REPORT NO.
UCB/SESM-81/03

STRUCTURAL ENGINEERING AND
STRUCTURAL MECHANICS

PENALTY FORMULATIONS FOR RUBBER-LIKE ELASTICITY

by

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Report to:
Malaysian Rubber Producers Research Association

NOVEMBER 1981

DEPARTMENT OF CIVIL ENGINEERING
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA

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ABSTRACT

The analysis of incompressible rubber-like materials is considered by a penalty function approach. In particular the necessary compatibility conditions for developing penalty forms for isotropic nonlinear elasticity are addressed, and the commonly used form of strain energy functional leading to a penalty formulation of the incompressibility constraint extended. The Mooney-Rivlin model is used to show how previous developments can lead to physically meaningless situations. For a certain class of incompressible materials, which includes the important Mooney-Rivlin model, a new and particularly simple formulation is proposed. The finite element implementation of this penalty formulation, is considered in the Appendices A. and B. of this report.

ACKNOWLEDGEMENTS

Support for this research was provided by the Malaysian Rubber Producers Research Association of Brichendonbury, England, as part of a program of research on the use of rubber bearings for the seismic protection of buildings. This support is gratefully acknowledged.

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1. Introduction

The penalty function method, originally proposed by Courant [1], provides a simple and effective procedure of reducing a constrained minimization problem, to one without constraints.

The method has a simple geometrical interpretation, when placed in the general context of optimization theory in Banach spaces [2]. Furthermore, in contrast with most of duality methods, it can be justified without invoking any convexity assumptions on the functional to be minimized [2], [3]. This remarkable fact, makes the method particularly attractive in problems in nonlinear elastostatics, where the total potential energy functional is rarely convex *.

In the context of nonlinear elastostatics, the penalty function method provides a procedure of enforcing the incompressibility constraint, without restraining the configuration space to isochoric deformations. For this class of problems, the application of the method hinges on a suitable extension to the compressible range, of the constitutive model for the given incompressible material.

The most widely used extension [3]-[6], assumes as strain energy in the compressible range, the sum of the strain energy potential of the incompressible material plus a penalty term enforcing the incompressibility constraint.

The purpose of this paper is to discuss the form of this penalty term and moreover, show that this form of strain energy is not the only possible one leading to a penalty formulation of the incompressibility constraint.

For a certain class of incompressible materials, to which the Mooney-Rivlin model belongs, an alternative form of the strain energy potential in the compressible range is proposed, which leads to a simpler form of the elasticity tensor. This form is, therefore, particularly useful in the context of a finite element formulation.

* As a matter of fact, convexity implies uniqueness and therefore precludes buckling [12]. See also [8] and references therein.

1. Isotropic non-linear elastostatics.

Consider a hyperelastic, isotropic body B, assumed to be identified with its reference configuration $B \subset R^3$, a bounded open set with smooth boundary ∂B. Let $\Phi : B \rightarrow R^3$ be any (finite) deformation, and $\partial B_d \subset \partial B$ the part of the boundary where Φ is prescribed. Similarly $\partial B_t \subset \partial B$ is that part of the boundary where the Kirchhoff stress vector $\mathbf{P}\hat{\mathbf{N}}=\bar{\mathbf{t}}$ is prescribed. The configuration space C is then the set of diffeomorphisms defined by

$$C = \{ \Phi: B \rightarrow R^3 \mid \; det(\nabla\Phi) > 0 \; and \; \Phi|_{\partial B_d} = \tilde{g} \} \tag{1}$$

where $\partial B_d \cap \partial B_t = \varnothing$ and $\partial B_d \cup \partial B_t = \partial B$

Denote by $\mathbf{F} = \nabla\Phi$ the deformation gradient and let $\mathbf{C}=\mathbf{F}'\mathbf{F}$ be the right Cauchy-Green tensor, with principal invariant $I_i, (i=1,2,3)$. If we let $J = det(\mathbf{F})$ then ρ_o and $\rho = \frac{\rho_o}{J}$ are the densities in the undeformed and deformed configurations respectively. Since the body is assumed to be hyperelastic and isotropic the strain energy functional $W: C \rightarrow R$ exists and is given by:

$$\Phi \rightarrow W(\Phi) = \overline{W}(\mathbf{F}) = \overline{\overline{W}}(\mathbf{C}) = \tilde{W}(I_1, I_2, I_3) \tag{2}$$

and the first Piola-Kirchhoff and Cauchy stress tensors, denoted by $\mathbf{P}$ and σ respectively are computed from the strain energy W according to

$$\mathbf{P} = \frac{\partial \overline{W}}{\partial \mathbf{F}} \tag{3}$$

and

$$\sigma = \frac{1}{J}\mathbf{P}\mathbf{F}' \tag{4}$$

Since $I_1 = tr(\mathbf{C})$, $I_2 = \frac{1}{2}\{I_1^2 - tr(\mathbf{C}^2)\}$, and $I_3 = det(\mathbf{C}) = J^2$, application of the chain rule and the Cayley-Hamilton theorem gives for the Cauchy stress tensor, the constitutive equation [9]

$$\sigma = \tilde{\beta}_o\mathbf{I} + \tilde{\beta}_1\mathbf{B} + \tilde{\beta}_2\mathbf{B}^{-1} \tag{5}$$

where $\mathbf{B} = \mathbf{F}\mathbf{F}'$ is the left Cauchy-Green tensor, and the functions $\tilde{\beta}_i(I_1, I_2, I_3)$ $(i=1,2,3)$ are given by

$$\tilde{\beta}_o = \frac{2}{J}\left\{ I_2\frac{\partial \tilde{W}}{\partial I_2} + I_3\frac{\partial \tilde{W}}{\partial I_3} \right\}, \;\; \tilde{\beta}_1 = \frac{2}{J}\frac{\partial \tilde{W}}{\partial I_1}, \;\; \tilde{\beta}_2 = -\frac{2}{J}I_3\frac{\partial \tilde{W}}{\partial I_2} \tag{6}$$

However, the response functions $\tilde{\beta}_i$, $(i=1,2,3)$ are not independent. They are related by the key compatibility conditions :

$$\begin{aligned} \tfrac{1}{2}\tilde{\beta}_1 + I_3\frac{\partial\tilde{\beta}_1}{\partial I_3} - I_2 I_3^{-1}\frac{\partial\tilde{\beta}_2}{\partial I_1} &= \frac{\partial\tilde{\beta}_o}{\partial I_1} \\ \tfrac{1}{2}\tilde{\beta}_2 + I_3\frac{\partial\tilde{\beta}_2}{\partial I_3} + I_2\frac{\partial\tilde{\beta}_2}{\partial I_2} &= -I_3\frac{\partial\tilde{\beta}_o}{\partial I_2} \\ I_3\frac{\partial\tilde{\beta}_1}{\partial I_2} + \frac{\partial\tilde{\beta}_2}{\partial I_1} &= 0 \end{aligned} \tag{7}$$

which can be easily checked using (6).

Denote by **A** the fourth order elasticity tensor, defined in terms of the strain energy functional by

$$\mathbf{A} = \frac{\partial^2 \overline{W}}{\partial \mathbf{F}\partial \mathbf{F}} = \frac{\partial \mathbf{P}}{\partial \mathbf{F}} \tag{8}$$

with components A_{iIjJ}, with respect to the standard basis $\{\mathbf{E}_I\}$ and $\{\mathbf{e}_i\}$ in the configurations B and $\Phi(B)$ respectively.

