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Volatility and Mispricing: Robust Variance Estimation and Black-Scholes Call Option Pricing

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"Volatility and Mispricing: Robust Variance Estimation and Black-Scholes Call Option Pricing"

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I. Introduction

Extensive empirical investigation has established that the Black-Scholes [1973] option pricing model yields values which differ systematically from market prices with respect to the volatility of the underlying common stock (Black and Scholes [1972], Geske and Roll [1984]), time until expiration (Black [1975], Rubinstein [1985]) and exercise price (Black [1975], MacBeth and Merville [1979], Rubinstein [1985]). Other research has established that stock returns are not perfectly normally distributed [Fama (1965)]. However, the Black-Scholes model assumes that stock returns are (instantaneously) normally distributed. Given this systematic option mispricing and the observed deviations of stock returns from normality, alternative option pricing models have been developed by relaxing Black-Scholes' assumption that stock returns are normally distributed (Cox [1975], Cox and Ross [1976], Merton [1976], Geske [1979], Rubinstein [1983]).

The observed deviations from normality exhibited by stock returns may be due to measurement error in stock prices, or may reflect the fact that the true underlying distribution of stock returns is nonnormal. Even if the true distribution is nonnormal, the issue is whether these deviations are "large" enough so that an option pricing model based on an alternative distributional assumption can be consistently more accurate than the Black-Scholes model. Extant empirical evidence [Ball and Torous (1985)] suggests that observed statistically significant deviations from normality exhibited by common stock returns are not large enough to result in economically significant differences between Black-Scholes model prices and prices from currently available models based on alternative stock return specifications.

However, if common stock returns are not normally distributed, the sample standard deviation is no longer an efficient estimator of volatility. Lengthening the tails of the underlying distribution explodes the variability of the sample standard deviation (Tukey [1960]). Hence, even slight deviations from normality exhibited by actual common stock returns will systematically affect volatility estimates, possibly contributing to the observed mispricing with respect to these volatility estimates. By contrast, robust estimators of volatility are insensitive to slight deviations from normality (Huber [1977]). Robust techniques ensure that the deviations from normality exhibited by common stock returns will have a minimal impact on the performance of the resultant estimator of common stock volatility, and thereby possibly improving the model's pricing properties.

The purpose of this paper is to (1) empirically investigate the impact of common stock returns' deviations from normality upon the observed mispricing of the Black-Scholes call option model, and (2) re-examine the model's pricing properties when robust statistical procedures are used to estimate the volatility of common stock returns. As noted by Geske and Roll [1984], and others, investigations of variance-related mispricing cannot rely on implied variances which vary across options with the same expiration date written on the same underlying common stock and, furthermore, explicitly assume the validity of the Black-Scholes model.

Using robust estimates of volatility, we eliminate the mispricing of the Black-Scholes model with respect to the skewness and the kurtosis of the distribution of underlying common stock returns. Concomitantly, we reduce the model's systematic mispricing with respect to both volatility and time until expiration. However, robust statistical procedures do not alter significantly the systematic mispricing of the Black-Scholes model with respect to exercise price.

The plan of this paper is as follows. In Section II robust estimation is discussed. Section III details the data employed in this study. In Section IV, estimating volatility by the sample standard deviation of the underlying common stock returns, we empirically examine the mispricing of the Black-Scholes model with respect to volatility, time until expiration, and exercise price. We also empirically document that the model systematically misprices options with respect to the skewness and kurtosis of the distribution of underlying daily common stock returns. Furthermore, the mispricing of the Black-Scholes model with respect to these deviations from normality contributes to the model's observed mispricing with respect to volatility, time until expiration, and to a lesser extent, exercise price. Section V presents our robust estimator of the volatility of daily common stock returns. In deriving this alternative estimator, we take into account the fact that the distribution of daily common stock returns is not only leptokurtic relative to the normal distribution, but also, in general, skewed. In Section VI, we re-examine the pricing properties of the Black-Scholes option pricing model when robust estimates of the volatility of underlying daily common stock returns are employed. Section VII presents our summary and conclusions.

II. Robust Estimation

> "Everyone believed in the normal distribution, the mathematicians because they thought it was experimental fact, the experimenters because they thought it was a mathematical theorem." Poincare'

Robust statistics has as its primary goal improved estimation and inference when the observed distribution is not exactly equivalent to the assumed distribution. A model and its tests are based on an assumed distribution which may differ from the observed distribution. While the normal distribution is the most frequent assumption, many models are based on alternative distributions. Robust statistical techniques offer methods for dealing with observed deviations from any parametric assumption.

The observed deviations from any assumed parametric model may be attributable to measurement error, or the true distribution may actually differ from the distribution assumed. In either case robust statistics are relevant for improving the quality of both estimation and inference.

