COMMITTEE ON ECONOMETRICS AND MATHEMATICAL ECONOMICS

Working Paper IP-182

THREE NOTES ON EQUILIBRIUM PRICES AND THE EQUILIBRIUM PRICE CORRESPONDENCE IN PURE EXCHANGE ECONOMIES*

Andreu Mas-Colell

May 1973

*Research for this paper has been supported by NSF Grants GS-35890X and GS-3274. Thanks are due to Professor G. Debreu for putting me on the track of discovering a faulty formulation of some hypotheses. Of course, he is not responsible for their final form. These notes gather and develop some of the results originally in "On the Uniqueness of Competitive Equilibrium and the Jacobian Condition" and "Local Uniqueness and Global Properties of Equilibria" (Minneapolis, May 1972).

INSTITUTE OF BUSINESS AND ECONOMIC RESEARCH

CENTER FOR RESEARCH IN MANAGEMENT SCIENCE

UNIVERSITY OF CALIFORNIA, BERKELEY

The common feature of the present notes is that they are concerned with the equilibrium prices of pure exchange smooth economies; differentiability is the key technical concept. They have two main themes: (i) the search of conditions for the uniqueness of equilibrium in terms of the sign of the Jacobian determinant of the excess demand functions; and (ii) the study of continuity properties in the large of the equilibrium correspondence.

Note I investigates the first question. A very general result is given. It is motivated by the work of K. Arrow and F. Hahn [2], I. Pierce and J. Wise [11], and E. Dierker [7].

Notes II and III, which should be read together, introduce initial endowments explicitly and pose problems of the following type: how does the equilibrium correspondence (i.e., the correspondence which assigns to every initial endowment allocation its equilibrium price vectors) behave over paths connecting any two arbitrary initial endowments configurations? Which connectedness properties does it have? As a conclusion one could say that the results obtained, as well as the obtainable ones, are very limited, indeed.

Admittedly, paths in the initial endowment space lack interpretation in the Walrasian general equilibrium framework in which the problem is discussed. The point is, however, that instructive conclusions can be drawn from the weak nature of the results arrived at. One feels that if this analysis cannot be pursued very far in the simplest of the models, then pessimism is called for about its possibilities in the temporary equilibrium context, where it properly belongs.

Note I*

Excess demand functions are naturally endowed with boundary conditions. Can one exploit this fact to obtain uniqueness of equilibrium theorems in terms of the sign of the Jacobian determinant of those functions? This problem has recently been studied by I. Pierce and J. Wise [11] and by K. Arrow and F. Hahn [2]. In this note we give a very general result along this line. In essence, it amounts to no more than a reformulation of a theorem of E. Dierker [7], but since our boundary condition and general set-up are slightly different, we provide a proof.

In substance the result says that if the Jacobian determinant has uniform nonzero sign on the equilibrium price set, then this set is a singleton. Focusing attention on the sign over the equilibrium price set, rather than the whole price domain, has the decisive advantage that, as it should be, the conclusion is independent of the normalization procedure.

We state the proposition for excess demand functions in which a good plays the role of numeraire. Besides following tradition we proceed in this manner because this case is of interest in itself (sometimes a numeraire is naturally distinguished) and because, as it will be justified, there is no loss of generality in doing so.

The commodity space is $\mathbb{R}^{\ell}$.[1] The ℓ-th commodity is a numeraire.

---

*I am indebted to Y. Younès for comments and encouragement.

[1]The following notation shall be maintained throughout the paper: subscripts denote vectors (regarded as columns), superscripts components of vectors; $x >> y$ means $x^i > y^i$ for every i, $x > y$ means $x^i \geq y^i$ for every i and $x \neq y$; $x \geq y$ means $x > y$ or $x = y$; x' is the transpose of x; $\mathbb{R}^n_{++} = \{x \in \mathbb{R}^n: x >> 0\}$; $xy = \sum_i x^i y^i$, $x, y \in \mathbb{R}^n$. $Dg(x)$ denotes the derivative map of $g: A \to \mathbb{R}^m$, $A \subset \mathbb{R}^n$ (assumed to be continuously differentiable). Sequences are denoted by $< >$. The boundary of $A \subset \mathbb{R}^n$ is ∂A. $B_\varepsilon(x) = \{x' \in \mathbb{R}^n: \|x - x'\| < \varepsilon\}$, $x \in \mathbb{R}^n$; $I = [0,1]$.

Let $f: \mathbb{R}^{\ell-1}_{++} \to \mathbb{R}^{\ell-1}$ be a C^1 (excess demand) function satisfying:

(1.1) There is a real number k such that, for every $1 \leq i \leq \ell - 1$ and $p \in \mathbb{R}^{\ell-1}_{++}$, $k \leq f^i(p)$ and $-pf(p) \geq k$.

