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Positive Intermediate Ricci Curvature with Symmetries

A Dissertation submitted in partial satisfaction
of the requirements for the degree of

Doctor of Philosophy

in

Mathematics

by

Lawrence Gerasin Mouillé

June 2020

Dissertation Committee:

Dr. Frederick Wilhelm, Chairperson
Dr. Reinhard Schultz
Dr. Bun Wong

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The Dissertation of Lawrence Gerasin Mouillé is approved:

Committee Chairperson

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Abstract of the Dissertation

Positive Intermediate Ricci Curvature with Symmetries

by

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Doctor of Philosophy, Graduate Program in Mathematics
University of California, Riverside, June 2020
Dr. Frederick Wilhelm, Chairperson

In this dissertation, we study manifolds that have positive k^{th}-intermediate Ricci curvature, which we abbreviate as $\mathrm{Ric}_k > 0$. This condition interpolates between having positive sectional curvature and having positive Ricci curvature. Specifically, we study this curvature condition in the presence of isometric group actions.

First, we show that if M is a positively curved homogeneous space, then $M \times M$ admits a metric with $\mathrm{Ric}_2 > 0$. We also construct metrics with $\mathrm{Ric}_2 > 0$ on several other products of homogeneous spaces. It follows from these examples that the Hopf Conjectures, Petersen-Wilhelm Conjecture, Berger Fixed Point Theorem, and Hsiang-Kleiner Theorem for positive sectional curvature do not hold in the setting of $\mathrm{Ric}_2 > 0$.

Second, we establish the following: In a manifold with $\mathrm{Ric}_k > 0$, if a submanifold N has a tangential subspace on which the intrinsic k^{th}-intermediate Ricci curvatures are non-positive, then the dimension of that subspace is bounded above by $\mathrm{codim}(N) + k$. As a consequence, we obtain a local symmetry rank bound for manifolds with $\mathrm{Ric}_k > 0$.

Finally, we establish that if there are $k+1$ commuting Killing fields on a compact manifold with $\mathrm{Ric}_k > 0$, then there exists a point at which the Killing fields are linearly dependent. Using this, we establish a cohomogeneity obstruction and a symmetry rank bound for compact manifolds with $\mathrm{Ric}_k > 0$.

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Chapter 1:

Introduction

A famous open problem in Riemannian geometry is to classify manifolds with positive curvature. To address this problem, Grove suggested in the 1990's that researchers attempt to classify positively curved manifolds with "large isometry groups". This initiative, now referred to as the Grove Symmetry Program, has proven to be incredibly fruitful, sparking many ground-breaking results in the last few decades. Because of the success of this program, we seek to study Riemannian manifolds with *positive intermediate Ricci curvature* under the presence of isometric group actions in this work.

Definition. *Suppose (M, g) is a Riemannian manifold and $k \in \{1, \ldots, \dim(M) - 1\}$. We say (M, g) has* ***positive*** $\mathbf{k}^{\text{th}}$***-intermediate Ricci curvature*** *if for all sets of orthonormal vectors $\{u, e_1, \ldots, e_k\}$, we have*

$$\sum_{i=1}^{k} \sec(u, e_i) > 0.$$

We abbreviate this by writing $\mathrm{Ric}_k(M, g) > 0$, omitting M or g when they are understood.

Notice if $k = 1$, this condition is equivalent to having positive sectional curvature, and if $k = \dim(M) - 1$, it is equivalent to positive Ricci curvature. Furthermore, if $\mathrm{Ric}_k(M, g) > 0$

for some k, then $\mathrm{Ric}_\ell(M, g) > 0$ for all $\ell \geq k$. In this respect, positive intermediate Ricci curvature is a natural condition that interpolates between positive sectional curvature and positive Ricci curvature. However, it appears that there have been no attempts to systematically document examples of manifolds with positive intermediate Ricci curvature. For a few elementary examples, see Section 2.1.1.

1.1 Statements of results

In Chapter 3, we address this scarcity of examples by establishing the following:

Theorem A. *If M is a positively curved homogeneous space, then $M \times M$ admits a metric g_ℓ such that* $\mathrm{Ric}_2(M \times M, g_\ell) > 0$.

The metric constructed in Theorem A is a Cheeger deformation of the product metric on $M \times M$ with respect to the diagonal action of any group that acts isometrically and transitively on M. In fact, the result that we prove is more general than Theorem A; see Theorem 3.3 in Section 3.1 for more details. In Section 3.2, we show that the projection to either factor of the product in Theorem 3.3 is a Riemannian submersion. In Section 3.3, we demonstrate how these results indicate that the class of manifolds with $\mathrm{Ric}_2 > 0$ is vastly different from the class of manifolds with positive sectional curvature. Specifically, we show the Euler characteristics, free group actions, fundamental groups, and Riemannian submersions that may occur within these classes are surprisingly different.

In Chapter 4, we present a restriction on the intrinsic curvatures of submanifolds:

Theorem B. *Let M be a Riemannian manifold, let $k \in \{1, \dots, \dim(M)-1\}$, and let $N \subset M$ be a submanifold through a point p. Suppose $\mathcal{S}$ is a subspace of T_pN on which all intrinsic k^{th}-intermediate Ricci curvatures are non-positive while the extrinsic k^{th}-intermediate Ricci curvatures are positive. Then*

$$\dim(\mathcal{S}) \leq \dim(M) - \dim(N) + k.$$

The proof of Theorem B, which can be found in Section 4.1, is local in nature and relies only on the Gauss equation.

Now recall that the *symmetry rank* of a Riemannian manifold (M, g), denoted $\operatorname{symrank}(M, g)$, is the rank of its isometry group. In other words, $\operatorname{symrank}(M, g)$ is the maximal dimension of a torus that can act isometrically and effectively on (M, g). The argument for Theorem B is inspired by Wilking's argument for his symmetry rank bound for manifolds with quasi-positive curvature [49], which can be found in [12]. Related to this, we have the following consequence of Theorem B:

Corollary C. *Let M be a Riemannian manifold, and let $k \in \{1, \dots, \dim(M)-2\}$. Suppose $N \subset M$ is a submanifold through a point p such that T_pN is spanned by commuting Killing fields of N. If all extrinsic k^{th}-intermediate Ricci curvatures are positive on T_pN, then*

$$\dim(N) \leq \left\lfloor \frac{\dim(M) + k}{2} \right\rfloor.$$

We call this a "local symmetry rank" bound because when a torus acts isometrically on a manifold, the orbits are spanned by commuting Killing fields. We will say that a manifold M has *k-maximal local symmetry rank* at a point p if it contains a submanifold N through p that satisfies the hypotheses of Corollary C and has $\dim(N) = \lfloor \frac{\dim(M)+k}{2} \rfloor$. Not only is it possible to construct examples of manifolds with k-maximal local symmetry rank, but we establish the following:

Theorem D. *Let M be a smooth manifold of dimension ≥ 3, let $k \in \{1, \ldots, \dim(M) - 2\}$, and choose $p \in M$. Every Riemannian metric g on M is arbitrarily close in the C^1-topology to a metric $\tilde{g}$ such that $(M, \tilde{g})$ has k-maximal local symmetry rank at p.*

The proof of Theorem D can be found in Section 4.2.1. The idea is to construct metrics on $\mathbb{R}^n$ that have k-maximal local symmetry rank at the origin, and then glue a small neighborhood of the origin into the original manifold while only slightly affecting the original metric in the C^1-topology.

Now applying Corollary C to the orbits of a torus action, we obtain the following global symmetry rank bound:

Corollary E. *Suppose (M, g) is a connected Riemannian n-manifold. If M contains a point at which all k^{th}-intermediate Ricci curvatures are positive for some $k \in \{1, \ldots, n-2\}$, then*

$$\operatorname{symrank}(M, g) \leq \left\lfloor \frac{n+k}{2} \right\rfloor.$$

We prove Corollary E and provide context for it in Section 4.3. In Section 4.3.1, we also highlight ramifications of this result in the setting on non-negative sectional curvature, specifically related to the Maximal Symmetry Rank Conjecture.

In Chapter 5, we establish the following symmetry obstruction:

Theorem F. *Suppose (M, g) is a closed Riemannian manifold with $\mathrm{Ric}_k(M, g) > 0$ for some $k \in \{1, \ldots, \dim(M) - 1\}$. If there are $k+1$ commuting Killing fields on M, then they must be linearly dependent at some point in M.*

In particular, Theorem F implies that if a torus T^r acts isometrically on M with $r \geq k+1$, then there is a codimension k subtorus $T^{r-k} \subset T^r$ with non-empty fixed point set in M. This result generalizes the Isotropy Rank Lemma from the setting of positive sectional curvature. The proof of Theorem F, found in Section 5.1, is a generalization of the argument for the Berger Fixed Point Theorem in [5].

Given an isometric action by a Lie group G on a manifold M, the *cohomogeneity* of the action, denoted $\mathrm{cohom}(M, G)$, is the dimension of the orbit space M/G. Equivalently, $\mathrm{cohom}(M, G)$ is the codimension of the principal orbits in M. Theorem F provides the following obstruction on the cohomogeneity of isometric group actions:

Corollary G. *Suppose* (M, g) *is a closed Riemannian manifold with* $\mathrm{Ric}_k(M, g) > 0$ *for some* $k \in \{1, \ldots, \dim(M) - 1\}$*. If a Lie group* G *acts isometrically on* M *with principal isotropy subgroup* $H \subset G$*, then*

$$\mathrm{rank}(G) - \mathrm{rank}(H) \leq \mathrm{cohom}(M, G) + k.$$

We prove Corollary G in Section 5.2. The last result that we present is the following symmetry rank bound for compact manifolds:

Theorem H. *Suppose* (M, g) *is a closed, connected,* n*-dimensional Riemannian manifold.*

1. *If* $\mathrm{Ric}_2(M, g) > 0$*, then*
$$\mathrm{symrank}(M, g) \leq \left\lfloor \frac{n+1}{2} \right\rfloor.$$
2. *If* $\mathrm{Ric}_k(M, g) > 0$ *for* $k \in \{3, \ldots, n-1\}$*, then*
$$\mathrm{symrank}(M, g) \leq \left\lfloor \frac{n+k}{2} \right\rfloor - 1.$$

Theorem H generalizes the symmetry rank bound obtained by Grove and Searle in [17]. The argument, presented in Section 5.3, is an adaptation of Grove and Searle's argument. However, we incorporate the fact that manifolds with positive Ricci curvature in dimensions ≥ 4 cannot admit isometric torus actions of cohomogeneity-one; See Lemma 5.14.

Chapter 2:

Background

In this chapter, we provide background information related to positive intermediate Ricci curvature, Cheeger deformations, and positively curved homogeneous spaces.

2.1 Positive intermediate Ricci curvature

Recall from Chapter 1 that

$$\mathrm{Ric}_k(M,g) > 0 \quad \Longleftrightarrow \quad \sum_{i=1}^{k} \sec(u,e_i) > 0 \text{ for all orthonormal } u, e_1, \ldots, e_k.$$

More generally, we can define the intermediate Ricci curvature of a given flag in the tangent bundle of M. To that end, let $\mathrm{Fl}(1,k+1;TM)$ denote the *partial flag bundle* consisting of signature-$(1,k+1)$ flags tangent to M. In other words, elements of $\mathrm{Fl}(1,k+1;TM)$ are pairs of subspaces $(\ell,\mathcal{V})$ in a given tangent space such that $\dim(\ell)=1$, $\dim(\mathcal{V})=k+1$, and $\ell \subset \mathcal{V}$.

Definition 2.1. *Suppose (M,g) is a Riemannian manifold and $k \in \{1,\ldots,\dim(M)-1\}$. Given a flag $(\ell,\mathcal{V}) \in \mathrm{Fl}(1,k+1;TM)$, let $R^{\mathcal{V}}$ denote the type-$(1,3)$ curvature tensor R*

restricted to $\mathcal{V}$ and composed with orthogonal projection to $\mathcal{V}$. Then the $\mathbf{k}^{\mathrm{th}}$***-intermediate Ricci curvature*** *of the flag $(\ell, \mathcal{V})$ is defined to be*

$$\mathrm{Ric}_k(\ell, \mathcal{V}) \coloneqq \mathrm{trace}\left(x \mapsto R^{\mathcal{V}}(x, u)u\right) = \sum_{i=1}^{k} \sec(u, e_i),$$

where u is a unit vector in the line ℓ, and $e_1, \ldots, e_k$ form an orthonormal basis for the orthogonal complement of ℓ in $\mathcal{V}$. When convenient, given $u, e_1, \ldots, e_k$ as above, we may write

$$\mathrm{Ric}_k(u, \mathcal{V}) \coloneqq \mathrm{Ric}_k(\mathrm{span}\{u\}, \mathcal{V}),$$

$$\mathrm{Ric}_k(u; e_1 \ldots, e_k) \coloneqq \mathrm{Ric}_k(\mathrm{span}\{u\}, \mathrm{span}\{u, e_1, \ldots, e_k\}).$$

Notice that $\mathrm{Ric}_1(u, \mathcal{V})$ is the sectional curvature of the 2-plane $\mathcal{V}$, and $\mathrm{Ric}_{n-1}(u, T_pM)$ is the Ricci curvature $\mathrm{Ric}(u, u)$. Also notice that the value of $\mathrm{Ric}_k(\ell, \mathcal{V}) = \mathrm{Ric}_k(u; e_1, \ldots, e_k)$ is independent of the choice of unit vector $u \in \ell \subset \mathcal{V}$ and orthonormal $e_1, \ldots, e_k \in \mathcal{V} \cap \ell^{\perp}$.

For structure results on manifolds with lower bounds on intermediate Ricci curvature, see [7], [18], [19], [20], [21], [24, Theorem 6.1], [39], [40], [46], [48, Remark 2.4], or [52].

2.1.1 Elementary sources of examples

In this section, we present elementary methods for generating examples of manifolds with positive intermediate Ricci curvature. Namely, we discuss positive intermediate Ricci curvature fo compact symmetric spaces, Riemannian products, and Riemannian submersions.

Compact symmetric spaces

Nash proved in [26] that a compact homogeneous space G/H admits a metric with positive Ricci curvature if and only if its fundamental group is finite. In addition, if G/H is a locally symmetric space of rank r, then $\mathrm{Ric}_{r-1}(G/H) \not> 0$. Thus, for any compact locally symmetric space G/H with finite fundamental group, we have $\mathrm{Ric}_k(G/H) > 0$ for some $k \in \{\mathrm{rank}(G/H), \dots, \dim(G/H) - 1\}$.

Product metrics

Next, we have the following for Riemannian products of positively curved manifolds:

Proposition 2.2. *Given Riemannian manifolds (M^m, g_M) and (N^n, g_N), consider their product $M^m \times N^n$ equipped with the product metric g_{prod}. If (M^m, g_M) and (N^n, g_N) are both positively curved, then*

$$\mathrm{Ric}_k(M^m \times N^n, g_{\mathrm{prod}}) > 0 \text{ for } k \geq \max\{m, n\} + 1.$$

Proof. Choose a signature-$(1, k+1)$ flag $(\ell, \mathcal{V}) \in \mathrm{Fl}(1, k+1; T(M \times N))$ for some $k \geq \max\{m, n\} + 1$, and consider the projections $\pi_M : T(M \times N) \to TM$ and $\pi_N : T(M \times N) \to TN$. Notice that either $\pi_M(\ell) \neq \{0\}$ or $\pi_N(\ell) \neq \{0\}$. Without loss of generality, suppose $\pi_M(\ell)$ is non-trivial, and choose a unit vector $(u^M, u^N) \in \ell$.

Because $k \geq \max\{m,n\}+1$, we also have

$$\begin{aligned}\dim(\pi_M(\mathcal{V})) \geq \dim(\mathcal{V}) - \dim(N) &= k+1-n \\ &\geq (n+1)+1-n \\ &= 2.\end{aligned}$$

So we can choose a unit vector $({e_1}^M, {e_1}^N) \in \mathcal{V} \cap \ell^\perp$ such that the vectors ${e_1}^M$ and u^M are linearly independent in $\pi_M(\mathcal{V})$. Because (M, g_M) is positively curved, $\mathrm{curv}_M(u^M, {e_1}^M) > 0$, where $\mathrm{curv}_M(v,w) \coloneqq R_M(v,w,w,v)$. Furthermore, because $\mathrm{curv}_N \geq 0$,

$$\sec_{M\times N}((u^M,u^N),({e_1}^M,{e_1}^N)) = \mathrm{curv}_M(u^M,{e_1}^M) + \mathrm{curv}_N(u^N,{e_1}^N) > 0.$$

Now choose any vectors $\{({e_i}^N, {e_i}^N)\}_{i=2}^k$ that, together with (u^M, u^N) and $({e_1}^M, {e_1}^N)$, form an orthonormal basis of $\mathcal{V}$. Because $(M \times N, g_{\mathrm{prod}})$ is non-negatively curved, we have

$$\begin{aligned}\mathrm{Ric}_k(\ell,\mathcal{V}) &= \sum_{i=1}^k \sec_{M\times N}((u^M,u^N),({e_i}^M,{e_i}^N)) \\ &\geq \sec_{M\times N}((u^M,u^N),({e_1}^M,{e_1}^N)) \\ &> 0.\end{aligned}$$

■

Riemannian submersions

Finally, we have the following consequence of O'Neill's Fundamental Equations of Riemannian Submersions [27]:

Corollary 2.3. *If* $\pi : M \to B$ *is a Riemannian submersion and* $\mathrm{Ric}_k > 0$ *on the horizontal distribution of* π *in* M *for* $1 \le k \le \dim(B) - 1$, *then* $\mathrm{Ric}_k(B) > 0$.

