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Dynamics of Electricity Markets with Unknown Utility Functions: An Extremum Seeking Control Approach*

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Abstract— **This paper studies the dynamics of electricity markets with unknown utility functions (i.e., the profit functions of consumers and the cost functions of suppliers). The market clearing procedure is formulated as a social welfare optimization problem, and the dynamics of electricity markets is modeled as primal-dual dynamics based on gradient estimation. The gradients of the unknown utility functions are estimated by adding and multiplying periodic signals at the measurable input and output, respectively. We prove the semi-globally practically asymptotically (SPA) stability of the market dynamics. Numerical results demonstrate the SPA stability and the balance between supply and demand.**

I. INTRODUCTION

Matching supply with demand has been an active topic in operating electricity markets. Traditionally, we need to achieve the balance between supply and demand, which requires substantial infrastructure to be idle for all but a few hours a year. Recently, demand response is proposed to control the load of consumers and provide more flexible balance strategies. In that case, the market operator can schedule the supply and the demand simultaneously. With the development of smart grid, which enables reliable and real-time communications between the suppliers and the consumers, the supply and the demand can be scheduled within much shorter intervals. The communications between the market operator and the consumers are dependent on an advanced metering infrastructure (AMI) [1], which supports to collect the electricity consumption and publish the electricity price, such as time of use (TOU), critical peak pricing (CPP), and real time pricing (RTP) [2].

Convex optimization was utilized to model the market clearing procedure in smart grid. The objective of the market operator is to maximize the social welfare, which is defined by subtracting the total cost to the suppliers from the total profits of the consumers. These works dealt with different constraints in the optimization model. Specifically, ideal power balance constraints with and without statistical demand elasticity were considered in [3] and [4], respectively, and the market dynamics was modeled as a distributed iterative algorithm. The constraints on the operations of appliances were considered in the optimization model [5], [6], which can be solved by the primal-dual dynamics. Furthermore, the volatility of electricity markets under RTP was studied based on the primal-dual dynamics [7]. The above works assumed that the profit functions and the cost functions were known to the consumers and the suppliers. Some quadratic functions were used for approximating the profits of the consumers and the cost to the suppliers [8]–[10]. However, the accurate model of the profits and cost are hard to obtain because of many uncertainties, such as the changing environment factors and operating conditions. In fact, the market dynamics are dependent on the gradients of the utility functions, which are unknown to the market participants. On the other hand, extremum seeking control (ESC) is an adaptive learning method to search for the optimal solution of an unknown function [11]. Some reference have dealt with the extremum seeking for the non-cooperative game [12], [13] and constrained convex optimization problems [14], [15]. However, the optimization problems and the seeking algorithms can not used for modeling the dynamics of electricity markets.

In this study, we use ESC to study the dynamics of electricity markets, which does not require the market participants to know the utility functions. The market participants only need to measure the input and output values of the utility functions and search for the optimal strategies with estimated gradients. The novelty of this work is two-fold. First, we use ESC to develop a type of primal-dual dynamics based on gradient estimation. Second, we apply the ESC-based primal-dual dynamics to model the behaviors of the participants in electricity markets. To the best of our knowledge, this is the first work to apply ESC to model the dynamics of electricity markets.

The rest of the paper is organized as follows. Preliminaries are given in Section II. The dynamics of electricity markets are formulated in Section III. Section IV proves the stability of the dynamics based on gradient estimation. Numerical results are shown in Section V. Finally, we draw conclusions in Section VI.

II. PRELIMINARIES

In this section, we first introduce the notations and definitions used in the paper. Given a vector x, we define $\|x\|$

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denotes the Euclidean norm and $x \in L_\infty$ denotes $\|x\|_{L_\infty} = \text{ess}\sup_{t\geq 0}\|x(t)\| < \infty$.

Definition 1: [16] A continuous function $\beta : R_{\geq 0} \times R_{\geq 0} \to R_{\geq 0}$ is of class $\mathscr{KL}$ if it is nondecreasing in its first argument and converging to zero in its second argument.

Definition 2: [17] A vector function $f(x,\varepsilon) \in R^n$ is said to be $O(\varepsilon)$ if for any compact set $\mathscr{D}$ if there exist positive constants k and ε such that $\|f(x,\varepsilon)\| \leq k\varepsilon$, for $\varepsilon \in (0,\varepsilon^*]$, $x \in \mathscr{D}$.

