbb{R}_+ \setminus \mathfrak{Z}^H \,=\, \bigcup_{m \in \mathbb{N}} \mathcal{C}^H_m\,.$$

In order to be able to permute the indices in a proper and consistent way, let us define the particular permutation matrices

$$\mathfrak{P}_{1,2} \,:=\, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \mathfrak{P}_{2,3} \,:=\, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{7.22}$$

Here $\mathfrak{P}_{1,2}$ represents the permutation between the first and second elements, and $\mathfrak{P}_{2,3}$ represents the permutation between the second and third elements.

We enlarge the probability space by introducing I.I.D. random (permutation) matrices $\{\Xi^G_{\ell,m},\, \ell \in \mathbb{N},\, m \in \mathbb{N}\}$ and $\{\Xi^H_{\ell,m},\, \ell \in \mathbb{N},\, m \in \mathbb{N}\}$, independent of each other and of the filtration $\mathbb{F}^R(\cdot)$ generated by the rank process $(R_1(\cdot), R_2(\cdot), R_3(\cdot))'$. Here, for each (ℓ, m), the random matrix $\Xi^G_{\ell,m}$ takes each of the values in $\{\mathcal{I},\, \mathfrak{P}_{1,2}\}$ with probability $1/2$; whereas $\Xi^H_{\ell,m}$ takes each of the values in $\{\mathcal{I},\, \mathfrak{P}_{2,3}\}$ with probability $1/2$.

With these ingredients we introduce the simple, matrix-valued process

$$\begin{aligned} \boldsymbol{\eta}(\cdot) \,:=\, \sum_{\ell \in \mathbb{N}} \sum_{m \in \mathbb{N}} \mathbf{1}_{\mathcal{C}^G_\ell \cap \mathcal{C}^H_m \cap [\sigma_0, \infty)}(\cdot) \Big(\big(\Xi^G_{\ell,m} - \mathcal{I} \big) \cdot \mathbf{1}_{\{\inf \mathcal{C}^G_\ell > \inf \mathcal{C}^H_m\}} \\ + \big(\Xi^H_{\ell,m} - \mathcal{I} \big) \cdot \mathbf{1}_{\{\inf \mathcal{C}^G_\ell < \inf \mathcal{C}^H_m\}} \Big) \end{aligned} \tag{7.23}$$

and then define the matrix-valued process $Z(\cdot)$ as the solution to the stochastic integral equation

$$Z(\cdot) \,=\, \mathcal{I} + \int_0^\cdot Z(t)\, \mathrm{d}\boldsymbol{\eta}(t)\,. \tag{7.24}$$

For the construction of this solution, we proceed via an approximating sequence as in (7.28)-(7.29) below. The definition of the process $\boldsymbol{\eta}(\cdot)$ in (7.23), after σ_0, is understood as follows:

(i) On the interval $\mathcal{C}_\ell^G \cap \mathcal{C}_m^H$ of the excursion which starts from a point in $\mathfrak{Z}^G$ (i.e., $\inf \mathcal{C}_\ell^G > \inf \mathcal{C}_m^H$), the simple process $\boldsymbol{\eta}(\cdot)$ assigns the non-zero matrix

$$\mathfrak{P}_{1,2} - \mathcal{I} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{7.25}$$

with probability $1/2$, or the 0 matrix with probability $1/2$.
(ii) On the interval $\mathcal{C}_\ell^G \cap \mathcal{C}_m^H$ of the excursion which starts from a point in $\mathfrak{Z}^H$ (i.e., $\inf \mathcal{C}_\ell^G < \inf \mathcal{C}_m^H$), the simple process $\boldsymbol{\eta}(\cdot)$ assigns the non-zero matrix

$$\mathfrak{P}_{2,3} - \mathcal{I} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \tag{7.26}$$

with probability $1/2$, or the 0 matrix with probability $1/2$.
(iii) When the excursion starts from the corner $\{t \ge 0 : G(t) = H(t) = 0\}$ (that is, $\inf \mathcal{C}_\ell^G = \inf \mathcal{C}_m^H$ for some ℓ and m), then the process $\boldsymbol{\eta}(\cdot)$ assigns the 0 matrix to this excursion.

The value $Z(t)$ of the process defined in (7.24) represents the product of (countably many, random) permutations listed in (7.22), until time $t \ge 0$. Since products of permutations are also permutations, the process $Z(\cdot)$ takes values in the collection of permutation matrices.

Finally, with the rank process $R(\cdot) = (R_1(\cdot), R_2(\cdot), R_3(\cdot))'$ constructed as in (7.17), let us define the vector process $X(\cdot) := (X_1(\cdot), X_2(\cdot), X_3(\cdot))'$ by

$$X(\cdot) := Z(\cdot) R(\cdot) . \tag{7.27}$$

Now we introduce the enlarged filtration $\mathbb{F} := \{\mathfrak{F}(t),\, t \ge 0\}$ via $\mathfrak{F}(t) := \widetilde{\mathfrak{F}}(t) \vee \mathfrak{F}^Z(t)$. Since the sequences of I.I.D. random matrices $\{\Xi_{\ell,m}^G;\, \ell \in \mathbb{N},\, m \in \mathbb{N}\}$ and $\{\Xi_{\ell,m}^H;\, \ell \in \mathbb{N},\, m \in \mathbb{N}\}$ are independent of $\mathbb{F}^R$, it can be shown as in PROKAJ (2009) that both triples $(W_1(\cdot), W_2(\cdot), W_3(\cdot))$ and $(R_1(\cdot), R_2(\cdot), R_3(\cdot))$ are semimartingales with respect to this enlarged filtration $\mathbb{F}$.

Proposition 7.1. *There exists a weak solution, unique in the sense of the probability distribution, for the obverse system (7.1) with the requirements (7.2).*

This solution is pathwise unique and strong, up until the first time $\mathcal{S}$ a triple collision occurs; however, the solution fails to be strong after $\mathcal{S}$.

Proof. We split the argument in three distinct parts.

(i) Existence: We show that, on a suitable filtered probability space, the process $X(\cdot)$ defined by (7.27), with $Z(\cdot)$ in (7.24) and $\boldsymbol{\eta}(\cdot)$ in (7.23), satisfies (7.1) for suitable independent Brownian motions $B_1(\cdot), B_2(\cdot), B_3(\cdot)$, as well as (7.2). The proof is based on the unfolding of semimartingales as in ICHIBA ET AL. (2018) in the context of WALSH semimartingales.

We start by defining recursively the sequence $\{\tau_\ell^\varepsilon, \ell \in \mathbb{N}_0\}$ of stopping times by $\tau_0^\varepsilon := 0$,

$$\begin{aligned} \tau_{2\ell+1}^\varepsilon &:= \inf\{t > \tau_{2\ell}^\varepsilon : G(t) \wedge H(t) \ge \varepsilon\}, \\ \tau_{2\ell+2}^\varepsilon &:= \inf\{t > \tau_{2\ell+1}^\varepsilon : G(t) \wedge H(t) = 0\}, \end{aligned} \tag{7.28}$$

as well as the approximating processes $X^\varepsilon(\cdot) := Z^\varepsilon(\cdot) R(\cdot)$, where

$$Z^\varepsilon(\cdot) = \mathcal{I} + \int_0^\cdot Z^\varepsilon(t) \mathrm{d}\boldsymbol{\eta}^\varepsilon(t), \qquad \boldsymbol{\eta}^\varepsilon(\cdot) := \sum_{\ell \in \mathbb{N}} \boldsymbol{\eta}(\cdot)\, \mathbf{1}_{[\tau_{2\ell+1}^\varepsilon, \tau_{2\ell+2}^\varepsilon)}(\cdot) \tag{7.29}$$

for every $\varepsilon > 0$. Then for the approximating processes we have by the product rule

$$X^\varepsilon(\cdot) = \int_0^\cdot \mathrm{d}\big(Z^\varepsilon(t) R(t)\big) = \int_0^\cdot Z^\varepsilon(t)\, \mathrm{d}R(t) + \int_0^\cdot \mathrm{d}Z^\varepsilon(t)\, R(t)$$

Now, as $\varepsilon \downarrow 0$, the process $X^\varepsilon(\cdot)$ converges to $X(\cdot) = Z(\cdot)R(\cdot)$ in (7.27), and the first term on the right-hand side converges in probability to the stochastic integral $\int_0^\cdot Z(t)\,\mathrm{d}R(t)$.