Proof. Choose an orthonormal set $\{u, e_1, \ldots, e_k\}$ on B, and consider their horizontal lifts $\{\hat{u}, \hat{e}_1, \ldots, \hat{e}_k\}$ in M. Then by the O'Neill Horizontal Curvature Equation,

$$\sum_{i=1}^{k} \sec_B(u, e_i) = \sum_{i=1}^{k} \sec_M(\hat{u}, \hat{e}_i) + 3|A_u e_i|^2 > 0.$$

■

Pro and Wilhelm prove in [33] that Riemannian submersions need not preserve positive Ricci curvature. Specifically, they construct examples of Riemannian submersions $(M^n, \hat{g}) \to (S^2, \check{g})$ for all dimensions $n \ge 4$ such that $\mathrm{Ric}(M^n, \hat{g}) > 0$ while $(S^2, \check{g})$ has planes of negative curvature. In other words, the base of the submersion does not have positive Ricci curvature. Because $\mathrm{Ric}_k > 0$ implies that $\mathrm{Ric} > 0$, their result shows that some restriction on $\dim(B)$ is necessary for a Riemannian submersion $M \to B$ to preserve $\mathrm{Ric}_k > 0$ in general.

2.2 Cheeger deformations

We now review Cheeger deformations, which were introduced by Cheeger in [8]. We will follow many of the notational conventions used in [36], but we adapt them slightly so that we can use left-invariant metrics instead of bi-invariant metrics.

Consider a Riemannian manifold (M, g) on which a compact Lie group G acts isometrically. Let $K_M : \mathfrak{g} \to \Gamma(TM)$ denote the action field map; i.e. $K_M(x)$ is the Killing field on M generated by $x \in \mathfrak{g}$ via the G-action. Given $p \in M$, define the linear map $K_{M,p} : \mathfrak{g} \to T_pM$ such that $K_{M,p}(x)$ for $x \in \mathfrak{g}$ is the vector field $K_M(x)$ evaluated at p.

Now fix a left-invariant metric g_{left} on G. Given $\ell > 0$, consider the one-parameter family of metrics $\hat{g}_\ell = \ell^2 g_{\text{left}} + g$ on $G \times M$. Then G acts isometrically and freely on $(G \times M, \hat{g}_\ell)$:

$$a \cdot (b, p) = (ab, a \cdot p), \qquad \text{for all } a, b \in G \text{ and } p \in M.$$

The orbit space of this action on $G \times M$ is diffeomorphic to M, and the quotient map $q : G \times M \to M$ is given by

$$q(a, p) = a^{-1} \cdot p.$$

Because this action on $G \times M$ is free, the quotient M admits a metric g_ℓ such that the quotient map $q : (G \times M, \hat{g}_\ell) \to (M, g_\ell)$ is a Riemannian submersion. The family of Riemannian manifolds $\{(M, g_\ell)\}_{\ell > 0}$ is called a *Cheeger deformation* of (M, g) with respect to the G-action and the left-invariant metric g_{left}.

Remark 2.4. *Typically, Cheeger deformations are defined using a fixed bi-invariant metric on G. When this is the case, the G-action on (M, g_ℓ) is by isometries. However, if instead a left-invariant metric on G is used, the G-action on (M, g_ℓ) may not be by isometries. See Remark 3.4 for information on how this affects the examples constructed in Theorems 3.1 and 3.2.*

Cheeger reparametrization

To more easily track the behavior of curvatures during Cheeger deformations, we use the bundle isomorphism $\mathcal{C}_\ell : TM \to TM$ called the *Cheeger reparametrization*. To define it, consider a vector $v \in T_pM$. Let $\hat{v}_\ell \in TG \times TM$ be the vector that is horizontal with respect to the Riemannian submersion $q : (G \times M, \hat{g}_\ell) \to (M, g_\ell)$ such that $\hat{v}_\ell$ projects to $v \in T_pM$ under the differential of the projection to the second factor $\pi_2 : G \times M \to M$. Then $\mathcal{C}_\ell : T_pM \to T_pM$ is defined by

$$\mathcal{C}_\ell(v) = dq(\hat{v}_\ell).$$

One useful aspect of the Cheeger reparametrization is that one can use it to relate the Cheeger-deformed metric g_ℓ to the original metric g according to the following:

Lemma 2.5 (Proposition 6.3 in [31]). *Let g_ℓ denote a Cheeger deformation of a metric g on a manifold M. For all points $p \in M$ and vectors $u, v \in T_pM$,*

$$g_\ell(\mathcal{C}_\ell(u), v) = g(u, v).$$

In particular, given a distribution $\mathcal{D}$ on M, if $\mathcal{D}_g^\perp$ denotes the distribution orthogonal to $\mathcal{D}$ with respect to the original metric g, then the distribution orthogonal to $\mathcal{D}$ with respect to the Cheeger-deformed metric g_ℓ is given by

$$\mathcal{D}_{g_\ell}^\perp = \{\mathcal{C}_\ell(x) : x \in \mathcal{D}_g^\perp\}.$$

Because every G-orbit in $G \times M$ has a unique point of the form (e, p), when we consider vectors tangent to $G \times M$, we assume that the footpoint is of this form. Notice that the kernel of $dq_{(e,p)} : T_{(e,p)}(G \times M) \to T_pM$ is given by

$$\mathcal{V}_{(e,p)} = \{(z, K_{M,p}(z)) : z \in \mathfrak{g}\} \,.$$

When $\ell = 1$ and $v \in T_pM$, denote the G-factor of $\hat{v}_1 \in T_{(e,p)}(G \times M)$ by $\kappa_p(v)$. In other words, $\kappa_p(v)$ is defined so that $\hat{v}_1 = (\kappa_p(v), v)$. Because $\hat{v}_1$ is required to be perpendicular to $\mathcal{V}_{(e,p)}$ with respect to the metric $\hat{g}_1 = g_{\text{left}} + g$, we have that $\kappa_p(v)$ must satisfy the equation

$$g_{\text{left}}(\kappa_p(v), z) = -g(v, K_{M,p}(z)), \tag{2.1}$$

for all $v \in T_pM$ and $z \in \mathfrak{g}$. For any $\ell > 0$, because $\hat{v}_\ell$ must be perpendicular to $\mathcal{V}_{(e,p)}$ with respect to $\hat{g}_\ell = \ell^2 g_{\text{left}} + g$, it then follows from Equation 2.1 that

$$\hat{v}_\ell = \left(\tfrac{1}{\ell^2}\kappa_p(v), v\right) .$$

Then by the definition of $\mathcal{C}_\ell$ and q, for all $v \in T_pM$, we have

$$\mathcal{C}_\ell(v) = dq(\hat{v}_\ell) = -\tfrac{1}{\ell^2}K_{M,p}(\kappa_p(v)) + v. \tag{2.2}$$

Because we will primarily work with homogeneous spaces, we may omit the point p in the notation above when the dependence on p is insignificant for a given argument. The following will be useful for working with the maps K_M and κ:

Lemma 2.6 (Proposition 2.1 in [36]). *Fix $p \in M$, and let $G_p \leq G$ denote the isotropy subgroup at p. Consider the associated Lie subalgebra $\mathfrak{g}_p \subseteq \mathfrak{g}$, and let $\mathfrak{g}_p^\perp$ be the orthogonal complement of $\mathfrak{g}_p$ with respect to the left-invariant metric g_{left} on G. Also let $G \cdot p$ denote the G-orbit containing p in M. Then:*

1. *$K_{M,p} : \mathfrak{g} \to T_pM$ takes values in $T_p(G \cdot p)$, and restricting $K_{M,p}$ to $\mathfrak{g}_p^\perp$ gives a linear isomorphism $\mathfrak{g}_p^\perp \to T_p(G \cdot p)$.*

2. *$\kappa_p : T_pM \to \mathfrak{g}$ takes values in $\mathfrak{g}_p^\perp$, and restricting κ_p to $T_p(G \cdot p)$ gives a linear isomorphism $T_p(G \cdot p) \to \mathfrak{g}_p^\perp$.*

Generic plane principle

Using the Cheeger reparametrization, Petersen and Wilhelm established the Generic Plane Principle, which serves as a means for tracking positively curved planes during Cheeger deformations; see Propositions 6.1 and 6.2 in [31]. Because the Cheeger deformations they consider depend on bi-invariant metrics on the group G, we adapt the Generic Plane Principle to allow for Cheeger deformations dependent upon left-invariant metrics on G. First, let curv_g denote the un-normalized sectional curvature with respect to a metric g. In other words, if R_g denotes the Riemann curvature tensor, then $\mathrm{curv}_g(x, y) = R_g(x, y, y, x)$.

Lemma 2.7 (Generic Plane Principle)**.** *Let (M, g_ℓ) be a Cheeger deformation of a non-negatively curved manifold (M, g) with respect to a G-action on M and a left-invariant metric g_{left} on G. If $\mathrm{curv}_{g_{\text{left}}}(\kappa(\mathcal{P})) \geq 0$ for all planes $\mathcal{P}$ tangent to M, then we have the following:*

1. *(M, g_ℓ) has non-negative sectional curvature.*
2. *If a plane $\mathcal{P}$ is positively curved with respect to g, then $\mathcal{C}_\ell(\mathcal{P})$ is positively curved with respect to g_ℓ for all $\ell > 0$.*
3. *Suppose $\mathrm{curv}_{g_{\text{left}}}(\kappa(u), \kappa(v)) > 0$ for some $u, v \in T_pM$. If $\mathcal{P} = \mathrm{span}\{u, v\}$, then $\mathcal{C}_\ell(\mathcal{P})$ is positively curved with respect to g_ℓ for all $\ell > 0$.*

Proof. Consider the Riemannian submersion $q : (G \times M, \hat{g}_\ell) \to (M, g_\ell)$ which defines the Cheeger deformed metric g_ℓ, where $\hat{g}_\ell = \ell^2 g_{\text{left}} + g$. Recall that for $v \in T_pM$,

$$\mathcal{C}_\ell(v) = Dq(\hat{v}_\ell) = Dq\left(\tfrac{1}{\ell^2}\kappa(v), v\right).$$

So given $u, v \in T_pM$, O'Neill's Horizontal Curvature Equation [27] implies

$$\begin{aligned}
\operatorname{curv}_{g_\ell}(\mathcal{C}_\ell(u), \mathcal{C}_\ell(v)) &\geq \operatorname{curv}_{\hat{g}_\ell}\left(\left(\tfrac{1}{\ell^2}\kappa(u), u\right), \left(\tfrac{1}{\ell^2}\kappa(v), v\right)\right) \\
&= \operatorname{curv}_{\ell^2 g_{\text{left}}}\left(\tfrac{1}{\ell^2}\kappa(u), \tfrac{1}{\ell^2}\kappa(v)\right) + \operatorname{curv}_g(u, v) \\
&= \tfrac{1}{\ell^6}\operatorname{curv}_{g_{\text{left}}}(\kappa(u), \kappa(v)) + \operatorname{curv}_g(u, v).
\end{aligned}$$

Because $\sec_g \geq 0$ and $\operatorname{curv}_{g_{\text{left}}}(\kappa(\mathcal{P})) \geq 0$ for all planes $\mathcal{P}$ tangent to M, it follows that $\operatorname{curv}_{g_\ell} \geq 0$. In addition, if either summand above is positive, then $\operatorname{curv}_{g_\ell}(\mathcal{C}_\ell(u), \mathcal{C}_\ell(v)) > 0$. Thus, the result follows. ■

2.3 Positively curved homogeneous spaces

We now review general facts about positively curved homogeneous spaces. The classification of compact, simply connected, positively curved homogeneous spaces was carried out by Berger [4], Wallach [45], Aloff-Wallach [1], and Bérard Bergery [3], with an omission in [4] that was corrected by Wilking in [47]. See Tables 2.1 and 2.2 for a complete list of these homogeneous spaces. For an overview of the classification, see [50].

$\mathbf{G}$	$\mathbf{H}$	$\mathbf{G/H}$
$\mathsf{SO}(n+1)$	$\mathsf{SO}(n)$	S^n
$\mathsf{SU}(n+1)$	$\mathsf{U}(n)$	$\mathbb{C}\mathrm{P}^n$
$\mathsf{Sp}(n+1)$	$\mathsf{Sp}(n)\times\mathsf{Sp}(1)$	$\mathbb{H}\mathrm{P}^n$
F_4	$\mathsf{Spin}(9)$	$\mathbb{O}\mathrm{P}^2$
$\mathsf{Sp}(2)$	$\mathsf{Sp}(1)_{\max}$	B^7
$\mathsf{SU}(5)$	$\mathsf{Sp}(2)\times S^1$	B^{13}
$\mathsf{SU}(3)\times\mathsf{SO}(3)$	$\mathsf{U}(2)$	$W^7_{1,1}$
$\mathsf{SU}(n+1)$	$\mathsf{SU}(n)$	S^{2n+1}
$\mathsf{Sp}(n+1)$	$\mathsf{Sp}(n)$	S^{4n+3}
$\mathsf{Sp}(n+1)$	$\mathsf{Sp}(n)\times\mathsf{U}(1)$	$\mathbb{C}\mathrm{P}^{2n+1}$
$\mathsf{Spin}(9)$	$\mathsf{Spin}(7)$	S^{15}

Table 2.1: Simply connected normal homogeneous spaces G/H with positive sectional curvature.

$\mathbf{G}$	$\mathbf{K}$	$\mathbf{H}$	$\mathbf{G/H}$
$\mathsf{SU}(3)$	$\mathsf{U}(2)$	T^2	W^6
$\mathsf{Sp}(3)$	$\mathsf{Sp}(2)\times\mathsf{Sp}(1)$	$\mathsf{Sp}(1)^3$	W^{12}
F_4	$\mathsf{Spin}(9)$	$\mathsf{Spin}(8)$	W^{24}
$\mathsf{SU}(3)$	T^2	$S^1_{p,q}$	$W^7_{p,q}$

Table 2.2: Simply connected, positively curved homogeneous spaces G/H that are not normal, and the subgroups $K < G$ used to obtain the metrics of positive curvature on G/H.

All simply connected, positively curved homogeneous spaces admit a homogeneous metric of the following form:

Consider closed subgroups $H \subseteq K \subseteq G$ with corresponding Lie algebras $\mathfrak{h} \subseteq \mathfrak{k} \subseteq \mathfrak{g}$, and fix a bi-invariant metric g_{bi} on G. First, we Cheeger deform (G, g_{bi}) with respect to the action of K by right multiplication and the bi-invariant metric $g_{\mathrm{bi}}|_K$. Thus, we obtain a new metric on $(g_{\mathrm{bi}})_\ell$ on G for which K acts isometrically by right multiplication. The metric $(g_{\mathrm{bi}})_\ell$ is in fact left-invariant, so we will denote it by g_{left}. Recall from Section 2.2 that we have the

Riemannian submersion

$$q : (K \times G, (\hat{g}_{\mathrm{bi}})_\ell) \to (G, g_{\mathrm{left}}).$$

Here, $(\hat{g}_{\mathrm{bi}})_\ell = \ell^2 g_{\mathrm{bi}}|_K + g_{\mathrm{bi}}$. Now, the quotient for the action of $H \subseteq K$ on G by right multiplication induces a homogeneous metric g_{hom} on G/H via the projection

$$\pi : (G, g_{\mathrm{left}}) \to (G/H, g_{\mathrm{hom}}).$$

Composing these quotient maps, we have that $(G/H, g_{\mathrm{hom}})$ is the base of a Riemannian submersion from a Lie group with a bi-invariant metric:

$$\pi \circ q : (K \times G, (\hat{g}_{\mathrm{bi}})_\ell) \to (G/H, g_{\mathrm{hom}}).$$

Let $\mathfrak{h}^\perp \subseteq \mathfrak{g}$ denote the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to g_{left}. Then $\mathfrak{h}^\perp$ is the horizontal distribution for π. Let $\mathfrak{p} \subseteq \mathfrak{k}$ denote the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{k}$ with respect to g_{left}. Then the horizontal distribution of $\pi \circ q$ is given by

$$\mathcal{H}_{\pi\circ q} = \{(0, x) : x \in \mathfrak{k}^\perp\} \oplus \left\{\left(-\tfrac{1}{\ell^2} y, y\right) : y \in \mathfrak{p}\right\}.$$

Remark 2.8. *All of the homogeneous spaces G/H in Tables 2.1 and 2.2 admit positively curved homogeneous metrics g_{hom} that can be described as above. Notice that if $(G/H, g_{\mathrm{hom}})$ is normal homogeneous, then in the construction outline above, K can be taken to be G, and the left-invariant metric g_{left} on G is in fact a rescaling of the original bi-invariant metric g_{bi}.*

To prove Theorem 3.1, we will use the following:

Lemma 2.9 (Tapp [44])**.** *If $\pi : (G, g_{\mathrm{bi}}) \to (M, g)$ is a Riemannian submersion, then every horizontal zero-curvature plane in G projects to a zero-curvature plane in M.*

Applying Lemma 2.9 to the homogeneous spaces constructed above, we can summarize the discussion from this section as follows:

Corollary 2.10. *Suppose $(G/H, g_{\text{hom}})$ is a homogeneous space with positive sectional curvature. Then with respect to the associated left-invariant metric g_{left} on G, $\sec_{g_{\text{left}}}(\mathcal{P}) > 0$ for all planes $\mathcal{P} \subseteq \mathfrak{h}^{\perp}$.*

Proof. If G/H is a homogeneous space which admits a positively curved metric, then it admits a homogeneous metric g_{hom} as described above. By the contrapositive of Lemma 2.9, every horizontal plane with respect to the Riemannian submersion $\pi \circ q : (K \times G, (\hat{g}_{\text{bi}})_{\ell}) \to (G/H, g_{\text{hom}})$ is positively curved. Because dq maps $\mathcal{H}_{\pi \circ q}$ onto $\mathfrak{h}^{\perp}$, we have that all planes in $\mathfrak{h}^{\perp}$ are positively curved with respect to g_{left} by O'Neill's Horizontal Curvature Equation [27]. ■

Chapter 3:

New examples of positive intermediate Ricci curvature

In this chapter, we use Cheeger deformations to establish that some products of positive curved homogeneous spaces admit metrics with $\mathrm{Ric}_2 > 0$. For example, we establish Theorem A, which we restate here for convenience:

Theorem 3.1. *If M is a positively curved homogeneous space, then $M \times M$ admits a metric g_ℓ such that $\mathrm{Ric}_2(M \times M, g_\ell) > 0$.*

For a complete list of simply connected, positively curved homogeneous spaces, see Tables 2.1 and 2.2 in Section 2.3. Given a positively curved homogeneous metric g_{hom} on M, the metric in Theorem 3.1 is a Cheeger deformation of the product metric on $M \times M$ under the diagonal action by any group that acts isometrically and transitively on (M, g_{hom}). The Cheeger deformation of $S^2 \times S^2$ with respect to the diagonal action of $\mathsf{SO}(3)$ was considered by Müter [25]. Bettiol deformed this metric on $S^2 \times S^2$ further to construct a metric of positive biorthogonal curvature on $S^2 \times S^2$ [6].