Definition 3: [17] Given a parameterized family of systems:

$$\dot{x} = f(t,x,\varepsilon), \tag{1}$$

where $x \in R^n$, $t \in R_+$ and $\varepsilon \in R^l_+$ are the state vector, time variable and parameter vector, respectively. The system (1) is said to be semi-globally practically asymptotically (SPA) stable, uniformly in $(\varepsilon_1,\dots,\varepsilon_j), j \in \{1,\dots,l\}$, if there exists $\beta \in \mathscr{KL}$ and constructed parameters $\varepsilon = (\varepsilon_1,\dots,\varepsilon_l)$ such that

$$\|x\| \leq \beta(\|x(0)\|,(\varepsilon_1 \cdot \varepsilon_2 \cdot \dots \cdot \varepsilon_l)(t-t_0)) + \nu, \tag{2}$$

for all $t \geq t_0$, where the constructed parameters ε and the initial state vector $x_0 = x(t_0)$ satisfy: For each pair of strictly positive real numbers (Δ,ν), the initial state $\|x(0)\| \leq \Delta$ and there exist real numbers $\varepsilon_k^* = \varepsilon_k^*(\Delta,\nu) > 0, k = 1,\dots,j$ and for each fixed $\varepsilon_k \in (0,\varepsilon_k^*), k = 1,\dots,j$ there exist $\varepsilon_i = \varepsilon_i(\varepsilon_1,\dots,\varepsilon_{i-1},\Delta,\nu)$, with $i = j+1,\dots,l$.

Lemma 1: [16] Suppose that $W : [0,\infty) \to R$ satisfies

$$D^\dagger W(t) \leq -\alpha W(t) + \gamma(t), \tag{3}$$

where $D^\dagger$ denotes the upper Dini derivative, α is a positive constant, and $\gamma(t) \in L_\infty$. Then,

$$\|W(t)\| \leq e^{-\alpha t}\|W(0)\| + \alpha^{-1}\|\gamma(t)\|_{L_\infty}. \tag{4}$$

III. PROBLEM FORMULATION

We consider an electricity market consisting of suppliers, consumers and a market operator, which is usually named as independent system operator (ISO). Both the suppliers and the consumers can communicate with the ISO in real time and schedule the electricity consumption and production according to the market clearing price. The electricity consumption of the consumers are denoted as $x = (x_1,\dots,x_i,\dots,x_N)^{\mathrm{T}}$, where x_i is the electricity consumption of consumer $i \in \mathbb{N} = \{1,\dots,N\}$. Each consumer is associated with a profit function $u_i(x_i) : R^+ \to R$, where $u_i(x_i)$ denotes the individual profit obtained from consuming x_i units of electricity. Similarly, the electricity production of the suppliers are denoted as $q = (q_1,\dots,q_j,\dots,q_M)^{\mathrm{T}}$, where q_j is the electricity production of supplier $j \in \mathbb{M} = \{1,\dots,M\}$. Each supplier is associated with a cost function $c_j(q_j) : R^+ \to R$, where $c_j(q_j)$ denotes the cost of producing q_j units of electricity. In general, the market clearing procedure can be formulated as

$$\begin{aligned} \text{maximize} \quad & \sum_{i=1}^N u_i(x_i) - \sum_{j=1}^M c_j(q_j) \\ \text{subject to} \quad & \sum_{i=1}^N x_i \leq \sum_{j=1}^M q_j. \end{aligned} \tag{5}$$

Before solving (5), we assume that the profit function $u_i(x_i)$ is concave in x_i and the cost function $c_j(q_j)$ is convex in q_j, which are common assumptions in the market dynamics based on convex optimization [3]–[7]. Then, (5) has a unique optimal solution and can be solved by its lagrangian dual with no dual gap [18]. The Lagrangian function of (5) is defined as

$$L(x,\lambda) = \sum_{i=1}^N u_i(x_i) - \sum_{j=1}^M c_j(q_j) - \lambda(\sum_{i=1}^N x_i - \sum_{j=1}^M q_j), \tag{6}$$

where λ is the lagrangian multiplier. Since the Lagrangian function is concave in x and q and convex in λ, the saddle point of the Lagrangian function is the optimal solution of (5). Then, we can transform (5) into the following individual optimization problems:

$$x_i^* = \arg\max u_i(x_i) - \lambda x_i, \tag{7}$$

and

$$q_j^* = \arg\max \lambda q_j - c_j(q_j). \tag{8}$$

The corresponding dual optimization problem is defined as

$$\lambda^* = \arg\max_{\lambda>0} D(p), \tag{9}$$

where $D(p) = L(x^*,\lambda)$ is the dual function. Then, the primal-dual gradient dynamics to obtain the optimal solution are given as

$$\dot{x}_i = k_i(\frac{\mathrm{d}u_i(x_i)}{\mathrm{d}x_i} - \lambda), \tag{10}$$

$$\dot{q}_j = h_j(\lambda - \frac{\mathrm{d}c_j(q_j)}{\mathrm{d}q_j}), \tag{11}$$

$$\dot{\lambda} = g[\sum_{i=1}^N x_i - \sum_{j=1}^M q_j]^+_\lambda. \tag{12}$$

Equations (10), (11), and (12) model the dynamics of the market participants, including the consumption dynamics of the consumers, the production dynamics of the suppliers, and the price dynamics of the ISO. The above primal-dual dynamics need the gradient information of the profit functions and the cost functions, which are easy to obtain when the profit functions and the cost functions are accurately known to the consumers and the suppliers, respectively. In electricity markets, these two functions are difficult to obtain. However, we can establish a profit/cost calculation model as a reference system for the consumers and the suppliers and measure the profits and the cost in real time. Next, we use ESC to model the dynamics of the electricity markets with unknown profit functions and cost functions. The core idea is to estimate the gradient by adding and multiplying periodic signals to the input and output of the unknown utility functions, respectively. We give the ESC-based market dynamics in Fig. 1 and the implementations in Fig. 2. The