Let us analyze the semimartingale dynamics of this last integral. Since $Z(\cdot)$ is a permutation-matrix-valued process, the instantaneous drift components of $\int_0^\cdot Z(t)\,\mathrm{d}R(t)$ are

$$\int_0^\cdot Z(t) \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{pmatrix} \mathrm{d}t \,=\, \int_0^\cdot \sum_{k=1}^3 \begin{pmatrix} \delta_k \mathbf{1}_{\{X_1(t) = R_k(t)\}} \\ \delta_k \mathbf{1}_{\{X_2(t) = R_k(t)\}} \\ \delta_k \mathbf{1}_{\{X_3(t) = R_k(t)\}} \end{pmatrix} \mathrm{d}t$$

for $t \ge 0$. Similarly, the martingale components of $\int_0^\cdot Z(t)\,\mathrm{d}R(t)$ are given by

$$\begin{aligned} \int_0^\cdot Z(t) \begin{pmatrix} \mathrm{d}W_1(t) \\ 0 \\ \mathrm{d}W_3(t) \end{pmatrix} &= \int_0^\cdot \begin{pmatrix} \mathbf{1}_{\{X_1(t) = R_1(t)\}}\mathrm{d}W_1(t) + \mathbf{1}_{\{X_1(t) = R_3(t)\}}\mathrm{d}W_3(t) \\ \mathbf{1}_{\{X_2(t) = R_1(t)\}}\mathrm{d}W_1(t) + \mathbf{1}_{\{X_2(t) = R_3(t)\}}\mathrm{d}W_3(t) \\ \mathbf{1}_{\{X_3(t) = R_1(t)\}}\mathrm{d}W_1(t) + \mathbf{1}_{\{X_3(t) = R_3(t)\}}\mathrm{d}W_3(t) \end{pmatrix} \\ &= \int_0^\cdot \begin{pmatrix} (\mathbf{1}_{\{X_1(t)=R_1(t)\}} + \mathbf{1}_{\{X_1(t)=R_3(t)\}})\mathrm{d}B_1(t) \\ (\mathbf{1}_{\{X_2(t)=R_1(t)\}} + \mathbf{1}_{\{X_2(t)=R_3(t)\}})\mathrm{d}B_2(t) \\ (\mathbf{1}_{\{X_3(t)=R_1(t)\}} + \mathbf{1}_{\{X_3(t)=R_3(t)\}})\mathrm{d}B_3(t) \end{pmatrix}, \end{aligned} \tag{7.30}$$

where by the P. LÉVY theorem the processes

$$B_i(\cdot) := \sum_{k=1}^3 \int_0^\cdot \mathbf{1}_{\{X_i(t) = R_k(t)\}}\mathrm{d}W_k(t)\,, \qquad i \,=\, 1,2,3 \tag{7.31}$$

are independent Brownian motions with respect to $\mathbb{F}$ (we recall that $W_i(\cdot)$, $i = 1,2,3$ are independent $\mathbb{F}-$Brownian motions). Finally, the local time components contributed by the term $\int_0^\cdot Z(t)\mathrm{d}R(t)$ are

$$\begin{aligned} &\int_0^\cdot Z(t) \begin{pmatrix} (1/2)\,\mathrm{d}L^G(t) \\ -(1/2)\,\mathrm{d}L^G(t) + (1/2)\,\mathrm{d}L^H(t) \\ -(1/2)\,\mathrm{d}L^H(t) \end{pmatrix} \\ &= \frac{1}{2}\int_0^\cdot Z(t) \left(\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \mathrm{d}L^G(t) + \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \mathrm{d}L^H(t) \right). \end{aligned} \tag{7.32}$$

On the other hand, in the limit of the term $\int_0^\cdot \mathrm{d}Z^\varepsilon(t) R^X(t)$ as $\varepsilon \downarrow 0$, local time components appear and cancel those in (7.32). More precisely, by (7.24), we have

$$\int_0^T \mathrm{d}Z^\varepsilon(t)\, R(t) \,=\, \int_0^T Z^\varepsilon(t)\big(\mathrm{d}\boldsymbol{\eta}^\varepsilon(t)\big)R(t)\,, \tag{7.33}$$

and

$$\int_0^T \mathrm{d}\eta^\varepsilon(t) R(t) \,=\, \sum_{\{\ell \,:\, \tau^\varepsilon_{2\ell+1} \le T\}} \boldsymbol{\eta}^\varepsilon(\tau^\varepsilon_{2\ell+1})\, R(\tau^\varepsilon_{2\ell+1})\,.$$

The random vector $\boldsymbol{\eta}^\varepsilon(\tau^\varepsilon_{2\ell+1})\, R(\tau^\varepsilon_{2\ell+1})$ can take values

$$\big(\mathfrak{P}_{1,2} - \mathcal{I}\big)R(\tau^\varepsilon_{2\ell+1}) \,=\, \begin{pmatrix} -R_1(\tau^\varepsilon_{2\ell+1}) + R_2(\tau^\varepsilon_{2\ell+1}) \\ R_1(\tau^\varepsilon_{2\ell+1}) - R_2(\tau^\varepsilon_{2\ell+1}) \\ 0 \end{pmatrix} \,=\, \varepsilon \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

or 0, each with equal probability $1/2$, if it corresponds to the excursion from $\mathfrak{Z}^G$ for sufficiently small $\varepsilon > 0$. It can take values

$$\big(\mathfrak{P}_{2,3} - \mathcal{I}\big) R(\tau^{\varepsilon}_{2\ell+1}) = \begin{pmatrix} 0 \\ -R_2(\tau^{\varepsilon}_{2\ell+1}) + R_3(\tau^{\varepsilon}_{2\ell+1}) \\ R_2(\tau^{\varepsilon}_{2\ell+1}) - R_3(\tau^{\varepsilon}_{2\ell+1}) \end{pmatrix} = \varepsilon \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

or 0, each with probability $1/2$, if it corresponds to the excursion from $\mathfrak{Z}^H$, for sufficiently small $\varepsilon > 0$. We are deploying here (7.23), (7.25)-(7.26), and the almost-sure continuity of the sample paths of $R(\cdot)$.

Thus, by the law of large numbers and the excursion-theoretic characterization of local time via "upcrossings" (Theorem VI.1.10 of REVUZ & YOR (1999)), we obtain the limit

$$\int_0^T \mathrm{d}\boldsymbol{\eta}^{\varepsilon}(t) R(t) \underset{\varepsilon \downarrow 0}{\longrightarrow} \int_0^T \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \mathrm{d}L^G(t) + \int_0^T \frac{1}{2} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \mathrm{d}L^H(t)$$

in probability. Combining this limit with (7.33), we obtain the convergence in probability

$$\int_0^T \mathrm{d}Z^{\varepsilon}(t) R(t) \underset{\varepsilon \downarrow 0}{\longrightarrow} \frac{1}{2} \int_0^T Z(t) \left[\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \mathrm{d}L^G(t) + \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \mathrm{d}L^H(t) \right]; \tag{7.34}$$

hence the local time components in (7.32) are cancelled by the limit (7.34) of $\int_0^T \mathrm{d}Z^{\varepsilon}(t) R(t)$.