Expanding upon Theorem 3.1, we also establish the following:

Theorem 3.2. *Given compact Lie subgroups $H \leq K \leq G$, if G/H admits a homogeneous metric of positive curvature, then $G/K \times G/H$ admits a metric with* $\mathrm{Ric}_2 > 0$.

If $H = K$ in Theorem 3.2, then the statement is equivalent to Theorem 3.1. Table 3.1 below shows some examples of product manifolds $G/K \times G/H$ with $H \neq K$ that admit $\mathrm{Ric}_2 > 0$ by Theorem 3.2.

G	**K**	**H**	**G/K × G/H**
$\mathsf{SU}(n+1)$	$\mathsf{U}(n)$	$\mathsf{SU}(n)$	$\mathbb{C}\mathrm{P}^n \times S^{2n+1}$
$\mathsf{Sp}(n+1)$	$\mathsf{Sp}(n) \times \mathsf{Sp}(1)$	$\mathsf{Sp}(n)$	$\mathbb{H}\mathrm{P}^n \times S^{4n+3}$
$\mathsf{Sp}(n+1)$	$\mathsf{Sp}(n) \times \mathsf{Sp}(1)$	$\mathsf{Sp}(n) \times \mathsf{U}(1)$	$\mathbb{H}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^{2n+1}$
$\mathsf{SU}(3)$	$\mathsf{U}(2)$	$S^1_{p,q}$	$\mathbb{C}\mathrm{P}^2 \times W^7_{p,q}$
$\mathsf{SU}(3)$	$\mathsf{U}(2)$	T^2	$\mathbb{C}\mathrm{P}^2 \times W^6$
$\mathsf{SU}(3)$	T^2	$S^1_{p,q}$	$W^6 \times W^7_{p,q}$
$\mathsf{Sp}(3)$	$\mathsf{Sp}(2) \times \mathsf{Sp}(1)$	$\mathsf{Sp}(1)^3$	$\mathbb{H}\mathrm{P}^2 \times W^{12}$
F_4	$\mathsf{Spin}(9)$	$\mathsf{Spin}(8)$	$\mathbb{O}\mathrm{P}^2 \times W^{24}$

Table 3.1: Products of simply connected homogeneous spaces $G/K \times G/H$ that admit metrics with $\mathrm{Ric}_2 > 0$ by Theorem 3.2.

In fact, Theorem 3.2 holds if we only assume that the identity components H_0 and K_0 of the Lie groups H and K satisfy $H_0 \leq K_0$. We incorporate this observation in Theorem 3.3, which we prove in Section 3.1 below. See Remark 3.5 for a description of which planes *may not* be positively curved under the metrics in Theorems 3.1 and 3.2.

Suppose $M \times N$ is a product manifold that admits a metric g_ℓ with $\mathrm{Ric}_2 > 0$ by Theorems 3.1, 3.1, or 3.3. In Section 3.2, we prove the projections to the factors $(M \times N, g_\ell) \to M$ and $(M \times N, g_\ell) \to N$ are Riemannian submersions; see Theorem 3.8.

In Section 3.3, we explore several consequences of Theorems 3.1, 3.1, and 3.3. Namely, we highlight several free isometric actions on our new examples in Corollary 3.12, we demonstrate groups that can be realized as fundamental groups of manifolds with $\mathrm{Ric}_2 > 0$ in Corollary 3.13, and we exhibit quotients by free diagonal actions that admit $\mathrm{Ric}_2 > 0$ in Corollary 3.15. Along the way, we contrast the results of this chapter with structure results and famous conjectures from the setting of positively curved manifolds.

3.1 $\mathrm{Ric}_2 > 0$ ON PRODUCTS OF HOMOGENEOUS SPACES

In this section, we prove the following generalization of Theorems 3.1 and 3.2:

Theorem 3.3. *Suppose H and K are closed subgroups of a compact Lie group G such that their identity components satisfy $H_0 \leq K_0$. Suppose further that $M = G/K$ and $N = G/H$ both admit positively curved homogeneous metrics g_M and g_N induced by a fixed left-invariant metric g_{left} on G. Let g_{prod} denote the product metric on $M \times N$, and consider the Cheeger deformation, $(M \times N, g_\ell)$, of $(M \times N, g_{\mathrm{prod}})$ with respect to the diagonal G-action and the left-invariant metric g_{left}. Then*

$$\mathrm{Ric}_2(M \times N, g_\ell) > 0 \textit{ for all } \ell > 0.$$

Remark 3.4. *When M and N are normal homogeneous (Table 2.1), then the left-invariant metric g_{left} in Theorem 3.3 is in fact bi-invariant. It then follows from Remark 2.4 that the diagonal G-action on $M \times N$ is by isometries of the Cheeger-deformed metric g_ℓ. Otherwise, the diagonal G-action on $M \times N$ may not be by isometries of g_ℓ.*

Remark 3.5. *The Riemannian manifolds* $(M \times N, g_\ell)$ *from Theorem 3.3 are non-negatively curved. If a plane* $\mathcal{P}$ *has curvature zero with respect to* g_ℓ*, then it can be written as*

$$\mathcal{P} = \mathrm{span}\{(K_M(x), 0), (0, K_N(x))\}$$

for some $x \in \mathfrak{k}^\perp \subseteq \mathfrak{g}$*. Notice that this is a necessary condition, but it may not be a sufficient one. In particular, the collection of planes that may have curvature zero with respect to* g_ℓ *can be parametrized by the unit sphere in* $\mathfrak{k}^\perp$*. Because* $M = G/K$*, this sphere has dimension* $\dim(M) - 1$.

Let curv_g denote the un-normalized sectional curvature with respect to a metric g. In other words, if R_g denotes the type-$(0,4)$ Riemann curvature tensor associated with g, then $\mathrm{curv}_g(x, y) = R_g(x, y, y, x)$. First, we start by establishing which planes have curvature zero in $(M \times N, g_{\mathrm{prod}})$:

Lemma 3.6. *Suppose* (M, g_M) *and* (N, g_N) *are positively curved manifolds. A plane* $\mathcal{P}$ *tangent to* $M \times N$ *has curvature zero with respect to the product metric* g_{prod} *if and only if it can be written as* $\mathcal{P} = \mathrm{span}\{(u, 0), (0, v)\}$ *for some* $u \in TM$ *and* $v \in TN$*.*

Proof. Choose vectors $(u_1, v_1), (u_2, v_2) \in T(M \times N)$ that span a plane $\mathcal{P}$. Letting $g_{\mathrm{prod}} = g_M + g_N$ denote the product metric on $M \times N$, notice that

$$\begin{aligned}\mathrm{curv}_{g_{\mathrm{prod}}}(\mathcal{P}) &= \mathrm{curv}_{g_{\mathrm{prod}}}\big((u_1, v_1), (u_2, v_2)\big)\big) \\ &= \mathrm{curv}_{g_M}(u_1, u_2) + \mathrm{curv}_{g_N}(v_1, v_2).\end{aligned}$$

Because $\sec_{g_M} > 0$ and $\sec_{g_N} > 0$, the expression above is zero if and only if the respective sets $\{u_1, u_2\}$ and $\{v_1, v_2\}$ are linearly dependent. This implies that $\mathcal{P}$ can be written as $\mathcal{P} = \text{span}\{(u, 0), (0, v)\}$ for some non-zero vectors u, v. ■

Now let H, K, and G be as in Theorem 3.3, and let $\mathfrak{h}$, $\mathfrak{k}$, and $\mathfrak{g}$ denote the associated Lie algebras. Since $H_0 \leq K_0$, we have that $\mathfrak{h} \subseteq \mathfrak{k}$. Let g_{left} be the left-invariant metric on G corresponding to the positively curved homogeneous metrics g_M on $M = G/K$ and g_N on $N = G/H$. Let $\kappa : T(M \times N) \to \mathfrak{g}$ be the map defined in Section 2.2 associated with the Cheeger deformation of $(M \times N, g_{\text{prod}})$ with respect to the diagonal G-action and the left-invariant metric g_{left} on G. Now, we establish which planes in $M \times N$ "correspond" to zero-curvature planes in (G, g_{left}):

Lemma 3.7. *Choose vectors $(K_M(x), 0), (0, K_N(y)) \in T(M \times N)$ for some $x \in \mathfrak{k}^\perp$ and $y \in \mathfrak{h}^\perp$. Then $\text{curv}_{g_{\text{left}}}(\kappa(K_M(x), 0), \kappa(0, K_N(y))) = 0$ if and only if x and y are linearly dependent in $\mathfrak{k}^\perp$.*

Proof. First, notice it follows from Equation 2.1 that in this setting, $\kappa : T(M \times N) \to \mathfrak{g}$ must satisfy the equation

$$g_{\text{left}}(\kappa(u, v), z) = -g_{\text{prod}}\left((u, v), K_{M \times N}(z)\right)$$

for all $(u, v) \in T(M \times N)$ and $z \in \mathfrak{g}$. So given $z \in \mathfrak{g}$,

$$\begin{aligned} g_{\text{left}}\big(\kappa(K_M(x), 0), z\big) &= -g_{\text{prod}}\big((K_M(x), 0), K_{M \times N}(z)\big) \\ &= -g_{\text{prod}}\big((K_M(x), 0), (K_M(z), K_N(z))\big) \\ &= -g_M(K_M(x), K_M(z)). \end{aligned}$$

Thus, because $M = G/K$ and $x \in \mathfrak{k}^\perp$, it follows that $\kappa(K_M(x), 0) \in \mathfrak{k}^\perp$. Similarly,

$$\begin{aligned} g_{\text{left}}\big(\kappa(0, K_N(y)), z\big) &= -g_{\text{prod}}\big((0, K_N(y)), K_{M \times N}(z))\big) \\ &= -g_{\text{prod}}\big((0, K_N(y)), (K_M(z), K_N(z))\big) \\ &= -g_N(K_N(y), K_N(z)). \end{aligned}$$

Hence, because $N = G/H$ and $y \in \mathfrak{h}^\perp$, it follows that $\kappa(0, K_N(y)) \in \mathfrak{h}^\perp$. In particular, because $\mathfrak{k}^\perp \subseteq \mathfrak{h}^\perp$,

$$\text{span}\{\kappa(K_M(x), 0), \kappa(0, K_N(y))\} \subseteq \mathfrak{h}^\perp.$$

By Corollary 2.10, $\sec_{g_{\text{left}}}(\mathcal{P}) > 0$ for all planes $\mathcal{P} \subseteq \mathfrak{h}^\perp$. Therefore, we have that

$$\text{curv}_{g_{\text{left}}}(\kappa(K_M(x), 0), \kappa(0, K_N(y))) = 0$$

$$\Updownarrow$$

$$\kappa(K_M(x), 0) \text{ and } \kappa(0, K_N(y)) \text{ are linearly dependent},$$

which implies that both $\kappa(K_M(x), 0)$ and $\kappa(0, K_N(y))$ lie in $\mathfrak{k}^\perp \subseteq \mathfrak{h}^\perp$. Now notice for all $x \in \mathfrak{k}^\perp$ and $z \in \mathfrak{h}^\perp$,

$$\begin{aligned} g_{\text{left}}\big(\kappa(K_M(x), 0), z\big) &= -g_M(K_M(x), K_M(z)) \\ &= -g_{\text{left}}(x, z). \end{aligned}$$

Also, for all $y, z \in \mathfrak{h}^\perp$,

$$g_{\mathrm{left}}\big(\kappa(0, K_N(y)), z\big) = -g_N(K_N(y), K_N(z))$$
$$= -g_{\mathrm{left}}(y, z).$$

Thus, it follows that $\kappa(K_M(x), 0)$ and $\kappa(0, K_N(y))$ are linearly dependent in $\mathfrak{k}^\perp$ if and only if x and y are linearly dependent in $\mathfrak{k}^\perp$. Therefore, the result follows. ∎

Finally, we use Lemma 2.7 to prove Theorem 3.3, and hence Theorems 3.1 and 3.2:

Proof of Theorem 3.3. Let $(M \times N, g_\ell)$ denote the Cheeger deformation of $(M \times N, g_{\mathrm{prod}})$ with respect to the diagonal G-action and the left-invariant metric g_{left}. Notice that $(M \times N, g_{\mathrm{prod}})$ is non-negatively curved, and recall from Corollary 2.10 that $\sec_{g_{\mathrm{left}}} > 0$ for all planes in $\mathfrak{h}^\perp$. Then by Lemma 2.7, $\sec_{g_\ell} \geq 0$, and if $\sec_{g_\ell}(\mathcal{C}_\ell(\mathcal{P})) = 0$ for a plane $\mathcal{P}$ tangent to $M \times N$, then $\sec_{g_{\mathrm{prod}}}(\mathcal{P}) = 0$ and $\mathrm{curv}_{g_{\mathrm{left}}}(\kappa(\mathcal{P})) = 0$. By Lemmas 3.6 and 3.7, these conditions imply that

$$\mathcal{P} = \mathrm{span}\{(K_M(x), 0), (0, K_N(x))\}$$

for some $x \in \mathfrak{k}^\perp$. In particular, given any unit vector u tangent to $M \times N$, there is at most one unit vector e_1 such that $\sec_{g_\ell}(u, e_1) = 0$. Therefore, because $(M \times N, g_\ell)$ is non-negatively curved, it follows that $\mathrm{Ric}_2(M \times N, g_\ell) > 0$ for all $\ell > 0$. ∎

3.2 Projections to factors are Riemannian submersions

In this section, we prove the following:

Theorem 3.8. *Suppose a closed Lie Group G acts isometrically and transitively on manifolds M and N. Assume that the metrics g_M, g_N are induced by a fixed left-invariant metric g_{left} on G. Let g_ℓ denote the Cheeger deformation of the product metric g_{prod} on $M \times N$ by the diagonal G-action with respect to g_{left}. Then M and N admit metrics with respect to which the projections to the factors $(M \times N, g_\ell) \to M$ and $(M \times N, g_\ell) \to N$ are Riemannian submersions.*

Proof. Without loss of generality, we prove that the projection $\pi : (M \times N, g_\ell) \to M$ is a Riemannian submersion. We will call a vector field on $M \times N$ *projectable* if it is π-related to a vector field on M. Note that the condition of being projectable is metric-independent. Now, given $(p, q) \in M \times N$, the kernel of the differential $d\pi_{(p,q)} : T_pM \times T_qN \to T_pM$ is given by

$$\mathcal{V}_{(p,q)} = \{(0, v) : v \in T_qN\}.$$

To prove Theorem 3.8, we will show that given projectable vector fields that are g_ℓ-orthogonal to the distribution $\mathcal{V}$, their inner product with respect to g_ℓ is constant along the fibers of $\pi : M \times N \to M$.