ESC-based market dynamics can be modeled as

$$\begin{aligned}
\dot{\hat{x}}_i &= k_i(\zeta_i - \hat{\lambda}), && (13)\\
\dot{\hat{q}}_j &= h_j(\hat{\lambda} - \xi_j), && (14)\\
\dot{\hat{\lambda}} &= g[\sum_{i=1}^{N} \hat{x}_i - \sum_{j=1}^{M} \hat{q}_j]^{+}_{\hat{\lambda}}, && (15)\\
\dot{\zeta}_i &= -\hat{\omega}_i^c(\zeta_i - \frac{2}{a}u_i(\hat{x}_i + a\sin(\omega t))\sin(\omega t)), && (16)\\
\dot{\xi}_j &= -\hat{\omega}_j^s(\xi_j - \frac{2}{a}c_j(\hat{q}_j + a\sin(\omega t))\sin(\omega t)), && (17)
\end{aligned}$$

where k_i, h_j, and g are the adaptive gains of the market dynamics, and $\hat{\omega}_i^c$ and $\hat{\omega}_j^s$ are the frequencies of the low-pass filters. To separate (16) and (17) from (13)–(15), we choose $\hat{\omega}_i^c$ and $\hat{\omega}_j^s$ such that $k_i = \delta\hat{\omega}_i^c$ and $h_j = \delta\hat{\omega}_j^s$ with small scalar δ. In the ESC-based market dynamics (13)–(17), we only need to measure the input and output values of the utility functions.

IV. MAIN RESULTS

In this section, we will prove the SPA stability of the ESC-based market dynamics. We first define $\hat{\omega}_i^c = \omega_L\omega_i^c, \hat{\omega}_j^s = \omega_L\omega_j^s, g = \delta\omega_L\omega^p$, where ω_L is a positive, real number, and ω_i^c, ω_j^s and ω^p are positive, rational numbers. We define $\omega_{\min}^c = \min\{\omega_1^c, \ldots, \omega_i^c, \ldots, \omega_N^c\}$, $\omega_{\min}^s = \min\{\omega_1^s, \ldots, \omega_j^s, \ldots, \omega_M^s\}$, $\omega_{\max}^c = \max\{\omega_1^c, \ldots, \omega_i^c, \ldots, \omega_N^c\}$, $\omega_{\max}^s = \max\{\omega_1^s, \ldots, \omega_j^s, \ldots, \omega_M^s\}$ and assume that $\omega_{\min}^c/\omega^p \gg \delta, \omega_{\min}^s/\omega^p \gg \delta, \omega_{\min}^c/\omega_{\max}^s \gg \delta, \omega_{\min}^s/\omega_{\max}^c \gg \delta$, which ensure that (16) and (17) can be separated from (13)–(15). Next, we prove the SPA stability in the following theorem:

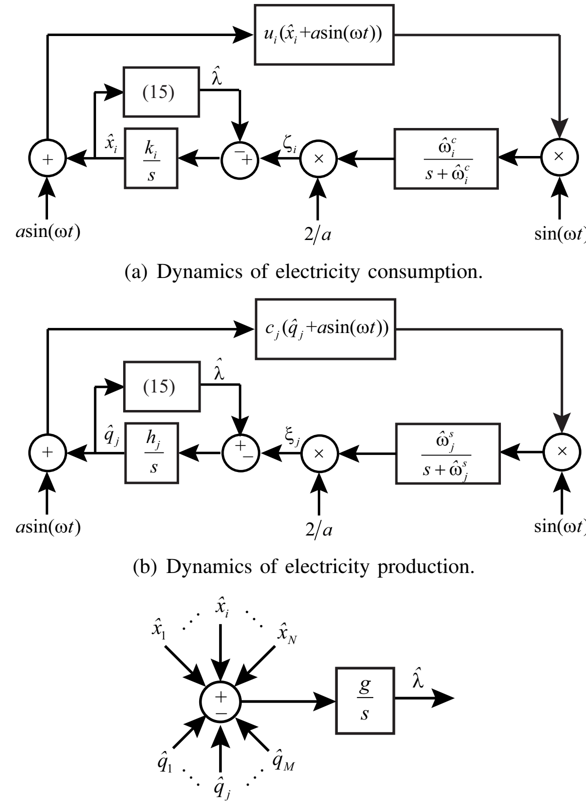
Theorem 1: The ESC-based market dynamics (13)–(17) are SPA stable with respect to δ, a, and ω_L, if the following conditions are satisfied:

- The profit function $u_i(x_i)$ is concave and Lipschitz continuous in x_i, for $i = 1, \ldots, N$, and the cost function $c_j(q_j)$ is convex and Lipschitz continuous in q_j, for $j = 1, \ldots, M$.
- $\mathrm{d}^2u_i(x_i)/\mathrm{d}x_i^2 \leq -\eta_1$ and $\mathrm{d}^2c_j(q_i)/\mathrm{d}q_j^2 \geq \eta_2$, where η_1 and η_2 are positive real numbers.