Therefore, the process $X(\cdot)$ in (7.27) satisfies the requirements of the system (7.1)-(7.2). Consequently, a weak solution $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, $(X(\cdot), B(\cdot))$ exists for the system (7.1)-(7.2).

Remark: Let us point out that, in contrast to the situation in (7.5), (7.6), where only $W_1(\cdot)$ and $W_3(\cdot)$ appear, all three "rank-specific" Brownian motions $W_1(\cdot)$, $W_2(\cdot)$ and $W_3(\cdot)$ are needed here for constructing the "driving", or "name-specific", Brownian motions $B_1(\cdot)$, $B_2(\cdot)$, $B_3(\cdot)$ in (7.31). This is common in situations where the quadratic variation of a driving local martingale can vanish, and an additional, independent randomness is needed to "re-ignite" the motion — as for instance in the proof of the DOOB representation of continuous local martingales with quadratic variation which is absolutely continuous with respect to LEBESGUE measure.

(ii) Uniqueness in Distribution: Suppose that there are two probability measures $\mathbb{P}_j(\cdot)$, $j = 1, 2$ under which $X(\cdot)$ in (7.27) satisfies (7.1)-(7.2) and $B(\cdot)$ is a three-dimensional, independent Brownian motion. For $j = 1, 2$ we have $\mathbb{P}_j(\mathcal{S} < \infty) = 1$. By complete analogy with the discussion in subsection 4.3, up to the first time $\mathcal{S}$ of triple collision defined as in (4.14), this solution is pathwise unique, thus also *strong;* that is, adapted to the filtration $\mathbb{F}^{(B_1, B_2, B_3)}$ generated by the 3-D Brownian motion $(B_1(\cdot), B_2(\cdot), B_3(\cdot))$. Hence, its probability distribution is uniquely determined over the interval $[0, \mathcal{S})$; in other words, $\mathbb{P}_1(\cdot) \equiv \mathbb{P}_2(\cdot)$ on $\mathcal{F}(\mathcal{S}-)$.

At $t = \mathcal{S}$, we have $X_1(\mathcal{S}) = X_2(\mathcal{S}) = X_3(\mathcal{S})$, $\mathbb{P}_j$-a.e., and ties are resolved in favor of the lowest index for $j = 1, 2$. For $t > \mathcal{S}$, each name appears in rank equally likely, since the system (7.1)-(7.2) is invariant under permutations; in particular, for every $t > 0$,

$$\mathbb{P}_j\big(X_i(t) = R^X_k(t) \,|\, t > \mathcal{S}\big) = \frac{1}{3}; \qquad (i, k) \in \{1, 2, 3\}, \quad j = 1, 2. \tag{7.35}$$

Here the probability distribution of the rank process $R^X_k(\cdot)$, $k = 1, 2, 3$ in (7.3) is uniquely determined through (7.17) by the probability distribution of the reflected Brownian motion $(G(\cdot), H(\cdot))$ in section 7.2. Since the probability distribution of $X(t)$, $t > \mathcal{S}$ is determined by the rank process $R^X(t)$ and the name-rank correspondence, it is uniquely determined for $t \ge \mathcal{S}$.

Standard arguments based on the MARKOV property allow us now to extend these considerations to the finite-dimensional distributions of $t \geq \mathcal{S}$, i.e., $\mathbb{P}_1(\,\cdot\,, t \geq \mathcal{S}) \equiv \mathbb{P}_2(\,\cdot\,, t \geq \mathcal{S})$ for every $t > 0$. Therefore, combining with the uniqueness in distribution before $\mathcal{S}$, we deduce that the weak solution we constructed is unique in distribution, that is, $\mathbb{P}_1(\cdot) \equiv \mathbb{P}_2(\cdot)$.

(iii) Failure of Pathwise Uniqueness and of Strength: In the construction of the matrix-valued processes $\boldsymbol{\eta}(\cdot)$ in (7.23) and $Z(\cdot)$ in (7.24), the excursion starting from the corner $\{t \geq 0 : G(t) = H(t) = 0\}$ *does not* appear explicitly, because the triple collision local time $L^{G+H}(\cdot) = L^{R_1^X - R_3^X}(\cdot)$ is identically equal to zero, as in (7.18). The corresponding construction of $X(\cdot)$ does not change the name-rank correspondence immediately before and after the triple collision. Since the triple collision local time $L^{G+H}(\cdot)$ does not grow, one may perturb in the above construction the weak solution, by randomly permuting the names of particles immediately after the triple collision time $\mathcal{S}$ – but still obtain the same stochastic dynamics (7.1)-(7.2), and hence, the same probability distribution.

Then the resulting sample path of $X(\cdot)$ is different from the original sample path, and pathwise uniqueness fails. But here we have uniqueness of distribution, so the solution of (7.1)-(7.2) cannot be strong after the first triple collision $\mathcal{S}$; this is because of the "dual" YAMADA-WATANABE theorem (ENGELBERT (1991), CHERNY (2001)). □

Remark 7.2. The above approach to solving (7.1)-(7.2) is reminiscent of the construction of the WALSH Brownian motion, and of the splitting stochastic flow of the TANAKA equation. It would be interesting to examine the solvability of (7.1)-(7.2) via the spectral measures of classical/non-classical noises, and via the theory of stochastic flows developed by TSIREL'SON (1997), WARREN (2002), LE JAN & RAIMOND (2004a, 2004b) and WATANABE (2000) (see also AKAHORI, IZUMI & WATANABE (2009) and the references listed there).

7.3 Local Time Considerations: The Case of Equal Drifts

When $\delta_1 = \delta_2 = \delta_3$, it is possible to describe the local behavior of the semimartingale reflected Brownian motion $(G(\cdot), H(\cdot))$ at the corner of the quadrant, and in the manner of WILLIAMS (1987), as follows.

Let us denote by $(\boldsymbol{\varrho}_\cdot, \boldsymbol{\vartheta}_\cdot)$ the system (7.10)-(7.11) in polar coordinate in $\mathbb{R}^2$, i.e., $0 \leq G(\cdot) = \boldsymbol{\varrho}_\cdot \cos(\boldsymbol{\vartheta}_\cdot)$, $0 \leq H(\cdot) = \boldsymbol{\varrho}_\cdot \sin(\boldsymbol{\vartheta}_\cdot)$. In the notation of VARADHAN & WILLIAMS (1985), this system corresponds to planar Brownian motion reflected on the faces of the nonnegative quadrant at angles $\theta_1 \equiv \theta_2 := \arctan(1/2)$ relative to the interior normals there, thus

$$\alpha := \frac{\theta_1 + \theta_2}{\Xi} = \frac{2}{\pi} \arctan\big(4/3\big) \in \left(\frac{1}{2}, \frac{2}{3}\right).$$