Consider the following example. An experienced option market maker observing that returns of some optioned stocks are not normally distributed could attribute the deviations from normality to measurement error. This market maker could know from experience that the true stock return distribution is normal, and could condition the observed distribution with a priori knowledge. Alternatively, another less experienced market maker without knowledge of the true distribution but cognizant of the observed deviations from normality could still improve the estimate of stock return volatility by using robust statistics.[1]

To motivate robust statistical methods in option pricing, we appeal to Huber's [1977] approach based on the formalism of a two-person, zero-sum game.[2] Let G denote the true distribution of stock returns assumed to underly observed option prices. For example, Black-Scholes assume that G is a normal distribution. However, if the observed distribution of stock returns, F, differs from the assumed distribution, G, then F may lie in a neighborhood of G. More formally, the observed distribution, F, may be regarded as a mixture of the assumed distribution, G, and an alternate distribution, H, such that

$$F = (1 - \epsilon)\, G + \epsilon\, H \qquad (1)$$

where $\epsilon > 0$ is a known fraction and H is an unknown distribution which modifies the assumed distribution. The unknown distribution H can have many interpretations. For stock returns, H may reflect the fact that deviations from the assumed normality in stock returns may be due to violations of independence, nonstationarity, approximations to the central limit theorem, gross data errors from recording, rounding or grouping errors since stock prices are recorded in 1/8's or 1/16's or since trades occur at the bid or the ask.

The two-person zero-sum game involves the statistician and Nature, which generates the observed data. In particular, Nature chooses a distribution F in the neighborhood of the assumed distribution G, the statistician chooses a variance estimator Ψ, and the gain to Nature and the loss to the statistician is the estimator's asymptotic variance, $V(F, \Psi)$. Huber establishes that there exists a saddle point to this game where Nature chooses the least favorable distribution and the statistician uses a minimax strategy to select an estimator which minimizes asymptotic variance for this least favorable distribution. For observed data $\{x_1, x_2, \ldots, x_n\}$, Huber's robust variance estimator is that value s^2 which satisfies

$$\frac{1}{n-1} \sum_{i=1}^{n} \psi_H \left(\frac{x_i - T}{\sqrt{s^2}}\right) = E[\psi_H^2(z) \mid z \sim N(0,1)] \qquad (2)$$

where $$\psi_H(u) = \begin{cases} -b & u < -b \\ u & -b \le u \le b \\ b & u > b \end{cases}$$

for some $b > 0$, where T is a location estimate of the sample. Notice that when $\psi_H(u) = u$ and $T = \bar{x}$, the sample mean, the solution to the preceding expression is given by the sample variance. Deviations from normality have minimal effect on the performance of Huber's minimax estimator since the influence of observations far from the center of the sample is limited.

By contrast, the performance of the sample standard deviation deteriorates in the presence of slight deviations from normality. For example,[3] compare the sample standard deviation, s_n, with the mean absolute deviation, d_n, given in formulas (3) and (4) below:

$$s_n = [n^{-1} \sum_i (x_i - \bar{x})^2]^{1/2} \tag{3}$$

and

$$d_n = n^{-1} \sum_i |x_i - \bar{x} |. \tag{4}$$

When the data are normal, as n get large, s_n converges to the population standard deviation σ, while d_n converges to $\sqrt{2/\pi} \ \sigma \doteq 0.8 \ \sigma$. In this case the asymptotic relative efficiency of s_n to d_n, $[\text{Var } (s_n)/E(s_n)^2 \div \text{Var } (d_n)/E(d_n)^2]$, is approximately .876. However, for just slight deviation from normality, which are common in even the best samples, the efficiency of the sample standard deviation is significantly reduced. For example, with reference to expression (1), if $G \equiv N(\mu,\sigma^2)$ and $H \equiv N(\mu,9\sigma^2)$, and if the observed data is characterized by ϵ equal to 0.2% or 5%, the asymptotic relative efficiency is 1.016 and 2.035, respectively. Thus, the relative efficiency of robust volatility estimates given slight deviations from normality will improve the pricing properties of the Black-Scholes option pricing model.

In the next section we examine the data and report the observed deviations from the assumed normality.

III. Data

To investigate the Black-Scholes model's pricing properties, data on the various inputs to the model must be measured accurately. Otherwise, resultant measurement error, not attributable to the model, will lead to the apparent mispricing of options.

In this study, transactions data for call options listed on the Chicago Board Options Exchange (CBOE) are compiled from the Berkeley Options Data Base. By employing transactions data, we minimize simultaneity problems between option prices and underlying stock prices. Call options are sampled for one date per month, the date randomly chosen from amongst the dates following that month's expiration date, over the period August 1976 to October 1977. We include only options that traded closest to but after 12:00 pm and traded no later than 2:30 pm to minimize problems associated with opening and closing prices. Furthermore, to better focus on the pricing properties of the Black-Scholes model, we restrict our attention to that subsample of call options written on non-dividend paying stocks. To that end, the CRSP daily return file is employed to identify those underlying common stocks which did not pay dividends during their options' remaining time until expiration. As a result, we have a sample of 2323 call options written on non-dividend paying common stock spanning fifteen calendar months.