With respect to the product metric g_{prod}, the distribution orthogonal to $\mathcal{V}$ is given by $\{(x, 0) : x \in TM\}$. So by Lemma 2.5, the distribution orthogonal to $\mathcal{V}$ with respect to the

Cheeger-deformed metric g_ℓ can be written as

$$\mathcal{H} = \operatorname{span}\{\mathcal{C}_\ell(X,0) : X \text{ is a vector field on } M\}.$$

First, we will show that given a vector field $(X,0)$ on $M \times N$, the field $\mathcal{C}_\ell(X,0)$ is also projectable. Given $z \in \mathfrak{g}$, the map $M \times N \to \mathbb{R}$ given by $(p,q) \mapsto g_{\text{prod}}\left((X,0),(K_{M,p}(z),K_{N,q}(z))\right)$ is constant along the fibers of $\pi : M \times N \to M$. Furthermore, by Equation 2.1, the map $\kappa : T(M \times N) \to \mathfrak{g}$ satisfies

$$g_{\text{left}}(\kappa_{(p,q)}(X,0),z) = -g_{\text{prod}}((X,0),(K_{M,p}(z),K_{N,q}(z)))$$

for all $z \in \mathfrak{g}$. Hence, it follows that the map $M \times N \to \mathfrak{g}$ given by $(p,q) \mapsto \kappa_{(p,q)}(X,0)$ is also constant along the fibers of π. By Equation 2.2, $\mathcal{C}_\ell(X,0)$ can be expressed as

$$\mathcal{C}_\ell(X,0)|_{(p,q)} = -\tfrac{1}{\ell^2}\left(K_{M,p}(\kappa_{(p,q)}(X,0)),K_{N,q}(\kappa_{(p,q)}(X,0))\right) + (X|_p,0).$$

Thus, because $(p,q) \mapsto \kappa_{(p,q)}(X,0)$ is constant along the fibers of π, the first summand in the expression above is a projectable field. So because $(X,0)$ is also projectable, we have shown that $\mathcal{C}_\ell(X,0)$ is a sum of projectable fields, and hence is projectable. In particular, the horizontal distribution $\mathcal{H}$ for $\pi : (M \times N, g_\ell) \to M$ is spanned by projectable vector fields.

Now notice that for vector fields X and Y on M,

$$\begin{aligned}
&g_\ell(\mathcal{C}_\ell(X,0),\mathcal{C}_\ell(Y,0))_{(p,q)} \\
&\qquad = \hat{g}_\ell\left(\widehat{(X,0)}_\ell,\widehat{(Y,0)}_\ell\right)_{(e,p,q)} \\
&\qquad = \tfrac{1}{\ell^2}g_{\text{left}}(\kappa_{(p,q)}(X,0),\kappa_{(p,q)}(Y,0)) + g_{\text{prod}}((X,0),(Y,0))_{(p,q)}.
\end{aligned}$$

In particular, for all vector fields X and Y on M, the map $M \times N \to \mathbb{R}$ given by $(p,q) \mapsto g_\ell(\mathcal{C}_\ell(X,0), \mathcal{C}_\ell(Y,0))_{(p,q)}$ is constant along the fibers of $\pi : M \times N \to M$. Thus, we have shown that with respect to the metric g_ℓ, the inner product of horizontal, projectable fields is constant along the fibers of π. Therefore, M admits a metric with respect to which $\pi : (M \times N, g_\ell) \to M$ is a Riemannian submersion. ■

3.3 Context & Consequences

In the context of positive intermediate Ricci curvature, $\mathrm{Ric}_2 > 0$ is a strong condition, second only to positive sectional curvature. Despite the proximity of $\mathrm{Ric}_2 > 0$ and $\sec > 0$ in this hierarchy, we describe below how these conditions have wildly different implications on the topology of the underlying manifolds as a consequence of the results in this chapter. In particular, we show the Euler characteristics, free isometric actions, fundamental groups, and Riemannian submersions that can occur for the class of manifolds with $\mathrm{Ric}_2 > 0$ are vastly different from that of the class of manifolds with $\sec > 0$.

Euler characteristics

Given $n \geq 2$, Theorem 3.1 establishes that $S^n \times S^n$ admits a metric with $\mathrm{Ric}_2 > 0$. This relates to famous conjectures in the setting of positive sectional curvature that are attributed to Hopf:

Conjecture 3.9 (Hopf Conjectures)**.**

1. *$S^2 \times S^2$ cannot admit a metric of strictly positive sectional curvature.*

2. *Any compact, even-dimensional manifold with positive sectional curvature has positive Euler characteristic.*

Theorem 3.1 applied to $S^2 \times S^2$ shows that Hopf Conjecture 1 does not hold if "positive sectional curvature" is replaced with "$\mathrm{Ric}_2 > 0$". Furthermore, because $\chi(S^{2n-1} \times S^{2n-1}) = 0$, Theorem 3.1 also shows the conclusion of Hopf Conjecture 2 does not hold for $\mathrm{Ric}_2 > 0$ in dimensions $\equiv 2 \mod 4$.

Recall also the following theorem proved by Hsiang and Kleiner in [23]:

Theorem 3.10 (Hsiang-Kleiner Theorem [23])**.** *Suppose M is a compact, orientable, 4-dimensional manifold with positive sectional curvature. If M admits a non-trivial Killing field, then $\chi(M) \leq 3$. In particular, M is homeomorphic to either S^4 or $\mathbb{C}\mathrm{P}^2$.*

The metrics on $S^2 \times S^2$ with $\mathrm{Ric}_2 > 0$ from Theorem 3.1 are invariant under the diagonal S^1-action; see Remark 3.4. Thus the action induces a non-trivial Killing field on $S^2 \times S^2$. Therefore, because $\chi(S^2 \times S^2) = 4$, Theorem 3.1 shows that the conclusion of the Hsiang-Kleiner Theorem does not hold for $\mathrm{Ric}_2 > 0$.

Free isometric actions by connected groups

We now highlight free isometric actions by connected groups on some of the Riemannian manifolds constructed in Theorem 3.1.

Consider a positively curved homogeneous space $M = G/H$. If M is normal homogeneous (Table 2.1), then the diagonal action of any subgroup $K \le G$ continues to be by isometries under the metric constructed in Theorem 3.1; see Remark 3.4. In particular, the $\mathrm{Ric}_2 > 0$ metrics on $S^{2n-1} \times S^{2n-1}$ from Theorem 3.1 are invariant under the respective free diagonal S^1-actions. This relates to the following fixed point theorem proved by Berger in [5]:

Theorem 3.11 (Berger Fixed Point Theorem [5]). *If M is a closed, even-dimensional, manifold with positive sectional curvature, then any Killing field on M has a zero.*

Because free isometric S^1-actions induce Killing fields that are nowhere zero, $S^{2n-1} \times S^{2n-1}$ are even-dimensional manifolds that admit $\mathrm{Ric}_2 > 0$ with non-vanishing Killing fields. Thus the conclusion of the Berger Fixed Point Theorem does not hold for $\mathrm{Ric}_2 > 0$ in dimensions $\equiv 2 \mod 4$.

Shankar observed in [38] that the positively curved normal homogeneous Aloff-Wallach space $W^7_{1,1}$ from [1, 47] admits a free isometric $\mathsf{SO}(3)$-action. Thus, the free diagonal $\mathsf{SO}(3)$-action on $W^7_{1,1} \times W^7_{1,1}$ is by isometries of the metric constructed in Theorem 3.1; see Remark 3.4.

In addition, the free component-wise actions of $S^1 \times S^1$ on $S^{2n+1} \times S^{2n+1} \subset \mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$ and $S^3 \times S^3$ on $S^{4n+3} \times S^{4n+3} \subset \mathbb{H}^{n+1} \times \mathbb{H}^{n+1}$ by right multiplication are by isometries of the respective metrics constructed Theorem 3.1. Thus, to summarize, we have the following:

Corollary 3.12.

1. *$W^7_{1,1} \times W^7_{1,1}$ admits a metric with $\mathrm{Ric}_2 > 0$ that is invariant under the free diagonal $\mathsf{SO}(3)$-action.*
2. *$S^{2n+1} \times S^{2n+1}$ for $n \geq 1$ admits a metric with $\mathrm{Ric}_2 > 0$ that is invariant under a free isometric T^2-action.*
3. *$S^{4n+3} \times S^{4n+3}$ for $n \geq 0$ admits a metric with $\mathrm{Ric}_2 > 0$ that is invariant under a free isometric $(S^3 \times S^3)$-action.*

In contrast with Corollary 3.12, manifolds with positive sectional curvature cannot admit free isometric T^2-actions [5, 42]. Hence, such manifolds also cannot admit free isometric $(S^3 \times S^3)$-actions.

Fundamental groups

In this section, we demonstrate fundamental groups that can be achieved by even-dimensional manifolds with $\mathrm{Ric}_2 > 0$. We show that many of these examples consequently cannot admit metrics with positive sectional curvature by the Synge Theorem.

Corollary 3.12 implies that any finite subgroup of $\mathsf{SO}(3)$, T^2, or $S^3 \times S^3$ can be realized as the fundamental group of a closed, even-dimensional manifold with $\mathrm{Ric}_2 > 0$ by considering the corresponding quotients of $W^7_{1,1} \times W^7_{1,1}$ or $S^n \times S^n$. The finite subgroups of $\mathsf{SO}(3)$ are, up to conjugacy,

$$\mathbb{Z}_n \text{ for } n \geq 1, \quad S_4, \quad A_4, \quad A_5, \quad \text{or} \quad D_m \text{ for } m \geq 2;$$

see, for example, [51, Theorem 2.6.5]. Here, S_n denotes the permutation group on n letters, $A_n < S_n$ is the subgroup of even permutations, and D_m is the dihedral group of order $2m$. The finite subgroups of T^2 are

$$\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \text{ for } n_1, n_2 \geq 1.$$

The finite subgroups of $S^3 \times S^3$ include products of any two of the groups

$$\mathbb{Z}_n \text{ for } n \geq 1, \quad 2S_4, \quad 2A_4, \quad 2A_5, \quad \text{or} \quad 2D_m \text{ for } m \geq 2;$$

see, for example, [51, Theorem 2.6.7]. Here, given a subgroup $\Gamma < \mathsf{SO}(3)$, 2Γ denotes the lift of Γ to $S^3 \cong \mathsf{SU}(2)$ through the double cover $\mathsf{SU}(2) \to \mathsf{SO}(3)$.

In addition, let M^7 and N^7 each denote a quotient of $W^7_{1,1}$ by one of S_4, A_4, A_5, $\mathbb{Z}_n$ for $n \geq 1$, or D_m for $m \geq 2$. Then M^7 and N^7 are positively curved homogeneous spaces. Furthermore, their product $M^7 \times N^7$ admits a metric with $\mathrm{Ric}_2 > 0$ by Theorem 3.3, which generalizes Theorem 3.2. In summary, we have the following:

Corollary 3.13. *The product of any two of the following groups can be realized as the fundamental group of a closed, even-dimensional manifold with* $\mathrm{Ric}_2 > 0$*:*

$$\mathbb{Z}_n \text{ for } n \geq 1, \quad S_4, \quad A_4, \quad A_5, \quad \text{or} \quad D_m \text{ for } m \geq 2.$$

Furthermore, the same is true for the product of any two of the following:

$$\mathbb{Z}_n \text{ for } n \geq 1, \quad 2S_4, \quad 2A_4, \quad 2A_5, \quad \text{or} \quad 2D_m \text{ for } m \geq 2.$$

In contrast with Corollary 3.13, we have the following fundamental result established by Synge [43] for positively curved manifolds:

Theorem 3.14 (Synge Theorem [43])**.** *Let M be a compact manifold with positive sectional curvature.*

1. *If M is even-dimensional, then:*
   - $\pi_1(M) \cong 0$ *if M is orientable, and*
   - $\pi_1(M) \cong \mathbb{Z}_2$ *if M is non-orientable.*
2. *If M is odd-dimensional, then M is orientable.*

In particular, for groups Γ listed in Corollary 3.13 that are not isomorphic to the trivial group or $\mathbb{Z}_2$, the associated manifold with fundamental group Γ admits $\mathrm{Ric}_2 > 0$ but cannot admit a metric of positive sectional curvature by Synge's Theorem. It is yet to be determined if there are simply connected manifolds that admit $\mathrm{Ric}_2 > 0$ but do not admit $\sec > 0$.

Wilhelm showed in [46] that the conclusion of the Synge Theorem holds if, instead of positive sectional curvature, one assumes that $\mathrm{Ric}_k(M) \geq k$ and the first systole, i.e. the length of the shortest, closed, non-contractible curve, is strictly greater than $\pi\sqrt{(k-1)/k}$. Therefore, the manifolds referenced in Corollary 3.13 that have $\pi_1 \ncong 0$ or $\mathbb{Z}_2$ must have first systole $\leq \pi/\sqrt{2}$ when scaled so that $\mathrm{Ric}_2 \geq 2$.

Riemannian submersions

We now describe odd-dimensional quotients of previous examples which admit $\mathrm{Ric}_2 > 0$, and we relate Theorem 3.8 to the Petersen-Wilhelm fiber dimension conjecture for positively curved manifolds.

Recall from Corollary 2.3 that if $\mathrm{Ric}_k(M) > 0$, $M \to B$ is a Riemannian submersion, and $\dim(B) \geq k+1$, then $\mathrm{Ric}_k(B) > 0$. By Corollaries 3.12 and 2.3, given $n \geq 1$, the quotient of $S^{2n+1} \times S^{2n+1}$ by the free diagonal S^1-action and the quotient of $S^{4n-1} \times S^{4n-1}$ by the free diagonal S^3 action each admit metrics with $\mathrm{Ric}_2 > 0$. In addition, the quotients of $W^7_{1,1} \times W^7_{1,1}$ by either the free diagonal S^1-action or the free diagonal $\mathsf{SO}(3)$-action each admit a metric with $\mathrm{Ric}_2 > 0$. In summary, we have the following:

Corollary 3.15. *The following quotient maps are Riemannian submersions and the base spaces admit metrics with* $\mathrm{Ric}_2 > 0$*:*

$$
\begin{aligned}
S^{2n+1} \times S^{2n+1} &\to (S^{2n+1} \times S^{2n+1})/\Delta S^1 && (\dim = 4n+1 \textit{ with } n \geq 1),\\
S^{4n-1} \times S^{4n-1} &\to (S^{4n-1} \times S^{4n-1})/\Delta S^3 && (\dim = 8n-5 \textit{ with } n \geq 1),\\
W^7_{1,1} \times W^7_{1,1} &\to (W^7_{1,1} \times W^7_{1,1})/\Delta S^1 && (\dim = 13),\\
W^7_{1,1} \times W^7_{1,1} &\to (W^7_{1,1} \times W^7_{1,1})/\Delta \mathsf{SO}(3) && (\dim = 11).
\end{aligned}
$$

Now recall Theorem 3.8 asserts that the projections $M \times N \to M$ and $M \times N \to N$ are Riemannian submersions for the examples $M \times N$ constructed in Theorem 3.2. Theorem 3.8 and Corollary 3.15 relate to the fiber dimension conjecture of Petersen and Wilhelm:

Conjecture 3.16 (Petersen-Wilhelm Conjecture)**.** *If M is a compact manifold with positive sectional curvature and $\pi : M \to B$ is a Riemannian submersion with fiber F, then*

$$\dim(F) < \dim(B).$$

By work of Amann and Kennard [2] and González-Álvaro and Radeschi [16], the Petersen-Wilhelm Conjecture has been verified for all known examples of closed manifolds with positive sectional curvature. For progress toward the conjecture in the general case, we refer the reader to [15, 41].

In contrast, Theorem 3.8 and Corollary 3.15 both provide examples of Riemannian submersions for which the domain has $\mathrm{Ric}_2 > 0$ while the dimension of the fiber is larger than the dimension of the base. The most striking of these examples is the projection $\mathbb{O}\mathrm{P}^2 \times W^{24} \to \mathbb{O}\mathrm{P}^2$. By Theorem 3.2, $\mathbb{O}\mathrm{P}^2 \times W^{24}$ admits a metric g_ℓ with $\mathrm{Ric}_2 > 0$, and by Theorem 3.8, $\mathbb{O}\mathrm{P}^2$ admits a metric with respect to which the projection $(\mathbb{O}\mathrm{P}^2 \times W^{24}, g_\ell) \to \mathbb{O}\mathrm{P}^2$ is a Riemannian submersion. Because the dimensions of the fiber $F = W^{24}$ and the base $B = \mathbb{O}\mathrm{P}^2$ of this submersion satisfy $\dim(F) = \dim(B) + 6$, the Petersen-Wilhelm Conjecture with "positive sectional curvature" replaced by "$\mathrm{Ric}_2 > 0$" fails to hold by a large margin.

Chapter 4:

Intermediate Ricci curvature restrictions on submanifolds

In this chapter, we establish a relationship between intrinsic and extrinsic intermediate Ricci curvatures of submanifolds. Our main result is Theorem 4.1 in Section 4.1. In Section 4.2, we apply Theorem 4.1 to locally-defined commuting Killing fields to obtain a local symmetry rank bound for manifolds with positive intermediate Ricci curvatures. We discuss consequences of this, including ramifications in the setting of non-negative curvature, and we show that the local symmetry rank bound is optimal in the sense that any metric is close in the C^1-topology that achieves the local symmetry rank bound.