From the theory of VARADHAN & WILLIAMS (1985), we know that the process $(G(\cdot), H(\cdot))$, started in the interior of the quadrant, hits eventually the vertex (0,0) with probability one, but does not get absorbed there: it manages to escape from the vertex, though it hits immediately the boundary of the quadrant (cf. WILLIAMS (1987), section 3). We define the function

$$\varphi(\rho, \theta) = \rho^\alpha \cos(\alpha\theta - \theta_1), \qquad 0 \leq \rho < \infty, \quad 0 \leq \theta \leq \pi/2,$$

and note $(2/\sqrt{5}) \leq \cos(\alpha\theta - \theta_1) \leq 1$ for $0 \leq \theta \leq \pi/2$. Always with $\delta_1 = \delta_2 = \delta_3$, the process $\varphi(\boldsymbol{\varrho}_\cdot, \boldsymbol{\vartheta}_\cdot)$ is a nonnegative, continuous local submartingale; the continuous, adapted, non-decreasing process in its DOOB-MEYER decomposition is a constant multiple of

$$0 \leq \Lambda^\bullet(\cdot) := \frac{\alpha(2-\alpha)}{2} \lim_{\varepsilon \downarrow 0} \varepsilon^{1-(2/\alpha)} \int_0^\cdot \Big(\cos\big(\alpha\boldsymbol{\vartheta}_t - \theta_1\big)\Big)^{(2/\alpha)-2} \cdot \mathbf{1}_{[0,\varepsilon)}\big(\varphi(\boldsymbol{\varrho}_t, \boldsymbol{\vartheta}_t)\big)\, \mathrm{d}t\,, \quad (7.36)$$

(Lemma 2.8 in WILLIAMS (1987), p. 305), in the sense of convergence in probability. The continuous, increasing, additive functional $\Lambda^\bullet(\cdot)$ is supported on $\big\{t \ge 0 : G(t) = H(t) = 0\big\}$.

The expression in (7.36) provides a measure of how this kind of occupation time grows, as a function of the angles of reflection in this quadrant. Since $\alpha < 1$, we deduce from (7.36) that the semimartingale $\varrho.$ does not accumulate semimartingale local time at the origin, i.e., $L^{\boldsymbol{\varrho}}(\cdot) \equiv 0$; see the Remark 7.4 below. Likewise, the semimartingale $\varphi(\boldsymbol{\varrho}., \boldsymbol{\vartheta}.)$ does not accumulate semimartingale local time at the origin, i.e., $L^{\varphi(\boldsymbol{\varrho},\boldsymbol{\vartheta})}(\cdot) \equiv 0$; and this, despite the fact that we can find the continuous, increasing additive functional $\Lambda^\bullet(\cdot)$ in (7.36), also called *"local time for $\varphi(\boldsymbol{\varrho}., \boldsymbol{\vartheta}.)$ at the corner"*. For a similar phenomenon in BESSEL processes of dimension $\delta \in (1,2)$, cf. Exercise XI (1.25) of REVUZ & YOR (1999), Appendix A.1 in ICHIBA ET AL. (2011).

Remark 7.3. Let us denote by $\mathbb{P}^\bullet$ (*respectively* $\mathbb{E}^\bullet$) the probability measure (*respectively* expectation) induced by the system (7.10)-(7.11) with $\delta_1 = \delta_2 = \delta_3$, $G(0) = H(0) = 0$. Let us rescale $\Lambda^\bullet(\cdot)$ by

$$L^\bullet(t) := \Big(\mathbb{E}^\bullet\Big[\int_0^\infty e^{-s}\mathrm{d}\Lambda^\bullet(s)\Big]\Big)^{-1}\Lambda^\bullet(t)\ ; \qquad 0 \le t < \infty\,.$$

In this case the right continuous inverse $\tau^\bullet(u) := \inf\{t \ge 0 : L^\bullet_t > u\}$ of the map $t \mapsto L^\bullet(t)$ is a stable subordinator of index $\kappa = \alpha/2$ and rate 1 under $\mathbb{P}^\bullet$, i.e.,

$$\log \mathbb{E}^\bullet\big(\exp\big(-\lambda\tau^\bullet(u)\big)\big) = -u\,\lambda^{\alpha/2}\,, \qquad t, u > 0\,.$$

As a result, the set $\big\{t \ge 0 : G(t) = H(t) = 0\big\}$ has HAUSSDORFF dimension $\kappa = \alpha/2$, and its HAUSSDORFF measure is known. For the details of excursions of the semimartingale reflected Brownian motion from the corner of the quadrant see WILLIAMS (1987) and ROGERS (1989). WILLIAMS (1987) also shows that the measure with the density right below is invariant for the process $(\boldsymbol{\varrho}., \boldsymbol{\vartheta}.)$:

$$f(\rho,\theta) = \rho^{-\alpha}\cos(\alpha\theta - \theta_1)\,, \qquad 0 \le \rho < \infty\,, \quad 0 \le \theta \le \pi/2\,.$$

Remark 7.4. Suppose that $\delta_1 = \delta_2 = \delta_3$; that there exist a smooth function $\widetilde{\varphi}(\rho,\theta)$ for $0 < \rho < \infty$, $0 \le \theta \le \pi/2$ and real constants $r_0 > 0$, $c_1 > 0$, $c_2 < 1$, $c_3 > 0$, $p > 0$ such that $c_0 := (2/\alpha) - 1 - p(1-c_2) > 0$, $[c_3\,\varphi(\rho,\theta)]^p \le \widetilde{\varphi}(\rho,\theta)$ for every $0 \le \rho \le r_0$, $0 \le \theta \le \pi\,/\,2$; and that $\widetilde{\boldsymbol{\varphi}}. := \widetilde{\varphi}(\boldsymbol{\varrho}., \boldsymbol{\vartheta}.)$ is a semimartingale with quadratic variation $\langle\widetilde{\boldsymbol{\varphi}}\rangle.$ and

$$\int_0^\cdot \mathbf{1}_{[0,r_0)}(\widetilde{\boldsymbol{\varphi}}_t)\,\mathrm{d}\langle\widetilde{\boldsymbol{\varphi}}\rangle_t \le c_1\int_0^\cdot \mathbf{1}_{[0,r_0)}(\widetilde{\boldsymbol{\varphi}}_t)\cdot|\widetilde{\varphi}(\boldsymbol{\varrho}_t,\boldsymbol{\vartheta}_t)|^{c_2}\,\mathrm{d}t\,.$$

Then it follows from (7.36) that *the semimartingale local time $L^{\widetilde{\varphi}(\boldsymbol{\varrho},\boldsymbol{\vartheta})}(\cdot)$ for $\widetilde{\varphi}(\boldsymbol{\varrho}., \boldsymbol{\vartheta}.)$ does not accumulate at the origin, i.e., $L^{\widetilde{\boldsymbol{\varphi}}}(\cdot) \equiv L^{\widetilde{\varphi}(\boldsymbol{\varrho},\boldsymbol{\vartheta})}(\cdot) \equiv 0$.*