The market price of a call option written on a non-dividend paying common stock is taken to be the average of the lowest bid or transaction price and the highest ask or transaction price quoted during a constant common stock price interval. The corresponding risk-free interest rate is compiled from the CRSP bond file and is measured by the yield of a Treasury bill with maturity date closest to the option's expiration date.

For each of the underlying common stocks, we collect daily return data from the CRSP daily return file for 30 days, 60 days, 120 days, and 240 days prior to the dates for which options data are available. Estimating volatility on the basis of varying sample sizes allows us to empirically investigate the effect of non-stationarity in the underlying security return process upon the pricing properties of the Black-Scholes model. From Tables 1 and 2, we conclude that these underlying daily common stock returns, on average, exhibit significant deviations from normality. Notice in Table 1 that as the sample size increases from 30 days to 240 days, the proportion of the sampled common stock returns exhibiting statistically significant leptokurtosis increases. This is consistent with non-stationarity in the return distribution or with more power in the test statistic due to a larger sample size. The results of Table 2 are consistent with both statistically significant positive and negative skewness in the sampled common stock returns. We investigate the effects of these types of deviations from normality upon the mispricing of the Black-Scholes model in the next section.

IV. Pricing Properties of the Black Scholes Model: Volatility Estimated by Sample Standard Deviation

Given our sample of call options written on non-dividend paying common stocks, we now examine empirically the pricing properties of the Black-Scholes model when volatility is estimated by the underlying common stock returns' sample standard deviation. We also examine empirically the model's mispricing with respect to the deviations from normality exhibited by the underlying common stock returns. The effects of these deviations from normality upon the model's mispricing with respect to volatility, time until expiration, and exercise price are also investigated.

In Table 3 we document the mispricing of the Black-Scholes model with respect to the sample standard deviation, sample skewness, and the sample kurtosis of the underlying common stock returns. Here and throughout, mispricing is defined as the dollar difference between the Black-Scholes model price and the corresponding market price. Following Rubinstein [1985], we employ a nonparametric procedure, the Spearman rank correlation coefficient, to assess the degree of association between the model's mispricing and the characteristics of the distribution of underlying common stock returns. Nonparametric procedures assume nothing about the population from which the observed sample is drawn, all observations are weighted equally.

When volatility is estimated by the underling common stock returns' sample standard deviation, the Black-Scholes model not only systematically misprices calls with respect to volatility, but also systematically misprices calls with respect to skewness and kurtosis. As previously documented by others, the larger (smaller) the sample standard deviation, on average, the larger the model's overvaluing (undervaluing). This result holds for all sample sizes and is highly significant throughout. In addition, the more negatively skewed the distribution of underlying common stock returns, on average, the greater the model's overpricing. This result holds for all sample sizes, except the 30 day sample. Also, the more leptokurtic the distribution of underlying common stock returns, on average, the greater the model's overpricing. This result is highly significant across all sample sizes.

The magnitude of the model's mispricing with respect to volatility and the effect of deviations from normality upon this mispricing are presented in

Table 4. We classify the sampled call options as to whether they are written on common stock with sample standard deviation, s, of $s < .20$, $.20 < s < .30$, or, $s > .30$, and provide the resultant number of options, the corresponding mean mispricing, the standard error of the mean mispricing, as well as the mean sample skewness and mean sample kurtosis of the underlying common stock returns. As expected, the model underprices options written on low volatility stocks, and overprices options written on high volatility stocks. The magnitude of the underpricing of options written on low volatility stocks diminishes with increasing sample size. However, the magnitude of the overpricing of options written on high volatility stocks does not diminish with increasing sample size.

We also note that the magnitude of the overpricing of options written on high volatility stocks exceeds the magnitude of the underpricing of options written on low volatility stocks. This result is consistent with sampling error in the sample standard deviation contributing to the observed volatility mispricing of the Black-Scholes model. High sample standard deviations tend to reflect positive sampling error, which is unbounded from above, while low sample standard deviations tend to reflect negative sampling error, which is bounded from below by zero. Furthermore, the high volatility stocks exhibit, on average, significant deviations from normality whereas the low volatility stocks, on average, do not. This suggests that the overpricing of options written on high volatility stocks may simply reflect the mispricing of the Black-Scholes model with respect to deviations from normality.

The Black-Scholes model also systematically misprices the sampled call options with respect to their time until expiration. From Table 5 we conclude that, on average, the longer an option's time until expiration, the greater the model's overpricing. This result holds for all sample sizes except the 30 day sample where the time until expiration mispricing does not appear to be economically significant.[4] Concurrently, for each sample size, as the time until expiration increases, on average, the mean sample skewness of the underlying common stock returns becomes more negative, while the mean sample kurtosis increases. Notice that the deviations from normality do not appear to be as dramatic for the 30 day sample. Observing greater deviations from normality with increasing time until expiration is consistent with common stock returns to firms which do not as a policy pay dividends exhibiting greater deviations from normality than common stock returns to firms which temporarily suspend dividend payments. It is also consistent with non-stationarity. Again these results suggest that the mispricing of options with respect to their time until expiration may simply reflect the mispricing of the Black-Scholes model with respect to deviations from normality.