Proof: Let $\tau = \omega_L t$, we obtain the market dynamics in the new time scale τ,

$$\begin{aligned}
\frac{\mathrm{d}\hat{x}_i}{\mathrm{d}\tau} &= \delta\omega_i^c(\zeta_i - \hat{\lambda}), && (18)\\
\frac{\mathrm{d}\hat{q}_j}{\mathrm{d}\tau} &= \delta\omega_j^s(\hat{\lambda} - \xi_j), && (19)\\
\frac{\mathrm{d}\hat{\lambda}}{\mathrm{d}\tau} &= \delta\omega^p[\sum_{i=1}^{N} \hat{x}_i - \sum_{j=1}^{M} \hat{q}_j]^{+}_{\hat{\lambda}}, && (20)\\
\frac{\mathrm{d}\zeta_i}{\mathrm{d}\tau} &= -\omega_i^c(\zeta_i - \frac{2}{a}u_i(\hat{x}_i + a\sin(\omega t))\sin(\omega t)), && (21)\\
\frac{\mathrm{d}\xi_j}{\mathrm{d}\tau} &= -\omega_j^s(\xi_j - \frac{2}{a}c_j(\hat{q}_j + a\sin(\omega t))\sin(\omega t)). && (22)
\end{aligned}$$

According to the averaging theory [16], the dynamic system with periodic disturbance can be approximated by



(a) Dynamics of electricity consumption.

(b) Dynamics of electricity production.

(c) Market clear pricing.

Fig. 1. ESC-Based Market Dynamics.

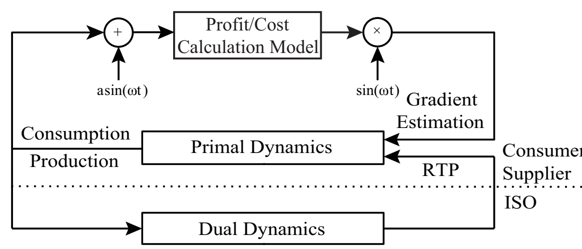


Fig. 2. Implementations in Electricity Markets.

its average system,

$$\begin{aligned}
\frac{\mathrm{d}\hat{x}_i^A}{\mathrm{d}\tau} &= \delta\omega_i^c(\zeta_i^A - \hat{\lambda}^A), && (23)\\
\frac{\mathrm{d}\hat{q}_j^A}{\mathrm{d}\tau} &= \delta\omega_j^s(\hat{\lambda}^A - \xi_j^A), && (24)\\
\frac{\mathrm{d}\hat{\lambda}^A}{\mathrm{d}\tau} &= \delta\omega^p[\sum_{i=1}^{N} \hat{x}_i^A - \sum_{j=1}^{M} \hat{q}_j^A]^{+}_{\hat{\lambda}^A}, && (25)\\
\frac{\mathrm{d}\zeta_i^A}{\mathrm{d}\tau} &= -\omega_i^c(\zeta_i^A - \frac{2}{a}Q_i^A), && (26)\\
\frac{\mathrm{d}\xi_j^A}{\mathrm{d}\tau} &= -\omega_j^s(\xi_j^A - \frac{2}{a}F_j^A), && (27)
\end{aligned}$$

where Q_i^A and F_j^A are defined as

$$\begin{aligned}
Q_i^A &= \frac{1}{2\pi}\int_0^{2\pi} u_i(\hat{x}_i + a\sin(\omega t))\sin(\omega t)\mathrm{d}t, && (28)\\
F_j^A &= \frac{1}{2\pi}\int_0^{2\pi} c_j(\hat{q}_j + a\sin(\omega t))\sin(\omega t)\mathrm{d}t. && (29)
\end{aligned}$$