Indeed, since $2\,/\sqrt{5} \le \cos(\alpha\theta - \theta_1)) \le 1$ for $0 \le \theta \le \pi/2$ and

$$\frac{1}{u}\int_0^\cdot \mathbf{1}_{[0,u)}(\widetilde{\boldsymbol{\varphi}}_t)\mathrm{d}\langle\widetilde{\boldsymbol{\varphi}}\rangle_t \le \frac{c_1}{\varepsilon^{1-c_2}}\int_0^\cdot \mathbf{1}_{[0,u)}([c_3\varphi(\boldsymbol{\varrho}_t,\boldsymbol{\vartheta}_t)]^p)\mathrm{d}t = \frac{c_1}{\varepsilon^{1-c_2}}\int_0^\cdot \mathbf{1}_{[0,\ u^{1/p}\,/\,c_3)}(\varphi(\boldsymbol{\varrho}_t,\boldsymbol{\vartheta}_t))\mathrm{d}t$$
$$= \frac{c_1}{(c_3\,\varepsilon)^{p(1-c_2)}}\int_0^\cdot \mathbf{1}_{[0,u)}(\varphi(\boldsymbol{\varrho}_t,\boldsymbol{\vartheta}_t))\mathrm{d}t \le c_1 c_3^{-p(1-c_2)}\varepsilon^{c_0+1-(2/\alpha)}\int_0^\cdot \mathbf{1}_{[0,\varepsilon)}(\varphi(\boldsymbol{\varrho}_t,\boldsymbol{\vartheta}_t))\mathrm{d}t\,,$$

where $\varepsilon := u^{1/p}\,/\,c_3$ for $0 < u \le r_0$, combining these estimates with (7.36), letting $u \downarrow 0$, and hence $\varepsilon \downarrow 0$, we obtain the convergence in probability

$$L^{\widetilde{\varphi}(\boldsymbol{\varrho},\boldsymbol{\vartheta})}(\cdot) := \lim_{u\downarrow 0}\frac{1}{2u}\int_0^\cdot \mathbf{1}_{[0,u)}(\widetilde{\boldsymbol{\varphi}}_t)\mathrm{d}\langle\widetilde{\boldsymbol{\varphi}}\rangle_t = 0\,. \tag{7.37}$$

With this claim we can verify $L^{\boldsymbol{\varrho}}(\cdot) \equiv 0$ by choosing $\widetilde{\varphi}(\rho, \theta) := \rho$ for $0 \le \rho < \infty$, $0 \le \theta \le \pi/2$ with $p := 1/\alpha > 0$, $c_1 := 1$, $c_2 := 0$, $c_3 := 1$, $r_0 := 1$ and $c_0 := (1/\alpha) - 1 > 0$, since $0 < \alpha < 1$. Similarly, it can be verified that $L^{\varphi(\boldsymbol{\varrho},\boldsymbol{\vartheta})}(\cdot) \equiv 0$ because $|\nabla\varphi(\rho,\theta)|^2 = \alpha^2\rho^{2\alpha-2}$, $\mathrm{d}\langle\varphi(\boldsymbol{\varrho},\boldsymbol{\vartheta})\rangle_t / \mathrm{d}t = \alpha^2 \boldsymbol{\varrho}_t^{2\alpha-2}$, $c_1 := (\sqrt{5}/2)^{c_2}\alpha^2 > 0$, $c_2 := 2\alpha - 2 < 1$, $c_3 = 1$, $p := 1$, $r_0 = 1$, $c_0 := (2/\alpha) - 1 - (3 - 2\alpha) = (2/\alpha) + 2\alpha - 4 \approx 0.568 > 0$.

Remark 7.5. For a general choice of drifts $(\delta_1, \delta_2, \delta_3)$ we may use GIRSANOV's change of measure on top of the procedure described above. In fact, the rank system (7.17) is rewritten as

$$\begin{aligned} R_1(t) &= x_1 + \delta_2 t + \widetilde{W}_1(t) + \frac{1}{2}A(t)\,, \\ R_2(t) &= x_2 + \delta_2 t - \frac{1}{2}A(t) + \frac{1}{2}\Gamma(t)\,, \\ R_3(t) &= x_3 + \delta_2 t + \widetilde{W}_3(t) - \frac{1}{2}\Gamma(t)\,, \end{aligned} \tag{7.38}$$

where $\widetilde{W}_1(t) := (\delta_1 - \delta_2)t + W_1(t)$ and $\widetilde{W}_3(t) := (\delta_3 - \delta_2)t + W_3(t)$, $t \ge 0$ are independent Brownian motion under a new probability measure equivalent to the original measure by the Girsanov theorem. Thus we may deal with a general choice of drifts.

8 The System of (2.1) with Skew-Elastic Collisions

We shall study in this section a variant of the system (2.1) — with the same purely ballistic motions for the leader and laggard particles, and the same diffusive motion for the middle particle — but now with skew-elastic collisions as in FERNHOLZ, ICHIBA & KARATZAS (2013a) between the second- and third-ranked particles, namely,

$$\begin{aligned} X_i(\cdot) \,=\, x_i &+ \sum_{k=1}^{3} \delta_k \int_0^{\cdot} \mathbf{1}_{\{X_i(t) = R_k^X(t)\}}\,\mathrm{d}t + \int_0^{\cdot} \mathbf{1}_{\{X_i(t) = R_2^X(t)\}}\,\mathrm{d}B_i(t) \\ &+ \int_0^{\cdot} \mathbf{1}_{\{X_i(t) = R_2^X(t)\}}\,\mathrm{d}L^{R_2^X - R_3^X}(t) + \int_0^{\cdot} \mathbf{1}_{\{X_i(t) = R_3^X(t)\}}\,\mathrm{d}L^{R_2^X - R_3^X}(t)\,, \quad i = 1,2,3 \end{aligned} \tag{8.1}$$

in the notation of (2.4), (3.6). Here again, δ_1, δ_2, δ_3 and $x_1 > x_2 > x_3$ are given real numbers. This system was first introduced and studied in FERNHOLZ (2011).

We shall try to find a weak solution to this system; in other words, construct a filtered probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, $\mathbb{F} = \{\mathfrak{F}(t)\}_{0 \le t < \infty}$ rich enough to accommodate independent Brownian motions $B_1(\cdot)$, $B_2(\cdot)$, $B_3(\cdot)$ and continuous semimartingales $X_1(\cdot)$, $X_2(\cdot)$, $X_3(\cdot)$ so that, with probability one, the equations of (8.1) are satisfied along with the "non-stickiness" and "soft triple collision" requirements

$$\int_0^{\infty} \mathbf{1}_{\{R_k^X(t) = R_\ell^X(t)\}}\,\mathrm{d}t \,=\, 0\,, \quad \forall\ k < \ell\,; \qquad L^{R_1^X - R_3^X}(\cdot) \equiv 0\,. \tag{8.2}$$

8.1 Analysis

Assuming that such a weak solution to the system of (8.1), (8.2) has been constructed, the ranked processes $R_k^X(\cdot)$ as in (2.4) are continuous semimartingales with decompositions

$$R_1^X(t) \,=\, x_1 + \delta_1\, t + \frac{1}{2}\Lambda^{(1,2)}(t)\,, \qquad R_3^X(t) \,=\, x_3 + \delta_3\, t + \frac{1}{2}\,\Lambda^{(2,3)}(t) \tag{8.3}$$

$$R_2^X(t) \,=\, x_2 + \delta_2\, t + W(t) - \frac{1}{2}\,\Lambda^{(1,2)}(t) + \frac{3}{2}\,\Lambda^{(2,3)}(t) \tag{8.4}$$

by analogy with (3.1)-(3.3). We are using here the exact same notation for the standard Brownian motion $W(\cdot)$ as in (3.4), and the exact same notation for the collision local times $\Lambda^{(k,\ell)}(\cdot)\,,\ k<\ell$ as in (3.5). For the gaps $G(\cdot)=R_1^X(\cdot)-R_2^X(\cdot)\,, H(\cdot)=R_2^X(\cdot)-R_3^X(\cdot)$ we have the SKOROKHOD-type representations of the form (3.10)-(3.11), now with

$$U(t)=x_1-x_2+\big(\delta_1-\delta_2\big)\,t-W(t)-\frac{3}{2}\,L^H(t)\,,\quad V(t)=x_2-x_3+\big(\delta_2-\delta_3\big)\,t+W(t)-\frac{1}{2}\,L^G(t)\,.$$