Let V be the space of kinematically admissible variations, that is, the tangent space to the configuration space C. Thus V is defined by :

$$V = \{\ \mathbf{v}\colon B \rightarrow R^3 \mid \ \mathbf{v}|_{\partial B_d} = \mathbf{o}\ \} \tag{9}$$

With this notation, the condition $\nabla\mathbf{v}.(\mathbf{A}.\nabla\mathbf{w}) = \nabla\mathbf{w}.(\mathbf{A}.\nabla\mathbf{v})$ for any $\mathbf{v},\mathbf{w}\in V$ is the necessary and sufficient condition for the potential $W\colon C \rightarrow R$ to exist [8]-[9]. This condition is equivalent to the symmetry of elasticity tensor; i.e

$$A_{iIjJ} = A_{jJiI} \tag{10}$$

It can be shown, that the symmetry condition (10) holds, provided the compatibility conditions given by (7) hold. Therefore (7) are just the integrability conditions for $\mathbf{P} = J\sigma\mathbf{F}^{-t}$ to be derivable from the potential W.

1.1. Incompressibility. The Mooney-Rivlin model.

In the case of an incompressible hyperelastic material, the configurations $\Phi\colon B \rightarrow R^3$ are restricted by the constraint $J = det(\mathbf{F}) = 1$. The constrained configuration space $C_{inc} \subset C$ is then defined by

$$C_{inc} = \{ \Phi: B \rightarrow R^3 \mid \Phi|_{\partial B_d} = \bar{g}, \ and\ J = 1 \} \tag{11}$$

and the constrained tangent space of admissible variations :

$$V_{inc} = \{ \mathbf{v}: B \rightarrow R^3 \mid \mathbf{v}|_{\partial B_d} = \mathbf{o}, \ and\ div(\mathbf{v}) = 0 \} \tag{12}$$

The strain energy functional, denoted now by $\hat{W}: C_{inc} \rightarrow R$, depends only on the first and second invariant I_1 and I_2. Then instead of equations (3) and (5), the appropriate expressions for $\mathbf{P}$ and σ are [9],[10]

$$\mathbf{P} = -\hat{p}\mathbf{F}^{-t} + \frac{\partial W}{\partial \mathbf{F}} \tag{13}$$

and

$$\sigma = -\hat{p}\mathbf{I} + \hat{\beta}_1\mathbf{B} + \hat{\beta}_2\mathbf{B}^{-1} \tag{14}$$

where $\hat{p}: B \rightarrow R$ is the hydrostatic pressure, a bounded function to be determined by the boundary conditions. The response functions $\hat{\beta}_\alpha$, (α=1,2), depend only on the invariants I_1 and I_2. They are given now by:

$$\hat{\beta}_1 = 2\frac{\partial \hat{W}}{\partial I_1}, \quad \hat{\beta}_2 = -2\frac{\partial \hat{W}}{\partial I_2} \tag{15}$$

and the integrability conditions (7) for the response functions $\hat{\beta}_\alpha$, (α=1,2) reduce to

$$\frac{\partial \hat{\beta}_1}{\partial I_2} = -\frac{\partial \hat{\beta}_2}{\partial I_1} \tag{16}$$

The Mooney-Rivlin model corresponds to the assumption $\hat{\beta}_1 = a_1$ and $\hat{\beta}_2 = -a_2$, where a_1 and a_2 are given constants independent of I_1 and I_2, $a_1>0$ and $a_2>0$ [10]. By redefining the hydrostatic pressure to be $p = \hat{p}+\hat{\beta}_1+\hat{\beta}_2$, equation (14) takes the form :

$$\sigma = -\Big(p+(a_1-a_2)\Big)\mathbf{I} + \hat{\sigma} \tag{17.a}$$

where the extra stress $\hat{\sigma}$ is given by

$$\hat{\sigma} = a_1\mathbf{B} - a_2\mathbf{B}^{-1} \tag{17.b}$$

The strain energy potential, for this model is

$$\hat{W} = \tfrac{1}{2}\Big(a_1(I_1-3) + a_2(I_2-3) \Big) \tag{18}$$

3. Penalty formulation for nonlinear elastostatics.

In elastostatics, the variational formulation of the incompressibility constraint by a penalty function procedure involves the extension of the constitutive model for the incompressible material, to the compressible range.

In the linearized theory, this extension is immediate. The constitutive equation $\sigma = -p\mathbf{I} + 2\mu\mathbf{e}$, where $\mathbf{e}$ is the deviatoric part of the infinitesimal strain tensor ϵ, is extended to the compressible range by considering instead $\sigma = K div(\mathbf{u}) + 2\mu\mathbf{e}$. This is the same as adopting for the strain energy $\tilde{W}$ in the compressible range a form $\tilde{W} = \hat{W} + \frac{1}{2}K\{div(\mathbf{u})\}^2$, where $\hat{W} = \mu\, tr(\mathbf{e}.\mathbf{e})$ is the distortional part. A penalty formulation makes use of this compressible model, with the bulk modulus K as a penalty parameter [11].

In the non linear theory however, the situation is quite different. A first formulation [3]-[6] considers a strain energy potential of the form

$$\tilde{W}(I_1,I_2,I_3) = \hat{W}(I_1,I_2) + U(I_3) \tag{19}$$

in the compressible range. $\hat{W}(I_1,I_2)$ is the strain energy of the given incompressible material, and the term $U(I_3)$ has the structure of a penalty function enforcing the incompressibility constraint. A suitable form for this term will be discussed in 3.1.

The type of strain energy functional given by (19), is not the only possible one leading to a penalty formulation. Alternative expressions to (19), more convenient for computational purposes, will be considered in 3.2.

In the sequel, the following convention will be adopted: variables with a superimposed "^" will always be associated with the incompressible model, while those with a superimposed "~" will be associated with the constitutive model extended to the compressible range.

3.1. General formulation

Let us assume a relationship between the strain energies $\hat{W}(I_1,I_2)$ and $\tilde{W}(I_1,I_2,I_3)$ of the form (19). The associated response functions $\hat{\beta}_\alpha(I_1,I_2)$ and $\tilde{\beta}_\alpha(I_1,I_2,I_3)$ are related by

$$\tilde{\beta}_1 = \frac{\hat{\beta}_1}{J} \;,\; \tilde{\beta}_2 = \hat{\beta}_2 J \tag{20}$$

which follow from the definitions (6) and (15), and equation (19).

Since $\frac{2}{J} I_3 \frac{\partial}{\partial I_3} = \frac{\partial}{\partial J}$, it is convenient for simplicity to regard U in (19) as a function of J rather than of I_3. A suitable form for the term $U(I_3)$ in (19), is provided by

$$U(I_3) = \tfrac{1}{2}\lambda\{G(J)\}^2 + cH(J) \tag{21}$$

where G: $(0,\infty) \rightarrow R$ is a penalty function with the property that $G(J) = 0$ if and only if $J = 1$ and $\lambda > 0$ is the penalty parameter. The constant c and the function $H(J)$ with the property $H(J) = 0$ and $H'(J) = 1$ iff $J = 1$, guarantee that the reference configuration $\Phi = \mathit{Identity}$ is stress free. The usual way in which this condition is enforced [6],[7], amounts to taking $H(J) = J$.

From equation (21), and the definition of $\tilde{\beta}_o(I_1,I_2,I_3)$ given by (6), it follows that

$$\tilde{\beta}_o = \lambda G(J)\frac{dG(J)}{dJ} + \left\{c\frac{dH(J)}{dJ} - \frac{I_2}{J}\hat{\beta}_2\right\} \tag{22}$$

The response functions $\tilde{\beta}_i, (i=0,1,2)$ defined by equations (20) and (22) satisfy the compatibility equations (7), and completely characterize the constitutive equation (5) for the Cauchy stress tensor. The constant c is given by

$$c = \{2\hat{\beta}_2 - \hat{\beta}_1\}|_{\Phi=\mathit{Identity}} \tag{23}$$

for the reference configuration to be stress free. The corresponding expression for the first Piola-Kirchhoff tensor follows from (4), and the elasticity tensor **A** can then be computed using (8).