Approximating $u_i(\hat{x}_i + a\sin(\omega t))$ and $c_j(\hat{q}_j + a\sin(\omega t))$ with the Taylor series, we have

$$\begin{aligned}\frac{2}{a}Q_i^A &= \frac{1}{a\pi}\int_0^{2\pi}(u_i(\hat{x}_i) + a\sin(\omega t)\frac{\mathrm{d}u_i(\hat{x}_i)}{\mathrm{d}\hat{x}_i} \\ &\quad + \sum_{n=2}^{\infty}\frac{(a\sin(\omega t))^n}{n!}\frac{\mathrm{d}^n u_i(\hat{x}_i)}{\mathrm{d}(\hat{x}_i)^n})\sin(\omega t)\mathrm{d}t \\ &= \frac{\mathrm{d}u_i(\hat{x}_i)}{\mathrm{d}\hat{x}_i} + O_i^x(a^2). \end{aligned} \tag{30}$$

$$\begin{aligned}\frac{2}{a}F_j^A &= \frac{1}{a\pi}\int_0^{2\pi}(c_j(\hat{q}_j) + a\sin(\omega t)\frac{\mathrm{d}c_j(\hat{q}_j)}{\mathrm{d}\hat{q}_j} \\ &\quad + \sum_{n=2}^{\infty}\frac{(a\sin(\omega t))^n}{n!}\frac{\mathrm{d}^n c_j(\hat{q}_j)}{\mathrm{d}(\hat{q}_j)^n})\sin(\omega t)\mathrm{d}t \\ &= \frac{\mathrm{d}c_j(\hat{q}_j)}{\mathrm{d}\hat{q}_j} + O_j^q(a^2). \end{aligned} \tag{31}$$

Let $\alpha = \tau\delta$ and substitute (30) and (31) into the average system (23)–(27), we have the dynamic system in time scale α,

$$\frac{\mathrm{d}\hat{x}_i^A}{\mathrm{d}\alpha} = \omega_i^c(\zeta_i^A - \hat{\lambda}^A), \tag{32}$$

$$\frac{\mathrm{d}\hat{q}_j^A}{\mathrm{d}\alpha} = \omega_j^s(\hat{\lambda}^A - \xi_j^A), \tag{33}$$

$$\frac{\mathrm{d}\hat{\lambda}^A}{\mathrm{d}\alpha} = \omega^p[\sum_{i=1}^{N}\hat{x}_i^A - \sum_{j=1}^{M}\hat{q}_j^A]^+_{\hat{\lambda}^A}, \tag{34}$$

$$\delta\frac{\mathrm{d}\zeta_i^A}{\mathrm{d}\alpha} = -\omega_i^c(\zeta_i^A - \frac{\mathrm{d}u_i(\hat{x}_i)}{\mathrm{d}\hat{x}_i} - O_i^x(a^2)), \tag{35}$$

$$\delta\frac{\mathrm{d}\xi_i^A}{\mathrm{d}\alpha} = -\omega_j^s(\xi_j^A - \frac{\mathrm{d}c_j(\hat{q}_j)}{\mathrm{d}\hat{q}_j} - O_j^q(a^2)). \tag{36}$$

The system (32)–(36) is a standard singular perturbation form with fast dynamics, ζ_i^A and ξ_j^A when δ is small. "Freezing" the dynamics (35) and (36) at the equilibrium $\zeta_i^{A*} = \frac{\mathrm{d}u_i(\hat{x}_i)}{\mathrm{d}\hat{x}_i} + O_i^x(a^2)$ and $\xi_j^{A*} = \frac{\mathrm{d}c_j(\hat{q}_j)}{\mathrm{d}\hat{q}_j} + O_j^q(a^2)$, we obtain the reduced system.

$$\frac{\mathrm{d}\hat{x}_i^r}{\mathrm{d}\alpha} = \omega_i^c(\frac{\mathrm{d}u_i(\hat{x}_i^r)}{\mathrm{d}\hat{x}_i^r} + O_i^x(a^2) - \hat{\lambda}^r)), \tag{37}$$

$$\frac{\mathrm{d}\hat{q}_j^r}{\mathrm{d}\alpha} = \omega_j^s(\hat{\lambda}^r - \frac{\mathrm{d}c_j(\hat{q}_j^r)}{\mathrm{d}\hat{q}_j^r} - O_j^q(a^2)), \tag{38}$$

$$\frac{\mathrm{d}\hat{\lambda}^r}{\mathrm{d}\alpha} = \omega^p[\sum_{i=1}^{N}\hat{x}_i^r - \sum_{j=1}^{M}\hat{q}_j^r]^+_{\hat{\lambda}^r}. \tag{39}$$

Let $\tilde{x}^r = \hat{x}^r - \hat{x}^{r*}$, $\tilde{q}^r = \hat{q}^r - \hat{q}^{r*}$, and $\tilde{\lambda}^r = \hat{\lambda}^r - \hat{\lambda}^{r*}$, we select the following Lyapunov function:

$$V = V_1 + V_2 + V_3 = \frac{1}{2}\tilde{x}^{r\mathrm{T}}\Phi^{-1}\tilde{x}^r + \frac{1}{2}\tilde{q}^{r\mathrm{T}}\Psi^{-1}\tilde{q}^r + \frac{1}{2\omega^p}(\tilde{\lambda}^r)^2, \tag{40}$$

where $\Phi = \mathrm{diag}\{\omega_i^c\}$ and $\Psi = \mathrm{diag}\{\omega_j^s\}$ are diagonal matrices. Defining $O^x = (O_1^x, \ldots, O_i^x, \ldots, O_N^x)^{\mathrm{T}}$, $O^q = (O_1^q, \ldots, O_j^q, \ldots, O_M^q)^{\mathrm{T}}$, $R_N = (1, \ldots, 1)^T$ with $|R_N| = N$ and $R_M = (1, \ldots, 1)^T$ with $|R_M| = M$. Then, the derivative of the Lyapunov function along the reduced system (37)-(39) is denoted as