4.1 Intrinsic versus extrinsic curvature

In this section, we prove Theorem B, which we restate here with more accuracy:

Theorem 4.1. *Let N be a submanifold of M, let $k \in \{1, \ldots, \dim(M) - 1\}$, and let $\mathcal{S}$ be a subspace of T_pN with $\dim(\mathcal{S}) \geq k+1$. Assume for all flags $(\ell, \mathcal{V})$ of signature $(1, k+1)$ in $\mathcal{S}$ that the intrinsic intermediate Ricci curvature $\mathrm{Ric}_k^N(\ell, \mathcal{V})$ is non-positive while the extrinsic intermediate Ricci curvature $\mathrm{Ric}_k(\ell, \mathcal{V})$ is positive. Then*

$$\dim(\mathcal{S}) \leq \dim(M) - \dim(N) + k.$$

Assume that M, N, and $\mathcal{S}$ are as in Theorem 4.1. Let $\mathrm{I\!I}$ denote the second fundamental form for $N \subset M$, i.e. for $u, v \in T_pN$, $\mathrm{I\!I}(u, v) = (\nabla_u V)^\perp \in T_pN^\perp$, where V is any extension of v to a vector field. Given a unit vector $u \in T_pN$, let $\mathcal{O}_u$ denote the orthogonal complement of $\mathrm{span}\{u\}$ in $\mathcal{S}$. Now considering $\mathrm{I\!I}(u, \cdot)$ as a linear map $\mathcal{O}_u \to T_pN^\perp$, we have the following:

Lemma 4.2. *If $\dim(\mathcal{S}) > \dim(M) - \dim(N) + k$, then $\dim(\ker \mathrm{I\!I}(u, \cdot)) \geq k$.*

Proof. If $\dim(\mathcal{S}) > \dim(M) - \dim(N) + k$, then

$$\begin{aligned}
\dim(\ker \mathrm{I\!I}(u, \cdot)) &= \dim(\mathcal{O}_u) - \dim(\mathrm{Im}\, \mathrm{I\!I}(u, \cdot)) \\
&\geq (\dim(\mathcal{S}) - 1) - (\dim(M) - \dim(N)) \\
&> (\dim(M) - \dim(N) + k - 1) - (\dim(M) - \dim(N)).
\end{aligned}$$

Therefore, $\dim(\ker \mathrm{I\!I}(u, \cdot)) > k - 1$. ∎

We now prove the main theorem of this chapter using Lemma 4.2 and the Gauss equation:

Proof of Theorem 4.1. Suppose $\dim(\mathcal{S}) > \dim(M) - \dim(N) + k$, and fix a unit vector $u \in \mathcal{S}$ such that $|\,\mathrm{I\!I}(u,u)| \geq 0$ is minimal. By Lemma 4.2, we can choose orthonormal vectors $e_1, \ldots, e_k \in \mathcal{O}_u$ such that $\mathrm{I\!I}(u, e_i) = 0$. For $i = 1, \ldots, k$, define $f_i : \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned} f_i(\theta) &= |\mathrm{I\!I}\,(\cos(\theta)\,u + \sin(\theta)\,e_i, \cos(\theta)\,u + \sin(\theta)\,e_i)|^2 \\ &= \cos^4(\theta)|\,\mathrm{I\!I}(u,u)|^2 + 2\cos^2(\theta)\sin^2(\theta)\langle \mathrm{I\!I}(u,u), \mathrm{I\!I}(e_i,e_i)\rangle + \sin^4(\theta)|\,\mathrm{I\!I}(e_i,e_i)|^2. \end{aligned}$$

Then because $f_i(0) = |\,\mathrm{I\!I}(u,u)|^2$ is minimal,

$$f_i''(0) = 4\left(\langle \mathrm{I\!I}(u,u), \mathrm{I\!I}(e_i,e_i)\rangle - |\,\mathrm{I\!I}(u,u)|^2\right) \geq 0,$$

and so we must have $\langle \mathrm{I\!I}(u,u), \mathrm{I\!I}(e_i,e_i)\rangle \geq 0$. Now let $\mathcal{V} \coloneqq \mathrm{span}\{u, e_1, \ldots, e_k\} \subseteq \mathcal{S}$ and $\ell \coloneqq \mathrm{span}\{u\} \subset \mathcal{V}$. Because $\mathrm{I\!I}(u, e_i) = 0$ and $\langle \mathrm{I\!I}(u,u), \mathrm{I\!I}(e_i,e_i)\rangle \geq 0$ for all i, we have by the Gauss equation that

$$\begin{aligned} \mathrm{Ric}_k(\ell, \mathcal{V}) &= \sum_{i=1}^{k} \sec_M(u, e_i) \\ &= \sum_{i=1}^{k} \left[\sec_N(u, e_i) + |\,\mathrm{I\!I}(u, e_i)|^2 - \langle \mathrm{I\!I}(u,u), \mathrm{I\!I}(e_i,e_i)\rangle\right] \\ &\leq \sum_{i=1}^{k} \sec_N(u, e_i) \\ &= \mathrm{Ric}_k^N(\ell, \mathcal{V}). \end{aligned}$$

By assumption, $\mathrm{Ric}_k^N(\ell, \mathcal{V}) \leq 0$. Therefore we have shown that if $\dim(\mathcal{S}) > \dim(M) - \dim(N) + k$, then there exists a flag $(\ell, \mathcal{V})$ such that $\mathrm{Ric}_k(u, \mathcal{V}) \leq 0$, thus proving Theorem 4.1 by contraposition. ∎

4.2 Local symmetry rank bound

Definition 4.3. *The* ***symmetry rank*** *of a Riemannian manifold* (M, g)*, which we denote* $\mathrm{symrank}(M, g)$*, is the rank of its isometry group, i.e. the maximal dimension of a torus that can act isometrically and effectively on* (M, g)*.*

Grove and Searle established the following symmetry rank bound for manifolds with positive sectional curvature:

Theorem 4.4 (Maximal Symmetry Rank Theorem [17])**.** *Any closed, connected Riemannian* n*-manifold* (M, g) *with positive sectional curvature has*

$$\mathrm{symrank}(M, g) \leq \left\lfloor \frac{n+1}{2} \right\rfloor,$$

and in the case of equality, M *is diffeomorphic to* S^n*,* $\mathbb{R}\mathrm{P}^n$*,* $\mathbb{C}\mathrm{P}^{n/2}$*, or a lens space.*

The symmetry rank bound from the Grove-Searle Maximal Symmetry Rank Theorem [17] is proven using global arguments. It relies on globally defined torus actions and a theorem of Berger stating that any Killing field on an even dimensional positively curved manifold has a zero. Wilking obtained the same bound for quasi-positive curvature using only the Gauss equation:

Theorem 4.5 (Symmetry Rank Bound for Quasi-positive Curvature [49])**.** *Suppose (M, g) is a connected Riemannian n-manifold. If M contains a point at which all sectional curvatures are positive, then*

$$\mathrm{symrank}(M, g) \leq \left\lfloor \frac{n+1}{2} \right\rfloor.$$

Galaz-García included Wilking's argument for Theorem 4.5 in [12]. Searle and Wilhelm noticed that Wilking's argument only requires commuting Killing fields and positive sectional curvature for planes spanned by the Killing fields [37]. This observation inspired the results of this chapter.

Now, if a tangent space of a submanifold is spanned by commuting Killing fields, then intrinsic curvature of that submanifold at that point is identically zero. In particular, applying Theorem 4.1 in this setting, we obtain Corollary C, which we restate here for convenience:

Corollary 4.6. *Suppose that M is a Riemannian n-manifold, $N \subset M$ is a submanifold through a point p, the tangent space T_pN is spanned by commuting Killing fields of N, and $k \in \{1, \ldots, n-2\}$. If $\mathrm{Ric}_k(\ell, \mathcal{V}) > 0$ for all flags $(\ell, \mathcal{V})$ of signature $(1, k+1)$ in T_pN, then*

$$\dim(N) \leq \left\lfloor \frac{n+k}{2} \right\rfloor.$$

Proof. Because T_pN is spanned by commuting Killing fields of N, the intrinsic curvature for T_pN are zero. In particular, applying Theorem 4.1 with $\mathcal{S} \coloneqq T_pN$, we have that

$$\begin{aligned}
\dim(N) &= \dim(T_pN) \\
&\leq \dim(M) - \dim(N) + k \\
&= n - \dim(N) + k.
\end{aligned}$$

Therefore, it follows that $\dim(N) \leq \frac{n+k}{2}$. ■

Note that the submanifold N in Corollary 4.6 is not assumed to be complete; it is only required to be defined in a neighborhood of the point p. Also notice the utility of Corollary 4.6 is that it is local in nature, and thus can be applied in many scenarios.

4.2.1 k-maximal symmetry rank

Now we introduce terminology for a manifold that achieves the upper bound in Corollary 4.6:

Definition 4.7. *Suppose M has a submanifold N such that for some point $p \in N$, T_pN is spanned by commuting Killing fields of N, and $\mathrm{Ric}_k(\ell, \mathcal{V}) > 0$ for all signature-$(1, k+1)$ flags $(\ell, \mathcal{V})$ in T_pN. If $\dim(N) = \lfloor \frac{n+k}{2} \rfloor$, then we say M has* ***k-maximal local symmetry rank*** *at p.*

Notice that no manifold admits $(n-1)$-maximal local symmetry rank: If $k = n-1$, then $\lfloor \frac{n+k}{2} \rfloor = n - 1 < n = k + 1$. So when $k = n - 1$, there can be no $(k+1)$-dimensional subspaces $\mathcal{V}$ of T_pN while $\dim(N) = \lfloor \frac{n+k}{2} \rfloor$.

Recall from the Maximal Symmetry Rank Theorem that the list of manifolds which admit positive sectional curvature and maximal global symmetry rank is restrictive, only consisting of S^n, $\mathbb{R}\mathrm{P}^n$, $\mathbb{C}\mathrm{P}^{n/2}$, or lens spaces. In stark contrast, we have Theorem D, which we restate here for convenience:

Theorem 4.8. *Let M be an n-manifold, $n \geq 3$, $k \in \{1, \ldots, n-2\}$, and $p \in M$. Every Riemannian metric g on M is arbitrarily close in the C^1-topology to a metric $\tilde{g}$ such that $(M, \tilde{g})$ has k-maximal local symmetry rank at p.*

Theorem 4.8 shows that the upper bound in Corollary 4.6 can be realized on any manifold. We prove Theorem 4.8 by sewing in a model metric on a small ball around the point p and showing that the resulting metric can be made arbitrarily close in the C^1-distance to the original. These model metrics are constructed on $\mathbb{R}^n$ and have k-maximal local symmetry rank at the origin.

Construction of k-maximal local symmetry rank

In this section, we prove the following:

Proposition 4.9. *There exist metrics g^{model} on $\mathbb{R}^n$ that have k-maximal local symmetry rank at the origin for all $n \geq 3$ and $k \in \{1, \ldots, n-2\}$.*

Throughout this section, let $n \geq 3$, $k \in \{1, \ldots, n-2\}$, and $d = \lfloor \frac{n+k}{2} \rfloor$. To prove Proposition 4.9, we will construct metrics on $\mathbb{R}^{n-d} \times \mathbb{R}^d$ such that the coordinate vector fields for the $\mathbb{R}^d$ factor are Killing fields and $\mathrm{Ric}_k(\ell, \mathcal{V}) > 0$ for all signature-$(1, k+1)$ flags $(\ell, \mathcal{V})$ in $T_0\mathbb{R}^d$. We do not claim that these metrics are complete as this will not be necessary for proving Theorem 4.8 in Section 4.2.1. First, we establish the following computational simplification:

Lemma 4.10. *Given any Riemannian manifold M and a natural number $d \leq \dim(M) - 1$, suppose that $\{K_i\}_{i=1}^d$ is an orthonormal basis of a subspace $\mathcal{K} \subseteq T_pM$, $k \in \{1, \ldots, d-1\}$, and there exist $\mu, \nu \in [0, \infty)$ such that the following hold:*

1. *$R(K_i, K_j)K_\ell = 0$ when the indices i, j, ℓ are mutually distinct.*

2. *$\sec(K_i, K_j) \in \{-\nu, \mu\}$ for all $i \neq j$.*

3. *For each i, there exist at most $k-1$ indices $j \neq i$ such that*

   $\sec(K_i, K_j) = -\nu$.

4. *$\mu - (k-1)\nu > 0$.*

Then $\mathrm{Ric}_k(\ell, \mathcal{V}) > 0$ for all signature-$(1, k+1)$ flags $(\ell, \mathcal{V})$ in $\mathcal{K}$.

Proof. Let $\mathrm{Ric}^{\mathcal{K}} : \mathcal{K} \to \mathcal{K}$ denote the Ricci $(1,1)$-tensor restricted to $\mathcal{K}$ and composed with projection onto $\mathcal{K}$. Then by definition, $\langle \mathrm{Ric}^{\mathcal{K}}(u), u\rangle = \mathrm{Ric}_{d-1}(u, \mathcal{K})$ for any unit vector $u \in \mathcal{K}$. By Property 1, we get that $\mathrm{Ric}^{\mathcal{K}}$ is diagonalized by $\{K_i\}_{i=1}^d$; see, for example, [30, Proposition 4.1.3]. Then by Properties 2 and 3, we have that

$$\mathrm{Ric}_{d-1}(u, \mathcal{K}) \geq \min_{i=1,\dots,d}\{\mathrm{Ric}_{d-1}(K_i, \mathcal{K})\} \geq (d-k)\mu - (k-1)\nu,$$

for all $u \in \mathcal{K}$. Applying Property 4, we get

$$\mathrm{Ric}_{d-1}(u, \mathcal{K}) > (d-k-1)\mu. \tag{4.1}$$

Now, define an operator $\mathfrak{R}^{\mathcal{K}} : \Lambda^2\mathcal{K} \to \Lambda^2\mathcal{K}$ by

$$\left\langle \mathfrak{R}^{\mathcal{K}}\left(\textstyle\sum_i X_i \wedge Y_i\right) \ , \ \textstyle\sum_j V_j \wedge W_j \right\rangle = \sum_{i,j} R(X_i, Y_i, W_j, V_j),$$

which is the curvature operator restricted to the subspace $\mathcal{K}$. Then by Property 1, $\mathfrak{R}^{\mathcal{K}}$ is diagonalized by $\{K_i \wedge K_j\}_{i,j}$; see, for example, [30, Proposition 4.1.2]. Recall that by Property 2, on $\mathcal{K}$ we have

$$\sec \leq \mu. \tag{4.2}$$

Thus, given a $(k+1)$-dimensional subspace $\mathcal{V} \subseteq \mathcal{K}$ and unit vector $u \in \mathcal{V}$, choose orthonormal vectors $\{e_i\}_{i=1}^{d-1}$ such that $\{u, e_1, \dots, e_k\}$ is an orthonormal basis for $\mathcal{V}$ and $\{u, e_1, \dots, e_k, e_{k+1}, \dots, e_{d-1}\}$ is an orthonormal basis for $\mathcal{K}$.

Then by Inequalities 4.1 and 4.2, we have

$$\begin{aligned}
\mathrm{Ric}_k(u, \mathcal{V}) &= \sum_{i=1}^{k} \sec(u, e_i) \\
&= \mathrm{Ric}_{d-1}(u, \mathcal{K}) - \sum_{i=k+1}^{d-1} \sec(u, e_i) \\
&> [(d-k-1)\mu] - [(d-k-1)\mu] = 0.
\end{aligned}$$

■

Now we use Lemma 4.10 to construct metrics with k-maximal local symmetry rank:

Proof of Proposition 4.9. Consider $\mathbb{R}^{n-d} \times \mathbb{R}^d$ with coordinates $(x_1, \ldots, x_{n-d}, y_1, \ldots, y_d)$. Given positive smooth functions $\phi_i : \mathbb{R}^{n-d} \to \mathbb{R}$, define the metric g^{model} on $\mathbb{R}^{n-d} \times \mathbb{R}^d$ by

$$g^{\mathrm{model}} = dx_1{}^2 + \cdots + dx_{n-d}{}^2 + \phi_1{}^2 dy_1{}^2 + \cdots + \phi_d{}^2 dy_d{}^2.$$

Setting $K_i = \frac{1}{\phi_i(0)} \frac{\partial}{\partial y_i}$, the fields $\{K_i\}_{i=1}^d$ are the desired commuting Killing fields under this metric, and they are orthonormal at the origin. We will choose the ϕ_i such that $\left(\mathbb{R}^n, g^{\mathrm{model}}\right)$ and K_i together satisfy the hypotheses of Lemma 4.10.

Let $\mathcal{K} = \mathrm{span}\{K_i|_0\}_{i=1}^d = T_0\mathbb{R}^d$, and denote the orthogonal complement by $\mathcal{K}^\perp = T_0\mathbb{R}^{n-d}$. Let $\mathrm{I\!I} : \mathcal{K} \times \mathcal{K} \to \mathcal{K}^\perp$ denote the second fundamental form for the submanifold $\{(0, \ldots, 0)\} \times \mathbb{R}^d$. Then $\mathrm{I\!I}(K_i, K_j) = 0$ for $i \neq j$, and combining this with the fact that K_i are commuting Killing fields, we get that Property 1 of Lemma 4.10 is satisfied. We will show that the ϕ_i can be chosen such that Properties 2, 3, and 4 are satisfied at the origin. Notice that at the origin, $\mathrm{I\!I}(K_i, K_i) = -\frac{\operatorname{grad}\phi_i}{\phi_i(0)}$. Thus by the Gauss equation, because the submanifold $\{(0,\ldots,0)\} \times \mathbb{R}^d$ has zero intrinsic curvature, we have that the extrinsic curvatures are

$$\sec(K_i, K_j) = -\frac{\langle \operatorname{grad}\phi_i, \operatorname{grad}\phi_j \rangle}{\phi_i(0)\phi_j(0)}$$

when $i \neq j$. We now break the construction into two cases: $k = n-2$ and $k \leq n-3$.