In order to discuss appropriate forms for the functions $G(J)$ and $H(J)$ in equation (21), it is convenient to consider separately the effect of the penalty term $\lambda\{G(J)\}^2$. Thus, we rewrite equation (19) in the form:

$$\tilde{W}(I_1,I_2,I_3) = \tfrac{1}{2}\lambda\{G(J)\}^2 + W^* \tag{24.a}$$
$$W^* = cH(J) + \hat{W} \tag{24.b}$$

3.1.1. The choice of the penalty term

In the context of any numerical procedure, like the finite element method, the condition

$\lambda \rightarrow \infty$ can only be enforced in a relative sense.

It is therefore appropriate to view equation (24) in physical terms, as the strain energy functional of a compressible material, which exhibits incompressible behavior as the parameter $\lambda \rightarrow \infty$. Accordingly, equation (24) should remain physically meaningful, for the widest possible range of values of λ.

The behavior of the strain energy given by (24), can be easily examined as λ varies in $(0,\infty)$, by considering the simple problem of the extension of a cube with unit length sides. We shall take the stretching ratio of its side, as the control variable. Thus, any possible deformation $\Phi \in C$ has a deformation gradient

$$\mathbf{F} = \gamma \mathbf{I} \tag{25}$$

with determinant $J = \gamma^3$. The associated first Piola-Kirchhoff tensor (The force required to perform the extension) is a hydrostatic pressure †

$$\mathbf{P} = p(\gamma)\mathbf{I} \tag{26}$$

For a given value of $\lambda \in (0,\infty)$, the following conditions are expected to hold:

c.1) The strain energy $\tilde{W} \rightarrow \infty$, as the stretching ratio $\gamma \rightarrow 0$ or $\gamma \rightarrow \infty$.

c.2) $p(\gamma)$ is a monotone function in $(0,\infty)$ with $p(1) = 0$, $p \rightarrow -\infty$ as $\gamma \rightarrow 0$, and $p \rightarrow +\infty$ as $\gamma \rightarrow +\infty$.

We shall consider in the sequel, the special case of the Mooney-Rivlin model. For this type of material, the strain energy and first Piola-Kirchhoff stresses for the extension problem are

$$\begin{aligned}
\tilde{W} &= \tfrac{1}{2}\lambda\{G(\gamma^3)\}^2 + W^* \\
W^* &= -(2a_2+a_1)H(\gamma^3) + \frac{3}{2}\left(a_1(\gamma^2-1)+a_2(\gamma^4-1)\right) \\
\mathbf{P} &= \left\{2a_2\left(\gamma^3-\gamma^2H'(\gamma^3)\right) + a_1\left(\gamma-\gamma^2H'(\gamma^3)\right)\right\}\mathbf{I}
\end{aligned} \tag{27}$$

† Taking the stresses as control variables, leads to the classical example which shows lack of uniqueness in nonlinear elasticity. Seven solutions for a simple Neo-Hookean material are possible [9].

(a) For $\lambda = 0$ we have the limiting case of no penalty term and $\tilde{W} = W^*$. The choice:

$$H(J) = \ln(J) \tag{28}$$

produces the strain energy $\tilde{W}$ and first Piola-Kirchhoff stresses $\mathbf{P}$ plotted in fig.1 and fig.2 respectively. (Curves λ=0). Both $\tilde{W}$ and $\mathbf{P}$ satisfying conditions c.1) and c.2).

(b) For $\lambda > 0$, the response is affected by the penalty term. Two common forms of penalty function are $G(J) = (J-1)$ [3]-[6], and $G(J) = \ln(J)$ [7]. The first one, together with $H(J) = J$ [6], produces the family of curves plotted in fig.3. Notice that an instability appears around J=.5, and that $\mathbf{P}$ vanish when γ=0. A similar instability phenomenon is expected to happen for $G(J)=\ln(J)$, although around the very large value of $J=10^3$. However, the combined function

$$\{G(J)\}^2 = (\ln J)^2 + (J-1)^2 \tag{29}$$

is a suitable form of penalty function, and produces the family of curves plotted in fig.1 and fig.2 for different values of λ. For each $\lambda \in (0,\infty)$, $\tilde{W}$ and $\mathbf{P}$ satisfy conditions c.1) and c.2).

The functions $H(J)$ and $G(J)$ given by (28) and (29) do not introduce any special computational effort. In fact, the corresponding response function $\tilde{\beta}_o$ is

$$\tilde{\beta}_o = \frac{1}{J}\{\lambda(J(J-1)+\ln J)+c-I_2\hat{\beta}_2\} \tag{30}$$

3.2. Alternative formulation

Instead of considering a relationship of the type given by (19) between strain energy potentials, attention will now be focussed in the constitutive equations (5) and (14) for the Cauchy stresses, in the compressible and incompressible cases respectively.

We shall consider first, the case of the Mooney-Rivlin model. Extensions to more general incompressible models will be examined later.

Let g: $(0,\infty) \rightarrow R$ be a smooth real valued function, such that $g(x) = 0$ if and only if $x = 1$.

Consider first the replacement of the term $-p\mathbf{I}$ in equation (17.a) by $\lambda g(J)$, where $\lambda > 0$ is a real constant. Now let $\lambda \to \infty$; i.e consider a sequence $\{\lambda_n\}$ such that $\lim_{n\to\infty} \lambda_n = \infty$. Since $p\colon B \to R$ is a bounded function (say in the uniform norm) and $g(J) = \frac{p}{\lambda}$, it follows that if we let $\lambda \to \infty$ then $g(J) \to 0$. By the assumptions on $g(J)$, $J \to 1$ as $\lambda \to \infty$.

The described replacement of hydrostatic preassure $-p\mathbf{I}$, shows that the incompressible case can be viewed as the limit of a sequence of compressible cases. However, in contrast with the linearized theory, this formulation requires the modification of the term $\hat{\sigma}$ in (17.a). In fact, if $\hat{\sigma}$ remains unchanged one is led to the constitutive model

$$\sigma = \{g(J)-(a_1-a_2)\}\mathbf{I} + a_1\mathbf{B} - a_2\mathbf{B}^{-1} \tag{31}$$

and substitution into the general integrability conditions (7) yields:

$$\tfrac{1}{2}a_1 = \frac{\partial\tilde{\beta}_o}{\partial I_1} = 0,\ \tfrac{1}{2}a_2 = -\frac{\partial\tilde{\beta}_o}{\partial I_2} = 0$$

Thus $\hat{\sigma}=0$, and we arrive to the remarkable conclusion that the only possible hyperelastic compressible model, compatible with the assumption of constants response functions $\tilde{\beta}_\alpha$, (α=1,2) is the hydrostatic pressure $\sigma = g(J)\mathbf{I}$.

Since the response functions $\tilde{\beta}_1$ and $\tilde{\beta}_2$ can not be a pair of constants in the compressible range, although they must reduce to the Mooney-Rivlin constants a_1 and a_2 whenever $J = 1$, we are led to the assumption:

$$\beta_1 = \tilde{\beta}_1(I_3),\ \ \beta_2 = \tilde{\beta}_2(I_3) \tag{32}$$

Substitution into the integrability conditions (7) yields a pair of ordinary differential equations which have the simple solution

$$\tilde{\beta}_1 = \frac{a_1}{J},\ \tilde{\beta}_2 = \frac{a_2}{J} \tag{33}$$

Therefore, a consistent extrapolation of the Mooney-Rivlin model to the compressible range is given by

$$\sigma = \{\lambda g(J)-(a_1-a_2)h(J)\}\mathbf{I} + \frac{a_1}{J}\mathbf{B} - \frac{a_2}{J}\mathbf{B}^{-1} \tag{34}$$

where for more generality, the function $h\colon (0,\infty) \to R$ with the property $h(x)=1$ iff $x=1$, has been introduced.

