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Discrete ergodic Jacobi matrices: Spectral properties and Quantum dynamical bounds

DISSERTATION

submitted in partial satisfaction of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

in Mathematics

by

Rui Han

Dissertation Committee:
Professor Svetlana Jitomirskaya, Chair
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2017

Dedication

This thesis is dedicated to my beloved parents and wife.

For their endless love, support and encouragement.

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Acknowledgements

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Publications and Preprints

- R. Han, Continuity of measure of the spectrum for Schrödinger operators with potentials driven by shfits and skew-shifts on higher dimensional tori. preprint
- R. Han, Sch'nol's theorem and the spectrum of long range operators. arXiv:1704.04603
- R. Han, C.A.Marx, Large coupling asymptotics of the Lyapunov exponent for analytic quasiperiodic potentials. arXiv:1612.04321
- R. Han, S. Jitomirskaya, Full measure reducibility and localization for quasi-periodic Jacobi operators: a topological criterion. arXiv:1608.01032
- R. Han, S. Jitomirskaya, Quantum dynamical bounds for ergodic potentials with underlying dynamics of zero topological entropy. arXiv:1607.08576
- R. Han, F. Yang, Generic continuous spectrum for multi-dimensional quasiperiodic Schrödinger operators with rough potentials. Journal of spectral theory, to appear.
- R. Han, Absence of point spectrum for the self-dual extended Harper's model, Int. Math. Res. Not. to appear.
- R. Han, Dry Ten Martini Problem for the non-self-dual extended Harper's model, Tran. Amer. Math. Soc. to appear.
- R. Han, Uniform localization is always uniform, Proc. Amer. Math. Soc. 144 (2016), 609-612.

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- Perspectives of Mathematics in the 21st Century: Conference in Celebration of the 90th Anniversary of Mathematics Department of Tsinghua University
  Apr 22-24, 2017
- Young Researchers Symposium "Methods of Modern Mathematical Physics", Fields Institute, Toronto, Canada. Aug 22-26, 2016
- Great Lakes Mathematical Physics Meeting, The Institute of Mathematical and Theoretical Physics of MSU, East Lansing, MI. June 17-19, 2016
- Simons Center's Workshop: Between Dynamics and Spectral Theory, Simons Center for Geometry and Physics, Stony Brook, NY. June 5-10, 2016
- Special session at AMS Fall Western Sectional Meeting, Fullerton, CA
  Oct 24-25 2015
- Workshop: Spectral Properties of Quasicrystals via Analysis, Dynamics, and Geometric Measure Theory, BIRS, Oaxaca, Mexico Sept 27-Oct 2 2015
- Almost Periodic and Other Ergodic Problems, Isaac Newton Institute, University of Cambridge, UK Mar 20-May 20 2015
- 33nd Annual Western States Math-Physics Meeting, Caltech, Pasadena, CA
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- Special Session at USA-Uzbekistan Conference on Analysis and Mathematical Physics, CSUF, Fullerton ,CA May 20-24 2014

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- Dynamical System seminar, Nanjing University, China. Apr 25, 2017
- Mathematical Physics seminar, Caltech. Mar 8, 2017
- Mathematical Physics and Probability seminar, UC Davis. Feb 1, 2017

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Hyperbolic plane.	Jan 2014
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Abstract of the Dissertation

Discrete ergodic Jacobi matrices: Spectral properties and Quantum dynamical bounds

By

Rui Han

Doctor of Philosophy in Mathematics
University of California, Irvine, 2017
Professor Svetlana Jitomirskaya, Chair

In this thesis we study discrete quasiperiodic Jacobi operators as well as ergodic operators driven by more general zero topological entropy dynamics. Such operators are deeply connected to physics (quantum Hall effect and graphene) and have enjoyed great attention from mathematics (e.g. several of Simon's problems). The thesis has two main themes. First, to study spectral properties of quasiperiodic Jacobi matrices, in particular when off-diagonal sampling function has non-zero winding number or singularities. Second, to address the consequences of positive Lyapunov exponent for Schrödinger operators with a class of potentials of bounded discrepancy, prime example being those driven by shifts and skew-shifts on multi-dimensional tori.

Within the first theme, one of our results provides an if and only if topological criterion for obtaining localization from reducibility of the dual Jacobi cocycles. As an application of this result to the extended Harper's model, we obtain sharp arithmetic spectral transition in the positive Lyapunov exponent regime. Two other results about the extended Harper's model include a proof of non-degeneracy of all possible spectral gaps (known as Dry Ten Martini Problem) for the non-self-dual regions, and an arithmetic result on purely continuous spectrum for the self-dual region that is optimal and improves on a recent work by Avila-Jitomirskaya-Marx, who proved a measure-theoretic version.

The most important contribution among the second group is a general localization-type result for ergodic potentials of bounded discrepancy. As concrete applications of our general result, we build the first arithmetic localization-type results for potentials defined by shifts and (the first non-perturbative ones for) skew-shifts of higher-dimensional tori. Similar consideration also leads to the continuity of spectral data, for Schrödinger operators with such underlying dynamics in the positive Lyapunov exponent regime.

Introduction

0.1 Discrete Jacobi matrices

Let $(\mathcal{M}, g)$ be a compact Riemannian manifold equipped with metric g. Let Vol_g be its Riemannian volume density. Let f be an invertible uniquely ergodic volume preserving map on $\mathcal{M}$.

Consider the following 1-dimensional discrete Jacobi matrix on $l^2(\mathbb{Z})$ given by

$$(H_{c,v,f}(\theta)u)_n = c(f^n\theta)u_{n+1} + \overline{c(f^{n-1}\theta)}u_{n-1} + v(f^n\theta)u_n, \tag{1}$$

where $\theta \in \mathcal{M}$ is called phase and c, v are functions on $\mathcal{M}$. In particular, we will assume v be a real-valued function.

Two prime examples of $(\mathcal{M}, f)$ are: shifts $f = f_{s,\alpha} : \theta \to \theta + \alpha$ on $\mathbb{T}^d$ and skew-shifts $f = f_{ss,\alpha} : (\theta_1, \theta_2, ..., \theta_d) \to (\theta_1 + \alpha, \theta_2 + \theta_1, ..., \theta_d + \theta_{d-1})$ on $\mathbb{T}^d$. When $f = f_{s,\alpha}$ operator $H_{c,v,f_{s,\alpha}}$, which we denote by $H_{c,v,\alpha}$ for short, is called discrete quasiperiodic Jacobi matrix. We also mention that when $c(\theta) \equiv 1$, $H_{1,v,f}$ is referred to as discrete Schrödinger operator.

0.2 Motivation

Quasiperiodic Jacobi matrices arise naturally from the study of tight-binding electrons on a two-dimensional lattice exposed to a perpendicular magnetic field. The most prominent example of such operators is the Harper's equation, mathematically known as the almost Mathieu operator (AMO):

$$(Hu)_n = u_{n+1} + u_{n-1} + 2\lambda \cos 2\pi(\theta + n\alpha)u_n. \tag{2}$$

In early 1970s, numerical studies of the spectrum of AMO (when $\lambda = 1$) generated the first fractal in the physics literature, the "Hofstadter butterfly". From that time on, this operator has continuously drawn great attention from both mathematics and physics societies. Up to now, it has already been connected to the work related to three Nobel prizes (quantum Hall effect (1998, 2016) and graphene (2010)) and one Fields medal (Ten Martini Problem).

After three decades of active research, much is known about the AMO. In this thesis we mainly explore generalizations of AMO in two different directions: one-frequency Jacobi matrices (with non-constant $c(\theta)$) and multi-frequency Schrödinger operator (multi-dimensional α).

In the first part of the thesis (Chapters 2-4), we explore the influence of non-constant sampling function c on the spectral theory of quasiperiodic operators. It turns out the spectral properties will be largely different from the Schrödinger case, especially when c has non-zero winding number or singularities.

One prime example of (non-Schrödinger) Jacobi matrices is the extended Harper's model (EHM):

$$(H_{\lambda,\alpha}(\theta)u)_n = c_\lambda(\theta + n\alpha)u_{n+1} + \tilde{c}_\lambda(\theta + (n-1)\alpha)u_{n-1} + 2\cos 2\pi(\theta + n\alpha)u_n. \tag{3}$$

where $c(\theta) = \lambda_1 e^{-2\pi i(\theta+\frac{\alpha}{2})} + \lambda_2 + \lambda_3 e^{2\pi i(\theta+\frac{\alpha}{2})}$. This model was first proposed by D.J. Thouless in 1983 [67] and arises when 2D electrons are allowed to hop to both nearest neighboring (expressed through λ_2) and the next-nearest lattice sites (expressed through λ_1 and λ_3). It includes AMO as a special case (when $\lambda_1 = \lambda_3 = 0$). An interesting feature of EHM is that it accounts for different lattice geometries: not only square lattice (as AMO), but also triangular lattice (when one of λ_1, λ_3 vanishes). Thus it is more closely connected to the Graphene, whose hexagonal lattice can be viewed as two interlacing triangular lattices.

In the last decades, there have been many important advances about EHM, including the explicit formula of the Lyapunov exponent [37], localization in the positive Lyapunov exponent region for Diophantine frequencies [43] and spectral decomposition for all frequencies in the zero Lyapunov exponent regions [8].

While in the positive Lyapunov exponent region, there is definitely a different behavior for the Diophantine [43] and Liouville [57] α, thus it is interesting to determine a transition. In the last few years, there have been several remarkable developments, where in models with classical small denominator problems leading to arithmetic transitions in spectral behavior, sharp results were obtained, with analysis up to the very arithmetic threshold. For the Maryland model, the spectral phase diagram was determined exactly for all α, θ in [52]. For the AMO, the transition in α (conjectured in 1994 [51]) was recently proved in [12]. Even more recently, pure point spectrum up to the transition was established by a different method in [47, 48] with also an arithmetic condition on θ. Our result presented in Chapter 2 adds to this growing collection by establishing a sharp transition in α for the extended Harper's operator. This is achieved by establishing an if-and-only-if topological criterion for the reducibility of general quasiperiodic Jacobi cocycles to imply pure point spectrum of the dual model.

In Chapter 3, we answer a question by Avila-Jitomirskaya-Marx. Specifically, we obtain an arithmetic result on purely continuous spectrum for the self-dual EHM that improves on [8], where a measure-theoretic version was proved.

Aside from the aforementioned developments on the spectral decomposition of EHM, little was known about the spectrum as a set. In Chapter 4, we study the Cantor structure of the spectrum for non-self-dual EHM, and prove the Dry Ten Martini Problem in the Diophantine case.

In the second part of the thesis (Chapters 5, 6), we explore the consequences of positive Lyapunov exponent for Schrödinger operators with potentials driven by shift and skew-shifts on multi-dimensional tori.

Positive Lyapunov exponents are generally viewed as a signature of localization. While it is known that they can coexist even with almost ballistic transport [62] [27], vanishing of certain dynamical exponents has been identified as a reasonable expected consequence of hyperbolicity of the corresponding transfer-matrix cocycle. Results in this direction were obtained in [25, 26] for one-frequency trigonometric polynomials, and recently in [45], for one-frequency quasiperiodic potentials under very mild assumptions on regularity of the sampling function. In Chapter 5 we identify a general property responsible for positive Lyapunov exponents implying vanishing of the dynamical quantities in the rather general case of underlying dynamics defined by volume preserving maps of Riemannian manifolds with zero topological entropy, and under very minimal regularity assumptions. This work presents the first localization-type results that hold in such generality. Our general results allow us, in particular, to obtain localization-type statements for potentials defined by shifts and (the first non-perturbative ones for) skew-shifts of higher-dimensional tori.

A natural approach to quasiperiodic operators is through periodic approximants, obtained by replacing the irrational frequency by a sequence of rationals. In particular, since the spectrum at rational frequencies can be obtained numerically and are easier to study, continuity in frequency allows us to study the spectrum at irrational frequencies via rational approximation. While many recent significant advances in discrete Schrödinger operators, see e.g. [15, 40, 2], require one dimensional torus shift and analytic potentials, our results presented in Chapter 6 reveal that continuity of the spectrum is a much more general phenomenon: it holds for both shifts and skew-shift of higher dimensional tori and also Hölder continuous potentials.

Chapter 1

Preliminaries

1.1 Notations

For $x \in \mathbb{R}$, let $\|x\|_{\mathbb{T}} = \text{dist}(x, \mathbb{Z})$. For a bounded analytic function f defined on a strip $\{|\text{Im}\theta| < \epsilon\}$ we let $\|f\|_\epsilon = \sup_{|\text{Im}\theta|<\epsilon} |f(\theta)|$. If f is a bounded continuous function on $\mathbb{R}$, we let $\|f\|_0 = \sup_{\theta \in \mathbb{R}} |f(\theta)|$. For a set $U \subset \mathbb{R}^d$ let $|U|$ be the Lebesgue measure of U.

1.2 Rational approximation of α

First, let us introduce the Diophantine condition on $\mathbb{T}^d$:

$$\text{DC}(\tau) = \cup_{c>0}\text{DC}(c,\tau) = \cup_{c>0}\{(\alpha_1, ..., \alpha_d) | \|\langle \vec{h}, \alpha \rangle\|_{\mathbb{T}} \geq \frac{c}{r(\vec{h})^\tau} \text{ for any } \vec{0} \neq \vec{h} \in \mathbb{Z}^d\}$$

where $r(\vec{h}) = \prod_{i=1}^d \max(|h_i|, 1)$. It is well-known that when $\tau > 1$, $DC(\tau)$ is a full measure set.

We also introduce the weak Diophantine condition:

$$\text{WDC}(\tau) = \cup_{c>0}\text{WDC}(c,\tau) = \cup_{c>0}\{(\alpha_1, ..., \alpha_d) | \max\{\|h\alpha_i\|_{\mathbb{T}}\} \geq \frac{c}{|h|^\tau} \text{ for any } 0 \neq h \in \mathbb{Z}\}.$$

It is well-known that when $\tau > \frac{1}{d}$, $\text{WDC}(\tau)$ is a full measure set.

Clearly, in general $\text{DC}(\tau) \subseteq \text{WDC}(\tau)$, while in the single frequency case $\text{DC}(\tau) = \text{WDC}(\tau)$.

1.2.1 Single frequency

Let α be an irrational number and let $\{\frac{p_n}{q_n}\}$ be its continued fraction approximants. The following properties (see e.g.[68]) are well-known:

$$\frac{1}{2q_{n+1}} \leq \|q_n\alpha\|_{\mathbb{T}} \leq \frac{1}{q_{n+1}}. \tag{1.1}$$

$$\|k\alpha\|_{\mathbb{T}} > \|q_n\alpha\|_{\mathbb{T}} \text{ for } q_n < |k| < q_{n+1}. \tag{1.2}$$

Let $\beta(\alpha) \in [0, \infty]$ be given by

$$\beta(\alpha) = \limsup_{n\to\infty} \frac{\ln q_{n+1}}{q_n}. \tag{1.3}$$

Roughly speaking, $\beta(\alpha)$ being large means α can be approximated well by rational numbers. We also mention that in some papers, Diophantine condition refers to $\beta(\alpha) = 0$, we denote it by $\alpha \in \mathrm{DC}$. Clearly, $\mathrm{DC}(\tau) \subset \mathrm{DC}$ for any $\tau > 0$.

1. If $\alpha \in \mathrm{DC}(c, \tau)$ for some $c > 0$, we have

$$\|k\alpha\|_{\mathbb{T}} \geq \frac{c}{|k|^\tau} \text{ for any } k \neq 0. \tag{1.4}$$

   In particular, combining (1.1) with (1.4) we have

$$cq_{n+1} \leq q_n^\tau. \tag{1.5}$$

2. If $\alpha \notin \mathrm{DC}(\tau)$, there exists a subsequence of the continued fraction approximants $\{\frac{p_{n_k}}{q_{n_k}}\}$ so that

$$q_{n_k+1} > q_{n_k}^\tau. \tag{1.6}$$

1.2.2 Multiple frequencies

Let $\alpha = (\alpha_1, \alpha_2, ..., \alpha_d)$ be a set of irrational frequencies. Let $\{\frac{\vec{p}_n}{q_n}\}$ be its best simultaneous approximation with respect to the Euclidean norm on $\mathbb{T}^d$, namely,

$$\sum_{j=1}^d \|q_n\alpha_j\|_{\mathbb{T}}^2 < \sum_{j=1}^d \|k\alpha_j\|_{\mathbb{T}}^2 \text{ for any } 0 < |k| < q_n.$$

Clearly, by the pigeonhole principle, we have

$$\sqrt{\sum_{j=1}^{d} \|q_n\alpha_j\|_{\mathbb{T}}^2} \leq \frac{2\Gamma(\frac{d}{2}+1)^{\frac{1}{d}}}{\sqrt{\pi}q_{n+1}^{\frac{1}{d}}}. \tag{1.7}$$

By the definition of Diophantine and weak-Diophantine condition.

1. If $\alpha \in \mathrm{DC}(c,\tau)$, then

$$\|\langle \vec{k}, \alpha\rangle\|_{\mathbb{T}} \geq \frac{c}{r(\vec{k})^{\tau}} \ \text{ for any } \vec{k} \in \mathbb{Z}^d\backslash\{\vec{0}\}. \tag{1.8}$$

2. If $\alpha \in \mathrm{WDC}(c,\tau)$, then

$$\max_{1\leq j\leq d} \|k\alpha_j\|_{\mathbb{T}} \geq \frac{c}{|k|^{\tau}} \ \text{ for any } k \in \mathbb{Z}\backslash\{\vec{0}\}. \tag{1.9}$$

   In particular combining (1.7) with (1.9), we have for $\alpha \in \mathrm{WDC}(\tau)$,

$$c' q_{n+1}^{\frac{1}{d}} \leq q_n^{\tau} \ \text{ for some constant } \ c'. \tag{1.10}$$

3. If $\alpha \notin \mathrm{WDC}(\tau)$, there exists a subsequence of the best simultaneous Diophantine approximation $\{\frac{\vec{p}_{n_k}}{q_{n_k}}\}$ so that

$$\lim_{k\to\infty} q_{n_k}^{\tau} \max_{1\leq j\leq d} \|q_{n_k}\alpha_j\|_{\mathbb{T}} = 0. \tag{1.11}$$

1.3 Cocycles and Lyapunov exponent

For a given $z \in \mathbb{C}$, a formal solution u of $H_{c,v,f}u = zu$ can be reconstructed via the following equation.

$$\begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix} = A_{c,z}(f^n\theta) \begin{pmatrix} u(n) \\ u(n-1) \end{pmatrix},$$

where

$$A_{c,z}(\theta) = \frac{1}{c(\theta)} \begin{pmatrix} z - v(\theta) & -\overline{c(f^{-1}\theta)} \\ c(\theta) & 0 \end{pmatrix}$$

Let $A_{c,z,k}(\theta)$ be the product of consecutive transfer matrices:

$$A_{c,z,k}(\theta) = A_{c,z}(f^{k-1}\theta)\cdots A_{c,z}(f\theta)A_{c,z}(\theta) \text{ for } k>0, \quad A_{c,z,0}(\theta) = \mathrm{Id}, \text{ and}$$
$$A_{c,z,k}(\theta) = (A_{c,z,-k}(f^k\theta))^{-1} \text{ for } k<0.$$

Then for any $k \in \mathbb{Z}$ we have the following.

$$\begin{pmatrix} u(k) \\ u(k-1) \end{pmatrix} = A_{c,z,k}(\theta) \begin{pmatrix} u(0) \\ u(-1) \end{pmatrix}.$$

The linear skew-product $(f, A_{c,z}(\cdot))$, defined below, is called the associated cocycle to $H_{c,v,f}$ at energy z.

$$\begin{aligned} (f, A_{c,z}(\cdot)) : (\mathcal{M},\ \mathbb{C}^2) &\to (\mathcal{M},\ \mathbb{C}^2) \\ (\theta, x) &\to (f\theta,\ A_{c,z}(\theta)x) \end{aligned}$$

We define the Lyapunov exponent

(1.12)
$$L(f,z) = \lim_{k\to\infty} \frac{1}{k}\int_{\mathcal{M}} \ln \|A_{c,z,k}(\theta)\| \ \mathrm{dVol}_g(\theta) = \inf_k \frac{1}{k}\int_{\mathcal{M}} \ln \|A_{c,z,k}(\theta)\| \ \mathrm{dVol}_g(\theta).$$

Furthermore, $L(f,z) = \lim_{k\to\infty} \frac{1}{k} \ln \|A_{c,z,k}(\theta)\|$ for Vol_g-*a.e.* $\theta \in \mathcal{M}$.

In the case of quasiperiodic operators, we will always write $(\alpha, A_{c,z}(\cdot))$ instead of $(f_{s,\alpha}, A_{c,z}(\cdot))$ and $L(\alpha, z)$ rather than $L(f_{s,\alpha}, z)$ for simplicity.

1.4 Spectral measures and Integrated density of states

Let $\mu_{c,v,f,\theta}$ be the spectral measure of $H_{c,v,f}(\theta)$ associated to δ_0, namely for any Borel set U, we have

$$\mu_{c,v,f,\theta}(U) = (\delta_0, \chi_U(H_{c,v,f}(\theta))\delta_0).$$

We define the density of states measure $dN_{H_{c,v,f}}$ by

$$dN_{H_{c,v,f}}(U) = \int_{\mathbb{T}} \mu_{c,v,f,\theta}(U) \ \mathrm{d}\theta.$$

$N_{H_{c,v,f}}(E) := N_{H_{c,v,f}}(-\infty, E)$ is called the integrated density of states (IDS) of $H_{c,v,f}(\theta)$.

1.5 Reducibility and rotation number

A quasiperiodic cocycle (α, A) is called *L^2-reducible* (or *C^ω-reducible*) if there exists a matrix function $B \in L^2(\mathbb{T}, SL(2,\mathbb{R}))$ (or $B \in C^\omega(\mathbb{T}, PSL(2,\mathbb{R}))$ respectively) and a constant matrix A_* such that

$$B(\theta+\alpha)A(\theta)B^{-1}(\theta) = A_* \quad \text{for a.e. } \theta \in \mathbb{T}, \tag{1.13}$$

Let

$$R_x = \begin{pmatrix} \cos 2\pi x & -\sin 2\pi x \\ \sin 2\pi x & \cos 2\pi x \end{pmatrix}.$$

Any $A \in C^0(\mathbb{T}, PSL(2,\mathbb{R}))$ is homotopic to $\theta \to R_{\frac{n}{2}x}$ for some $n \in \mathbb{Z}$, called the degree of A, denoted by $\deg A = n$.

Assume now that $A \in C^0(\mathbb{T}, SL(2,\mathbb{R}))$ is homotopic to identity. Then there exists $\psi : \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ and $u : \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}^+$ such that

$$A(x) \cdot \begin{pmatrix} \cos 2\pi y \\ \sin 2\pi y \end{pmatrix} = u(x,y) \begin{pmatrix} \cos 2\pi(y+\psi(x,y)) \\ \sin 2\pi(y+\psi(x,y)) \end{pmatrix}.$$

The function ψ is called a lift of A. Let μ be any probability on $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ which is invariant by the continuous map $T : (x,y) \mapsto (x+\alpha, y+\psi(x,y))$, projecting over Lebesgue measure on the first coordinate (for instance, take μ as any accumulation point of $\frac{1}{n}\sum_{k=0}^{n-1} T_*^k \nu$ where ν is Lebesgue measure on $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$). Then the number

$$\rho(\alpha, A) = \int \psi d\mu \mod \mathbb{Z} \tag{1.14}$$

does not depend on the choices of ψ and μ, and is called the fibered rotation number of (α, A).

It can be proved directly by the definition that

$$|\rho(\alpha, A) - x| < C\|A - R_x\|_0. \tag{1.15}$$

The fibered rotation number is invariant under real conjugacies which are homotopic to the identity. In general, if $(\alpha, A^{(1)})$ and $(\alpha, A^{(2)})$ are real conjugate, namely there exists $B \in C^0(\mathbb{T}, PSL(2,\mathbb{R}))$ so that $B^{-1}(x+\alpha)A^{(2)}(x)B(x) = A^{(1)}(x)$ and $\deg B = k$, then

$$\rho(\alpha, A^{(1)}) = \rho(\alpha, A^{(2)}) - k\alpha/2. \tag{1.16}$$

We say that (α, A) is uniformly hyperbolic if there exists continuous splitting $\mathbb{C}^2 = E^s(x) \bigoplus E^u(x)$, $x \in \mathbb{T}$ such that for some constant $C, \eta > 0$ and all $n \geq 0$, $\|A_n(x)v\| \leq Ce^{-\eta n}\|v\|$ for $v \in E^s(x)$ and $\|A_{-n}(x)v\| \leq Ce^{-\eta n}\|v\|$ for $v \in E^u(x)$.

For uniformly hyperbolic cocycles there is the following well-known result.

Theorem 1.5.1 *Let (α, A) be a uniformly hyperbolic cocycle, with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Then $2\rho(\alpha, A) \in \alpha\mathbb{Z} + \mathbb{Z}$.*

1.5.1 Normalized cocycle

Given a quasiperiodic Jacobi matrix $H_{c,v,\alpha}$ with continuous function v and analytic c. Let $\tilde{c}(\theta) = \overline{c(\theta)}$ on $\mathbb{T}$ and its analytic extension off $\mathbb{T}$. Let $|c|(\theta) = \sqrt{c(\theta)\tilde{c}(\theta)}$, then $|c|(\theta)$ is an analytic function which coincides with $|c(\theta)|$ on $\mathbb{T}$.

We define the normalized transfer matrix $\tilde{A}_{c,z}(\theta)$ as below.

$$\tilde{A}_{|c|,z}(\theta) = \frac{1}{\sqrt{|c|(\theta)|c|(\theta-\alpha)}} \begin{pmatrix} z - v(\theta) & -|c|(\theta-\alpha) \\ |c|(\theta) & 0 \end{pmatrix}. \tag{1.17}$$

The advantage of $\tilde{A}_{|c|,z}$ over $A_{c,z}$ is that it is a $\mathrm{SL}(2,\mathbb{R})$ matrix homotopic to identity.

The cocycle $(\alpha, \tilde{A}_{c,z}(\cdot))$ will be called the normalized cocycle.

By (1.14), we can define the rotation number of the normalized cocycle $(\alpha, \tilde{A}_{|c|,E})$, denoted by $\rho(\alpha, \tilde{A}_{|c|,E})$. It is a non-increasing continuous function of energy E.

Let $H_{|c|,v,\alpha}$ be the normalized Jacobi matrix

$$(H_{|c|,v,\alpha}(\theta)u)_n = |c|(\theta+n\alpha)u_{n+1} + |c|(\theta+(n-1)\alpha)u_{n-1} + v(\theta+n\alpha)u_n. \tag{1.18}$$

The following relation between $\rho(\alpha, \tilde{A}_{|c|,E})$ and $N_{H_{|c|,v,\alpha}}(E)$ is well-known.

$$N_{H_{|c|,v,\alpha}}(E) = 1 - 2\rho(\alpha, \tilde{A}_{|c|,E}). \tag{1.19}$$

Note that $N_{H_{|c|,v,\alpha}}(E) = N_{H_{c,v,\alpha}}(E)$ since $H_{|c|,v,\alpha}(\theta)$ and $H_{c,v,\alpha}(\theta)$ differ by a unitary conjugation.

1.6 Extended Harper's model

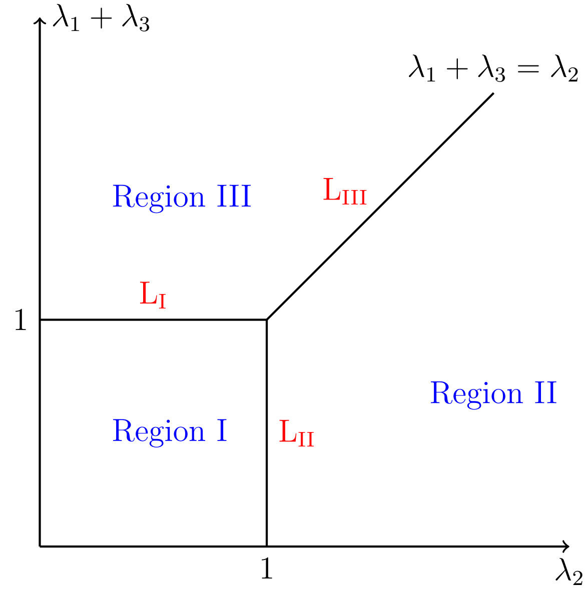
The extended Harper's model (EHM), defined as below, is a prime example of quasiperiodic Jacobi matrix.

$$(H_{\lambda,\alpha}(\theta)u)_n = c_\lambda(\theta + n\alpha)u_{n+1} + \tilde{c}_\lambda(\theta + (n-1)\alpha)u_{n-1} + 2\cos 2\pi(\theta + n\alpha)u_n.$$

where $c_\lambda(\theta) = \lambda_1 e^{-2\pi i(\theta+\frac{\alpha}{2})} + \lambda_2 + \lambda_3 e^{2\pi i(\theta+\frac{\alpha}{2})}$.

Note that in order to distinguish from general Jacobi matrix, we denote the extended Harper's model by $H_{\lambda,\alpha}$.

According to the duality transformation $\sigma : \lambda = (\lambda_1, \lambda_2, \lambda_3) \to \hat{\lambda} = (\frac{\lambda_3}{\lambda_2}, \frac{1}{\lambda_2}, \frac{\lambda_1}{\lambda_2})$, the coupling constant parameter space is naturally divided into three regions.



Region I $0 \le \lambda_1 + \lambda_3 \le 1, 0 < \lambda_2 \le 1$,

Region II $0 \le \lambda_1 + \lambda_3 \le \lambda_2, 1 \le \lambda_2$,

Region III $\max\{1, \lambda_2\} \le \lambda_1 + \lambda_3,\ \lambda_2 > 0$.

1.6.1 Aubry duality of the extended Harper's model

Observation 1.6.1 *σ is a bijective map on* $0 \le \lambda_1 + \lambda_3,\ 0 < \lambda_2$.

(i) $\sigma(\mathrm{I}^\circ) = \mathrm{II}^\circ$, $\sigma(\mathrm{III}^\circ) = \sigma(\mathrm{III}^\circ)$

(ii) *Letting* $\mathrm{L_I} := \{\lambda_1 + \lambda_3 = 1, 0 < \lambda_2 \le 1\}$, $\mathrm{L_{II}} := \{0 \le \lambda_1 + \lambda_3 \le 1, \lambda_2 = 1\}$, *and* $\mathrm{L_{III}} := \{1 \le \lambda_1 + \lambda_3 = \lambda_2\}$, $\sigma(\mathrm{L_I}) = \mathrm{L_{III}}$ *and* $\sigma(\mathrm{L_{II}}) = \mathrm{L_{II}}$.

As σ bijectively maps $\mathrm{III} \cup \mathrm{L_{II}}$ onto itself, the literature refers to $\mathrm{III} \cup \mathrm{L_{II}}$ as the *self-dual regime*. We further divide III into $\mathrm{III}_{\lambda_1=\lambda_3}$ (*isotropic self-dual regime*) and $\mathrm{III}_{\lambda_1\neq\lambda_3}$ (*anisotropic self-dual regime*).

Let $\Sigma_{\lambda,\alpha,\theta}$ be the spectrum of $H_{\lambda,\alpha,\theta}$, and $\Sigma_{\lambda,\alpha} = \cup_{\theta\in\mathbb{T}}\Sigma_{\lambda,\alpha,\theta}$. It is a well-known result that $\Sigma_{\lambda,\alpha,\theta}$ is independent of θ if α is irrational.

By Aubry duality, the spectrum of $H_{\lambda,\alpha}$ and the spectrum of its dual model $H_{\hat{\lambda},\alpha}$ are connected to each other in the following way.

$$\Sigma_{\lambda,\alpha} = \lambda_2 \Sigma_{\hat{\lambda},\alpha}. \tag{1.20}$$

Moreover, the integrated density of states $N_{\lambda,\alpha}(E)$ of $H_{\lambda,\alpha}$ coincides with the IDS $N_{\hat{\lambda},\alpha}(E/\lambda_2)$ of $H_{\hat{\lambda},\alpha,\theta}$.

1.6.2 Lyapunov exponent of the extended Harper's model

An remarkable feature of the extended Harper's model is that Lyapunov exponents when restricted to the spectrum are constant and depend only on λ. Indeed, according to Avila's Global theory, we have

Theorem 1.6.2 *[[37], see also Appendix A] The extended Harper's model is super-critical in region* I^{o} *and sub-critical in region* II^{o}*. Indeed, if* λ *belongs to region* I^{o},

- *for any* $E \in \Sigma_{\lambda,\alpha}$,

$$L(\alpha, E) \equiv L(\lambda) = \ln \frac{1 + \sqrt{1 - 4\lambda_1\lambda_3}}{\max(\lambda_1+\lambda_3, \lambda_2) + \sqrt{\max(\lambda_1+\lambda_3,\lambda_2)^2 - 4\lambda_1\lambda_3}} > 0. \tag{1.21}$$

- $\hat{\lambda}$ *belongs to region II, furthermore, for any* $E \in \Sigma_{\hat{\lambda},\alpha}$,

$$L(\alpha, E) = L(\alpha, A_{c_{\hat{\lambda}},E}(\cdot + i\epsilon)) = L(\alpha, \tilde{A}_{|c_{\hat{\lambda}}|,E}(\cdot + i\epsilon)) = 0 \quad \text{for} \quad |\epsilon| \le \frac{L(\lambda)}{2\pi}. \tag{1.22}$$

1.6.3 Presence of singularities

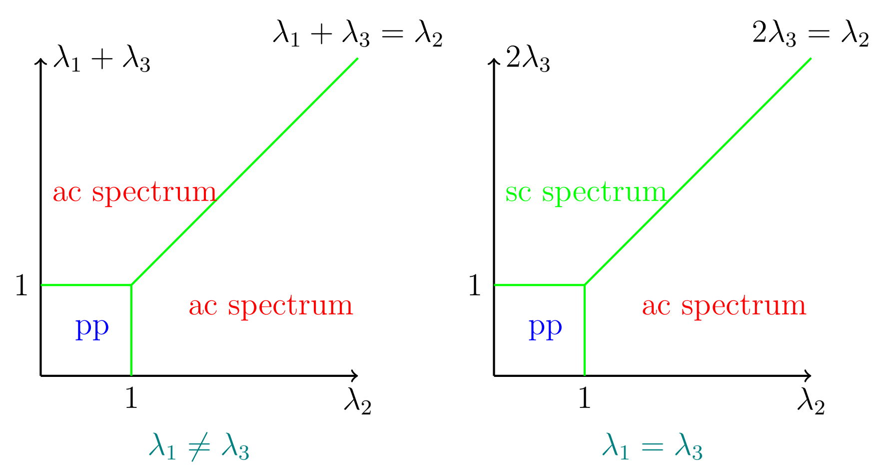
Another interesting phenomenon of extended Harper's model is the presence of singularities.

Observation 1.6.3 *(e.g. [8]) $c_\lambda(\theta)$ could take zero value when the parameters λ satisfy some certain conditions. Indeed,*

- *when $\lambda_1 = \lambda_3 \geq \frac{\lambda_2}{2}$, singular points are $\theta_1 = \frac{1}{2\pi}\arccos\left(-\frac{\lambda_2}{2\lambda_1}\right) - \frac{\alpha}{2}$ and $\theta_2 = -\frac{1}{2\pi}\arccos\left(-\frac{\lambda_2}{2\lambda_1}\right) - \frac{\alpha}{2}$ (notice that when $\lambda_1 = \frac{\lambda_2}{2}$ there is a single singular point $\theta_1 = \theta_2 = \frac{1}{2} - \frac{\alpha}{2}$).*
- *when $\lambda_1 \neq \lambda_3$ and $\lambda_1 + \lambda_3 = \lambda_2$, the singular point is $\theta_1 = \frac{1}{2} - \frac{\alpha}{2}$.*

1.6.4 Spectral properties of the extended Harper's model

Due to [43, 8], we have the following spectral properties for the extended Harper's model.



- pp for Diophantine α and a.e. θ due to Jitomirskaya-Koslover-Schulteis [43].
- ac and sc for any irrational α and a.e. θ due to Avila-Jitomirskaya-Marx [8].

Chapter 2

Full measure reducibility and localization for Jacobi operators: a topological criterion

2.1 Introduction

In this chapter we study the general class of Jacobi operators

$$(H_{c,v,\alpha}(\theta)u)_n = c(\theta+n\alpha)u_{n+1} + \tilde{c}(\theta+(n-1)\alpha)u_{n-1} + v(\theta+n\alpha)u_n, \tag{2.1}$$

where $c(\theta) = \sum_k \hat{c}_k e^{2\pi i k(\theta+\frac{\alpha}{2})} \in C^\omega(\mathbb{T})$, $\tilde{c}(\theta) \in C^\omega(\mathbb{T})$, $\tilde{c}(\theta) = \overline{c(\theta)}$ on $\mathbb{T}$, and $v(\theta) = \sum_k \hat{v}_k e^{2\pi i k\theta} \in C^\omega(\mathbb{T})$. We will assume $\hat{v}_k = \overline{\hat{v}_{-k}}$, $\hat{c}_k \in \mathbb{R}$. Such operators arise as effective Hamiltonians in a tight-binding description of a crystal subject to a weak external magnetic field, with c, v reflecting the lattice geometry and the allowed electron hopping between lattice sites. The prime example, both in math and in physics literature, is the extended Harper's model, see (3).

The Aubry dual of $H_{c,v,\alpha}$ is an operator $\tilde{H}_{c,v,\alpha}$ defined by

$$(\tilde{H}_{c,v,\alpha}(x)u)_m = \sum_{m'} d_{m'}(c,v)(x)u_{m-m'}, \tag{2.2}$$

where $d_{m'}(c,v)(x) = \hat{c}_{m'} e^{2\pi i(x-\frac{m'}{2}\alpha)} + \hat{v}_{-m'} + \hat{c}_{-m'} e^{-2\pi i(x-\frac{m'}{2}\alpha)}$.

The Aubry duality can be explained by the magnetic nature and corresponding gauge invariance of operators $H_{c,v,\alpha}$ [63] and has been formulated and explored on different levels, e.g. [63], [32], [7]. The dynamical formulation of Aubry duality is an observation that if $\tilde{H}_{c,v,\alpha}(\theta)$ has an eigenvalue at E with respective eigenvector $\{u_n\}$, then, considering its Fourier transform, $u(x) := \sum_{n\in\mathbb{Z}} u_n \mathrm{e}^{2\pi i n x} \in L^2(\mathbb{T}) \setminus \{0\}$ and letting

$$M_\theta(x) = \begin{pmatrix} u(x) & u(-x) \\ \mathrm{e}^{-2\pi i\theta} u(x-\alpha) & \mathrm{e}^{2\pi i\theta} u(-(x-\alpha)) \end{pmatrix}, \tag{2.3}$$

M_θ provides an L^2 semiconjugacy between the transfermatrix cocycle of $H_{c,v,\alpha}$ and the rotation $R_\theta = \begin{pmatrix} \mathrm{e}^{2\pi i\theta} & 0 \\ 0 & \mathrm{e}^{-2\pi i\theta} \end{pmatrix}$. For θ that are not α-rational, $\det M_\theta(x)$ doesn't vanish for a.e. x [8], leading to reducibility of the transfermatrix cocycle of $H_{c,v,\alpha}$ to a constant rotation R_θ. In particular, pure point spectrum for a.e. θ of $\tilde{H}_{c,v,\alpha}(\theta)$ leads to reducibility for cocycles of $H_{c,v,\alpha}$ for a.e. E with respect to the density of states [65, 7], with the quality of reducibility governed by the rate of decay of u_n. As there are well developed methods to prove localization (thus exponential decay of the eigenfunctions) in various applications, this can be used to establish further interesting consequences [7, 8, 34].

With the development of recent powerful methods [9, 5, 3] to establish non-perturbative reducibility directly and independently of localization for the dual model, the reverse direction: obtaining localization for $\tilde{H}_{c,v,\alpha}$ from reducibility of $H_{c,v,\alpha}$, first used in a more restricted form back in [20], started gaining prominence. In the Schrödinger case, reducibility provides a direct construction of eigenfunctions for the dual model (with the decay governed by the quality of reducibility), so their completeness becomes the main issue. This has been considered a nontrivial question even for the almost Mathieu family. It had been conjectured for a long time [51] that $\lambda = e^\beta$, where β is the upper rate of exponential growth of denominators of continued fractions approximants to α (see (5.1.2)), is the phase transition line from purely singular continuous spectrum to pure point spectrum. A combination of the almost reducibility conjecture [3] and techniques of [5, 35, 70] led to establishing reducibility throughout the dual of the entire conjectured localization region, yet completeness of the resulting eigenfunctions remained a problem. This was recently resolved in [12] where the authors used delicate quantitive information on the reducibility and therefore dual eigenfunctions with certain rate of decay to prove the pure point spectrum part of the conjecture. More recently, in [42], the authors obtained an elementary proof of complete localization for the dual model under the assumption of only certain L^2-reducibility of the Schrödinger cocycle for $H(\theta)$ for almost all energies with respect to the density of states measure.

For the Jacobi case the situation is more problematic. It was noticed (albeit in a different form) in [63] that for $c \not\equiv 1$ the existence of reducibility at E for the cocycle of $H_{c,v,\alpha}$ may not lead to E being an eigenvalue of $\tilde{H}_{c,v,\alpha}$. The difficulty is also reflected in the extended Harper's model (see Section 1.6). On the positive side, in the dual regions I and II, we do in general have purely absolutely continuous spectrum, which is always associated to reducibility, in region II, and pure point spectrum in region I [43, 8]. However on the negative side, purely absolutely continuous spectrum for a.e. θ has been proved throughout the whole self-dual region III in the anisotropic case [8]. Thus whether reducibility implies localization for the dual model could depend on c, v, α, and even the existence of dual eigenvectors, automatic in the Schrödinger case, becomes an issue.

In this paper, we answer this question for analytic c. We establish an if-and-only-if topological criterion in terms of the function c only, for the reducibility for $H_{c,v,\alpha}$ to imply pure point spectrum of $\tilde{H}_{c,v,\alpha}$. Thus we extend the result of [42] to the Jacobi setting in a sharp way and also describe exactly what happens in the region to which it does not extend. It turns out the *winding number* $w(c)$ of $c(\theta)$ (see (2.4)) is the key quantity.

With the normalized transfer matrix cocycle $(\alpha, \tilde{A}_{|c|,E})$ defined in (1.17), we have

Theorem 2.1.1 *Suppose for $c(\theta) \in C^{\omega}_{\frac{h}{2\pi}}(\mathbb{T}, \mathbb{C}\backslash\{0\})$, $\beta(\alpha) < h$, the normalized cocycles $(\alpha, \tilde{A}_{|c|,E})$ are L^2-reducible for a.e. E with respect to the density of states measure. Then for a.e. $x \in \mathbb{T}$*

- *if* $w(c) = 0$, *the spectra of* $\tilde{H}_{c,v,\alpha}(x)$ *are pure point.*

- *if* $w(c) \neq 0$, *the spectra of* $\tilde{H}_{c,v,\alpha}(x)$ *are purely absolutely continuous.*

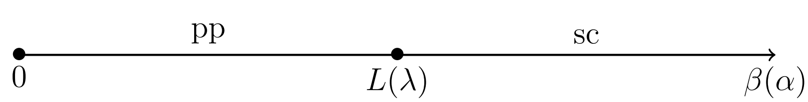
As an important application, we obtain sharp arithmetic phase transition result for the extended Harper's model (see (3)) in the positive Lyapunov exponent region.

Theorem 2.1.2 *When* $L(\lambda) > 0$,

- *if* $\beta(\alpha) < L(\lambda)$, $H_{\lambda,\alpha}(\theta)$ *has pure point spectrum for a.e.* θ.

- *if* $L(\lambda) < \beta(\alpha)$, $H_{\lambda,\alpha}(\theta)$ *has purely singular continuous spectrum for a.e.* θ.

Remark 2.1.1 $L(\lambda) > 0$ *if and only if* $1 > \max(\lambda_1 + \lambda_3, \lambda_2)$, *see Theorem 1.6.2.*



The second statement of Theorem 2.1.2 does not require a specific form of $c(\theta)$ and holds for general analytic c that are even allowed to vanish on $\mathbb{T}$. Namely, one can define a coefficient $\delta_c(\alpha,\theta) \in [-\infty, \infty]$, dependent on $c(\theta)$ through its zeros on $\mathbb{T}$ only, see (2.36), and satisfying $\delta_c(\alpha,\theta) = \beta(\alpha)$ for a.e. θ, so that

Theorem 2.1.3 *For any Lipshitz* v, $H_{c,v,\alpha}(\theta)$ *has no eigenvalues on* $\{E : L(E) < \delta_c(\alpha,\theta)\}$. *In particular, for a.e.* θ *it has no eigenvalues on* $\{E : L(E) < \beta(\alpha)\}$.

This immediately implies

Corollary 2.1.1 *If* $L(E) > 0$ *for a.e.* E *(in particular, if there exists* $\theta_0 \in \mathbb{T}$ *with* $c(\theta_0) = 0$*), then* $H_{c,v,\alpha}(\theta)$ *has purely singular continuous spectrum on* $\{E : L(E) < \delta_c(\alpha,\theta)\}$. *If* $L(E) > 0$ *for a.e.* E *and* c *does not vanish on* $\mathbb{T}$, *then* $H_{c,v,\alpha}(\theta)$ *has purely singular continuous spectrum on* $\{E : L(E) < \beta(\alpha)\}$, *for all* θ.

We prove Theorem 2.1.1 in Section 2.3. We first show that zero $w(c)$ ensures that elements of the reducibility matrix can be used to construct eigenfunctions for the dual model. Then we employ an argument of [42] to show completeness of those eigenfunctions. To prove the second part we establish a unitary conjugacy in $L^2(\mathbb{T} \times \mathbb{Z})$ between $\tilde{H}_{c,v,\alpha}$ and $\tilde{H}_{s,v,\alpha}$ for a certain s with $w(s) = 0$, ensuring that, by part one, $\tilde{H}_{s,v,\alpha}(x)$ has pure point spectrum for a.e. x. We then use those eigenfunctions to construct a large family of vector-valued functions $\psi_x^{q,\ell,j}(\cdot)$ such that for a.e. x the corresponding spectral measures are constant in x. Finally, we prove their absolute continuity based again on the reducibility for the original model and argue completeness. Once Theorem 3.1.4 is proved, to establish the pure point part of Theorem 2.1.2 all we need is the dual reducibility which follows quickly from a combination of [3, 5], similarly to the argument of [12]. This is done in Section 2.5. In fact, Theorem 2.1.2 is an extension of the main theorem of [12], and specializes to it when $\lambda_1 = \lambda_3 = 0$.