Let $E_f = \{p \in \mathbb{R}^{\ell-1}_{++}: f(p) = 0\}$; f is <u>regular</u> if $|Df(p)| \neq 0$ for every $p \in E_f$. We state a (rather weak) boundary condition which, roughly, means that every commodity is desirable:

(1.2) (i) If $\langle p_n \rangle \to p \in \partial\mathbb{R}^{\ell-1}_{++}$, $p_n \in \mathbb{R}^{\ell-1}_{++}$, then there is N such that, whenever $n > N$, $f^i(p_n) > 0$ for some i with $p^i = 0$.

(ii) If $\langle \|p_n\| \rangle \to \infty$ and $\langle \|f(p_n)\| \rangle \to 0$, then there is N such that, whenever $n > N$, $f^i(p_n) < 0$ for some i.

Note that in (i) and (ii) above i may depend on n.

<u>Proposition 1</u>. Let f be regular and satisfy (1.2). If $p_1, p_2 \in E_f$ *imply* sign $|Df(p_1)|$ = sign $|Df(p_2)|$ *then* E_f *is a one-element set.*

<u>Remark 1</u>. Of course, if f is regular, the stated condition is a necessary one. Note that Proposition 1 also gives an existence result.

We now justify our claim that normalization matters are irrelevant. Suppose that f derives from a nonnormalized C^1 excess demand function $F: \mathbb{R}^{\ell}_{++} \to \mathbb{R}^{\ell}$ satisfying:

(1.3) $PF(P) = 0$ and $F(\lambda P) = F(P)$ for every $P \in \mathbb{R}^{\ell}_{+}$ and $\lambda > 0$;

(1.4) There is $r \in \mathbb{R}^{\ell}$ such that $F(P) \gg r$ for every $P \in \mathbb{R}^{\ell}_{+}$;

(1.5) If $\langle P_n \rangle \to P \in \partial\mathbb{R}^{\ell}_{++} \sim \{0\}$, then $\langle \|F(P_n)\| \rangle \to \infty$.

The last condition ((1.5)), with (1.3) and (1.4), is equivalent to the desirability hypothesis used by G. Debreu [5] and by K. Arrow and F. Hahn [1, p. 31]. The latter authors prove (Theorem 4.8) that it is always satisfied by excess demand functions generated by consumers with strictly positive initial endowments and convex, strictly monotone preferences. We have:

(1.6) f satisfies (1.2).

Proof. If $p \in \mathbb{R}^{\ell-1}_{++}$, let $P = (p,1) \in \mathbb{R}^{\ell}_{++}$. Obviously, by (1.4) and (1.5), (1.2(i)) holds. Let $<\|p_n\|> \to \infty$, $<\|f(p_n)\|> \to 0$; then $<\frac{P_n}{\|P_n\|}> \to \partial\mathbb{R}^{\ell}_{++} \sim \{0\}$ which implies $<\|F(P_n)\|> \to \infty$, or $<|F^{\ell}(P_n)|> \to \infty$; by (1.4), $<F^{\ell}(P_n)> \to +\infty$. Since $P_n F(P_n) = 0$, there is $N > 0$ such that if $n > N$, then $f^i(p_n) = F^i(P_n) < 0$ for some $0 \leq i \leq \ell - 1$. This ends the proof of (1.6).

Let α: $\mathbb{R}^{\ell}_{++} \to R$ be an arbitrary C^1 function such that $\alpha^{-1}(1) \neq \phi$ and $D\alpha(P) >> 0$ for every $P \in \mathbb{R}^{\ell}_{++}$. Define the C^1 manifold $A_\alpha = \alpha^{-1}(1)$ and, for $P \in A_\alpha$, let $\pi_{T_P(A_\alpha)}$ denote the perpendicular projection map of $\mathbb{R}^{\ell}$ on $T_P(A_\alpha)$ the tangent space of A_α at P. Define F_α: $A_\alpha \to \mathbb{R}^{\ell}$ by $F_\alpha(P) = \pi_{T_P(A_\alpha)}F(P)$. Note that F_α is a vector field on A_α i.e., $F_\alpha(P) \in T_P(A_\alpha)$ for every P. Therefore, for every $P \in A_\alpha$, the determinant $|DF_\alpha(P)|$ of the linear transformation $DF_\alpha(P)$: $T_P(A_\alpha) \to T_P(A_\alpha)$ is well defined. Every (smooth) normalization procedure can be expressed in this manner. In particular, the one used by I. Pierce and J. Wise [11] corresponds to the case where α is linear with strictly positive coefficient vector and the one in Proposition 1 above to the case where α is linear with coefficient vector $(0,\ldots,0,1)$. We say that F is regular and sign invariant if for some α with the above prescribed properties one has sign $|DF_\alpha(P_1)|$ = sign $|DF_\alpha(P_2)| \neq 0$ whenever $P_1, P_2 \in A_\alpha$ and $F_\alpha(P_1) = F_\alpha(P_2) = 0$. Then:

(1.7) Regularity and sign invariance of F are well defined, i.e., independent of the α chosen.