If we take $g(J)=\frac{1}{2}\frac{d}{dJ}\left[G(J)\right]^2$ and $h(J)=\frac{d}{dJ}H(J)$, where the functions $G(J)$ and $H(J)$ have the same meaning as in the previous section, equation (34) corresponds to a strain energy potential of the form:

$$\tilde{W}(I_1,I_2,I_3) = U(I_3) + \tfrac{1}{2}\left[a_1(I_1-3) + a_2(I_2I_3^{-1}-3)\right] \tag{35}$$

where $U(I_3)$ is again given by (21), with the constant $c = a_1-a_2$.

This strain energy potential has a structure different from that considered in equation (19) and discussed earlier. In contrast with the previous formulation, the resulting first Piola-Kirchhoff tensor does not contain the invariant I_2, and I_3 only appears in $\tilde{\beta}_o$. Thus, it leads to a simpler expression of the elasticity tensor **A**, more convenient for computational purposes.

The form of strain energy given by (35), also leads to a penalty formulation of the incompressibility constraint. This point is examined next.

3.2.1. Penalty formulation for the Mooney-Rivlin model.

Let us rewrite equation (35) in a form analogous to (24.a), with W^* given now by

$$W^* = (a_2-a_1)H(J) + \tfrac{1}{2}a_1(I_1-3) + \tfrac{1}{2}a_2(I_2I_3^{-1}-3) \tag{36}$$

The total potential energy functional corresponding to the strain energy W^* is, for any configuration $\Phi \in C$, given by

$$\Pi^*(\Phi) = \int_B W^* dV - \int_B \bar{\mathbf{b}}.\mathbf{x}dV - \int_{B_t} \bar{\mathbf{t}}.\mathbf{x}dS \tag{37}$$

where $\mathbf{x}$ is the vector from the origin to $x = \Phi(X)$, and $\bar{\mathbf{b}}$ the body forces per unit of mass in the reference configuration .

With this notation, the variational formulation of the boundary value problem for an elastic material with strain energy given by (36), is then

Find $\Phi \in C$ *such that*

$$\Pi(\Phi) = \min_{\Lambda \in C}\left\{\Pi^*(\Lambda) + \tfrac{1}{2}\lambda\int_B \left[G(J)\right]^2 dV\right\} \tag{38}$$

For isochoric deformations $\Phi \in C_{inc}$ $I_3=1$ and (36) reduces to the strain energy for the Mooney-Rivlin model. By the assumptions on $G(J)$, the condition $\Phi \in C_{inc}$ is equivalent to $G(J(\Phi)) = 0$. Therefore, the variational formulation of the boundary value problem for the Mooney-Rivlin model is given by

find $\Phi \in C_{inc}$ *such that*

$$\Pi^*(\Phi) = \min_{\Lambda \in C} \left\{ \Pi^*(\Lambda) \right\} \tag{39}$$

Subjected to: $G(J(\Lambda)) = 1$

If we let $\lambda \rightarrow \infty$, equation (38) is just the penalty formulation of the constrained problem given by (39). [1]-[3]

3.2.2. Extension to more general constitutive models.

Let us examine for what class of incompressible materials, a simple relationship between response functions of the form:

$$\tilde{\beta}_\alpha(I_1,I_2,I_3) = \hat{\beta}_\alpha(I_1,I_2)\phi_\alpha(J), \quad (\alpha=1,2) \tag{40}$$

and

$$\tilde{\beta}_o(I_1,I_2,I_3) = \hat{\beta}(J) \tag{41}$$

analogous to that found for the Mooney-Rivlin model, is in general possible.

Substitution of (40) and (41) into the integrability conditions (7) and the use of condition (12) yields, after some manipulation the system of equations:

$$\begin{aligned}
&\tfrac{1}{2}\frac{d}{dJ}\left(J\phi_1(J)\right)\hat{\beta}_1 = -\phi_1(J)I_2\frac{\partial\hat{\beta}_1}{\partial I_2}\\
&\tfrac{1}{2}\frac{d}{dJ}\left(J\phi_2(J)\right)\hat{\beta}_2 = -\phi_2(J)I_2\frac{\partial\tilde{\beta}_2}{\partial I_2}\\
&J^2\phi(J)\frac{\partial\hat{\beta}_1}{\partial I_2} + \phi(J)\frac{\partial\hat{\beta}_2}{\partial I_1} = 0
\end{aligned} \tag{42}$$

The solution of the system of equations (42) together with equation (12) is given by

$$\phi_1(J) = \frac{1}{J}, \quad \phi_2(J) = \frac{1}{J^{2\alpha-1}} \tag{43}$$

which corresponds to a strain energy potential of the form:

$$\hat{W}(I_1,I_2) = \hat{f}(I_1) + AI_2^{\alpha} + B \tag{44}$$

where A and B are constants.

For this class of constitutive model, exactly the same formulation as that used for the Mooney-Rivlin model is applicable.

Equation (44) includes some forms of strain energy commonly used in rubber elasticity, like

$$\hat{W} = \hat{f}(I_1-3) + (I_2-3) \tag{45}$$

4. Final remarks.

The penalty formulation of the incompressibility constraint in finite elasticity, amounts to considering an extended constitutive model which, although no longer incompressible, exhibits incompressible behavior in the limit as the penalty parameter tends to infinity.

It has been shown in this report that consideration of the physical significance of this extended model leads to a proper choice of the penalty term.

Finally, for a certain class of materials a simpler extension to the compressible range was proposed as an alternative to the one commonly used [3]-[7]. This class of materials includes some frequently used in rubber elasticity, in particular the important Mooney-Rivlin model.

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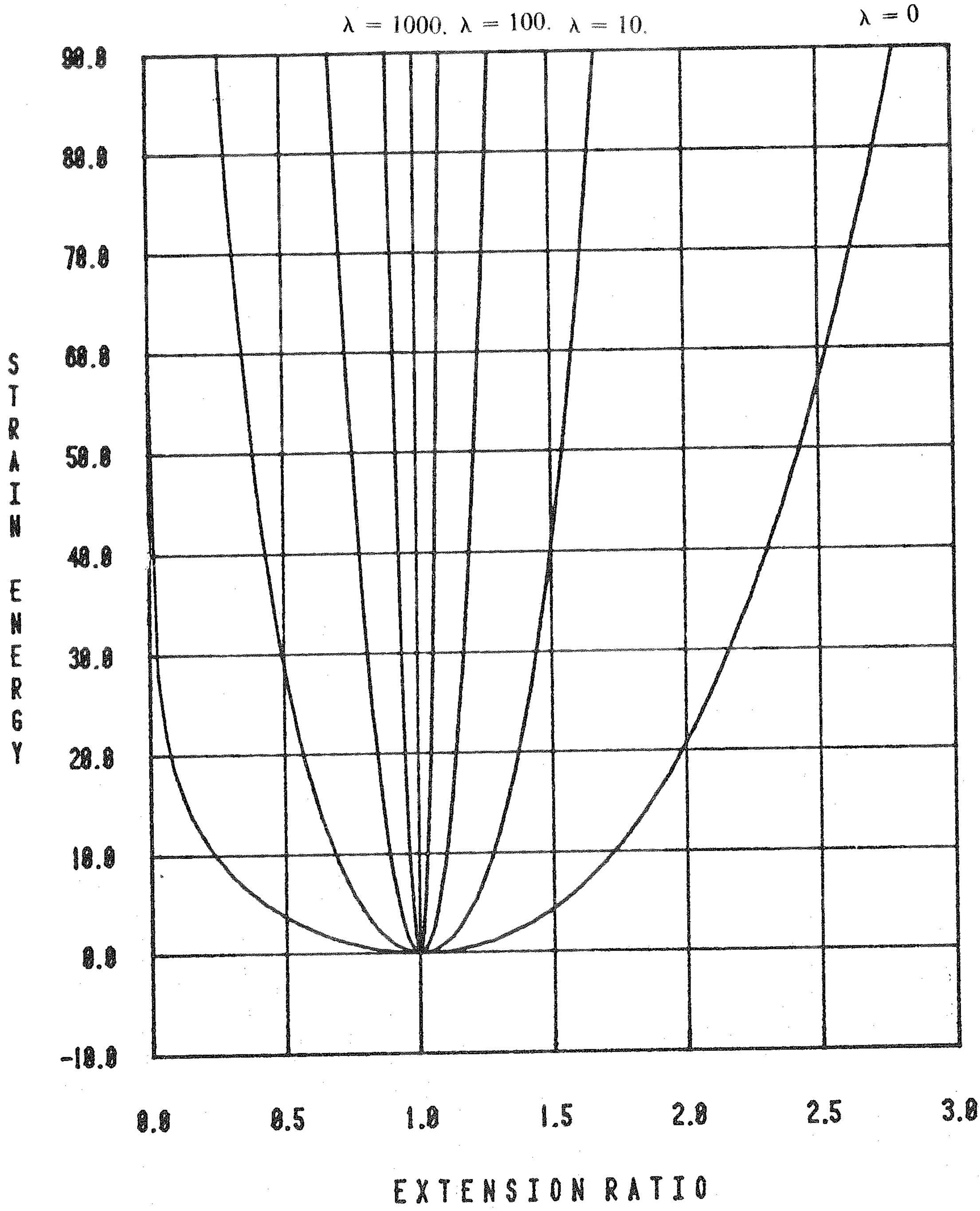