The singular continuous part as well as the general Theorem 2.1.3 are proved in Section 2.4. The result is similar in spirit to the recent theorems on meromorphic potentials [52, 53]. The non-singular case is simpler and could follow similarly to the singular continuous part of [12] but we choose to treat it together with the more involved singular case. While the Jacobi situation is quite different, the common feature of singular Jacobi and meromorphic cocycles is their singularity, leading to certain shared phenomena. It is an interesting question whether the first statement of the Corollary 2.1.1 is sharp at least in some situations, so whether like in the Maryland model [52], there is pure point spectrum for the complementary set of θ. It is also interesting to see whether the second statement is sharp for general analytic potentials (something still far from reach even in the Schrödinger case).

2.2 Preliminaries

2.2.1 Winding number

For $c(\theta) \in C^\omega(\mathbb{T}, \mathbb{C}\backslash\{0\})$ on $\mathbb{T}$, let

$$w(c) = \int_{\mathbb{T}} \frac{c'(\theta)}{c(\theta)}\, \mathrm{d}\theta \tag{2.4}$$

be the winding number of $c(\theta)$. It describes how many times does the graph of $c(\theta)$ circle around the origin when θ goes along $\mathbb{T}$.

2.2.2 Integrated density of states of dual operators

The Aubry duality between $H_{c,v,\alpha}$ and $\tilde{H}_{c,v,\alpha}$ implies the following relation between their density of states measures, see e.g. a particular case of Theorem 2 in [63],

$$dN_{H_{c,v,\alpha}}(E) = dN_{\tilde{H}_{c,v,\alpha}}(E). \tag{2.5}$$

2.2.3 Cocycles and Lyapunov exponent

We have the following uniform control of the norm of transfer matrices.

Lemma 2.2.1 *(e.g.[7]) Let (α, A) be a continous cocycle, then for any $\delta > 0$ there exists $C_\delta > 0$ such that for any $n \in \mathbb{N}$ and $\theta \in \mathbb{T}$ we have*

$$\|A_n(\theta)\| \leq C_\delta e^{(L(\alpha,A)+\delta)n}.$$

Remark 2.2.1 *If we apply the previous lemma to one dimensional cocycle, we have that for any continuous function z, if $\ln|z(\theta)| \in L^1(\mathbb{T})$ then for any $\epsilon > 0$ there exists constant $C > 0$ so that for any $a \leq b \in \mathbb{Z}$.*

$$\prod_{k=a}^{b} |z(\theta + k\alpha)| \leq C e^{(b-a+1)(\int_{\mathbb{T}} \ln|z(\theta)|d\theta+\epsilon)} \text{ for any } \theta \in \mathbb{T}.$$

2.3 Proof of Theorem 2.1.1

The proof of Theorem 3.1.4 mainly relies on Lemmas 2.3.1 and 2.3.6.

2.3.1 Proof of the first part of Theorem 3.1.4

Lemma 2.3.1 *Let $s(\theta) = c(\theta)e^{-2\pi i k_0(\theta+\frac{\alpha}{2})}$, where $k_0 = w(c)$. Then under the conditions of Theorem 3.1.4, the spectra of the dual Hamiltonians $\tilde{H}_s(x)$ are pure point for a.e. x.*

Proof: We start with

Lemma 2.3.2 *Suppose* $s(\theta) = \sum_{k\in\mathbb{Z}} \hat{s}_k e^{2\pi i k(\theta+\frac{\alpha}{2})}$, $\hat{s}_k \in \mathbb{R}$. *Suppose* $s(\theta)$ *is analytic and nonzero on* $|\mathrm{Im}\theta| \leq \frac{h}{2\pi}$ *and* $w(s) = 0$. *Then if* $\beta(\alpha) < h$ *there exists analytic funtion* $f(\theta)$ *such that*

$$\frac{s(\theta)}{|s|(\theta)} = e^{f(\theta+\alpha)-f(\theta)}.$$

Proof: Since $w(s(\cdot + i\epsilon)) \equiv 0$, we can properly define $\log s(\theta)$ and $\arg s(\theta)$ on $|\mathrm{Im}\theta| \leq \frac{h}{2\pi}$. Now that obviously $\tilde{s}(\theta) = s(-\theta-\alpha)$, we have

$$\int_{\mathbb{T}} \ln|s(\theta)|\ \mathrm{d}\theta = \int_{\mathbb{T}} \ln|\tilde{s}(\theta)|\ \mathrm{d}\theta. \tag{2.6}$$

and

$$\int_{\mathbb{T}} \arg s(\theta)\ \mathrm{d}\theta - \int_{\mathbb{T}} \arg \tilde{s}(\theta)\ \mathrm{d}\theta = \int_{\mathbb{T}} \arg s(\theta)\ \mathrm{d}\theta - \int_{\mathbb{T}} \arg s(-\theta-\alpha)\ \mathrm{d}\theta = 0. \tag{2.7}$$

Combining (2.6), (2.7) with $\beta(\alpha) < h$ we are able to solve a coholomogical equation, hence there exists an analytic function $g(\theta)$ so that

$$g(\theta+\alpha) - g(\theta) = \ln s(\theta) - \ln \tilde{s}(\theta).$$

This clearly implies

$$\frac{s(\theta)}{\tilde{s}(\theta)} = e^{g(\theta+\alpha)-g(\theta)}.$$

Hence

$$\frac{s(\theta)}{|s|(\theta)} = e^{f(\theta+\alpha)-f(\theta)},$$

where $f(\theta) = \frac{1}{2}g(\theta)$. □

Now let us come back to the proof of Lemma 2.3.1. We have for a.e. E with respect to the density of states measure $dN_{v,c,\alpha}$ [1] , there is $B_E \in L^2(\mathbb{T}, SL(2,\mathbb{R}))$ so that

$$B_E^{-1}(\theta+\alpha)\tilde{A}_{|c|,E}(\theta)B_E(\theta) = A_*, \tag{2.8}$$

[1]It is the same as $dN_{H_{s,v,\alpha}} = dN_{H_{|c|,v,\alpha}} = dN_{H_{|s|,v,\alpha}}$, since $H_{c,v,\alpha}(\theta)$, $H_{s,v,\alpha}(\theta)$, $H_{|c|,v,\alpha}(\theta)$ and $H_{|s|,v,\alpha}(\theta)$ differ from each other by unitary conjugations for any fixed $\theta \in \mathbb{T}$.

where A_* is a constant matrix. Since for $\theta \in \mathbb{T}$, $\tilde{A}_{|c|,E}(\theta) = \tilde{A}_{|s|,E}(\theta)$, we have

$$B_E^{-1}(\theta+\alpha)\tilde{A}_{|s|,E}(\theta)B_E(\theta) = A_*. \tag{2.9}$$

By [5] (see Lemma 1.4 therein), if $(\alpha, \tilde{A}_{|s|,E})$ is L^2-reducible for a.e. E with respect to the density of states measure, then $(\alpha, \tilde{A}_{|s|,E})$ is C^ω-reducible for $E \in U$ where U is a set with $dN_{v,|s|,\alpha}(U) = 1$. Thus we could assume (2.9) holds for $B_E \in C^\omega(\mathbb{T}, PSL(2,\mathbb{R}))$.

Next we are going to show the following:

Lemma 2.3.3 *For a.e. E with respect to the density of states measure, there exists $\tilde{B}_E(\theta) \in C^\omega(\mathbb{T}, SL(2,\mathbb{R}))$ so that*

$$\tilde{B}_E^{-1}(\theta+\alpha)\tilde{A}_{|s|,E}(\theta)\tilde{B}_E(\theta) = R_{\rho_{|s|}(E)}. \tag{2.10}$$

Proof: By (2.9), we already have

$$B_E^{-1}(\theta+\alpha)\tilde{A}_{|s|,E}(\theta)B_E(\theta) = A_*,$$

where $B_E(\theta) \in C^\omega(\mathbb{T}, PSL(2,\mathbb{R}))$ and A_* is a constant matrix. We could assume $A_* = R_\phi$ or $A_* = J_\pm = \begin{pmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{pmatrix}$ or $A_* = \pm\mathrm{Id}$. If E is such that $A_* = J_\pm$ or $\pm\mathrm{Id}$, we get $\rho_{|s|}(E) \in \mathbb{Z}\alpha + \mathbb{Z}$. Therefore such E's form a measure zero set with respect to the density of states measure. Now, let's consider the case $A_* = R_\phi$. By (1.16), this implies $\phi = \rho_{|s|}(E) - k\alpha/2$, where $k = \deg B_E$. Now let $\tilde{B}_E(\theta) = B_E(\theta)R_{-\frac{k}{2}\theta}$. We have

$$\tilde{B}_E^{-1}(\theta+\alpha)\tilde{A}_{|s|,E}(\theta)\tilde{B}_E(\theta) = R_{\rho_{|s|}(E)}.$$

Note that $\deg \tilde{B}_E = 0$, thus $\tilde{B}_E(\theta) \in C^\omega(\mathbb{T}, SL(2,\mathbb{R}))$. □

Now by Lemma 2.3.3, we could assume there exists $\tilde{B}_E(\theta) \in C^\omega(\mathbb{T}, GL(2,\mathbb{C}))$ so that

$$\tilde{B}_E^{-1}(\theta+\alpha)\tilde{A}_{|s|,E}(\theta)\tilde{B}_E(\theta) = \begin{pmatrix} e^{2\pi i \rho_{|s|}(E)} & 0 \\ 0 & e^{-2\pi i \rho_{|s|}(E)} \end{pmatrix}. \tag{2.11}$$

By Lemma 2.3.2, there exists analytic $f(\theta)$ so that $s(\theta) = |s|(\theta)e^{f(\theta+\alpha)-f(\theta)}$. Then by (2.11), we have

$$
\begin{aligned}
&\begin{pmatrix} e^{2\pi i\rho_{|s|}(E)} & 0 \\ 0 & e^{-2\pi i\rho_{|s|}(E)} \end{pmatrix} \\
=&B_E^{-1}(\theta+\alpha)\tilde{A}_{|s|,E}(\theta)B_E(\theta) \\
=&\frac{s(\theta)}{\sqrt{|s|(\theta)|s|(\theta-\alpha)}}\{M_s(\theta+\alpha)B_E(\theta+\alpha)\}^{-1}A_{s,E}(\theta)M_s(\theta)B_E(\theta) \\
=&\left\{\frac{M_s(\theta+\alpha)B_E(\theta+\alpha)e^{-\frac{f(\theta+\alpha)}{2}}}{\sqrt{|s|(\theta)}}\right\}^{-1} A_{s,E}(\theta)\left\{\frac{M_s(\theta)B_E(\theta)e^{-\frac{f(\theta)}{2}}}{\sqrt{|s|(\theta-\alpha)}}\right\}.
\end{aligned} \tag{2.12}
$$

Let $\tilde{D}_E(\theta) = \begin{pmatrix} D_{E,11}(\theta) & D_{E,12}(\theta) \\ D_{E,21}(\theta) & D_{E,22}(\theta) \end{pmatrix} := \{\frac{M_s(\theta)B_E(\theta)e^{-\frac{f(\theta)}{2}}}{\sqrt{|s|(\theta-\alpha)}}\}$. (2.12) yields that

(2.13)

$$(E-v(\theta))D_{E,11}(\theta) = e^{2\pi i\rho_{|s|}(E)}s(\theta)D_{E,11}(\theta+\alpha) + e^{-2\pi i\rho_{|s|}(E)}\tilde{s}(\theta-\alpha)D_{E,11}(\theta-\alpha),$$

(2.14)

$$(E-v(\theta))D_{E,21}(\theta) = e^{-2\pi i\rho_{|s|}(E)}s(\theta)D_{E,21}(\theta+\alpha) + e^{2\pi i\rho_{|s|}(E)}\tilde{s}(\theta-\alpha)D_{E,21}(\theta-\alpha).$$

We now can follow the argument of [42]. We are going to show that

Lemma 2.3.4 *For a.e. x, $\tilde{H}_{s,v,\alpha}(x)$ has a complete set of normalized eigenfunctions with simple eigenvalues.*

Proof: As mentioned, this proof is essentially from [42], we include it here for completeness. Since $\rho_{|s|} : \mathbb{R} \to [0, \frac{1}{2}]$ is bijective on the spectrum, for each $x \in [0, \frac{1}{2}]$ there exists $E(x)$ such that $\rho_{|s|}(E(x)) = x$. By (2.13) and a straightforward computation, there is F_1 with $|F_1| = 0$ so that for $x \in [0, \frac{1}{2}] \setminus F_1$, $\tilde{H}_{s,v,\alpha}(x)$ has a normalized eigenfunction $\{u_k(x)\}_k = \left\{\frac{\hat{D}_{E(x),11}(k)}{\|\hat{D}_{E(x),11}\|_{L^2(\mathbb{T})}}\right\}_k$ at energy $E(x)$. Also for $x \in [-\frac{1}{2}, 0] \setminus F_2$, $|F_2| = 0$, $\tilde{H}_{s,v,\alpha}(x)$ has a normalized eigenfunction $\{u_k(x)\}_k = \left\{\frac{\hat{D}_{E(x),12}(k)}{\|\hat{D}_{E(x),12}\|_{L^2(\mathbb{T})}}\right\}_k$ at energy $E(-x)$. Let

$$F = (F_1 + \mathbb{Z}\alpha) \cup (F_2 + \mathbb{Z}\alpha) \cup \{x \in [-\frac{1}{2}, \frac{1}{2}] \mid 2x \in \mathbb{Z}\alpha + \mathbb{Z}\}. \tag{2.15}$$

Clearly, $|F| = 0$. Now for every $x \in F^c$, every $n \in \mathbb{Z}$, $\tilde{H}_{s,v,\alpha}(x+n\alpha)$ has a normalized eigenfunction $\{u_k(x+n\alpha)\}_k$ at energy $E(x+n\alpha)$. Also for different m and n, $E(x+m\alpha) \neq E(x+n\alpha)$, since otherwise we would have $x + m\alpha = -(x+n\alpha) \mod \mathbb{Z}$, which is impossible due to our definition of F, (2.15). Let $E_n(x) := E(x+n\alpha)$, $P_n(x)$ be the spectral projection of $\tilde{H}_{s,v,\alpha}(x)$ onto $E_n(x)$ and $P(x) = \sum_{n\in\mathbb{Z}} P_n(x)$. Notice that $\tilde{H}_{s,v,\alpha}(x+n\alpha) = T^{-n}\tilde{H}_{s,v,\alpha}(x)T^n$, where $(Tu)_k = u_{k-1}$. Thus

$$P_n(x)T^n u(x+n\alpha) = T^n u(x+n\alpha),$$

in other words, $T^n u(x)$ is in the range of $P_n(x-n\alpha)$. Thus for any $l \in \mathbb{Z}$, $\langle \delta_l, P_n(x-n\alpha)\delta_l\rangle \geq |\langle \delta_l, T^n u(x)\rangle|^2$, therefore $\sum_{n\in\mathbb{Z}}\langle \delta_l, P_n(x-n\alpha)\delta_l\rangle \geq 1$. We have

$$1 \geq \int_{F^c} \langle \delta_l, P(x)\delta_l\rangle = \int_{F^c} \sum_{n\in\mathbb{Z}} \langle \delta_l, P_n(x-n\alpha)\delta_l\rangle \geq 1.$$

This implies for a.e. x, $\langle \delta_l, P(x)\delta_l\rangle = 1$ for every l, therefore $P(x) = 1$. Thus for a.e. $x \in \mathbb{T}$, $\cup_{n\in\mathbb{Z}} T^n u(x+n\alpha)$ forms a complete set of eigenfunctions and $\cup_{n\in\mathbb{Z}} E(x+n\alpha)$ forms the eigenvalues. □

Note that this immediately implies Lemma 2.3.1, and thus also the first part of Theorem 3.1.4, since when $w(c) = 0$, we have $\tilde{H}_{s,v,\alpha} = \tilde{H}_{c,v,\alpha}$. □

As a byproduct of full measure C^ω-reducibility (2.9), we could obtain the following result about the absolute continuity of the density of states measure which will play an important role in the proof of Lemma 2.3.6.

Lemma 2.3.5 *The density of states measure of $H_{c,v,\alpha}$ (and thus of $H_{s,v,\alpha}$, $\tilde{H}_{c,v,\alpha}$ and $\tilde{H}_{s,v,\alpha}$) is absolutely continuous.*

Proof: By subordinacy theory [54], $(\alpha, \tilde{A}_{|c|,E})$ being analytically reducible for $E \in U$ implies that for any θ, the singular part of the spectral measure $\mu_{|c|,v,\alpha,\theta}$ of $H_{|c|,v,\alpha}(\theta)$ gives zero weight to U. By Lemma 2.3.3, this implies $dN_{H_{|c|,v,\alpha}} = dN_{H_{c,v,\alpha}} = dN_{H_{s,v,\alpha}}$ are absolutely continuous. By (2.5) and Footnote 1, we get that $dN_{\tilde{H}_{c,v,\alpha}} = dN_{\tilde{H}_{s,v,\alpha}}$, the density of states measures of the dual Hamiltonians $\tilde{H}_{c,v,\alpha}$ and $\tilde{H}_{s,v,\alpha}$, are absolutely continuous. □

2.3.2 Proof of the second part of Theorem 3.1.4

Lemma 2.3.6 *If $w(c) = k_0 \neq 0$, the spectra of $\tilde{H}_{c,v,\alpha}$ are purely absolutely continuous for a.e. x.*

Proof of Lemma 2.3.6

The plan of the proof is to find a unitary transformation of $L^2(\mathbb{T} \times \mathbb{Z})$ relating $\tilde{H}_{c,v,\alpha}$ to $\tilde{H}_{s,v,\alpha}$, and prove that the (already established in the first part) a.e. pure point spectrum of $\tilde{H}_{s,v,\alpha}(x)$ for a.e. x leads to absolutely continuous spectrum of $\tilde{H}_{c,v,\alpha}(x)$ for a.e. x.

Let us introduce two unitary transformations on $\mathcal{H} = L^2(\mathbb{T} \times \mathbb{Z})$,

$$(U_R\psi)(x,n) = \int_0^1 e^{2\pi i\beta n} \sum_{p\in\mathbb{Z}} \psi(\beta,p)e^{2\pi ip(x+n\alpha)}\, d\beta. \tag{2.16}$$

$$(U_k\phi)(x,n) = e^{2\pi i(nk(\frac{n\alpha}{2}+x))}\psi(x,n). \tag{2.17}$$

U_R, first introduced in [23], is just the Aubry duality transformation, also given in a more compact form as

$$U_R\psi(x,n) = \hat{\psi}(n, x+\alpha n), \tag{2.18}$$

where $\hat{\psi} \in L^2(\mathbb{Z}\times\mathbb{T})$ is the Fourier transform. Operator U_k, first introduced in [63], is unitary on each fiber. We have

$$(U_R^{-1}\psi)(x,n) = \int_0^1 e^{-2\pi i\beta n} \sum_{p\in\mathbb{Z}} \psi(\beta,p)e^{-2\pi ip(x+n\alpha)}\, d\beta. \tag{2.19}$$

Define $(S_m\psi)(x,n) = \psi(x+m\alpha, n-m)$. Then $(S_m v_{l,j})(x,n) = v_{l,j}(x+m\alpha, n-m) = v_{l,m+j}(x,n)$.

Lemma 2.3.7 *The following hold*

$$(U_RS_l\psi)(x,n) = e^{2\pi ilx}(U_R\psi)(x,n) \tag{2.20}$$

$$(U_R^{-1}S_l\psi)(x,n) = e^{-2\pi ilx}(U_R^{-1}\psi)(x,n) \tag{2.21}$$

$$(U_kS_l\psi)(x,n) = e^{2\pi ilk(x+\frac{l\alpha}{2})}(S_lU_k\psi)(x,n). \tag{2.22}$$

Proof: Straightforward computation. □

Define operators $H_{c,v,\alpha}$, $\tilde{H}_{c,v,\alpha}$ as acting on $\mathcal{H}$ via direct integrals in x of $H_{c,v,\alpha}(x)$ and $\tilde{H}_{c,v,\alpha}(x)$. Then one way to formulate the Aubry duality is

Lemma 2.3.8

$$\tilde{H}_{c,v,\alpha} = U_R^{-1} H_{c,v,\alpha} U_R. \tag{2.23}$$

Proof: A computation using Lemma 2.3.7. □

Now we establish a connection between $\tilde{H}_{s,v,\alpha}$ and $\tilde{H}_{c,v,\alpha}$. It is given by the following

Lemma 2.3.9

$$\tilde{H}_{c,v,\alpha} = (U_R^{-1} U_{k_0} U_R) \tilde{H}_{s,v,\alpha} (U_R^{-1} U_{k_0} U_R)^{-1}. \tag{2.24}$$

Proof: A more involved computation using Lemma 2.3.7. □

By Lemma 2.3.4, for $x \in F^c$ with $|F| = 0$, $\tilde{H}_{s,v,\alpha}(x)$ has a complete set of normalized eigenfunctions with simple eigenvalues. First, we are going, following [32], to prove there is a covariant measurable enumeration of this set.

For any $x \in F^c$, let $u(x, \cdot)$ be one of its normalized eigenfunctions. Define $j(u(x))$ be the leftmost maximum for $|u(x, \cdot)|$. We fix $u(x, \cdot)$ by requiring $u(x, j) > 0$ and say it is attached to j. The key observation is that the argument of Section 2 of [32], while formulated there for discrete one dimensional Schrödinger operators, works verbatim for any dicrete one-dimensional operator with simple eigenvalues [2]. Thus we get for a.e. x a complete set of eigenfunctions $\{v_{l,j}(x, \cdot)\}_{l,j}$ with eigenvalues $\{e_{l,j}(x)\}$ so that

1. for each fixed l, j, $v_{l,j}(x, \cdot)$ and $e_{l,j}(x)$ are measurable functions of x.
2. $\{v_{l,j}(x, \cdot)\}_j$ are attached to j.
3. $v_{l,j}(x, j) \geq v_{l+1,j}(x, j)$. If the equality holds then $e_{l,j}(x) > e_{l+1,j}(x)$. [3]

[2] The existence of measurable enumeration of eigenfunctions was proved, in great generality in [31]. However, since we need a covariant representation satisfying (2.27) the argument of [32] is better suited to our needs

[3] For fixed l, j, generally, $v_{l,j}(x)$ may vanish identically on a positive measure set of x

By simplicity of the eigenvalues, for any $(l,j) \neq (l',j')$ we have

$$\sum_{n\in\mathbb{Z}} \overline{v_{l,j}(x,n)} v_{l',j'}(x,n) = 0. \tag{2.25}$$

Since $\tilde{H}_{s,v,\alpha}(x+p\alpha) = T^{-p}\tilde{H}_{s,v,\alpha}(x)T^{p}$, where $T\psi(n) = \psi(n-1)$, we have

$$v_{l,j}(x+p\alpha, \cdot - p) = v_{l,j+p}(x,\cdot). \tag{2.26}$$

Therefore by (2.25) and (2.26), for any l, l', any $p \neq 0$,

$$\sum_{n} \overline{v_{l,j}(x+p\alpha, n-p)} v_{l',j}(x,n) = 0. \tag{2.27}$$

Fix any l, j and $f_q(x) \in L^2(\mathbb{T})$. Let $\psi_x^{q,l,j}(n) = (U_R^{-1}U_{k_0}U_R f_q v_{l,j})(x,n) \in l^2(\mathbb{Z})$. Let $\mu_x^{q,l,j}$ be the spectral measure of $\tilde{H}_{c,v,\alpha}(x)$ associated to $\psi_x^{q,l,j}(\cdot)$.

Lemma 2.3.10 *$d\mu_x^{q,l,j}$ is a.e. independent of x.*

Proof: Take any continuous function F and $m \neq 0$. By the definition of spectral measure we have, by (2.24),

$$\begin{aligned}
\mathcal{I} \triangleq & |\int_{\mathbb{T}} e^{2\pi i m x} \int F(E)\, d\mu_x^{q,l,j}(E)\, dx| \\
= & |\int_{\mathbb{T}} e^{2\pi i m x} \langle \psi_x^{q,l,j}, F(\tilde{H}_{c,v,\alpha}(x))\psi_x^{q,l,j}\rangle_{l^2(\mathbb{Z})}\, dx| \\
= & |\langle U_R^{-1}U_{k_0}U_R f_q v_{l,j}, e^{2\pi i m x} U_R^{-1}U_{k_0}U_R F(\tilde{H}_{s,v,\alpha}) f_q v_{l,j}\rangle_{\mathcal{H}}|
\end{aligned}$$

Applying (2.20), (2.21) and (2.22) to this inner product we have

$$\begin{aligned}
\mathcal{I} = & |\langle U_R^{-1}U_{k_0}U_R f_q v_{l,j}, U_R^{-1}S_{-m}U_{k_0}U_R F(\tilde{H}_{s,v,\alpha}) f_q v_{l,j}\rangle_{\mathcal{H}}| \\
= & |\langle U_{k_0}U_R f_q v_{l,j}, e^{2\pi i m k_0 (x - \frac{m\alpha}{2})} U_{k_0} S_{-m} U_R F(\tilde{H}_{s,v,\alpha}) f_q v_{l,j}\rangle_{\mathcal{H}}| \\
= & |\langle U_R f_q v_{l,j}, e^{2\pi i m k_0 x} S_{-m} U_R F(\tilde{H}_{s,v,\alpha}) f_q v_{l,j}\rangle_{\mathcal{H}}| \\
= & |\langle U_R f_q v_{l,j}, S_{-m} e^{2\pi i m k_0 x} U_R F(\tilde{H}_{s,v,\alpha}) f_q v_{l,j}\rangle_{\mathcal{H}}| \\
= & |\langle S_m U_R f_q v_{l,j}, e^{2\pi i m k_0 x} U_R F(\tilde{H}_{s,v,\alpha}) f_q v_{l,j}\rangle_{\mathcal{H}}| \\
= & |\langle S_m U_R f_q v_{l,j}, U_R S_{m k_0} F(\tilde{H}_{s,v,\alpha}) f_q v_{l,j}\rangle_{\mathcal{H}}| \\
= & |\langle U_R^{-1} S_m U_R f_q v_{l,j}, S_{m k_0} F(\tilde{H}_{s,v,\alpha}) f_q v_{l,j}\rangle_{\mathcal{H}}| \\
= & |\langle S_{-m k_0} e^{-2\pi i m x} f_q v_{l,j}, F(\tilde{H}_{s,v,\alpha}) f_q v_{l,j}\rangle_{\mathcal{H}}|
\end{aligned}$$

Thus, by (2.27),

(2.28)

$$
\begin{aligned}
\mathcal{I} =& \left| \int_{x\in\mathbb{T}} \sum_{n\in\mathbb{Z}} \overline{e^{-2\pi i m(x-mk_0\alpha)} f_q(x-mk_0\alpha) v_{l,j}(x-mk_0\alpha, n+mk_0)} F(e_{l,j}(x)) f_q(x) v_{l,j}(x,n)\ dx \right| \\
=& \left| \int_{x\in\mathbb{T}} e^{2\pi i m x} \overline{f_q(x-mk_0\alpha)} f_q(x) F(e_{l,j}(x)) \sum_{n\in\mathbb{Z}} \overline{v_{l,j}(x-mk_0\alpha, n+mk_0)} v_{l,j}(x,n)\ dx \right| \\
=& 0,
\end{aligned}
$$

This result implies $\int F(E)\ d\mu_x^{q,l,j}(E)$ is a.e. independent of x for all continuous functions F. Since the set of continuous function is separable, we conclude that $d\mu_x^{q,l,j}$ is a.e. independent of x. □

Lemma 2.3.10 is similar to the analogous (but much simpler) statement in [32] for the Aubry duality transformation U_R. After that the argument of [32] for absolute continuity of the dual measures relies on the application of Deift-Simon theorem [29] (the latter is still unproved for the zero $L(E)$ case leading to a gap in [32], but correct in case of $L(E) > 0$.) Here we however cannot employ this line of reasoning since Deift-Simon theorem requires a second order operator while our $\tilde{H}_{s,v,\alpha}$ is generally long-range. Thus we employ a different strategy to obtain absolute continuity, which has an additional advantage of being somewhat universal.

Lemma 2.3.11 *For a.e. x, $d\mu_x^{q,l,j}$ is absolutely continuous.*

Proof: Note that by the definition of spectral measure, for any Borel set $\mathcal{A}$ we have

(2.29)

$$
\begin{aligned}
& \int_{\mathbb{T}} d\mu_x^{q,l,j}(\mathcal{A})\ dx \\
=& \int_{\mathbb{T}} \langle \psi_x^{q,l,j}, \chi_{\mathcal{A}}(\tilde{H}_{c,v,\alpha}(x)) \psi_x^{q,l,j} \rangle_{l^2(\mathbb{Z})}\ dx \\
=& \langle U_R^{-1} U_{k_0} U_R f_q v_{l,j}, \chi_{\mathcal{A}}(\tilde{H}_{c,v,\alpha}) U_R^{-1} U_{k_0} U_R f_q v_{l,j} \rangle_{\mathcal{H}} \\
=& \langle U_R^{-1} U_{k_0} U_R f_q v_{l,j}, U_R^{-1} U_{k_0} U_R \chi_{\mathcal{A}}(\tilde{H}_{s,v,\alpha}) f_q v_{l,j} \rangle_{\mathcal{H}} \\
=& \langle f_q v_{l,j}, \chi_{\mathcal{A}}(\tilde{H}_{s,v,\alpha}) f_q v_{l,j} \rangle_{\mathcal{H}} \\
=& d\mu_{f_q v_{l,j}}(\mathcal{A}),
\end{aligned}
$$

where $d\mu_{f_q v_{l,j}}$ is the spectral measure of $\tilde{H}_{s,v,\alpha} : \mathcal{H} \to \mathcal{H}$ associated to the vector $f_q(x)v_{l,j}(x,\cdot) \in \mathcal{H}$. Since $d\mu_x^{q,l,j}$ is a.e. independent of x, by (2.29) we get

$$d\mu_x^{q,l,j}(\mathcal{A}) = d\mu_{f_q v_{l,j}}(\mathcal{A}) \text{ for a.e. } x. \tag{2.30}$$

Now it suffices to show that for zero Lebesgue measure set $\mathcal{A}$, $d\mu_{f_q v_{l,j}}(\mathcal{A}) = 0$. Note that again by the definition of spectral measure,

$$\begin{aligned} & d\mu_{f_q v_{l,j}}(\mathcal{A}) \\ = & \int_{\mathbb{T}} \sum_{n\in\mathbb{Z}} \overline{f_q v_{l,j}(x,n)} (\chi_{\mathcal{A}}(\tilde{H}_{s,v,\alpha}) f_q v_{l,j})(x,n)\ dx. \end{aligned} \tag{2.31}$$

For a.e. $x \in \mathbb{T}$,

$$(\tilde{H}_{s,v,\alpha} v_{l,j})(x,n) = e_{l,j}(x) v_{l,j}(x,n), \tag{2.32}$$

thus $(\tilde{H}_{s,v,\alpha} f_q v_{l,j})(x,n) = e_{l,j}(x) f_q(x) v_{l,j}(x,n)$. By (3.6),

$$\begin{aligned} & d\mu_{f_q v_{l,j}}(\mathcal{A}) \\ = & \int_{\mathbb{T}} \sum_{n\in\mathbb{Z}} \overline{f_q(x) v_{l,j}(x,n)} (\chi_{\mathcal{A}}(\tilde{H}_{s,v,\alpha}) f_q v_{l,j})(x,n)\ dx \\ = & \int_{\mathbb{T}} \sum_{n\in\mathbb{Z}} \overline{f_q(x) v_{l,j}(x,n)} \chi_{\mathcal{A}}(e_{l,j}(x)) f_q(x) v_{l,j}(x,n)\ dx \\ = & \int_{\mathbb{T}} \chi_{\mathcal{A}}(e_{l,j}(x)) |f_q(x)|^2\ dx. \end{aligned} \tag{2.33}$$

It is thus enough to show for any $q, l, j \in \mathbb{Z}$ that (2.33)=0. We can prove this using the absolutely continuity of the density of states measure. Note that for any $k \in \mathbb{Z}$,

$$dN_{\tilde{H}_{s,v,\alpha}}(\mathcal{A}) = \int_{\mathbb{T}} d\mu_{\delta_k,x}(\mathcal{A})\ dx, \tag{2.34}$$

where $d\mu_{\delta_k,x}$ is the spectral measure of $\tilde{H}_{s,v,\alpha}(x)$ associated to the vector $\delta_k \in l^2(\mathbb{Z})$. Since for a.e. x, $v_{l,j}(x,\cdot)$ is an orthonormal basis of $l^2(\mathbb{Z})$, we have that

$$\delta_k(\cdot) = \sum_{l,j} \langle \delta_k(\cdot), v_{l,j}(x,\cdot)\rangle v_{l,j}(x,\cdot) = \sum_{l,j} v_{l,j}(x,k) v_{l,j}(x,\cdot).$$

By (2.25) and (2.32), this means for a.e. x,

$$d\mu_{\delta_k,x}(\mathcal{A}) = \langle \delta_k, \chi_{\mathcal{A}}(\tilde{H}_{s,v,\alpha}(x))\delta_k\rangle = \sum_{l,j} |v_{l,j}(x,k)|^2 \chi_{\mathcal{A}}(e_l(x,j)). \tag{2.35}$$

By Lemma 2.3.5, $dN_{\tilde{H}_{s,v,\alpha}}(\mathcal{A}) = 0$. Thus combining (2.34) with (2.35) we get

$$\sum_{l,j} \int \chi_{\mathcal{A}}(e_{l,j}(x))\ dx = \sum_{k\in\mathbb{Z}} \int_{\mathbb{T}} d\mu_{\delta_k,x}(\mathcal{A})\ dx = 0.$$

This implies in particular

$$\chi_{\mathcal{A}}(e_{l,j}(x)) = 0 \text{ for a.e. } x \text{ and any } l, j.$$

By (2.33), $d\mu_{f_q v_{l,j}}(\mathcal{A}) = 0$. By (2.30), we conclude that for a.e. x, $\{d\mu_x^{q,l,j}\}_{q,l,j}$ are absolutely continuous. □

Let $\{f_q(x)\}_q$ be an orthonormal basis for $L^2(\mathbb{T})$. Note that the non-vanishing $\{v_{l,j}(x,\cdot)\}$ form an orthonormal basis for $\mathcal{H}$. It follows that $\{f_q(x)v_{l,j}(x,\cdot)\}_{q,l,j}$ form a complete orthogonal set in $\mathcal{H}$ (but not necessarily orthonormal). Since $U_R^{-1}U_{k_0}U_R$ is unitary, it follows that $\{(U_R^{-1}U_{k_0}U_R f_q v_{l,j})(x,\cdot)\}_{q,l,j}$ is a complete orthogonal set in $\mathcal{H}$. Thus for a.e. x, $\{\psi_x^{q,l,j}(\cdot) = (U_R^{-1}U_{k_0}U_R f_q v_{l,j})(x,\cdot)\}_{q,l,j}$ is a complete set in $l^2(\mathbb{Z})$. Since $\psi_x^{q,l,j}$ is a complete set, we get that $\tilde{H}_{c,v,\alpha}(x)$ only has absolutely continuous spectrum for a.e. x. □

2.4 Absence of eigenvalues. Proof of Theorem 2.1.3 and the second part of Theorem 2.1.2

2.4.1 Preparation for the proof of Theorem 2.1.3

Consider a general Jacobi operator $(H_{c,v,\alpha}(\theta)u)_n = c(\theta + n\alpha)u_{n+1} + \tilde{c}(\theta + (n-1)\alpha)u_{n-1} + v(\theta + n\alpha)u_n$, where $v(\theta)$ is Lipshitz and $c(\theta) \in \mathbb{C}^\omega(\mathbb{T})$ is allowed to have zeros on $\mathbb{T}$. Let $c(\theta) = f(\theta)g(\theta)$, where $f(\theta) = \prod_{j=1}^m (e^{2\pi i\theta} - e^{2\pi i\theta_j})$ with $\{\theta_j\}_{j=1}^m$ being zeros of $c(\theta)$ counting multiplicities, and $g(\theta) \neq 0$ on $\mathbb{T}$.

Let us define

$$\delta_c(\alpha,\theta) = \limsup_{n\to\infty} \frac{\sum_{j=1}^m \ln \|q_n(\theta - \theta_j)\| + \ln q_{n+1}}{q_n}. \tag{2.36}$$

Note that for a.e. θ, $\delta_c(\alpha,\theta) = \beta(\alpha)$.

We will assume θ does not belong to the following countable set (otherwise the operator is not well defined).

$$\theta \notin \Theta \triangleq \cup_{j=1}^{m} \theta_j + \mathbb{Z}\alpha + \mathbb{Z} \tag{2.37}$$

Fix any $\theta \notin \Theta$ and energy E satisfying $L(E) < \delta_c(\alpha, \theta)$. Recall that a formal solution to $H_c(\theta)u = Eu$ can be reconstructed via the following equation:

$$\begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = A_{c,E}(\theta + n\alpha) \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix},$$

where $A_{c,E}(\theta) = \begin{pmatrix} \frac{E-v(\theta)}{c(\theta)} & -\frac{\tilde{c}(\theta-\alpha)}{c(\theta)} \\ 1 & 0 \end{pmatrix}$. We separate the singular and regular parts of $A_{c,E}$ and rewrite it in the following way:

$$A_{c,E}(\theta) = \frac{1}{f(\theta)} \begin{pmatrix} \frac{E-v(\theta)}{g(\theta)} & -\frac{\tilde{c}(\theta-\alpha)}{g(\theta)} \\ f(\theta) & 0 \end{pmatrix} \triangleq \frac{1}{f(\theta)} F_{c,E}(\theta). \tag{2.38}$$

From now on we will omit the dependence of these matrices on c, E and denote $A(\theta) = A_{c,E}(\theta)$ and $F(\theta) = F_{c,E}(\theta)$. Let $A^k = A(\theta + k\alpha)$, $F^k = F(\theta + k\alpha)$. For any function $z(\theta)$ on $\mathbb{T}$ let $z_k = z(\theta + k\alpha)$, for simplicity. Note that clearly we have $\int_{\mathbb{T}} \ln |f(\theta)| d\theta = 0$, hence $L(E) = L(\alpha, A) = L(\alpha, F)$.

The first step is standard in Gordon-type methods. For $A \in GL(2, \mathbb{C})$ we have the following Caley-Hamilton equations:

$$A^2 - \mathrm{Tr}A \cdot A + \det A \cdot \mathrm{Id} = 0, \tag{2.39}$$

$$A - \mathrm{Tr}A \cdot \mathrm{Id} + \det A \cdot A^{-1} = 0. \tag{2.40}$$

Fix any $0 < \epsilon < (\delta_c(\alpha, \theta) - L(E))/4$. By the definition of $\delta_c(\alpha, \theta)$, there exists a subsequence $\{q_{n_l}\}$ of $\{q_n\}$ such that

$$\prod_{j=1}^{m} \|q_{n_l}(\theta - \theta_j)\| \geq \frac{e^{(\delta_c - \epsilon)q_{n_l}}}{q_{n_l+1}}. \tag{2.41}$$

We will use the following estimate.

Lemma 2.4.1 *[53]*

$$\prod_{j=0}^{q_{n_l}-1} |f(\theta + j\alpha)| \geq \frac{e^{(\delta_c - \epsilon)q_{n_l}}}{q_{n_l+1}}.$$

2.4.2 Proof of Theorem 2.1.3

Assume u is a bounded solution to $H_{c,v,\alpha}(\theta)u = Eu$. We could scale u so that $\| \begin{pmatrix} u_0 \\ u_{-1} \end{pmatrix} \| = 1$. We will prove

Lemma 2.4.2 *For q_{n_l} large enough*

$$\max \left(\| \begin{pmatrix} u_{q_{n_l}} \\ u_{q_{n_l}-1} \end{pmatrix} \|, \| \begin{pmatrix} u_{2q_{n_l}} \\ u_{2q_{n_l}-1} \end{pmatrix} \|, \| \begin{pmatrix} u_{-q_{n_l}} \\ u_{-q_{n_l}-1} \end{pmatrix} \| \right) \geq \frac{1}{4},$$

Proof of Lemma 2.4.2

If $\| \begin{pmatrix} u_{q_{n_l}} \\ u_{q_{n_l}-1} \end{pmatrix} \| = \|A_{q_{n_l}}(\theta) \begin{pmatrix} u_0 \\ u_{-1} \end{pmatrix} \| < \frac{1}{4}$, we divide the discussion into 2 cases:

Case 1: $|\mathrm{Tr} A_{q_{n_l}}(\theta)| \leq \frac{1}{2}$.

Note that since $|\det A_{q_{n_l}}(\theta)| = |\frac{c(\theta-\alpha)}{c(\theta+(q_{n_l}-1)\alpha)}| \to 1$, (2.39) implies $\|A^2_{q_{n_l}}(\theta) \begin{pmatrix} u_0 \\ u_{-1} \end{pmatrix} \| \geq \frac{7}{8}$ for q_{n_l} large enough. By telescoping,

$$\begin{aligned}
&(A^2_{q_{n_l}}(\theta) - A_{2q_{n_l}}(\theta)) \begin{pmatrix} u_0 \\ u_{-1} \end{pmatrix} \\
=&(\prod_{k=0}^{q_{n_l}-1} A^k - \prod_{k=q_{n_l}}^{2q_{n_l}-1} A^k) A_{q_{n_l}}(\theta) \begin{pmatrix} u_0 \\ u_{-1} \end{pmatrix} \\
=& \sum_{i=0}^{q_{n_l}-1} \left(\prod_{k=i+1}^{q_{n_l}-1} A^k \right) \cdot \left(A^i - A^{q_{n_l}+i} \right) \cdot \left(\prod_{k=q_{n_l}}^{q_{n_l}+i-1} A^k \right) \begin{pmatrix} u_{q_{n_l}} \\ u_{q_{n_l}-1} \end{pmatrix} \\
=& \sum_{i=0}^{q_{n_l}-1} \left(\prod_{k=i+1}^{q_{n_l}-1} A^k \right) \cdot \left(A^i - A^{q_{n_l}+i} \right) \cdot \begin{pmatrix} u_{q_{n_l}+i} \\ u_{q_{n_l}+i-1} \end{pmatrix}
\end{aligned}$$

(2.42)

$$= \sum_{i=0}^{q_{n_l}-1} \left(\prod_{k=i+1}^{q_{n_l}-1} \frac{F^k}{f_k} \right) \cdot \left(\frac{F_i - F_{q_{n_l}+i}}{f_i} \cdot \begin{pmatrix} u_{q_{n_l}+i} \\ u_{q_{n_l}+i-1} \end{pmatrix} + \frac{f_{q_{n_l}+i} - f_i}{f_i} \cdot \begin{pmatrix} u_{q_{n_l}+i+1} \\ u_{q_{n_l}+i} \end{pmatrix} \right).$$

Note that by our assumption u is a bounded solution, so there exists a constant

$C_1 > 0$ so that

$$(2.43) \qquad \| \begin{pmatrix} u_t \\ u_{t-1} \end{pmatrix} \| \leq C_1 \text{ for any } t \in \mathbb{Z}.$$

Clearly

$$(2.44) \qquad |f_{q_{n_l}+i} - f_i| \leq \frac{C_2}{q_{n_l+1}} \quad \text{for some constant } C_2,$$

and since v is Lipshitz we have

$$(2.45) \qquad \|F_i - F_{q_{n_l}+i}\| \leq \frac{C_3}{q_{n_l+1}} \text{for some constant } C_3.$$

Also by Lemmas 2.4.1, 2.2.1 and Remark 2.2.1, we have

$$(2.46) \qquad \frac{\| \prod_{k=i+1}^{q_{n_l}-1} F^k\|}{|\prod_{k=i}^{q_{n_l}-1} f_k|} = \frac{\| \prod_{k=i+1}^{q_{n_l}-1} F^k\| \cdot |\prod_{k=0}^{i-1} f_k|}{|\prod_{k=0}^{q_{n_l}-1} f_k|} \leq q_{n_l+1} e^{(L(E)-\delta_c+3\epsilon)q_{n_l}}.$$

Now we combine (2.42), (2.43), (5.5), (2.44) with (2.46),

$$\|(A^2_{q_{n_l}}(\theta) - A_{2q_{n_l}}(\theta)) \begin{pmatrix} u_0 \\ u_{-1} \end{pmatrix} \| < e^{(L(E)-\delta_c+4\epsilon)q_{n_l}} \to 0.$$

Hence $\|A_{2q_{n_l}-1}(\theta) \begin{pmatrix} u_0 \\ u_{-1} \end{pmatrix} \| \sim \|A^2_{q_{n_l}}(\theta) \begin{pmatrix} u_0 \\ u_{-1} \end{pmatrix} \| \geq \frac{7}{8}$.

Case 2: $|\mathrm{Tr} A_{q_{n_l}}(\theta)| > \frac{1}{2}$.

Then $|\det A_{q_{n_l}}(\theta)| = |\frac{c(\theta-\alpha)}{c(\theta+(q_{n_l}-1)\alpha)}| \to 1$ and (2.40) imply $\|A^{-1}_{q_{n_l}} \begin{pmatrix} u_0 \\ u_{-1} \end{pmatrix} \| \geq \frac{1}{4}$ for q_{n_l} large enough. Similar to *Case 1*, we can prove $\|A_{-q_{n_l}}(\theta) \begin{pmatrix} u_0 \\ u_{-1} \end{pmatrix} \| \sim \|A^{-1}_{q_{n_l}}(\theta) \begin{pmatrix} u_0 \\ u_{-1} \end{pmatrix} \| \geq \frac{1}{4}$. □

By Lemma 2.4.2, $H_{c,v,\alpha}(\theta)$ has no decaying solutions on $\{E : L(E) < \delta_c(\alpha, \theta)\}$, therefore no eigenvalues. □

2.4.3 Proof of the second part of Theorem 2.1.2

Now let's come back to the extended Harper's model, where $c_\lambda(\theta) = \lambda_1 e^{-2\pi i(\theta+\frac{\alpha}{2})} + \lambda_2 + \lambda_3 e^{2\pi i(\theta+\frac{\alpha}{2})}$. Note that $c_\lambda(\theta)$ could take zero value when the parameters λ satisfy some certain conditions. In fact, recall Observation 1.6.3,

- when $\lambda_1 = \lambda_3 \geq \frac{\lambda_2}{2}$, singular points are $\theta_1 = \frac{1}{2\pi}\arccos\left(-\frac{\lambda_2}{2\lambda_1}\right) - \frac{\alpha}{2}$ and $\theta_2 = -\frac{1}{2\pi}\arccos\left(-\frac{\lambda_2}{2\lambda_1}\right) - \frac{\alpha}{2}$ (notice that when $\lambda_1 = \frac{\lambda_2}{2}$ there is a single singular point $\theta_1 = \theta_2 = \frac{1}{2} - \frac{\alpha}{2}$).

- when $\lambda_1 \neq \lambda_3$ and $\lambda_1 + \lambda_3 = \lambda_2$, the singular point is $\theta_1 = \frac{1}{2} - \frac{\alpha}{2}$.