Proof. It is immediately seen that if this were not true the following situation would necessarily arise: for some α with the prescribed properties and $P \in A_\alpha$ such that $F_\alpha(P) = 0$, one has $|DF_\alpha(P)| = 0$, i.e., $\pi_{T_P(A_\alpha)}DF(P)v = 0$ for some $v \neq 0$ such that $D\alpha(P)v = 0$. Therefore, either $DF(P)v \geq 0$ or $DF(P)v \leq 0$. Differentiating Walras' identity, $F(P) + PDF(P) = PDF(P) = 0$; hence $DF(P)v = 0$ and, since v is linearly independent of P, rk $DF(P) < \ell - 1$ which implies $|DF_{\alpha'}(P)| = 0$ for any admissible α', a contradiction.

Therefore, by Proposition 1, (1.6), and (1.7), if F satisfies (1.3)-(1.5) and is regular and sign invariant, then, up to scalar multiplication, an equilibrium vector is uniquely determined.

Proposition 1 includes, as special cases, the analogous theorems in K. Arrow and F. Hahn [2] and in I. Pierce and J. Wise [11]. The boundary condition we use is weaker and more natural than the one postulated in [11] (see K. Arrow and F. Hahn [2, Ch. 2, p. 29] and H. Nikaido [10, Ch. 6, p. 324] for discussion and examples on boundary conditions for excess demand functions); Theorem 3 in [11] and 15 in [2, p. 236] follow the roundabout (and nonindependent of normalization) method of establishing the invertibility of the whole excess demand function; Theorem 14 in [2] is closer in spirit to Proposition 1 (or, rather, vice versa) but the condition given there involves the signs of the minors of the Jacobian matrix.

Remark 2. For later reference we point out that it is proved that if f is regular and satisifes (1.2), then $\sum_{p \in E_f} \text{sign } |Df(p)| = (-1)^{\ell-1}$.

Proof of Proposition 1. By (1.2) E_f is compact. Let $\|p\| < s$ for every $p \in E_f$. Suppose there was a sequence $\langle p_n \rangle$ such that $\langle \|p_n\| \rangle \to \infty$ and $\frac{f(p_n)}{\|f(p_n)\|} = \frac{p_n}{\|p_n\|}$ for every n. Then, for every n, $\|p_n\| \|f(p_n)\| = p_n f(p_n) < -k$ (by (1.1)). Hence $\lim \|f(p_n)\| = 0$ which, by (1.2 (ii)) would imply p_M^i for some i and M, a contradiction. Therefore there is $r > s$ such that if $\|p\| < r$, $p \in \mathbb{R}_{++}^{\ell-1}$, then $\frac{f(p)}{\|f(p)\|} \neq \frac{p}{\|p\|}$. An analogous argument (using (1.2 (i))) permits us now to conclude that there are $\bar{p} \in \mathbb{R}_{++}^{\ell-1}$ and $\varepsilon > 0$ such that, defining $L = \{p \in \mathbb{R}_{++}^{\ell-1}: p^i \geq \varepsilon \text{ for all } i \text{ and } \|p\| < r\}$ and $g: L \to \mathbb{R}^{\ell-1}$ by $g(p) = p - \bar{p}$, one has $\bar{p} \in \text{Int.}L$, $E_f \subset L$ and $\frac{g(p)}{\|g(p)\|} \neq \frac{f(p)}{\|f(p)\|}$ for every $p \in \partial L$. This means that we can find a C^1 function $\hat{f}(p): L \to R_+^{\ell-1}$ such that $\hat{f}$ equals f in a neighborhood of E_f, $\hat{f}(p) = 0$ only if $p \in E_f$, and $\hat{f}|\partial L = g|\partial L$ (take $\hat{f}(p) = \gamma(p)f(p) + (1 - \gamma(p))g(p)$ for an appropriate $\gamma: L \to [0,1]$).

The proof is concluded. Since $\hat{f}|\partial L = g|\partial L$, $\hat{f}$ and $\hat{g}$ have the same algebraic number of fixed points, i.e., $\sum_{p \in E_f} \text{sign } |D\hat{f}(p)| = \text{sign } |Dg(\bar{p})|$ (see P. Alexandrov [1, Theorem 2.22, vol. 3, p. 135] and J. Milnor [8, Ch. 6]). Hence, using the hypothesis, $\#(E_f) = \left| \sum_{p \in E_f} \text{sign}|Df(p)| \right| = |\text{sign } |Dg(\bar{p})|| = 1$.