Fig. 1. Strain energy in isotropic extension. Proposed model

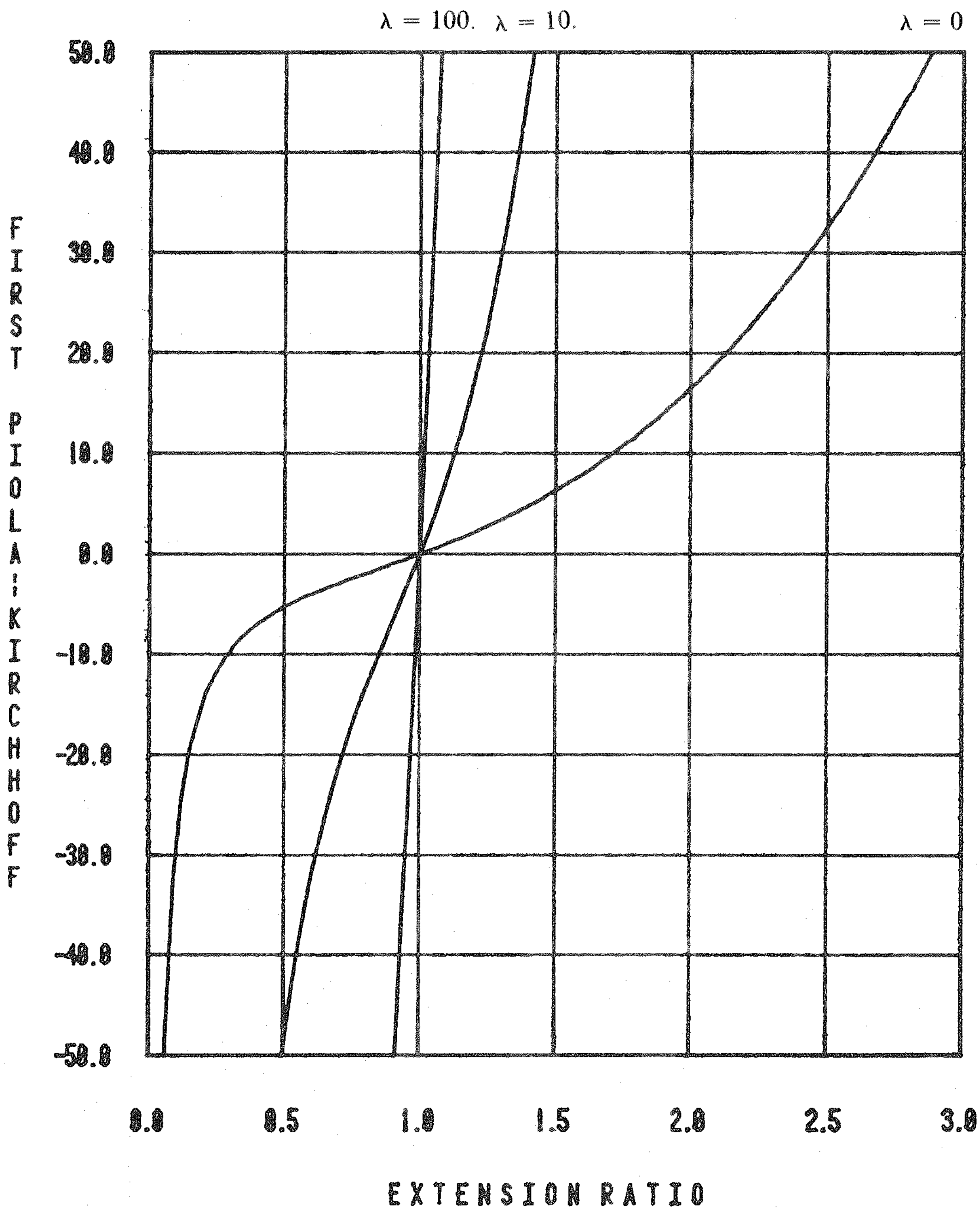

Fig. 2. Piola-Kirchhoff hydrostatic pressure. Proposed model

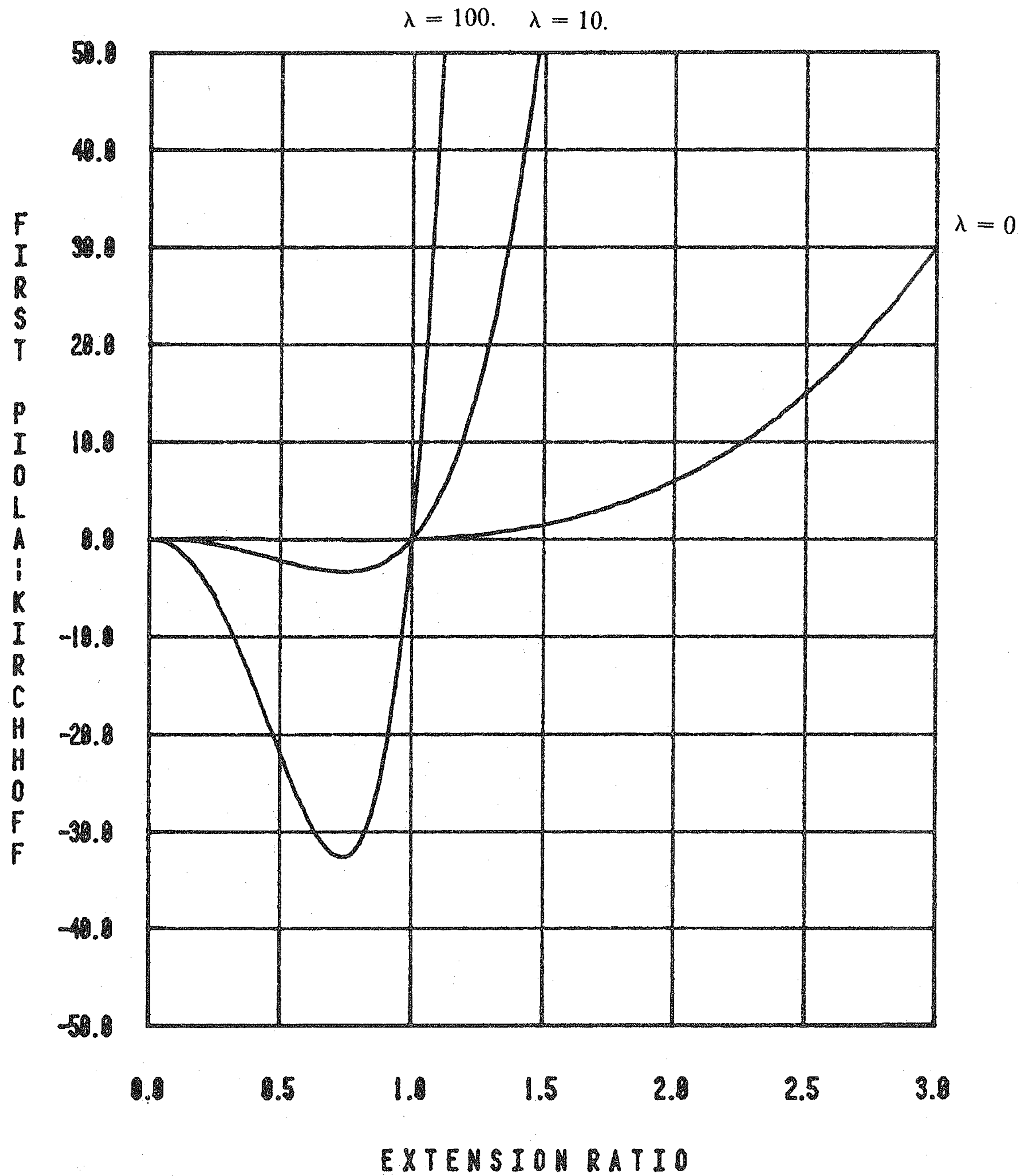

Fig. 3. Piola-Kirchhoff hydrostatic pressure. Other models.

$$\mathbf{K}\mathbf{U} = \mathbf{R} \tag{A.7}$$

from which the incremental deformation is computed.

The iterative solution procedure outlined in the previous section, amounts to solving at each intermediate configuration, say $\Phi^{(n)}$, the linear system of equations (A.7) for the incremental displacements $\mathbf{U}^{(n)}$. The norm $||\mathbf{R}^{(n)}||$ of the residual is then compared with a pre-established tolerance to assess convergence. This procedure is equivalent to the Newton-Ralphson method.[A.1],[A.3].

a.3. Numerical implementation.

The structure of the tangent stiffness matrix and the residual force vector, for a typical element Ω_e with n_e nodes, is the following †

$$\mathbf{K} = \begin{bmatrix} \mathbf{k}^{11} & \cdots & \mathbf{k}^{1n_e} \\ \cdot & \cdots & \cdot \\ \mathbf{k}^{n_e1} & \cdots & \mathbf{k}^{n_en_e} \end{bmatrix} \tag{A.8}$$

and

$$\mathbf{R} = \begin{bmatrix} \mathbf{r}^1 \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{r}^{n_e} \end{bmatrix} \qquad \mathbf{U} = \begin{bmatrix} \mathbf{u}^1 \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{u}^{n_e} \end{bmatrix} \tag{A.9}$$

The dimensions of any submatrix $\mathbf{k}^{\alpha\beta}$,$(\alpha,\beta=1,n_e)$ is (dxd), and (dx1) that of the vectors $\mathbf{r}^\alpha$, and $\mathbf{u}^\alpha$, d being the spatial dimension of the problem. In addition, the following convention will be used: the element (i,j) of the matrix $\mathbf{k}^{\alpha\beta}$ will be denoted by $K_{ij}^{\alpha\beta}$, and by R_i^α the i component of the vector $\mathbf{r}^\alpha$.

With this notation, the tangent stiffness and the residual force vector for a typical element are given by

$$K_{ij}^{\alpha\beta} = \int_{\Omega_e} A_{iIjJ} N,_I^\alpha N,_J^\beta dV \tag{A.10}$$

† The superindex e will be dropped in what follows.

$$R_i^\alpha = \int_{\Omega_e} \bar{b}_i N^\alpha dV + \int_{\partial\Omega_{e_t}} \bar{t}_i N^\alpha dV - \int_{\Omega_e} P_{Ii} N,_I^\alpha dV \tag{A.11}$$

where a "," denotes partial differentiation with respect to material coordinates.

Equations (A.10) and (A.11) show that the elasticity tensor **A** and the first Piola-Kirchhoff stress tensor **P**, is all that is required to carry out the analysis.