Thus for the extended Harper's model, the proper definition of $\delta_c(\alpha, \theta)$ depends on the parameters:

- *Case 1*: (non-singular case): When (1) $\lambda_1 \neq \lambda_3$ and $\lambda_1 + \lambda_3 \neq \lambda_2$ or (2) $\lambda_1 = \lambda_3 < \frac{\lambda_2}{2}$, we have $\delta_c(\alpha, \theta) = \beta(\alpha)$ for all θ.

- *Case 2*: When $\lambda_1 + \lambda_3 = \lambda_2$, then $\delta_c(\alpha, \theta) = \limsup_{n\to\infty} \frac{\ln \|q_n(\theta-\theta_1)\| + \ln q_{n+1}}{q_n} = \beta(\alpha)$ for a.e. θ, where $\theta_1 \triangleq \frac{1}{2} - \frac{\alpha}{2}$.

- *Case 3*: When $\lambda_1 = \lambda_3 > \frac{\lambda_2}{2}$, then $\delta(\alpha, \theta) = \limsup_{n\to\infty} \frac{\sum_{j=1,2} \ln \|q_n(\theta-\theta_j)\| + \ln q_{n+1}}{q_n} = \beta(\alpha)$ for a.e. θ, where $\theta_1 = \frac{1}{2\pi}\arccos\left(-\frac{\lambda_2}{2\lambda_1}\right) - \frac{\alpha}{2}$ and $\theta_2 = -\frac{1}{2\pi}\arccos\left(-\frac{\lambda_2}{2\lambda_1}\right) - \frac{\alpha}{2}$.

Note that for each of the three cases, $L(\lambda) < \beta(\alpha)$ implies absence of eigenvalues for either all or an arithmetic explicit full measure set of θ. Thus the purely singular continuous part simply comes from the fact that $L(\lambda) > 0$. $\square$

2.5 Pure point spectrum. Proof of the first part of Theorem 2.1.2

2.5.1 Preparation

Note that if $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ is in region I$^{\circ}$, its dual $\hat{\lambda} = \left(\frac{\lambda_3}{\lambda_2}, \frac{1}{\lambda_2}, \frac{\lambda_1}{\lambda_2}\right)$ belongs to region II$^{\circ}$. By Theorem 1.6.2, for λ in region I$^{\circ}$ and any $E \in \Sigma_{\lambda,\alpha}$, $L(E) \equiv L(\lambda) > 0$; also for any $E \in \Sigma_{\hat{\lambda},\alpha}$, $(\alpha, \tilde{A}_{|c_{\hat{\lambda}}|,E})$ is subcritical on $|\mathrm{Im}\theta| \leq \frac{L(\lambda)}{2\pi}$. It is straightforward that $w(c_{\hat{\lambda}}) = 0$, since $c_{\hat{\lambda}}(\theta)$ is explicitly given by

$$c_{\hat{\lambda}}(\theta) = \frac{\lambda_1}{\lambda_2} e^{-2\pi i(\theta+\frac{\alpha}{2})} \left(e^{2\pi i(\theta+\frac{\alpha}{2})} - \frac{-1+\sqrt{1-4\lambda_1\lambda_3}}{2\lambda_1}\right) \left(e^{2\pi i(\theta+\frac{\alpha}{2})} - \frac{-1-\sqrt{1-4\lambda_1\lambda_3}}{2\lambda_1}\right).$$

The following theorems provide full measure reducibility of $(\alpha, \tilde{A}_{|c_{\hat{\lambda}}|,E})$.

Theorem 2.5.1 *[3] For $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ such that $\beta(\alpha) > 0$, if a cocycle (α, A) is subcritical on $|\mathrm{Im}\theta| \leq \frac{h}{2\pi}$, then for every $0 < h' < h$, there exists $C > 0$ such that if $\delta > 0$ is sufficiently small, then there exist a subsequence $\{\frac{p_{n_k}}{q_{n_k}}\}$ of the continued fraction approximants of α, sequences of matrices $B_{n_k} \in C^{\omega}_{\frac{h'}{2\pi}}(\mathbb{T}, \mathrm{PSL}(2,\mathbb{R}))$ and $R_{n_k} \in \mathrm{SO}(2,\mathbb{R})$ such that $\|B_{n_k}\|_{\frac{h'}{2\pi}} \leq e^{C\delta q_{n_k}}$ and $\|B_{n_k}(\theta+\alpha)A(\theta)B^{-1}_{n_k}(\theta) - R_{n_k}\|_{\frac{h'}{2\pi}} \leq e^{-\delta q_{n_k}}$.*

Theorem 2.5.2 *[5, 35, 70] Let $(\alpha, A) \in \mathbb{R}\backslash\mathbb{Q} \times C^{\omega}_{\frac{h}{2\pi}}(\mathbb{T}, \mathrm{SL}(2,\mathbb{R}))$ with $0 < \tilde{h} < h'$, $R \in \mathrm{SO}(2, R)$, for every $\tau > 0$, $\gamma > 0$, if $\mathrm{rot}_f(\alpha, A) \in \mathrm{DC}_\alpha(\tau, \gamma)$, where*

$$\mathrm{DC}_\alpha(\tau,\gamma) = \{\phi \in \mathbb{T} | \|2\phi - m\alpha\|_{\mathbb{T}} \geq \frac{\gamma}{(1+|m|)^\tau}, \text{ for any nonzero } m \in \mathbb{Z}\}$$

then there exists $T = T(\tau)$, $\kappa = \kappa(\tau)$ such that if

$$\|A(\theta) - R\|_{\frac{h'}{2\pi}} < T(\tau)\gamma^{\kappa(\tau)}(h' - \tilde{h})^{\kappa(\tau)}, \tag{2.47}$$

then there exists $B \in C^{\omega}_{\frac{\tilde{h}}{2\pi}}(\mathbb{T}, \mathrm{SL}(2,\mathbb{R}))$ and $\varphi \in C^{\omega}_{\frac{\tilde{h}}{2\pi}}(\mathbb{T}, \mathbb{R})$ such that

$$B(\theta+\alpha)A(\theta)B^{-1}(\theta) = R_{\varphi(\theta)},$$

with estimates $\|B(\theta) - \mathrm{Id}\|_{\frac{\tilde{h}}{2\pi}} \leq \|A(\theta) - R\|^{\frac{1}{2}}_{\frac{h}{2\pi}}$ and $\|\varphi(\theta) - \hat{\varphi}(0)\|_{\frac{\tilde{h}}{2\pi}} \leq 2\|A(\theta) - R\|_{\frac{h}{2\pi}}$. Moreover if $\beta(\alpha) < \tilde{h}$, (α, A) is reducible.

2.5.2 Proof of the pure point part of Theorem 2.1.2

This proof follows that of Proposition 4.2 in [12], however some modifications are needed. We include it here for reader's convenience. Let us consider energy $E \in \Sigma_{\hat{\lambda},\alpha}$ so that $\rho(\alpha, \tilde{A}_{|c_{\hat{\lambda}}|,E}) \in \cup_{\gamma>0} DC_\alpha(\tau,\gamma)$ for some $\tau > 1$. Note that since $|\cup_{\gamma>0} DC_\alpha(\tau,\gamma)| = 1$, this is a full density of states measure set of energies. Fix $\epsilon > 0$ small enough so that $\beta(\alpha) < L(\lambda) - 2\epsilon$.

First, by Theorem 2.5.1, for $h = L(\lambda)$ and $h' = L(\lambda) - \epsilon$, there exists constant $C > 0$ so that for $\delta > 0$ small there exists a subsequence $\{\frac{p_{n_k}}{q_{n_k}}\}$ of the continued fraction approximants and $B_{n_k} \in C^{\omega}_{\frac{L(\lambda)-\epsilon}{2\pi}}(\mathbb{T}, \mathrm{PSL}(2,\mathbb{R}))$, $R_{n_k} \in \mathrm{SO}(2,\mathbb{R})$, such that $\|B_{n_k}\|_{\frac{L(\lambda)-\epsilon}{2\pi}} \leq e^{C\delta q_{n_k}}$ and

$$\|B_{n_k}(\theta+\alpha)\tilde{A}_{|c_{\hat{\lambda}}|,E}(\theta)B^{-1}_{n_k}(\theta) - R_{n_k}\|_{\frac{L(\lambda)-\epsilon}{2\pi}} \leq e^{-\delta q_{n_k}}. \tag{2.48}$$

As is pointed out in [12], one could consult footnote 5 of [3] to prove the following estimate on the $\deg B_{n_k}$

$$|\deg B_{n_k}| \leq C(\lambda, \epsilon) q_{n_k} \tag{2.49}$$

Clearly by (1.16),

$$\rho(\alpha, B_{n_k}(\theta+\alpha)\tilde{A}_{|c_{\hat{\lambda}}|,E}(\theta)B_{n_k}^{-1}(\theta)) = \rho(\alpha, \tilde{A}_{|c_{\hat{\lambda}}|,E}) + \deg B_{n_k}\alpha. \tag{2.50}$$

Thus since $\rho(\alpha, \tilde{A}_{|c_{\hat{\lambda}}|,E}) \in DC_\alpha(\tau, \gamma)$ for some $\gamma > 0$, by (2.49) and (2.50) we have

$$\begin{aligned} &\|\rho(\alpha, B_{n_k}(\theta+\alpha)\tilde{A}_{|c_{\hat{\lambda}}|,E}(\theta)B_{n_k}^{-1}(\theta)) + m\alpha\|_{\mathbb{T}} \\ =&\|\rho(\alpha, \tilde{A}_{|c_{\hat{\lambda}}|,E}) + (\deg B_{n_k} + m)\alpha\|_{\mathbb{T}} \\ \geq&\frac{\gamma}{(1+Cq_{n_k}+|m|)^\tau} \\ \geq&\frac{(1+Cq_{n_k})^{-\tau}\gamma}{(1+|m|)^\tau}. \end{aligned}$$

This implies $\rho(\alpha, B_{n_k}(\theta+\alpha)\tilde{A}_{|c_{\hat{\lambda}}|,E}(\theta)B_{n_k}^{-1}(\theta)) \in DC_\alpha(\tau, (1+Cq_{n_k})^{-\tau}\gamma)$.

Secondly, fix $\tilde{h} = L(\lambda) - 2\epsilon$. For q_{n_k} large enough, in particular when the following holds, with $T(\tau), \kappa(\tau)$ from (2.47),

$$(1+Cq_{n_k})^{\tau\kappa(\tau)} < T(\tau)e^{\frac{1}{2}\delta q_{n_k}}(\gamma\epsilon)^{\kappa(\tau)}, \tag{2.51}$$

we have by (2.48)

$$\|B_{n_k}(\theta+\alpha)\tilde{A}_{|c_{\hat{\lambda}}|,E}(\theta)B_{n_k}^{-1}(\theta) - R_{n_k}\|_{\frac{L(\lambda)-\epsilon}{2\pi}} < T(\tau)(1+Cq_{n_k})^{-\tau\kappa(\tau)}(\gamma\epsilon)^{\kappa(\tau)}. \tag{2.52}$$

Thus by Theorem 4.2.3, since $\beta(\alpha) < \tilde{h} = L(\lambda) - 2\epsilon$, we get $(\alpha, \tilde{A}_{|c_{\hat{\lambda}}|,E})$ is reducible. Note that this provides us with the requirement to apply our Theorem 3.1.4, and taking into account that $w(c_{\hat{\lambda}}) = 0$ we get that $H_{\lambda,\alpha}(\theta)$ has pure point spectrum for a.e. θ. □

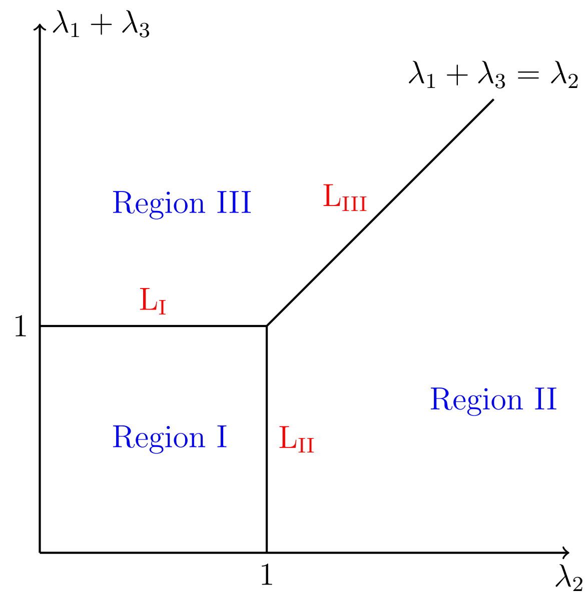
Chapter 3

Absence of point spectrum for the self-dual extended Harper's model

3.1 Introduction

In this chapter we present a new way to exclude point spectrum for self-dual extended Harper's model.

Recall that we have the following decomposition of the parameter space.



Region I $0 \leq \lambda_1 + \lambda_3 \leq 1, 0 < \lambda_2 \leq 1,$

Region II $0 \leq \lambda_1 + \lambda_3 \leq \lambda_2, 1 \leq \lambda_2$,

Region III $\max\{1, \lambda_2\} \leq \lambda_1 + \lambda_3,\ \lambda_2 > 0$.

According to the action of the *duality transformation* $\sigma : \lambda = (\lambda_1, \lambda_2, \lambda_3) \to \hat{\lambda} = (\frac{\lambda_3}{\lambda_2}, \frac{1}{\lambda_2}, \frac{\lambda_1}{\lambda_2})$, we have the following observation [37]:

Observation 3.1.1 *σ is a bijective map on $0 \leq \lambda_1 + \lambda_3,\ 0 < \lambda_2$.*

(i) $\sigma(\mathrm{I}^\circ) = \mathrm{II}^\circ$*,* $\sigma(\mathrm{III}^\circ) = \sigma(\mathrm{III}^\circ)$

(ii) Letting $\mathrm{L}_{\mathrm{I}} := \{\lambda_1 + \lambda_3 = 1, 0 < \lambda_2 \leq 1\}$*,* $\mathrm{L}_{\mathrm{II}} := \{0 \leq \lambda_1 + \lambda_3 \leq 1, \lambda_2 = 1\}$*, and* $\mathrm{L}_{\mathrm{III}} := \{1 \leq \lambda_1 + \lambda_3 = \lambda_2\}$*,* $\sigma(\mathrm{L}_{\mathrm{I}}) = \mathrm{L}_{\mathrm{III}}$ *and* $\sigma(\mathrm{L}_{\mathrm{II}}) = \mathrm{L}_{\mathrm{II}}$*.*

As σ bijectively maps $\mathrm{III} \cup \mathrm{L}_{\mathrm{II}}$ onto itself, the literature refers to $\mathrm{III} \cup \mathrm{L}_{\mathrm{II}}$ as the *self-dual regime*. We further divide III into $\mathrm{III}_{\lambda_1=\lambda_3}$ (*isotropic self-dual regime*) and $\mathrm{III}_{\lambda_1\neq\lambda_3}$ (*anisotropic self-dual regime*).

Recently, a complete understanding of the spectral properties of the extended Harper's model for a.e. θ has been established:

Theorem 3.1.2 *[8]The following Lebesgue decomposition of the spectrum of $H_{\lambda,\alpha}(\theta)$ holds for a.e. θ.*

- *For all Diophantine α, for Region* I^o*, $H_{\lambda,\alpha}(\theta)$ has pure point spectrum.*
- *For all irrational α, for Regions* II^o*,* $\mathrm{III}^\mathrm{o}_{\lambda_1\neq\lambda_3}$*, $H_{\lambda,\alpha}(\theta)$ has purely absolutely continuous spectrum.*
- *For all irrational α, for Regions* $\mathrm{III}^\mathrm{o}_{\lambda_1=\lambda_3} \cup \mathrm{L}_{\mathrm{I}} \cup \mathrm{L}_{\mathrm{II}} \cup \mathrm{L}_{\mathrm{III}}$*, $H_{\lambda,\alpha}(\theta)$ has purely singular continuous spectrum.*

As pointed out in [8], the main missing link between [37, 38] and Theorem 3.1.2 is the following theorem, excluding eigenvalues in the self-dual regime. We say θ is α-rational if $2\theta \in \mathbb{Z}\alpha + \mathbb{Z}$, otherwise we say θ is α-irrational.

Theorem 3.1.3 *[8] For all irrational α,*

- *for* $\lambda \in \mathrm{III}_{\lambda_1 \neq \lambda_3} \cup \mathrm{L}_{\mathrm{II}}$*,* $H_{\lambda,\alpha}(\theta)$ *has empty point spectrum for all* α*-irrational* θ*.*

- *for* $\lambda \in \mathrm{III}_{\lambda_1 = \lambda_3}$*,* $H_{\lambda,\alpha}(\theta)$ *has empty point spectrum for a.e.* θ*.*

In [8] the authors had to exclude more phases than α-rational θ in the isotropic self-dual regime.

In this paper we give a simple proof of the following theorem.

Theorem 3.1.4 *For all irrational* α*, for* $\lambda \in \mathrm{III}$*,* $H_{\lambda,\alpha}(\theta)$ *has empty point spectrum for all* α*-irrational* θ*.*

Remark 3.1.1 *Our result for the isotropic self-dual regime* $\mathrm{III}_{\lambda_1=\lambda_3}$ *is sharp. Indeed, according to Proposition 5.1 in [8], for* α*-rational* θ*, for a dense set of coupling constants,* $H_{\lambda,\alpha}(\theta)$ *has point spectrum.*

We organize this chapter in the following way: in Section 2 we include some preliminaries, in Section 3 we present two lemmas that will be used in Section 5, then we deal with $\mathrm{III}_{\lambda_1=\lambda_3}$ and $\mathrm{III}_{\lambda_1\neq\lambda_3} \cap \{\lambda_1+\lambda_3 = 1\}$ in Section 4 and $\mathrm{III}_{\lambda_1\neq\lambda_3} \cap \{\lambda_1+\lambda_3 > 1\}$ in Section 5.

3.2 Preliminaries

3.2.1 Singularities of the self-dual extended Harper's model

The presence of singularities of $c_\lambda(\theta)$ is explicit. Indeed, recall Observation 1.6.3,

For $\lambda \in \mathrm{III}$, necessary conditions for real roots of $c_\lambda(\theta)$ are $\lambda \in \mathrm{III}_{\lambda_1=\lambda_3}$ or $\lambda \in \mathrm{III}_{\lambda_1\neq\lambda_3} \cap \{\lambda_1 + \lambda_3 = \lambda_2\}$. Moreover,

- for $\lambda \in \mathrm{III}_{\lambda_1=\lambda_3}$, $c_\lambda(\theta)$ has real roots determined by

  $$2\lambda_3 \cos 2\pi(\theta + \frac{\alpha}{2}) = -\lambda_2, \tag{3.1}$$

  and giving rise to a double root at $\theta = \frac{1}{2} - \frac{\alpha}{2}$ if $\lambda \in \mathrm{III}_{\lambda_1=\lambda_3} \cap \{\lambda_1 + \lambda_3 = \lambda_2\}$.

- for $\lambda \in \mathrm{III}_{\lambda_1\neq\lambda_3} \cap \{\lambda_1+\lambda_3 = \lambda_2\}$, $c_\lambda(\theta)$ has only one simple real root at $\theta = \frac{1}{2} - \frac{\alpha}{2}$.

Remark 3.2.1 *By the definition of the duality transformation* σ*:* $\lambda \to \hat{\lambda}$*,* $c_{\hat{\lambda}}(\theta)$ *has singular points if and only if* $\lambda \in \mathrm{III}_{\lambda_1=\lambda_3}$ *or* $\lambda \in \mathrm{III}_{\lambda_1\neq\lambda_3} \cap \{\lambda_1+\lambda_3=1\}$*.*

It will be clear in Section 4 that presence of singularities of $c_{\hat{\lambda}}(\theta)$ indeed simplifies the proof of empty point spectrum of $H_{\lambda,\alpha}(\theta)$.

3.3 Lemmas

Lemma 3.3.1 *For* $\lambda \in \mathrm{III}_{\lambda_1\neq\lambda_3} \cap \{\lambda_1+\lambda_3>1\}$*, when* $\lambda_3 > \lambda_1$*, we have*

$$\frac{c_{\hat{\lambda}}(\theta)}{|c|_{\hat{\lambda}}(\theta)} = e^{-2\pi i(\theta+\frac{\alpha}{2})+if(\theta)} \quad \text{and} \quad \frac{\tilde{c}_{\hat{\lambda}}(\theta)}{|c|_{\hat{\lambda}}(\theta)} = e^{2\pi i(\theta+\frac{\alpha}{2})-if(\theta)},$$

for a real analytic function $f(\theta)$ *on* $\mathbb{T}$ *with* $\int_{\mathbb{T}} f(\theta)\mathrm{d}\theta = 0$*.*

Proof: By the definition of $c_{\hat{\lambda}}(\theta)$ we have

$$c_{\hat{\lambda}}(\theta) = \frac{\lambda_3}{\lambda_2}e^{-2\pi i(\theta+\frac{\alpha}{2})} + \frac{1}{\lambda_2} + \frac{\lambda_1}{\lambda_2}e^{2\pi i(\theta+\frac{\alpha}{2})} \tag{3.2}$$

$$= \frac{\lambda_1}{\lambda_2}e^{-2\pi i(\theta+\frac{\alpha}{2})}(e^{2\pi i(\theta+\frac{\alpha}{2})} - y_+)(e^{2\pi i(\theta+\frac{\alpha}{2})} - y_-), \tag{3.3}$$

where $y_\pm = \frac{-1\pm\sqrt{1-4\lambda_1\lambda_3}}{2\lambda_1}$. Note that

(3.4)

$$y_+ = \overline{y_-} \text{ with } |y_+| = |y_-| = \sqrt{\frac{\lambda_3}{\lambda_1}} > 1, \quad \text{when } 1 \leq 2\sqrt{\lambda_1\lambda_3},$$

(3.5)

$$y_+, y_- \in \mathbb{R} \text{ with } |y_+| > |y_-| = \frac{2\lambda_3}{\lambda_1 + \sqrt{1-4\lambda_1\lambda_3}} > 1, \quad \text{when } \lambda_1+\lambda_3 > 1 > 2\sqrt{\lambda_1\lambda_3}.$$

Note that

$$\frac{c_{\hat{\lambda}}(\theta)}{|c|_{\hat{\lambda}}(\theta)} = \sqrt{\frac{c_{\hat{\lambda}}(\theta)}{\tilde{c}_{\hat{\lambda}}(\theta)}} = e^{-2\pi i(\theta+\frac{\alpha}{2})}\sqrt{\frac{(e^{2\pi i(\theta+\frac{\alpha}{2})} - y_+)(e^{2\pi i(\theta+\frac{\alpha}{2})} - y_-)}{(e^{-2\pi i(\theta+\frac{\alpha}{2})} - y_+)(e^{-2\pi i(\theta+\frac{\alpha}{2})} - y_-)}}. \tag{3.6}$$

By (3.4), we have

$$\int_{\mathbb{T}} \arg \frac{(e^{2\pi i(\theta+\frac{\alpha}{2})} - y_+)(e^{2\pi i(\theta+\frac{\alpha}{2})} - y_-)}{(e^{-2\pi i(\theta+\frac{\alpha}{2})} - y_+)(e^{-2\pi i(\theta+\frac{\alpha}{2})} - y_-)}\mathrm{d}\theta = 0, \tag{3.7}$$

and

$$(3.8) \qquad \left|\frac{(e^{2\pi i(\theta+\frac{\alpha}{2})}-y_+)(e^{2\pi i(\theta+\frac{\alpha}{2})}-y_-)}{(e^{-2\pi i(\theta+\frac{\alpha}{2})}-y_+)(e^{-2\pi i(\theta+\frac{\alpha}{2})}-y_-)}\right| \equiv 1.$$

Thus there exists a real analytic function $g(\theta)$ on $\mathbb{T}$ such that

$$(3.9) \qquad \frac{(e^{2\pi i(\theta+\frac{\alpha}{2})}-y_+)(e^{2\pi i(\theta+\frac{\alpha}{2})}-y_-)}{(e^{-2\pi i(\theta+\frac{\alpha}{2})}-y_+)(e^{-2\pi i(\theta+\frac{\alpha}{2})}-y_-)} = e^{ig(\theta)},$$

with $\int_{\mathbb{T}} g(\theta)\mathrm{d}\theta = 0$. Taking $f(\theta) = g(\theta)/2$ yields the desired the result. □

Lemma 3.3.2 *There is a subsequence $\{\frac{p_{m_l}}{q_{m_l}}\}$ of the continued fraction approximants of α so that for any analytic function f on $\mathbb{T}$ with $\int_{\mathbb{T}} f(\theta)\mathrm{d}\theta = 0$, we have*

$$\lim_{l\to\infty} f(x) + f(x+\alpha) + \cdots + f(x + q_{m_l}\alpha - \alpha) = 0$$

uniformly in $x \in \mathbb{T}$.

Proof: Suppose f is analytic on $|\mathrm{Im}\theta| \le \delta_0$, then $|\hat{f}(n)| \le ce^{-2\pi\delta_0|n|}$ for some constant $c > 0$.

Case 1 If $\beta(\alpha) = 0$, then by solving the coholomogical equation we get $f(x) = h(x+\alpha) - h(x)$ for some analytic $h(x)$. Then

$$\begin{aligned}&\lim_{m\to\infty}(f(x) + f(x+\alpha) + \cdots + f(x+q_m\alpha-\alpha))\\ =&\lim_{m\to\infty}(h(x+q_m\alpha) - h(x)) = 0\end{aligned}$$

uniformly in x.

Case 2 If $\beta(\alpha) > 0$, choose a sequence m_l such that $q_{m_l+1} \ge e^{\frac{\beta}{2}q_{m_l}}$. Then

$$\begin{aligned}&|f(x) + f(x+\alpha) + \cdots + f(x+q_{m_l}\alpha-\alpha)|\\ =&\Big|\sum_{|n|\ge 1}\hat{f}(n)(1+e^{2\pi in\alpha}+\cdots+e^{2\pi in(q_{m_l}-1)\alpha})e^{2\pi inx}\Big|\\ =&\Big|\sum_{|n|\ge 1}\hat{f}(n)\frac{1-e^{2\pi inq_{m_l}\alpha}}{1-e^{2\pi in\alpha}}e^{2\pi inx}\Big|\\ \le&\sum_{1\le|n|\le q_{m_l}-1} c\left|\frac{1-e^{2\pi inq_{m_l}\alpha}}{1-e^{2\pi in\alpha}}\right| + \sum_{|n|\ge q_{m_l}} ce^{-2\pi\delta_0|n|}q_{m_l}\\ \le& c\frac{q_{m_l}^3}{q_{m_l+1}} + cq_{m_l}e^{-2\pi\delta_0 q_{m_l}} \to 0 \quad \text{as } l\to\infty\end{aligned}$$

uniformly in x. □

3.4 Consequence of point spectrum

This part follows from [8]. We present the material here for completeness and readers' convenience.

Suppose $\{u_n\}$ is an $l^2(\mathbb{Z})$ solution to $H_{\lambda,\alpha}(\theta)u = Eu$, where $\lambda = (\lambda_1, \lambda_2, \lambda_3)$. This means

$$c_\lambda(\theta + n\alpha)u_{n+1} + \tilde{c}_\lambda(\theta + (n-1)\alpha)u_{n-1} + 2\cos(2\pi(\theta + n\alpha))u_n = Eu_n. \tag{3.10}$$

Let $u(x) = \sum_{n\in\mathbb{Z}} u_n e^{2\pi inx} \in L^2(\mathbb{T})$. Multiplying (3.10) by $e^{2\pi inx}$ and then summing over n, we get

$$e^{2\pi i\theta} c_{\hat{\lambda}}(x)u(x+\alpha) + e^{-2\pi i\theta}\tilde{c}_{\hat{\lambda}}(x-\alpha)u(x-\alpha) + 2\cos 2\pi x\ u(x) = \frac{E}{\lambda_2}u(x), \tag{3.11}$$

where $\hat{\lambda} = (\frac{\lambda_3}{\lambda_2}, 1, \frac{\lambda_1}{\lambda_2})$. If we multiply (3.10) by $e^{-2\pi inx}$ and sum over n, we get

$$e^{-2\pi i\theta} c_{\hat{\lambda}}(x)u(-x-\alpha) + e^{2\pi i\theta}\tilde{c}_{\hat{\lambda}}(x-\alpha)u(-x+\alpha) + 2\cos 2\pi x\ u(-x) = \frac{E}{\lambda_2}u(-x). \tag{3.12}$$

Thus writing (3.11), (3.12) in terms of matrices, we get

$$\frac{1}{c_{\hat{\lambda}}(x)}\begin{pmatrix} \frac{E}{\lambda_2} - 2\cos 2\pi x & -\tilde{c}_{\hat{\lambda}}(x-\alpha) \\ c_{\hat{\lambda}}(x) & 0 \end{pmatrix}\begin{pmatrix} u(x) & u(-x) \\ e^{-2\pi i\theta}u(x-\alpha) & e^{2\pi i\theta}u(-(x-\alpha)) \end{pmatrix}$$

$$= \begin{pmatrix} u(x+\alpha) & u(-(x+\alpha)) \\ e^{-2\pi i\theta}u(x) & e^{2\pi i\theta}u(-x) \end{pmatrix}\begin{pmatrix} e^{2\pi i\theta} & 0 \\ 0 & e^{-2\pi i\theta} \end{pmatrix} \tag{3.13}$$

Let $M_\theta(x) \in L^2(\mathbb{T})$ be defined by

$$M_\theta(x) = \begin{pmatrix} u(x) & u(-x) \\ e^{-2\pi i\theta}u(x-\alpha) & e^{2\pi i\theta}u(-(x-\alpha)) \end{pmatrix}.$$

Let

$$A_{\hat{\lambda},E/\lambda_2}(x) = \frac{1}{c_{\hat{\lambda}}(x)}\begin{pmatrix} \frac{E}{\lambda_2} - 2\cos 2\pi x & -\tilde{c}_{\hat{\lambda}}(x-\alpha) \\ c_{\hat{\lambda}}(x) & 0 \end{pmatrix}$$

be the transfer matrix associated to $H_{\hat{\lambda},\alpha}(\theta)$ and

$$R_\theta = \begin{pmatrix} e^{2\pi i\theta} & 0 \\ 0 & e^{-2\pi i\theta} \end{pmatrix}$$

be the constant rotation matrix. Then (3.13) becomes

$$A_{\hat{\lambda},E}(x)M_\theta(x) = M_\theta(x+\alpha)R_\theta. \tag{3.14}$$

Taking determinant, we have the following proposition.

Proposition 3.4.1 *[8] If θ is α-irrational, then*

$$|\det M_\theta(x)| = \frac{b}{|c|_{\hat{\lambda}}(x-\alpha)} \tag{3.15}$$

for some constant $b>0$ and a.e. $x\in\mathbb{T}$.

3.5 Regions $\mathrm{III}_{\lambda_1=\lambda_3}$ and $\mathrm{III}_{\lambda_1\neq\lambda_3}\cap\{\lambda_1+\lambda_3=1\}$

We will show the following lemma.

Lemma 3.5.1 *If θ is α-irrational, then for $\lambda\in\mathrm{III}_{\lambda_1=\lambda_3}$ or $\lambda\in\mathrm{III}_{\lambda_1\neq\lambda_3}\cap\{\lambda_1+\lambda_3=1\}$, $H_{\lambda,\alpha,\theta}$ has no point spectrum.*

Proof: According to Remark 3.2.1, we have $c_{\hat{\lambda}}(x_0)=0$ for some $x_0\in\mathbb{T}$. Note that presence of singularity implies $\frac{1}{c_{\hat{\lambda}}(x)}\notin L^1(\mathbb{T})$. Thus by (3.15), $\det M_\theta(x)\notin L^1(\mathbb{T})$. This contradicts with $M_\theta(x)\in L^2(\mathbb{T})$. □

3.6 Region $\mathrm{III}_{\lambda_1\neq\lambda_3}\cap\{\lambda_1+\lambda_3>1\}$

Without loss of generality, we assume $\lambda_3>\lambda_1$. Fix θ. Denote $\det M_\theta(x)=g(x)$ for simplicity.

Lemma 3.6.1 *If θ is α-irrational, then $H_{\lambda,\alpha}(\theta)$ has no point spectrum in the anisotropic self-dual region.*

Proof: Taking determinant in (3.14), we get:

$$\frac{\tilde{c}_{\hat{\lambda}}(x-\alpha)}{c_{\hat{\lambda}}(x)}g(x) = g(x+\alpha).$$

This implies

$$g(x+k\alpha)=\frac{\tilde{c}_{\hat{\lambda}}(x+k\alpha-2\alpha)\cdots\tilde{c}_{\hat{\lambda}}(x)\ \tilde{c}_{\hat{\lambda}}(x-\alpha)}{c_{\hat{\lambda}}(x+k\alpha-\alpha)\cdots c_{\hat{\lambda}}(x+\alpha)c_{\hat{\lambda}}(x)}g(x). \tag{3.16}$$

Taking $k=q_{m_l}$, as in Lemma 3.3.2, on one hand, since $g(x)$ is an L^1 function, as the determinant of an L^2 matrix, and $\lim_{l\to\infty}\|q_{m_l}\alpha\|_{\mathbb{T}}=0$, we have

$$\lim_{l\to\infty}\|g(x+q_{m_l}\alpha)-g(x)\|_{L^1}=0.$$

By (3.16), this implies

$$0=\lim_{l\to\infty}\|g(x+q_{m_l}\alpha)-g(x)\|_{L^1}=\lim_{l\to\infty}\int\left|1-\frac{\prod_{j=-1}^{q_{m_l}-2}\tilde{c}_{\hat{\lambda}}(x+j\alpha)}{\prod_{j=0}^{q_{m_l}-1}c_{\hat{\lambda}}(x+j\alpha)}\right|\cdot|g(x)|\mathrm{d}x. \tag{3.17}$$

On the other hand, by Lemma 3.3.1

$$\begin{aligned}
&\lim_{l\to\infty}\int\left|1-\frac{\prod_{j=-1}^{q_{m_l}-2}\tilde{c}_{\hat{\lambda}}(x+j\alpha)}{\prod_{j=-1}^{q_{m_l}-1}c_{\hat{\lambda}}(x+j\alpha)}\right|\cdot|g(x)|\mathrm{d}x\\
=&\lim_{l\to\infty}\int\left|1-\frac{|c|_{\hat{\lambda}}(x-\alpha)}{|c|_{\hat{\lambda}}(x+q_{m_l}\alpha-\alpha)}e^{-i\left(\sum_{j=-1}^{q_{m_l}-2}f(x+j\alpha)+\sum_{j=0}^{q_{m_l}-1}f(x+j\alpha)\right)}e^{4\pi i q_{m_l}x}e^{2\pi i q_{m_l}(q_{m_l}-1)\alpha}\right|\cdot|g(x)|\mathrm{d}x\\
\geq&\liminf_{l\to\infty}\left(\int|1-e^{4\pi i q_{m_l}x+2\pi i q_{m_l}^2\alpha}||g(x)|\mathrm{d}x\right.\\
&\qquad\left.-\int\left|1-\frac{|c|_{\hat{\lambda}}(x-\alpha)}{|c|_{\hat{\lambda}}(x+q_{m_l}\alpha-\alpha)}e^{-i\left(\sum_{j=-1}^{q_{m_l}-2}f(x+j\alpha)+\sum_{j=0}^{q_{m_l}-1}f(x+j\alpha)\right)}e^{-2\pi i q_{m_l}\alpha}\right|\cdot|g(x)|\mathrm{d}x\right)\\
:=&\liminf_{l\to\infty}(I_1-I_2).
\end{aligned}\tag{3.18}$$

Combining the fact $\|q_{m_l}\alpha\|_{\mathbb{T}}\to 0$ with Lemma 3.3.2, we get pointwise convergence,

$$\frac{|c|_{\hat{\lambda}}(x-\alpha)}{|c|_{\hat{\lambda}}(x+q_{m_l}\alpha-\alpha)}e^{-i\left(\sum_{j=-1}^{q_{m_l}-2}f(x+j\alpha)+\sum_{j=0}^{q_{m_l}-1}f(x+j\alpha)\right)}e^{-2\pi i q_{m_l}\alpha}\to 1\quad\text{as } l\to\infty.$$

Then by dominated convergence theorem, we get $\lim_{l\to\infty}I_2=0$. Then (3.18) implies that for any small constant $\delta>0$,

$$\begin{aligned}
&\lim_{l\to\infty}\|g(x+q_{m_l}\alpha)-g(x)\|_{L^1}\\
\geq&\liminf_{l\to\infty}I_1\\
\geq&\liminf_{l\to\infty}\int_{\|2q_{m_l}x+q_{m_l}^2\alpha\|_{\mathbb{T}}\geq\delta}4\delta|g(x)|\mathrm{d}x,
\end{aligned}$$

where $|\{x : \|2q_{m_l}x + q_{m_l}^2\alpha\| \geq \delta\}| \triangleq |F_{m_l,\delta}| = 1 - 2\delta$. Thus

$$\begin{aligned}&\lim_{l\to\infty} \|g(x+q_{m_l}\alpha) - g(x)\|_{L^1}\\ \geq &\liminf_{l\to\infty}(4\delta\|g\|_{L^1} - 4\delta\int_{F^c_{m_l,\delta}} |g(x)|\mathrm{d}x)\\ \geq &\liminf_{l\to\infty}(4\delta\|g\|_{L^1} - 8\delta^2\|g\|_{L^\infty}).\end{aligned}$$

By (3.15) $|g(x)| = \frac{b}{|c|_{\hat{\lambda}}(x-\alpha)}$ for some constant $b > 0$, thus $\|g\|_{L^1}$, $\|g\|_{L^\infty}$ are positive finite numbers, so one can choose $\delta \sim 0$ such that $4\delta\|g\|_{L^1} - 8\delta^2\|g\|_{L^\infty}$ is strictly positive. This contradicts with (3.17). $\square$

Chapter 4

Dry Ten Martini Problem for the non-self-dual extended Harper's model

4.1 Introduction

For the almost Mathieu operator, it was proved in [6] that the spectrum is a Cantor set for any $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $\lambda \neq 0$. This is the *Ten Martini Problem* dubbed by Barry Simon, after an offer of Mark Kac. A much more difficult problem, known as the dry version of the Ten Martini Problem, is to prove that the spectrum is not only a Cantor set, but that all gaps predicted by the Gap-Labelling theorem [13], [39] are open. The first result was obtained for Liouvillean α [21], and later it was proved for a set of (λ, α) of positive Lebesgue measure [65]. The most recent result is [7], in which they were able to deal with all Diophantine frequencies and $\lambda \neq 1$. A solution for all irrational frequencies and $\lambda \neq 1$ was also recently announced in [12].

In this chapter we prove the dry version of the Ten Martini Problem for the extended Harper's model in the non-self-dual regions (I° and II°) under the Diophantine condition.

Let p_n/q_n be the continued fraction appoximants of $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Let

$$\beta(\alpha) = \limsup_{n\to\infty} \frac{\ln q_{n+1}}{q_n}.$$

Throughout this chapter, we say α satisfies the Diophantine condition if $\beta(\alpha) = 0$, denoted by $\alpha \in \mathrm{DC}$.

It is known that when energy E is in the closure of a spectral gap, the integrated density of states $N_{\lambda,\alpha}(E) \in \alpha\mathbb{Z} + \mathbb{Z}$ (see [13, 39]). Here we prove the inverse is true.

Theorem 4.1.1 *If* $\alpha \in \mathrm{DC}$ *and* λ *belongs to region* I^{o} *or region* II^{o}*, all possible spectral gaps are open.*

Remark 4.1.1 *We note the Dry Ten Martini problem has not yet been solved for the self-dual AMO. In the self-dual region* III*, Cantor spectrum is known in the isotropic case (when* $\lambda_1 = \lambda_3$*), see Fact* 2.1 *in [8]. In fact one could prove the operator has zero Lebesgue measure spectrum for all frequencies.*

In order to prove Theorem 4.1.1, we first establish almost localization (see section 4.3.1) in region I^{o}, then a quantitative version of Aubry duality to obtain almost reducibility (see section 4.3.2) in region II^{o} which enables us to deal with all energies whose rotation numbers are α-rational.

Thus the strategy follows that of [7], but we need to extend the almost localization and quantitative duality, as well as the final argument to our Jacobi setting, which is non-trivial on a technical level. At the same time unlike [7], we only deal with a short-range dual operator, leading to a significant streamlining of some arguments of [7].

We organize the paper as follows: in section 4.2 we present some preliminaries, in section 4.3 we state our main results about almost localization and almost reducibility, relying on which we provide a proof of Theorem 4.1.1. In section 4.4 and 4.5 we prove the main results that we present in section 4.3.

4.2 Preliminaries

4.2.1 Generalized eigenfunctions for every phase

By Sch'nol's theorem and Aubry duality, we have the following.

Theorem 4.2.1 *[14], [66] For any λ, θ, there exists a dense set of $E \in \Sigma_{\lambda,\alpha}$ such that there exists a non-zero solution of $H_{\hat{\lambda},\alpha}(\theta)u = \frac{E}{\lambda_2}u$ with $|u_k| \leq 1 + |k|$.*

4.2.2 Bounded eigenfunction for every energy

The next result from [7] allows us to pass from a statement of every θ to every E.

Theorem 4.2.2 *[7] If $E \in \Sigma_\lambda$ then there exists $\theta(E) \in \mathbb{T}$ and a bounded solution of $H_{\hat{\lambda},\alpha}(\theta)u = \frac{E}{\lambda_2}u$ with $u_0 = 1$ and $|u_k| \leq 1$.*

4.2.3 Localization and reducibility

Theorem 4.2.3 *Given α irrational, $\theta \in \mathbb{R}$ and λ in region II°, fix $E \in \Sigma_{\lambda,\alpha}$, and suppose $H_{\hat{\lambda},\alpha}(\theta)u = \frac{E}{\lambda_2}u$ has a non-zero exponentially decaying eigenfunction $u = \{u_k\}_{k\in\mathbb{Z}}$, $|u_k| \leq e^{-c|k|}$ for $|k|$ large enough. Then the following statements hold:*

- *(A) If $2\theta \notin \alpha\mathbb{Z} + \mathbb{Z}$, then there exists $M : \mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{R})$ analytic, such that*

$$M^{-1}(x+\alpha)\tilde{A}_{|c_{\hat{\lambda}}|,E}(x)M(x) = R_{\pm\theta}.$$

  In this case $\rho(\alpha, \tilde{A}_{|c_{\hat{\lambda}}|,E}) = \pm\theta + \frac{m}{2}\alpha \ \mathrm{mod}\mathbb{Z}$, where $m = \deg M$ (here since $M \in \mathrm{SL}(2,\mathbb{R})$, we have that m is an even number) and $2\rho(\alpha, \tilde{A}_{|c_{\hat{\lambda}}|,E}) \notin \alpha\mathbb{Z} + \mathbb{Z}$.

- *(B) If $2\theta \in \alpha\mathbb{Z} + \mathbb{Z}$ and $\alpha \in \mathrm{DC}$, then there exists $M : \mathbb{R}/\mathbb{Z} \to \mathrm{PSL}(2,\mathbb{R})$ analytic, such that*

$$M^{-1}(x+\alpha)\tilde{A}_{|c_{\hat{\lambda}}|,E}(x)M(x) = \begin{pmatrix} \pm1 & a \\ 0 & \pm1 \end{pmatrix}$$

  with $a \neq 0$. In this case $\rho(\alpha, \tilde{A}_{|c_{\hat{\lambda}}|,E}) = \frac{m}{2}\alpha \ \mathrm{mod}\mathbb{Z}$, where $m = \deg M$, i.e. $2\rho(\alpha, \tilde{A}_{|c_{\hat{\lambda}}|,E}) \in \alpha\mathbb{Z} + \mathbb{Z}$.

Proof: Let $u(x) = \sum_{k\in\mathbb{Z}} \hat{u}_k e^{2\pi ikx}$, $U(x) = \begin{pmatrix} e^{2\pi i\theta}u(x) \\ u(x-\alpha) \end{pmatrix}$. Then

$$A_{c_\lambda,E}(x)U(x) = e^{2\pi i\theta}U(x+\alpha),$$

$$\tilde{A}_{|c_{\hat{\lambda}}|,E}(x)\tilde{U}(x) = e^{2\pi i\theta}\tilde{U}(x+\alpha).$$

Notice $\tilde{U}(x) = Q_\lambda(x)U(x)$ is analytic in $|\text{Im}x| < \frac{\tilde{c}}{2\pi}$, where $\tilde{c} = \min(L(\hat{\lambda}), c)$, and Q_λ as in $A.0.2$. Define $\overline{\tilde{U}(x)}$ to be the complex conjugate of $\tilde{U}(x)$ on $\mathbb{T}$ and its analytic extension to $|\text{Im}x| < \frac{\tilde{c}}{2\pi}$. Let $M(x)$ be the matrix with columns $\tilde{U}(x)$ and $\overline{\tilde{U}(x)}$. Then,

$$\tilde{A}_{|c_{\hat{\lambda}}|,E}(x)M(x) = M(x+\alpha)\begin{pmatrix} e^{2\pi i\theta} & 0 \\ 0 & e^{-2\pi i\theta} \end{pmatrix} \quad \text{on } \mathbb{T}.$$

Then since $\det M(x+\alpha) = \det M(x)$, we know $\det M(x)$ is a constant on $\mathbb{T}$.