It is important to recognize that the overpricing, on average, of options which are written on common stock which exhibit significant deviations from normality is consistent with increased sampling error in the sample standard deviation due to the non-normality of common stock returns. Increased sampling error in the sample standard deviation also implies that the distribution of the sample standard deviation becomes more positively skewed. As a result, there is a probability of a large positive sampling error and, as such, large overpricing by the Black-Scholes model. By contrast, there is a lesser probability of a correspondingly large negative sampling error and, as such, large underpricing by the Black-Scholes model since the distribution of the sample standard deviation is bounded from below by zero. On average, then, the Black-Scholes model will overprice options written on common stock which exhibit significant deviations from normality.

In Table 6 we examine the mispricing of the Black-Scholes option pricing model with respect to exercise price. As previously documented by others, we find that for this sample period the model tends to overprice deep in-the-money ($S/X > 1.15$) calls. However, this overpricing of deep in-the-money calls does not appear to be economically significant. Furthermore, contrary to previous results, we find that across all sample sizes the mispricing of deep out-of-the-money ($S/X < .85$) calls is not statistically significant. For these deep out-of-the-money calls, notice that for all sample sizes except the 30 day sample, the underlying common stock returns exhibit significant deviations from normality. That is, on average, the deep out-of-the-money calls are written on common stock which have experienced sizeable price declines. The fact that we do not, on average, misprice these calls requires the Black-Scholes model, on average, to underprice deep out-of-the-money calls written on common stock which do not exhibit deviations from normality. This then would offset the model's overpricing of deep out-of-the-money calls written on common stock which do exhibit significant deviations from normality. Finally, for all sample sizes except the 240 day sample, the model tends to underprice at-the-money ($.85 < S/X < 1.15$) calls, the magnitude of the underpricing increasing with decreases in the sample size.

To further investigate the effect of common stock returns' deviations from normality upon mispricing of the Black-Scholes model, we now turn our attention to the subsample of calls written on common stock which do not exhibit statistically significant deviations from normality. Comparing the pricing properties of the Black-Scholes model in a sample that includes options written on common stock which exhibit deviations from normality with a subsample which does not, allows us to isolate the pricing effects of these deviations from normality.

Table 7 summarizes our results. In Panel A we document the model's mispricing with respect to sample standard deviation. While, as before, the model still underprices options written on common stock with low volatility, notice that the model's overpricing of options written on common stock with high volatility is reduced significantly. In other words, the fact that in the entire sample we observe the Black-Scholes model, on average, overpricing options written on common stock with high volatility simply reflects the model's overpricing, on average, of options written on common stock which exhibit significant deviations from normality.

From Panel B, we conclude that the Black-Scholes model no longer overprices long-term options. In fact, it appears that for options written on common stock which do not exhibit significant deviations from normality, the model tends to underprice long-term options, especially in the 30 day sample. Again, the overpricing of long-term options that we observe in the entire sample is simply a consequence of the Black-Scholes model's overpricing of options written on common stock which exhibit significant deviations from normality.

Panel C presents the Black-Scholes model's pricing properties with respect to exercise price. The model's exercise price bias does not appear to be economically significant. However, even in the absence of deviations from normality, the model still tends to overprice deep in-the-money options and underprice at-the-money options. Furthermore, for options written on common stock which do not exhibit deviations from normality, we now detect a statistically significant underpricing of deep out-of-the-money options. The overpricing of deep out-of-the-money options written on common stock which do exhibit deviations from normality offset this underpricing so that for the entire sample we observe no mispricing of deep out-of-the-money options.

Deviations from normality exhibited by common stock returns do contribute to the observed pricing properties of the Black-Scholes model. In particular, these deviations from normality contribute to the model's observed overpricing of options written on high volatility stocks and the model's observed overpricing of long-term options. Furthermore, these deviations from normality contribute to the observed lack of mispricing of deep out-of-the-money options. If deviations from normality effect the Black-Scholes model by increasing the sampling error of the sample standard deviation, robust estimation of the volatility of common stock returns will improve the model's pricing properties since the performance of robust statistical procedures is much less affected by these deviations.

V. A Robust Estimator of Common Stock Return Volatility

The sample standard deviation is notably nonrobust to slight deviations from normality. By contrast, these slight deviations from normality will cause only a slight change in the performance of a robust estimator of common stock return volatility. Our robust estimator of volatility is based upon Lax's [1985] A estimators of volatility. His Monte Carlo analysis establishes that across a variety of long-tailed symmetric distributions, the minimum efficiency of this class of estimators is highest amongst more than 150 alternative volatility estimators. However, we must modify Lax's A estimators of volatility to take into account the fact that the distribution of daily common stock returns is not only long-tailed but also, in general, skewed.