let us consider the penalty formulation of the Mooney-Rivlin model discussed in section 3.2. Introducing the notation

$$\beta(J) = \lambda g(J) - (a_1 - a_2) h(J) \tag{A.12}$$

the Cauchy stress tensor given by (34) takes the form

$$\sigma = \beta(J)\mathbf{I} + \frac{a_1}{J}\mathbf{B} - \frac{a_2}{J}\mathbf{B}^{-1} \tag{A.13}$$

and from equations (3) and (8) the components of **P** and **A** are given by

$$P_{iI} = \{J\beta(J)\} F_{Ii}^{-1} + a_1 F_{iI} - a_2 B_{ia}^{-1} F_{Ia}^{-1} \tag{A.14}$$

and

$$A_{iIjJ} = J\frac{d}{dJ}\{J\beta(J)\} F_{Ii}^{-1} F_{Jj}^{-1} - \{J\beta(J)\} F_{Ji}^{-1} F_{Ij}^{-1} + a_1 \delta_{ij} \delta_{IJ}$$
$$+ a_2 \{ C_{IJ}^{-1} B_{ij}^{-1} + F_{Ij}^{-1} C_{JA}^{-1} F_{iA}^{-1} + F_{Ji}^{-1} C_{IA}^{-1} F_{Ai}^{-1} \} \tag{A.15}$$

Equations (A.14) and (A.15) completely define the tangent stiffness and the residual for an arbitrary element. The assembly and solution of the global system of equations (A.7) can be carried out with any standard finite element program.

The integrations involved in the computation of the stiffness matrix and the residual, are performed numerically using Gauss quadrature formulae. In order to avoid the well known "locking" phenomenon [A.1],[A.4] that appears for high values of the penalty parameter λ, use of reduce integration techniques must be made [A.1].

In the case of plane strain, the Mooney-Rivlin model reduces to a Neo-Hookean material and equations (A.13) to (A.15) take a particularly simple form. The constant a_1 is replaced by $\mu = \frac{1}{2}(a_1 + a_2)$, and the terms containing a_2 can be eliminated. This case has been implemented in the FORTRAN subroutine listed at the end of this report, designed for the general purpose computer program FEAP listed in chap.24 of reference [A.1].

a.4. Numerical examples.

The performance of the element is illustrated in the following two examples for which exact solutions are available. The results obtained for the problem of a cube subjected to an homogeneous extension are plotted in Fig.A.1, and those corresponding to a case of pure shear in Fig.A.2. In both examples a 4-node isoparametric element was the the type of element used. A 1x1 Gauss quadrature for the volumetric part of the stiffness and the residual, the terms containing the penalty parameter λ, and a 2x2 Gauss quadrature for the deformational part, was the reduced integration procedure adopted. The agreement between the finite element solution, and the exact solution is evident.