Case 1. If $\det M(x) \neq 0$, then let $M(x) = \tilde{M}(x)\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$.

$$\tilde{M}^{-1}(x+\alpha)\tilde{A}_{|c_{\hat{\lambda}}|,E}(x)\tilde{M}(x) = R_\theta = \begin{pmatrix} \cos 2\pi\theta & -\sin 2\pi\theta \\ \sin 2\pi\theta & \cos 2\pi\theta \end{pmatrix}.$$

Case 2. If $\det M(x) = 0$, then if we denote $\tilde{U}(x) = \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix}$, then $\det M(x) = 0$ means there exists $\eta(x)$ such that $u_1(x) = \eta(x)\overline{u_1(x)}$ and $u_2(x) = \eta(x)\overline{u_2(x)}$. This implies that $\eta(x) \in \mathbb{C}^\omega(\mathbb{T}, \mathbb{C})$, and $|\eta(x)| = 1$ on $\mathbb{T}$. Therefore there exists $\phi(x) \in \mathbb{C}^\omega(\mathbb{R}/2\mathbb{Z}, \mathbb{C})$ such that $\phi^2(x) = \eta(x)$ and $|\phi(x)| = 1$. It is easy to see $\overline{\phi(x)}u_1(x) = \phi(x)\overline{u_1(x)}$ and $\overline{\phi(x)}u_2(x) = \phi(x)\overline{u_2(x)}$. Then we define $W(x) = \begin{pmatrix} \overline{\phi(x)}u_1(x) \\ \overline{\phi(x)}u_2(x) \end{pmatrix}$, it is a real vector on $\mathbb{R}/2\mathbb{Z}$ with $W(x+1) = \pm W(x)$, and $\tilde{U}(x) = \phi(x)W(x)$. Now let us define $\tilde{M}(x)$ to be the matrix with columns $W(x)$ and $\frac{1}{\|W(x)\|^{-2}}R_{\frac{1}{4}}W(x)$, then $\det \tilde{M}(x) = 1$ and $\tilde{M}(x) \in \text{PSL}(2,\mathbb{R})$. Since

$$\tilde{A}_{|c_{\hat{\lambda}}|,E}(x)W(x) = \frac{e^{2\pi i\theta}\phi(x+\alpha)}{\phi(x)}W(x+\alpha).$$

We have

$$\tilde{A}_{|c_{\hat{\lambda}}|,E}(x)\tilde{M}(x) = \tilde{M}(x+\alpha)\begin{pmatrix} d(x) & \tau(x) \\ 0 & d(x)^{-1} \end{pmatrix}$$

where $d(x) = \frac{e^{2\pi i\theta}\phi(x+\alpha)}{\phi(x)}$, $|d(x)| = 1$ and $d(x)$ being real number, therefore $d(x) = \pm 1$. Also $\tau(x) \in \mathbb{C}^\omega(\mathbb{R}/2\mathbb{Z}, \mathbb{C})$. But in fact $\tilde{M}^{-1}(x+\alpha)\tilde{A}_{|c_{\hat{\lambda}}|,E}(x)\tilde{M}(x)$ is well-defined on $\mathbb{T}$. Therefore $\tau(x) \in \mathbb{C}^\omega(\mathbb{T}, \mathbb{C})$. Now since we assumed $\alpha \in \mathrm{DC}$, we can further reduce $\tau(x)$ to the constant $\tau = \int_{\mathbb{T}} \tau(x)\mathrm{d}x$. In fact there exists $\psi(x) \in \mathbb{C}^\omega(\mathbb{T}, \mathbb{C})$ such that $-\psi(x+\alpha) + \psi(x) + \tau(x) = \int_{\mathbb{T}} \tau(x)\mathrm{d}x$. This implies

$$\begin{pmatrix} 1 & -\psi(x+\alpha) \\ 0 & 1 \end{pmatrix} \tilde{M}^{-1}(x+\alpha)\tilde{A}_{|c_{\hat{\lambda}}|,E}(x)\tilde{M}(x) \begin{pmatrix} 1 & \psi(x) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \pm 1 & \tau \\ 0 & \pm 1 \end{pmatrix}.$$

In fact if $\det M(x) = 0$, then $\frac{e^{2\pi i\theta}\phi(x+\alpha)}{\phi(x)} = \pm 1$, which implies that $2\theta \in \alpha\mathbb{Z} + \mathbb{Z}$. Therefore if $2\theta \notin \alpha\mathbb{Z} + \mathbb{Z}$, we must be in case (A). If on the other hand, $2\theta \in \alpha\mathbb{Z} + \mathbb{Z}$, $2\theta = k\alpha + n$, suppose $\tilde{M}^{-1}(x+\alpha)\tilde{A}_{|c_{\hat{\lambda}}|,E}(x)\tilde{M}(x) = R_\theta$, then $R_{-\frac{k}{2}(x+\alpha)}\tilde{M}^{-1}(x+\alpha)\tilde{A}_{|c_{\hat{\lambda}}|,E}(x)\tilde{M}(x)R_{\frac{k}{2}x} = R_{\frac{n}{2}} = \pm I$ leading to a contradiction. Therefore if $2\theta \in \alpha\mathbb{Z} + \mathbb{Z}$, we must be in case (B). $\square$

4.2.4 Characterization of the Diophantine condition

Recall that the Diophantine condition of α is $\beta(\alpha) = \limsup_{n\to\infty} \frac{\ln q_{n+1}}{q_n} = 0$. Thus for any $\xi > 0$, there exists $C_\xi > 0$ such that

$$\|k\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq C_\xi e^{-\xi|k|} \quad \text{for any } k \neq 0. \tag{4.1}$$

4.2.5 Rational approximation

Lemma 4.2.1 *[6] Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $x \in \mathbb{R}$ and $0 \leq l_0 \leq q_n - 1$ be such that $|\sin\pi(x + l_0\alpha)| = \inf_{0\leq l\leq q_n-1} |\sin\pi(x + l\alpha)|$, then for some absolute constant $C_1 > 0$,*

$$-C_1 \ln q_n \leq \sum_{0\leq l\leq q_n-1, l\neq l_0} \ln|\sin\pi(x + l\alpha)| + (q_n - 1)\ln 2 \leq C_1 \ln q_n$$

Lemma 4.2.2 *[7] Let $1 \leq r \leq [q_{n+1}/q_n]$. If $p(x)$ has essential degree at most $k = rq_n - 1$ and $x_0 \in \mathbb{R}/\mathbb{Z}$, then for some absolute constant C_2,*

$$\|p(x)\|_0 \leq C_2 q_{n+1}^{C_2 r} \sup_{0 \leq j \leq k} |p(x_0 + j\alpha)|.$$

4.2.6 Cocycles and Lyapunov exponent

Recall the following uniform control of norm of transfer matrices.

Lemma 4.2.3 *(e.g.[7]) Let (α, A) be a continous cocycle, then for any $\delta > 0$ there exists $C_\delta > 0$ such that for any $n \in \mathbb{N}$ and $\theta \in \mathbb{T}$ we have*

$$\|A_n(\theta)\| \leq C_\delta e^{(L(\alpha,A)+\delta)n}.$$

4.3 Main estimates and proof of Theorem 4.1.1

4.3.1 Almost localization for every θ

Definition 4.3.1 *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $\theta \in \mathbb{R}$, $\epsilon_0 > 0$. We say that k is an ϵ_0-resonance of θ if $\|2\theta - k\alpha\| \leq e^{-\epsilon_0 |k|}$ and $\|2\theta - k\alpha\| = \min_{|l| \leq |k|} \|2\theta - l\alpha\|$.*

Definition 4.3.2 *Let $0 = |n_0| < |n_1| < \ldots$ be the ϵ_0-resonances of θ. If this sequence is infinite, we say θ is ϵ_0-resonant, otherwise we say it is ϵ_0-non-resonant.*

Definition 4.3.3 *We say the extended Harper's model $\{H_{\lambda,\alpha}(\theta)\}_\theta$ exhibits almost localization if there exists $C_0, C_3, \epsilon_0, \tilde{\epsilon}_0 > 0$, such that for every solution u to $H_{\lambda,\alpha}(\theta)u = Eu$ satisfying $u(0) = 1$ and $|u(m)| \leq 1 + |m|$, and for every $C_0(1 + |n_j|) < |k| < C_0^{-1}|n_{j+1}|$, we have $|u(k)| \leq C_3 e^{-\tilde{\epsilon}_0 |k|}$ (where n_j are the ϵ_0-resonances of θ).*

Theorem 4.3.1 *If λ belongs to region II^{o}, $\{H_{\hat{\lambda},\alpha}(\theta)\}_\theta$ is almost localized for every $\alpha \in \mathrm{DC}$.*

Remark 4.3.1 *It is clear from Theorem 4.3.1 that almost localization implies localization for non-resonant θ.*

We will actually prove the following explicit lemma:

Lemma 4.3.1 *Let λ be in region* II^{o}. *Let C_4 be the absolute constant in Lemma* 4.4.3, *then for any $0 < \epsilon_0 < \frac{L(\hat{\lambda})}{100C_4}$, there exists constant $C_3 > 0$, which depends on λ, α and ϵ_0, so that for every solution u of $H_{\hat{\lambda},\alpha,\theta}u = Eu$ satisfying $u(0) = 1$ and $|u_k| \leq 1 + |k|$, if $3(|n_j| + 1) < |k| < \frac{1}{3}|n_{j+1}|$, then $|u_k| \leq C_3 e^{-\frac{L(\hat{\lambda})}{5}|k|}$, where $\{n_j\}$ are the ϵ_0-resonances of θ.*

The proof of Lemma 4.3.1 (and thus of Theorem 4.3.1) is given in Section 4.4.

4.3.2 Almost reducibility

Let λ be in region II^{o}. For every $E \in \Sigma_{\lambda,\alpha}$, let $\theta(E) \in \mathbb{T}$ be given in Theorem 4.2.2. Let $0 < \epsilon_0 < \frac{L(\hat{\lambda})}{100C_4}$ and $\{n_j\}$ be the set of ϵ_0- resonances of $\theta(E)$. Then for some positive constants N_0, C and c, independent of E and θ, we have the following theorem:

Theorem 4.3.2 *For any fixed j, with $N_0 < n = |n_j| + 1 < \infty$, let $N = |n_{j+1}|$, $L^{-1} = \|2\theta - n_j\alpha\|$. Then there exists $W : \mathbb{T} \to \mathrm{SL}(2, \mathbb{R})$ analytic such that $|\deg W| \leq Cn$, $\|W\|_0 \leq CL^C$ and $\|W^{-1}(x + \alpha)\tilde{A}_{|c_{\hat{\lambda}}|,E}(x)W(x) - R_{\mp\theta}\| \leq Ce^{-cN}$.*

Remark 4.3.2 *Notice that this theorem requires $n > N_0$, which is not always ensured when $\theta(E)$ is non-resonant, however in that case we have localization for $H_{\hat{\lambda},\alpha,\theta}$ instead of almost localization. We will prove Theorem* 4.3.2 *in Section 5.*

4.3.3 Spectral consequences of Almost reducibility

Let C_4 be as in Lemma 4.3.1.

Theorem 4.3.3 *Assume $\alpha \in$* DC. *For λ in region* II^{o}, *fix $E \in \Sigma_{\lambda,\alpha}$. Assume $\theta(E) \in \mathbb{T}$ is such that $H_{\hat{\lambda},\alpha}(\theta)u = \frac{E}{\lambda_2}u$ has solution satisfying $u_0 = 1$ and $|u_k| \leq 1$. Let C be the constant in Theorem* 4.3.2. *Then $\theta(E)$ and $\rho(\alpha, \tilde{A}_{|c_{\hat{\lambda}}|,E})$ have the following relation:*

- *(A) If θ is ϵ_0-non-resonant for some $\frac{L(\hat\lambda)}{100C_4} > \epsilon_0 > 0$, then $2\theta \in \mathbb{Z}\alpha + \mathbb{Z}$ if and only if $2\rho(\alpha, \tilde{A}_{|c_{\hat\lambda}|,E}) \in \mathbb{Z}\alpha + \mathbb{Z}$.*
- *(B) If θ is ϵ_0-resonant for some $\frac{L(\hat\lambda)}{100C_4} > \epsilon_0 > 0$, then $\rho(\alpha, \tilde{A}_{|c_{\hat\lambda}|,E})$ is $\frac{\epsilon_0}{C+2}$-resonant.*

Proof of Theorem 4.3.3

(A): When θ is ϵ_0-non-resonant for some $\frac{L(\hat\lambda)}{100C_4} > \epsilon_0 > 0$, Theorem 4.3.1 implies $H_{\hat\lambda,\alpha}(\theta)$ has exponentially decaying eigenfunction. Then applying Theorem 4.2.3 we get $2\theta \in \mathbb{Z}\alpha + \mathbb{Z}$ if and only if $2\rho(\alpha, \tilde{A}_{|c_{\hat\lambda}|,E}) \in \mathbb{Z}\alpha + \mathbb{Z}$.

(B): Assume θ is ϵ_0-resonant for some $\frac{L(\hat\lambda)}{100C_4} > \epsilon_0 > 0$. Fix any $\xi < \frac{\epsilon_0}{2C+2}$, then there exists $C_\xi > 0$ such that for any $k \neq 0$ we have $\|k\alpha\| \geq C_\xi e^{-\xi|k|}$. Now take an ϵ_0-resonance n_j of θ such that $n = |n_j| > \max\left(\frac{-\ln C_\xi/2}{\epsilon_0-(2C+2)\xi}, N_0\right)$. Then there exists $|m| \leq Cn$ such that $2\rho(\alpha, \tilde{A}_{|c_{\hat\lambda}|,E}) - m\alpha = -2\theta$. Then

$$\|2\rho(\alpha, \tilde{A}_{|c_{\hat\lambda}|,E}) - (m-n_j)\alpha\| = \|2\theta - n_j\alpha\| < e^{-\epsilon_0 n} \leq e^{-\frac{\epsilon_0}{C+2}|m-n_j|}.$$

Take any $|l| \leq |m-n_j|$, $l \neq m-n_j$. Then

$$\|(l-(m-n_j))\alpha\| \geq C_\xi e^{-2\xi|m-n_j|} > 2e^{-\epsilon_0 n} > 2\|2\rho(\alpha, \tilde{A}_{|c_{\hat\lambda}|,E}) - (m-l_0)\alpha\|.$$

Thus $\|2\rho(\alpha, \tilde{A}_E) - l\alpha\| > \|2\rho(\alpha, \tilde{A}_{|c_{\hat\lambda}|,E}) - (m-n_j)\alpha\|$ for any $|l| \leq |m-n_j|$, $l \neq m-n_j$. This by definition means $\rho(\alpha, \tilde{A}_{|c_{\hat\lambda}|,E})$ is $\frac{\epsilon_0}{C+2}$-resonant. □

Now based on Theorem 4.3.3, we can complete the proof of the dry version of Ten Martini Problem for extended Harper's model in regions I° and II°.

Proof of Theorem 4.1.1

It is enough to consider λ in region II°. Let $E \in \Sigma_{\lambda,\alpha}$ be such that $N_{\lambda,\alpha}(E) \in \mathbb{Z}\alpha + \mathbb{Z}$. We are going to show E belongs to the boundary of a component of $\mathbb{R} \setminus \Sigma_{\lambda,\alpha}$. Now by (1.19) we have $2\rho(\alpha, \tilde{A}_{|c_{\hat\lambda}|,E}) \in \alpha\mathbb{Z} + \mathbb{Z}$, thus by Theorem 4.3.3, $2\theta(E) \in \alpha\mathbb{Z} + \mathbb{Z}$. By Theorem 4.2.3, this means there exist $M(x) \in C_h^\omega(\mathbb{T}, \mathrm{PSL}(2,\mathbb{R}))$ such that $M^{-1}(x+\alpha)\tilde{A}_{|c_{\hat\lambda}|,E}(x)M(x) = \begin{pmatrix} \pm 1 & a \\ 0 & \pm 1 \end{pmatrix}$. Without loss of generality, we assume

$M^{-1}(x+\alpha)\tilde{A}_{|c_{\hat{\lambda}}|,E}(x)M(x) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$. Let $\tilde{M}(x) = \frac{M(x)}{\sqrt{|c|(x-\alpha)}}$, then

$$\tilde{M}^{-1}(x+\alpha)\begin{pmatrix} \frac{E-v(x)}{|c|(x)} & -\frac{|c|(x-\alpha)}{|c|(x)} \\ 1 & 0 \end{pmatrix}\tilde{M}(x) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}.$$

Now let $\tilde{M}(x) = \begin{pmatrix} M_{11}(x) & M_{12}(x) \\ M_{21}(x) & M_{22}(x) \end{pmatrix}$. Then $M_{21}(x) = M_{11}(x-\alpha)$ and $M_{22}(x) = M_{12}(x-\alpha) - aM_{11}(x-\alpha)$ and

$$\begin{aligned}
&\tilde{M}^{-1}(x+\alpha)\begin{pmatrix} \frac{E+\epsilon-v(x)}{|c|(x)} & -\frac{|c|(x-\alpha)}{|c|(x)} \\ 1 & 0 \end{pmatrix}\tilde{M}(x) \\
=&\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} + \epsilon\begin{pmatrix} M_{11}(x)M_{12}(x) - aM_{11}^2(x) & M_{12}^2(x) - aM_{11}(x)M_{12}(x) \\ -M_{11}^2(x) & -M_{11}(x)M_{12}(x) \end{pmatrix}. \\
\triangleq& M_0 + \epsilon M_1(x).
\end{aligned}$$

Now we look for $Z_\epsilon(x)$ of the form $e^{\epsilon Y(x)}$ such that

$$Z_\epsilon^{-1}(x+\alpha)(M_0 + \epsilon M_1(x))Z_\epsilon(x) = M_0 + \epsilon[M_1] + O(\epsilon^2).$$

We then just need to solve the equation:

$$(I - \epsilon Y(x+\alpha) + O(\epsilon^2))(M_0 + \epsilon M_1(x))(I + \epsilon Y(x) + O(\epsilon^2)) = M_0 + \epsilon[M_1] + O(\epsilon^2).$$

It is sufficient to solve the coholomogical equation:

$$Y(x+\alpha)M_0 - M_0 Y(x) = M_1(x) - [M_1],$$

which is guaranteed by the Diophantine condition on α. Thus

$$\begin{aligned}
&(M(x+\alpha)Z_\epsilon(x+\alpha))^{-1}\tilde{A}_{|c_{\hat{\lambda}}|,E}(x)(M(x)Z_\epsilon(x)) \\
=&\begin{pmatrix} 1 + \epsilon[M_{11}M_{12}] - a\epsilon[M_{11}^2] & a + \epsilon[M_{12}^2] - a\epsilon[M_{11}M_{12}] \\ -\epsilon[M_{11}^2] & 1 - \epsilon[M_{11}M_{12}] \end{pmatrix} + O(\epsilon^2) \\
\triangleq& M_\epsilon + O(\epsilon^2).
\end{aligned}$$

Notice that $\tilde{A}_{|c_{\hat{\lambda}}|,E}$ is uniformly hyperbolic iff $\text{Trace}(M_\epsilon) > 2$ which is fulfilled when $-a\epsilon[M_{11}^2] > 0$. Thus for ϵ small, satisfying $-a\epsilon[M_{11}^2] > 0$, $E + \epsilon \notin \Sigma_{\lambda,\alpha}$, which means this spectral gap is open. □

4.4 Almost localization in region I$^{\text{o}}$

In this section we will prove Lemma 4.3.1. For fixed λ in region II$^{\text{o}}$ and E, let $D_{\hat{\lambda},E}(\theta) = c_{\hat{\lambda}}(\theta)A_{\hat{\lambda},E}(\theta)$.

Recall the following result in [37]. For any $E \in \Sigma_{\hat{\lambda},\alpha}$, we have,

$$L(\alpha, A_{\hat{\lambda},E}) = L(\alpha, D_{\hat{\lambda},E}) - \int_{\mathbb{T}} \ln|c_{\hat{\lambda}}(\theta)| \mathrm{d}\theta \triangleq \tilde{L} - \int \ln|c_{\hat{\lambda}}| > 0,$$

where $\tilde{L} = \ln \frac{\lambda_2+\sqrt{\lambda_2^2-4\lambda_1\lambda_3}}{2\lambda_2}$ and $\int \ln|c_{\hat{\lambda}}| = \ln \frac{\max(\lambda_1+\lambda_3,1)+\sqrt{\max(\lambda_1+\lambda_3,1)^2-4\lambda_1\lambda_3}}{2\lambda_2}$.

Proof of of Lemma 4.3.1

Suppose u is a solution satisfying the condition of Lemma 4.3.1. For an interval $I = [x_1, x_2]$, let Γ_I be the coupling operator between I and $\mathbb{Z} \setminus I$:

$$\Gamma_I(i,j) = \begin{cases} \tilde{c}_{\hat{\lambda}}(\theta + (x_1-1)\alpha), & (i,j) = (x_1, x_1 - 1) \\ c_{\hat{\lambda}}(\theta + (x_1-1)\alpha), & (i,j) = (x_1 - 1, x_1) \\ \tilde{c}_{\hat{\lambda}}(\theta + x_2\alpha), & (i,j) = (x_2 + 1, x_2) \\ c_{\hat{\lambda}}(\theta + x_2\alpha), & (i,j) = (x_2, x_2 + 1) \\ 0 & \text{otherwise.} \end{cases}$$

Let $H_I = R_I H_{\hat{\lambda},\alpha}(\theta) R_I^*$ be the restricted operator of $H_{\hat{\lambda},\alpha}(\theta)$ to I. Then for $x \in I$, we have $(H_I + \Gamma_I - E)u(x) = 0$. Thus $u(x) = G_I \Gamma_I u(x)$, where $G_I = (E - H_I)^{-1}$. By matrix multiplication:

$$\begin{aligned} u(x) &= \sum_{y \in I, (y,z) \in \Gamma_I} G_I(x,y)\Gamma_I(y,z)u(z) \\ &= \tilde{c}_{\hat{\lambda}}(\theta + (x_1-1)\alpha)G_I(x,x_1)u(x_1-1) + c_{\hat{\lambda}}(\theta + x_2\alpha)G_I(x,x_2)u(x_2+1). \end{aligned}$$

Let us denote $P_k(\theta) = \det(E - H_{[0,k-1]}(\theta))$. Then the $k-$step matrix $D_{\hat{\lambda},E,k}(\theta)$ satisfies:

$$D_{\hat{\lambda},E,k}(\theta) = \begin{pmatrix} P_k(\theta) & -\tilde{c}(\theta-\alpha)P_{k-1}(\theta+\alpha) \\ c(\theta+(k-1)\alpha)P_{k-1}(\theta) & -\tilde{c}(\theta-\alpha)c(\theta+(k-1)\alpha)P_{k-2}(\theta+\alpha) \end{pmatrix}.$$

This relation between $P_k(\theta)$ and $D_{\hat{\lambda},E,k}(\theta)$ gives a general upper bound of $P_k(\theta)$ in terms of $\tilde{L}$. Indeed by Lemma 4.2.3, for any $\epsilon > 0$ there exists $C(\epsilon) > 0$ so that

$$|P_n(\theta)| \le C(\epsilon) e^{(\tilde{L}+\epsilon)n} \quad \text{for any } n \in \mathbb{N}.$$

By Cramer's rule:

$$|G_I(x_1, y)| = \prod_{j=x_1}^{y-1} |c_{\hat{\lambda}}(\theta + j\alpha)| \left| \frac{\det(E - H_{[y+1,x_2]}(\theta))}{\det(E - H_I(\theta))} \right| = \prod_{j=x_1}^{y-1} |c_{\hat{\lambda}}(\theta + j\alpha)| \left| \frac{P_{x_2-y}(\theta + (y+1)\alpha)}{P_k(\theta + x_1\alpha)} \right|,$$
$$|G_I(y, x_2)| = \prod_{j=y+1}^{x_2} |c_{\hat{\lambda}}(\theta + j\alpha)| \left| \frac{\det(E - H_{[x_1,y-1]}(\theta))}{\det(E - H_I(\theta))} \right| = \prod_{j=y+1}^{x_2} |c_{\hat{\lambda}}(\theta + j\alpha)| \left| \frac{P_{y-x_1}(\theta + x_1\alpha)}{P_k(\theta + x_1\alpha)} \right|.$$

Notice that $P_k(\theta)$ is an even function about $\theta + \frac{k-1}{2}\alpha$, it can be written as a polynomial of degree k in $\cos 2\pi(\theta + \frac{k-1}{2}\alpha)$. Let $P_k(\theta) = Q_k(\cos 2\pi(\theta + \frac{k-1}{2}\alpha))$. Let $M_{k,r} = \{\theta \in \mathbb{T},\ |Q_k(\cos 2\pi\theta)| \le e^{(k+1)r}\}$.

Definition 4.4.1 *Fix $m > 0$. A point $y \in \mathbb{Z}$ is called $(k, m)-$regular if there exists an interval $[x_1, x_2]$ containing y, where $x_2 = x_1 + k - 1$ such that*

$$|G_I(y, x_i)| \le e^{-m|y-x_i|} \text{ and } \operatorname{dist}(y, x_i) \ge \frac{1}{3}k \text{ for } i = 1, 2,$$

otherwise y is called $(k, m)-$singular.

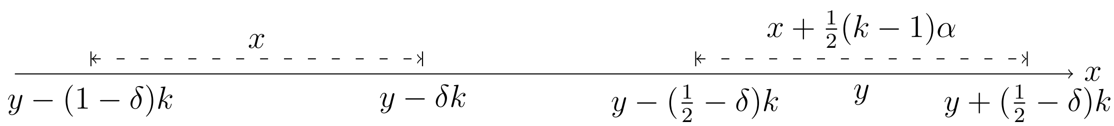
Lemma 4.4.1 *Suppose $y \in \mathbb{Z}$ is $(k, \tilde{L} - \int \ln|c_{\hat{\lambda}}| - \rho)-$singular. Then for any $\epsilon > 0$ and any $x \in \mathbb{Z}$ satisfying $y - \frac{2}{3}k \le x \le y - \frac{1}{3}k$, we have $\theta + (x + \frac{1}{2}(k-1))\alpha$ belongs to $M_{k,\tilde{L}-\frac{1}{3}\rho+\epsilon}$ for $k > k(\lambda, \epsilon, \rho)$.*

Proof: Suppose there exists $\epsilon > 0$ and x_1: $y - (1-\delta)k \le x_1 \le y - \delta k$, such that $\theta + (x_1 + \frac{1}{2}(k-1))\alpha$ does not belong to $M_{k,\tilde{L}-\frac{1}{3}\rho+\epsilon}$, that is $|P_k(\theta + x_1\alpha)| > e^{(k+1)(\tilde{L}-\rho\delta+\epsilon)}$,

$$|G_I(x_1, y)| \le \prod_{j=x_1}^{y-1} |c_{\hat{\lambda}}(\theta + j\alpha)| e^{(k-|x_1-y|)(\tilde{L}+\epsilon)} e^{-(k+1)(\tilde{L}-\frac{1}{3}\rho+\epsilon)}$$
$$< e^{-(\tilde{L}-\int \ln|c_{\hat{\lambda}}|-\rho)|y-x_1|} \quad \text{for } k > k(\lambda, \epsilon, \rho).$$

Similarly

$$|G_I(x_2, y)| \le e^{-(\tilde{L}-\int \ln|c_{\hat{\lambda}}|-\rho)|y-x_2|}.$$



□

Definition 4.4.2 *We say that the set* $\{\theta_1, ..., \theta_{k+1}\}$ *is* $\gamma-$*uniform if*

$$\max_{x\in[-1,1]} \max_{i=1,...,k+1} \prod_{j=1,j\neq i}^{k+1} \frac{|x-\cos 2\pi\theta_j|}{|\cos 2\pi\theta_i - \cos 2\pi\theta_j|} < e^{k\gamma}$$

Lemma 4.4.2 *Let* $\gamma_1 < \gamma$*. If* $\theta_1, ..., \theta_{k+1} \in M_{k,\tilde{L}-\gamma}$*, then* $\{\theta_1, ..., \theta_{k+1}\}$ *is not* γ_1-*uniform for* $k > k(\gamma, \gamma_1)$*.*

Proof: Otherwise, using Lagrange interpolation form we can get $|Q_k(x)| < e^{k\tilde{L}}$ for all $x \in [-1, 1]$. This implies $|P_k(x)| < e^{k\tilde{L}}$ for all x. But by Herman's subharmonic function argument, $\int_{\mathbb{R}/\mathbb{Z}} \ln |P_k(x)| dx \geq k\tilde{L}$. This is impossible. □

Now take ξ and ϵ_0 such that $0 < 1000\xi < \epsilon_0$. Then for $|n_{j+1}| > N(\xi)$ we have

$$2e^{-4\xi|n_{j+1}|} \leq C_\xi e^{-2\xi|n_{j+1}|} \leq \|(n_{j+1} - n_j)\alpha\| = \|n_{j+1}\alpha - 2\theta + 2\theta - n_j\alpha\| \leq 2\|2\theta - n_j\alpha\| \leq 2e^{-\epsilon_0|n_j|},$$

which yields that

$$|n_{j+1}| > \frac{\epsilon_0}{4\xi}|n_j| > 250|n_j|. \tag{4.2}$$

Without loss of generality, assume $3(|n_j| + 1) < y < \frac{|n_{j+1}|}{3}$ and $y > N(\xi)$. Select n such that $q_n \leq \frac{y}{8} < q_{n+1}$ and let s be the largest positive integer satisfying $sq_n \leq \frac{y}{8}$. Set $I_1, I_2 \subset \mathbb{Z}$ as follows

$$I_1 = [1 - 2sq_n, 0] \textit{ and } I_2 = [y - 2sq_n + 1, y + 2sq_n], \text{ if } n_j < 0$$

$$I_1 = [0, 2sq_n - 1] \textit{ and } I_2 = [y - 2sq_n + 1, y + 2sq_n], \text{ if } n_j \geq 0$$

Lemma 4.4.3 *Let* $\theta_j = \theta + j\alpha$*, then set* $\{\theta_j\}_{j\in I_1\cup I_2}$ *is* $C_4\epsilon_0 + C_4\xi-$*uniform for some absolute constant* C_4 *and* $y > y(\alpha, \epsilon_0, \xi)$*.*

Proof: Without loss of generality, we assume $n_j > 0$. Take $x = \cos 2\pi a$. Now it suffices to estimate

$$\sum_{j\in I_1\cup I_2,\ j\neq i} (\ln|\cos 2\pi a - \cos 2\pi\theta_j| - \ln|\cos 2\pi\theta_i - \cos 2\pi\theta_j|) \triangleq \sum_1 - \sum_2 .$$

Lemma 4.2.1 reduces this problem to estimating the minimal terms.

First we estimate $\sum_1$:

$$\begin{aligned}
\sum_1 &= \sum_{j\in I_1\cup I_2, j\neq i} \ln|\cos 2\pi a - \cos 2\pi\theta_j| \\
&= \sum_{j\in I_1\cup I_2, j\neq i} \ln|\sin\pi(a+\theta_j)| + \sum_{j\in I_1\cup I_2, j\neq i} \ln|\sin\pi(a-\theta_j)| + (6sq_n - 1)\ln 2 \\
&\triangleq \sum_{1,+} + \sum_{1,-} + (6sq_n - 1)\ln 2.
\end{aligned}$$

We cut $\sum_{1,+}$ or $\sum_{1,-}$ into $6s$ sums and then apply Lemma 4.2.1, we get that for some absolute constant C_1:

$$\sum_1 \leq -6sq_n \ln 2 + C_1 s \ln q_n.$$

Next, we estimate $\sum_2$.

$$\begin{aligned}
\sum_2 &= \sum_{j\in I_1\cup I_2, j\neq i} \ln|\cos 2\pi\theta_j - \cos 2\pi\theta_i| \\
&= \sum_{j\in I_1\cup I_2, j\neq i} \ln|\sin\pi(2\theta + (i+j)\alpha)| + \sum_{j\in I_1\cup I_2, j\neq i} \ln|\sin\pi(i-j)\alpha| + (6sq_n - 1)\ln 2 \\
&\triangleq \sum_{2,+} + \sum_{2,-} + (6sq_n - 1)\ln 2.
\end{aligned}$$

We need to carefully estimate the minimal terms. For $\sum_{2,+}$, we use the property of resonant set; and for $\sum_{2,-}$, we use the Diophantine condition on α.

For any $0 < |j| < q_{n+1}$, we have $\|j\alpha\| \geq \|q_n\alpha\| \geq C_\xi e^{-\xi q_n}$. Therefore

$$\max(\ln|\sin x|, \ln|\sin(x + \pi j\alpha)|) \geq -2\xi q_n \quad \text{for } y > y(\alpha,\xi).$$

This means in any interval of length sq_n, there can be at most one term which is less than $-2\xi q_n$. Then there can be at most 6 such terms in total.

For the part $\sum_{2,-}$, since $\|(i-j)\alpha\| \geq C_\xi e^{-\xi|i-j|} \geq e^{-20\xi s q_n}$, these 6 smallest terms must be bounded by $-20\xi s q_n$ from below. Hence $\sum_{2,-} \geq -6 s q_n \ln 2 - C\xi s q_n - C s \ln q_n$ for $y > y(\xi)$ and some absolute constant C.

For the part $\sum_{2,+}$, notice $|i+j| \leq 2y + 4 s q_n < 3y < |n_{j+1}|$ and $i+j > 0 > -n_j$. Suppose $\|2\theta + k_0\alpha\| = \min_{j\in I_1\cup I_2} \|2\theta + (i+j)\alpha\| \leq e^{-100\epsilon_0 s q_n} < e^{-\epsilon_0|k_0|}$. Then for any $|k| \leq |k_0| \leq 40 s q_n$ (including $|n_j|$),

$$\|2\theta - k\alpha\| \geq \|(k+k_0)\alpha\| - \|2\theta + k_0\alpha\| > \|2\theta + k_0\alpha\| \quad \text{for } y > y(\alpha, \epsilon_0, \xi).$$

This means $-k_0$ must be a ϵ_0-resonance, therefore $|k_0| \leq |n_{j-1}|$. Then

$$\|2\theta - n_j\alpha\| \geq \|(n_j + k_0)\alpha\| - \|2\theta + k_0\alpha\| \geq C_\xi e^{-12\xi s q_n} - e^{-100\epsilon_0 s q_n} > e^{-100\epsilon_0 s q_n} \geq \|2\theta + k_0\alpha\|$$

leads to a contradiction. Thus the smallest terms must be greater than $-100\epsilon_0 s q_n$. We can bound $\sum_{2,+}$ by $-6 s q_n \ln 2 - 600\epsilon_0 s q_n - 12\xi s q_n - C s \ln q_n$ from below. Therefore $\sum_2 \geq -6 s q_n \ln 2 - C\epsilon_0 s q_n - C\xi s q_n - C s \ln q_n$. Thus the set $\{\theta_j\}_{j\in I_1\cup I_2}$ is $C_4\epsilon_0 + C_4\xi-$uniform for $y > y(\alpha, \epsilon_0, \xi)$ and some absolute constant C_4. $\square$

Now let C_4 be the absolute constant in Lemma 4.4.3. Choose $0 < 1000\xi < \epsilon_0 < \frac{L(\hat{\lambda})}{100 C_4}$. Combining Lemma 4.4.2 and Lemma 4.4.3, we know that when $y > y(\alpha, \epsilon_0, \xi)$, $\{\theta_j\}_{j\in I_1\cup I_2}$ can not be inside the set $M_{6sq_n-1, \tilde{L}-2C_4\epsilon_0}$ at the same time. Therefore 0 and y can not be $(6sq_n - 1, \tilde{L} - \int \ln|c_{\hat{\lambda}}| - 9C_4\epsilon_0)$ at the same time. However 0 is $(6sq_n - 1, \tilde{L} - \int \ln|c_{\hat{\lambda}}| - 9C_4\epsilon_0)-$singular given n large enough. Therefore

$$\{\theta_j\}_{j\in I_1} \subset M_{6sq_n-1, \tilde{L}-2C_4\epsilon_0}.$$

Thus y must be $(6sq_n - 1, \tilde{L} - \int \ln|c_{\hat{\lambda}}| - 9C_4\epsilon_0)-$regular. This implies

$$|u(y)| \leq e^{-(\tilde{L} - \int \ln|c_{\hat{\lambda}}| - 9C_4\epsilon_0)\frac{1}{4}|y|} < e^{-\frac{L(\hat{\lambda})}{5}|y|} \quad \text{for } |y| \geq y(\lambda, \alpha, \epsilon_0, \xi).$$

Thus there exists $C_3 = C_{\lambda,\alpha,\epsilon_0,\xi}$ such that $|u(y)| \leq C_3 e^{-\frac{L(\hat{\lambda})}{5}|y|}$ for any $3|n_j| \leq |y| \leq \frac{1}{3}|n_{j+1}|$ and $j \in \mathbb{N}$.

4.5 Almost reducibility in region II$^{\mathrm{o}}$

Proof of Theorem 4.3.2

For any $E \in \Sigma_{\lambda,\alpha}$, take $\theta(E)$ and $\{u_k\}$ as in Theorem 4.2.2. Let C_4 be the absolute constant from Lemma 4.4.3, and C_2 be the absolute constant from Lemma 4.2.2. Fix $\max(32C_2\xi, 1000\xi) < \epsilon_0 < \min(\frac{L(\hat{\lambda})}{200}, \frac{L(\hat{\lambda})}{100C_4})$. By Lemma 4.3.1, there exists C depending on λ and α such that for any $3|n_j| < |k| < \frac{1}{3}|n_{j+1}|$, we have $|u_k| \leq Ce^{-\frac{L(\hat{\lambda})}{5}|k|}$.

For any n, $9|n_j| < n < \frac{1}{9}|n_{j+1}|$, of the form

$$n = rq_m - 1 < q_{m+1}. \text{ [1]} \tag{4.3}$$

Let $u(x) = u^I(x) = \sum_{k\in I} u_k e^{2\pi ikx}$ with $I = [-[\frac{n}{2}], [\frac{n}{2}]] = [x_1, x_2]$. Define

$$U(x) = \begin{pmatrix} e^{2\pi i\theta}u(x) \\ u(x-\alpha) \end{pmatrix}.$$

Let $A(\theta) = A_{c_\lambda,E}(\theta)$ and $\tilde{A}(\theta) = \tilde{A}_{|c_{\hat{\lambda}}|,E}(\theta)$. By direct computation:

$$A(x)U(x) = e^{2\pi i\theta}U(x+\alpha) + \begin{pmatrix} g(x) \\ 0 \end{pmatrix} \triangleq e^{2\pi i\theta}U(x+\alpha) + G(x).$$

The Fourier coefficients of $g(x)$ are possibly nonzero only at four points x_1, x_2, $x_1 - 1$ and $x_2 + 1$. Since $|u_k| \leq C_1 e^{-\frac{L(\hat{\lambda})}{5}|k|}$ when $3|n_j| < |k| < \frac{1}{3}|n_{j+1}|$, we know that $\|G(x)\|_{\frac{L(\hat{\lambda})}{20\pi}} \leq C_1 e^{-\frac{L(\hat{\lambda})}{20}n}$.

Combining Theorem 1.6.2 and 4.2.3, we have exponential control of the growth of the transfer matrix, for any $\delta > 0$ there exists $C_\delta > 0$ such that

$$\|\tilde{A}_k(x)\|_{\frac{L(\hat{\lambda})}{2\pi}} \leq C_\delta e^{\delta|k|}, \quad \text{for any } k.$$

With some effort we are able to get the following significantly improved upper bound:

Theorem 4.5.1 *For some $C > 0$ depending on λ and α,*

$$\|\tilde{A}_k(x)\|_{\mathbb{T}} \leq C(1+|k|)^C.$$

[1] The existence of such n comes from (4.2).

Proof: Let $\tilde{U}(x) = Q(x)U(x)$, $\tilde{G}(x) = Q(x+\alpha)G(x)$, where $Q = Q_\lambda$ is given in $(A.0.2)$. Since

$$\max\left(\|Q(x)\|_{\frac{L(\hat{\lambda})}{20\pi}}, \|Q^{-1}(x)\|_{\frac{L(\hat{\lambda})}{20\pi}}\right) \leq C,$$

we have

$$\tilde{A}(x)\tilde{U}(x) = e^{2\pi i\theta}\tilde{U}(x+\alpha) + \tilde{G}(x),$$

where $\|\tilde{G}(x)\|_{\frac{L(\hat{\lambda})}{20\pi}} \leq Ce^{-\frac{L(\hat{\lambda})}{20}n}$.

Lemma 4.5.1 *Let C_2 be the constant from Lemma 4.2.2, then for any δ, $2C_2\xi < \delta < \frac{\epsilon_0}{16}$, we have*

$$\inf_{|\mathrm{Im}(x)| \leq \frac{L(\hat{\lambda})}{20\pi}} \|\tilde{U}(x)\| \geq e^{-2\delta n},$$

for $n > n(\alpha, \delta)$.

Proof: We will prove the statement by contradiction. Suppose for some $x_0 \in \{|\mathrm{Im}(x)| \leq \frac{L(\hat{\lambda})}{20\pi}\}$ we have $\|\tilde{U}(x_0)\| < e^{-2\delta n}$. Notice that for any $l \in \mathbb{N}$,

$$e^{2\pi il\theta}\tilde{U}(x_0+l\alpha) = \tilde{A}_l(x_0)\tilde{U}(x_0) - \sum_{m=1}^{l} e^{2\pi i(m-1)\theta}\tilde{A}_{l-m}(x_0+m\alpha)\tilde{G}(x_0+(m-1)\alpha).$$

This implies for $n > n(\delta)$ large enough and for any $0 \leq l \leq n$, $\|\tilde{U}(x_0+l\alpha)\| \leq e^{-\delta n}$, thus $\|u(x_0+l\alpha)\| \leq C_\delta e^{-\delta n}$. By Lemma 4.2.2, $\|u(x+i\mathrm{Im}(x_0))\|_{\mathbb{T}} \leq C_2 C_\delta e^{C_2\xi n}e^{-\delta n} \leq e^{-\frac{\delta}{2}n}$. This contradicts with $\int_{\mathbb{T}} u(x+i\mathrm{Im}(x_0))\mathrm{d}x = u_0 = 1$. □

Lemma 4.5.2 *[3] Let $V : \mathbb{T} \to \mathbb{C}^2$ be analytic in $|\mathrm{Im}(x)| < \eta$. Assume that $\delta_1 < \|V(x)\| < \delta_2^{-1}$ holds on $|\mathrm{Im}(x)| < \eta$. Then there exists $M : \mathbb{T} \to \mathrm{SL}(2,\mathbb{C})$ analytic on $|\mathrm{Im}(x)| < \eta$ with first column V and $\|M\|_\eta \leq C\delta_1^{-2}\delta_2^{-1}(1-\ln(\delta_1\delta_2))$.*

Applying Lemma 4.5.2, let $M(x)$ be the matrix with first column $\tilde{U}(x)$. Then $e^{-2\delta n} \leq \|\tilde{U}(x)\|_{\frac{\delta}{\pi}} \leq e^{\delta n}$ and hence $\|M(x)\|_{\frac{\delta}{\pi}} \leq Ce^{6\delta n}$. Therefore

$$M^{-1}(x+\alpha)\tilde{A}(x)M(x) = \begin{pmatrix} e^{2\pi i\theta} & 0 \\ 0 & e^{-2\pi i\theta} \end{pmatrix} + \begin{pmatrix} \beta_1(x) & b(x) \\ \beta_3(x) & \beta_4(x) \end{pmatrix}$$

where $\|\beta_1(x)\|_{\frac{\delta}{\pi}},\ \|\beta_3(x)\|_{\frac{\delta}{\pi}},\ \|\beta_4(x)\|_{\frac{\delta}{\pi}} \le Ce^{-\frac{L(\hat\lambda)}{40}n}$, and $\|b(x)\|_{\frac{\delta}{\pi}} \le Ce^{13\delta n}$. Let

$$\Phi(x) = M(x)\begin{pmatrix} e^{\frac{L(\hat\lambda)}{160}n} & 0 \\ 0 & e^{-\frac{L(\hat\lambda)}{160}n} \end{pmatrix}.$$

Then we would have:

$$\Phi(x+\alpha)^{-1}\tilde A(x)\Phi(x) = \begin{pmatrix} e^{2\pi i\theta} & 0 \\ 0 & e^{-2\pi i\theta} \end{pmatrix} + H(x),$$

where $\|H(x)\|_{\frac{\delta}{\pi}} \le Ce^{-\frac{L(\hat\lambda)}{160}n}$, and $\|\Phi(x)\|_{\frac{\delta}{\pi}} \le Ce^{\frac{L(\hat\lambda)}{80}n}$. Thus

$$\sup_{0\le s\le e^{\frac{L(\hat\lambda)}{320}n}} \|\tilde A_s(x)\|_{\mathbb{T}} \le e^{\frac{L(\hat\lambda)}{20}n}$$

for $n \ge n(\lambda,\alpha)$ satisfying (4.3). For s large, there always exists $9|n_j| < n < \frac{1}{9}|n_{j+1}|$ satisfying (4.3) such that $cn \le \frac{320}{L(\hat\lambda)}\ln s \le n$ with some absolute constant c. Thus there exists C depending on λ and α such that $\|\tilde A_k(x)\|_{\mathbb{T}} \le C(1+|k|)^C$. □

Now we come back to the proof of Theorem 4.3.2. Fix some $n = |n_j|$, and $N = |n_{j+1}|$. Let $u(x) = u^{I_2}(x)$ with $I_2 = [-[\frac{N}{9}], [\frac{N}{9}]]$ and $U(x) = \begin{pmatrix} e^{2\pi i\theta}u(x) \\ u(x-\alpha) \end{pmatrix}$. Then

$$A(x)U(x) = e^{2\pi i\theta}U(x+\alpha) + G(x) \ \ \textit{with} \ \ \|G(x)\|_{\frac{L(\hat\lambda)}{20\pi}} \le Ce^{-\frac{L(\hat\lambda)}{90}N}.$$

Define $U_0(x) = e^{\pi i n_j x}U(x)$. Notice that if n_j is even, then $U_0(x)$ is well-defined on $\mathbb{T}$, otherwise $U_0(x+1) = -U_0(x)$.

$$\tilde A(x)\tilde U_0(x) = e^{2\pi i\tilde\theta}\tilde U_0(x+\alpha) + H(x),$$

where $\tilde\theta = \theta - \frac{n_j}{2}\alpha$, $\tilde U_0(x) = Q(x)U_0(x)$ and $\|H(x)\|_{\frac{L(\hat\lambda)}{20\pi}} \le Ce^{-\frac{L(\hat\lambda)}{100}N}$. Consider the matrix $W(x)$ with $\tilde U_0(x)$ and $\overline{\tilde U_0(x)}$ being its two columns. Then

$$\tilde A(x)W(x) = W(x+\alpha)\begin{pmatrix} e^{2\pi i\tilde\theta} & 0 \\ 0 & e^{-2\pi i\tilde\theta} \end{pmatrix} + \tilde H(x).$$

Theorem 4.5.2 *Let $L^{-1} = \|2\theta - n_j\alpha\|$. Then for $n > N_0(\lambda,\alpha)$ we have*

$$|\det W(x)| \ge L^{-4C} \ \ \text{for any } x\in\mathbb{T},$$

where C is the constant appeared in Theorem 4.5.1.