Note II

In this note we introduce initial endowments explicitly and study the configuration of equilibrium prices over arbitrarily separated initial endowments patterns. We deal with a C^2 excess demand function $F\colon \mathbb{R}^{\ell}_{++} \times \mathbb{R}^{\ell N}_{++} \to \mathbb{R}^{\ell}$ satisfying

(2.1) $PF(P,\omega) = 0$ for every $(P,\omega) \in \mathbb{R}^{\ell}_{++} \times \mathbb{R}^{\ell N}_{++}$;

(2.2) $F(\lambda P,\omega) = F(P,\omega)$ for every $(P,\omega) \in \mathbb{R}^{\ell}_{++} \times \mathbb{R}^{\ell N}_{++}$ and $\lambda \in \mathbb{R}_{++}$:

(2.3) If $<P_n,\omega_n> \to (P_0,\omega) \in \partial\mathbb{R}^{\ell}_{++} \sim \{0\} \times \mathbb{R}^{\ell}_{++}$, then $<\|F(P_n,\omega_n)\|> \to \infty$.

Let $f\colon \mathbb{R}^{\ell-1}_{++} \times \mathbb{R}^{\ell N}_{++} \to \mathbb{R}^{\ell-1}$ be the function derived from F by taking the ℓ-th good as numeraire. Define the equilibrium correspondence $W\colon \mathbb{R}^{\ell N}_{++} \to \mathbb{R}^{\ell-1}_{++}$ by $W(\omega) = \{p \in \mathbb{R}^{\ell-1}_{++}\colon f(p,\omega) = 0\}$. The <u>critical set of endowments</u> is $C = \{\omega' \in \mathbb{R}^{\ell N}_{+}\colon$ for some $p' \in W(\omega')$, $|D_p f(p',\omega')| = 0\}$.

It is known (see G. Debreu [5]) that W behaves well locally. Namely, for almost every initial endowment ω there is a neighborhood $\mathcal{O} \subset R^{\ell N}_{+}$ of ω and $q(<\infty)$ C^1 1-1 functions $\varphi_j\colon \mathcal{O} \to \mathbb{R}^{\ell-1}_{++}$ such that $\bigcup_{j=1}^{q} f_j(\mathcal{O}) = W(\mathcal{O})$ and $\varphi_j(\mathcal{O}) \cap \varphi_i(\mathcal{O}) = \phi$ if $i \neq j$. One refers to this property as "stability" of W on $\mathcal{O}$. It implies that if $\nu\colon I \to \mathcal{O}$ is continuous then a $p_1 \in W(\nu(1))$ is uniquely determined by a choice of $p_0 \in W(\nu(0))$ and a continuity requirement (i.e., that prices do not jump). It is simple to see that this does not necessarily hold for arbitrary displacements of the initial ω. The problem is that although the set of exceptional economies (i.e., C) has Lebesgue measure zero it is still a highly troublesome one. The purpose of this note is to investigate which, however weak, global properties the correspondence W may still have.

Suppose that the economy is initially at ω_0 and that, being free to choose $p_0 \in W(\omega_0)$, we want to reach in a continuous manner (and staying in equilibrium) some predetermined ω_1 and $p_1 \in W(\omega_1)$ (for example, the pair ω_1, p_1 may maximize a given Social Welfare Function); this is possible because, under very general conditions, the graph of W is arcwise connected (for a formal proof see P. Balasko [3]). However, it is realistic to assume that not every initial endowment path is allowable and it is, therefore, of interest to know if analogous results hold when constraints are imposed.

For the case in which $W(\omega_1)$ is unique an answer is embodied in:

<u>*Proposition 2.*</u> *Let* $A \subset \mathbb{R}^{\ell N}_{++}$ *be an open, arcwise connected set and* $\omega_0, \omega_1 \in A \sim C$. *Then there are continuous functions* $\omega(\cdot): I \to A$, $p(\cdot): I \to \mathbb{R}^{\ell-1}_{++}$ *such that* $\omega(0) = \omega_0$, $\omega(1) = \omega_1$ *and* $p \in W(\omega(t))$ *for every* t.

<u>*Remark 3.*</u> It is always true that, under the hypothesis made, for every continuous function $\underline{\omega}: I \to \mathbb{R}^{\ell N}_{++}$ the graph of $W|\,\underline{\omega}$ has a component intersecting $\mathbb{R}^{\ell N}_{++} \times \{0\}$ and $\mathbb{R}^{\ell N}_{++} \times \{1\}$ (F. Browder [4, Theorem 2]). What we prove is that for "almost every" path in the initial endowment space this component is an embedding of I.