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}\alpha} &= \tilde{x}^{r\mathrm{T}}(u'(\hat{x}^r) + O^x(a^2) - \hat{\lambda}^r R_N) \\ &\quad + \tilde{q}^{r\mathrm{T}}(\hat{\lambda}^r R_M - c'(\hat{q}^r) - O^q(a^2)) \\ &\quad + \tilde{\lambda}^r[\tilde{x}^{r\mathrm{T}}R_N - \tilde{q}^{r\mathrm{T}}R_M]^+_{\hat{\lambda}_r}, \end{aligned} \tag{41}$$

where $u'(\hat{x}^r) = (\mathrm{d}u_1(\hat{x}_1^r)/\mathrm{d}\hat{x}_1^r, \ldots, \mathrm{d}u_i(\hat{x}_i^r)/\mathrm{d}\hat{x}_i^r, \ldots, \mathrm{d}u_N(\hat{x}_N^r)/\mathrm{d}\hat{x}_N^r)^{\mathrm{T}}$ and $c'(\hat{q}^r) = (\mathrm{d}c_1(\hat{q}_1^r)/\mathrm{d}\hat{q}_1^r, \ldots, \mathrm{d}c_j(\hat{q}_j^r)/\mathrm{d}\hat{q}_j^r, \ldots, \mathrm{d}c_M(\hat{q}_M^r)/\mathrm{d}\hat{q}_M^r)^{\mathrm{T}}$. We note that $\tilde{\lambda}^r[\tilde{x}^{r\mathrm{T}}R_N - \tilde{q}^{r\mathrm{T}}R_M]^+_{\hat{\lambda}^r} \leq \tilde{\lambda}^r(\tilde{x}^{r\mathrm{T}}R_N - \tilde{q}^{r\mathrm{T}}R_M)$, and then

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}\alpha} &\leq \tilde{x}^{r\mathrm{T}}(u'(\hat{x}^r) + O^x(a^2) - \hat{\lambda}^r R_N) \\ &\quad + \tilde{q}^{r\mathrm{T}}(\hat{\lambda}^r R_M - c'(\hat{q}^r) - O^q(a^2)) \\ &\quad + \tilde{\lambda}^r(\tilde{x}^{r\mathrm{T}}R_N - \tilde{q}^{r\mathrm{T}}R_M) \\ &= \tilde{x}^{r\mathrm{T}}u'(\hat{x}^r) - \hat{\lambda}^{r*}\tilde{x}^{r\mathrm{T}}R_N + \tilde{x}^{r\mathrm{T}}O^x(a^2) \\ &\quad - \tilde{q}^{r\mathrm{T}}c'(\hat{q}^r) + \hat{\lambda}^{r*}\tilde{q}^{r\mathrm{T}}R_M - \tilde{q}^{r\mathrm{T}}O^q(a^2) \\ &= \tilde{x}^{r\mathrm{T}}(u'(\hat{x}^r) - u'(\hat{x}^{r*})) + \tilde{x}^{r\mathrm{T}}O^x(a^2) \\ &\quad - \tilde{q}^{r\mathrm{T}}(c'(\hat{q}^r) - c'(\hat{q}^{r*})) - \tilde{q}^{r\mathrm{T}}O^q(a^2). \end{aligned} \tag{42}$$

Using the Mean Value Theorem [19] and conditions $\mathrm{d}^2u_i(x_i)/\mathrm{d}x_i^2 \leq -\eta_1$ and $\mathrm{d}^2c_j(q_i)/\mathrm{d}q_j^2 \geq \eta_2$, we obtain

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}\alpha} &\leq -\eta_1\|\tilde{x}^r\|^2 - \eta_2\|\tilde{q}^r\|^2 + \|\tilde{x}^r\|\|O^x(a^2)\| \\ &\leq -2\eta_1\omega_{\min}^c V_1 - 2\eta_2\omega_{\min}^s V_2 + \sqrt{2\omega_{\max}^c}\|O^x(a^2)\|\sqrt{V_1}. \end{aligned} \tag{43}$$