Proof: First, we fix $\xi_1 < \frac{\epsilon_0}{1600}$ so that $\|k\alpha\| \geq C_{\xi_1} e^{-\xi_1|k|}$ for any $k \neq 0$. We have the following estimate about L:

Lemma 4.5.3 $e^{\epsilon_0 n} \leq L \leq e^{4\xi_1 N}$.

$$e^{-2\xi_1 N} \leq \|(n_{j+1}-n_j)\alpha\| \leq 2\|n_j\alpha - 2\theta\| = 2L^{-1} \leq 2e^{-\epsilon_0 n} \quad \text{for } n \geq N(\xi_1).$$

Now we prove by contradiction. Suppose there exists κ and $x_0 \in \mathbb{T}$ such that $\|\tilde{U}_0(x_0) - \kappa\overline{\tilde{U}_0(x_0)}\| < L^{-4C}$. Then

$$\begin{aligned}
&\|\tilde{U}_0(x_0+l\alpha)e^{2\pi i l\tilde{\theta}} - \kappa\overline{\tilde{U}_0(x_0+l\alpha)}e^{-2\pi i l\tilde{\theta}}\| \\
\leq &\|\sum_{m=0}^{l-1}\tilde{A}_{l-m}(x_0+m\alpha)H(x_0+m\alpha) - \kappa\sum_{m=0}^{l-1}\tilde{A}_{l-m}(x_0+m\alpha)\overline{H(x_0+m\alpha)}\| + \|A_l(x_0)\|L^{-4C} \\
\leq &CL^{2C}e^{-\frac{L(\hat{\lambda})}{100}N} + CL^{-2C} < L^{-C}.
\end{aligned}$$

for $0 \leq |l| \leq L^2$. If we take $j = \frac{L}{4}$, then

$$\|\tilde{U}_0(x_0+\frac{L}{4}\alpha) + \kappa\overline{\tilde{U}_0(x_0+\frac{L}{4}\alpha)}\| < L^{-1}. \tag{4.4}$$

Next since $\|U_0(x)\|_{\mathbb{T}} \leq n$, we have $\|\tilde{U}_0(x)\|_{\mathbb{T}} \leq Cn$. Thus

$$\|\tilde{U}_0(x_0+l\alpha) - \kappa\overline{\tilde{U}_0(x_0+l\alpha)}\| < L^{-\frac{1}{3}} \quad \text{for } 0 \leq |l| \leq L^{\frac{1}{2}}.$$

For any analytic function $f(x) = \sum_{k\in\mathbb{Z}}\hat{f}_k e^{2\pi i kx}$, define $f_{[-m,m]}(x) = \sum_{|k|\leq m}\hat{f}_k e^{2\pi i kx}$. For any column vector $V(x) = \begin{pmatrix} v^{(1)}(x) \\ v^{(2)}(x)\end{pmatrix}$, let $V_{[-m,m]}(x) = \begin{pmatrix} v^{(1)}_{[-m,m]}(x) \\ v^{(2)}_{[-m,m]}(x)\end{pmatrix}$. Now let us define $\tilde{U}_0^{[9n]}(x) = Q(x)e^{\pi i n_j x}U_{[-9n,9n]}(x)$. Then

$$\|\tilde{U}_0^{[9n]}(x) - \tilde{U}_0(x)\|_{\mathbb{T}} \leq Ce^{-\frac{9}{5}L(\hat{\lambda})n}.$$

Consider $[e^{-\pi i n_j x}\tilde{U}_0^{[9n]}(x)]_{[-18n,18n]}(x)e^{\pi i n_j x}$. This function differs from a polynomial with essential degree $36n$ only by a multiple of $e^{\pi i n_j x}$. Notice that $Q(x)$ is analytic in $\{x : |\mathrm{Im}(x)| \leq \frac{L(\hat{\lambda})}{4\pi}\}$, thus $|\hat{Q}(k)| \leq Ce^{-\frac{L(\hat{\lambda})}{2}|k|}$. Then

$$|\widehat{e^{-\pi i n_j x}\tilde{U}_0^{[9n]}}(k)| \leq \sum_{|m|\leq 9n}|\hat{Q}(k-m)\hat{U}(m)| \leq Cne^{-\frac{L(\hat{\lambda})}{2}(|k|-9n)} \quad \text{for } |k| \geq 18n.$$

Thus

$$\|e^{-\pi i n_j x}\tilde{U}_0^{[9n]}(x) - [e^{-\pi i n_j x}\tilde{U}_0^{[9n]}]_{[-18n,18n]}(x)\|_{\mathbb{T}} \leq e^{-4L(\hat{\lambda})n},$$

$$\|\tilde{U}_0(x) - [e^{-\pi i n_j x}\tilde{U}_0^{[9n]}]_{[-18n,18n]}(x)e^{\pi i n_j x}\|_{\mathbb{T}} \leq e^{-4L(\hat{\lambda})n}.$$

Hence

$$\begin{aligned}&\|[e^{-\pi i n_j x}\tilde{U}_0^{[9n]}]_{[-18n,18n]}(x_0+l\alpha)e^{2\pi i n_j(x_0+l\alpha)} - \kappa\overline{[e^{-\pi i n_j x}\tilde{U}_0^{[9n]}]_{[-18n,18n]}(x_0+l\alpha)}\|_{\mathbb{T}}\\ <&2L^{-\frac{1}{3}} + e^{-4L(\hat{\lambda})n},\end{aligned}$$

for $|l| \leq L^{\frac{1}{2}}$. Notice that

$$[e^{-\pi i n_j x}\tilde{U}_0^{[9n]}]_{[-18n,18n]}(x)e^{2\pi i n_j x} - \kappa\overline{[e^{-\pi i n_j x}\tilde{U}_0^{[9n]}]_{[-18n,18n]}(x)}$$

is a polynomial whose essential degree is at most $37n$. Thus by Lemma 4.2.2, we would have

$$\|[e^{-\pi i n_j x}\tilde{U}_0^{[9n]}]_{[-18n,18n]}(x)e^{\pi i n_j x} - \kappa\overline{[e^{-\pi i n_j x}\tilde{U}_0^{[9n]}]_{[-18n,18n]}(x)e^{\pi i n_j x}}\|_{\mathbb{T}} < L^{-\frac{1}{4}} + e^{-2L(\hat{\lambda})n}.$$

Hence $\|\tilde{U}_0(x) - \kappa\overline{\tilde{U}_0(x)}\|_{\mathbb{T}} < L^{-\frac{1}{4}} + 2e^{-2L(\hat{\lambda})n}$. But combining with (9.1) we would get $\|\tilde{U}_0(x_0 + \frac{L}{4}\alpha)\| < 2L^{-\frac{1}{4}} + 2e^{-2L(\hat{\lambda})n}$, but this contradicts with $\inf_{x\in\mathbb{T}} \|\tilde{U}_0(x)\| > e^{-2\delta n}$ since $\delta < \frac{\epsilon_0}{16}$. □

Now for $n > N_0(\lambda, \alpha)$, take $S(x) = \mathrm{Re}\tilde{U}_0(x)$ and $T(x) = \mathrm{Im}\tilde{U}_0(x)$. Let $W_1(x)$ be the matrix with columns $S(x)$ and $T(x)$. Notice that $\det W_1(x)$ is well-defined on $\mathbb{T}$ and $\det W_1(x) \neq 0$ on $\mathbb{T}$, hence without loss of generality we could assume $\det W_1(x) > 0$ on $\mathbb{T}$, otherwise we simply take $W_1(x)$ to be the matrix with columns $S(x)$ and $-T(x)$. Then

$$\|\tilde{A}(x)W_1(x) - W_1(x+\alpha)R_{-\tilde{\theta}}\|_{\mathbb{T}} \leq Ce^{-\frac{L(\hat{\lambda})}{45}N}.$$

By taking determinant, we get

$$\det W_1(x) = \det W_1(x+\alpha) + O(e^{-\frac{L(\hat{\lambda})}{50}N}) \ \text{ on } \mathbb{T}.$$

Since $\det W_1(x)$ is analytic on $|\mathrm{Im}x| \le \frac{L(\hat{\lambda})}{20\pi}$, by considering the Fourier coefficients we could get

$$\det W_1(x) = w_0 + O(e^{-\frac{L(\hat{\lambda})}{100}N}) \text{ on } \mathbb{T},$$

where $w_0 \ge L^{-5C}$. Thus $\det W_1(x)$ is almost a positive constant.

Define $W_2(x) = \det W_1(x)^{-\frac{1}{2}} W_1(x)$. Then $W_2(x) \in C^\omega(\mathbb{T})$ and $\det W_2(x) = 1$. We have

$$W_2^{-1}(x+\alpha)\tilde{A}(x)W_2(x) = \frac{\det W_1(x+\alpha)^{\frac{1}{2}}}{\det W_1(x)^{\frac{1}{2}}} R_{-\tilde{\theta}} + O(e^{-\frac{L(\hat{\lambda})}{100}N}) \text{ on } \mathbb{T},$$

$$W_2^{-1}(x+\alpha)\tilde{A}(x)W_2(x) = R_{-\tilde{\theta}} + O(e^{-\frac{L(\hat{\lambda})}{200}N}) \text{ on } \mathbb{T}.$$

Now let's prove $\deg W_2(x) \le 36n$. $\deg W_2(x)$ is the same as the degree of its columns. For

$M : \mathbb{R}/2\mathbb{Z} \to \mathbb{R}^2$, we say $\deg M = k$ if M is homotopic to $\begin{pmatrix} \cos k\pi x \\ \sin k\pi x \end{pmatrix}$.

For some constant $c > 0$, we obviously have

$$\int_{\mathbb{T}} \|S(x)\| \,\mathrm{d}x + \int_{\mathbb{T}} \|T(x)\| \,\mathrm{d}x \ge \int_{\mathbb{T}} \|S(x) + iT(x)\| \,\mathrm{d}x = \int_{\mathbb{T}} \|\tilde{U}_0(x)\| \,\mathrm{d}x \ge c.$$

Without loss of generality we could assume $\int_{\mathbb{T}} \|S(x)\| \,\mathrm{d}x > \frac{c}{2}$. Also

$$\tilde{A}(x)S(x) = S(x+\alpha)\cos 2\pi\tilde{\theta} - T(x+\alpha)\sin 2\pi\tilde{\theta} + O(e^{-\frac{L(\hat{\lambda})}{45}N}) \text{ on } \mathbb{T}.$$

Then since $\|2\tilde{\theta}\| = L^{-1}$,

$$\tilde{A}(x)S(x) = S(x+\alpha) + O(L^{-\frac{1}{2}}) \text{ on } \mathbb{T}.$$

First we prove $\inf_{x\in\mathbb{T}} \|S(x)\| \ge e^{-2L(\hat{\lambda})n}$. Suppose otherwise. Then there exists $x_0 \in \mathbb{T}$, so that $\|S(x_0)\| < e^{-2L(\hat{\lambda})n}$. Then $\|\mathrm{Re}\tilde{U}_0(x_0 + l\alpha)\| < e^{-\frac{\epsilon_0}{8}n}$ for $|l| < e^{\frac{\epsilon_0}{4C}n}$, where C is the constant that appeared in Theorem 4.5.1. We have already shown that

$$\|\tilde{U}_0(x) - [e^{-\pi i n_j x}\tilde{U}_0^{[9n]}]_{[-18n,18n]} e^{\pi i n_j x}\|_{\mathbb{T}} < e^{-4L(\hat{\lambda})n}.$$

Thus

$$\|\mathrm{Re}[e^{-\pi i n_j x}\tilde{U}_0^{[9n]}]_{[-18n,18n]}(x_0 + l\alpha)\| < e^{-\frac{\epsilon_0}{16}n}$$

for $|l| < e^{\frac{\epsilon_0}{4C}n}$. However $\mathrm{Re}[e^{-\pi i n_j x}\tilde{U}_0^{[9n]}]_{[-18n,18n]}$ is a polynomial with essential degree at most $36n$. Using Lemma 4.2.2 we are able to get $\|\mathrm{Re}[e^{-\pi i n x}\tilde{U}_0^{[9n]}]_{[-18n,18n]}e^{\pi i n_j x}\|_{\mathbb{T}} < e^{-\frac{\epsilon_0}{32}n}$, and thus $\|\mathrm{Re}\tilde{U}_0(x)\|_{\mathbb{T}} < e^{-\frac{\epsilon_0}{64}n}$ which is a contradiction to $\int_{\mathbb{T}} \|\mathrm{Re}\tilde{U}_0(x)\| \, \mathrm{d}x > \frac{c}{2}$. At the meantime, we also get $\|S(x) - \mathrm{Re}[e^{-\pi i n_j x}\tilde{U}_0^{[9n]}]_{[-18n,18n]}(x)e^{\pi i n_j x}\|_{\mathbb{T}} \triangleq \|S(x) - h(x)\|_{\mathbb{T}} \leq e^{-4L(\hat{\lambda})n}$. The first column of $W_2(x)$ is $\det W_1(x)^{-\frac{1}{2}}S(x)$. We have

$$
\begin{aligned}
&\|\frac{S(x)}{\det W_1(x)^{\frac{1}{2}}} - \frac{h(x)}{w_0{}^{\frac{1}{2}}}\| \\
\leq &\frac{1}{|\det W_1(x)^{\frac{1}{2}}|}\|S(x) - h(x) + (1 - \frac{\det W_1(x)^{\frac{1}{2}}}{w_0{}^{\frac{1}{2}}})h(x)\| \\
\leq &L^{2C}(e^{-4L(\hat{\lambda})n} + L^{8C}e^{-\frac{L(\hat{\lambda})}{100}N}) \\
\leq &e^{-3L(\hat{\lambda})n} < \|\frac{S(x)}{\det W_1(x)^{\frac{1}{2}}}\| \quad \text{on } \mathbb{T}.
\end{aligned}
$$

Thus by Rouché's theorem $|\deg W_2(x)| = |\deg h(x)| \leq 19n$. Notice that

$$|\rho(\alpha, W_2^{-1}\tilde{A}W_2) + \tilde{\theta}| < Ce^{-\frac{L(\hat{\lambda})}{200}N}.$$

Then, by 1.16 for some $|m| \leq 19n$:

$$|\rho(\alpha, \tilde{A}) - \frac{m}{2}\alpha + \tilde{\theta}| < Ce^{-\frac{L(\hat{\lambda})}{200}N}.$$

□

Chapter 5

Quantum dynamical bounds for ergodic potentials with underlying dynamics of zero topological entropy

5.1 Introduction

Positive Lyapunov exponents are generally viewed as a signature of localization. While it is known that they can coexist even with almost ballistic transport [62] [27], vanishing of certain dynamical exponents has been identified as a reasonable expected consequence of hyperbolicity of the corresponding transfer-matrix cocycle. Results in this direction were obtained in [25] [26] for one-frequency trigonometric polynomials, and recently in [45], for one-frequency quasiperiodic potentials under very mild assumptions on regularity of the sampling function. In this paper we identify a general property responsible for positive Lyapunov exponents implying vanishing of the dynamical quantities in the rather general case of underlying dynamics defined by volume preserving maps of Riemannian manifolds with zero topological entropy, and under very minimal regularity assumptions. This work presents the first localization-type results that hold in such generality. We expect that positive topological entropy should also lead to vanishing of the dynamical quantities for a.e. (but not every!) phase, but this should be approached by completely different methods and will be explored in a future work.

Our general results allow us, in particular, to obtain localization-type statements for potentials defined by shifts and skew-shifts of higher-dimensional tori. Pure point spectrum with exponentially decaying eigenfunctions has been obtained for a.e. multi-frequency shifts in the regime of positive Lyapunov exponents in [17] and for the skew-shift on $\mathbb{T}^2$ with a perturbative condition in [18], both very delicate results. While bounds on transport exponents are certainly weaker than dynamical localization that often (albeit not always [49]) accompanies pure point spectrum [19], we note that pure point spectrum can be destroyed by generic rank one perturbations [28] while vanishing of the transport exponents is robust in this respect. Finally, our results are the first ones for both of these families that hold under purely arithmetic conditions and the first non-perturbative ones for the skew-shift.

Let $(\mathcal{M}, g)$ be a d-dimensional compact (smooth) Riemannian manifold with a metric g. Let Vol_g be its Riemannian volume density (see (5.3)). Let f be a uniquely ergodic volume preserving map on $\mathcal{M}$, which means Vol_g is its unique invariant probability measure. We will study the dynamical properties of the Schrödinger operator acting on $l^2(\mathbb{Z})$:

$$H_{v,f}(\theta)u(n) = u(n+1) + u(n-1) + v(f^n\theta)u(n). \tag{5.1}$$

The time dependent Schrödinger equation

$$i\partial_t u = H_{v,f}(\theta)u,$$

leads to a unitary dynamical evolution

$$u(t) = e^{-itH_{v,f}(\theta)}u(0).$$

Under the time evolution, the wavepacket will in general spread out with time. For operators with absolutely continuous spectrum, scattering theory already leads to a good understanding of the quantum dynamics. In this paper we will study the spreading of the wavepacket under positive Lyapunov exponent assumption, which automatically implies the absence of absolutely continuous spectrum.

Let $e^{-itH_{v,f(\theta)}}\delta_0$ be the time evolution with the localized initial state δ_0. Let

$$a_\theta(n,t) = |\langle e^{-itH_{v,f(\theta)}}\delta_0, \delta_n\rangle|^2.$$

$a_\theta(n,t)$ describes the probability of finding the wavepacket at site n at time t. We denote the p-th moment of $a_\theta(n,t)$ by

$$\langle |X|_\theta^p(t)\rangle = \sum_n (1+|n|)^p a_\theta(n,t).$$

Dynamical localization is defined as boundedness of $\langle |X|_\theta^p(t)\rangle$ in time t. This implies purely point spectrum, therefore for general operators with positive Lyapunov exponent such a strong control of the wavepacket is not possible. Thus we need to define proper transport exponents which decribe the rate of the spreading of the wavepacket. For $p>0$ define the upper and lower transport exponents

$$\beta_\theta^+(p) = \limsup_{t\to\infty} \frac{\ln\langle |X|_\theta^p(t)\rangle}{p\ln t}; \quad \beta_\theta^-(p) = \liminf_{t\to\infty} \frac{\ln\langle |X|_\theta^p(t)\rangle}{p\ln t}.$$

Obtaining upper bounds for the two transport exponents above implies a power-law control of the spreading rate of the entire wavepacket.

It is also interesting to consider a portion of the wavepacket. For a nonnegative function $A(t)$ of time, let

$$\langle A(t)\rangle_T = \frac{2}{T}\int_0^\infty e^{-2t/T}A(t)\ dt$$

be its time average. Set

$$P_{\theta,T}(L) = \sum_{|n|\le L}\langle a_\theta(n,t)\rangle_T.$$

Roughly speaking, $P_{\theta,T}(T^a) > \tau$ means that, in average, over time T, a portion of the wavepacket stays inside a box of size T^a. Let us introduce two other scaling exponents:

$$\overline{\xi_\theta} = \lim_{\tau\to 0}\limsup_{T\to\infty}\frac{\ln\inf\{L|P_{\theta,T}(L)+P_{f\theta,T}(L)>\tau\}}{\ln T}$$

$$\underline{\xi_\theta} = \lim_{\tau\to 0}\liminf_{T\to\infty}\frac{\ln\inf\{L|P_{\theta,T}(L)+P_{f\theta,T}(L)>\tau\}}{\ln T}$$

The vanishing of $\beta^{\pm}$ and $\overline{\xi}$, $\underline{\xi}$ can be viewed as localization-type statements. For $\mathcal{M} = \mathbb{T}$ the one-dimensional torus, $f = f_{s,\alpha} : \theta \to \theta + \alpha$ the irrational rotation by α, the Lebesgue measure m is the unique invariant probability measure of f. It was first proved in [25], [26] that for v being trigonometric polynomial, under the assumption of positive Lyapunov exponent, $\beta_\theta^+(p) = 0$ for all $p > 0$, all θ and Diophantine α; $\beta_\theta^- = 0$ for all $p > 0$, all θ and all α. It was recently proved in [45] that under very mild restrictions on regularity of the potential, under the assumption of positivity and continuity of the Lyapunov exponent, $\beta_\theta^+(p) = 0$ for all $p > 0$, all θ and Diophantine α; $\beta_\theta^-(p) = 0$ for all $p > 0$, all θ and all α. It was also proved in [45] that for piecewise Hölder function, under the assumption of positive Lyapunov exponent, $\overline{\xi}_\theta = 0$ for a.e.θ and Diophantine α, $\underline{\xi}_\theta = 0$ for a.e.θ and all α.

Remark 5.1.1 *The two Diophantine sets of α are different between [25], [26] and [45]. They are both full measure sets, but [45] covers slightly thinner set of frequencies because they need to handle potentials with weaker regularity.*

In this paper we consider d-dimensional compact Riemannian manifold $\mathcal{M}$ and uniquely ergodic volume preserving map f. We consider maps with the following volume scaling property. For $1 \le l \le d$, let $\Sigma(l)$ be the set of C^∞ mappings $\sigma : Q^l \to \mathcal{M}$ where Q^l is the l-dimensional unit cube. Let $\mathrm{Vol}_{g,l}(\sigma)$ be the induced l-dimensional volume of the image of σ in $\mathcal{M}$ counted with multiplicity, i.e. if σ is not one-to-one, and the image of one part coincides with that from another part, then we will count the set as many times as it iscovered. For $n = 1, 2, ...$ and $1 \le l \le d$, let

$$(5.2) \qquad V_l(f) = \sup_{\sigma \in \Sigma(l)} \limsup_{n \to \infty} \frac{1}{n} \log \mathrm{Vol}_{g,l}(f^n \sigma) \quad \text{and} \quad V(f) = \max_l V_l(f).$$

Volume preserving f always satisfies $V_d(f) = V_d(f^{-1}) = 0$. Here we need to make an extra assumption that $V(f) = V(f^{-1}) = 0$. It is known that for smooth invertible map f, $V(f) = V(f^{-1})$ is equal to the *topological entropy* of f [69], thus our class of maps includes all smooth maps with zero topological entropy. In particular, it includes both the irrational rotation and the skew-shift.

For such maps we will assume that f has a bounded discrepancy.

Let $J_N(\theta) = J(\theta, f\theta, ..., f^{N-1}\theta)$ (see (5.9)) be the isotropic discrepancy function of the sequence $\{f^n\theta\}_{n=0}^{N-1}$. For $\delta > 0$, we will say f has *strongly δ-bounded isotropic discrepancy* if $J_N(\theta) \le |N|^{-\delta}$ uniformly in θ for $|N| > N_0$; f has *weakly δ-bounded isotropic discrepancy* if there exists a sequence $\{N_j\}$ such that $J_{N_j}(\theta) \le |N_j|^{-\delta}$ uniformly in θ. It turns out many concrete dynamical systems feature these properties. We will show in Lemmas 5.3.6 - 5.3.8 that the following holds.

- For the shifts of higher dimensional tori, $f = f_{s,\alpha} : \theta \to \theta + \alpha$ has strongly bounded isotropic discrepancy for Diophantine α;

- For the skew-shift $f = f_{ss,\alpha} : (y_1, y_2, ..., y_d) \to (y_1 + \alpha, y_2 + y_1, ..., y_d + y_{d-1})$, has strongly bounded isotropic discrepancy for Diophantine α, and weakly bounded isotropic discrepancy for Liouvillean α.

Under the assumption of boundedness of discrepancy and scaling property of f, we are ready to formulate the following two abstract results.

Let μ_θ be the spectral measure of H_θ corresponding to δ_0. Let $N = \int_{\mathcal{M}} \mu_\theta \, \mathrm{dVol}_g$ be the integrated density of states.

Theorem 5.1.1 *Let v be a piecewise Hölder function, suppose $L(E)$ is positive on a Borel subset U with $N(U) > 0$. Suppose f is a uniquely ergodic volume preserving map satisfying $V(f) = V(f^{-1}) = 0$ and for some $\delta > 0$*

- *f has weakly δ-bounded isotropic discrepancy, then $\underline{\xi_\theta} = 0$ for Vol_g-a.e. $\theta \in \mathcal{M}$;*

- *f has strongly δ-bounded isotropic discrepancy, then $\overline{\xi_\theta} = 0$ for Vol_g-a.e. $\theta \in \mathcal{M}$.*

Remark 5.1.2 *The full measure set of θ appearing in Theorem 5.1.1 is precisely the set $\{\theta : \mu_\theta + \mu_{f\theta}(U) > 0\}$.*

Theorem 5.1.2 *Let v be a piecewise Hölder function, suppose $L(E)$ is continuous in E and $L(E) > 0$ for every $E \in \mathbb{R}$. Suppose f is a uniquely ergodic volume preserving map satisfying $V(f) = V(f^{-1}) = 0$ and for some $\delta > 0$*

- *f has weakly δ-bounded isotropic discrepancy, then $\beta_\theta^-(p) = 0$ for all $\theta \in \mathcal{M}$ and $p > 0$;*

- *f has strongly δ-bounded isotropic discrepancy, then $\beta_\theta^+(p) = 0$ for all $\theta \in \mathcal{M}$ and $p > 0$.*

Theorems 5.1.1, 5.1.2 extend the results of [45] from irrational rotations of the circle to general uniquely ergodic maps of compact Riemannian manifolds with zero topologogical entropy and bounded discrepancy. One key to achieving such generality is a new argument that does not rely on harmonic analysis/ approximation by trigonometric polynomials.

By [24], $\beta_\theta^-(p) \geq p \dim_H(\mu_\theta)$ where $\dim_H(\mu)$ is the Hausdorff dimension of μ. Thus as a consequence of $\beta_\theta^-(p) = 0$ we have the following

Corollary 5.1.1 *Under the assumption of Theorem 5.1.2, $\dim_H(\mu_\theta) = 0$ for all $\theta \in \mathcal{M}$.*

Remark The point here is that we obtain zero Hausdorff dimension of the spectral measure for *all* rather than a.e. $\theta \in \mathcal{M}$ (the latter follows for general ergodic potentials). The following Theorems 5.1.3 - 5.1.6 are all corollaries of our abstract results. Theorems 5.1.7 and 5.1.8 depend on a somewhat different technique (bypassing the discrepancy considerations), which allow us to cover more frequencies in case of the shift of $\mathbb{T}^2$. To our knowledge, Theorems 5.1.3 -5.1.8 are the first arithmetic localization-type results.

Theorem 5.1.1 reduces vanishing of (upper or lower) ξ_θ to bounds on the isotropic discrepancy. As corollaries, we obtain

Theorem 5.1.3 *Let $(\mathcal{M}, f) = (\mathbb{T}^d, f_{s,\alpha})$. For piecewise Hölder v, suppose $L(E)$ is positive on a Borel subset U with $N(U) > 0$. Then if $\alpha \in DC(\tau) \subset \mathbb{T}^d$, $\overline{\xi_\theta} = 0$ for a.e. $\theta \in \mathbb{T}^d$.*

Remark 5.1.3 *The Diophantine condition is essential for the vanishing of $\overline{\xi}$ [50].*

Theorem 5.1.4 *Let $(\mathcal{M}, f) = (\mathbb{T}^d, f_{ss,\alpha})$. For piecewise Hölder v, suppose $L(E)$ is positive on a Borel subset U with $N(U) > 0$. Then*

- *for all irrational α, $\underline{\xi_{y_1,y_2,\ldots,y_d}} = 0$ for a.e. $(y_1, y_2, \ldots, y_d) \in \mathbb{T}^d$,*

- *if* $\alpha \in DC(\tau)$ *for some* $\tau > 1$, $\overline{\xi_{y_1,y_2,...,y_d}} = 0$ *for a.e.* $(y_1, y_2, ..., y_d) \in \mathbb{T}^d$.

Remark 5.1.4 *The full measure set appearing in Theorems 5.1.3 and 5.1.4 are precisely the set* $\{\theta : \mu_\theta + \mu_{f\theta}(U) > 0\}$.

Similarly, for systems with continuous Lyapunov exponent, Theorem 5.1.2 reduces vanishing of $\beta_\theta^\pm(p)$ to the same discrepancy bounds, and we obtain

Theorem 5.1.5 *Let* $(\mathcal{M}, f) = (\mathbb{T}^d, f_{s,\alpha})$. *For piecewise Hölder* ϕ, *suppose* $L(E)$ *is continuous in* E *and* $L(E) > 0$ *for every* $E \in \mathbb{R}$. *Then if* $\alpha \in DC(\tau) \subset \mathbb{T}^d$, $\beta_\theta^+(p) = 0$ *for all* $\theta \in \mathbb{T}^d$, $p > 0$.

Corollary 5.1.2 *Under the assumption of Theorem 5.1.5, if* $\alpha \in DC(\tau)$, $\dim_H(\mu_\theta) = 0$ *for all* $\theta \in \mathbb{T}^d$.

Remark 5.1.5 *The Diophantine condition is essential for* $\beta^+ = 0$ *[50].*

Theorem 5.1.6 *Let* $(\mathcal{M}, f) = (\mathbb{T}^d, f_{ss,\alpha})$. *For piecewise Hölder* v, *suppose* $L(E)$ *is continuous in* E *and* $L(E) > 0$ *for every* $E \in \mathbb{R}$. *Then*

- *for all irrational* α, $\beta^-_{y_1,y_2,...,y_d}(p) = 0$ *for all* $(y_1, y_2, ..., y_d) \in \mathbb{T}^d$, $p > 0$,
- *if* $\alpha \in DC(\tau)$ *for some* $\tau > 1$, $\beta^+_{y_1,y_2,...,y_d}(p) = 0$ *for all* $(y_1, y_2, ..., y_d) \in \mathbb{T}^d$, $p > 0$.

Corollary 5.1.3 *Under the assumption of Theorem 5.1.6, for all irrational* α, $\dim_H(\mu_\theta) = 0$ *for all* $(y_1, y_2, ..., y_d) \in \mathbb{T}^d$.

Finally, for the case of the irrational shift $\mathbb{T}^2$ we can make two more delicate statements, using a different technique to obtain arithmetic estimates.

Theorem 5.1.7 *Let* $(\mathcal{M}, f) = (\mathbb{T}^2, f_{s,\alpha})$. *For piecewise Hölder* v, *suppose* $L(E)$ *is positive on a Borel subset* U *with* $N(U) > 0$. *Then if* $\alpha = (\alpha_1, \alpha_2) \in \cup_{\tau>1} WDC(\tau)$, $\underline{\xi_\theta} = 0$ *for a.e.* $\theta \in \mathbb{T}^2$.

Remark 5.1.6 *The full measure set appearing in Theorem 5.1.7 is precisely the set* $\{\theta : \mu_\theta + \mu_{f\theta}(U) > 0\}$.

Theorem 5.1.8 *Let* $(\mathcal{M}, f) = (\mathbb{T}^2, f_{s,\alpha})$. *For piecewise Hölder* v, *suppose* $L(E)$ *is continuous in* E *and* $L(E) > 0$ *for every* $E \in \mathbb{R}$. *Then if* $\alpha = (\alpha_1, \alpha_2) \in \cup_{\tau>1} WDC(\tau)$, $\beta_\theta^-(p) = 0$ *for all* $\theta \in \mathbb{T}^2$, $p > 0$.

Corollary 5.1.4 *Under the assumption of Theorem 5.1.8, if* $\alpha \in \cup_{\tau>1} WDC(\tau)$, $\dim_H(\mu_\theta) = 0$ *for all* $\theta \in \mathbb{T}^2$.

The most technically complex part of the paper consists in obtaining arithmetic estimates on covering of the torus by the trajectory of a small ball in a polynomial (in the inverse radius) time, which we obtain by estimating the discrepancy in Theorems 5.1.3 - 5.1.6, and by the bounded remainder set technique in Theorems 5.1.7, 5.1.8. The discrepancy estimates are standard for the Diophantine shifts and are ideologically similar to the known results on equidistribution of $n^k\alpha$, for the case of higher dimensional Diophantine skew shifts. We still develop the proof for the Diophantine skew shift case in full detail because we did not find it in the literature and also because it serves as a good preparation to the Liouville higher dimensional skew shift, for which to the best of our knowledge, our estimates are new. We note that for the *Diophantine* skew shift of $\mathbb{T}^2$ and shifts of $\mathbb{T}^d$ the results on the covering of the torus by a trajectory of a ball are shown in [4] by a completely different technique, through solving the cohomological equation. By the nature of the cohomological equation that technique is not extandable to the Liouville or weakly Diophantine case.

We organize this paper as follows: in section 2 we introduce some basic definitions. Some of them have been mentioned in the introduction but not in details. In section 3 we will present some key lemmas and proofs of Theorems 5.1.1 - 5.1.8. In sections 4-8 we prove the key lemmas that are listed in section 3.

5.2 Preparation

5.2.1 Riemannian manifold

Let $\mathcal{M}$ be a d-dimensional compact Riemannian manifold with a Riemannian metric g.

Let K be a compact set in some coordinate patch $(U, x^1, ..., x^d)$. We define the volume of K to be

$$\mathrm{Vol}_g(K) := \int_{x(K)} \sqrt{|G \circ x^{-1}|} dx^1 \cdots dx^d,$$

where $G = \det g_{ij}$, $g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$ and $dx^1 \cdots dx^d$ is the Lebesgue measure on $\mathbb{R}^d$. This definition is free of choice of coordinate. If K is not contained in a single coordinate patch, one could apply partition of unity to define $\mathrm{Vol}_g(K)$. More precisely, we pick an atlas $(U_\alpha, x^1_\alpha, ..., x^d_\alpha)$ of $\mathcal{M}$ and a partition of unity $\{\rho_\alpha\}$ subordinate to this atlas. Now we can set

$$\mathrm{Vol}_g(K) = \sum_\alpha \int_{x^\alpha(K \cap U_\alpha)} (\rho_\alpha \sqrt{|G^\alpha|}) \circ (x^\alpha)^{-1} dx^1_\alpha \cdots dx^d_\alpha.$$

The *Riemannian volume density* (see e.g.[64], section 3.4) on $(\mathcal{M}, g)$ is

$$\mathrm{dVol}_g = \sum_\alpha (\rho_\alpha \sqrt{|G^\alpha|}) \circ (x^\alpha)^{-1} dx^1_\alpha \cdots dx^d_\alpha. \tag{5.3}$$

In the above definition, we do not assume $\mathcal{M}$ to be oriented. If $\mathcal{M}$ is oriented, then the volume density is actually a positive n-form, called the volume form.

If $\varrho : [a, b] \to \mathcal{M}$ is a continuously differentiable curve in the Riemannian manifold $\mathcal{M}$, then we define its length $l(\varrho)$ by

$$l(\varrho) = \int_a^b \sqrt{g_{\varrho(t)}(\dot{\varrho}(t), \dot{\varrho}(t))}\, dt,$$

where $g_{\varrho(t)}$ is the inner product g at the point $\varrho(t)$. One could define the distance between any two point x, $y \in \mathcal{M}$ as follows

$$\begin{aligned} &dist(x, y) \\ =& \inf\{l(\varrho) : \varrho \text{ is a continuous, piecewise continuously differentiable curveconnecting } x \text{ and } y\}. \end{aligned}$$

With the definition of distance, *geodesics* in a Riemannian manifold are then the locally distance-minimizing paths.

Let $v \in \mathrm{T}_x\mathcal{M}$ be a tangent vector to the manifold $\mathcal{M}$ at x. Then there is a unique geodesic ϱ_v satisfying $\varrho_v(0) = x$ with initial tangent vector $\dot{\varrho}_v(0) = v$. The corresponding *exponential map* is defined by $\exp_x(v) = \varrho_v(1)$.

Let $B_r(x) = \{y \in \mathcal{M} : dist(x, y) < r\}$ be a *geodesic ball* centered at $x \in \mathcal{M}$ with radius r. It is known that $B_r(x) = \exp_x(B(0, r))$ where $B(0, r) = \{v \in \mathrm{T}_x\mathcal{M} : g_x(v, v) < r\}$.

Proposition 5.2.1 *There exists $r_g > 0$ so that for all $r < r_g$, there exists positive constants C_g and c_g which are independent of $x \in \mathcal{M}$ so that*

$$c_g r^d \leq \mathrm{Vol}_g(B_r(x)) \leq C_g r^d \text{ for any } x \in \mathcal{M}. \tag{5.4}$$

Proof: We will discuss about the proof briefly. We could identify the tangent space $\mathrm{T}_x\mathcal{M}$ isometrically with $\mathbb{R}^d$. Now $\exp_x : \mathbb{R}^d \to \mathcal{M}$ is a diffeomorphism on some small ball $B_{\mathbb{R}^d}(0, r)$. On this ball, straight lines are mapped to length-minimizing geodesics ([22], Proposition 3.6), and thus Euclidean balls are mapped to geodesic balls of the same radius. Taking r smaller if necessary, we can assume the Jacobian of $\exp_x$ is bounded away from 0 and ∞ on $B_{\mathbb{R}^d}(0, r)$, thus for $r < r_x$ we have that $c_{g_x} r^d \leq \mathrm{Vol}_g(B_r(x)) \leq C_{g_x} r^d$. Since $\mathcal{M}$ is a compact manifold, we could take r_x, c_{g_x}, C_{g_x} independent of $x \in \mathcal{M}$. □

We call a subset C of $\mathcal{M}$ is said to be a *geodesically convex set* if, given any two points in C, there is a minimizing geodesic contained within C that joins those two points.

The *convexity radius at a point* $x \in \mathcal{M}$ is the supremum (which may be $+\infty$) of $r_x \in \mathbb{R}$ such that for all $r < r_x$ the geodesic ball $B_{r_x}(x)$ is geodesically convex. The *convexity radius of* $(\mathcal{M}, g)$ is the infimum over the points $x \in \mathcal{M}$ of the convexity radii at these points.

Proposition 5.2.2 *[16] For compact manifold $\mathcal{M}$, the convexity radius r'_g of $(\mathcal{M}, g)$ is positive.*

This clearly implies for any $x \in \mathcal{M}$, any $r < r'_g$, $B_r(x)$ is geodesically convex.

5.2.2 Piecewise Hölder function

Let $L_\gamma(\mathcal{M})$ be the space of γ-Lipschitz functions on $\mathcal{M}$. For $v \in L_\gamma(\mathcal{M})$ define

$$\|v\|_{L_\gamma} = \|v\|_\infty + \sup_{\theta_1,\theta_2\in M} \frac{|v(\theta_1)-v(\theta_2)|}{dist(\theta_1,\theta_2)^\gamma}. \tag{5.5}$$

We say v is piecewise Hölder if there exists $\gamma > 0$, positive integer K and $\{v_j\}_{j=1}^K \subset L_\gamma(\mathcal{M})$ so that

$$v(\theta) = \sum_{j=1}^K \chi_{S_j}(\theta) v_j(\theta)$$

where $\{S_j\}_{j=1}^M$ are sets with "good boundary", namely $\{\partial S_j\}_{j=1}^K$ are $d-1$ dimensional smooth submanifolds of $\mathcal{M}$. Clearly the discontinuity set J_v of v is $\cup_{j=1}^K \partial S_j$, and

$$\mathrm{Vol}_{g,d-1}(J_v) \le \sum_{j=1}^K \mathrm{Vol}_{g,d-1}(\partial S_j) < \infty. \tag{5.6}$$

Clearly for any two points θ_1, θ_2 so that $dist(\theta_i, J_v) \ge r$, if $dist(\theta_1,\theta_2) < r$ then we have

$$|v(\theta_1)-v(\theta_2)| \le dist(\theta_1,\theta_2)^\gamma \sum_{j=1}^K \|v_j\|_{L_\gamma}. \tag{5.7}$$

5.2.3 Spectral measure and integrated density of states

Let μ_θ be the spectral measure of H_θ corresponding to δ_0 defined by

$$\langle (H_\theta - z)^{-1}\delta_0, \delta_0\rangle = \int_{\mathbb{R}} \frac{d\mu_\theta(x)}{x-z}.$$

Then clearly $\mu_{f\theta}$ is the spectral measure of H_θ corresponding to δ_1. Let $N = \int_{\mathcal{M}} \frac{\mu_\theta+\mu_{f\theta}}{2}\, \mathrm{dVol}_g(\theta)$ be the integrated density of states. Clearly $N(U) > 0$ for some set U implies $\frac{\mu_\theta+\mu_{f\theta}}{2}(U) > 0$ for Vol_g-$a.e.$ $\theta \in \mathcal{M}$.

5.2.4 Discrepancy

Let $\vec{x}_1, ..., \vec{x}_N \in \mathcal{M}$, for a subset C of $\mathcal{M}$, let the counting function

$$A(C; \{\vec{x}_n\}_{n=1}^N) = \sum_{n=1}^N \chi_C(\vec{x}_n) \tag{5.8}$$

The *isotropic discrepancy* $J_N(\{\vec{x}_n\}_{n=1}^N)$ is defined as

$$J_N(\{\vec{x}_n\}_{n=1}^N) = \sup_{C\in\mathscr{C}} |\frac{A(C;\{\vec{x}_n\}_{n=1}^N}{N} - \mathrm{Vol}_g(C)|, \tag{5.9}$$

where $\mathscr{C}$ is the family of all geodesically convex subsets of $\mathcal{M}$.

For a point $\theta \in \mathcal{M}$, let $J_N(\theta) = J(\{f^n\theta\}_{n=0}^{N-1})$. We say a map $f : \mathcal{M} \to \mathcal{M}$ has *strongly δ-bounded isotropic discrepancy* if for some $N > N_0$, $J_N(\theta) \le N^{-\delta}$ uniformly in $\theta \in \mathcal{M}$. We say f has *weakly δ-bounded isotropic discrepany* if there is a subsequence $\{N_j\}$ such that $J_{N_j}(\theta) \le N_j^{-\delta}$ uniformly in $\theta \in \mathcal{M}$.

If $\mathcal{M} = \mathbb{T}^d$ be the d-dimensional torus, we define the *discrepancy* $D_N(\{\vec{x}_n\}_{n=1}^N)$ as follows

$$D(\{\vec{x}_n\}_{n=1}^N) = \sup_{C\in\mathscr{J}} |\frac{A(C;\{\vec{x}\}_{n=1}^N)}{N} - m(C)|, \tag{5.10}$$

where $\mathscr{J}$ is the family of any subinterval C of the form $C = \{(\theta_1, ..., \theta_d) \in \mathbb{T}^d : \beta_i \le \theta_i < \kappa_i \text{ for } 1 \le i \le d\}$.

For a point $\theta \in \mathbb{T}^d$, let $D_N(\theta) = D(\{f^n\theta\}_{n=0}^{N-1})$. We say a map $f : \mathbb{T}^d \to \mathbb{T}^d$ has *strongly δ-bounded discrepancy* if for some $N > N_0$, $D_N(\theta) \le N^{-\delta}$ uniformly in $\theta \in \mathbb{T}^d$. We will f has *weakly δ-bounded discrepany* if there is a subsequence $\{N_j\}$ such that $D_{N_j}(\theta) \le N_j^{-\delta}$ uniformly in $\theta \in \mathbb{T}^d$.

When $\mathcal{M} = \mathbb{T}^d$, the isotropic discrepancy and discrepancy can be tightly controled by each other:

Lemma 5.2.1 *([58], Theorem 1.6 in Chapter 2) For any sequence $\{\vec{x}_n\}_{n=1}^N$ in $\mathbb{T}^d$, we have*

$$D_N(\{\vec{x}_n\}_{n=1}^N) \le J_N(\{\vec{x}_n\}_{n=1}^N) \le (4d\sqrt{d}+1) D_N(\{\vec{x}_n\}_{n=1}^N)^{\frac{1}{d}}. \tag{5.11}$$

Therefore, by (5.11), when $\mathcal{M} = \mathbb{T}^d$,

Proposition 5.2.3 *f has strong (weak) δ-bounded isotropic discrepancy for some $\delta > 0$ is equivalent to f has strong (weak) $\tilde{\delta}$-bounded discrepancy for some $\tilde{\delta} > 0$.*

In section 5 and 6 we are going to apply the following two inequalities to estimate the upper bound of discrepancy.

Lemma 5.2.2 *[56] [Erdös-Turán-Koksma* inequality] *For any positive integer H_0, we have*

$$D(\{\vec{x}_n\}_{n=1}^N) \leq C_d(\frac{1}{H_0} + \sum_{0<|\vec{h}|\leq H_0} \frac{1}{r(\vec{h})}|\frac{1}{N}\sum_{n=1}^N e^{2\pi i\langle \vec{h},\vec{x}_n\rangle}|) \tag{5.12}$$

where $|\vec{h}| = \max_{j=1}^d |h_j|$.

Lemma 5.2.3 (*e.g.[58], Lemma 3.1 in Chapter 1*) [*Van der Corput's* Fundamental Inequality] *For any integer* $1 \leq H \leq N$*, we have*

$$|\frac{1}{N}\sum_{n=1}^N u_n|^2 \leq \frac{N+H-1}{N^2H}\sum_{n=1}^N |u_n|^2 + \frac{2(N+H-1)}{N^2H^2}\sum_{k=1}^{H-1}(H-k)\mathrm{Re}\sum_{n=1}^{N-k} u_n\overline{u_{n+k}}. \tag{5.13}$$

5.3 Key lemmas and proofs of Theorem 5.1.1 - 5.1.8

5.3.1 Covering $\mathcal{M}$ with the orbit of a geodesic ball and Proofs of Theorem 5.1.1, 5.1.7, 5.1.2 and 5.1.8

Lemma 5.3.1 *Let v be a piecewise Hölder function with $1 \geq \gamma > 0$. Suppose $L(E)$ is positive on a Borel subset U with $N(U) > 0$.*

1. *If there exists a sequence $r_k \to 0$ so that any geodesic ball in $\mathcal{M}$ with radius r_k covers the whole $\mathcal{M}$ in r_k^{-M} steps, then $\underline{\xi_\theta} = 0$ for* Vol_g*-a.e.* $\theta \in \mathcal{M}$;
2. *If for any small $r > 0$, any geodesic ball with radius r covers the whole $\mathcal{M}$ in r^{-M} steps, then $\overline{\xi_\theta} = 0$ for* Vol_g*-a.e.* $\theta \in \mathcal{M}$;

Lemma 5.3.2 *Let v be a piecewise Hölder function with $1 \geq \gamma > 0$. Suppose $L(E)$ is continuous in E and $L(E) > 0$ for every $E \in \mathbb{R}$.*

1. *If there exists a sequence $r_k \to 0$ so that any geodesic ball in $\mathcal{M}$ with radius r_k covers the whole $\mathcal{M}$ in r_k^{-M} steps, then $\beta_\theta^-(p) = 0$ for all $\theta \in \mathcal{M}$ and $p > 0$;*

2. *If for any small $r > 0$, any geodesic ball with radius r covers the whole $\mathcal{M}$ in r^{-M} steps, then $\beta^+_\theta(p) = 0$ for all $\theta \in \mathcal{M}$ and $p > 0$.*

Lemmas 5.3.1 and 5.3.2 are key to our abstract argument. They are proved in section 4. The connection to bounded discrepancy comes in the following

Let r_g be as in Proposition 5.2.1 and r'_g as in Proposition 5.2.2.