The intuition underlying this class of volatility estimators is that the variance of a location estimator can serve as a volatility estimator. For example, if $x_i \sim$ iid $N(0,\sigma^2)$ and $\bar{x} = n^{-1} \sum_{i=1}^{n} x_i$, then as $n \to \infty$,

$$n \operatorname{var}(\bar{x}) \to \sigma^2. \qquad (4)$$

However, the sample mean is also notably nonrobust to slight deviations from normality (Tukey [1960]). Consequently, an A estimator of volatility is defined, analogously to expression (4), from the asymptotic variance of a robust estimator of location. Given a robust estimator of location, T, under appropriate regularity conditions, as $n \to \infty$,

$$n \operatorname{var}(T) \to A(T), \qquad (5)$$

where A(T) represents the asymptotic variance of the location estimator T. A finite sample approximation to the asymptotic variance of the robust location estimator gives an A estimator of volatility.

Lax's A estimators of volatility assume the underlying distribution is long-tailed but symmetric. If the underlying distribution is skewed, then A(T), which is appropriate if the distribution is symmetric, will underestimate the variance of T (see Carroll [1979], Theorem 1, page 625). The bias increases as the degree of asymmetry increases. Since T is a smooth function of the underlying distribution, T being a robust estimator of location, implies that jackknifing (Miller [1974]) will be effective in estimating the variance of T. As the variance of a robust location estimator serves as a robust volatility estimator, we modify Lax's A estimators to take into account possible asymmetry in the distribution of common stock returns by employing jackknifed variance estimates of a robust location estimator as robust volatility estimates.

In order to calculate the jackknifed variance of a robust location estimator T, we split our sample of size n into g groups of size h each, n=gh. Letting T_j be the robust location estimate based on the sample of size (g-1)h, where the jth group of size h has been deleted, we define

$$\bar{T} = g^{-1} \sum_{j=1}^{g} T_j . \qquad (6)$$

The jackknifed estimate of the variance of T is then given by

$$(g - 1)^{-1} \sum_{j=1}^{g} (T_j - \bar{T})^2 . \qquad (7)$$

Our robust common stock return volatility estimator is based on the biweight estimator of location. That is, given data $\{x_1, x_2, \ldots, x_n\}$, the robust location estimate is calculated as the solution, T, of the equation

$$\sum_{i=1}^{n} \psi (u_i) = 0 \qquad (8)$$

where $u_i = (x_i - T)/cMAD$, with c = 9 and MAD = median $\{|x_i - \text{median}(x_i)|\}$, and the bisquare function ψ is defined by

$$\psi(u_i) = \begin{cases} u_i(1-u_i^2)^2 & \text{for } |u_i| < 1 \\ 0 & \text{for } |u_i| \geq 1. \end{cases} \qquad (9)$$

Lax's Monte Carlo analysis confirms that across a variety of long-tailed and symmetric distributions, the minimum efficiency of the A estimator of volatility based on this particular specification is higher than that of any other A estimator. A Newton-Raphson iterative procedure is employed to solve equation (8). Using the sample median as the initial estimate of location, in each case convergence was achieved in no more than five iterations.

In Table 8, we contrast the properties of our robust estimates of common stock return volatility with the corresponding volatility estimates provided by the sample standard deviation. For the sample of all common stock returns across all days which do not exhibit statistically significant deviations from normality, the robust estimates of volatility are not biased relative to the sample standard deviation estimates. There is little or no difference between the average robust estimates of volatility and the average volatility estimates provided by the sample standard deviation.

Furthermore, in this case the sampling variance of the robust estimates is slightly larger than the sampling variance of the sample standard deviation estimates. This follows since for normally distributed data the sample standard deviation is the optimal estimate of volatility. However, for the sample of all common stock returns across all days which do exhibit statistically significant deviations from normality, the sampling variance of the robust estimates is significantly smaller than the sampling variance of the sample standard deviation estimates. The efficiency gains of our robust procedures tend to increase with decreasing sampling size. Also, the average sample standard deviation estimates exceed the average robust estimates of volatility across all sample sizes reflecting the relative inefficiency of the sample standard deviation in the presence of non-normal data.

VI. Pricing Properties of the Black Scholes Model: Volatility Estimated by Robust Techniques

Robust statistical techniques provide more efficient estimates of volatility given deviations from normality in common stock returns. We now examine empirically the pricing properties of the Black-Scholes model when robust estimates of common stock return volatility are employed.