There exists a positive saclar η^* such that

$$\eta^* V = 2\eta_1\omega_{\min}^c V_1 + 2\eta_2\omega_{\min}^s V_2. \tag{44}$$

When $\eta \in [0, \eta^*]$, we have

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}\alpha} &\leq -\eta V + \sqrt{2\omega_{\max}^c}\|O^x(a^2)\|\sqrt{V} \\ &= -\eta V + 2\theta\sqrt{V}, \end{aligned} \tag{45}$$

where θ is defined by

$$\theta = \sqrt{\frac{\omega_{\max}^c}{2}}\|O^x(a^2)\|. \tag{46}$$

Setting $W = \sqrt{V}$, we obtain

$$D^{\dagger}W \leq -\frac{\eta}{2}W + \theta, \tag{47}$$

which, from Lemma 1, implies that

$$\|W\| \leq e^{-\frac{\eta}{2}\alpha}\|W(0)\| + \frac{2}{\eta}\theta. \tag{48}$$

Let $\tilde{z}^r = (\tilde{x}_1^r, \ldots, \tilde{x}_N^r, \tilde{q}_1^r, \ldots, \tilde{q}_M^r, \tilde{\lambda}^r)^{\mathrm{T}}$, we obtain

$$\begin{aligned}\|\tilde{z}^r\| &\leq \sqrt{2\omega_{\max}}\|W\| \\ &\leq \sqrt{2\omega_{\max}}(e^{-\frac{\eta}{2}\alpha}\|W(0)\| + \frac{2}{\eta}\theta), \end{aligned} \tag{49}$$

where $\omega_{\max} = \max\{\omega_1^c, \ldots, \omega_N^c, \omega_1^s, \ldots, \omega_M^s, \omega^p\}$. Thus, the reduced system (37)–(39) is SPA stable with respect to a in the α-time scale.

Defining the boundary system as $e_i^x = \zeta_i^A - \frac{2}{a} Q_i^A$ for all $i = 1, \ldots, N$ and $e_j^q = \xi_j^A - \frac{2}{a} F_j^A$ for all $j = 1, \ldots, M$. From (35) and (36), we see that the boundary system is globally asymptotically stable. Combining with the SPA stability of the reduced system and Lemma 2 in [20], the average system (23)–(27) is SPA stable with respect to a and δ in the τ-time scale. Finally, using the Lemma 1 in [20], the original system (13)–(17) is SPA stable with respect to a, δ, and ω_L in the t-time scale. ∎

V. Numerical Results

In this section, we consider an electricity market consisting of 10 consumers and 5 suppliers. The profit functions and the cost functions are assumed to be some quadratic functions with coefficients randomly selected.

$$\begin{cases} u_1(x_1) = -2x_1^2 + 9x_1 \\ u_2(x_2) = 1.5x_2^2 + 6x_2 \\ u_3(x_3) = -0.5x_3^2 + 7x_3 \\ u_4(x_4) = -0.6x_4^2 + 6x_4 \\ u_5(x_5) = -x_5^2 + 7x_5 \\ u_6(x_6) = -2x_6^2 + 10x_6 \\ u_7(x_7) = -0.5x_7^2 + 5x_7 \\ u_8(x_8) = -0.9x_8^2 + 8x_8 \\ u_9(x_9) = -0.6x_9^2 + 10x_9 \\ u_{10}(x_{10}) = -1.5x_{10}^2 + 8x_{10} \end{cases} \tag{50}$$

and

$$\begin{cases} c_1(q_1) = 0.2q_1^2 - 2q_1 + 5 \\ c_2(q_2) = 0.18q_2^2 - 1.8q_2 + 3 \\ c_3(q_3) = 0.15q_3^2 - 1.7q_3 + 1 \\ c_4(q_4) = 0.13q_4^2 - 1.5q_4 + 3 \\ c_5(q_5) = 0.1q_5^2 - 1.2q_5 + 5 \end{cases} \tag{51}$$

The optimal solutions of (5) are obtained as $x^* = \{2.1, 1.8, 6.4, 4.5, 3.2, 2.4, 4.4, 2.1, 7.9, 2.5\}$ and $q^* = \{6.4, 6.6, 7.6, 7.9, 8.8\}$. The adaptive gains of the market dynamics are defined as $k_i = 0.2$, $h_j = 0.2$ and $g = 0.1$. We set the parameters of the ESC dynamics as $a = 0.1$, $\delta = 0.1$, $\omega = 20$, $\hat{\omega}_i^c = 2$, $\hat{\omega}_j^s = 2$ and $\hat{\omega}^p = 1$. Fig. 3–Fig. 5 show the ESC-based market dynamics converge to a neighborhood around the optimal solutions of (5) (i.e., SPA stability). To evaluate the balance between supply and demand, we define the absolute matching errors (AME) as

$$\text{AME} = \sum_{i=1}^{N} \hat{x}_i - \sum_{j=1}^{M} \hat{q}_j, \tag{52}$$

and the relative matching errors (RME) as

$$\text{RME} = \frac{\sum_{i=1}^{N} \hat{x}_i - \sum_{j=1}^{M} \hat{q}_j}{\sum_{i=1}^{N} \hat{x}_i} \times 100\%. \tag{53}$$

The AME and the RME versus the iterations of the ESC-based market dynamics are shown in Fig. 6 and Fig. 7, respectively. Both the AME and the RME converge to a neighborhood around 0 with rapid speed. The amplitude of the fluctuations is within [-0.2, 0.2] for the AME and within [-1%, 1%] for the RME at the equilibrium.