Whereas, from the theory of the SKOROKHOD reflection problem we obtain now the relationships linking the two local time processes $L^G(\cdot)$ and $L^H(\cdot)$, namely

$$L^G(t)\;=\;\max_{0\le s\le t}\big(-U(s)\big)^+\;=\;\max_{0\le s\le t}\Big(x_2-x_1+\big(\delta_2-\delta_1\big)\,s+W(s)+\frac{3}{2}\,L^H(s)\Big)^+\,,\tag{8.5}$$

$$L^H(t)\;=\;\max_{0\le s\le t}\big(-V(s)\big)^+\;=\;\max_{0\le s\le t}\Big(x_3-x_2+\big(\delta_3-\delta_2\big)\,s-W(s)+\frac{1}{2}\,L^G(s)\Big)^+\tag{8.6}$$

- The resulting system

$$G(t)\;=\;x_1-x_2+\big(\delta_1-\delta_2\big)\,t-W(t)-\frac{3}{2}\,L^H(t)+L^G(t)\,,\qquad 0\le t<\infty\tag{8.7}$$

$$H(t)\;=\;x_2-x_3+\big(\delta_2-\delta_3\big)\,t+W(t)-\frac{1}{2}\,L^G(t)+L^H(t)\,,\qquad 0\le t<\infty\tag{8.8}$$

for the two nonnegative gap processes is again of the HARRISON & REIMAN (1981) type (3.14): it amounts to reflecting along the faces of the nonnegative outhant the degenerate, two-dimensional Brownian motion $\mathfrak{Z}(\cdot)$ as in (3.15), with drift vector $\boldsymbol{m}=\big(\delta_1-\delta_2,\,\delta_2-\delta_3\big)'$ and covariance matrix

$$\mathcal{C}\;=\begin{pmatrix}1&-1\\-1&1\end{pmatrix}$$

as in (3.16), but now with reflection matrix

$$\mathcal{R}\;:=\mathcal{I}-\mathcal{Q}\,,\qquad \mathcal{Q}\;=\begin{pmatrix}0&3/2\\1/2&0\end{pmatrix}\,,\qquad\text{thus}\quad \mathcal{R}^{-1}\boldsymbol{m}\;=\;2\begin{pmatrix}2\,\delta_1+\delta_2-3\,\delta_3\\ \delta_1+\delta_2-2\,\delta_3\end{pmatrix}\,.\tag{8.9}$$

It is important to note here that the matrix $\mathcal{Q}$ has zero elements on its diagonal and spectral radius strictly less than 1, and that the *skew-symmetry condition*

$$\mathcal{R}+\mathcal{R}'\;=\;2\,\mathcal{C}\tag{8.10}$$

of HARRISON & WILLIAMS (1987) is satisfied by these covariance and reflection matrices.

8.2 Synthesis

Let us start now with given real numbers $\delta_1,\,\delta_2,\,\delta_3$, and $x_1>x_2>x_3$, and construct a filtered probability space $(\Omega,\mathfrak{F},\mathbb{P}),\ \mathbb{F}=\big\{\mathfrak{F}(t)\big\}_{0\le t<\infty}$ rich enough to support a standard Brownian motion $W(\cdot)$. By analogy with (8.5)-(8.6), we consider the system of equations

$$A(t)\;=\;\max_{0\le s\le t}\Big(x_2-x_1+\big(\delta_2-\delta_1\big)\,s+W(s)+\frac{3}{2}\,\Gamma(s)\Big)^+\,,\quad 0\le t<\infty\tag{8.11}$$

$$\Gamma(t)\;=\;\max_{0\le s\le t}\Big(x_3-x_2+\big(\delta_3-\delta_2\big)\,s-W(s)+\frac{1}{2}\,A(s)\Big)^+\,,\quad 0\le t<\infty\tag{8.12}$$

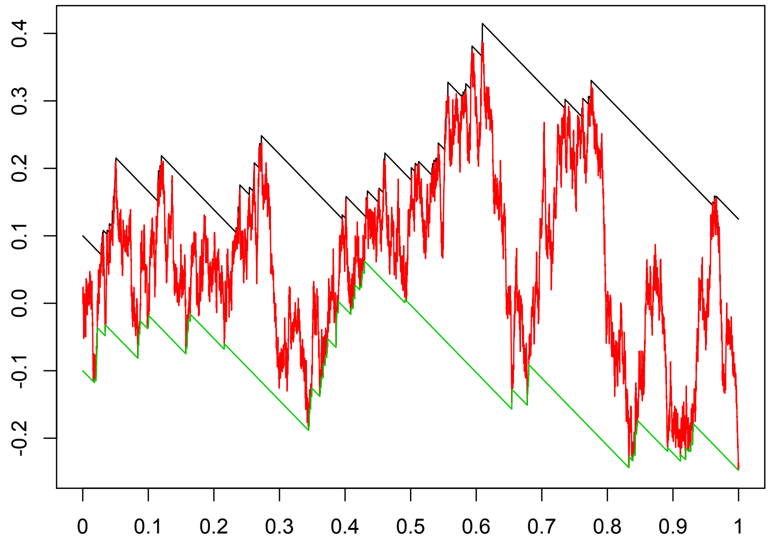

Figure 4: Simulated processes; Black $= R_1(\cdot)$, Red $= R_2(\cdot)$, Green $= R_3(\cdot)$. Here we have taken $\delta_1 = -1$, $\delta_2 = -2$ and $\delta_3 = -1$ in (8.1). We are indebted to Dr. E.R. FERNHOLZ for this picture.

for two continuous, nondecreasing and adapted processes $A(\cdot)$ and $\Gamma(\cdot)$ with $A(0) = \Gamma(0) = 0$. Theorem 1 of HARRISON & REIMAN (1981) guarantees that this system has a unique continuous solution $(A(\cdot), \Gamma(\cdot))$, adapted to the smallest filtration $\mathbb{F}^W$ to which the driving Brownian motion $W(\cdot)$ is itself adapted. With this solution in place, we construct the continuous semimartigales

$$U(t) := x_1 - x_2 + \big(\delta_1 - \delta_2\big)\, t - W(t) - \frac{3}{2}\,\Gamma(t)\,, \quad V(t) := x_2 - x_3 + \big(\delta_2 - \delta_3\big)\, t + W(t) - \frac{1}{2}\, A(t)\,, \tag{8.13}$$

and then "fold" them to obtain their SKOROKHOD reflections

$$G(t) \;:=\; U(t) + \max_{0 \le s \le t} \big(- U(s) \big)^+ \;=\; x_1 - x_2 + \big(\delta_1 - \delta_2\big)\, t - W(t) - \frac{3}{2}\,\Gamma(t) + A(t) \;\ge\; 0 \tag{8.14}$$

$$H(t) \;:=\; V(t) + \max_{0 \le s \le t} \big(- V(s) \big)^+ \;=\; x_2 - x_3 + \big(\delta_2 - \delta_3\big)\, t + W(t) - \frac{1}{2}\, A(t) + \Gamma(t) \;\ge\; 0 \tag{8.15}$$

for $t \in [0, \infty)$. As before, for these two continuous, nonnegative semimartingales the theories of the SKOROKHOD reflection problem and of semimartingale local time give, respectively,

$$\int_0^\infty \mathbf{1}_{\{G(t)>0\}}\, \mathrm{d}A(t) \;=\; 0\,, \qquad \int_0^\infty \mathbf{1}_{\{H(t)>0\}}\, \mathrm{d}\Gamma(t) \;=\; 0 \tag{8.16}$$

and

$$\int_0^\infty \mathbf{1}_{\{G(t)=0\}}\, \mathrm{d}t \;=\; 0\,, \qquad \int_0^\infty \mathbf{1}_{\{H(t)=0\}}\, \mathrm{d}t \;=\; 0\,. \tag{8.17}$$

We claim the additional properties

$$\int_0^\infty \mathbf{1}_{\{G(t)=0\}}\, \mathrm{d}\Gamma(t) \;=\; 0\,, \qquad \int_0^\infty \mathbf{1}_{\{H(t)=0\}}\, \mathrm{d}A(t) \;=\; 0\,. \tag{8.18}$$

Indeed, focusing on the first one (the second is established similarly), we see that $\int_0^\infty \mathbf{1}_{\{G(t)=0\}}\, \mathrm{d}\Gamma(t) = \int_0^\infty \mathbf{1}_{\{G(t)=H(t)=0\}}\, \mathrm{d}\Gamma(t) = 0$ holds, on the strength of the second equality in (8.16) and of Theorem 1 in REIMAN & WILLIAMS (1988).