From Table 9, notice that across all sample sizes we eliminate the mispricing of the Black-Scholes model with respect to the sample kurtosis of the distribution of underlying common stock returns. Furthermore, except for the 240 day sample, the model's mispricing with respect to the sample skewness of the distribution of underlying common stock returns is no longer statistically significant. In addition, with robust estimates of common stock return volatility, we reduce but do not eliminate the Black-Scholes model's systematic mispricing with respect to volatility. Robust procedures are least effective in reducing systematic volatility mispricing in the 30 day sample since for this sample size a majority of common stock returns do not exhibit statistically significant deviations from normality.

We further investigate this volatility mispricing in Table 10. As in the sample of options written on common stock which do not exhibit statistically significant deviations from normality, the model's systematic mispricing with respect to robust volatility estimates is due primarily to the underpricing of options written on low volatility stocks. By contrast, we reduce significantly the model's overpricing of options written on high volatility stocks. Only for the 30 day sample does the overpricing of these options appear to be economically significant. This reduction in the overpricing of options written on high volatility stocks follows since the robust estimation of common stock return volatility eliminates the Black-Scholes model's mispricing with respect to sample skewness and sample kurtosis.

The robust estimation of the Black-Scholes model minimizes the model's systematic overpricing of options with long terms until expiration. In Table 11 we document the model's resultant mispricing of options with respect to time until expiration. Only for the 240 day sample is the overpricing of the long term options statistically significant. For all other sample sizes, the mispricing of these options is not statistically significant. Again, the reduction in the overpricing of these options reflects the elimination of the model's mispricing with respect to sample skewness and sample kurtosis.

Finally, Table 12 presents the Black-Scholes model's mispricing with respect to exercise price when we employ robust estimates of common stock return volatility. As in the sample of options written on common stock which do not exhibit statistically significant deviations from normality, the model's overpricing of deep in-the-money options, and underpricing of at-the-money and deep out-of-the-money options is statistically significant throughout. However, as before, this exercise price bias does not appear to be economically significant.

VII. Summary and Conclusions

This study uses option transactions data to investigate the relation between volatility estimation and call option mispricing using the Black-Scholes formula. Estimating volatility by the sample standard deviation, the Black-Scholes model systematically misprices our large sample of call options written on non-dividend paying common stock with respect to the sample skewness and the sample kurtosis of the underlying distribution of common stock returns. This systematic mispricing with respect to sample skewness and sample kurtosis contributes to the model's observed mispricing with respect to volatility, time until expiration, and, to a lesser extent, exercise price. Robust estimation of common stock return volatility eliminates the model's systematic mispricing with respect to sample skewness and sample kurtosis. As a result, robust estimation of common stock return volatility minimizes the Black-Scholes model's mispricing with respect to time until expiration and reduces the model's mispricing with respect to volatility. However, even in the absence of deviations from normality, the Black-Scholes model still undervalues options written on low volatility stocks and systematically misvalues options with respect to exercise price. However, the exercise price bias does not appear to be economically significant.

Table 1

day	total number of underlying common stocks	number of underlying common stock exhibiting positive kurtosis at 5% significance level			
	sample size :	240 days	120 days	60 days	30 days
1	26	18	17	11	9
2	59	44	32	24	14
3	23	20	14	10	6
4	48	37	32	17	9
5	59	44	33	22	16
6	68	54	39	27	17
7	37	31	22	17	14
8	64	55	37	23	17
9	25	21	16	8	7
10	31	24	18	8	10
11	63	50	35	21	14
12	24	19	13	9	6
13	29	25	20	17	11
14	58	47	36	25	13
15	65	53	43	29	15

Table 2

day	total number of underlying common stocks	number of underlying common stock exhibiting positive skewness at 5% significance level sample size : 240 days				number of underlying common stock exhibiting negative skewness at 5% significance level sample size : 240 days			
		240 days	120 days	60 days	30 days	240 days	120 days	60 days	30 days
1	26	9	10	3	4	1	3	5	6
2	59	28	19	11	7	4	6	6	5
3	23	9	8	7	6	2	4	4	1
4	48	17	9	7	5	4	10	11	5
5	59	26	9	12	13	4	12	6	8
6	68	20	10	14	9	7	12	11	10
7	37	6	5	5	5	4	9	9	11
8	64	14	9	10	6	7	13	12	6
9	25	5	4	1	4	7	6	4	4
10	31	3	5	3	3	9	8	4	4
11	63	4	8	6	4	12	20	14	7
12	24	3	2	3	0	7	6	7	1
13	29	6	3	7	4	4	6	6	5
14	58	10	10	14	5	14	12	8	7
15	65	18	12	15	7	17	15	11	7

Table 3

sample size	correlation between mispricing and volatility	correlation between mispricing and sample kurtosis	correlation between mispricing and sample skewness
240 days	.192**	.187**	-.128**
120 days	.190**	.164**	-.106**
60 days	.242**	.170**	-.154**
30 days	.188**	.096**	-.020