<u>*Remark 4.*</u> The result in the Proposition is a weak one. In particular, one would like to predetermine the initial endowment path which is not possible in general. Another limitation is the following: once the path of initial endowments and p_0 have been picked the requirement that the equilibrium price path be continuous does not uniquely determine it. Without "central intervention" the economy may "derail" when crossing the critical set C.

Proof of Proposition 2. Let $\mathcal{A}$ be the set of C^1 functions $\alpha\colon I \to R^{\ell N}_{++}$ topologized by the C^1 norm: $\|\alpha - \alpha'\|_1 = \sup_t \|\alpha(t) - \alpha'(t)\| + \sup_t \|D\alpha(t) - D\alpha'(t)\|$.

Let $\mathcal{B}$ be the subset of $\mathcal{A}$ formed by the elements α for which we can find $\alpha' \in \mathcal{A}$ and a continuous $p(\cdot)\colon I \to R^{\ell-1}_{++}$ with $\alpha'(I) = \alpha(I)$ and $p(t) = W(\alpha'(t))$ for every t.

For every $\alpha \in \mathcal{A}$ define $f_\alpha\colon R^{\ell-1}_{++} \times I \to \mathbb{R}^{\ell-1}$ by $f_\alpha(p,t) = f(p,\alpha(t))$. Let $\hat{\mathcal{B}} = \{\alpha \in \mathcal{B}\colon 0$ is a regular value of $f_\alpha\}$; $\hat{\hat{\mathcal{B}}} = \{\alpha \in \hat{\mathcal{B}}\colon \alpha(0),\alpha(1) \notin C\}$.

(2.4) $\hat{\hat{\mathcal{B}}}$ is a dense subset of $\mathcal{A}$.

Proof. We prove that $\hat{\mathcal{B}}$ is dense. The proof for $\hat{\hat{\mathcal{B}}}$ is completely analogous.

Let $\delta > 0$ and $\alpha \in \mathcal{A}$ be given. Since any α can be C^1 approximated by a C^2 function (see J. Munkres [9, p. 39]) we can assume that α is C^2. For every $c \in \mathbb{R}^\ell_{++}$ let $c' = (c,0,\ldots,0) \in \mathbb{R}^{\ell N}_{++}$. Define $\bar{\alpha}\colon \mathbb{R}^\ell_{++} \times I \to \mathbb{R}^{\ell N}_{++}$ by $\bar{\alpha}(c,t) = \alpha(t) + c'$; $\alpha_c = \bar{\alpha}(\cdot,c)$. Clearly, $\alpha_c \in \mathcal{A}$ and for c' small enough $\|\alpha - \alpha_{c'}\|_1 < \delta$.

Define $G\colon \mathbb{R}^{\ell-1}_{++} \times \mathbb{R}^\ell_{++} \times I \to \mathbb{R}^{\ell-1}$ by $G(p,c,t) = f(p,\bar{\alpha}(c,t))$. It is easily seen (see, for example, E. Dierker and H. Dierker [6]) that G is a regular map. Therefore $G^{-1}(0)$ is a C^2 $(\ell+1)$-manifold and we have, by Sard's Theorem (see J. Milnor [8, p. 16]) that for almost every $c \in R^\ell_{++}$, 0 is a regular value of f_{α_c}.[2] Hence $\hat{\mathcal{B}}$ is a dense subset of $\mathcal{A}$.

---

[2]More precisely, Sard's Theorem asserts that almost every $c \in R^\ell_+$ is a regular value of the projection map: $\tau\colon G^{-1}(0) \to R^\ell_+$, $\tau(p,c,t) = c$. But c is a regular value of τ if and only if 0 is a regular value of $f(\cdot,c,\cdot)$, i.e., of f_{α_c}.

Remark. $\hat{\hat{\mathscr{B}}}$ is also open.

(2.5) $\hat{\hat{\hat{\mathscr{B}}}} \subset \mathscr{B}$.

Proof. Let $\alpha \in \hat{\hat{\hat{\mathscr{B}}}}$. Then $f_\alpha^{-1}(0) \subset \mathbb{R}_{++}^{\ell-1} \times I$ is a C^1 manifold so that (J. Milnor [8, p. 13])

$$\partial f_\alpha^{-1}(0) = \partial(\mathbb{R}_{++}^{\ell-1} \times I) \cap f_\alpha^{-1}(0) =$$

$$= (f_\alpha^{-1}(0) \cap (\mathbb{R}_{++}^{\ell-1} \times \{0\})) \cup (f_\alpha^{-1}(0) \cap (\mathbb{R}_{++}^{\ell-1} \times \{1\})).$$

By (2.3), $f^{-1}(0)$ is compact. Therefore (see J. Milnor [8, Appendix 1]) it is a finite disjoint union of segments and circles, i.e., $f_\alpha^{-1}(0) = \bigcup_{i=1}^{k} I_i \cup \bigcup_{i=k+1}^{q} U_i$, where the I_i's and U_i's are diffeomorphic images of I and the unit circle, respectively.