Case $\mathbf{k = n - 2}$: If $k = n-2$, then $d = \lfloor \frac{n+k}{2} \rfloor = n-1$, and so $\mathcal{K}^\perp$ is 1-dimensional. Choose a unit vector U in $\mathcal{K}^\perp$. We will define the ϕ_i such that for some constants $a, b \in (0, \infty)$,

1. $\operatorname{grad}\phi_1 = \cdots = \operatorname{grad}\phi_k = U$,
2. $\phi_1(0) = \cdots = \phi_k(0) = a$,
3. $\operatorname{grad}\phi_d = -U$, and
4. $\phi_d(0) = b$

Then by construction, Properties 2 and 3 of Lemma 4.10 are satisfied with

$$\mu = \frac{|U|^2}{ab} = \frac{1}{ab},$$
$$-\nu = -\frac{|U|^2}{a^2} = -\frac{1}{a^2}.$$

Thus, to satisfy Property 4, we need

$$\frac{1}{ab} - \frac{k-1}{a^2} > 0.$$

Therefore, choosing any values for a and b such that $a > (k-1)b$, the $k = n-2$ case of Proposition 4.9 follows from Lemma 4.10.

Case $\mathbf{k \leq n-3}$: If $k \leq n-3$, then $\mathcal{K}^\perp$ has dimension $n-d \geq 2$. In the unit sphere $S^{n-d-1} \subset \mathcal{K}^\perp$, choose a vector U. Letting $U^\perp$ denote the orthogonal complement of $\operatorname{span}\{U\}$ in $\mathcal{K}^\perp$, consider the equatorial sphere $S^{n-d-2} = U^\perp \cap S^{n-d-1}$. Now inscribe a regular $(n-d-1)$-simplex in S^{n-d-2}, and define vectors $V_1(0), \ldots, V_{n-d}(0) \in S^{n-d-2}$ to be the vertices of this simplex. Hence

$$\langle V_i(0), V_j(0) \rangle = -\frac{1}{n-d-1}$$

for all distinct $i, j \in \{1, \ldots, n-d\}$. Now for $\theta \in [0, \frac{\pi}{2}]$ define

$$V_i(\theta) = \cos(\theta) V_i(0) - \sin(\theta) U.$$

Notice that

$$\langle V_i(\theta), V_j(\theta) \rangle = -\frac{\cos^2 \theta}{n-d-1} + \sin^2 \theta$$

is strictly increasing on $[0, \frac{\pi}{2}]$ and takes the value 0 at $\theta = \xi \in (0, \frac{\pi}{2})$ given by

$$\xi = \arctan\left(\sqrt{\frac{1}{n-d-1}}\right).$$

Hence, $\langle V_i(\theta), V_j(\theta) \rangle$ is negative for $\theta \in [0, \xi)$. We will now choose the ϕ_i such that

1. $\operatorname{grad} \phi_1 = \cdots = \operatorname{grad} \phi_k = U$,

2. $\phi_1(0) = \cdots = \phi_k(0) = a$,

3. $\operatorname{grad} \phi_{k+j} = V_j(\theta)$ for $j = 1, \ldots, n-d$, and

4. $\phi_{k+1}(0) = \cdots = \phi_d(0) = b$

for some values of $a, b \in (0, \infty)$ and $\theta \in (0, \xi)$. See Figure 4.1 for an illustration of how the $\operatorname{grad} \phi_i$ may be arranged in S^{n-d-1}.

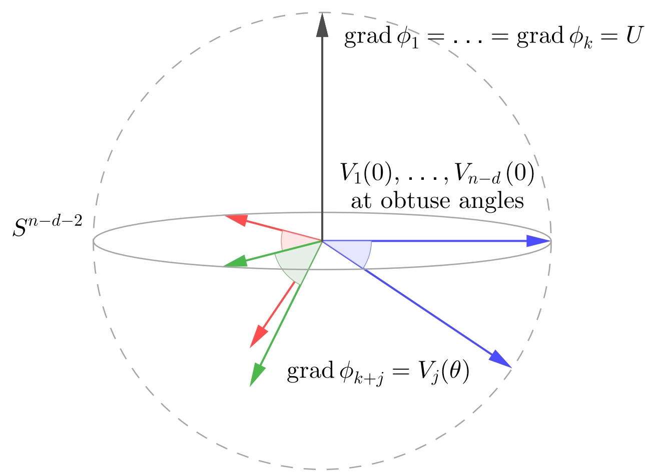

Figure 4.1: grad ϕ_i in S^{n-d-1}

Then because $\theta < \xi$, the negative curvatures among $\{\sec(K_i, K_j)\}_{i,j=1}^{d}$ correspond to distinct values of $i, j \in \{1, \ldots, k\}$, and these curvatures all have the same value

$$-\nu := -\frac{|U|^2}{a^2} = -\frac{1}{a^2}.$$

So for each i, there exist at most $k-1$ indices $j \neq i$ such that $\sec(K_i, K_j) = -\nu$, and thus Property 3 of Lemma 4.10 is satisfied. To satisfy Property 2, we need to choose a, b, θ so that for all $\ell \in \{1, \ldots, k\}$ and for all distinct $i, j \in \{1, \ldots, n-d\}$,

$$\sec(K_\ell, K_{k+i}) = \sec(K_{k+i}, K_{k+j}).$$

This common value is given by

$$\mu := -\frac{\langle U, V_i(\theta)\rangle}{ab} = -\frac{\langle V_i(\theta), V_j(\theta)\rangle}{b^2}. \tag{4.3}$$

Furthermore, to satisfy Property 4, we need

$$\mu - \frac{k-1}{a^2} > 0. \tag{4.4}$$

Now choose any value for $a > 0$. Then, once $\theta \in (0, \xi)$ is chosen, we will define b by

$$b = \frac{a\langle V_i(\theta), V_j(\theta)\rangle}{\langle U, V_i(\theta)\rangle}.$$

Thus Equation 4.3 will be satisfied with μ taking the value

$$\mu = -\frac{\langle U, V_i(\theta)\rangle^2}{a^2\langle V_i(\theta), V_j(\theta)\rangle}.$$

Finally, Inequality 4.4 holds if θ is chosen such that

$$\langle U, V_i(\theta)\rangle^2 > -(k-1)\langle V_i(\theta), V_j(\theta)\rangle.$$

Notice that $\langle U, V_i(\theta)\rangle^2$ approaches a positive constant dependent on n and k as θ approaches ξ, while $-\langle V_i(\theta), V_j(\theta)\rangle$ approaches 0 as θ approaches ξ. Therefore, there exists a value θ such that Inequality 4.4 holds. Therefore, applying Lemma 4.10, Proposition 4.9 is proven. ■

Sewing and the C^1-topology

In this section, we establish a general Sewing Theorem for changing a Riemannian metric within metric ball around a point while remaining close to the original metric in the C^1-topology. We then use the Sewing Theorem to Prove Theorem 4.8. First, we discuss the C^ℓ-norm for tensors.

Recall that two smooth maps $F_1, F_2 : M \to N$ are ϵ-close in the *weak C^ℓ-topology* if their values and partial derivatives up to order ℓ are ϵ-close with respect to fixed atlases on M and N; see Chapter 2 of [22]. If the atlases are finite, this leads to a notion of *C^ℓ-distance.*

Now, given vector bundles E_1 and E_2, Euclidean metrics on E_1 and E_2, and a bundle map $\phi : E_1 \to E_2$, the C^ℓ-*norm* of ϕ is defined as follows: If E_1^1 denotes the unit sphere bundle under the Euclidean metric on E_1, then $|\phi|_{C^\ell}$ is the C^ℓ-distance from $\phi|_{E_1^1}$ to the zero bundle map $E_1 \to E_2$. The C^ℓ-norm of a tensor ω is the C^ℓ-distance from ω to the zero-section. Notice that these definitions depend on the choice of Euclidean metrics.

Throughout this section, let M be a fixed manifold, and let g be a fixed Riemannian metric on M. All C^ℓ-norms will be defined in terms of the fixed metric g on TM.

We now establish the following:

Theorem 4.11 (Sewing Theorem). *Let (M, g) be a Riemannian manifold, let p be a point in M, and let g^* be a Riemannian metric defined on a neighborhood of p such that $g(\gamma', \cdot) = g^*(\gamma', \cdot)$ for all geodesic rays γ emanating from p. For every $\epsilon > 0$, there exists a Riemannian metric $\tilde{g}$ on M that is ϵ-close to g in the C^1-distance such that $\tilde{g} \equiv g^*$ on an open ball centered at p.*

The author has been informed that Searle, Solórzano, and Wilhelm have proven if $\sec(M, g) \geq K$, then $\tilde{g}$ in Theorem 4.11 can be made to satisfy $\sec(M, \tilde{g}) \geq \tilde{K}$ for any $\tilde{K} < K$ [35].

Now define t to be the value of $\mathrm{dist}_g(p, \cdot)$, and let ∂_t denote $\mathrm{grad}(\mathrm{dist}_g(p, \cdot))$. We begin by proving the following:

Lemma 4.12 (Converse Gauss Lemma). *Let (M, g) be a Riemannian manifold, let $p \in M$, and let g^* be as in Theorem 4.11.*

1. $\mathrm{dist}_g(p, \cdot) = \mathrm{dist}_{g^*}(p, \cdot)$, *and the integral curves of ∂_t are geodesics emanating from p with respect to both metrics g and g^*.*

2. *Along geodesic rays emanating from p, the family of Jacobi fields that vanish at p are the same for both metrics g and g^*.*

For a more general version of Lemma 4.12, see [31, Proposition 2.2].

Proof. Because $g^*(\partial_t, \partial_t) = g(\partial_t, \partial_t) = 1$, the integral curves of ∂_t are also geodesics under g^*, and hence Part (1) follows. Now notice that along a geodesic ray emanating from p, a Jacobi field J that vanishes at p under g is realized as the variation field of a variation by g-geodesics emanating from p. Because these curves are also geodesics under g^*, J is also a Jacobi field under g^*. ∎

Now letting $\mathrm{inj\,rad}_p$ denote the injectivity radius of (M, g) at p, choose $\delta \in (0, \frac{1}{2}\,\mathrm{inj\,rad}_p)$ small enough such that g^* is defined on the closed ball $\overline{B(p, 2\delta)} \subset M$.

Lemma 4.13. *There is a symmetric $(0, 2)$-tensor r defined on $\overline{B(p, 2\delta)}$ such that*

$$g - g^* = t^2 r.$$

Furthermore, for any $\ell \in \mathbb{N}$, there exists a constant C such that $|r|_{C^\ell} < C$ with respect to the metric g.

Proof. Choose normal coordinates $(x^1, \ldots, x^n)$ around p, and let $\partial_1, \ldots, \partial_n$ denote the corresponding coordinate vector fields. Consider the rotational fields $J = x^j\partial_i - x^i\partial_j$. These fields are normal to $\partial_t = \mathrm{grad}(\mathrm{dist}(p, \cdot))$, and they vanish at p. Now consider any geodesic $\gamma : [0, 2\delta] \to M$ emanating from p. Then J are also Jacobi fields along γ because the Lie derivative $\mathcal{L}_{\partial_t} J = [\partial_t, J] = 0$. Notice that for every point $\gamma(t_0)$ with $t_0 > 0$, there is a neighborhood around $\gamma(t_0)$ such that every tangent space is spanned by ∂_t and $n-1$ of the rotational fields J. We choose $n-1$ such rotational fields and denote them by $J_1, \ldots, J_{n-1}$. We will construct the tensor r in this neighborhood of $\gamma(t_0)$ by defining its values on $\left\{\frac{J_1}{|J_1|}, \ldots, \frac{J_{n-1}}{|J_{n-1}|}, \partial_t\right\}$. By the Gauss Lemma, $(g - g^*)(\partial_t, \cdot) \equiv 0$, and so we set $r(\partial_t, \cdot) \equiv 0$. Informed by [10, Chapter 5 Proposition 2.7], we now compute the Taylor expansion of $(g - g^*)(J_i, J_j)_{\gamma(t)}$ centered at $t = 0$, where γ is a geodesic emanating from p. First notice that because the J_i are Jacobi and $J_i(0) = 0$, we have

$$J_i''(0) = -\left(R_g(J_i, \partial_t)\partial_t\right)(0) = 0.$$

Second, for any vector field V along γ, we have at $t = 0$

$$\begin{aligned} g\left(\nabla_{\partial_t}(R_g(J_i, \partial_t)\partial_t), V\right) &= \partial_t g\left(R_g(J_i, \partial_t)\partial_t, V\right) - g\left(R_g(J_i, \partial_t)\partial_t, V'\right) \\ &= \partial_t g\left(R_g(V, \partial_t)\partial_t, J_i\right) \\ &= g\left(\nabla_{\partial_t}(R_g(V, \partial_t)\partial_t), J_i\right) + g\left(R_g(V, \partial_t)\partial_t, J_i'\right) \\ &= g\left(R_g(J_i', \partial_t)\partial_t, V\right). \end{aligned}$$

Using these facts and that $J_i(0) = 0 = J_j(0)$, we have the following at $t = 0$:

$$
\begin{aligned}
g(J_i, J_j) &= 0, \\
g(J_i, J_j)' &= g(J_i', J_j) + g(J_i, J_j') = 0, \\
g(J_i, J_j)'' &= g(J_i'', J_j) + 2g(J_i', J_j') + g(J_i, J_j'') = 2g(J_i', J_j'), \\
g(J_i, J_j)''' &= g(J_i''', J_j) + 3g(J_i'', J_j') + 3g(J_i', J_j'') + g(J_i, J_j''') = 0, \\
g(J_i, J_j)'''' &= g(J_i'''', J_j) + 4g(J_i''', J_j') + 6g(J_i'', J_j'') + 4g(J_i', J_j''') + g(J_i, J_j'''') \\
&= -8R_g(J_i', \partial_t, \partial_t, J_j').
\end{aligned}
$$

By Lemma 4.12, γ is a geodesic for both g and g^*, and J_i, J_j are Jacobi fields along γ for either metric. Hence, we have that the equations above also hold with g^* substituted for g. Thus, by applying these calculations to the Taylor expansions of $g(J_i, J_j)_{\gamma(t)}$ and $g^*(J_i, J_j)_{\gamma(t)}$ centered at $t = 0$, and using the fact that the inner products induced by g and g^* on T_pM agree, we have

$$
(g - g^*)(J_i, J_j)_{\gamma(t)} = -\frac{t^4}{3}(R_g - R_{g^*})(J_i', \partial_t, \partial_t, J_j')_p + O(t^5) \qquad \text{as } t \to 0
$$

By normalizing this equation, we get

$$
(g - g^*)\left(\frac{J_i}{|J_i|}, \frac{J_j}{|J_j|}\right)_{\gamma(t)} = \frac{t^4}{3|J_i(t)||J_j(t)|}(R_{g^*} - R_g)(J_i', \partial_t, \partial_t, J_j')_p + O(t^3),
$$

where $|J_i|$ is measured using g. It follows from the computations of derivatives $g(J_i, J_j)$ above that $|J_i(t)| = t + O(t^2) = |J_j(t)|$ as $t \to 0$. Thus, we have

$$
\lim_{t \to 0} \frac{d}{dt}\left[(g - g^*)\left(\frac{J_i}{|J_i|}, \frac{J_j}{|J_j|}\right)_{\gamma(t)}\right] = 0. \tag{4.5}
$$

Now, recall that if a smooth function $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(0) = 0 = f'(0)$, then there is a smooth function $h : \mathbb{R} \to \mathbb{R}$ such that $f(t) = t^2 h(t)$. Thus

$$(g - g^*) \left(\frac{J_i}{|J_i|}, \frac{J_j}{|J_j|} \right)_{\gamma(t)} = t^2 r_{i,j}(t)$$

for some smooth function $r_{i,j}$. Thus we define $r\big(\frac{J_i}{|J_i|}, \frac{J_j}{|J_j|}\big) := r_{i,j}$ so that $g - g^* = t^2 r$. Because the inner products induced by g and g^* on T_pM agree, it follows from Equation 4.5 and Taylor's theorem that r can be smoothly extended to $\overline{B(p, 2\delta)}$. Finally, the inequality $|r|_{C^\ell} < C$ follows from the fact that r is smooth on the compact set $\overline{B(p, 2\delta)}$. ∎

We are now prepared to prove the Sewing Theorem.