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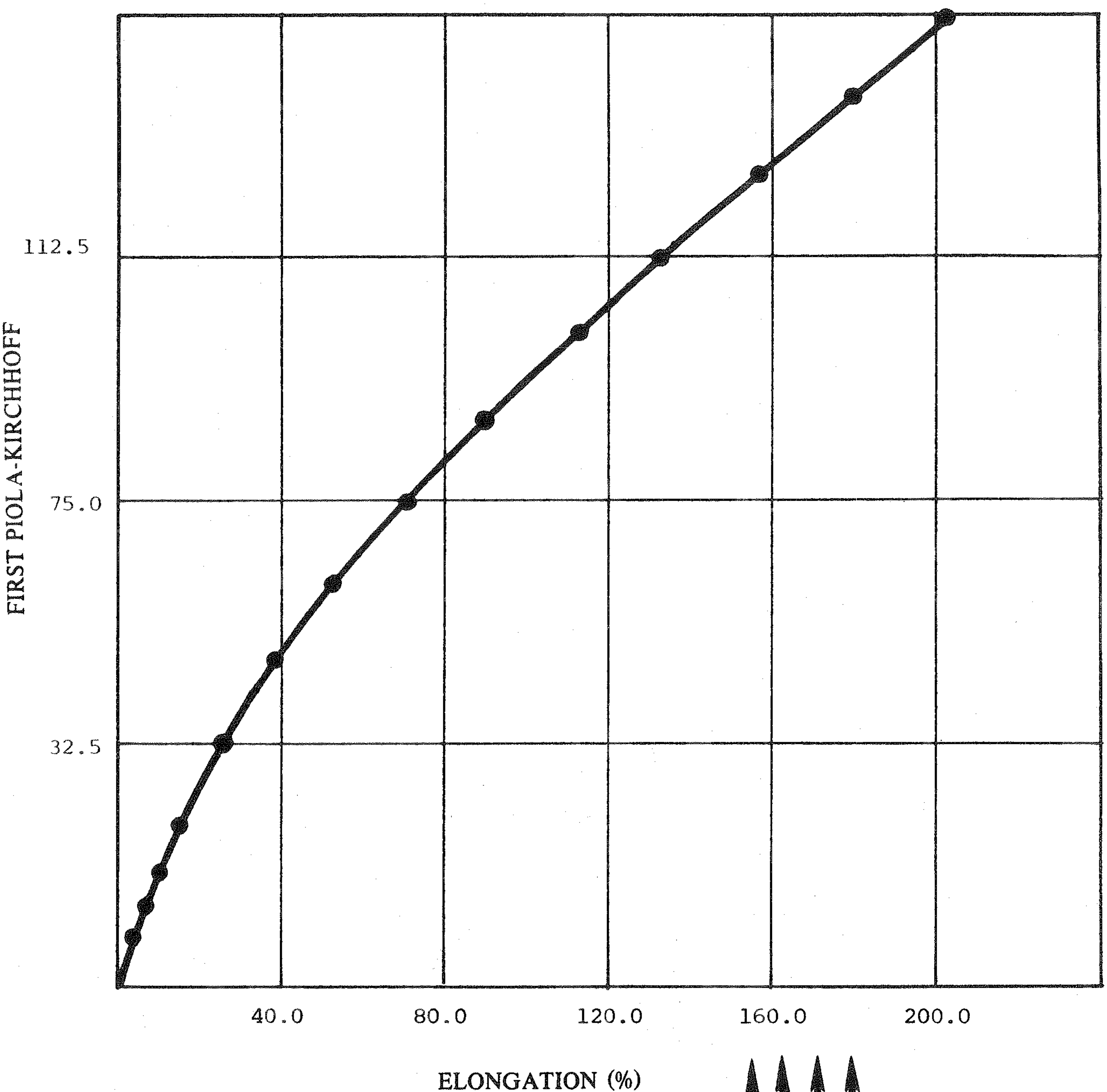


$\lambda = 10000.$

$\mu = 50.$

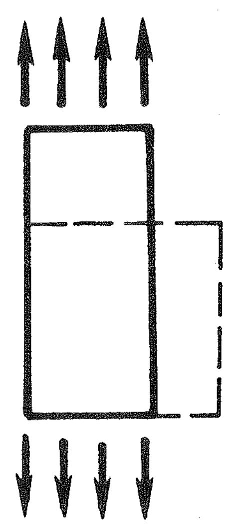
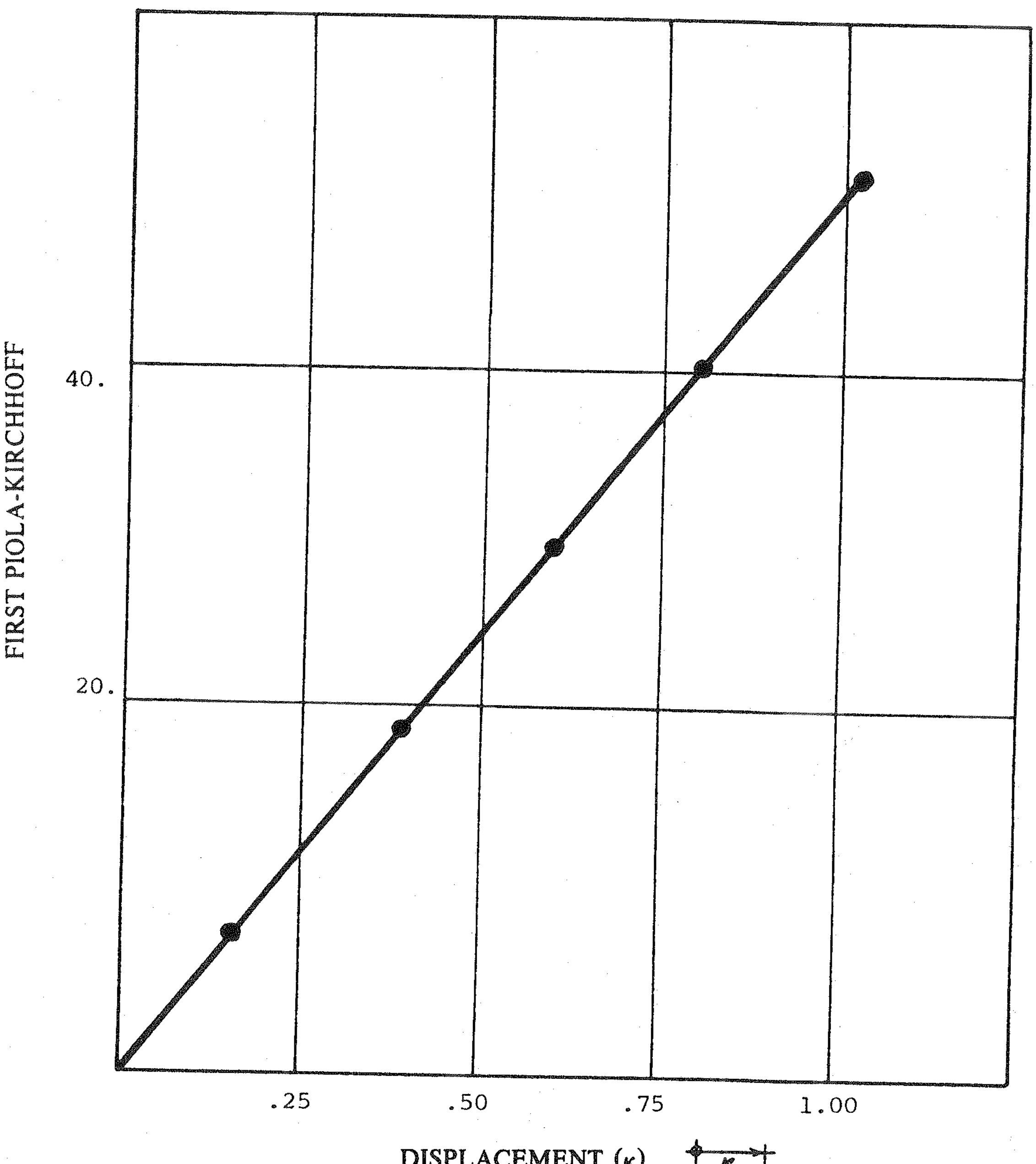
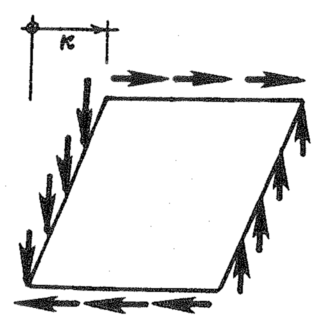


Fig. A.1. Homogeneous Extension of a cube. Plane strain.