Let (p_i^0,t_i^0), (p_i^1,t_i^1), $i = 1,\dots,k$, designate the end points of the segments I_i. Since $(p_i^h,t_i^h) \in \partial f_\alpha^{-1}(0)$, $i = 1,\dots,k$, $h = 1,2$, it follows that $t_i^h = 0$ or $t_i^h = 1$. Therefore $p_i^h \in W(\alpha(0))$ or $p_i^h \in W(\alpha(1))$. By hypothesis, $\alpha(0) \notin C$, which implies that if $p \in W(\alpha(0))$, then $(p,0) = (p_i^h,0)$ for some $1 \leq i \leq k$, $h = 0,1$. Since $\#W(\alpha(0))$ is odd (see Note I, Remark 2) we have that $k > 0$ and for some $0 \leq i' \leq k$, $p_{i'}^0$, $p_{i'}^1$ do not belong to the same of the sets $W(\alpha(0))$, $W(\alpha(1))$. Suppose $p_{i'}^0 \in W(\alpha(0))$ and $p_{i'}^1 \in W(\alpha(1))$.

The proof is completed for if $g\colon I \to I_{i'}$ is a diffeomorphism and π_1, π_2 the projections of $R_{++}^{\ell-1} \times I$ onto $R_{+}^{\ell-1}$ and I respectively, we have that $\hat{\alpha} = \mu_2 \circ g$ and $p(\cdot) = \mu_1 \circ g(\cdot)$ are as desired.

To conclude the proof of the proposition let $\mathcal{O}_1, \mathcal{O}_2 \subset A$ be such that $\omega_0 \in \mathcal{O}_1$, $\omega_1 \in \mathcal{O}_2$, $\mathcal{O}_1 \cup \mathcal{O}_2 \subset \sim C$. Then by (2.4) and (2.5) we can find $\alpha \in \mathscr{B}$ with $\alpha(0) \in \mathcal{O}_1$, $\alpha(1) \in \mathcal{O}_2$ and $\alpha(I) \subset A$; the proposition follows, then, by the definition of $\mathscr{B}$ and the stability of W on $\mathcal{O}_1$ and $\mathcal{O}_2$ (see G. Debreu [5]).

Note III

In this note the definition and assumptions of the previous note are maintained. We discuss the following problem (see Remark 4 of Note II for motivation):

(3.1) Let $\omega_0, \omega_1 \in \mathbb{R}^{\ell N}_{++}$ be given. Find continuous functions $\bar{\omega}: I \to \mathbb{R}^{\ell N}_{+}$, $\bar{p}: I \to \mathbb{R}^{\ell-1}_{++}$ such that:

(i) $\bar{\omega}(0) = \omega_0$, $\bar{\omega}(1) = \omega_1$;

(ii) $\bar{p}(t) \in W(\bar{\omega}(t))$, $t \in I$;

(iii) There is $\varepsilon > 0$ such that, for every $0 \leq t \leq 1$, if $p' \in W(\bar{\omega}(t))$ and $\|p' - p(t)\| < \varepsilon$ then $p' = \bar{p}(t)$.

In contrast to Note II, here no constraints are imposed on the initial endowment path, but strong demands are made on the equilibrium price one.

We conjecture that the solvability, in general, of problem (3.1) is not guaranteed by our assumptions. A counterexample is lacking. In the sequel we pursue a line of attack which is natural and yields some (very weak) results.

Let $f_i: \mathbb{R}^{\ell-1}_{++} \times \mathbb{R}^{\ell}_{++} \to \mathbb{R}^{\ell-1}$ denote the excess demand function of the i-th consumer and define $\bar{f}: \mathbb{R}^{\ell-1}_{++} \times \mathbb{R}^{\ell N}_{++} \to \mathbb{R}^{\ell N}$ by $\bar{f}(p,\omega) = (f_1(p,\omega_1),\dots,f_N(p,\omega_N))$. We assume:

(3.2) Every f_i is continuously differentiable and is originated by maximization of a preference relation.