- We need to identify the regulating processes $A(\cdot)$, $\Gamma(\cdot)$ as local times; we start by observing

$$L^G(\cdot) = \int_0^{\cdot} \mathbf{1}_{\{G(t)=0\}}\, \mathrm{d}G(t) = \int_0^{\cdot} \mathbf{1}_{\{G(t)=0\}} \Big[\mathrm{d}A(t) - \frac{3}{2} \mathrm{d}\Gamma(t) - \mathrm{d}W(t) + \big(\delta_1 - \delta_2\big)\, \mathrm{d}t \Big].$$

from (3.6) and (8.14). The last (LEBESGUE) and next-to-last (ITÔ) integrals in this expression vanish on the strength of (8.17), whereas the third-to-last integral vanishes on account of (8.18); so we are left with the identification $L^G(\cdot) = \int_0^{\cdot} \mathbf{1}_{\{G(t)=0\}}\, \mathrm{d}A(t) \equiv A(\cdot)$, where the last equality comes on the heels of (8.16). We establish similarly the identification $L^H(\cdot) \equiv \Gamma(\cdot)$.

- By analogy with (8.3)-(8.4), we construct now the $\mathbb{F}^W$−adapted *processes of ranks*

$$R_1(t) \,:=\, x_1 + \delta_1\, t + \frac{1}{2}\, A(t) \tag{8.19}$$

$$R_2(t) \,:=\, x_2 + \delta_2\, t + W(t) - \frac{1}{2}\, A(t) + \frac{3}{2}\, \Gamma(t) \tag{8.20}$$

$$R_3(t) \,:=\, x_3 + \delta_3\, t + \frac{1}{2}\, \Gamma(t) \tag{8.21}$$

and note $R_1(\cdot) - R_2(\cdot) = G(\cdot) \geq 0$, $R_2(\cdot) - R_3(\cdot) = H(\cdot) \geq 0$, thus $R_1(\cdot) \geq R_2(\cdot) \geq R_3(\cdot)$ and

$$R_1(t) - R_3(t) \,=\, G(t) + H(t) \,=\, x_1 - x_3 + \big(\delta_1 - \delta_3\big)\, t + \frac{1}{2} \big[\, A(t) - \Gamma(t) \,\big].$$

In particular, the continuous process $R_1(\cdot) - R_3(\cdot) \geq 0$ is of finite first variation on compact intervals, so its local time at the origin vanishes, as posited in (8.2): $L^{R_1 - R_3}(\cdot) \equiv 0$. The other properties posited there are direct consequences of (8.17). Finally, the identifications $A(\cdot) \equiv L^G(\cdot) \equiv L^{R_1-R_2}(\cdot)$, $\Gamma(\cdot) \equiv L^H(\cdot) \equiv L^{R_2-R_3}(\cdot)$ show, in conjunction with (8.20), that the rank vector process $(R_1(\cdot), R_2(\cdot), R_3(\cdot))$ and the scalar, standard Brownian motion $W(\cdot)$ generate the exact same filtration.

- We can construct now on a suitable filtered probability space independent Brownian motions $B_1(\cdot)$, $B_2(\cdot)$, $B_3(\cdot)$ and continuous, adapted processes $X_1(\cdot)$, $X_2(\cdot)$, $X_3(\cdot)$ so that, with probability one, the equations of (8.1) are satisfied, along with those of (8.2), up until the first time of a triple collision

$$\mathcal{S} \,:=\, \big\{ t \geq 0 \,:\, X_1(t) = X_2(t) = X_3(t) \big\}\,, \tag{8.22}$$

as well as $R_k^X(t) = R_k(t)$, $0 \leq t < \mathcal{S}$ for $k = 1, 2, 3$. Just as before, this is done by considering the particles two-by-two in the manner of ICHIBA ET AL. (2013), and applying the results in FERNHOLZ ET AL. (2013a) and (2013b).

It is an open question, whether $\mathbb{P}(\mathcal{S} = \infty) = 1$ *is valid in this case.*

8.3 Invariant Distribution

Under the conditions

$$3\,\delta_3 \,>\, 2\,\delta_1 + \delta_2\,, \qquad 2\,\delta_3 \,>\, \delta_1 + \delta_2\,, \tag{8.23}$$

we have that both components of the vector

$$\boldsymbol{\lambda} \equiv \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} := -\,\mathcal{R}^{-1}\boldsymbol{m} \,=\, 2\,\begin{pmatrix} 3\,\delta_3 - 2\,\delta_1 - \delta_2 \\ 2\,\delta_3 - \delta_1 - \delta_2 \end{pmatrix} \tag{8.24}$$

are positive numbers. Repeating the reasoning in section 5, we deduce that here again the two-dimensional process $\big(G(\cdot), H(\cdot)\big)$ of gaps is positive recurrent, has a unique invariant measure $\boldsymbol{\pi}$ with $\boldsymbol{\pi}((0,\infty)^2) = 1$, and converges to this probability measure in distribution as $t \to \infty$.

For instance, we have the analogue

$$\mathrm{d}\big(G^2(t) + 3G(t)H(t) + 3H^2(t)\big) \,=\, \Big[1 - \frac{1}{2}\big(\lambda_1 G(t) + 3\lambda_2 H(t)\big)\Big]\mathrm{d}t + \big(3H(t) + G(t)\big)\,\mathrm{d}W(t)$$

of the dynamics of (5.14) in the present context; and the function $V(g,h) = \exp\big\{\sqrt{g^2 + 3gh + 3h^2}\,\big\}$ is a LYAPOUNOV function for the semimartingale reflecting Brownian motion $(G(\cdot), H(\cdot))$, which is thus seen to be positive recurrent and to have a unique invariant distribution. We also deduce, just as before, the Strong Laws of Large Numbers

$$\lim_{t\to\infty}\frac{L^G(t)}{t} \,=\, \lambda_1 \,=\, 2\,\big(3\,\delta_3 - 2\,\delta_1 - \delta_2\big)\,, \qquad \lim_{t\to\infty}\frac{L^H(t)}{t} \,=\, \lambda_2 \,=\, 2\,\big(2\,\delta_3 - \delta_1 - \delta_2\big)\,.$$

On the other hand, in view of the fact that the covariance matrix $\mathcal{C}$ and the reflection matrix $\mathcal{R}$ satisfy the skew-symmetry condition of (8.10), the results of O'CONNELL & ORTMANN (2012) suggest that *the invariant probability measure for the vector process* $\big(G(\cdot), H(\cdot)\big)$ *of gaps should be the product of exponentials*

$$\boldsymbol{\pi}\big(\mathrm{d}g, \mathrm{d}h\big) \,=\, 4\,\lambda_1\,\lambda_2\,e^{-2\,\lambda_1\,g\,-2\,\lambda_2\,h}\,\mathrm{d}g\,\mathrm{d}h\,, \qquad (g,h) \in (0,\infty)^2\,. \tag{8.25}$$