Lemma 5.3.3 *If f has weakly δ-bounded isotropic discrepancy, then there exists $r_k \to 0$ as $k \to \infty$ such that any geodesic ball in $\mathcal{M}$ with radius r_k will cover the whole $\mathcal{M}$ in $r_k^{-\frac{2d}{\delta}}$ steps.*

Proof: There exists a sequence $\{N_k\}$ and $k_0 > 0$ such that for any $k > k_0$ we have $J_{N_k}(\{f^n\theta\}_{n=0}^{N-1}) \leq N_k^{-\delta}$. This means for any geodesically convex set $C \subset \mathcal{M}$, $\frac{\sum_{n=0}^{N_k-1} \chi_C(f^n\theta)}{N_k} - \mathrm{Vol}_g(C) \geq -N_k^{-\delta}$ holds for all $\theta \in \mathcal{M}$. Thus if we take $r_k = N_k^{-\frac{\delta}{2d}} < \min(r_g, r'_g)$, then by Proposition 5.2.2, we know $B_{r_k}(\theta)$ is geodesically convex. By Proposition 5.2.1, $\mathrm{Vol}_g(B_{r_k}(\theta)) \geq c_g r_k^d = c_g N_k^{-\frac{\delta}{2}} > N_k^{-\delta}$. Thus $\sum_{n=0}^{r_k^{-\frac{2d}{\delta}}-1} \chi_{B_{r_k}(\theta)}(f^n\theta) > 0$ for any $\theta \in \mathcal{M}$. □

Lemma 5.3.4 *If f has strongly δ-bounded isotropic discrepancy, then for any $0 < r < \min(r_g, r'_g)$, any geodesic ball in $\mathcal{M}$ with radius r will cover the whole $\mathcal{M}$ in $r^{-\frac{2d}{\delta}}$ steps.*

Proof: There exists N_0 such that for any $N > N_0$ we have $J_N(\{f^n\theta\}_{n=0}^{N-1}) \leq N^{-\delta}$ for all $\theta \in \mathcal{M}$. This means for any $0 < r < \min(r_g, r'_g)$, any geodesic ball $B_r(\theta)$ (it is geodesically convex by Proposition 5.2.2) and $N = r^{-\frac{2d}{\delta}}$ we have $\frac{\sum_{n=0}^{r^{-\frac{2d}{\delta}}-1} \chi_{B_r(\theta)}(f^n\theta)}{r^{-\frac{2d}{\delta}}} - \mathrm{Vol}_g(B_r(\theta)) \geq -r^{2d}$. Since by Proposition 5.2.1, $\mathrm{Vol}_g(B_r(\theta)) \geq c_g r^d > r^{2d}$, we have $\sum_{n=0}^{r^{-\frac{2d}{\delta}}-1} \chi_{B_r(\theta)}(f^n\theta) > 0$ for any $\theta \in \mathcal{M}$. □

In the case of 2-dimensional irrational rotation, we also have

Lemma 5.3.5 *For any $(\alpha_1, \alpha_2) \in \cup_{\tau>1} WDC(\tau)$, there exists $r_k(\alpha_1, \alpha_2, \tau) \to 0$ as $k \to \infty$ such that any Euclidean ball with radius r_k covers the whole $\mathbb{T}^2$ in $r_k^{-800\tau^4}$ steps.*

Remark 5.3.1 *This lemma will be proved in section 8.*

We are now ready to complete the proof of the main Theorems.

Proof of Theorem 5.1.1

Combining Lemma 5.3.3, 5.3.4 with Lemma 5.3.1.

Proof of Theorem 5.1.7

Combining Lemma 5.3.5 with Lemma 5.3.1.

Proof of Theorem 5.1.2

Combining Lemma 5.3.3, 5.3.4 with Lemma 5.3.2.

Proof of Theorem 5.1.8

Combining Lemma 5.3.5 with Lemma 5.3.2.

5.3.2 Estimation of Discrepancy and Proofs of Theorems 5.1.3, 5.1.5, 5.1.4 and 5.1.6

For irrational rotation and skew-shift, we have the following control of their discrepancies.

Lemma 5.3.6 *If $\alpha \in DC(\tau)$, then for some constant $\delta > 0$, $D_N(\{\theta + n\alpha\}_{n=0}^{N-1}) \leq N^{-\delta}$ uniformly in $\theta \in \mathbb{T}^d$.*

Let $\vec{Y}_n = (y_1 + \binom{n}{1}\alpha,\ y_2 + \binom{n}{1}y_1 + \binom{n}{2}\alpha,\ ...,\ y_d + \binom{n}{1}y_{d-1} + \cdots + \binom{n}{d}\alpha)$.

Lemma 5.3.7 *If $\alpha \in DC(\tau)$, then for some constant $\delta > 0$, $D_N(\{\vec{Y}_n\}_{n=1}^{N}) \leq N^{-\delta}$ uniformly in $(y_1, ..., y_d) \in \mathbb{T}^d$.*

Lemma 5.3.8 *If $\alpha \notin DC(d)$, then for some constant $\delta > 0$ there exists a sequence $\{N_j\}$ so that $D_{N_j}(\{\vec{Y}_n\}_{n=1}^{N_j}) \leq N_j^{-\delta}$ uniformly in $(y_1, ..., y_d) \in \mathbb{T}^d$.*

Remark 5.3.2 *The proof of Lemma 5.3.6 will be given in section 5, the proofs of Lemma 5.3.7 and 5.3.8 will be given in section 6.*

Proof of Theorem 5.1.3, 5.1.5

Follows from Lemma 5.3.6 and Theorems 5.1.1, 5.1.2.

Proof of Theorem 5.1.4, 5.1.6

Follows from Lemma 5.3.7, 5.3.8 and Theorems 5.1.1, 5.1.2.

5.4 Proofs of Lemmas 5.3.1 and 5.3.2

5.4.1 Upper and lower bounds of transfer matrices

The following lemma about uniform upper bound of transfer matrix is essentially from [45]. We have adapted it into the following form for convenience.

Lemma 5.4.1 (*[45], Theorem 3.1*) *Let v be a function whose discontinuity set has measure 0 and f be a uniquely ergodic map on $\mathcal{M}$. Then*

5.4.1.1 *Let $L(E)$ be positive on a Borel set U and μ be a measure with $\mu(U) > 0$. Then for any $\zeta > 0$ there exists a number $D_\zeta > 0$, and for any $\epsilon > 0$ there exists a set $B_{\zeta,\epsilon}$ with $0 < \mu(B_{\zeta,\epsilon}) < \zeta$, and an integer $N_{\zeta,\epsilon}$ so that for any $E \in U \setminus B_{\zeta,\epsilon}$:*

1. *$L(E) \geq D_\zeta$,*
2. *for $n > N_{\zeta,\epsilon}$, $|z - E| < e^{-4\epsilon n}$ and $\theta \in \mathcal{M}$, we have $\frac{1}{n} \ln \|A_n(\theta, z)\| < L(E) + \epsilon$.*

5.4.1.2 *Furthermore, if $L(E)$ is continuous in E and U is a compact set, there exists $D > 0$ and for any $\epsilon > 0$ there exists an integer N_ϵ so that for any $E \in U$:*

1. *$L(E) \geq D$*
2. *for $n > N_\epsilon$, $|z - E| < e^{-4\epsilon n}$ and $\theta \in \mathcal{M}$, we have $\frac{1}{n} \ln \|A_n(\theta, z)\| < L(E) + \epsilon$.*

We are also able to formulate the following lower bound for the norm of transfer matrices.

Lemma 5.4.2 *Let v be a piecewise Hölder function with $1 \geq \gamma > 0$ and f be a uniquely ergodic volume preserving map on $\mathcal{M}$ with $V(f) = V(f^{-1}) = 0$. Then*

5.4.2.1 *Let $L(E)$ be positive on a Borel set U and μ be a measure with $\mu(U) > 0$. Then for any $\zeta, \epsilon > 0$, let D_ζ, $B_{\zeta,\epsilon}$ and $N_{\zeta,\epsilon}$ be defined as in* 5.4.1.1.

1. *If there exists a sequence $r_k \to 0$ so that any geodesic ball in $\mathcal{M}$ with radius r_k covers the whole $\mathcal{M}$ in r_k^{-M} steps, then there exists a sequence $\{n_k(\epsilon)\}$ such that for $k > k_{\zeta,\epsilon}$, any $E \in U \setminus B_{\zeta,\epsilon}$, $|z - E| < e^{-4\epsilon n_k}$ and $\theta \in \mathcal{M}$ we have*

$$\min_{\iota \in \{-1,1\}} \max_{\iota j = 0, \ldots, e^{\frac{5M\epsilon}{\gamma} n_k}} \|A_{n_k}(f^j\theta, z)\| \geq e^{n_k(L(E) - 3\epsilon)}.$$

2. *If for any small $r > 0$, any geodesic ball with radius r covers the whole $\mathcal{M}$ in r^{-M} steps, then for $n > N'_{\zeta,\epsilon}$, any $E \in U \setminus B_{\zeta,\epsilon}$, $|z - E| < e^{-4\epsilon n}$ and $\theta \in \mathcal{M}$ we have*

$$\min_{\iota \in \{-1,1\}} \max_{\iota j = 0, \ldots, e^{\frac{5M\epsilon}{\gamma} n}} \|A_n(f^j\theta, z)\| \geq e^{n(L(E) - 3\epsilon)}.$$

5.4.2.2 *Furthermore, if $L(E)$ is continuous in E and U is a compact set, let D be defined as in* 5.4.1.2 *and for any $\epsilon > 0$ let N_ϵ be defined as in* 5.4.1.2. *Then for any $E \in U$ we have $L(E) \geq D$ and for any $|z - E| < e^{-4\epsilon n}$ we have*

1. *if there exists a sequence $r_k \to 0$ so that any geodesic ball in $\mathcal{M}$ with radius r_k covers the whole $\mathcal{M}$ in r_k^{-M} steps, then there exists a sequence $\{n_k(\epsilon)\}$ such that for $k > k_\epsilon$ and any $\theta \in \mathcal{M}$,*

$$\min_{\iota \in \{-1,1\}} \max_{\iota j = 0, \ldots, e^{\frac{5M\epsilon}{\gamma} n_k}} \|A_{n_k}(f^j\theta, z)\| \geq e^{n_k(L(E) - 3\epsilon)}.$$

2. *if for any small $r > 0$, any geodesic ball with radius r covers the whole $\mathcal{M}$ in r^{-M} steps, then for $n > N'_\epsilon$ and any $\theta \in \mathcal{M}$,*

$$\min_{\iota \in \{-1,1\}} \max_{\iota j = 0, \ldots, e^{\frac{5M\epsilon}{\gamma} n}} \|A_n(f^j\theta, z)\| \geq e^{n(L(E) - 3\epsilon)}.$$

Proof of Lemma 5.4.2

We will focus on the proof of part (1) of 5.4.2.1. The other three proofs will be discussed briefly at the end of this section.

For any $E \in U \setminus B_{\zeta,\epsilon}$ and $n > N_{\zeta,\epsilon}$, by Lemma 5.4.1.1 we have $\frac{1}{n}\|A_n(\theta, E)\| < L(E) + \epsilon$. Since $\int_{\mathcal{M}} \frac{1}{n} \ln \|A_n(\theta, E)\| \, \mathrm{dVol}_g(\theta) \geq L(E)$, we have

$$\mathrm{Vol}_g(M_{n,E,L(E),\epsilon}) := \mathrm{Vol}_g(\{\theta \in \mathcal{M} : \frac{1}{n} \ln \|A_n(\theta, E)\| > L(E) - \epsilon\}) > \frac{1}{2}. \tag{5.14}$$

Now we take any $\theta \in M_{n,E,L(E),\epsilon}$ and $|z - E| < e^{-4\epsilon n}$. When $n > 2N_{\zeta,\epsilon} + 3$ by the standard telescoping we have,

$$\begin{aligned}
\|A_n(\theta, z)\| &\geq \|A_n(\theta, E)\| - \|A_n(\theta, z) - A_n(\theta, E)\| \\
&\geq e^{n(L(E)-\epsilon)} - (n + 2(N_{\zeta,\epsilon} + 1)\|A\|_\infty^{N_{\zeta,\epsilon}})e^{n(L(E)-3\epsilon)} \\
&> e^{n(L(E)-2\epsilon)}
\end{aligned}$$

for large enough $n > N'_{\zeta,\epsilon}$. This means

$$M_{n,E,L(E),\epsilon} \subset M_{n,z,L(E),2\epsilon}. \tag{5.15}$$

We know the discontinuity set of $\frac{1}{n} \ln \|A_n(\theta, z)\|$ is $J_n = \cup_{l=0}^{n-1} f^{-l}(J_v)$, where $J_v = \cup_{j=1}^{K} \partial S_j$ is defined in section 5.2.2. By our assumption (5.6) and the fact the $V_{d-1}(f^{-1}) = 0$ (by the definition (5.2) of $V(f^{-1})$). For n large enough, we have

$$\mathrm{Vol}_{g,d-1}(J_n) \leq e^{n\epsilon} \mathrm{Vol}_{g,d-1}(J_v), \tag{5.16}$$

note that the largeness depends only on f. Define

$$\tilde{M}_{n,z,L(E),2\epsilon} = M_{n,z,L(E),2\epsilon} \setminus \overline{F_{2e^{-5\epsilon n/\gamma}}(J_n)},$$

where a neighborhood is defined as

$$F_r(A) = \{\theta \in \mathcal{M} : dist(\theta, A) < r\}.$$

Then by (5.16),

$$\begin{aligned}
\mathrm{Vol}_g(\tilde{M}_{n,z,L(E),2\epsilon}) &\geq \mathrm{Vol}_g(M_{n,z,L(E),2\epsilon}) - 4e^{-5\epsilon n/\gamma} \mathrm{Vol}_{g,d-1}(J_n) \\
&\geq \mathrm{Vol}_g(M_{n,z,L(E),2\epsilon}) - 4e^{-n(\frac{5\epsilon}{\gamma} - \epsilon)} \mathrm{Vol}_{g,d-1}(J_v) > \frac{2}{5}.
\end{aligned}$$

In particular, it is a non-empty set. Now we take any $\tilde{\theta} \in \tilde{M}_{n,z,L(E),2\epsilon}$ and $\theta \in B_{e^{-5\epsilon n/\gamma}}(\tilde{\theta})$. We have, by telescoping, (5.7) and the fact that $V_1(f) = 0$ (by the definition (5.2) of $V(f)$),

$$\begin{aligned}
&\|A_n(\theta, z)\| \\
\geq &\|A_n(\tilde{\theta}, z)\| - \|A_n(\theta, z) - A_n(\tilde{\theta}, z)\| \\
\geq &e^{n(L(E)-2\epsilon)} - (\sum_{l=1}^{K} \|v_l\|_{L^\gamma})(n + 2(N_{\zeta,\epsilon}+1)\|A\|_\infty^{N_{\zeta,\epsilon}})e^{n(L(E)+\epsilon)} \max_{j=0,\ldots,n-1} (dist(f^j\theta, f^j\tilde{\theta}))^\gamma \\
\geq &e^{n(L(E)-2\epsilon)} - (\sum_{l=1}^{K} \|v_l\|_{L^\gamma})(dist(\theta, \tilde{\theta}))^\gamma (n + 2(N_{\zeta,\epsilon}+1)\|A\|_\infty^{N_{\zeta,\epsilon}})e^{n(L(E)+\epsilon+\gamma\epsilon)} \\
> &e^{n(L(E)-3\epsilon)}.
\end{aligned}$$

for $n > N''_{\zeta,\epsilon}$. This means

$$F_{e^{-5\epsilon n/\gamma}}(\tilde{M}_{n,z,L(E),2\epsilon}) \subset M_{n,z,L(E),3\epsilon}.$$

Hence for $E \in U \setminus B_{\zeta,\epsilon}$, $n > N''_{\zeta,\epsilon}$ and $|z - E| < e^{-4\epsilon n}$, $M_{n,z,L(E),3\epsilon}$ contains a geodesic ball with radius $e^{-\frac{5\epsilon}{\gamma}n}$. Then there exists a sequence $\{n_k(\epsilon)\}$ such that a geodesic ball with radius $e^{-\frac{5\epsilon}{\gamma}n_k} \sim r_k$ covers the whole $\mathcal{M}$ in at most $e^{\frac{5M\epsilon}{\gamma}n_k}$ steps. Thus for $E \in U \setminus B_{\zeta,\epsilon}$, $k > k_{\zeta,\epsilon}$ so that $n_k(\epsilon) > N''_{\zeta,\epsilon}$, any $|z - E| < e^{-4\epsilon n_k}$ and any $\theta \in \mathbb{T}^d$ we have

$$\min_{\iota \in \{-1,1\}} \max_{\iota j = 0,\ldots,e^{\frac{5M\epsilon}{\gamma}n_k}} \|A_{n_k}(f^j\theta, z)\| > e^{n_k(L(E)-3\epsilon)}.$$

Remark 5.4.1 *Notice that Part (2) of Lemma* 5.4.2.1 *follows without taking a subsequence* $\{n_k(\epsilon)\}$*,* 5.4.2.2 *follows without excluding the set* $B_{\zeta,\epsilon}$*.*

□

5.4.2 Dynamical bounds on ξ_θ

The key to estimate ξ_θ is to apply the following lemma by Killip, Kiselev and Last.

For $f : \mathbb{Z} \to H$ where H is a Banach space, the truncated l^2 norm in the positive and negative directions are defined by

$$\|f\|_L^2 = \sum_{n=1}^{\lfloor L \rfloor} |f(n)|^2 + (L - \lfloor L \rfloor)|f(\lfloor L \rfloor + 1)|^2 \text{ for } L > 0$$
$$\|f\|_L^2 = \sum_{n=0}^{\lfloor L \rfloor + 1} |f(n)|^2 + (\lfloor L \rfloor + 1 - L)|f(\lfloor L \rfloor)|^2 \text{ for } L < 0$$

The truncated l^2 norm in both directions is defined by

$$\|f\|_{L_1,L_2}^2$$
$$= \sum_{n=-\lfloor L_1 \rfloor}^{\lfloor L_2 \rfloor} |f(n)|^2 + (L_1 - \lfloor L_1 \rfloor)|f(-\lfloor L_1 \rfloor - 1)|^2 + (L_2 - \lfloor L_2 \rfloor)|f(\lfloor L_2 \rfloor + 1)|^2 \text{ for } L_1, L_2 \geq 1.$$

With $A_\bullet(\theta, z)$ being a function on $\mathbb{Z}$, define $\tilde{L}_\epsilon^+(\theta, z) \in \mathbb{R}^+$ and $\tilde{L}_\epsilon^-(\theta, z) \in \mathbb{R}^-$ by requiring

$$\|A_\bullet(\theta, z)\|_{\tilde{L}_\epsilon^\pm(\theta,z)} = 2\|A(\theta, z)\|\epsilon^{-1}.$$

Lemma 5.4.3 *([55], Theorem 1.5) Let H_θ be a Schrödinger operator and μ_θ be the spectral measure of H and δ_0. Let $T > 0$ and $L_1, L_2 > 2$, then*

(5.17)
$$\langle \frac{1}{2}(\|e^{-itH_\theta}\delta_0\|_{L_1,L_2}^2 + \|e^{-itH_\theta}\delta_1\|_{L_1,L_2}^2)\rangle_T > C\frac{\mu_\theta + \mu_{f\theta}}{2}(\{E : |\tilde{L}_{T^{-1}}^-| \leq L_1; \tilde{L}_{T^{-1}}^+ \leq L_2\})$$

where C is an universal constant.

This lemma directly implies $P_{\theta,T}(L) + P_{f\theta,T}(L) > C\frac{\mu_\theta+\mu_{f\theta}}{2}(\{E : \|A_\bullet(\theta, z)\|_{\pm L} > 2\|A(\theta, z)\|T\})$. The plan is to show that for any $\eta > 1$, any θ_0 satisfying $(\mu_{\theta_0} + \mu_{f\theta_0})(U) > 0$, we have $(\mu_{\theta_0} + \mu_{f\theta_0})(\{E : \|A_\bullet(\theta_0, z)\|_{\pm T} > T^\eta\}) \gtrsim (\mu_{\theta_0} + \mu_{f\theta_0})(U)$.

Proof of Lemma 5.3.1

We will prove part (1) in detail. Part (2) will be discussed briefly at the end of this proof.

Fix $\eta > 1$. Fix θ_0 such that $(\mu_{\theta_0} + \mu_{f\theta_0})(U) > 0$. Let $\zeta = \frac{1}{2}(\mu_{\theta_0} + \mu_{f\theta_0})(U)$, so a constant. Let $D = D_\zeta$ from Lemma 5.4.1. Let $\epsilon = \min\left(\frac{\gamma D}{40M\eta}, \frac{D}{6}\right)$. Then by Lemmas 5.4.1, there exists a set B, $0 < |B| < \frac{1}{2}(\mu_{\theta_0} + \mu_{V\theta_0})(U)$, and a sequence $\{n_k\}$, s.t. $L(E) \geq D$ on $U \setminus B$ and for $E \in U \setminus B$, $k \geq k_0$, $|z - E| < e^{-4\epsilon n_k}$ and any $\theta \in \mathcal{M}$,

$$\min_{\iota \in \{-1,1\}} \max_{\iota j = 0, \ldots, e^{\frac{5M\epsilon}{\gamma} n_k}} \|A_{n_k}(f^j\theta, z)\| > e^{n_k(L(E) - 3\epsilon)}.$$

Using that $A_{s+t}(\theta, z) = A_t(f^s(\theta), z)A_s(\theta, z)$, this implies, by the condition on ϵ,

$$\|A_\bullet(\theta, z)\|_{\pm e^{\frac{10M\epsilon}{\gamma} n_k}} > e^{\frac{n_k(L(E)-3\epsilon)}{2}} \geq e^{\frac{10M\epsilon}{\gamma} n_k \eta}.$$

If we take $T_k = e^{\frac{10M\epsilon}{\gamma} n_k}$, then $U \setminus B \subset \{E : \|A_\bullet(\theta, E)\|_{\pm T_k} > T_k^\eta\}$ for any θ, in particular θ_0. Then by (5.17),

$$P_{\theta_0, T_k^\eta}(T_k) + P_{f\theta_0, T_k^\eta}(T_k) \geq C\frac{\mu_{\theta_0} + \mu_{f\theta_0}}{2}(\{E : \|A_\bullet(\theta_0, E)\|_{\pm T_k} > T_k^\eta\}) \geq \tilde{C}\frac{\mu_{\theta_0} + \mu_{f\theta_0}}{2}(U).$$

This implies $\underline{\xi_\theta} = 0$ for all $\theta \in \mathcal{M}$ such that $(\mu_\theta + \mu_{f\theta})(U) > 0$.

Remark 5.4.2 *Using Lemmas 5.4.1.1 (2), 5.4.2.1 (2) instead of 5.4.1.1 (1), 5.4.2.1 (1), Part (2) can be proved without taking a subsequence n_k therefore the conclusion holds for all T large enough rather than a sequence T_k.* □

5.4.3 Bounds on β

The key to the bounds on β is to apply the following lemma by Damanik and Tcheremchansev.

Lemma 5.4.4 (*Theorem 1 of [25] plus Corollary 1 of [26]*) *Let H be the Schrödinger operator, with f real valued and bounded, and $K \geq 4$ such that $\sigma(H) \subset [-K+1, K-1]$. Suppose for all $\rho \in (0,1)$ we have*

$$\int_{-K}^{K} \left(\min_{\iota \in \{-1,1\}} \max_{1 \leq n \leq T^\rho} \|A_n(E + \frac{i}{T})\|^2 \right)^{-1} \mathrm{d}E = O(T^{-\eta}). \tag{5.18}$$

for any $\eta \geq 1$. Then $\beta^+(p) = 0$ for all $p > 0$. If (5.18) is satisfied for a sequence $T_k \to \infty$, then $\beta^-(p) = 0$ for all $p > 0$.

Proof of Lemma 5.3.2

We will prove part (1) in detail. A modification needed for part (2) is discussed briefly at the end of this proof.

It suffices to consider small $\rho \in (0,1)$. Fix any $\rho \in (0,1)$ small and $\eta \geq 1$. Aussme $\sigma(H) \subset [-K+1, K-1]$. Since $L(E)$ is continous in E on a compact set $[-K, K]$, we have $L(E) \geq D > 0$ on $[-K, K]$. Fix $\epsilon_\eta = \min\left(\frac{\rho\gamma D}{20M\eta}, \frac{D}{6}\right)$. By Lemma 5.4.2.2 there exists a sequence $\{n_{\eta,k}\}$ such that for any $E \in [-K, K]$, $k > k_\eta$, any $|z - E| < e^{-4\epsilon_\eta n_{\eta,k}}$ and any $\theta \in \mathcal{M}$,

$$\min_{\iota \in \{-1,1\}} \max_{\iota j = 0, \ldots, e^{\frac{5M\epsilon_\eta}{\gamma} n_{\eta,k}}} \|A_{n_{\eta,k}}(f^j\theta, z)\| > e^{n_{\eta,k}(L(E) - 3\epsilon_\eta)}.$$

Thus

$$\min_{\iota \in \{-1,1\}} \max_{j = 0, \ldots, e^{\frac{10M\epsilon_\eta}{\gamma} n_{\eta,k}}} \|A_j(\theta, z)\|^2 \geq e^{n_{\eta,k}(L(E) - 3\epsilon_\eta)} \geq e^{\frac{10M\epsilon_\eta}{\gamma\rho} n_{\eta,k}\eta}$$

holds for any $\theta \in \mathcal{M}$, any $E \in [-K, K]$ and $|z - E| < e^{-4\epsilon_\eta n_{\eta,k}}$. Now we take $T_{\eta,k} = e^{\frac{10M\epsilon_\eta}{\gamma\rho} n_{\eta,k}}$,

$$|E + \frac{i}{T_{\eta,k}} - E| = \frac{1}{T_{\eta,k}} < e^{-4\epsilon_\eta n_{\eta,k}}.$$

Thus

$$\min_{\iota \in \{-1,1\}} \max_{\iota j = 0, \ldots, T_{\eta,k}^\rho} \|A_j(\theta, E + \frac{i}{T_{\eta,k}})\|^2 \geq T_{\eta,k}^\eta$$

holds for any $E \in [-K, K]$. Therefore

$$\int_{-K}^{K} \left(\min_{\iota \in \{-1,1\}} \max_{1 \leq \iota n \leq T_{\eta,k}^\rho} \|A_n(\theta, E + \frac{i}{T_{\eta,k}})\|^2 \right)^{-1} dE \leq 2K T_{\eta,k}^{-\eta}.$$

Now take a sequence $\{k_i\}$ such that $T_{1,k_1} < T_{2,k_2} < \ldots$ Let $T_m = T_{m,k_m}$. Then

$$\int_{-K}^{K} \left(\min_{\iota \in \{-1,1\}} \max_{1 \leq \iota n \leq T_m^\rho} \|A_n(\theta, E + \frac{i}{T_m})\|^2 \right)^{-1} dE \leq 2K T_m^{-m}.$$

By (5.18), we have $\beta_\theta^-(p) \leq \rho$ for all $\theta \in \mathcal{M}$, any $\rho \in (0,1)$ and any $p > 0$, thus $\beta_\theta^-(p) = 0$ for all $\theta \in \mathcal{M}$ and any $p > 0$.

Remark 5.4.3 *Using Lemmas 5.4.1.2 (2) and 5.4.2.2 (2), part (2) follows without taking a subsequence $\{n_{\eta,k}\}$. Therefore the conclusion holds for all T large rather than a sequence T_k.* □

5.5 Irrational rotation with diophantine frequencies. Proof of Lemma 5.3.6

Lemma 5.3.6 is a standard result (see e.g. Chapter 2.3 in [58]). We include the proof here for completeness.

Proof of Lemma 5.3.6

For sufficiently small $\epsilon > 0$, fix an integer $H_0 \sim N^{1/(d(\tau-1)+1+d\epsilon)}$, define $g(n) = \frac{1}{n(n+1)}$ for $1 \leq n < H_0$ and $g(H_0) = \frac{1}{H_0}$. For $(n_1, ..., n_d) \in \mathbb{Z}^d$ with $1 \leq n_i \leq H_0$, define $f(n_1, ..., n_d) = \prod_{i=1}^{d} g(n_i)$. By Lemma 5.2.2, we have

$$
\begin{aligned}
D_N(\theta) &\leq C_d\Big(\frac{1}{H_0} + \sum_{0<|h|\leq H_0} \frac{1}{r(\vec{h})} \Big|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i \langle \vec{h}, \alpha\rangle n}\Big|\Big) \\
&\leq \tilde{C}_d\Big(\frac{1}{H_0} + \frac{1}{N}\sum_{0<|h|\leq H_0} \frac{1}{r(\vec{h})}\frac{1}{\|\langle \vec{h}, \alpha\rangle\|_{\mathbb{T}}}\Big) \\
&= \tilde{C}_d\Big(\frac{1}{H_0} + \frac{1}{N}\sum_{n_1,...,n_d=1}^{H_0} f(n_1, ..., n_d) \sum_{\vec{h}=(h_1,...,h_d)\neq\vec{0}, |h_j|\leq n_j} \frac{1}{\|\langle \vec{h}, \alpha\rangle\|_{\mathbb{T}}}\Big) \\
&\leq \tilde{C}_d\Big(\frac{1}{H_0} + \frac{1}{N}\sum_{n_1,...,n_d=1}^{H_0} f(n_1, ..., n_d) \sum_{j=1}^{3^d r(\vec{n})} \frac{r(\vec{n})^\tau}{j}\Big) \\
&\leq \tilde{C}_d\Big(\frac{1}{H_0} + \frac{1}{N}\sum_{n_1,...,n_d=1}^{H_0} f(n_1, ..., n_d) r(\vec{n})^\tau \log r(\vec{n})\Big) \\
&\leq \tilde{C}_d\Big(\frac{1}{H_0} + \frac{{H_0}^{d(\tau-1+\epsilon)}}{N}\Big) \\
&\lesssim N^{-1/(d(\tau-1)+1+d\epsilon)}.
\end{aligned}
$$

$\square$

5.6 Skew-shift. Proof of Lemmas 5.3.7 and 5.3.8

Skew-shift

Let f: $\mathbb{T}^d \to \mathbb{T}^d$ be defined as follows

$$f(y_1, y_2, ..., y_d) = (y_1 + \alpha, y_2 + y_1, ..., y_d + y_{d-1}).$$

Let $\vec{Y}_n = f^n(y_1, ..., y_d)$, then

$$\vec{Y}_n = (y_1 + \binom{n}{1}\alpha,\ y_2 + \binom{n}{1}y_1 + \binom{n}{2}\alpha,\ ...,\ y_d + \binom{n}{1}y_{d-1} + \cdots + \binom{n}{d}\alpha), \tag{5.19}$$

where $\binom{n}{m} = 0$ if $n < m$.

5.6.1 Preparation. Combinatorial identities

Lemma 5.6.1 *Let $r_t \in \mathbb{N}$ for $1 \le t \le s$, then we have*

$$\sum_{1\le t\le s}^{l_t=0,1} (-1)^{s-\sum_{t=1}^{s} l_t} \binom{\sum_{t=1}^{s} l_t r_t}{s-1} = 0, \tag{5.20}$$

$$\sum_{1\le t\le s}^{l_t=0,1} (-1)^{s-\sum_{t=1}^{s} l_t} \binom{\sum_{t=1}^{s} l_t r_t}{s} = \prod_{t=1}^{s} r_t. \tag{5.21}$$

Proof: Let us consider the coefficient C_a of x^a in the product $(1+x)^{r_1} \cdot (1+x)^{r_2} \cdot \cdots \cdot (1+x)^{r_s} = (1+x)^{\sum_{i=1}^{s} r_i}$. Let us denote

(5.22)

$$A^{(a)} = \{(\vec{j}_1, \vec{j}_2, ..., \vec{j}_s), \text{where } \vec{j}_t = (j_{t,1}, j_{t,2}, ..., j_{t,r_t}),\ j_{t,k} \in \{0,1\} | \sum_{t=1}^{s}\sum_{k=1}^{r_t} j_{t,k} = a\}$$

Each element in $A^{(a)}$ corresponds to one way of choosing 1 or x in each term of the product $(1+x)^{r_1} \cdot (1+x)^{r_2} \cdot \cdots \cdot (1+x)^{r_s}$ in order to get x^a, where $j_{t,k} = 0$ means we choose 1 out of the k-th $1+x$ from $(1+x)^{r_t}$, and $j_{t,k} = 1$ means we choose x instead of 1. Thus the capacity of $A^{(a)}$, denoted by $|A^{(a)}|$, is equal to $C_\alpha = \binom{\sum_{t=1} r_t}{a}$. Let us futher denote

$$A_t^{(a)} = A^{(a)} \cap \{\vec{j}_t = \vec{0}\} \tag{5.23}$$

For $a = s-1$, since it is impossible to obtain x^{s-1} with $\vec{j}_t \neq \vec{0}$ for any $1 \le t \le s$, we have

$$A^{(s-1)} \setminus (\cup_{t=1}^{s} A_t^{(s-1)}) = \emptyset. \tag{5.24}$$

For $a = s$,

$$A^{(s)} \setminus (\cup_{t=1}^{s} A_t^{(s)}) = D, \tag{5.25}$$

where

$$D = \{(\vec{j}_1, \vec{j}_2, ..., \vec{j}_t) | \sum_{k=1}^{r_t} j_{t,k} = 1 \text{ for } 1 \le t \le s\}. \tag{5.26}$$

Clearly,

$$| \cup_{t=1}^{s} A_t^{(a)} | = \sum_{i=1}^{s} (-1)^{i-1} \sum_{1 \le t_1 < t_2 < \cdots < t_i \le s} | \cap_{l=1}^{i} A_{t_l}^{(a)} |, \tag{5.27}$$

in which

$$\sum_{1 \le t_1 < t_2 < \cdots < t_i \le s} | \cap_{l=1}^{i} A_{t_l}^{(a)} | = \sum_{\sum_{t=1}^{s} l_t = s-i}^{l_t = 0,1} \binom{\sum_{t=1}^{s} l_t r_t}{a}. \tag{5.28}$$

Thus

$$\begin{aligned} |A^{(a)} \setminus (\cup_{t=1}^{s} A_t^{(a)})| &= \binom{\sum_{t=1}^{s} r_t}{a} + \sum_{i=1}^{s} (-1)^{i} \sum_{\sum_{t=1}^{s} l_t = s-i}^{l_t = 0,1} \binom{\sum_{t=1}^{s} l_t r_t}{a}, \\ &= \sum_{1 \le t \le s}^{l_t = 0,1} (-1)^{s - \sum_{t=1}^{s} l_t} \binom{\sum_{t=1}^{s} l_t r_t}{a}. \end{aligned} \tag{5.29}$$

For $a = s-1$, (5.20) follows directly from (5.24 and (5.29). While for $a = s$, (5.21) follows from (5.25), (5.29) and the fact that $|D| = \prod_{t=1}^{s} r_t$. □

5.6.2 Diophantine α. Proof of Lemma 5.3.7

For $\alpha \in DC(\tau)$, we take integers

$$H_j \sim N^{\frac{2^j}{(2^d-1)(\tau+\epsilon)}} \text{ for } 0 \le j \le d-1. \tag{5.30}$$

By Lemma 5.2.2,

$$D(\vec{Y}_1,...,\vec{Y}_N) \le C_d(\frac{1}{H_0} + \sum_{0<|\vec{h}|\le H_0} \frac{1}{r(\vec{h})}|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i\langle \vec{h},\vec{Y}_n\rangle}|)$$

$$= C_d(\frac{1}{H_0} + \sum_{0<|\vec{h}|\le H_0} \frac{1}{r(\vec{h})}|\frac{1}{N}\sum_{n=1}^{N} u_n^{(0)}|), \tag{5.31}$$

where

$$u_n^{(0)} = \exp\{2\pi i \sum_{j=1}^{d}(h_j\alpha + \sum_{r=1}^{d-j} h_{j+r}y_r)\binom{n}{j}\}. \tag{5.32}$$

For $1 \le s \le d-2$, let

$$u^{(s)}_{k_1,...,k_s,n} = \exp\left\{2\pi i \sum_{j=s+1}^{d}(h_j\alpha + \sum_{r=1}^{d-j} h_{j+r}y_r)\sum_{\substack{l_t=0,1\\ 1\le t\le s}}(-1)^{s-\sum_{t=1}^{s} l_t}\binom{n+\sum_{t=1}^{s} l_t k_t}{j}\right\} \tag{5.33}$$

Then by Lemma 5.2.3,

$$|\frac{1}{N-\sum_{t=1}^{s} k_s}\sum_{n=1}^{N-\sum_{t=1}^{s} k_t} u^{(s)}_{k_1,...,k_s,n}|^2$$
$$\lesssim \frac{1}{H_{s+1}} + \frac{1}{(N-\sum_{t=1}^{s} k_t)H_{s+1}^2}\sum_{k_{s+1}=1}^{H_{s+1}}(H_{s+1}-k_{s+1})|\sum_{n=1}^{N-\sum_{t=1}^{s+1} k_t} u^{(s)}_{k_1,...,k_s,n}\overline{u^{(s)}_{k_1,...,k_s,n+k_{s+1}}}|. \tag{5.34}$$

Here

$$
\begin{aligned}
&|\sum_{n=1}^{N-\sum_{t=1}^{s+1}k_t} u^{(s)}_{k_1,\dots,k_s,n}\overline{u^{(s)}_{k_1,\dots,k_s,n+k_{s+1}}}| \\
=&|\sum_{n=1}^{N-\sum_{t=1}^{s+1}k_t} \exp 2\pi i\sum_{j=s+1}^{d}(h_j\alpha+\sum_{r=1}^{d-j}h_{j+r}y_r)\sum_{1\le t\le s}^{l_t=0,1}(-1)^{s-\sum_{t=1}^{s}l_t}(\binom{n+\sum_{t=1}^{s}l_tk_t}{j}- \\
&\binom{n+k_{s+1}+\sum_{t=1}^{s}l_tk_t}{j})| \\
=&|\sum_{n=1}^{N-\sum_{t=1}^{s+1}k_t} \exp\left\{2\pi i\sum_{j=s+1}^{d}(h_j\alpha+\sum_{r=1}^{d-j}h_{j+r}y_r)\sum_{1\le t\le s+1}^{l_t=0,1}(-1)^{s+1-\sum_{t=1}^{s+1}l_t}\binom{n+\sum_{t=1}^{s+1}l_tk_t}{j}\right\}| \\
=&|\sum_{n=1}^{N-\sum_{t=1}^{s+1}k_t} \exp\left\{2\pi i\sum_{j=s+1}^{d}(h_j\alpha+\sum_{r=1}^{d-j}h_{j+r}y_r)\sum_{0\le t\le s+1}^{l_t=0,1}(-1)^{s+2-\sum_{t=0}^{s+1}l_t}\binom{l_0n+\sum_{t=1}^{s+1}l_tk_t}{j}\right\}|
\end{aligned}
$$

(5.35)

$$
\begin{aligned}
=&|\sum_{n=1}^{N-\sum_{t=1}^{s+1}k_t} \exp\left\{2\pi i\sum_{j=s+2}^{d}(h_j\alpha+\sum_{r=1}^{d-j}h_{j+r}y_r)\sum_{0\le t\le s+1}^{l_t=0,1}(-1)^{s+2-\sum_{t=0}^{s+1}l_t}\binom{l_0n+\sum_{t=1}^{s+1}l_tk_t}{j}\right\}| \\
=&|\sum_{n=1}^{N-\sum_{t=1}^{s+1}k_t} \exp\left\{2\pi i\sum_{j=s+2}^{d}(h_j\alpha+\sum_{r=1}^{d-j}h_{j+r}y_r)\sum_{1\le t\le s+1}^{l_t=0,1}(-1)^{s+1-\sum_{t=1}^{s+1}l_t}\binom{n+\sum_{t=1}^{s+1}l_tk_t}{j}\right\}|
\end{aligned}
$$

(5.36)

$$
=|\sum_{n=1}^{N-\sum_{t=1}^{s+1}k_t} u^{(s+1)}_{k_1,\dots,k_{s+1},n}|.
$$

Notice that in (5.35), we applied (5.21),

$$
\exp\left\{(h_{s+1}\alpha+\sum_{r=1}^{d-s-1}h_{s+1+r}y_r)\sum_{0\le t\le s+1}^{l_t=0,1}(-1)^{s+2-\sum_{t=0}^{s+1}l_t}\binom{l_0n+\sum_{t=1}^{s+1}l_tk_t}{s+1}\right\}=1.
$$

Combining (5.34) with (5.36), we get for any $0\le s\le d-3$,

(5.37)

$$
\begin{aligned}
&|\frac{1}{N-\sum_{t=1}^{s}k_s}\sum_{n=1}^{N-\sum_{t=1}^{s}k_t} u^{(s)}_{k_1,\dots,k_s,n}|^2 \\
\le&\frac{1}{H_{s+1}}+\frac{1}{(N-\sum_{t=1}^{s}k_t)H_{s+1}^2}\sum_{k_{s+1}=1}^{H_{s+1}}(H_{s+1}-k_{s+1})(N-\sum_{t=1}^{s+1}k_t)|\frac{1}{N-\sum_{t=1}^{s+1}k_t}\sum_{n=1}^{N-\sum_{t=1}^{s+1}k_t} u^{(s+1)}_{k_1,\dots,k_{s+1},n}|.
\end{aligned}
$$

By (5.34), for $s = d-2$,

(5.38)

$$
\begin{aligned}
&|\frac{1}{N-\sum_{l=1}^{d-2}k_l}\sum_{n=1}^{N-\sum_{l=1}^{d-2}k_l}u^{(d-2)}_{k_1,\ldots,k_{d-2},n}|^2\\
\lesssim&\frac{1}{H_{d-1}}+\frac{1}{(N-\sum_{l=1}^{d-2}k_l)H_{d-1}^2}\sum_{k_{d-1}=1}^{H_{d-1}}(H_{d-1}-k_{d-1})|\sum_{n=1}^{N-\sum_{l=1}^{d-1}k_l}u^{(d-2)}_{k_1,\ldots,k_{d-2},n}\overline{u^{(d-2)}_{k_1,\ldots,k_{d-2},n+k_{d-1}}}|\\
\lesssim&\frac{1}{H_{d-1}}+\frac{1}{(N-\sum_{l=1}^{d-2}k_l)H_{d-1}}\sum_{k_{d-1}=1}^{H_{d-1}}|\sum_{n=1}^{N-\sum_{l=1}^{d-1}k_l}u^{(d-2)}_{k_1,\ldots,k_{d-2},n}\overline{u^{(d-2)}_{k_1,\ldots,k_{d-2},n+k_{d-1}}}|,
\end{aligned}
$$

and

$$
\begin{aligned}
&|\sum_{n=1}^{N-\sum_{l=1}^{d-1}k_l}u^{(d-2)}_{k_1,\ldots,k_{d-2},n}\overline{u^{(d-2)}_{k_1,\ldots,k_{d-2},n+k_{d-1}}}|\\
=&|\sum_{n=1}^{N-\sum_{l=1}^{d-1}k_l}\exp\{2\pi i h_d\alpha\sum_{1\le l\le d-1}^{j_l=0,1}(-1)^{d-1-\sum_{l=1}^{d-1}j_l}\binom{n+\sum_{j=1}^{d-1}j_lk_l}{d}\}|\\
=&|\sum_{n=1}^{N-\sum_{l=1}^{d-1}k_l}\exp\{2\pi i h_d\alpha\sum_{0\le l\le d-1}^{j_l=0,1}(-1)^{d-\sum_{l=0}^{d-1}j_l}\binom{l_0n+\sum_{j=1}^{d-1}j_lk_l}{d}\}|\\
=&|\sum_{n=1}^{N-\sum_{l=1}^{d-1}k_l}\exp\{2\pi i h_d n\alpha\prod_{l=1}^{d-1}k_l\}| \qquad (5.39)\\
\lesssim&\frac{1}{\|h_d\alpha\prod_{l=1}^{d-1}k_l\|_{\mathbb{T}}}, \qquad (5.40)
\end{aligned}
$$

where in (5.39) we used (5.21).

Since $\alpha\in DC(\tau)$, by the property of Diophantine condition (1.4) and since $|h_i|\le H_0$, $1\le k_i\le H_i$ we have

$$\sum_{k_{d-1}=1}^{H_{d-1}}\frac{1}{\|h_d\alpha\prod_{l=1}^{d-1}k_l\|_{\mathbb{T}}}\le\sum_{j=1}^{H_{d-1}}\frac{m^\tau\prod_{l=1}^{d-1}H_l^\tau}{j}\le m^\tau H_{d-1}^{\tau+\epsilon}\prod_{l=1}^{d-2}H_l^\tau. \qquad (5.41)$$

Thus combining (5.38), (5.40) with (5.41), we have

$$|\frac{1}{N-\sum_{l=1}^{d-2}k_l}\sum_{n=1}^{N-\sum_{l=1}^{d-2}k_l}u^{(d-2)}_{k_1,\ldots,k_{d-2},n}|^2\lesssim\frac{1}{H_{d-1}}+\frac{m^\tau H_{d-1}^{\tau+\epsilon}\prod_{l=1}^{d-2}H_l^\tau}{H_{d-1}(N-\sum_{l=1}^{d-2}H_l)}\lesssim\frac{1}{H_{d-1}}=\frac{1}{H_{d-2}^2}.$$

Lemma 5.6.2 *For any $\alpha \in \mathbb{T}$, if for any $1 \le k_s \le H_{s,v,\alpha}$,*

$$|\frac{1}{N-\sum_{l=1}^{s} k_l} \sum_{n=1}^{N-\sum_{l=1}^{s} k_l} u^{(s)}_{k_1,\dots,k_s,n}|^2 \lesssim \frac{1}{H^2_{s,v,\alpha}},$$

then for any $0 \le t \le s-1$, $1 \le k_t \le H_t$ we have

$$|\frac{1}{N-\sum_{l=1}^{t} k_l} \sum_{n=1}^{N-\sum_{l=1}^{t} k_l} u^{(t)}_{k_1,\dots,k_t,n}|^2 \lesssim \frac{1}{H^2_t}.$$

Proof: For $t = s-1$, by (5.37),

$$\begin{aligned}
&|\frac{1}{N-\sum_{l=1}^{s-1} k_l} \sum_{n=1}^{N-\sum_{l=1}^{s-1} k_l} u^{(s-1)}_{k_1,\dots,k_{s-1},n}|^2 \\
\lesssim& \frac{1}{H_s} + \frac{1}{(N-\sum_{l=1}^{s-1} k_l)H_s^2} \sum_{k_s=1}^{H_s} (H_s-k_s)(N-\sum_{l=1}^{s} k_l)|\frac{\sum_{n=1}^{N-\sum_{l=1}^{s} k_l} u^{(s)}_{k_1,\dots,k_s,n}}{(N-\sum_{l=1}^{s} k_l)}| \\
\lesssim& \frac{1}{H_s} = \frac{1}{H^2_{s-1}}.
\end{aligned}$$

Then by reverse induction. □

At the final step we obtain

$$|\frac{1}{N}\sum_{n=1}^{N} u^{(0)}_n|^2 \lesssim \frac{1}{H_0^2}$$

Plugging it into (5.31), we have

$$D(\vec{Y}_1,\dots,\vec{Y}_N) \lesssim \frac{1}{H_0} + \sum_{0<|\vec{h}|\le H_0} \frac{1}{r(\vec{h})}\frac{1}{H_0} \lesssim \frac{1}{H_0^{1-\epsilon}} \sim N^{-\frac{1-\epsilon}{(2^d-1)(\tau+\epsilon)}}.$$

□

5.6.3 Liouvillean α, Proof of Lemma 5.3.8

For $\alpha \notin DC(d)$, by property (1.6), we could find a subsequence $\{\frac{p_n}{q_n}\}$ of the continued fraction approximants of α, so that $q_{n+1} > q_n^d$. In the following we will use q instead of q_n and $\tilde{q}$ instead of q_{n+1} for simplicity. Here we would like to show $D_q(\vec{Y}_1,\dots,\vec{Y}_q) \le q^{-\delta}$ for some $\delta > 0$. Take

$$H_j \sim q^{\frac{2^j}{2^d}} \text{ for } 0 \le j \le d-2 \quad \text{and} \quad H_{d-1} \sim q^{\frac{2^{d-1}(1+\epsilon)}{2^d}}, \tag{5.42}$$

where $\epsilon > 0$ is small enough so that

$$\prod_{l=0}^{d-1} H_l = q^{\frac{2^d-1+2^{d-1}\epsilon}{2^d}} < q. \tag{5.43}$$

Now by Lemma 5.2.2

(5.44)

$$D(\vec{Y}_1, ..., \vec{Y}_q) \leq C_d(\frac{1}{H_0} + \sum_{0<|\vec{h}|\leq H_0} \frac{1}{r(\vec{h})} \mid \frac{1}{q}\sum_{n=1}^{q} \exp\{2\pi i \sum_{j=1}^{d}(h_j\alpha + h_{j+1}y_1 + ... + h_d y_{d-j})\binom{n}{j}\} \mid)$$

Consider the following difference

$$\begin{aligned}
&\frac{1}{q}|\sum_{n=1}^{q} \exp\{2\pi i \sum_{j=1}^{d}(h_j\alpha + h_{j+1}y_1 + ... + h_d y_{d-j})\binom{n}{j}\} - \\
&\quad \sum_{n=1}^{q} \exp\{2\pi i \sum_{j=1}^{d}(h_j\frac{p}{q} + h_{j+1}y_1 + ... + h_d y_{d-j})\binom{n}{j}\}| \\
\leq &\frac{1}{q}\sum_{n=1}^{q} |\exp\{2\pi i \sum_{j=1}^{d} h_j(\alpha - \frac{p}{q})\binom{n}{j}\} - 1| \\
\lesssim &\frac{1}{q}\sum_{n=1}^{q}\sum_{j=1}^{d}\binom{n}{j} H_0|\alpha - \frac{p}{q}| \\
\lesssim &\frac{H_0}{q},
\end{aligned} \tag{5.45}$$

where in the last step we use (1.1), $|\alpha - \frac{p}{q}| \leq \frac{1}{q\tilde{q}} < \frac{1}{q^{d+1}}$.