For every $\omega \in \mathbb{R}^{\ell N}_{++}$, $p \in W(\omega)$ and $t \in I$ let $\omega_{p,t} = t(\omega + \bar{f}(p,\omega)) + (1 - t)\omega$. Note that $p \in W(\omega_{p,t})$ for all t. For every i, $\omega_i \in \mathbb{R}^{\ell}_{++}$, $p \in \mathbb{R}^{\ell-1}_{++}$, the $(\ell-1)\times(\ell-1)$ substitution term matrix and the income gradient vector at

$(p, \sum_{j=1}^{\ell-1} p^j \omega_i^j + \omega_i^\ell)$ of the i-th consumer demand function are denoted by $S_{\omega_i}(p)$, $b_i(p,\omega_i)$ respectively. Let $S_\omega(p) = \sum_{i=1}^{N} S_{\omega_i}(p)$. Thus:

(3.3) For every $\omega \in R_+^{\ell N}$, $p \in W(\omega)$:

$$D_p f(p,\omega_{p,t}) = S_\omega(p) - (1 - t) \sum_{i=1}^{N} f_i(p,\omega_i) b_i'(p,\omega_i)$$

$$= (1 - t) D_p f(p,\omega_{p,0}) + t D_p f(p,\omega_{p,1}).$$

Therefore $|D_p f(p,\omega_{p,t})|$ is an ℓ-th degree polynomial in t.

Under reasonable assumptions (namely, no "corners" on the indifference curves of at least one individual) problem (3.1) would have a solution if for every $\omega \in R_{++}^{\ell N}$ there was $p \in W(\omega)$ such that

(3.4) For some $\varepsilon > 0$ and $t \in I$, if $p' \in W(\omega_{p,t})$ and $\|p' - p\| \leq \varepsilon$ then $p' = p$.

Clearly, if given $\omega \in R_{++}^{\ell N}$, we could find a $p \in R_{++}^{\ell-1}$ satisfying:

(3.5) $p \in W(\omega)$, $|D_p f(p,\omega_{p,t})| \neq 0$ for every $t \in I$,
then (3.4) would also hold. In fact, (3.4) and (3.5) are almost equivalent:

(3.6) If $p \in W(\omega)$ and there are $t', t'' \in I$ such that $|D_p f(p,\omega_{p,t'})|$, $|D_p f(p,\omega_{p,t''})|$ have opposite signs then (3.4) does not hold.

Proof. Since $|D_p f(p,\omega_{p,t})|$ is a nonconstant polynomial in t, there is $0 < \bar{t} < 1$, $\delta > 0$ such that if $\bar{t} - \delta \leq t' < \bar{t}$, $\bar{t} < t'' \leq \bar{t} + \delta$, then $|D_p f(p,\omega_{p,t'})|$, $|D_p f(p,\omega_{p,t''})|$ have opposite signs. Suppose that (3.4) holds, then a straightforward argument yields the existence of $0 < \delta' < \delta$, $\gamma > 0$ and two maps $g', g'': \mathbb{R}^{\ell-1}_{++} \to \mathbb{R}^{\ell-1}$ such that $g' | \partial B_\gamma(p) = g'' | \partial B_\gamma(p) = f | \partial B_\gamma(p)$, $g' | B_{\gamma/2}(p) = f(\cdot, \omega_{p,\bar{t}-\delta}) | B_{\gamma/2}(p)$, $g'' | B_{\gamma/2}(p) = f(\cdot, \omega_{p,\bar{t}+\delta}) | B_{\gamma/2}(p)$, $f(p) \neq 0$ for every $p \in \partial B_\gamma(p)$ and p is the only zero of g' or g'' on $B_\gamma(p)$.

As in the proof of Proposition 1 this should imply $\operatorname{sign}|Df(p,\omega_{p,\bar{t}+\delta'})| = \operatorname{sign}|Dg''(p)| = \operatorname{sign}|Dg'(p)| = \operatorname{sign}|Df(p,\omega_{p,\bar{t}-\delta'})|$, a contradiction. Therefore (3.4) does not hold.

So, one is led to focus attention on condition (3.5). We know (see Note I, Remark 2) that if for $\omega \in \mathbb{R}^{\ell N}_{++}$, $f(\cdot,\omega)$ is regular, then there is a $p \in W(\omega)$ such that $\operatorname{sign}|D_p f(p,\omega)| = (-1)^{\ell-1}$. It is clear, then, that this is a necessary condition for the existence of a p satisfying (3.5) (if $|D_p f(p,\omega_{p,1})| \neq 0$ then by (3.2) $W(\omega_{p,1})$ is a singleton, so $\operatorname{sign}|D_p f(p,\omega_{p,1})| = (-1)^{\ell-1}$ and therefore, by (3.5), $\operatorname{sign}|D_p f(p,\omega)| = (-1)^{\ell-1}$). But it is far from sufficient; the following simple example shows that even if $D_p f(p,\omega)$ is a stable matrix, (3.5) may fail.