Proof of the claim in (8.25): This claim can be verified as in section 9 of HARRISON & WILLIAMS (1987); for completeness, we present now the details. As shown in that paper, and in DAI & KURTZ (2003), it is enough to find two measures $\boldsymbol{\nu}_1$ and $\boldsymbol{\nu}_2$ on $(0,\infty)$ so that the appropriate form of the Basic Adjoint Relationship for the system of (8.7), (8.8), namely

$$\int_0^\infty\!\!\int_0^\infty \Big(\big(D^2_{gg} + D^2_{hh} - 2D^2_{gh}\big) + 2\big(\delta_1 - \delta_2\big)D_g + 2\big(\delta_2 - \delta_3\big)D_h\Big) f(g,h)\,\boldsymbol{\pi}(\mathrm{d}g,\mathrm{d}h) + \tag{8.26}$$

$$+ \int_0^\infty \Big(D_g - \frac{1}{2}D_h\Big) f(0,h)\,\boldsymbol{\nu}_1(\mathrm{d}h) + \int_0^\infty \Big(D_h - \frac{3}{2}D_g\Big) f(g,0)\,\boldsymbol{\nu}_2(\mathrm{d}g) \,=\, 0\,,$$

holds for every function f of class $C^2\big((0,\infty)^2\big)$. Once again, selecting $f(g,h) = \exp\big(-\alpha_1 g - \alpha_2 h\big)$ for $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$ and substituting in (8.26), we obtain the equation

$$\Big((\alpha_1 - \alpha_2)^2 + 2(\delta_2 - \delta_1)\alpha_1 + 2(\delta_3 - \delta_2)\alpha_2\Big)\,\widehat{\boldsymbol{\pi}}(\alpha_1,\alpha_2) \,=\, \Big(\alpha_1 - \frac{\alpha_2}{2}\Big)\,\widehat{\boldsymbol{\nu}}_1(\alpha_2) + \Big(\alpha_2 - \frac{3\,\alpha_1}{2}\Big)\,\widehat{\boldsymbol{\nu}}_2(\alpha_1) \tag{8.27}$$

linking the LAPLACE transforms of the measures $\boldsymbol{\pi}$ and $\boldsymbol{\nu}_1$, $\boldsymbol{\nu}_2$.

In accordance with the guess (8.25) that we are trying to establish, let us posit a product-form expression

$$\boldsymbol{\pi}\big(\mathrm{d}g, \mathrm{d}h\big) \,=\, p_1(g)\,p_2(h)\,\mathrm{d}g\,\mathrm{d}h\,, \qquad (g,h) \in (0,\infty)^2 \tag{8.28}$$

for the invariant measure for the process of gaps; here $p_1(\cdot)$ and $p_2(\cdot)$ are probability density functions on the positive half-line. We denote by $\widehat{p}_1(\cdot)$ and $\widehat{p}_2(\cdot)$ the LAPLACE transforms of these density functions, set

$$c_j \,:=\, \lim_{\alpha\to\infty}\big(\alpha\,\widehat{p}_j(\alpha)\big)\,, \qquad j = 1, 2\,,$$

and note that (8.28) implies then $\widehat{\boldsymbol{\pi}}(\alpha_1, \alpha_2) = \widehat{p}_1(\alpha_1)\, \widehat{p}_2(\alpha_2)$. Now we divide the resulting expression (8.27) by $\alpha_1 > 0$ (respectively, by $\alpha_2 > 0$), then send α_1 (respectively, by α_2) to infinity; the results are, respectively,

$$\widehat{\boldsymbol{\nu}}_1(\alpha_2) \,=\, c_1\, \widehat{p}_2(\alpha_2)\,, \qquad \widehat{\boldsymbol{\nu}}_2(\alpha_1) \,=\, c_2\, \widehat{p}_1(\alpha_1)\,.$$

Substituting these expressions back into (8.27) gives

$$\widehat{p}_1(\alpha_1)\, \widehat{p}_2(\alpha_2) \Big((\alpha_1 - \alpha_2)^2 + 2(\delta_2 - \delta_1)\alpha_1 + 2(\delta_3 - \delta_2)\alpha_2 \Big) \,=$$

$$= c_1\, \widehat{p}_2(\alpha_2) \Big(\alpha_1 - \frac{\alpha_2}{2} \Big) + c_2\, \widehat{p}_1(\alpha_1) \Big(\alpha_2 - \frac{3\,\alpha_1}{2} \Big)\,;$$

whereas, setting $\alpha_2 = 0$ (respectively, $\alpha_1 = 0$) in this last equation, we obtain

$$c_1 \,=\, \widehat{p}_1(\alpha_1) \Big(\alpha_1 + 2\big(\delta_2 - \delta_1\big) + \frac{3}{2}\, c_2 \Big)\,, \qquad c_2 \,=\, \widehat{p}_2(\alpha_2) \Big(\alpha_2 + 2\big(\delta_3 - \delta_2\big) + \frac{1}{2}\, c_1 \Big).$$

On account of the rather obvious properties $\widehat{p}_1(0) = \widehat{p}_2(0) = 1$, we obtain the system of equations

$$c_1 \,=\, 2\big(\delta_2 - \delta_1\big) + \frac{3}{2}\, c_2\,, \qquad c_2 \,=\, 2\big(\delta_3 - \delta_2\big) + \frac{1}{2}\, c_1\,.$$

The solution to this system is now rather trivially $c_1 \,=\, 2\,\lambda_1$, $c_2 \,=\, 2\,\lambda_2$ in the notation of (8.24); this leads to the transforms $\widehat{p}_j(\alpha) \,=\, (2\lambda_j)/(\alpha + 2\lambda_j)$, $\alpha \geq 0$, and thence to the exponential probability density functions

$$p_1(g) \,=\, 2\,\lambda_1\, e^{-2\,\lambda_1\, g}\,, \quad g > 0 \qquad \text{and} \qquad p_2(h) \,=\, 2\,\lambda_2\, e^{-2\,\lambda_2\, h}\,, \quad h > 0 \tag{8.29}$$

and to the measures

$$\boldsymbol{\nu}_2(\mathrm{d}g) \,=\, 4\,\lambda_2\,\lambda_1\, e^{-2\,\lambda_1\, g}\, \mathrm{d}g\,, \qquad \text{and} \qquad \boldsymbol{\nu}_1(\mathrm{d}h) \,=\, 4\,\lambda_1\,\lambda_2\, e^{-2\,\lambda_2\, h}\, \mathrm{d}h \tag{8.30}$$

on the Borel sets of $(0, \infty)$. The total masses of these measures are respectively $\boldsymbol{\nu}_1((0,\infty)) \,=\, 2\,\lambda_1$ and $\boldsymbol{\nu}_2((0,\infty)) \,=\, 2\,\lambda_2$, just as in (6.11) and (6.12). For the two probability density functions of (8.29), the product measure (8.28) satisfies the Basic Adjoint Relationship (8.26), and is thus the invariant probability measure for the two-dimensional process $\big(G(\cdot), H(\cdot)\big)$ of gaps. $\square$

A picture of the paths of the resulting process $(R_1(\cdot), R_2(\cdot), R_3(\cdot))$ with $\delta_1 \,=\, -1,\ \delta_2 \,=\, -2$ and $\delta_3 = -1$ is depicted in Figure 4, reproduced here from FERNHOLZ (2011). Note that the conditions of (8.21) are satisfied in this case.

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