Then combining (5.44) with (5.45), we have

$$D(\vec{Y}_1, ..., \vec{Y}_q) \lesssim C_d(\frac{1}{H_0} + \sum_{0<|\vec{h}|\leq H_0} \frac{1}{r(\vec{h})} \mid \frac{1}{q}\sum_{n=1}^{q} u_n^{(0)} \mid) + \frac{H_0}{q}, \tag{5.46}$$

where $\tilde{u}_n^{(0)} = \exp\{2\pi i \sum_{j=1}^{d}(h_j\frac{p}{q} + h_{j+1}y_1 + ... + h_d y_{d-j})\binom{n}{j}\}$, that is $u_n^{(0)}$ as in (5.32) with α replaced with $\frac{p}{q}$. Thus with $\tilde{u}^{(s)}_{k_1,...,k_s,n}$ defined as in (5.33) with α replaced with $\frac{p}{q}$, similar to (5.38) and (5.39), we have

$$\begin{aligned}
&|\frac{1}{N-\sum_{l=1}^{d-2}k_l}\sum_{n=1}^{N-\sum_{l=1}^{d-2}k_l} \tilde{u}^{(d-2)}_{k_1,...,k_{d-2},n}|^2 \\
\lesssim &\frac{1}{H_{d-1}} + \frac{1}{(N-\sum_{l=1}^{d-2}k_l)H_{d-1}}\sum_{k_{d-1}=1}^{H_{d-1}} \mid \sum_{n=1}^{N-\sum_{l=1}^{d-1}k_l} \tilde{u}^{(d-2)}_{k_1,...,k_{d-2},n}\overline{\tilde{u}^{(d-2)}_{k_1,...,k_{d-2},n+k_{d-1}}}|,
\end{aligned} \tag{5.47}$$

and

$$
\begin{aligned}
&|\sum_{n=1}^{q-\sum_{l=1}^{d-1}k_l} \tilde{u}^{(d-2)}_{k_1,\dots,k_{d-2},n}\overline{\tilde{u}^{(d-2)}_{k_1,\dots,k_{d-2},n+k_{d-1}}}| \\
=&|\sum_{n=1}^{q-\sum_{l=1}^{d-1}k_l} \exp\{2\pi i h_d n\frac{p}{q}\prod_{l=1}^{d-1}k_l\}| \\
\lesssim&\frac{1}{\|h_d\frac{p}{q}\prod_{l=1}^{d-1}k_l\|_{\mathbb{R}/\mathbb{Z}}}. \qquad (5.48)
\end{aligned}
$$

Since $|h_d| \le H_0$, $1 \le k_i \le H_i$ and (5.43), for any $1 \le k \le H_{d-1}$ we have $\|kh_d\frac{p}{q}\prod_{l=1}^{d-2}k_l\|_{\mathbb{R}/\mathbb{Z}} \ge \frac{1}{q}$. Thus

$$
\sum_{k_{d-1}=1}^{H_{d-1}} \frac{1}{\|h_d\frac{p}{q}\prod_{l=1}^{d-1}k_l\|_{\mathbb{R}/\mathbb{Z}}} \lesssim \sum_{j=1}^{H_{d-1}}\frac{q}{j} \le q\ln H_{d-1}. \qquad (5.49)
$$

Then combining (5.47), (5.48) with (5.49), we get

(5.50)

$$
|\frac{1}{q-\sum_{l=1}^{d-2}k_l}\sum_{n=1}^{q-\sum_{l=1}^{d-2}k_l}\tilde{u}^{(d-2)}_{k_1,\dots,k_{d-2},n}|^2 \lesssim \frac{1}{H_{d-1}} + \frac{q\ln H_{d-1}}{(q-\sum_{l=1}^{d-2}H_l)H_{d-1}} \lesssim \frac{1}{H_{d-1}^{\frac{1}{1+\epsilon}}} = \frac{1}{H_{d-2}^2}.
$$

By Lemma 5.6.2,

$$
|\frac{1}{q}\sum_{n=1}^{q}\tilde{u}^{(0)}_n|^2 \lesssim \frac{1}{H_0}.
$$

Plugging it into (5.46), we get

$$
D(\vec{Y}_1,\dots,\vec{Y}_q) \lesssim \frac{1}{H_0} + \frac{(\log H_0)^d}{H_0} + \frac{H_0}{q} \lesssim \frac{1}{q^{\frac{1-\epsilon}{2^d}}}.
$$

5.7 Bounded remainder set

Most of the material covered in this section comes from [30]. We briefly discuss it here for completeness and readers' convenience. From now on we restrict our attention to irrational rotation on $\mathbb{T}^d$. For a measurable set $U \subset \mathbb{T}^d$, consider the function $A_N(U,\vec{x}) - N|U| := A(U,\{\vec{x}+n\alpha\}_{n=0}^{N-1}) - N|U| = \sum_{n=0}^{N-1}\chi_U(\vec{x}+n\alpha) - N|U|$. We will say U is a *bounded remainder set* (*BRS*) with respect to α if there exists a constant $C(U,\alpha) > 0$ such that $|A_N(U,\vec{x}) - N|U|| \le C(U,\alpha)$ for any N and a.e. $\vec{x} \in \mathbb{T}^d$. We will call a measurable function g on $\mathbb{T}^d$ a transfer function for U if its characteristic function satisfies

$$\chi_U(\vec{x}) - |U| = g(\vec{x}) - g(\vec{x} - \alpha) \text{ a.e.}$$

Obviously if g is a transfer function for U, then its Fourier coefficients satisfy

$$\hat{g}(\vec{m}) = \frac{\hat{\chi}_U(\vec{m})}{1 - e^{-2\pi i \langle \vec{m}, \alpha \rangle}}, \quad \vec{m} \neq 0. \tag{5.51}$$

Proposition 5.7.1 *[30] For a measurable set $U \subset \mathbb{T}^d$, the following are equivalent:*

- *U is a bounded remainder set.*
- *U has a bounded transfer function g.*

Theorem 5.7.1 *Any interval $I \subset \mathbb{T}$ of length $0 < |q\alpha - p| < 1$ is a BRS with respect to α, furthermore its transfer function h satisfies $\|g\|_\infty \leq |q|$.*

Proof: Without loss of generality, we consider an interval $I = [0, \kappa]$, where $\kappa = q\alpha - p > 0$. Then

$$\begin{aligned}
\chi_I(x) - |I| &= -\{x\} + \{x - \kappa\} \\
&= -\{x\} + \{x - q\alpha\} \\
&= (-\{x\} - ... - \{x - (q-1)\alpha\}) + (\{x - \alpha\} + ... + \{x - q\alpha\}) \\
&= g(x) - g(x - \alpha),
\end{aligned}$$

where $g(x) = -\sum_{j=0}^{q-1} \{x - j\alpha\}$, $\|g\|_\infty \leq |q|$. □

Theorem 5.7.2 *Let $\vec{v} = (v_1, v_2, ..., v_d) = q\alpha - \vec{p} \in \mathbb{Z}\alpha + \mathbb{Z}^d$, $v \notin \mathbb{Z}^d$, and let $\Sigma \in \mathbb{T}^{d-1}$ be a BRS with respect to the vector $(\frac{v_1}{v_d}, \frac{v_2}{v_d}, ... \frac{v_{d-1}}{v_d})$ with transfer function h. Then the set*

$$U = U(\Sigma, \vec{v}) = \{(\vec{x}, 0) + t\vec{v} : \vec{x} \in \Sigma, 0 \leq t < 1\},$$

is a BRS with respect to α, whose transfer function g satisfies $\|g\|_\infty \leq |q|(\|h\|_\infty + 1)$.

Proof: Let $\vec{v}_0 = (v_1, ..., v_{d-1})$ be the vector in $\mathbb{T}^{d-1}$, which consists of the first $d-1$ entries of $\vec{v}$. First, we wish to find a bounded function $\tilde{g}$ on $\mathbb{T}^d$ satisfying the cohomological equation

$$\chi_U(\vec{x}, y) - |U| = \tilde{g}(\vec{x}, y) - \tilde{g}(\vec{x} - \vec{v}_0, y - v_d) \quad \text{for a.e. } (\vec{x}, y) \in \mathbb{T}^{d-1} \times \mathbb{T}.$$

This means the Fourier coefficients satisfy the equation

(5.52)

$$\hat{\tilde{g}}(\vec{m}, n)(1 - e^{-2\pi i(\langle \vec{m}, \vec{v}_0\rangle + n v_d)}) = \int_0^{v_d} \int_{\Sigma + \frac{y}{v_d}\vec{v}_0} e^{-2\pi i \langle \vec{m}, \vec{x} + \frac{y}{v_d}\vec{v}_0\rangle} \mathrm{d}\vec{x}\ e^{-2\pi i n y} \mathrm{d}y, \quad (\vec{m}, n) \neq (\vec{0}, 0).$$

Which implies

$$\hat{\tilde{g}}(\vec{m}, n) = \frac{\hat{\chi}_\Sigma(\vec{m})}{2\pi i(\langle \vec{m}, \vec{v}_0\rangle / v_d + n)}, \quad (\vec{m}, n) \neq (\vec{0}, 0). \tag{5.53}$$

We know Σ is a BRS with respect to $\vec{v}_0/v_d$, by (5.51) its transfer function $h : \mathbb{T}^{d-1} \to \mathbb{R}$ satisfies

$$\hat{h}(\vec{m}) = \frac{\hat{\chi}_\Sigma(\vec{m})}{1 - e^{-2\pi i \langle \vec{m}, \vec{v}_0\rangle / v_d}}, \quad \vec{m} \neq 0.$$

It is straightforward to check that the bounded function $\tilde{g}$ defined by

$$\tilde{g}(\vec{x}, y) = h(\vec{x} - \frac{\vec{v}_0}{v_d}\{y\}) - |\Sigma| \cdot \{y\},$$

satisfies the coholomogical equation (5.53). Hence $\tilde{g}$ is a bounded transfer function for U with respect to $\vec{v}$.

Indeed, $\|\tilde{g}\|_\infty \leq \|h\|_\infty + 1$. Since $\vec{v} = q\alpha - \vec{p}$, letting $g(\vec{x}) = \tilde{g}(\vec{x}) + \tilde{g}(\vec{x} - \alpha) + ... + \tilde{g}(\vec{x} - (q-1)\alpha)$ we have that U is a BRS with respect to α with bounded transfer function g satisfying $\|g\|_\infty \leq |q| \|\tilde{g}\|_\infty \leq |q|(\|h\|_\infty + 1)$. □

The following corollary will be used several times in section 8.

Corollary 5.7.1 *Let $U \subset \mathbb{T}^2$ be the parallelogram spanned by two vectors $m(\alpha_1, \alpha_2) - (l_1, l_2)$ and $(q\frac{m\alpha_1 - l_1}{m\alpha_2 - l_2} - p,\ 0)$, then U is a BRS with respect to (α_1, α_2) with transfer function g satisfying $\|g\|_\infty \leq |m|(|q| + 1) \leq 2|mq|$.*

Proof: In this case v $= (v_1, v_2) = m(\alpha_1, \alpha_2) - (l_1, l_2) \in \mathbb{Z}\alpha + \mathbb{Z}^2$, $\Sigma = [0, q\frac{v_1}{v_2} - p] \times \{0\}$. We know the transfer function h of Σ with respect to v_1/v_2 satisfies $\|h\|_\infty \leq |q|$. Thus $\|g\|_\infty \leq |m|(|q| + 1) \leq 2|mq|$. □

5.8 2-dimensional irrational rotation with weak diophantine frequencies

In this section we deal with 2-dimensional weakly Diophantine frequencies. Our goal is to prove Lemma 5.3.5.

Proof of Lemma 5.3.5

Assume $(\alpha_1, \alpha_2) \in WDC(c_0, \tau/4)$, for some $\tau > 4$ and $c_0 > 0$. We divide the discussion into two parts.

First, we introduce the coprime Diophantine condition:

(5.54)

$$PDC(\tau) = \cup_{c>0} PDC(c,\tau) = \cup_{c>0}\{(\alpha_1,\alpha_2) | \|\langle \vec{h}, \alpha\rangle\|_{\mathbb{T}} \geq \frac{c}{|\vec{h}|^{\tau}} \text{ for any } \gcd(h_1,h_2)=1 \text{ or } h_1h_2=0 \text{ but } \vec{h}\neq\vec{0}\}.$$

Obviously if $\alpha \in PDC(c,\tau)$ both α_1 and α_2 belong to $DC(c,\tau)$.

Case A

$(\alpha_1, \alpha_2) \in PDC(c_1, \tau)$ for some $c_1 > 0$.

Let's take the best simultaneous approximation $\{(\frac{l_{1,n}}{m_n}, \frac{l_{2,n}}{m_n})\}$ of (α_1, α_2). They feature the following property.

Lemma 5.8.1 *([59], Theorem 3.5) If $\{1, \alpha_1, \alpha_2\}$ is linearly independent over $\mathbb{Q}$, then there are infinitely many n_k such that*

$$\begin{vmatrix} m_{n_k} & l_{1,n_k} & l_{2,n_k} \\ m_{n_k+1} & l_{1,n_k+1} & l_{2,n_k+1} \\ m_{n_k+2} & l_{1,n_k+2} & l_{2,n_k+2} \end{vmatrix} \neq 0$$

Now we take $r_k > 0$ such that

$$m_{n_k} \leq \frac{4}{\pi} r_k^{-2} < m_{n_k+1}. \tag{5.55}$$

By (1.7), the choice of r_k guarantees that for $n \geq n_k$,

$$(m_n\alpha_1 - l_{1,n}, m_n\alpha_2 - l_{2,n}) \in B_{r_k}(0,0), \tag{5.56}$$

where $B_r(x_1, x_2) := \{y = (y_1, y_2) \in \mathbb{T}^2 : \|y_1 - x_1\|_{\mathbb{T}}^2 + \|y_2 - x_2\|_{\mathbb{T}}^2 < r_k^2\}$. Let $\{\frac{p_{n,s}}{q_{n,s}}\}_{s=1}^{\infty}$ be the continued fraction approximants of $\frac{m_n\alpha_1 - l_{1,n}}{m_n\alpha_2 - l_{2,n}}$. For each n choose s_n such that

$$q_{n,s_n} \leq r_k^{-1} < q_{n,s_n+1}. \tag{5.57}$$

By (1.1), the choice of s_n guarantees that

$$(q_{n,s_n}\frac{m_n\alpha_1 - l_{1,n}}{m_n\alpha_2 - l_{2,n}} - p_{n,s_n}, 0) \in B_{r_k}(0,0). \tag{5.58}$$

By (1.7) and (1.9) we have

$$\frac{c_0}{m_n^{\tau/4}} \leq \max\{|m_n\alpha_1 - l_{1,n}|, |m_n\alpha_2 - l_{2,n}|\} \leq \frac{2}{\sqrt{\pi}\sqrt{m_{n+1}}}, \tag{5.59}$$

by (5.55) we have $m_{n_k} \leq \frac{4}{\pi} r_k^{-2}$, thus

$$\max(m_{n_k}, m_{n_k+1}, m_{n_k+2}) \leq C_{c_0,\tau} r_k^{-\frac{\tau^2}{2}}. \tag{5.60}$$

Case A.1

If for some $n \in \{n_k, n_k+1, n_k+2\}$, we have $q_{n,s_n+1} \leq r_k^{-2\tau^4}$.

Let U be the parallelogram spanned by the two vectors $m_n(\alpha_1, \alpha_2) - (l_{1,n}, l_{2,n})$ and $(q_{n,s_n}\frac{m_n\alpha_1 - l_{1,n}}{m_n\alpha_2 - l_{2,n}} - p_{n,s_n},\ 0)$. By (5.56) and (5.58), $U \subset B_{2r_k}(0,0)$. Corollary 5.7.1 implies that $|\sum_{j=0}^{M-1} \chi_U(x + j\alpha_1, y + j\alpha_2) - M|U|| \leq 4|m_n q_{n,s_n}|$ for *a.e.* (x,y). Thus as long as $M > \frac{4|m_n q_{n,s_n}|}{|U|}$, we should have $\cup_{j=0}^{M-1} U - (j\alpha_1, j\alpha_2)$ covers the whole $\mathbb{T}^2$ up to a measure zero set. Then

$$\mathbb{T}^2 \subseteq \cup_{j=0}^{M-1} B_{2r_k}(-j\alpha_1, -j\alpha_2) \text{ for } M > \frac{4|m_n q_{n,s_n}|}{|U|}. \tag{5.61}$$

Now we want to estimate $|U|$. Since $\alpha_2 \in DC(c_1, \tau)$, by (1.4) we have

$$|U| = |m_n\alpha_2 - l_{2,n}| \cdot |q_{n,s_n}\frac{m_n\alpha_1 - l_{1,n}}{m_n\alpha_2 - l_{2,n}} - p_{n,s_n}| \geq \frac{c_1}{|m_n|^\tau}\frac{1}{2q_{n,s_n+1}}.$$

Thus by (5.57) and (5.60),

$$\frac{4|m_n|q_{n,s_n}}{|S|} \leq \frac{8}{c_1}|m_n|^{1+\tau} q_{n,s_n} q_{n,s_n+1} \leq C_{c_0,c_1,\tau} r_k^{-3\tau^4}.$$

This means it takes $B_{2r_k}(0,0)$ at most $C_{\alpha_1,\alpha_2,\tau} r_k^{-3\tau^4}$ steps to cover the whole $\mathbb{T}^2$.

Case A.2

We will show now it is impossible to have $q_{n,s_n+1} > r_k^{-2\tau^4}$ for all $n \in \{n_k, n_k+1, n_k+2\}$. In this case by (1.1), (1.7) and (5.55), we have:

$$
\begin{aligned}
|q_{n,s_n} m_n \alpha_1 - p_{n,s_n} m_n \alpha_2 + M_n| =& |m_n\alpha_2 - l_{2,n}| \cdot |q_{n,s_n} \frac{m_n\alpha_1 - l_{1,n}}{m_n\alpha_2 - l_{2,n}} - p_{n,s_n}| \\
<& \frac{2}{\sqrt{\pi}\sqrt{|m_{n+1}|} q_{n,s_n}} < r_k^{2\tau^4+1} \qquad (5.62)
\end{aligned}
$$

where $M_n = p_{n,s_n} l_{2,n} - q_{n,s_n} l_{1,n}$.

We have the following estimates on the upper bounds of p_{n,s_n} and M_n. Combining (1.4), (5.55), (5.57), (5.59) with (5.60),

$$
|p_{n,s_n}| \le q_{n,s_n} |\frac{m_n\alpha_1 - l_{1,n}}{m_n\alpha_2 - l_{2,n}}| + \frac{1}{q_{n,s_n+1}} \le \frac{2q_{n,s_n}|m_n|^\tau}{c_1\sqrt{\pi}\sqrt{|m_{n+1}|}} + r_k^{2\tau^4} \le C_{c_0,c_1,\tau} r_k^{-\frac{\tau^3}{2}}. \qquad (5.63)
$$

By (5.62), (5.55), (5.60), (5.57) and (5.63),

$$
|M_n| < |q_{n,s_n} m_n \alpha_1 - p_{n,s_n} m_n \alpha_2| + r_k^{2\tau^4} \le C_{c_0,c_1,\tau} r_k^{-\tau^3}. \qquad (5.64)
$$

Case A.2.1

If $p_{n,s_n} = 0$ for some $n \in \{n_k, n_k+1, n_k+2\}$, then by (1.1), (1.7) and (5.54), (1.4),(5.55), (5.60)

$$
r_k^{2\tau^4} > \frac{1}{q_{n,s_n+1}} \ge |q_{n,s_n} \frac{m_n\alpha_1 - l_{1,n}}{m_n\alpha_2 - l_{2,n}}| \ge \frac{c_1\sqrt{\pi}\sqrt{|m_{n+1}|}}{2m_n^\tau} \ge C_{c_0,c_1,\tau} r_k^{\frac{\tau^3}{2}+1},
$$

contradiction.

Case A.2.2

If $M_n = 0$ for some $n \in \{n_k, n_k+1, n_k+2\}$, then by (5.62), (5.55), (5.63) and the fact that $(\alpha_1, \alpha_2) \in PDC(c_1, \tau)$,

$$
r_k^{2\tau^4} > |m_n||q_{n,s_n}\alpha_1 - p_{n,s_n}\alpha_2| \ge \frac{c_1|m_n|}{\max{(p_{n,s_n}, q_{n,s_n})}^\tau} \ge C_{c_0,c_1,\tau} r_k^{\frac{\tau^4}{2}},
$$

contradiction.

Case A.2.3

If $p_{n,s_n} \neq 0$ and $M_n \neq 0$ for any $n \in \{n_k, n_k+1, n_k+2\}$, then for any $i, j \in \{n_k, n_k+1, n_k+2\}$, we have:

$$\begin{aligned}(5.65)\qquad &|(q_{i,s_i}m_iM_j - q_{j,s_j}m_jM_i)\alpha_1 - (p_{i,s_i}m_iM_j - p_{j,s_j}m_jM_i)\alpha_2| \\ \leq &|(q_{i,s_i}m_i\alpha_1 - p_{i,s_i}m_i\alpha_2 + M_i)M_j| + |(q_{j,s_j}m_j\alpha_1 - p_{j,s_j}m_j\alpha_2 + M_j)M_i| \\ < &(|M_i| + |M_j|)r_k^{2\tau^4}.\end{aligned}$$

Case A.2.3.1

If $(q_{i,s_i}m_iM_j - q_{j,s_j}m_jM_i,\ p_{i,s_i}m_iM_j - p_{j,s_j}m_jM_i) \neq (0,0)$ for some $i, j \in \{n_k, n_k+1, n_k+2\}$.

Let $h = \gcd(q_{i,s_i}m_iM_j - q_{j,s_j}m_jM_i,\ p_{i,s_i}m_iM_j - p_{j,s_j}m_jM_i)$ be the greatest common divisor of the two numbers if they are both nonzero, let $h = 1$ otherwise. Then by (5.65),

$$|\frac{q_{i,s_i}m_iM_j - q_{j,s_j}m_jM_i}{h}\alpha_1 - \frac{p_{i,s_i}m_iM_j - p_{j,s_j}m_jM_i}{h}\alpha_2| < \frac{|M_i| + |M_j|}{h}r_k^{2\tau^4}.$$

However on one hand by (5.64),

$$\frac{|M_i| + |M_j|}{h}r_k^{2\tau^4} \leq (|M_i| + |M_j|)r_k^{2\tau^4} \leq C_{c_0,c_1,\tau}r_k^{2\tau^4 - \tau^3}.$$

On the other hand by the fact that $(\alpha_1, \alpha_2) \in PDC(c_1, \tau)$ and (5.55), (5.60), (5.63), (5.64),

$$\begin{aligned}&|\frac{q_{i,s_i}m_iM_j - q_{j,s_j}m_jM_i}{h}\alpha_1 - \frac{p_{i,s_i}m_iM_j - p_{j,s_j}m_jM_i}{h}\alpha_2| \\ \geq &\frac{c_1h^\tau}{|(q_{i,s_i}m_iM_j - q_{j,s_j}m_jM_i, p_{i,s_i}m_iM_j - p_{j,s_j}m_jM_i)|^\tau} \\ \geq &C_{c_0,c_1,\tau}r_k^{\frac{7}{4}\tau^4},\end{aligned}$$

contradiction.

Case A.2.3.2

If for any $i, j \in \{n_k, n_k+1, n_k+2\}$

$$q_{i,s_i} m_i M_j = q_{j,s_j} m_j M_i$$
$$p_{i,s_i} m_i M_j = p_{j,s_j} m_j M_i.$$

Then for $n = n_k$,

$$\frac{p_{n,s_n}}{q_{n,s_n}} = \frac{p_{n+1,s_{n+1}}}{q_{n+1,s_{n+1}}} = \frac{p_{n+2,s_{n+2}}}{q_{n+2,s_{n+2}}}.$$

Hence we can let $p = p_{n,s_n} = p_{n+1,s_{n+1}} = p_{n+2,s_{n+2}}$ and $q = q_{n,s_n} = q_{n+1,s_{n+1}} = q_{n+2,s_{n+2}}$. Then we would have (after plugging in $M_n = ql_{1,n} - pl_{2,n}$)

$$q(m_n l_{1,n+1} - m_{n+1} l_{1,n}) = p(m_n l_{2,n+1} - m_{n+1} l_{2,n}) \tag{5.66}$$

$$q(m_n l_{1,n+2} - m_{n+2} l_{1,n}) = p(m_n l_{2,n+2} - m_{n+2} l_{2,n}) \tag{5.67}$$

$$q(m_{n+1} l_{1,n+2} - m_{n+2} l_{1,n+1}) = p(m_{n+1} l_{2,n+2} - m_{n+2} l_{2,n+1}) \tag{5.68}$$

Then consider $(5.66) \cdot (-l_{1,n+2}) + (5.67) \cdot l_{1,n+1} + (5.68) \cdot (-l_{1,n})$, we get

$$p \cdot \begin{vmatrix} m_{n_k} & l_{1,n_k} & l_{2,n_k} \\ m_{n_k+1} & l_{1,n_k+1} & l_{2,n_k+1} \\ m_{n_k+2} & l_{1,n_k+2} & l_{2,n_k+2} \end{vmatrix} = q \cdot 0 = 0,$$

contradiction with the choice of n_k.

Case B

$(\alpha_1, \alpha_2) \notin PDC(\tau)$. By the definition of $PDC(\tau)$, the sequence $\vec{h}_n = (h_{1,n}, h_{2,n})$ for which (5.54) fails has to satisfy either $\gcd(h_{1,n}, h_{2,n}) = 1$ (Case B.1) or $h_{1,n}h_{2,n} = 0$ (Case B.2).

Case B.1

We can find a sequence $\{n_j\}$, such that $|\vec{h}_{n_j}| = \max(|h_{1,n_j}|, |h_{2,n_j}|) \to \infty$ as $j \to \infty$, $\gcd(h_{1,n_j}, h_{2,n_j}) = 1$ and $\|h_{1,n_j}\alpha_1 + h_{2,n_j}\alpha_2\|_{\mathbb{T}} < \frac{1}{|\vec{h}_{n_j}|^\tau}$.

Without loss of generality, we can assume $|h_{1,n_j}| = |\vec{h}_{n_j}|$. In this case we can take $r_{n_j} = \frac{1}{|h_{1,n_j}|}$. For simplicity we will replace n_j with n.

Now that $\|h_{1,n}\alpha_1 + h_{2,n}\alpha_2\|_{\mathbb{T}} < \frac{1}{|h_{1,n}|^\tau}$, we can find $l_{1,n}, l_{2,n} \in \mathbb{Z}$ such that $|h_{1,n}(\alpha_1 - l_{1,n}) + h_{2,n}(\alpha_2 - l_{2,n})| < \frac{1}{|h_{1,n}|^\tau}$. Since replacing (α_1, α_2) with $(\alpha_1 + l_{1,n}, \alpha_2 + l_{2,n})$ would not change anything, we will assume $|h_{1,n}\alpha_1 + h_{2,n}\alpha_2| < \frac{1}{|h_{1,n}|^\tau}$. Then

$$|\frac{\alpha_2}{\alpha_1} - (-\frac{h_{1,n}}{h_{2,n}})| < \frac{1}{|h_{1,n}|^\tau \alpha_1}. \tag{5.69}$$

We consider the following two lines on $\mathbb{T}^2$:

$$l_1(t) = (\{t\}, \{\frac{\alpha_2}{\alpha_1}t\}) \text{ and } l_2(t) = (\{t\}, \{-\frac{h_{1,n}}{h_{2,n}}t\}).$$

These two lines are close to each other in the sense that for $|t| \le |h_{1,n}|^{3\tau/4}$, by (5.69),

$$\|\{\frac{\alpha_2}{\alpha_1}t\} - \{-\frac{h_{1,n}}{h_{2,n}}t\}\|_{\mathbb{T}} \le |\frac{\alpha_2}{\alpha_1}t + \frac{h_{1,n}}{h_{2,n}}t| \le \frac{|t|}{|h_{1,n}|^\tau \alpha_1} \le \frac{1}{|h_{1,n}|^{\tau/4}\alpha_1}.$$

The graph of $l_2(t)$ is the hypotenuse of a right triangle with two legs of lengths $|h_{1,n}|$ and $|h_{2,n}|$ (mod $\mathbb{Z}^2$). We consider the orbit of $(\alpha_1, -\frac{h_{1,n}}{h_{2,n}}\alpha_1)$ under the rotation $(\alpha_1, -\frac{h_{1,n}}{h_{2,n}}\alpha_1)$. These points lie on $l_2(t)$. Under this rotation the point moves a distance $\frac{\sqrt{h_{1,n}^2 + h_{2,n}^2}}{|h_{2,n}|}\alpha_1$ at each step by a big interval with length $\sqrt{h_{1,n}^2 + h_{2,n}^2}$. Let $\{\frac{p_m}{q_m}\}_{m=1}^\infty$ be the continued fraction approximants of $\frac{\alpha_1}{h_{2,n}}$. Choose m such that

$$q_{m-1} \le |h_{1,n}|\sqrt{h_{1,n}^2 + h_{2,n}^2} < q_m. \tag{5.70}$$

Then it would take a point on $\mathbb{T}$ at most $q_m + q_{m-1}$ steps (under the $\frac{\alpha_1}{h_{2,n}}$ -rotation) to enter each interval of length $\frac{1}{|h_{1,n}|\sqrt{h_{1,n}^2 + h_{2,n}^2}}$ on $\mathbb{T}$ (e.g. [46]), which means it would take a point on $l_2(t)$ at most $q_m + q_{m-1} - 1$ steps (under the $\frac{\sqrt{h_{1,n}^2 + h_{2,n}^2}\alpha_1}{|h_{2,n}|}$-rotation) to enter each interval of length $\frac{1}{|h_{1,n}|} = r_n$ on the graph of $l_2(t)$. Moreover, it is easy to see that the distance from any $x \in \mathbb{T}^2$ to $l_2(t)$ is bounded by $\frac{1}{\sqrt{h_{1,n}^2 + h_{2,n}^2}} < r_n$. Thus

$$\mathbb{T}^2 \subseteq \cup_{k=0}^{q_m + q_{m-1}} B_{2r_n}(k\alpha_1, -\frac{h_{1,n}}{h_{2,n}}k\alpha_1). \tag{5.71}$$

By (1.1) and (5.69),

$$|p_{m-1}+q_{m-1}\frac{\alpha_2}{h_{1,n}}| = |p_{m-1}-q_{m-1}\frac{\alpha_1}{h_{2,n}}+q_{m-1}(\frac{\alpha_1}{h_{2,n}}+\frac{\alpha_2}{h_{1,n}})| \le \frac{1}{q_m}+\frac{q_{m-1}}{|h_{1,n}|^{\tau-1}}.$$

This implies, by (1.1) and (5.70),

$$\|q_{m-1}\alpha_1\|_{\mathbb{T}} \le |q_{m-1}\alpha_1 - h_{2,n}p_{m-1}| \le \frac{|h_{2,n}|}{q_m},$$
$$\|q_{m-1}\alpha_2\|_{\mathbb{T}} \le \frac{|h_{1,n}|}{q_m}+\frac{2}{|h_{1,n}|^{\tau-4}}.$$

Then by the fact that $\alpha \in WDC(c_0, \frac{\tau}{4})$ and (5.70),

$$\max\{\frac{|h_{2,n}|}{q_m}, \frac{|h_{1,n}|}{q_m}+\frac{2}{|h_{1,n}|^{\tau-4}}\} \ge \max\left(\|q_{m-1}\alpha_1\|_{\mathbb{T}}, \|q_{m-1}\alpha_2\|_{\mathbb{T}}\right) \ge \frac{c_0}{q_{m-1}^{\tau/4}} \ge \frac{c_0}{2^{\frac{\tau}{4}}|h_{1,n}|^{\tau/2}}.$$

This implies

$$q_m+q_{m-1} < 2q_m \le \frac{2^{\frac{\tau}{4}+2}}{c_0}|h_{1,n}|^{\tau/2+1}. \tag{5.72}$$

Since $0 \le k \le \frac{2^{\frac{\tau}{4}+2}}{c_0}|h_{1,n}|^{\tau/2+1} < r_n^{-\frac{3\tau}{4}}$, by (5.69) the points $(k\alpha_1, k\alpha_2)$ and $(k\alpha_1, -\frac{h_{1,n}}{h_{2,n}}k\alpha_1)$ differ at most by $r_n^{\frac{\tau}{4}}$ we obtain using (5.71) and (5.72),

$$\mathbb{T}^2 \subseteq \cup_{k=0}^{r_n^{-3\tau/4}} B_{3r_n}(k\alpha_1, k\alpha_2).$$

Case B.2

We can find a sequence $\{n_j\}$ such that $h_{2,n_j} \equiv 0$ and $|h_{1,n_j}| \to \infty$ such that

$$\|h_{1,n_j}\alpha_1\|_{\mathbb{T}} < \frac{1}{|h_{1,n_j}|^{\tau}}. \tag{5.73}$$

For simplicity we will replace n_j with n. We can find M_n such that $|h_{1,n}\alpha_1 - M_n| < \frac{1}{|h_{1,n}|^{\tau}}$. Let $d_n = \gcd(h_{1,n}, M_n)$ be the greatest common divisor. Let $\tilde{h}_{1,n} = \frac{h_{1,n}}{d_n}$ and $\tilde{M}_n = \frac{M_n}{d_n}$. We have

$$|\alpha_1 - \frac{\tilde{M}_n}{\tilde{h}_{1,n}}| < \frac{1}{|h_{1,n}|^{\tau+1}} \to 0. \tag{5.74}$$

If $\tilde{h}_{1,n}$ is bounded in n, then α_1 can be approximated arbitrarily closely by rationals with bounded donominators, which is impossible. Thus $|\tilde{h}_{1,n}| \to \infty$. Now take radius

$r_n = \frac{1}{|\tilde{h}_{1,n}|}$. For each $0 \le i \le \tilde{h}_{1,n} - 1$ consider $\{(i\alpha_1 + k\tilde{h}_{1,n}\alpha_1,\ i\alpha_2 + k\tilde{h}_{1,n}\alpha_2)\}_{k=0}^{\infty}$. Let $\{\frac{p_m}{q_m}\}_{m=1}^{\infty}$ be the continued fraction approximants of $\tilde{h}_{1,n}\alpha_2$. Choose m such that

$$q_{m-1} \le |\tilde{h}_{1,n}| = r_n^{-1} < q_m. \tag{5.75}$$

Then it takes any point on $\mathbb{T}$ at most $q_m + q_{m-1} - 1$ steps (under the $\tilde{h}_{1,n}\alpha_2-$rotation) to enter each interval of length r_n [46]. By (1.1),

$$|p_{m-1} - q_{m-1}\tilde{h}_{1,n}\alpha_2| \le \frac{1}{q_m}. \tag{5.76}$$

By (5.73), (5.75) and since $\tau > 4$, we have $\|q_{m-1}\tilde{h}_{1,n}\alpha_1\| \le \frac{q_{m-1}}{|\tilde{h}_{1,n}|^{\tau}} < \frac{c_0}{(q_{m-1}|\tilde{h}_{1,n}|)^{\tau/4}}$. By the fact that $\alpha \in WDC(c_0, \frac{\tau}{4})$, $\|q_{m-1}\tilde{h}_{1,n}\alpha_2\| \ge \frac{c_0}{(q_{m-1}|\tilde{h}_{1,n}|)^{\tau/4}}$. By (5.76) and (5.75), we have

$$q_m \le \frac{1}{c_0}|\tilde{h}_{1,n}|^{\frac{\tau}{2}}. \tag{5.77}$$

Now for $0 \le k \le q_m + q_{m-1} - 1$, by (5.74), (5.73) and (5.77), $\|i\alpha_1 + k\tilde{h}_{1,n}\alpha_1 - \frac{i\tilde{M}_n}{\tilde{h}_{1,n}}\|_{\mathbb{T}} \le \frac{C}{|\tilde{h}_{1,n}|^{\frac{\tau}{2}}} = Cr_n^{\frac{\tau}{2}}$. Since gcd $(\tilde{h}_{1,n}, \tilde{M}_n) = 1$, any interval of length $r_n = \frac{1}{|\tilde{h}_{1,n}|}$ contains $\frac{i\tilde{M}_n}{\tilde{h}_{1,n}}$ for some $0 \le i \le \tilde{h}_{1,n} - 1$. Thus

$$\mathbb{T}^2 \subseteq \cup_{k=0}^{(q_m+q_{m-1})|\tilde{h}_{1,n}|} B_{r_n}(k\alpha_1, k\alpha_2).$$

By (5.77), $(q_m + q_{m-1})|\tilde{h}_{1,n}| \le r_n^{-\tau}$, so we have

$$\mathbb{T}^2 \subseteq \cup_{k=0}^{r_n^{-\tau}} B_{r_n}(k\alpha_1, k\alpha_2). \tag{5.78}$$

$\square$.

Chapter 6

Continuity of measure of the spectrum for Schrödinger operators with potentials driven by shifts and skew-shifts on tori

6.1 Introduction

Consider Schrödinger operators acting on $l^2(\mathbb{Z})$:

$$H_{v,f}(\theta)u(n) = u(n+1) + u(n-1) + v(f^n\theta)u(n). \tag{6.1}$$

where v is the potential, $\theta \in \mathbb{T}^d$ is the phase and f is shift or skew-shift with frequency α on the torus $\mathbb{T}^d$. We study continuity of the spectra in frequency α. In particular, since the spectrum at rational frequencies can be obtained numerically and are easier to study, continuity in frequency allows us to study the spectrum at irrational frequencies via rational approximation. While many recent significant advances in discrete Schrödinger operators, see e.g. [15, 40, 2], require one dimensional torus shift and analytic potentials, our results reveal that continuity of the spectrum is a much more general phenomenon: it holds for both shift and skew-shift on higher dimensional torus and also Hölder continuous potentials. Our results can be viewed as a generalization of [44], where a similar result was obtained for $d = 1$ and f is a rotation of the circle.

Let $f_{s,\alpha} : \theta \to \theta + \alpha$ be the shift and $f_{ss,\alpha} : (\theta_1, \theta_2, ..., \theta_d) \to (\theta_1 + \alpha, \theta_2 + \theta_1, ..., \theta_d + \theta_{d-1})$ be the skew-shift. For a fixed $v, f_{*,\alpha}$, let us denote the spectrum of $H_{v,f_{*,\alpha},\theta}$ by $S(\alpha, \theta)$. Let $S(\alpha) = \cup_{\theta \in \mathbb{T}^d} S(\alpha, \theta)$. It is known that if α is irrational, $S(\alpha) = S(\alpha, \theta)$ for any $\theta \in \mathbb{T}^d$, while if α is rational, $S(\alpha, \theta)$ depends on θ and $S(\frac{\vec{p}_n}{q_n})$ is a union of at most q_n bands. We would like to establish that $\lim_{n\to\infty} S(\frac{\vec{p}_n}{q_n}) = S(\alpha)$ in the sense that $\lim_{n\to\infty} \chi_{S(\frac{\vec{p}_n}{q_n})}(E) = \chi_{S(\alpha)}(E)$ for a.e. $E \in \mathbb{R}$.

This question first arose from the Aubry-Andre conjecture [1] on the measure of the spectrum of the almost Mathieu operator ($d = 1$, $f = f_{s,\alpha}$ and $v(\theta) = 2\lambda \cos 2\pi\theta$) to be $4|1 - |\lambda||$. This conjecture has been proved for all irrational α, with partial results obtained in [10, 60, 61, 21, 47] and the extension to all irrational α was made in [41, 9] [1] (see e.g. [44] for a complete history). The proof of the Aubry-Andre conjecture contains two important ingredients: one is to obtain estimates about the rational frequencies [10, 61]: $|S(\frac{p_n}{q_n})| \to 4|1 - |\lambda||$; the other is to prove continuity of measure of the spectrum in frequency at irrationals. While the first ingredient clearly specializes to the almost Mathieu operator, the second ingredient, related to quantitative estimates on the Hausdorff continuity of the spectrum, have been studied for much more general potentials.

When $d = 1$ and $f = f_{s,\alpha}$, it was proved [41] that for any analytic f in the regime of positive Lyapunov exponent, $|S(\frac{p_n}{q_n})| \to |S(\alpha)|$ for every Diophantine α and its continued fraction approximants. Later, it was shown [37] that positivity of the Lyapunov exponent is not need for this result, in particular, $S(\frac{p_n}{q_n}) \to S(\alpha)$ for any analytic v and all irrational α. More recently, it has been proved [44] that under the condition of positive Lyapunov exponent, the regularity of v can be relaxed to Hölder continuity.

One of the key ingredients of the proof of [44] is strongly (weakly) M-dense property of the irrational rotation of the circle defined in the abstract form in Section 2.3. We say a dynamical system is strongly M-dense if any point will enter a ball with

[1]The argument of [9], applies to the critical value $\lambda = 1$, did not involve continuity in frequency

radius r within r^{-M} steps under the map as long as r is small, while the weak version requires only a sequence of lengths $r_k \to 0$. The strongly M-dense property for the irrational rotation of the circle is guaranteed by the Diophantine condition on α and proved using continued fraction expansion. For the higher dimensional shift and skew-shift, strongly M-dense properties have been studied using different methods in [4, 33], and some results on weakly M-dense property were obtained in [33]. These properties are important in our generalization of the results of [44] to both $(\mathbb{T}^d, f_{s,\alpha})$ and $(\mathbb{T}^d, f_{ss,\alpha})$ cases.

Let $L(\alpha, E)$ be the Lyapunov exponent of the operator $H_{v,f_{*,\alpha}}(\theta)$ at energy E (see (1.12)). Let $L_+(\alpha) = \{E : L(\alpha, E) > 0\}$ and $L_{\epsilon+}(\alpha) = \{E : L(\alpha, E) > \epsilon\}$.

With the Diophantine conditions defined in section 1.2, our main results are:

Theorem 6.1.1 *Let $f_{s,\alpha}$ be an irrational shift on $\mathbb{T}^d$. Let $1 \geq \gamma > \frac{d}{d+1}$ be a constant. Then if $\alpha \notin WDC(\frac{1}{\gamma})$ or $\alpha \in DC(\tau)$ for some $\tau > 1$, there exists a sequence of rationals $\frac{\vec{p}_n}{q_n} = (\frac{p_{1,n}}{q_n}, ..., \frac{p_{d,n}}{q_n}) \to \alpha$ such that for any $v \in C^\gamma(\mathbb{T}^d)$,*

$$\lim_{n\to\infty} S(\frac{\vec{p}_n}{q_n}) \cap L_+(\alpha) = S(\alpha) \cap L_+(\alpha).$$

Remark 6.1.1 *The sequence of rationals can be taken as the full sequence of best simultaneous approximation, of α (see section 2.2.2) when $\alpha \in DC(\tau)$, and a proper subsequence when $\alpha \notin WDC(\frac{1}{\gamma})$.*

A direct corollary is:

Corollary 6.1.1 *Let $\frac{\vec{p}_n}{q_n}$ be the chosen sequence of rationals as in Theorem 6.1.1, we have,*

$$\lim_{n\to\infty} |S(\frac{\vec{p}_n}{q_n}) \cap L_+(\alpha)| = |S(\alpha) \cap L_+(\alpha)|.$$

Theorem 6.1.2 *Let $f_{ss,\alpha}$ be a skew-shift on $\mathbb{T}^d$. For any $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. There exists a sequence of rationals $\frac{p_n}{q_n} \to \alpha$ such that for any $v \in C^\gamma(\mathbb{T}^d)$ with $1 \geq \gamma > \frac{1}{2}$,*

$$\lim_{n\to\infty} S(\frac{p_n}{q_n}) \cap L_+(\alpha) = S(\alpha) \cap L_+(\alpha).$$

Remark 6.1.2 *The sequence of rationals will the be full sequence of continued fraction approximants if* $\alpha \in DC(\tau)$ *for some* $\tau > 1$*, and a proper subsequence otherwise.*

A direct corollary is:

Corollary 6.1.2 *Let* $\frac{p_n}{q_n}$ *be the chosen sequence of rationals as in Theorem 6.1.2, we have,*

$$\lim_{n\to\infty} |S(\frac{p_n}{q_n}) \cap L_+(\alpha)| = |S(\alpha) \cap L_+(\alpha)|.$$

For shifts on two dimensional torus, as for the skew-shifts, we are able to cover all frequencies.

Theorem 6.1.3 *Let* $f_{s,\alpha}$ *be an irrational shift on* $\mathbb{T}^2$*. Let* $1 \geq \gamma > \frac{2}{3}$ *be a constant. Then for any* $\epsilon_0 > 0$ *and for any irrational* α*, there exists a sequence of rationals* $\frac{\vec{p}_n}{q_n} \to \alpha$ *(depending on* ϵ_0*) such that for any* $v \in C^\gamma(\mathbb{T}^2)$*,*

$$\lim_{n\to\infty} S(\frac{\vec{p}_n}{q_n}) \cap L_{\epsilon_0+}(\alpha) = S(\alpha) \cap L_{\epsilon_0+}(\alpha).$$

Similarly, we have

Corollary 6.1.3 *Let* $\frac{\vec{p}_n}{q_n}$ *be the chosen sequence of rationals as in Theorem 6.1.3, we have,*

$$\lim_{n\to\infty} |S(\frac{\vec{p}_n}{q_n}) \cap L_+(\alpha)| = |S(\alpha) \cap L_+(\alpha)|.$$

We organize this paper as follows: some preliminaries are presented in section 2, then the two key lemmas proved in section 3 prepare us for the proofs of main theorems in section 4.