Example. Let $N = 3$, $\ell = 3$. Initial endowments are $\omega_1 = (0,1,2)$, $\omega_2 = (1,30,0)$, $\omega_3 = (2,0,30)$. At the equilibrium price vector $P = (1,1,1)$ excess demands are, respectively, $(1.2,0,-1.2)$, $(0,-25.5,25.5)$, $(-1.2,25.5,-24.3)$; income gradient vectors are, respectively, $(\frac{1}{3},\frac{2}{3},0)$, $(\frac{2}{3},\frac{1}{3},0)$, $(\frac{1}{3},\frac{1}{3},\frac{1}{3})$; the (nonnormalized) substitution term matrix for every consumer is

$$\frac{1}{3}\begin{bmatrix} -1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -1 \end{bmatrix}.$$

It is clear that utility functions for the three individuals can be exhibited yielding these features. We have

$$D_p f(p,\omega) = \begin{bmatrix} -1 & -1 \\ 9 & -1 \end{bmatrix},$$

a stable matrix but

$$D_p f(p,\omega_{p,1/2}) = \begin{bmatrix} -1 & .3 \\ 4.75 & -1 \end{bmatrix}$$

has a negative determinant.

Sufficient conditions for (3.5) to hold can be given in very particular (and familiar) cases.

<u>*Proposition 3*</u>. *Let* $\omega \in \mathbb{R}^{\ell N}_{++}$, $p \in W(\omega)$. *Assume that* $\omega, \omega_{p,1} \notin C$. *Then (3.5) holds if any of the following hypotheses is fulfilled:*

(i) $D_p F(P,\omega)$ *has the (weak) gross substitute sign pattern;* $P = (p,1)$;

(ii) $\text{sign}|D_p f(p,\omega)| = (-1)^{\ell-1}$ *and either* $\ell = 2$ *or* $N = 2$;

(iii) *Every preference relation is homothetic at* $\omega_i + f_i(p,\omega_i)$ *(i.e.,* $b_i(p,\omega_i) = \lambda(\omega_i + f_i(p,\omega_i))$ *for some* $\lambda > 0$*) and there is* $c \in \mathbb{R}^{\ell}_{++}$ *such that for every* i, $\omega_i = \gamma_i c$ *for some* $\gamma_i > 0$;

(iv) $b_1(p,\omega_1) = \ldots = b_N(p,\omega_N)$.

<u>*Remark 5*</u>. Except for the case N = 2, all the hypotheses imply that, under a tatônnement process, p is a stable equilibrium.

<u>*Proof of Proposition 3*</u>. Since $S_\omega(p)$ $(= Df(p,\omega_{p,1}))$ is negative definite it is clear from (3.3) that if $D_p f(p,\omega)$ is negative quasi-semidefinite (i.e., a matrix H is

negative quasi-semidefinite if H + H' is negative semidefinite) then (3.5) holds. This is the case if (i) is satisfied: let $H \equiv D_p F(P,\omega)$, then by (2.1) and (2.2) $P(H + H') = 0$, so H + H' has a quasi-dominant negative diagonal and therefore $D_p f(p,\omega)$ is a stable matrix (see K. Arrow and F. Hahn [2, 12.4]); since it is symmetric it is negative definite. Likewise, if (ii) (resp., (iv)) holds, then $\sum_{i=1}^{N} f_i(p,\omega_i) b_i'(p,\omega_i)$ equals $\sum_{i=1}^{N} f_i(p,\omega_i) f_i'(p,\omega_i)$ (resp., 0); therefore, in both cases, $D_p f(p,\omega)$ is negative semidefinite. Finally, (3.5) will hold if $\text{sign}|D_p f(p,\omega)| = (-1)^{\ell-1}$ and $D_p f(p,\omega_{p,t})$ is a linear function of t. This is the case (see (3.3)) if $\ell = 2$ or if $N = 2$ (the coefficients of the powers of t greater than 1 vanish).

REFERENCES

[1] Alexandrov: Combinatorial Topology, vol. 3, Graylock Press, 1963.

[2] Arrow, K. and F. Hahn: General Competitive Analysis, Holden-Day, San Francisco, 1971.

[3] Balasko, P.: Equilibrium Space of a Family of Economies, mimeographed, August 1972, Paris.

[4] Browder, F.: On the Continuity of Fixed Points Under Deformations of Continuous Mappings, Summa Brasiliensis Mathematicae 4 (1960).

[5] Debreu, G.: Economies with a Finite Set of Equilibria, Econometrica, 38, (1970), 387-392.

[6] Dierker, E. and H. Dierker, On the Local Uniqueness of Equilibria, Econometrica, 40, 5 (September 1972).

[7] Dierker, E.: Two Remarks on the Number of Equilibria of an Economy, Econometrica, 40, 5 (September 1972).

[8] Milnor, J.: Topology from the Differentiable Viewpoint, 1965, The University Press of Virginia, Charlottesville.

[9] Munkres, J.: Elementary Differential Topology, Princeton University Press, Princeton, N.J., 1966.

[10] Nikaido, H.: Convex Structures and Economic Theory, 1968, Academic Press.

[11] Pierce, I. and J. Wise: On the Uniqueness of Competitive Equilibrium, to appear in Econometrica.