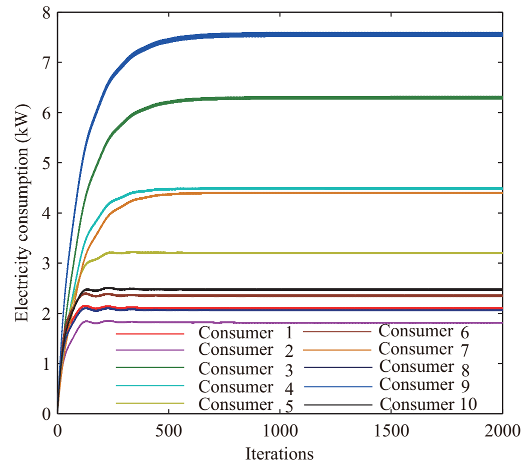


Fig. 3. Convergence of electricity consumption.

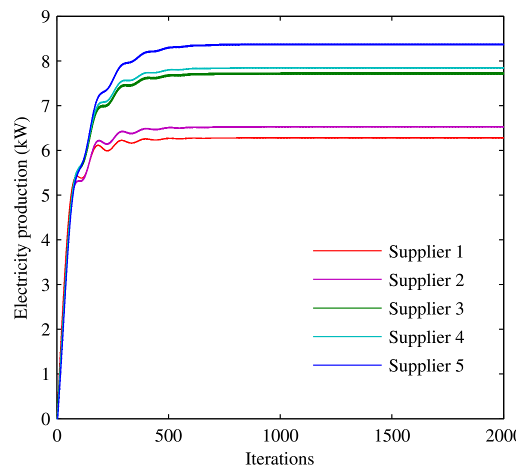


Fig. 4. Convergence of electricity production.

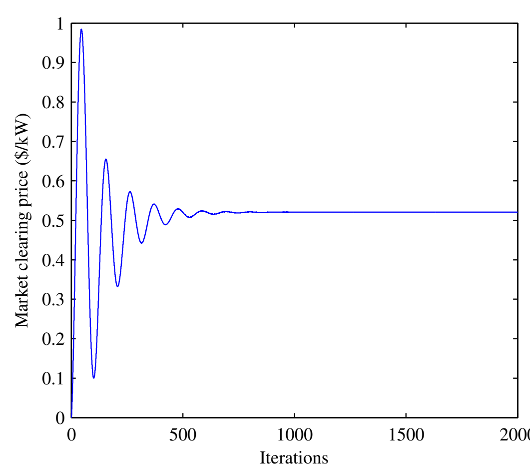


Fig. 5. Convergence of market clearing price.

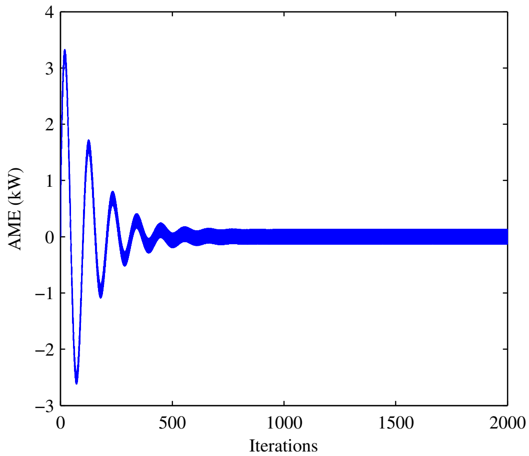


Fig. 6. Absolute matching errors.

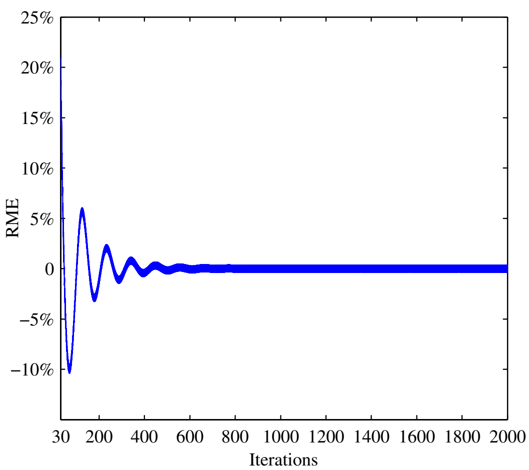


Fig. 7. Relative matching errors.

VI. CONCLUSIONS

In this study, we use ESC to model the dynamics of electricity markets with unknown utility functions. It is shown that the ESC-based market dynamics can converge to the equilibrium within a small neighborhood of the optimal solutions of the social-welfare optimization problem and achieves the balance between supply and demand. In the future, we will study the influence of the ESC dynamics on the price volatility in the electricity markets and consider more complicated market models with physical constraints and renewable power.

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