Proof of Theorem 4.11. Choose $\delta \in (0, \frac{1}{2} \operatorname{inj} \operatorname{rad}_p)$ such that g^* is defined on the closed ball $\overline{B(p, 2\delta)} \subset M$. By Lemma 7.3 in [32], there exists a smooth function $\phi : M \to [0, 1]$ such that

$$\begin{aligned} &\phi \equiv 1 \text{ where } \operatorname{dist}(p, \cdot) \leq \delta, \\ &\phi \equiv 0 \text{ where } \operatorname{dist}(p, \cdot) \geq 2\delta, \\ &|\phi|_{C^1} \leq \frac{2}{\delta}. \end{aligned}$$

On M, define the Riemannian metric $\tilde{g} = (1-\phi)g + \phi g^*$. Because $\tilde{g} \equiv g$ for $\operatorname{dist}(p, \cdot) \geq 2\delta$, assume $t = \operatorname{dist}(p, \cdot) < 2\delta$. Given unit vector fields U and V,

$$\begin{aligned}(g-\tilde{g})(U,V) &= g(U,V) - (1-\phi)g(U,V) - \phi g^*(U,V)\\ &= \phi(g-g^*)(U,V)\\ &= \phi t^2 r(U,V),\end{aligned}$$

where r is the tensor from Lemma 4.13. Furthermore, for a unit vector X, we have

$$\begin{aligned}X(g-\tilde{g})(U,V) &= (X\phi)t^2 r(U,V) + \phi X(t^2) r(U,V) + \phi t^2 X r(U,V)\\ &= (X\phi)t^2 r(U,V) + \phi(2t)(Xt) r(U,V) + \phi t^2 X r(U,V).\end{aligned}$$

Thus, because $\phi \leq 1$, $|\phi|_{C^1} \leq \frac{2}{\delta}$, $t < 2\delta$, and $|r|_{C^1} < C$, we have

$$\begin{aligned}|(g-\tilde{g})(U,V)| &< 4\delta^2 C,\\ |X(g-\tilde{g})(U,V)| &< 12\delta C + 4\delta^2 C.\end{aligned}$$

Therefore, for any $\epsilon > 0$, δ can be chosen so that $|g-\tilde{g}|_{C^1} < \epsilon$. ■

We now prove Theorem 4.8 using Theorem 4.11 and the metrics established in Proposition 4.9.

Proof of Theorem 4.8. Choose any Riemannian n-manifold (M, g) with $n \geq 3$. Let $p \in M$ and define g^* on a neighborhood of p to be the pull-back metric

$$g^* = \left(\exp_p^{-1}\right)^* \left(g^{\text{model}}\right),$$

where g^{model} is one of the metrics from Proposition 4.9 in Section 4.2.1. Then by the Gauss Lemma, $g(\gamma', \cdot) = g^*(\gamma', \cdot)$ for all geodesic rays γ emanating from p, and thus Theorem 4.11 can be applied to obtain a metric $\tilde{g}$ on M that has k-maximal local symmetry rank at p. Therefore, the space of metrics on M which have k-maximal local symmetry rank is dense under the C^1-topology. ■

4.3 Global symmetry rank bound

When a torus acts isometrically and effectively on a manifold, the principal orbits have the same dimension as the torus, and their tangent spaces are spanned by commuting Killing fields. Thus from Corollary 4.6, we obtain Corollary E, which we restate here for convenience:

Corollary 4.14. *Suppose (M, g) is a connected Riemannian n-manifold. If M contains a point at which all k^{th}-intermediate Ricci curvatures are positive for some $k \in \{1, \dots, n-2\}$, then*

$$\operatorname{symrank}(M, g) \leq \left\lfloor \frac{n+k}{2} \right\rfloor.$$

Proof. Supposed that a torus T^r with $r > \frac{n+k}{2}$ acts isometrically an effectively on a connected manifold M. Then, applying Corollary 4.6 at every point of each principal orbit, it follows that there is a flag $(\ell, \mathcal{V})$ at each of these points such that $\operatorname{Ric}_k(\ell, \mathcal{V}) \leq 0$. Because the set of principal orbits is a dense subset of M, it follows that every point in M has a flag $(\ell, \mathcal{V})$ such that $\operatorname{Ric}_k(\ell, \mathcal{V}) \leq 0$, thus proving Corollary 4.14 by contraposition. ■

Because $\mathrm{Ric}_1 > 0$ is equivalent to $\sec > 0$, this result generalizes the Grove-Searle symmetry rank bound for $\sec > 0$, Theorem 4.4, and the Wilking symmetry rank bound for quasi-positive curvature, Theorem 4.5.

If $k = n-2$, then Corollary 4.14 states that $\mathrm{symrank}(M,g) \leq \left\lfloor \frac{n+(n-2)}{2} \right\rfloor = n-1$. Cohomogeneity-one torus actions actions may occur on Ricci-positive manifolds in dimensions 2 and 3; e.g. the T^1-action on S^2 and the T^2-action on S^3. However, it is known that in dimensions ≥ 4, closed manifolds that admit cohomogeneity-one torus actions must have infinite fundamental group [28, 29]. Thus by the Bonnet-Myers theorem, such manifolds cannot admit invariant metrics of globally positive Ricci curvature, and hence cannot admit $\mathrm{Ric}_{n-2} > 0$ globally. Therefore, we have the following:

Remark 4.15. *If (M,g) is a closed, connected n-manifold for $n \geq 4$ and $\mathrm{Ric}_k(M,g) > 0$ for $k \geq n-2$, then*

$$\mathrm{symrank}(M,g) \leq n-2.$$

Corro and Galaz-García show in [9] that for each dimension $n \geq 6$, there exist examples of n-manifolds which admit a metric of positive Ricci curvature with $\mathrm{symrank}(M,g) = n-4$. It remains to be shown if this can be improved to give examples of positive Ricci curvature with $\mathrm{symrank}(M,g) = n-2$.

4.3.1 Ramifications in non-negative curvature

In this section, we discuss a consequence of Corollary 4.14 in the context of non-negative curvature. Specifically, Galaz-García and Searle state the Maximal Symmetry Rank Conjecture for non-negatively curved manifolds in [14], which has since been sharpened by Escher and Searle in [11]:

Conjecture 4.16 (Maximal Symmetry Rank Conjecture). *Let (M, g) be a closed, simply connected, n-dimensional Riemannian manifold with non-negative sectional curvature. Then*

1. $\operatorname{symrank}(M, g) \leq \left\lfloor \dfrac{2n}{3} \right\rfloor$, *and*

2. *in the case of equality, M is equivariantly diffeomorphic to a product of spheres or a quotient thereof by a free linear action of a torus of rank less than or equal to $2n \bmod 3$.*

It was shown that the Conjecture 4.16 holds for $4 \leq n \leq 6$ by Galaz-García and Searle in [14], it holds for $7 \leq n \leq 9$ by Escher and Searle in [11], and also Part (1) holds for $10 \leq n \leq 12$ in [11]. Furthermore, the Conjecture 4.16 has been confirmed in all dimensions for torus actions that are *isotropy-maximal* in [11]. With the assumption of non-negative curvature replaced with *rational ellipticity*, Part (1) has been established in all dimensions along with a rational homotopy theoretic version of Part (2) by Galaz-García, Kerin, and Radeschi in [13]. This result is relevant to the conjecture above because the Bott Conjecture claims that any non-negatively curved manifold is rationally elliptic.

Notice that a non-negatively curved manifold can have positive k^{th}-intermediate Ricci curvature for $k \geq 2$ without being positively curved. If $\mathrm{Ric}_k > 0$ at a point for $k \leq \frac{n}{3}$, then by Corollary 4.14, $\mathrm{symrank}(M, g) \leq \lfloor \frac{2n}{3} \rfloor$. In particular, we have the following:

Corollary 4.17. *Let (M, g) be a closed, simply connected Riemannian manifold with non-negative section curvature and dimension $n \geq 13$. If M contains a point at which $\mathrm{Ric}_{\lfloor n/3 \rfloor} > 0$, then the conclusion of Part (1) of the Maximal Symmetry Rank Conjecture holds.*

Chapter 5:

Torus actions on manifolds with positive intermediate Ricci curvature

In this chapter, we study isometric torus actions on closed manifolds with $\mathrm{Ric}_k > 0$. In Section 5.1, we establish the existence of fixed point sets for such actions, in Section 5.2, we establish a cohomogeneity restriction for general isometric actions, and in Section 5.3, we improve upon the symmetry rank bound established in Corollary 4.14.

5.1 Fixed point sets of torus actions

For studying isometric group actions on positively curved manifolds, the following has been an essential tool:

Lemma 5.1. *Suppose M is a closed manifold with positive sectional curvature on which a torus T^r acts isometrically.*

1. *If M is even-dimensional and $r \geq 1$, then the T^r-action on M has a fixed point.*
2. *If M is odd-dimensional and $r \geq 2$, then there exists a codimension-1 torus subgroup $T^{r-1} \subset T^r$ such that the T^{r-1}-action on M has a fixed point.*

The even-dimensional case in Lemma 5.1 is a consequence of Berger's Fixed Point Theorem from [5]. The odd-dimensional case was established by Sugahara in [42]. By adapting Berger's argument in [5], Lemma 5.1 can be reformulated in terms of commuting Killing fields:

Lemma 5.2 (Theorem 8.3.5 in [30])**.** *Suppose M is a closed n-manifold with positive sectional curvature. If there are two commuting Killing fields on M, then they must be linearly dependent at some point in M. Furthermore, if M is even-dimensional, then there is a point at which both Killing fields are zero.*

Adapting the argument further, we establish Theorem F, which we restate here for convenience:

Theorem 5.3. *Suppose M is a closed n-manifold and $\mathrm{Ric}_k(M, g) > 0$ for some $k \in \{1, \ldots, n-1\}$. If there are $k+1$ commuting Killing fields on M, then they must be linearly dependent at some point in M.*

Applying Theorem 5.3 to torus actions, we have the following:

Corollary 5.4. *Suppose M is a closed n-manifold and $\mathrm{Ric}_k(M,g) > 0$ for some $k \in \{1,\dots,n-1\}$. If a torus T^r acts isometrically on M with $r \geq k+1$, then there is a codimension-k torus subgroup $T^{r-k} \subset T^r$ such that the T^{r-k}-action on M has a fixed point.*

Example 5.5. *By Theorem 3.1, products of spheres $S^n \times S^n$ for $n \geq 2$ admit metrics g_ℓ with* $\mathrm{Ric}_2 > 0$*. So by Corollary 5.4, if a torus T^3 acts isometrically on $(S^n \times S^n, g_\ell)$, then there exists a circle subgroup $S^1 \subset T^3$ with non-empty fixed point set in $S^n \times S^n$. By Corollary 3.12, when n is odd, $(S^n, \times S^n, g_\ell)$ admits a free isometric T^2-action. Therefore, for $n \equiv 2$* mod 4 *and $k = 2$, Corollary 5.4 is optimal.*

We will now prove Theorem 5.3. Given vector fields $X, Y_1, \dots, Y_j$, set

$$\mathcal{Y}_j := \mathrm{span}\{Y_1, \dots, Y_j\},$$

and let $X_j^\perp$ denote the projection of X onto $\mathcal{Y}_j^\perp$. Then define

$$f_j := \frac{1}{2}\left|X_j^\perp\right|^2.$$

Lemma 5.6. *Let $X, Y_1, \dots, Y_k$ be linearly independent commuting Killing fields on M. Suppose there is a point $p \in M$ at which $Y_1|_p, \dots, Y_k|_p$ are orthonormal and $X|_p$ is orthogonal to the subspace $\mathcal{Y}_k|_p \subseteq T_pM$. Then for all $v \in T_pM$,*

$$\mathrm{Hess}\, f_k(v,v) = |\nabla_v X|^2 - \mathrm{curv}(X,v) - 4\sum_{j=1}^{k}\langle \nabla_v X, Y_j\rangle^2.$$

Proof. Set $\bar{Y}_1 \coloneqq Y_1$, and for $j = 2, \ldots, k$ define

$$\bar{Y}_j \coloneqq Y_j - \sum_{i=1}^{j-1} \frac{\langle Y_j, \bar{Y}_i\rangle}{|\bar{Y}_i|^2} \bar{Y}_i. \tag{5.1}$$

Then

$$f_k = \frac{1}{2}\left| X - \sum_{j=1}^{k} \frac{\langle X, \bar{Y}_j\rangle}{|\bar{Y}_j|^2}\bar{Y}_j \right|^2 = \frac{1}{2}\left(|X|^2 - \sum_{j=1}^{k} \frac{\langle X, \bar{Y}_j\rangle^2}{|\bar{Y}_j|^2} \right).$$

Thus, defining

$$h_k \coloneqq -\frac{\langle X, \bar{Y}_k\rangle^2}{2|\bar{Y}_k|^2},$$

we have that $f_k = f_{k-1} + h_k$. Consequently, we will prove Lemma 5.6 by induction on k. The base case, $k = 1$, is established in the proof of Theorem 8.3.5 in [30]. For the induction hypothesis, suppose that at p, for some $k \geq 2$,

$$\operatorname{Hess} f_{k-1}(v, v) = |\nabla_v X|^2 - \operatorname{curv}(X, v) - 4\sum_{j=1}^{k-1} \langle \nabla_v X, Y_j\rangle^2.$$

We must show that $\operatorname{Hess} f_k = \operatorname{Hess} f_{k-1} + \operatorname{Hess} h_k$ satisfies the conclusion of Lemma 5.6. Thus, it suffices to show that $\operatorname{Hess} h_k(v, v) = -4\langle \nabla_v X, Y_k\rangle^2$ at p. Now, applying Equation 5.1, we have

$$\begin{aligned}
h_k &= -\frac{\langle X, \bar{Y}_k\rangle^2}{2|\bar{Y}_k|^2} \\
&= -\frac{1}{2|\bar{Y}_k|^2}\left\langle X\ ,\ Y_k - \sum_{j=1}^{k-1}\frac{\langle Y_k, \bar{Y}_j\rangle}{|\bar{Y}_j|^2}\bar{Y}_j \right\rangle^2 \\
&= -\frac{1}{2|\bar{Y}_k|^2}\left(\langle X, Y_k\rangle - \sum_{j=1}^{k-1}\frac{\langle Y_k, \bar{Y}_j\rangle\langle X, \bar{Y}_j\rangle}{|\bar{Y}_j|^2} \right)^2 \\
&= -\frac{1}{2|\bar{Y}_k|^2}\left(\langle X, Y_k\rangle^2 - 2\sum_{j=1}^{k-1}\frac{\langle X, Y_k\rangle\langle Y_k, \bar{Y}_j\rangle\langle X, \bar{Y}_j\rangle}{|\bar{Y}_j|^2} + \left(\sum_{j=1}^{k-1}\frac{\langle Y_k, \bar{Y}_j\rangle\langle X, \bar{Y}_j\rangle}{|\bar{Y}_j|^2} \right)^2 \right).
\end{aligned}$$

Notice that the inner products $\langle X, Y_k\rangle$, $\langle Y_k, \bar{Y}_j\rangle$, and $\langle X, \bar{Y}_j\rangle$ all vanish at p, and the Hessian of a product of three or more functions that vanish at p will itself vanish at p. Thus defining

$$\tilde{h}_k := -\frac{\langle X, Y_k\rangle^2}{2|\bar{Y}_k|^2},$$

we have that at p,

$$\operatorname{Hess} h_k = \operatorname{Hess} \tilde{h}_k.$$

Now given $v \in T_pM$, because X and Y_k are commuting Killing fields, we have

$$v\langle X, Y_k\rangle = 2\langle \nabla_v X, Y_k\rangle = -2\langle \nabla_{Y_k} X, v\rangle. \tag{5.2}$$

Hence, the gradient of $\tilde{h}_k$ satisfies

$$\begin{aligned}\langle \nabla \tilde{h}_k, v\rangle &= v\left(-\frac{\langle X, Y_k\rangle^2}{2|\bar{Y}_k|^2}\right)\\ &= \frac{2\langle X, Y_k\rangle\langle \nabla_{Y_k} X, v\rangle}{|\bar{Y}_k|^2} - \frac{2\langle X, Y_k\rangle^2 v\left(|\bar{Y}_k|\right)}{|\bar{Y}_k|^3}.\end{aligned}$$

Again, because $\langle X, Y_k\rangle$ vanishes at p, terms in $\nabla\tilde{h}_k$ that have two or more factors of $\langle X, Y_k\rangle$ will vanish in the covariant derivative of $\nabla\tilde{h}_k$ at p. So applying Equation 5.2 again, we have that at p,

$$\begin{aligned}\operatorname{Hess} h_k(v,v) &= \operatorname{Hess} \tilde{h}_k(v,v)\\ &= \langle \nabla_v \nabla \tilde{h}_k, v\rangle\\ &= \left\langle \nabla_v\left(\frac{2\langle X, Y_k\rangle}{|\bar{Y}_k|^2}\nabla_{Y_k} X\right) + 0, v\right\rangle\\ &= v\left(\frac{2\langle X, Y_k\rangle}{|\bar{Y}_k|^2}\right)\langle \nabla_{Y_k} X, v\rangle\\ &= \left(\frac{4\langle \nabla_v X, Y_k\rangle}{|\bar{Y}_k|^2} - \frac{4\langle X, Y_k\rangle v(|\bar{Y}_k|)}{|\bar{Y}_k|^3}\right)\left(-\langle \nabla_v X, Y_k\rangle\right).\end{aligned}$$

Thus, because $\langle X, Y_k \rangle = 0$ and $|\bar{Y}_k| = 1$ at p, we have

$$\operatorname{Hess} h_k(v, v) = -4\langle \nabla_v X, Y_k \rangle^2.$$

Therefore, by induction, the result follows. ■

We can now use Lemma 5.6 to prove Theorem 5.3:

Proof of Theorem 5.3. Suppose there are $k+1$ linearly independent commuting Killing fields $X, Y_1, \ldots, Y_k$ on M. We will show that M must have a Ric_k that is non-positive.