$\lambda = 10000.$

$\mu = 50.$

Fig. A.2. Simple shear of a cube. Plane strain.

APPENDIX B.

Mooney-Rivlin plane strain element for FEAP.

b.1. Input of material properties.

The first card of the material data set is as described in reference [A.1] pp. 694. The element type number to be input in column 10, is 15. The second card containing the material property data is prepared according to

column	description
1 to 10	λ penalty parameter
11 to 20	μ generalized shear modulus
21 to 30	blank
31 to 35	k1 # integration points for bulk part
35 to 40	k2 # integration points for shear part

b.2. List of the element

```
330    l = l + ndf
       nn = 0
       do 360 jj = 1,ii
       do 340 i = 1,ndim
340    st(i) = shp(i,jj)
       l = 0
       do 355 i = 1,ndim
       do 350 j = 1,ndim
       do 350 k = 1,ndim
350    estif(i+nn,j+mm) = estif(i+nn,j+mm) + st(k)*b(k+l,j)
355    l = l + ndim
360    nn = nn + ndf
370    mm = mm + ndf
380    continue
c..... construct symmetric part
       do 390 i = 1,nstf
       do 390 j = i,nstf
390    estif(j,i) = estif(i,j)
       return
c
c..... compute the stress/strains and r.h.s.
     4 ki=d(6)
       if(isw.eq.4) ki=1
       do 495 kk=1,ki
       xlam=d(kk  )
       xnu =d(kk+2)
       iqs = d(kk+6)
       if(iqs**ndim.ne.lint) call pgauss(iqs,lint,sg,tg,wg)
       if(isw.eq.4) xnu = d(4)
       do 490 ll = 1,lint
       call shape(sg(ll),tg(ll),xl,shp,xjac,ndim,nel,ix,.false.)
       call con14(jcneg,ul,shp,xjac,ndf,ndim,nel,n,1)
       if(jcneg) return
       if(isw.eq.6) go to 420
       do 405 i = 1,ndim
       xx(i) = 0.
       do 405 j = 1,nel
405    xx(i) = xx(i) + xl(i,j)*shp(3,j)
       mct = mct - 1
       if(mct.gt.0) go to 410
       write(itp6,2001) o,head,time
       write(itp6,2002) ((shed(i,j),rst,j=1,ndim),i=1,ndim)
       mct = 10
410    write(itp6,2003) dm,n,ma,(xx(i),i=1,ndim)
       write(itp6,2004) pt,((t(i,j),j=1,ndim),i=1,ndim)
       write(itp6,2004) pp,((p(i,j),j=1,ndim),i=1,ndim)
       write(itp6,2004) ps,((s(i,j),j=1,ndim),i=1,ndim)
       write(itp6,2004) pe,((e(i,j),j=1,ndim),i=1,ndim)
       write(itp6,2004) pc,((c(i,j),j=1,ndim),i=1,ndim)
       write(itp6,2004) pf,((f(i,j),j=1,ndim),i=1,ndim)
       write(itp6,2005) detf
       go to 490
c..... compute forces
420    dv = wg(ll)*xjac
       mm = 0
       do 460 k = 1,nel
```

```
      do 430 i = 1,ndim
430   st(i) = shp(i,k)*dv
      do 450 i = 1,ndim
      temf = 0.0
      do 440 j = 1,ndim
440   temf = temf + p(j,i)*st(j)
450   force(i+mm) = force(i+mm) - temf
460   mm = mm + ndf
490   continue
495   continue
5     return
7     call plot9(ix,xl,ndim,nel)
      return
8     return
1000  format(3f10.0,2i5)
2000  format(/5x,40hfinite deformation 2d/3d elastic element//10x,
     1 9hlamda    =e15.5/10x,9hmu       =e15.5/
     2 10x,9hdensity =e15.5/
     3 10x,9hbulk pt = ,i5,/10x,10hshear pt =  ,i4,/1x )
2001  format(a1,20a4/5x,'time',e13.5/5x,16helement stresses//5x,
     1 17helement  material,'    1 coord.    2 coord.')
2002  format(13x,6htensor,3x,18a6)
2003  format(/2x,a5,i5,i7,3x,3f12.4)
2004  format(16x,a6,1p9e12.3)
2005  format(16x,6hdet(f) 1pe12.3)
      end
      subroutine con14(jcneg,ul,shp,xjac,ndf,nd,ne,nn,ip)
      implicit double precision (a-h,o-z)
      dimension ul(ndf,1),shp(3,1),fi(2,2),b(2,2)
      logical jcneg
      common /const1/xlam,xnu,f(2,2),e(2,2),c(2,2),s(2,2),p(2,2),t(2,2),
     1  a(4,4),detf
c
c.... compute deformation gradient f
      do 8 i = 1,3
      do 8 j = 1,3
 8    f(i,j) = 0.
      do 20 i = 1,nd
      f(i,i) = 1.
      do 20 j = 1,nd
      do 20 k = 1,ne
 20   f(i,j) = f(i,j) + ul(i,k)*shp(j,k)
c.... compute determinant of f
      tt = f(1,1)*f(2,2) - f(1,2)*f(2,1)
      if(tt.gt.0.0d0) go to 500
      jcneg = .true.
      write(6,2000) nn,tt
      return
 500  tl = dlog(tt)
c
c.... compute f inverse
      fi(1,1) = f(2,2)/tt
      fi(2,2) = f(1,1)/tt
      fi(1,2) = -f(1,2)/tt
      fi(2,1) = -f(2,1)/tt
c
```

```
c.... compute b,c,e
      do 40 i = 1,nd
      do 40 j = 1,nd
      b(i,j) = 0.
      c(i,j) = 0.
      do 35 k = 1,nd
      b(i,j) = b(i,j) + f(i,k)*f(j,k)
 35   c(i,j) = c(i,j) + f(k,i)*f(k,j)
 40   e(i,j) = .5*c(i,j)
      do 45 i = 1,nd
 45   e(i,i) = e(i,i) - .5
      if(ip) 600,600,700
c
c.... compute tangent tensor a
 600  c0 = xnu - xlam*(tl+tt*(tt-1.))
      c1 = xlam*(1.+(2.*tt-1.)*tt)
      m = 0
      do 80 i = 1,nd
      do 80 j = 1,nd
      m = m + 1
      n = 0
      do 80 k = 1,nd
      do 80 l = 1,nd
      n = n + 1
      tf = c1*fi(j,i)*fi(l,k) + c0*fi(j,k)*fi(l,i)
 80   a(m,n) = tf
      im = nd*nd
      do 100 i = 1,im
 100  a(i,i) = a(i,i) + xnu
      return
c
c.... compute p and t
 700  c0 = xlam*(tl+tt*(tt-1.))
      c1 = c0/tt
      c2 = xnu/tt
      do 200 i = 1,nd
      do 200 j = 1,nd
      p(i,j) = c0*fi(i,j) + xnu*(f(j,i) - fi(i,j))
 200  t(i,j) = c2*b(i,j)
      do 210 i = 1,nd
 210  t(i,i) = t(i,i) + c1 - c2
c.... compute s
 300  do 340 i = 1,nd
      do 340 j = 1,nd
      s(i,j) = 0.
      do 340 k = 1,nd
 340  s(i,j) = s(i,j) + p(i,k)*fi(j,k)
      detf=tt
      return
 2000 format(5x,39h**local volume in nonlinear element no. ,i5,
     1 16h  is negative = ,e14.7)
      end
```