6.2 Preparation

For $x \in \mathbb{R}^d$, let $\|x\|_{\mathbb{T}^d} = \text{dist}(x, \mathbb{Z}^d)$. For a Borel set $U \subseteq \mathbb{R}^d$, let $|U|$ be its Lebesgue measure. Let d_0 be the dimension of the frequency α and $d_1 = d - d_0 + 1$, hence we have $d_0 = d$ and $d_1 = 1$ when $f_{*,\alpha} = f_{s,\alpha}$, while $d_0 = 1$ and $d_1 = d$ when $f_{*,\alpha} = f_{ss,\alpha}$. Let $D_r(x) \subset \mathbb{T}^d$ be the Euclidean ball centred at x with radius r.

6.2.1 Covering $\mathbb{T}^d$ with the orbit of a ball

We say a point θ in $\mathbb{T}^d$ is (f, r, M)-*dense* for some $r > 0$, $M \geq 1$, if $\cup_{j=0}^{r^{-M}} D_r(f^j\theta) = \mathbb{T}^d$. This means, the ball $D_r(\theta)$ with radius r will cover the whole $\mathbb{T}^d$ in r^{-M} steps under the map f. We say $(\mathbb{T}^d, f)$ is *strongly M-dense* if there exists $r_0 > 0$ such that any point in $\mathbb{T}^d$ is (f, r, M)-dense. We say $(\mathbb{T}^d, f)$ is *weakly M-dense* if there exists a sequence $r_k \to 0$ as $k \to \infty$ such that any point in $\mathbb{T}^d$ is (f, r_k, M)-dense.

The following lemmas are extracted from section 3 of [33].

Lemma 6.2.1 *Let f_s be an irrational shift on $\mathbb{T}^d$ and f_{ss} be a skew-shift. We have,*

- *if $\alpha \in DC(\tau) \subset \mathbb{T}^d$, then $(\mathbb{T}^d, f_s)$ is strongly M-dense for some $M \geq 1$.*
- *if $\alpha \in DC(\tau) \subset \mathbb{T}$, then $(\mathbb{T}^d, f_{ss})$ is strongly M-dense for some $M \geq 1$.*
- *if $\alpha \notin DC(d) \subset \mathbb{T}$, then $(\mathbb{T}^d, f_{ss})$ is weakly M-dense for some $M \geq 1$.*
- *if $\alpha \in WDC(\tau) \subset \mathbb{T}^2$, then $(\mathbb{T}^2, f_s)$ is weakly M-dense for some $M \geq 1$.*

6.2.2 Upper and lower bounds on transfer matrices

The following lemma on the uniform upper bound of transfer matrix is essentially from [45], we have adapted it into the following form for convenience.

Lemma 6.2.2 *[45] Let v be a function whose discontinuity set has Lebesgue measure 0 and f be a uniquely ergodic map on $\mathbb{T}^d$. Let $L(E)$ be positive on a Borel set U and μ be a measure such that $\mu(U) > 0$. Then for any $\zeta, \epsilon > 0$ there exists a number $D_\zeta > 0$, a set $B_{\zeta,\epsilon}$ with $0 < \mu(B_{\zeta,\epsilon}) < \zeta$, and an integer $N_{\zeta,\epsilon}$ such that for any $E \in U \backslash B_{\zeta,\epsilon}$:*

- *$L(E) \geq D_\zeta$,*
- *for $n > N_{\zeta,\epsilon}$, $|z - E| < e^{-4\epsilon n}$ and $\theta \in \mathbb{T}^d$, we have $\frac{1}{n} \ln \|A_n(\theta, z)\| < L(E) + \epsilon$.*

We also have the following lemma on the lower bound of transfer matrix.

Lemma 6.2.3 *[33] Let $v \in C^\gamma(\mathbb{T}^d)$ with $1 \geq \gamma > 0$ and $f_{*,\alpha} = f_{s,\alpha}$ or $f_{ss,\alpha}$. Let $L(E)$ be positive on a Borel set U and a measure μ with $\mu(U) > 0$. For any ζ, ϵ, let $D_\zeta, B_{\zeta,\epsilon}$ and $N_{\zeta,\epsilon}$ be defined as in Lemma 6.2.2. Then*

1. *if* $(\mathbb{T}^d, f_{*,\alpha})$ *is strongly* M*-dense for some* $M > 0$*, then for* $n > N'_{\zeta,\epsilon}$*, any* $E \in U \backslash B_{\zeta,\epsilon}$*,* $|z - E| < e^{-4\epsilon n}$ *and* $\theta \in \mathbb{T}^d$ *we have*

$$\min_{\iota \in \{-1,1\}} \max_{\iota j = 0, \ldots, e^{\frac{5M\epsilon}{\gamma} n}} \|A_n(f^j_{*,\alpha}\theta, z)\| \geq e^{n(L(E)-3\epsilon)}.$$

2. *if* $(\mathbb{T}^d, f_{*,\alpha})$ *is weakly* M*-dense for some* $M > 0$*, then there exists a sequence* $\{n_k(\epsilon)\}$ *such that for any* $k > k_{\zeta,\epsilon}$*, any* $E \in U \backslash B_{\zeta,\epsilon}$*,* $|z - E| < e^{-4\epsilon n_k}$ *and* $\theta \in \mathbb{T}^d$ *we have*

$$\min_{\iota \in \{-1,1\}} \max_{\iota j = 0, \ldots, e^{\frac{5M\epsilon}{\gamma} n_k}} \|A_{n_k}(f^j_{*,\alpha}\theta, z)\| \geq e^{n_k(L(E)-3\epsilon)}.$$

6.2.3 Continuity of the spectrum for well approximated frequencies

The following lemma enables us to establish the continuity of the spectrum at frequencies that are well approximated by the rationals, it is an extension of the $(\mathbb{T}, f_{s,\alpha})$ case in [10, 44].

Lemma 6.2.4 *Let* $v \in C^\gamma(\mathbb{T}^d)$ *with* $1 \geq \gamma > 0$ *and* $f_{*,\alpha} = f_{s,\alpha}$ *or* $f_{ss,\alpha}$*. Then for each* $E \in S(\alpha)$*, for* $\|\alpha' - \alpha\|_{\mathbb{T}^{d_0}}$ *small enough, there exists* $E' \in S(\alpha')$ *such that*

$$|E - E'| < C_v \|\alpha - \alpha'\|_{\mathbb{T}^{d_0}}^{\frac{\gamma}{1+d_1\gamma}}. \tag{6.2}$$

Two direct corollaries of Lemma 6.2.4 are:

Lemma 6.2.5 *Let* $f = f_{s,\alpha}$*. If* $\alpha \notin WDC(\frac{1}{\gamma})$*, then there exists a proper subsequence of the best simultaneous approximation* $\{\frac{\vec{p}_{n_k}}{q_{n_k}}\}$ *of* α*, such that for any* $f \in C^\gamma(\mathbb{T}^d)$*, we have*

$$S(\alpha) \subseteq \liminf_{k \to \infty} S(\frac{\vec{p}_{n_k}}{q_{n_k}}). \tag{6.3}$$

Lemma 6.2.6 *Let* $f = f_{ss,\alpha}$*. If* $\alpha \notin DC(d - 1 + \frac{1}{\gamma})$*, then there exists a proper subsequence of the continued fraction approximants* $\{\frac{p_{n_k}}{q_{n_k}}\}$ *of* α*, such that for any* $f \in C^\gamma(\mathbb{T}^d)$*, we have*

$$S(\alpha) \subseteq \liminf_{k \to \infty} S(\frac{p_{n_k}}{q_{n_k}}). \tag{6.4}$$

The proofs of Lemmas 6.2.4, 6.2.5, 6.2.6 will be included in Section 6.5.

In the next sections, we therefore focus on the Diophantine α.

6.3 Key Lemmas

Lemma 6.3.1 *Let $v \in C^\gamma(\mathbb{T}^d)$ with $1 \geq \gamma > 0$ and $f_{*,\alpha} = f_{s,\alpha}$ or $f_{ss,\alpha}$. Recall that $d_0 = d, d_1 = 1$ for $f_{ss,\alpha}$ and $d_0 = 1, d_1 = d$ for $f_{ss,\alpha}$. Then*

1. *for any $\zeta, \epsilon > 0$, let $D_\zeta, B_{\zeta,\epsilon}$ and $N_{\zeta,\epsilon}$ be defined as in Lemma 6.2.2. If $(\mathbb{T}^d, f_{*,\alpha})$ is strongly M-dense, then for $n > N'_{\zeta,\epsilon}$, where $N'_{\zeta,\epsilon}$ is defined as in Lemma 6.2.3, $E \in S(\alpha) \cap L_+(\alpha) \setminus B_{\zeta,\epsilon}$ and $\|\alpha' - \alpha\|_{\mathbb{T}^{d_0}}$ small enough, there exists $E' \in S(\alpha')$ so that*

$$|E - E'| \leq Ce^{-n(\frac{D_\zeta}{4} - \frac{5M\epsilon}{\gamma})} + C_v\|\alpha - \alpha'\|^\gamma_{\mathbb{T}^{d_0}} e^{5M\epsilon d_1 n}, \tag{6.5}$$

   where C is an absolute constant.

2. *for any $\zeta, \epsilon > 0$, let $B_{\zeta,\epsilon}$ and $N_{\zeta,\epsilon}$ be defined as in Lemma 6.2.2. If $(\mathbb{T}^d, f_{*,\alpha})$ is weakly M-dense, then for $k > k_{\zeta,\epsilon}$, where $\{n_k(\epsilon)\}$ and $k_{\zeta,\epsilon}$ are defined as in Lemma 6.2.3, $E \in S(\alpha) \cap L_{\epsilon_0+}(\alpha) \setminus B_{\zeta,\epsilon}$ and $\|\alpha' - \alpha\|_{\mathbb{T}^{d_0}}$ small enough, there exists $E' \in S(\alpha')$ so that*

$$|E - E'| \leq Ce^{-n_k(\frac{\epsilon_0}{4} - \frac{5M\epsilon}{\gamma})} + C_v\|\alpha - \alpha'\|^\gamma_{\mathbb{T}^{d_0}} e^{5M\epsilon d_1 n_k}, \tag{6.6}$$

   where C is an absolute constant.

Proof of Lemma 6.3.1

We will prove part (2). Part (1) will be discussed briefly at the end of the proof. For $E \in S(\alpha) \cap L_{\epsilon_0+} \setminus B_{\zeta,\epsilon}$, by Lemma 6.2.2, for $n > N_{\zeta,\epsilon}$ and $|z - E| < e^{-4\epsilon n}$ we have

$$\|A_n(\theta, z)\| \leq e^{n(L(E)+\epsilon)}. \tag{6.7}$$

By Lemma 6.2.3, for $k > k_{\zeta,\epsilon}$, $|z - E| < e^{-4\epsilon n_k}$ and any $\theta \in \mathbb{T}^d$ we have

$$\min_{\iota\in\{-1,1\}} \max_{\iota j = 0,\ldots,e^{\frac{5M\epsilon}{\gamma} n_k}} \|A_{n_k}(f^j_{*,\alpha}\theta, z)\| \geq e^{n_k(L(E)-3\epsilon)}. \tag{6.8}$$

Let E_0 be a generalized eigenvalue of $H_{v,f_{*,\alpha},\theta}$ such that $|E - E_0| < e^{-n_k(L(E)+4\epsilon)}$, with generalized eigenvector ψ satisfying $|\psi(x)| = o((1 + |x|)^{1/2+\epsilon})$. Then there exists x_m

so that

$$|\psi(x_m)| \over 1+|x_m| = \max_x \frac{|\psi(x)|}{1+|x|}. \tag{6.9}$$

Let ψ be normalized so that

$$\frac{|\psi(x_m)|}{1+|x_m|} = 1. \tag{6.10}$$

For $k > k_{\zeta,\epsilon}$, let $Q_{n_k} = e^{\frac{5M\epsilon}{\gamma} n_k}$. There exists an x_1' with $x_m - Q_{n_k} - n_k \le x_1' \le x_m - n_k$ such that $\|A_{n_k}(f_{*,\alpha}^{x_1'}\theta, E_0)\| > e^{n_k(L(E)-3\epsilon)}$. Similarly there exists an x_3' with $x_m \le x_3' \le x_m + Q_{n_k}$ such that $\|A_{n_k}(f_{*,\alpha}^{x_3'}\theta, E_0)\| > e^{n_k(L(E)-3\epsilon)}$. In general, we have

$$A_n(f_{*,\alpha}^l\theta, z) = \begin{pmatrix} P_n(f_{*,\alpha}^l\theta, z) & -P_{n-1}(f_{*,\alpha}^{l-1}\theta, z) \\ P_{n-1}(f_{*,\alpha}^{l-1}\theta, z) & -P_{n-2}(f_{*,\alpha}^{l-2}\theta, z) \end{pmatrix}.$$

This implies for $x_1 = x_1'$ or $x_1' - 1$ and $k_l = n_k, n_k - 1$ or $n_k - 2$, we have

$$|P_{k_l}(f_{*,\alpha}^{x_1}\theta, E_0)| > \frac{1}{4} e^{k_l(L(E)-3\epsilon)}. \tag{6.11}$$

Similarly, for $x_3 = x_3'$ or $x_3' - 1$ and $k_r = n_k, n_k - 1$ or $n_k - 2$, we have

$$|P_{k_r}(f_{*,\alpha}^{x_3}\theta, E_0)| > \frac{1}{4} e^{k_r(L(E)-3\epsilon)}. \tag{6.12}$$

Let

$$x_l = x_1 + \left[\frac{k_l}{2}\right]; \quad x_r = x_3 + \left[\frac{k_r}{2}\right]. \tag{6.13}$$

Also set $x_2 = x_1 + k_l - 1$ and $x_4 = x_3 + k_r - 1$. By Cramer's rule and (6.7), (6.11),

$$|G_{[x_1,x_2]}^{E_0}(x_l, x_1)| = \left|\frac{P_{x_2-x_l}(f_{*,\alpha}^{x_l+1}\theta, E_0)}{P_{k_l}(f_{*,\alpha}^{x_1}\theta, E_0)}\right| \le \frac{e^{\frac{k_l}{2}(L(E)+\epsilon)}}{\frac{1}{4}e^{k_l(L(E)-3\epsilon)}} < e^{-\frac{n_k}{4}L(E)}. \tag{6.14}$$

Similarly

$$|G_{[x_3,x_4]}^{E_0}(x_r, x_3)| < e^{-\frac{n_k}{4}L(E)}. \tag{6.15}$$

For similar reasons, (6.14) holds if we replace (x_l, x_1) with (x_l, x_2), $(x_l - 1, x_1)$ or $(x_l - 1, x_2)$; (6.15) holds if we replace (x_r, x_3) with (x_r, x_4), $(x_r + 1, x_3)$ or $(x_r + 1, x_4)$.

Let $\Lambda = [x_l, x_r]$, we have $|\Lambda| < 3Q_{n_k} = 3e^{\frac{5M\epsilon}{\gamma}n_k}$. Let ψ_Λ be the truncation of ψ to Λ. For $x = x_i \pm 1$, $i = 1, 2, 3, 4$, by (6.9) and (6.10),

$$\frac{|\psi(x)|}{1+|x_m|} = \frac{|\psi(x)|}{1+|x|} \cdot \frac{1+|x|}{1+|x_m|} \leq \frac{1+|x_m|+|x_m-x|}{1+|x_m|} \leq 2e^{\frac{5M\epsilon}{\gamma}n_k}. \tag{6.16}$$

For $x_1 \leq x \leq x_2$,

$$\psi(x) = -G^{E_0}_{[x_1,x_2]}(x, x_1)\psi(x_1 - 1) - G^{E_0}_{[x_1,x_2]}(x, x_2)\psi(x_2 + 1). \tag{6.17}$$

Thus by (6.14) and (6.16),

$$|\psi(x_l)| \leq 4(1+|x_m|)e^{-n_k(\frac{L(E)}{4} - \frac{5M\epsilon}{\gamma})}.$$

Similarly

$$|\psi(x_r)| \leq 4(1+|x_m|)e^{-n_k(\frac{L(E)}{4} - \frac{5M\epsilon}{\gamma})}.$$

Hence the cut-off function satisfies

$$\|(H_{v,f_{*,\alpha},\theta} - E_0)\psi_\Lambda\| \leq C(1+|x_m|)e^{-n_k(\frac{L(E)}{4} - \frac{5M\epsilon}{\gamma})}.$$

Let $\phi_\Lambda = \frac{\psi_\Lambda}{\|\psi_\Lambda\|}$. Then by (6.10),

$$\|(H_{v,f_{*,\alpha},\theta} - E_0)\phi_\Lambda\| \leq Ce^{-n_k(\frac{L(E)}{4} - \frac{5M\epsilon}{\gamma})}. \tag{6.18}$$

For $f_{*,\alpha'}$, set $\theta' = f_{*,\alpha'}^{-\frac{x_l+x_r}{2}} f_{*,\alpha}^{\frac{x_l+x_r}{2}}\theta$. Then $f_{*,\alpha'}^{\frac{x_l+x_r}{2}}\theta' = f_{*,\alpha}^{\frac{x_l+x_r}{2}}\theta$, furthermore for $|k| \leq \frac{x_r - x_l}{2}$,

$$\|f_{*,\alpha'}^{k+\frac{x_l+x_r}{2}}\theta' - f_{*,\alpha}^{k+\frac{x_l+x_r}{2}}\theta\| = \|f^k_{*,\alpha'} f_{*,\alpha}^{\frac{x_l+x_r}{2}}\theta - f^k_{*,\alpha} f_{*,\alpha}^{\frac{x_l+x_r}{2}}\theta\| \leq C|k|^{d_1}\|\alpha - \alpha'\|_{\mathbb{T}^{d_0}}. \tag{6.19}$$

Thus since $f \in C^\gamma(\mathbb{T}^d)$,

$$\begin{aligned} \|(H_{v,f_{*,\alpha},\theta} - H_{v,f_{*,\alpha'},\theta'})\phi_\Lambda\| &\leq \max_{|k|\leq(x_r-x_l)/2} |v(f_{*,\alpha}^{k+\frac{x_l+x_r}{2}}\theta) - v(f_{*,\alpha'}^{k+\frac{x_l+x_r}{2}}\theta')| \\ &\leq C_v(|\Lambda|^{d_1}\|\alpha - \alpha'\|_{d_0})^\gamma \\ &= C_v\|\alpha - \alpha'\|^\gamma_{\mathbb{T}^{d_0}} e^{5M\epsilon d_1 n_k}. \end{aligned} \tag{6.20}$$

Then by the choice of E_0 and (6.18), (6.20),

(6.21)

$$\begin{aligned}\|(E-H_{v,f_{*,\alpha'},\theta'})\phi_\Lambda\| &\le |E-E_0| + \|(E_0-H_{v,f_{*,\alpha},\theta})\phi_\Lambda\| + \|(H_{v,f_{*,\alpha},\theta}-H_{v,f_{*,\alpha'},\theta'})\phi_\Lambda\| \\ &\le Ce^{-n_k(\frac{L(E)}{4}-\frac{5M\epsilon}{\gamma})} + C_v\|\alpha-\alpha'\|^{\gamma}_{\mathbb{T}^{d_0}} e^{5M\epsilon d_1 n_k}.\end{aligned}$$

This implies there exists $E' \in S(\alpha')$ so that

$$|E-E'| \le Ce^{-n_k(\frac{\epsilon_0}{4}-\frac{5M\epsilon}{\gamma})} + C_v\|\alpha-\alpha'\|^{\gamma}_{\mathbb{T}^{d_0}} e^{5M\epsilon d_1 n_k}.$$

Remark 6.3.1 *Part (1) can be proved by considering* $S(\alpha)\cap L_+(\alpha)$ *instead of* $S(\alpha)\cap L_{\epsilon_0+}(\alpha)$ *and without taking a subsequence* $\{n_k(\epsilon)\}$. □

Lemma 6.3.2 *Let* $v \in C^\gamma(\mathbb{T}^d)$ *with* $1 \ge \gamma > 0$ *and* $f_{*,\alpha} = f_{s,\alpha}$ *or* $f_{ss,\alpha}$.

1. *If* $(\mathbb{T}^d, f_{*,\alpha})$ *is strongly* M*-dense for some* $M > 1$*, then for any* $\zeta > 0$ *and* $\gamma > \beta > 0$ *there exists a set* B^β_ζ *with* $0 < |B^\beta_\zeta| < \zeta$ *such that for any* $E \in S(\alpha)\cap L_+(\alpha)\backslash B^\beta_\zeta$ *and* $\|\alpha'-\alpha\|_{\mathbb{T}^{d_0}}$ *small enough, there exists* $E' \in S(\alpha')$ *satisfying*

$$|E-E'| < C_v\|\alpha-\alpha'\|^\beta_{\mathbb{T}^{d_0}}.$$

2. *Let* $d = 2$*,* $f = f_{s,\alpha}$ *and* $\alpha \in WDC(\frac{1}{\gamma})$*. Then for any* $\epsilon_0 > 0$ *and* $\gamma > \beta > 0$ *there exists a sequence* $\frac{\vec{p}_{m_k}}{q_{m_k}} \to \alpha$*, with the property that for any* $\zeta > 0$ *there exists a set* $B^{\beta,\epsilon_0}_\zeta$ *with* $0 < |B^{\beta,\epsilon_0}_\zeta| < \zeta$ *such that for any* $E \in S(\alpha)\cap L_{\epsilon_0+}(\alpha)\backslash B^{\beta,\epsilon_0}_\zeta$ *there exists* $E' \in S(\frac{\vec{p}_{m_k}}{q_{m_k}})$ *satisfying*

$$|E-E'| < C_v\|\alpha-\frac{\vec{p}_{m_k}}{q_{m_k}}\|^\beta_{\mathbb{T}^2}.$$

Proof of Lemma 6.3.2

Part (1)

Given $\zeta > 0$, let $D_\zeta > 0$ be from Lemma 6.2.2. Fix $\epsilon = \epsilon(\zeta,\beta) = \frac{\gamma(\gamma-\beta)D_\zeta}{20M(\gamma-\beta+4d_1\gamma\beta)} < \frac{D_\zeta}{4}$. Let $B^\beta_\zeta := B_{\zeta,\epsilon(\zeta,\beta)}, N^\beta_\zeta := N_{\zeta,\epsilon(\zeta,\beta)}$ with $B_{\zeta,\epsilon}, N_{\zeta,\epsilon}$ as in Lemma 6.2.2. Let $\tilde{N}^\beta_\zeta := N'_{\zeta,\epsilon(\zeta,\beta)}$ be defined as in Lemma 6.2.3. By Lemma 6.3.1, for any $n > \tilde{N}^\beta_\zeta$, $E \in$

$S(\alpha)\cap L_+(\alpha)\setminus B_\zeta^\beta$ and $\|\alpha'-\alpha\|_{\mathbb{T}^{d_0}}$ small enough, there exists $E'\in S(\alpha')$ so that E' is close to E, namely,

$$|E-E'|\le Ce^{-n(\frac{D_\zeta}{4}-\frac{5M\epsilon}{\gamma})}+C_v\|\alpha-\alpha'\|_{\mathbb{T}^{d_0}}^{\gamma}e^{5M\epsilon d_1 n}. \tag{6.22}$$

There exists a small constant $\varrho_{\zeta,\beta}>0$ so that when $\|\alpha-\alpha'\|_{\mathbb{T}^{d_0}}<\varrho_{\zeta,\beta}$ we have

$$N'_{\zeta,\beta}<\frac{\gamma-\beta+2d_1\gamma\beta}{d_1\gamma D_\zeta}(-\ln\|\alpha-\alpha'\|_{\mathbb{T}^{d_0}}).$$

Then we could take $n>N'_{\zeta,\beta}$ satisfying

$$\frac{\gamma-\beta+4d_1\gamma\beta}{d_1\gamma D_\zeta}(-\ln\|\alpha-\alpha'\|_{\mathbb{T}^{d_0}})\le n\le\frac{4(\gamma-\beta+4d_1\gamma\beta)}{d_1\gamma D_\zeta}(-\ln\|\alpha-\alpha'\|_{\mathbb{T}^{d_0}}),$$

so that by (6.22) there exists $E'\in S(\alpha')$ with

$$|E-E'|<C_v\|\alpha-\alpha'\|_{\mathbb{T}^{d_0}}^{\beta}. \tag{6.23}$$

□

Part (2)

For $\epsilon_0>0$, fix a constant $\epsilon=\epsilon(\beta,\epsilon_0)=\frac{\gamma(\gamma-\beta)\epsilon_0}{20M(\gamma-\beta+4\beta)}<\frac{\epsilon_0}{4}$. For any $\zeta>0$, let $B_\zeta^{\beta,\epsilon_0}:=B_{\zeta,\epsilon(\beta,\epsilon_0)}$ and $N_\zeta^{\beta,\epsilon_0}:=N_{\zeta,\epsilon(\beta,\epsilon_0)}$ be as in Lemma 6.2.2. Let $\{n_k(\beta,\epsilon_0)\}:=\{n_k(\epsilon(\beta,\epsilon_0))\}$ and $k_\zeta^{\beta,\epsilon_0}:=k_{\zeta,\epsilon(\beta,\epsilon_0)}$ be as in Lemma 6.2.3. By Lemma 6.3.1, for any $k>k_\zeta^{\beta,\epsilon_0}$, $E\in S(\alpha)\cap L_{\epsilon_0+}(\alpha)\setminus B_\zeta^{\beta,\epsilon_0}$ and $\|\alpha'-\alpha\|_{\mathbb{T}^2}$ small enough, there exists $E'\in S(\alpha')$ so that

$$|E-E'|\le Ce^{-n_k(\frac{\epsilon_0}{4}-\frac{5M\epsilon}{\gamma})}+C_v\|\alpha-\alpha'\|_{\mathbb{T}^2}^{\gamma}e^{5M\epsilon n_k}. \tag{6.24}$$

$\alpha\in WDC(c,\frac{1}{\gamma})$ for some $c>0$. Take the sequence of best simultaneous approximation $\{\frac{\vec{p}_m}{q_m}\}$. By (1.10) we have $q_m\ge c^\gamma q_{m+1}^{\frac{\gamma}{2}}$. Combining this with (1.9) and (1.7), we have

$$\|\alpha-\frac{\vec{p}_{m+1}}{q_{m+1}}\|_{\mathbb{T}^2}\ge\frac{c}{q_{m+1}^{1+\frac{1}{\gamma}}}\ge c(\frac{1}{q_m}\frac{1}{\sqrt{q_{m+1}}})^{\frac{2}{\gamma}}\ge c\|\alpha-\frac{\vec{p}_m}{q_m}\|_{\mathbb{T}^2}^{\frac{2}{\gamma}}.$$

Which implies

$$-\ln\|\alpha-\frac{\vec{p}_m}{q_m}\|_{\mathbb{T}^2}<-\ln\|\alpha-\frac{\vec{p}_{m+1}}{q_{m+1}}\|_{\mathbb{T}^2}\lesssim-\frac{2}{\gamma}\ln\|\alpha-\frac{\vec{p}_m}{q_m}\|_{\mathbb{T}^2}.$$

Therefore for each $n_k(\beta, \epsilon_0)$ there must be a corresponding $m_k(\beta, \epsilon_0)$ such that that

$$\frac{\gamma\epsilon_0}{4(\gamma-\beta+4\beta)}n_k \leq -\ln\|\alpha - \frac{\vec{p}_{m_k}}{q_{m_k}}\|_{\mathbb{T}^2} \leq \frac{\epsilon_0}{\gamma-\beta+4\beta}n_k.$$

By (6.24) and the choice of m_k, there exists $E' \in S(\frac{\vec{p}_{m_k}}{q_{m_k}})$ so that

$$|E - E'| \leq C_v\|\alpha - \frac{\vec{p}_{m_k}}{q_{m_k}}\|_{\mathbb{T}^2}^{\beta}. \tag{6.25}$$

□

6.4 Proof of Theorems 6.1.1, 6.1.2 and 6.1.3

First of all, the continuity of $S(\alpha)$ in the Hausdorff metric implies that for any sequence $\frac{\vec{p}_n}{q_n} \to \alpha$,

$$\limsup_{n\to\infty} S(\frac{\vec{p}_n}{q_n}) \subseteq S(\alpha). \tag{6.26}$$

By (6.26) and Lemmas 6.2.5, 6.2.6, the proofs are all reduced to proving a statement of the following type

$$S(\alpha) \cap L_+(\alpha) \subseteq \liminf_{k\to\infty} S(\frac{\vec{p}_{n_k}}{q_{n_k}}).$$

Since the proofs for $(\mathbb{T}^d, f_{s,\alpha})$, $(\mathbb{T}^d, f_{ss,\alpha})$ and $(\mathbb{T}^2, f_{s,\alpha})$ (weakly M-dense) relying on Lemma 6.3.2 are quite similar, we will only give the proof for $(\mathbb{T}^d, f_{ss,\alpha})$ in detail. The other two proofs will be discussed briefly at the end of this section.

Proof of Theorem 6.1.2

Let $\frac{p_n}{q_n}$ be the full sequence of continued fraction approximants of α. Since $\gamma > \frac{1}{2}$, we could fix $\frac{1}{2} < \beta < \gamma$. By Lemma 6.3.2, for any $\zeta > 0$ there exists $B_\zeta := B_\zeta^\beta$, $0 < |B_\zeta| < \zeta$, such that for n large enough we have

$$S(\alpha) \cap L_+(\alpha) \setminus B_\zeta \subset \cup_{i=1}^{q_n'}[a_{n,i} - C_v\|\alpha - \frac{p_n}{q_n}\|_{\mathbb{T}}^{\beta},\ b_{n,i} + C_v\|\alpha - \frac{p_n}{q_n}\|_{\mathbb{T}}^{\beta}] := S(\frac{p_n}{q_n}) \cup F_n,$$

where $q_n' \leq q_n$ and

$$S(\frac{p_n}{q_n}) = \cup_{i=1}^{q_n'}[a_{n,i}, b_{n,i}].$$

This implies

$$S(\alpha) \cap L_+(\alpha) \setminus B_\zeta \subset \liminf_{n\to\infty} S(\frac{p_n}{q_n}) \cup F_n,$$

furthermore,

$$|S(\alpha) \cap L_+(\alpha) \setminus (\liminf_{n\to\infty} S(\frac{p_n}{q_n}) \cup F_n)| < \zeta. \tag{6.27}$$

By (1.1),

$$|F_n| \le 2C_v q_n \|\alpha - \frac{p_n}{q_n}\|_{\mathbb{T}}^\beta \le 2C_v q_{n+1}^{1-2\beta}, \tag{6.28}$$

which implies $\sum_n |F_n| < \infty$, thus $|\limsup_{n\to\infty} F_n| = 0$. This implies

$$|\liminf_{n\to\infty} S(\frac{p_n}{q_n}) \cup F_n| = |\liminf_{n\to\infty} S(\frac{p_n}{q_n})|. \tag{6.29}$$

Combining (6.27) with (6.29), we have

$$|S(\alpha) \cap L_+(\alpha) \setminus \liminf_{n\to\infty} S(\frac{p_n}{q_n})| < \zeta$$

for any $\zeta > 0$. Thus

$$S(\alpha) \cap L_+(\alpha) \subseteq \liminf_{n\to\infty} S(\frac{p_n}{q_n}). \tag{6.30}$$

□

Theorem 6.1.1 could be proved by taking $\frac{\vec{p}_n}{q_n}$ to be the full sequence of best simultaneous approximation. One needs to apply (1.8) to obtain the following (similar to (6.28))

$$|F_n| \le 2C_v q_{n+1}^{1-\frac{d+1}{d}\beta}. \tag{6.31}$$

Theorem 6.1.3 could be proved by applying part (2) of Lemma 6.3.2. □

6.5 Proofs of Lemmas 6.2.4, 6.2.5, 6.2.6

6.5.1 Lemma 6.2.4

The proof is very similar to that of [10, 44]. Given $\epsilon > 0$ and $E \in S(\alpha)$, there exists an approximate eigenfunction $\phi_\epsilon \in l^2(\mathbb{Z})$ such that $\|(H_{f_*,\alpha,\theta} - E)\phi_\epsilon\| < \epsilon\|\phi_\epsilon\|$. Set

$g_{j,L}(n) = \max(1 - \frac{|j-n|}{L}, 0)$. Avron-van Mouche-Simon [10] proved that for sufficiently large L, for any bounded $v : \mathbb{T}^d \to \mathbb{R}$ there exists j such that $g_{j,L}\phi_\epsilon \neq 0$ and for any $\epsilon > 0$,

$$\|(H_{f_{*,\alpha},\theta} - E)g_{j,L}\phi_\epsilon\|^2 \leq C(\epsilon^2 + L^{-2})\|g_{j,L}\phi_\epsilon\|^2, \tag{6.32}$$

where C is universal. Now let $\theta' = f_{*,\alpha'}^{-j} f_{*,\alpha}^{j}\theta$. By the Hölder assumption on v and $j - L \leq n \leq j + L$, we have

$$|v(f_{*,\alpha'}^n\theta') - v(f_{*,\alpha}^n\theta)| \leq C_v(L^{d_1}\|\alpha' - \alpha\|_{\mathbb{T}^{d_0}})^\gamma.$$

Thus,

$$\|(H_{f_{*,\alpha'}\theta'} - E)g_{j,L}\phi_\epsilon\| \leq \|(H_{f_{*,\alpha'}\theta'} - H_{f_{*,\alpha},\theta})g_{j,L}\phi_\epsilon\| + \|(H_{f_{*,\alpha},\theta} - E)g_{j,L}\phi_\epsilon\| \tag{6.33}$$

$$\leq (C_v(L^{d_1}\|\alpha' - \alpha\|_{\mathbb{T}^{d_0}})^\gamma + C(\epsilon^2 + L^{-2})^{\frac{1}{2}})\|g_{j,L}\phi_\epsilon\|. \tag{6.34}$$

Choosing $\epsilon = L^{-1} = C_v\|\alpha - \alpha'\|_{\mathbb{T}^{d_0}}^{-\frac{\gamma}{1+d_1\gamma}}$, we obtain the statement of Lemma 6.2.4. $\square$

6.5.2 Lemma 6.2.5

Assume $\alpha \notin WDC(\frac{1}{\gamma})$. Then by (1.11), there exists a subsequence of the best simultaneous Diophantine approximation $\{\frac{\vec{p}_{n_k}}{q_{n_k}}\}$ so that

$$\lim_{k\to\infty} q_{n_k}^{\frac{1}{1+\gamma}} \max_{1\leq j\leq d} \|q_{n_k}\alpha_j\|_{\mathbb{T}}^{\frac{\gamma}{1+\gamma}} = 0. \tag{6.35}$$

By Lemma 6.2.4, we have

$$S(\alpha) \subset \cup_{i=1}^{q'_{n_k}}[a_{n_k,i} - C_v\|\alpha - \frac{\vec{p}_{n_k}}{q_{n_k}}\|_{\mathbb{T}^d}^{\frac{\gamma}{1+\gamma}},\ b_{n_k,i} + C_v\|\alpha - \frac{\vec{p}_{n_k}}{q_{n_k}}\|_{\mathbb{T}^d}^{\frac{\gamma}{1+\gamma}}] := S(\frac{\vec{p}_{n_k}}{q_{n_k}}) \cup F_{n_k},$$

where $q'_{n_k} \leq q_{n_k}$ and

$$S(\frac{\vec{p}_{n_k}}{q_{n_k}}) = \cup_{i=1}^{q'_{n_k}}[a_{n_k,i}, b_{n_k,i}].$$

Thus, by (6.35),

$$S(\alpha) \subseteq \liminf_{k\to\infty} S(\frac{\vec{p}_{n_k}}{q_{n_k}}).$$

$\square$

6.5.3 Lemma 6.2.6

Assume $\alpha \notin DC(d-1+\frac{1}{\gamma})$. Then by (1.11), there exists a subsequence of the continued fraction approximants $\frac{p_{n_k}}{q_{n_k}}$ so that

$$\lim_{k\to\infty} q_{n_k}^{\frac{1+(d-1)\gamma}{1+d\gamma}} \|q_{n_k}\alpha\|_{\mathbb{T}}^{\frac{\gamma}{1+d\gamma}} = 0 \tag{6.36}$$

By Lemma 6.2.4, we have

$$S(\alpha) \subset \cup_{i=1}^{q'_{n_k}} [a_{n_k,i} - C_v\|\alpha - \frac{p_{n_k}}{q_{n_k}}\|_{\mathbb{T}}^{\frac{\gamma}{1+d\gamma}},\ b_{n_k,i} + C_v\|\alpha - \frac{p_{n_k}}{q_{n_k}}\|_{\mathbb{T}}^{\frac{\gamma}{1+d\gamma}}] := S(\frac{p_{n_k}}{q_{n_k}}) \cup F_{n_k},$$

where $q'_{n_k} \leq q_{n_k}$ and

$$S(\frac{p_{n_k}}{q_{n_k}}) = \cup_{i=1}^{q'_{n_k}} [a_{n_k,i}, b_{n_k,i}].$$

Thus, by (6.36),

$$S(\alpha) \subseteq \liminf_{k\to\infty} S(\frac{p_{n_k}}{q_{n_k}}).$$

□

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APPENDICES

Appendix A

When λ belongs to region II^{o}, let $\epsilon_2 = \ln \frac{\lambda_2+\sqrt{\lambda_2^2-4\lambda_1\lambda_3}}{\lambda_1+\lambda_3+\sqrt{(\lambda_1+\lambda_3)^2-4\lambda_1\lambda_3}} > L(\hat{\lambda})$. Then $c_\lambda(x)$ is analytic and nonzero on $|\mathrm{Im}(x)| < \frac{\epsilon_2}{2\pi}$. Furthermore, the winding number of $c_\lambda(\cdot + i\epsilon)$ is equal to zero when $|\epsilon| < \frac{\epsilon_2}{2\pi}$.

Lemma A.0.1 *When λ belongs to region* II^{o}*, we can find an analytic function $f(x)$ on $|\mathrm{Im}(x)| \leq \frac{L(\hat{\lambda})}{2\pi}$ such that $c_\lambda(x) = |c_\lambda|(x)e^{f(x+\alpha)-f(x)}$ and $\tilde{c}_\lambda(x) = |c_\lambda|(x)e^{-f(x+\alpha)+f(x)}$.*

Proof: Since the winding numbers of $c_\lambda(x)$ and $\tilde{c}_\lambda(x)$ are 0 on $|\mathrm{Im}(x)| \leq \frac{L(\hat{\lambda})}{2\pi}$, there exist analytic functions $g_1(x)$ and $g_2(x)$ on $|\mathrm{Im}(x)| \leq \frac{L(\hat{\lambda})}{2\pi}$, such that $c_\lambda(x) = e^{g_1(x)}$ and $\tilde{c}_\lambda(x) = e^{g_2(x)}$. Notice that

$$\int_{\mathbb{T}} \ln |c_\lambda(x)| \, \mathrm{d}x = \int_{\mathbb{T}} \ln |\tilde{c}_\lambda(x)| \, \mathrm{d}x$$
$$\int_{\mathbb{T}} \arg c_\lambda(x) \, \mathrm{d}x = \int_{\mathbb{T}} \arg \tilde{c}_\lambda(x) \, \mathrm{d}x,$$

so there exists an analytic function $f(x)$ such that $2f(x+\alpha) - 2f(x) = g_1(x) - g_2(x)$. Then $c_\lambda(x) = |c_\lambda|(x)e^{f(x+\alpha)-f(x)}$. □

Lemma A.0.2 *When λ belongs to region* II^{o}*, there exists an analytic matrix $Q_\lambda(x)$ defined on $|\mathrm{Im}(x)| \leq \frac{L(\hat{\lambda})}{2\pi}$ such that*

$$Q_\lambda^{-1}(x+\alpha)\tilde{A}_{|c_{\hat{\lambda}}|,E}(x)Q_\lambda(x) = A_{c_\lambda,E}(x).$$

Proof:

$$\begin{aligned}
\tilde{A}_{|c_{\hat{\lambda}}|,E}(x) =& \frac{1}{\sqrt{|c_\lambda|(x)|c_\lambda|(x-\alpha)}} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\frac{\tilde{c}_\lambda(x)}{c_\lambda(x)}} \end{pmatrix} \begin{pmatrix} E - v(x) & -\tilde{c}_\lambda(x-\alpha) \\ c_\lambda(x) & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\frac{c_\lambda(x-\alpha)}{\tilde{c}_\lambda(x-\alpha)}} \end{pmatrix} \\
=& \frac{c_\lambda(x)}{\sqrt{|c_\lambda|(x)|c_\lambda|(x-\alpha)}} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\frac{\tilde{c}_\lambda(x)}{c_\lambda(x)}} \end{pmatrix} A_{c_\lambda,E}(x) \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\frac{c_\lambda(x-\alpha)}{\tilde{c}_\lambda(x-\alpha)}} \end{pmatrix} \\
=& e^{f(x+\alpha)} \sqrt{|c_\lambda|(x)} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\frac{\tilde{c}_\lambda(x)}{c_\lambda(x)}} \end{pmatrix} A_{c_\lambda,E}(x) \left\{ e^{f(x)} \sqrt{|c_\lambda|(x-\alpha)} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\frac{\tilde{c}_\lambda(x-\alpha)}{c_\lambda(x-\alpha)}} \end{pmatrix} \right\}^{-1} \\
=& Q_\lambda(x+\alpha) A_{c_\lambda,E}(x) Q_\lambda^{-1}(x).
\end{aligned}$$

□

Lemma A.0.3 *If α is irrational, λ belongs to region* II^{o}*, $E \in \Sigma_{\lambda,\alpha}$, then $L(\alpha, A_{c_\lambda,E}(\cdot + i\epsilon)) = L(\alpha, \tilde{A}_{|c_{\hat{\lambda}}|,E}(\cdot + i\epsilon)) = 0$ for $|\epsilon| \leq \frac{L(\hat{\lambda})}{2\pi}$.*

Proof: $L(A_{c_\lambda,E}(\cdot + i\epsilon)) = L(D_{\lambda,E}(\cdot + i\epsilon)) - \int \ln |c_\lambda(x + i\epsilon)| \mathrm{d}x$

$$\begin{aligned}
& D_{\lambda,E}(x + i\epsilon) \\
=& \begin{pmatrix} E - e^{2\pi i(x+i\epsilon)} - e^{-2\pi i(x+i\epsilon)} & -\lambda_1 e^{2\pi i(x - \frac{\alpha}{2} + i\epsilon)} - \lambda_2 - \lambda_3 e^{-2\pi i(x - \frac{\alpha}{2} + i\epsilon)} \\ \lambda_1 e^{-2\pi i(x + \frac{\alpha}{2} + i\epsilon)} + \lambda_2 + \lambda_3 e^{2\pi i(x + \frac{\alpha}{2} + i\epsilon)} & 0 \end{pmatrix} \\
=& e^{2\pi\epsilon} \begin{pmatrix} -e^{2\pi i x} + o(1) & -\lambda_3 e^{-2\pi i(x - \frac{\alpha}{2})} + o(1) \\ \lambda_1 e^{-2\pi i(x + \frac{\alpha}{2})} + o(1) & 0 \end{pmatrix}.
\end{aligned}$$

Thus the asymptotic behaviour of $L(D_{\lambda,E}(\cdot + i\epsilon))$ is:

$$\begin{aligned}
L(D_{\lambda,E}(\cdot + i\epsilon)) &= \ln |\frac{1 + \sqrt{1 - 4\lambda_1\lambda_3}}{2}| + 2\pi\epsilon \quad \text{when } \epsilon \to \infty, \\
L(D_{\lambda,E}(\cdot + i\epsilon)) &= \ln |\frac{1 + \sqrt{1 - 4\lambda_1\lambda_3}}{2}| - 2\pi\epsilon \quad \text{when } \epsilon \to -\infty.
\end{aligned}$$

Then it suffices to calculate $\int \ln |c_\lambda(x + i\epsilon)| \mathrm{d}x$ in region II. We have

$$\begin{aligned}
& \int \ln |c_\lambda(x + i\epsilon)| \mathrm{d}x \\
=& \ln \lambda_3 - 2\pi\epsilon + \int \ln |e^{2\pi i x} - y_{1,\epsilon}| + \int \ln |e^{2\pi i x} - y_{2,\epsilon}|.
\end{aligned}$$

where $y_{1,\epsilon} = \frac{-\lambda_2+\sqrt{\lambda_2^2-4\lambda_1\lambda_3}}{2\lambda_3}e^{2\pi\epsilon}$ and $y_{2,\epsilon} = \frac{-\lambda_2-\sqrt{\lambda_2^2-4\lambda_1\lambda_3}}{2\lambda_3}e^{2\pi\epsilon}$.

$$\int \ln|c_\lambda(x+i\epsilon)|\mathrm{d}x = \begin{cases} 2\pi\epsilon + \ln\lambda_1 & \epsilon > \frac{1}{2\pi}\ln\frac{\lambda_2+\sqrt{\lambda_2^2-4\lambda_1\lambda_3}}{2\lambda_1}, \\ \ln\frac{\lambda_2+\sqrt{\lambda_2^2-4\lambda_1\lambda_3}}{2} & \frac{1}{2\pi}\ln\frac{\lambda_2-\sqrt{\lambda_2^2-4\lambda_1\lambda_3}}{2\lambda_1} \leq \epsilon \leq \frac{1}{2\pi}\ln\frac{\lambda_2+\sqrt{\lambda_2^2-4\lambda_1\lambda_3}}{2\lambda_1}, \\ -2\pi\epsilon + \ln\lambda_3 & \epsilon < \frac{1}{2\pi}\ln\frac{\lambda_2-\sqrt{\lambda_2^2-4\lambda_1\lambda_3}}{2\lambda_1}. \end{cases}$$

Thus $L(A_{c_\lambda,E}(\cdot+i\epsilon)) = 0$ when $|\epsilon| \leq \frac{1}{2\pi}\ln\frac{\lambda_2+\sqrt{\lambda_2^2-4\lambda_1\lambda_3}}{\max(1,\lambda_1+\lambda_3)+\sqrt{\max(1,\lambda_1+\lambda_3)^2-4\lambda_1\lambda_3}} = \frac{L(\hat{\lambda})}{2\pi}$. Since $\tilde{A}_{|c_{\hat{\lambda}}|,E}(x+i\epsilon) = Q_\lambda(x+\alpha+i\epsilon)A_{c_\lambda,E}(x+i\epsilon)Q_\lambda^{-1}(x+i\epsilon)$, the statement about $\tilde{A}_{|c_{\hat{\lambda}}|,E}$ is also true. □