** indicates significance at the 1% level

Table 4

sample size	volatility	s < .20	.20 < s < .30	s > .30
240 days	number of options	579	1146	598
	$\overline{mpr}$	-.024	.022	.117
	se($\overline{mpr}$)	.012	.006	.010
	$\overline{sk}$	.04	.00	-.21
	$\overline{kr}$	1.44	1.83	4.28
120 days	number of options	668	1156	499
	$\overline{mpr}$	-.062	.003	.063
	se($\overline{mpr}$)	.012	.006	.008
	$\overline{sk}$	-.03	-.09	-.31
	$\overline{kr}$	1.07	1.88	4.18
60 days	number of options	781	1097	445
	$\overline{mpr}$	-.081	-.018	.132
	se($\overline{mpr}$)	.012	.007	.013
	$\overline{sk}$	-.02	-.04	-.53
	$\overline{kr}$	.86	1.28	3.64
30 days	number of options	906	978	439
	$\overline{mpr}$	-.091	-.053	.140
	se($\overline{mpr}$)	.012	.008	.020
	$\overline{sk}$	-.04	.04	-.22
	$\overline{kr}$	.34	.64	1.55

Table 5

sample size	time until expiration	$\tau < 20$	$20 < \tau < 70$	$70 < \tau < 120$	$120 < \tau < 170$	$\tau > 170$
240 days	number of options	386	1498	189	107	143
	$\overline{mpr}$	-.003	.005	.055	.186	.337
	se($\overline{mpr}$)	.004	.005	.020	.031	.034
	$\overline{sk}$	.04	.01	-.24	-.41	-.42
	$\overline{kr}$	2.15	2.12	3.28	3.43	4.27
120 days	number of options	386	1498	189	107	143
	$\overline{mpr}$	-.007	-.022	-.025	.095	.187
	se($\overline{mpr}$)	.005	.005	.022	.026	.027
	$\overline{sk}$	-.07	-.07	-.34	-.44	-.32
	$\overline{kr}$	2.08	1.97	2.86	2.67	2.86
60 days	number of options	386	1498	189	107	143
	$\overline{mpr}$	-.007	-.036	-.003	.099	.157
	se($\overline{mpr}$)	.005	.006	.025	.045	.044
	$\overline{sk}$	-.09	-.08	-.25	-.47	-.35
	$\overline{kr}$	1.59	1.38	2.33	2.42	2.39
30 days	number of options	386	1498	189	107	143
	$\overline{mpr}$	-.014	-.045	-.009	-.052	.050
	se($\overline{mpr}$)	.005	.006	.025	.070	.058
	$\overline{sk}$	-.01	-.01	-.16	-.22	-.12
	$\overline{kr}$	.49	.63	1.05	1.19	1.15

Table 6

sample size	S/X	S/X < .85	.85 < S/X < 1.15	S/X > 1.15
240 days	number of options	537	1355	431
	$\overline{mpr}$	-.006	.038	.075
	se($\overline{mpr}$)	.005	.016	.009
	$\overline{sk}$	-.21	.00	-.02
	$\overline{kr}$	3.15	2.09	2.31
120 days	number of options	537	1355	431
	$\overline{mpr}$	-.007	-.023	.064
	se($\overline{mpr}$)	.005	.007	.008
	$\overline{sk}$	-.33	-.06	-.06
	$\overline{kr}$	3.12	1.80	2.00
60 days	number of options	537	1355	431
	$\overline{mpr}$	.014	-.042	.055
	se($\overline{mpr}$)	.008	.008	.009
	$\overline{sk}$	-.36	-.07	-.01
	$\overline{kr}$	2.38	1.36	1.33
30 days	number of options	537	1355	431
	$\overline{mpr}$	.009	-.074	.050
	se($\overline{mpr}$)	.010	.010	.010
	$\overline{sk}$	-.16	-.01	.02
	$\overline{kr}$	.93	.63	.61

Table 7

A:

sample size	volatility	$s < .20$	$.20 < s < .30$	$s > .30$
240 days	number of options	110	220	61
	$\overline{mpr}$	-.072	.011	-.040
	se($\overline{mpr}$)	.021	.013	.017
120 days	number of options	232	418	86
	$\overline{mpr}$	-.048	-.021	-.002
	se($\overline{mpr}$)	.015	.009	.019
60 days	number of options	428	557	131
	$\overline{mpr}$	-.067	-.037	.023
	se($\overline{mpr}$)	.011	.009	.012
30 days	number of options	612	636	220
	$\overline{mpr}$	-.075	-.052	-.021
	se($\overline{mpr}$)	.010	.010	.013

B:

sample size	time until expiration	$\tau < 20$	$20 < \tau < 70$	$70 < \tau < 170$	$\tau > 170$
240 days	number of options	69	255	51	16
	$\overline{mpr}$	.000	-.011	-.085	-.051
	se($\overline{mpr}$)	.011	.013	.041	.027
120 days	number of options	121	509	81	25
	$\overline{mpr}$	-.007	-.017	-.130	.003
	se($\overline{mpr}$)	.009	.008	.039	.088
60 days	number of options	180	761	122	53
	$\overline{mpr}$	-.001	-.043	-.096	-.037
	se($\overline{mpr}$)	.007	.008	.029	.053