Because $X, Y_1, \ldots, Y_k$ are linearly independent, $f_k = \frac{1}{2}|X_k^{\perp}|^2$ must attain a positive minimum at some point p. Replacing $Y_1, \ldots, Y_k$ with commuting Killing fields that span the same distribution $\mathcal{Y}_k$ and are orthonormal at p does not change the values of f_k. Furthermore, we can replace X with the Killing field that commutes with $Y_1, \ldots, Y_k$ such that $X|_p \in (\mathcal{Y}_k|_p)^{\perp}$, and this too will not change the values of f_k.

Now, with these new choices of $X, Y_1, \ldots, Y_k$, by Lemma 5.6, we know for $v \in T_pM$,

$$\operatorname{Hess} f_k(v, v) = |\nabla_v X|^2 - \operatorname{curv}(X, v) - 4\sum_{j=1}^{k} \langle \nabla_v X, Y_j \rangle^2.$$

Because f_k attains a minimum at p, we have $\operatorname{Hess} f_k(v, v) \geq 0$ for all $v \in T_pM$, and hence

$$\operatorname{curv}(X, v) \leq |\nabla_v X|^2 - 4\sum_{j=1}^{k} \langle \nabla_v X, Y_j \rangle^2, \tag{5.3}$$

for all $v \in T_pM$. With these choices of $X, Y_1, \ldots, Y_k$, we also have that

$$f_k \leq \frac{1}{2}|X|^2,$$

with equality at p. So defining

$$f_0 := \frac{1}{2}|X|^2,$$

we know that f_0 also attains a minimum at p, and hence $\nabla f_0 = -\nabla_X X = 0$ at p. Thus $\nabla X : T_pM \to T_pM$ is a skew-symmetric linear map with $\dim(\ker \nabla X) \geq 1$.

Suppose $\dim(\ker \nabla X) \geq k+1$. Then we may choose orthonormal $v_1, \ldots, v_k \in \ker \nabla X$ that are orthogonal to X. Therefore, applying Inequality 5.3, we have

$$\begin{aligned}
\mathrm{Ric}_k(X; v_1, \ldots, v_k) &= \sum_{i=1}^{k} \sec(X, v_i) \\
&= \frac{1}{|X|^2} \sum_{i=1}^{k} \mathrm{curv}(X, v_i) \\
&\leq \frac{1}{|X|^2} \sum_{i=1}^{k} \left(|\nabla_{v_i} X|^2 - 4 \sum_{j=1}^{k} \langle \nabla_{v_i} X, Y_j \rangle^2 \right) \\
&= 0.
\end{aligned}$$

Hence, if $\dim(\ker \nabla X) \geq k+1$, then M has a Ric_k that is non-positive.

Suppose now that $\dim(\ker \nabla X) = \ell + 1 \leq k$. Choose orthonormal $v_1, \ldots, v_\ell \in \ker \nabla X$ that are orthogonal to X, and let $\mathcal{V}$ denote $(\ker \nabla X)^\perp$. Given $u \in \ker \nabla X$ and $v \in \mathcal{V}$, because X is a Killing field,

$$\langle \nabla_v X, u \rangle = -\langle \nabla_u X, v \rangle = 0.$$

Thus $\mathrm{Im}(\nabla X|_{\mathcal{V}}) = \mathcal{V}$, and $\nabla X|_{\mathcal{V}} : \mathcal{V} \to \mathcal{V}$ is an isomorphism. Now because X is orthogonal to $\mathcal{Y}_k = \operatorname{span}\{Y_1, \ldots, Y_k\}$ and $\mathcal{V}$ at p, $\dim(\mathcal{Y}_k + \mathcal{V}) \leq n-1$, and hence

$$\begin{aligned}
\dim(\mathcal{Y}_k \cap \operatorname{Im}(\nabla X|_{\mathcal{V}})) &= \dim(\mathcal{Y}_k \cap \mathcal{V}) \\
&= \dim(\mathcal{Y}_k) + \dim(\mathcal{V}) - \dim(\mathcal{Y}_k + \mathcal{V}) \\
&\geq k + (n - \ell - 1) - (n-1) \\
&= k - \ell.
\end{aligned}$$

Thus, we can choose orthonormal $v_{\ell+1}, \ldots, v_k \in \mathcal{V}$ such that $\nabla_{v_{\ell+i}} X \in \mathcal{Y}_k$ for $i = 1, \ldots, k-\ell$. So for $= 1, \ldots, k-\ell$ and $j = i \ldots, k$, define scalars α_i^j so that

$$\nabla_{v_{\ell+i}} X = \sum_{j=1}^{k} \alpha_i^j Y_j$$

Then applying Inequality 5.3, we have

$$\begin{aligned}
\sum_{i=1}^{k} \sec(X, v_i) &= \frac{1}{|X|^2} \sum_{i=1}^{k} \operatorname{curv}(X, v_i) \\
&\leq \frac{1}{|X|^2} \sum_{i=1}^{k} \left(|\nabla_{v_i} X|^2 - 4 \sum_{j=1}^{k} \langle \nabla_{v_i} X, Y_j \rangle^2 \right) \\
&= \frac{1}{|X|^2} \sum_{i=1}^{k-\ell} \left(\sum_{j=1}^{k} \left(\alpha_i^j\right)^2 - 4 \sum_{j=1}^{k} \left(\alpha_i^j\right)^2 \right) \\
&= -\frac{3}{|X|^2} \sum_{i=1}^{k-\ell} \sum_{j=1}^{k} \left(\alpha_i^j\right)^2 \\
&< 0.
\end{aligned}$$

Thus, we have shown that if M has $k+1$ linearly independent commuting Killing fields, then M must have a Ric_k that is non-positive. Therefore, we have proven Theorem 5.3 by contraposition. ■

5.2 Rank difference and cohomogeneity

Given a smooth action by a Lie group G on a manifold M and a point $p \in M$, let $G_p \subset G$ denote the isotropy subgroup associated to p. By applying Corollary 5.4 to a maximal torus in G, the following is an immediate consequence:

Corollary 5.7 (Isotropy Rank Lemma for $\mathrm{Ric}_k > 0$)**.** *Suppose a Lie group G acts isometrically on a closed manifold M. If $\mathrm{Ric}_k(M) > 0$, then there exists a point $p \in M$ for which*

$$\mathrm{rank}(G) - \mathrm{rank}(G_p) \leq k.$$

In particular, applying Corollary 5.7 to compact homogeneous spaces, we have the following:

Corollary 5.8. *If a compact homogeneous space G/H has $\mathrm{Ric}_k(G/H) > 0$, then*

$$\mathrm{rank}(G) - \mathrm{rank}(H) \in \{ \quad k \quad , k-2, k-4, \dots \} \subset \mathbb{Z}_{\geq 0} \qquad \textit{if } k \equiv \dim(G/H) \mod 2,$$

$$\mathrm{rank}(G) - \mathrm{rank}(H) \in \{k-1, k-3, k-5, \dots \} \subset \mathbb{Z}_{\geq 0} \qquad \textit{if } k \not\equiv \dim(G/H) \mod 2.$$

Proof. By Corollary 5.7, $\mathrm{rank}(G) - \mathrm{rank}(H) \leq k$. Recall that $\dim(K) - \mathrm{rank}(K)$ is even for all compact Lie groups K. Thus,

$$\begin{aligned} \mathrm{rank}(G) - \mathrm{rank}(H) &\equiv \mathrm{rank}(G) - \dim(G) + \dim(G) - \dim(H) + \dim(H) - \mathrm{rank}(H) &\mod 2 \\ &\equiv \dim(G) - \dim(H) &\mod 2. \end{aligned}$$

Therefore, the result follows. ■

Definition 5.9. *Suppose a Lie group G acts smoothly on a manifold M. The* ***cohomogeneity*** *of the action is defined to be*

$$\mathrm{cohom}(M, G) \coloneqq \dim(M/G).$$

Equivalently, $\mathrm{cohom}(M, G)$ *is given by the codimension of the principal orbits in M.*

Using Lemma 5.1, Püttmann established the following:

Theorem 5.10 (Theorem C in [34])**.** *Suppose M is a closed, positively curved manifold. If G acts isometrically on M with principal isotropy subgroup H, then*

$$\mathrm{rank}(G) - \mathrm{rank}(H) \leq \mathrm{cohom}(M, G) + 1,$$

To prove this, Püttmann used the following:

Lemma 5.11 (Lemma 1.1 in [34])**.** *If a compact Lie group G acts isometrically on a manifold M with principal isotropy $H \subset G$, then for all points $p \in M$,*

$$\mathrm{rank}(G_p) - \mathrm{rank}(H) \leq \mathrm{cohom}(M, G).$$

Now, combining Lemma 5.11 with Corollary 5.7, we establish Corollary G, which we restate here for convenience:

Corollary 5.12. *Suppose (M, g) is a closed Riemannian manifold with $\mathrm{Ric}_k(M, g) > 0$ for some $k \in \{1, \ldots, \dim(M) - 1\}$. If a Lie group G acts isometrically on M with principal isotropy subgroup $H \subset G$, then*

$$\mathrm{rank}(G) - \mathrm{rank}(H) \leq \mathrm{cohom}(M, G) + k.$$

5.3 Symmetry rank bound for compact manifolds with $\mathrm{Ric}_k > 0$

Recall from the Grove-Searle Maximal Symmetry Rank Theorem, Theorem 4.4, that closed, connected, positively curved manifolds have symmetry rank $\leq \lfloor \frac{n+1}{2} \rfloor$. In Corollary 4.14, we established that connected manifolds with $\mathrm{Ric}_k > 0$ at a point have symmetry rank $\leq \lfloor \frac{n+k}{2} \rfloor$. If we now assume that the manifolds are closed and have $\mathrm{Ric}_k > 0$ globally, then we can apply Corollary 5.4 to improve the symmetry rank bound in Corollary 4.14:

Theorem 5.13. *Suppose M^n is a closed, connected, n-dimensional manifold.*

1. *If $\mathrm{Ric}_1(M^n) > 0$, i.e. $\sec(M^n) > 0$, then*

$$\mathrm{symrank}(M, g) \leq \left\lfloor \frac{n+1}{2} \right\rfloor.$$

2. *If $\mathrm{Ric}_2(M^n) > 0$ and n is odd, then*

$$\mathrm{symrank}(M, g) \leq \left\lfloor \frac{n+1}{2} \right\rfloor.$$

3. *If $\mathrm{Ric}_k(M^n) > 0$ for $k \in \{3, \ldots, n-1\}$ and n odd, then*

$$\mathrm{symrank}(M, g) \leq \left\lfloor \frac{n+k}{2} \right\rfloor - 1.$$

4. *If $\mathrm{Ric}_k(M^n) > 0$ for $k \in \{2, \ldots, n-1\}$ and n even, then*

$$\mathrm{symrank}(M, g) \leq \left\lfloor \frac{n+k}{2} \right\rfloor - 1.$$

Notice Theorem 5.13 is equivalent to Theorem H. We choose to state it differently here in order to assist with the formulation of our argument below. To prove Theorem 5.13, we will use the following:

Lemma 5.14. *Manifolds of dimension ≥ 4 cannot support metrics of positive Ricci curvature that are invariant under a cohomogeneity-one torus action.*

Lemma 5.14 follows from the work of Pak [28] and Parker [29], who showed that in dimensions ≥ 4, closed manifolds which admit cohomogeneity-one torus actions must have infinite fundamental group. Thus by the Bonnet-Myers theorem, such manifolds cannot admit invariant metrics of positive Ricci curvature. We will also use the following:

Lemma 5.15 (Proposition 8.3.8 in [30])**.** *Let M be compact and let X, Y be commuting Killing fields on M. If X and Y both vanish on a connected, totally geodesic submanifold $N \subset M$, then some linear combination of them vanishes on a larger submanifold in M. In particular, if N is fixed pointwise by an isometric T^2-action on M, then there is a circle subgroup $S^1 \subset T^2$ such that the component of its fixed point set which contains N has codimension $< \operatorname{codim}(N)$.*

We are now ready to establish our symmetry rank bound:

Proof of Theorem 5.13. Notice that Part 1 was established by Grove and Searle in Theorem 4.4. Also, Part 2 follows from Corollary 4.14, because if n is odd and $k = 2$, then

$$\left\lfloor \frac{n+k}{2} \right\rfloor = \left\lfloor \frac{n+2}{2} \right\rfloor = \left\lfloor \frac{n+1}{2} \right\rfloor .$$

We will prove Parts 3 and 4 of Theorem 5.13 using induction on the dimension of M. First, we establish the base cases, $\dim(M) = 4$ and $\dim(M) = 5$. If a 4-dimensional manifold M has $\mathrm{Ric}_2(M) > 0$ or $\mathrm{Ric}_3(M) > 0$ and T^3 acts isometrically on M, then the action must be at least S^1-ineffective by Lemma 5.14. Similarly, if a 5-dimensional manifold M has $\mathrm{Ric}_3(M) > 0$ or $\mathrm{Ric}_4(M) > 0$ and T^4 acts isometrically on M, then the action must be at least S^1-ineffective by Lemma 5.14.

Now, for the sake of induction, suppose that for some $n \geq 6$, Theorem 5.13 holds for all dimensions $\dim(M) \in \{4, 5, \ldots, n-1\}$. We wish to show that it holds for $\dim(M) = n$. So suppose M satisfies the hypothesis of Parts 3 or 4, and defining

$$r := \left\lfloor \frac{n+k}{2} \right\rfloor,$$

suppose that a torus T^r acts isometrically on M. We must show that this T^r-action on M is at least S^1-ineffective.

Case $\mathbf{k \geq n-2}$: If $k \geq n-2$, then

$$r = \left\lfloor \frac{n+k}{2} \right\rfloor \geq \left\lfloor \frac{n+(n-2)}{2} \right\rfloor = n-1.$$

Thus, the T^r-action must be at least S^1-ineffective by Lemma 5.14.

Case $\mathbf{k \leq n-3}$: If $k \leq n-3$, then

$$r = \left\lfloor \frac{n+k}{2} \right\rfloor \geq \left\lfloor \frac{(k+3)+k}{2} \right\rfloor = k+1. \tag{5.4}$$

Hence, by Corollary 5.4, there are circle subgroups of T^r which have non-empty fixed point sets. Among the collection of all components of fixed point sets for circle subgroups in T^r, choose an element N that has minimal codimension in M, and let S^1 denote the circle subgroup that fixes N. Notice that N is totally geodesic, N has even codimension in M, and $T^r/S^1 \cong T^{r-1}$ acts on N. We will prove that $N = M$.

Subcase dim(N) $\leq$ k: Assuming $\dim(N) \leq k$, we will show that N is in fact fixed by a torus $T^2 \subset T^r$. If $\dim(N) = k$, then because N has even codimension in M and because $k \leq n-3$, we get that $k \leq n-4$. In particular,

$$\begin{aligned}
\dim(N) &= k \\
&= (r-r)+k \\
&= r - \left\lfloor \frac{n+k}{2} \right\rfloor + k \\
&\leq r - \left\lfloor \frac{(k+4)+k}{2} \right\rfloor + k \\
&= r-(k+2)+k \\
&= r-2.
\end{aligned}$$

If $\dim(N) \leq k-1$, then by Inequality 5.4, we have

$$\begin{aligned}
\dim(N) &\leq k-1 \\
&= (r-r)+k-1 \\
&\leq r-(k+1)+k-1 \\
&= r-2.
\end{aligned}$$

Thus if $\dim(N) \leq k$, then $\dim(N) \leq r-2$. Hence the T^{r-1}-action on N must be at least S^1-ineffective. In particular, there is a torus subgroup $T^2 \subset T^r$ that fixes N. Therefore, it follows from Lemma 5.15 that N does not have minimal codimension, which is a contradiction.

Subcase $\mathbf{dim(N)} \geq \mathbf{k+1}$: Now assuming that $k+1 \leq \dim(N) \leq n-2$, we will show that N is fixed by a torus $T^2 \subset T^r$, again reaching a contradiction. Because $\dim(N) \geq k+1$ and N is totally geodesic, $\mathrm{Ric}_k(N) > 0$. Thus if $\dim(N) \leq n-2$, the induction hypothesis implies that

$$\begin{aligned}
\mathrm{symrank}(N) &\leq \left\lfloor \frac{\dim(N)+k}{2} \right\rfloor - 1 \\
&\leq \left\lfloor \frac{(n-2)+k}{2} \right\rfloor - 1 \\
&= \left\lfloor \frac{n+k}{2} - 1 \right\rfloor - 1 \\
&= r-2.
\end{aligned}$$

Thus the T^{r-1}-action on N must be S^1-ineffective. Hence if $\dim(N) \leq n-2$, there is a torus subgroup $T^2 \subset T^r$ that fixes N. Therefore, it follows again from Lemma 5.15 that N does not have minimal codimension in M, which is a contradiction.

Therefore, we must have $\dim(N) = n = \dim(M)$. This implies that the circle subgroup $S^1 \subset T^r$ that fixes N must fix all of M. This proves Theorem 5.13 by induction. ■

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