30 days	number of options	266	976	158	68
	$\overline{mpr}$	-.011	-.044	-.165	-.169
	se($\overline{mpr}$)	.006	.007	.029	.056

C: sample size	S/X	S/X < .85	.85 < S/X < 1.15	S/X > 1.15
240 days	number of options	62	253	76
	$\overline{mpr}$	-.042	-.034	.046
	se($\overline{mpr}$)	.005	.014	.022
120 days	number of options	133	452	151
	$\overline{mpr}$	-.032	-.053	.054
	se($\overline{mpr}$)	.005	.011	.015
60 days	number of options	227	666	223
	$\overline{mpr}$	-.033	-.074	.048
	se($\overline{mpr}$)	.004	.010	.013
30 days	number of options	326	860	282
	$\overline{mpr}$	-.041	-.096	.042
	se($\overline{mpr}$)	.004	.010	.011

TABLE 8

normal:

sample size	$\bar{s}$	$\bar{\hat{\sigma}}_A$	var(s) / var($\hat{\sigma}_A$)
240 days	.24	.24	.996
120 days	.23	.23	.979
60 days	.23	.23	.984
30 days	.23	.23	.978

non-normal:

sample size	$\bar{s}$	$\bar{\hat{\sigma}}_A$	var(s) / var($\hat{\sigma}_A$)
240 days	.27	.25	1.172
120 days	.26	.25	1.266
60 days	.27	.25	1.417
30 days	.27	.25	1.511

Table 9

sample size	correlation between mispricing and volatility	correlation between mispricing and sample kurtosis	correlation between mispricing and sample skewness
240 days	.101**	.030	-.070**
120 days	.092**	.000	-.031
60 days	.136**	.012	-.037
30 days	.151**	-.016	-.028

** indicates significance at the 1% level.

Table 10

sample size	volatility	$\hat{\sigma}_A < .20$	$.20 < \hat{\sigma}_A < .30$	$\hat{\sigma}_A > .30$
240 days	number of options	673	1133	517
	$\overline{mpr}$	-.034	.008	.024
	$se(\overline{mpr})$	.008	.006	.009
120 days	number of options	802	1066	455
	$\overline{mpr}$	-.069	-.016	-.005
	$se(\overline{mpr})$	.008	.006	.009
60 days	number of options	824	1083	416
	$\overline{mpr}$	-.068	-.031	.027
	$se(\overline{mpr})$	.009	.007	.009
30 days	number of options	913	969	441
	$\overline{mpr}$	-.094	-.059	.077
	$se(\overline{mpr})$	.009	.009	.017

Table 11

sample size	time until expiration	$\tau < 20$	$20 < \tau < 70$	$70 < \tau < 120$	$120 < \tau < 170$	$\tau > 170$
240 days	number of options	386	1498	189	107	143
	$\overline{mpr}$	-.006	-.014	-.008	.037	.120
	se($\overline{mpr}$)	.004	.005	.020	.028	.032
120 days	number of options	386	1498	189	107	143
	$\overline{mpr}$	-.011	-.040	-.075	-.029	.051
	se($\overline{mpr}$)	.005	.005	.022	.025	.026
60 days	number of options	386	1498	189	107	143
	$\overline{mpr}$	-.012	-.047	-.063	-.029	.014
	se($\overline{mpr}$)	.005	.006	.020	.035	.033
30 days	number of options	386	1498	189	107	143
	$\overline{mpr}$	-.014	-.053	-.032	-.099	-.060
	se($\overline{mpr}$)	.005	.006	.021	.069	.051

TABLE 12

sample size	S/X	S/X < .85	.85 < S/X < 1.15	S/X > 1.15
240 days	number of options	537	1355	431
	$\overline{mpr}$	-.028	-.010	.043
	se($\overline{mpr}$)	.004	.007	.010
120 days	number of options	537	1355	431
	$\overline{mpr}$	-.032	-.059	.054
	se($\overline{mpr}$)	.004	.007	.008
60 days	number of options	537	1355	431
	$\overline{mpr}$	-.018	-.072	.045
	se($\overline{mpr}$)	.005	.008	.009
30 days	number of options	537	1355	431
	$\overline{mpr}$	-.021	-.086	.043
	se($\overline{mpr}$)	.007	.010	.010

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FOOTNOTES

1. The fact that model performance can be improved by robust statistics does not rule out the existence of a better pricing model.

2. Alternate approaches to robust theory based on likelihood ratio tests or on influence functions offer additional means of relating this theory to general statistics.

3. This example is due to Tukey [1960].

4. Market makers have informed us that it is not profitable to attempt to arbitrage differences which are less than $0.10, on average.