

**UCLA**

**UCLA Electronic Theses and Dissertations**

**Title**

Higher spin Chern-Simons theory and KdV hierarchies

**Permalink**

<https://escholarship.org/uc/item/4hx647r6>

**Author**

Li, Yi

**Publication Date**

2018

Peer reviewed|Thesis/dissertation

UNIVERSITY OF CALIFORNIA  
Los Angeles

Higher spin Chern-Simons theory and KdV hierarchies

A dissertation submitted in partial satisfaction  
of the requirements for the degree  
Doctor of Philosophy in Physics and Astronomy

by

Yi Li

2018

© Copyright by  
Yi Li  
2018

# ABSTRACT OF THE DISSERTATION

Higher spin Chern-Simons theory and KdV hierarchies

by

Yi Li

Doctor of Philosophy in Physics and Astronomy

University of California, Los Angeles, 2018

Professor Michael Gutperle, Chair

This dissertation summarizes my research in the Lifshitz higher spin Chern-Simons theory and its relation to the integrable system KdV hierarchy as a Ph.D. candidate at UCLA. In Chapter 1, I briefly review the higher spin gravity theory and introduce the Chern-Simons theory as a realization of the Vasiliev theory in three dimensional spacetime. In Chapter 2, I review the KdV hierarchies. In Chapter 3, I discuss how to construct a solution to the Chern-Simons theory which yields a spacetime that exhibits Lifshitz scaling, I also calculate the boundary charge algebra and show the asymptotic Lifshitz symmetry is realized in terms of it. In Chapter 4, I reveal the relation between the Lifshitz Chern-Simons theory and the KdV hierarchies (in the non-supersymmetric case), a proof of the general correspondence is also given using the Drinfeld-Sokolov formalism. In Chapter 5, I work out the supersymmetric extension of this correspondence in a particular case, with the boundary charge algebra of the supersymmetric Chern-Simons theory and the second Hamiltonian structure of the super KdV identified. In Chapter 6, I discuss on the results of my study and possible directions of future research.

The dissertation of Yi Li is approved.

Per Kraus

Ciprian Manolescu

Michael Gutperle, Committee Chair

University of California, Los Angeles

2018

*To my family and teachers*

# TABLE OF CONTENTS

<b>Acknowledgments</b>		<b>vii</b>
<b>Vita</b>		<b>viii</b>
<b>1 Introduction to the higher spin Chern-Simons theory</b>		<b>1</b>
1.1 Poincare symmetry and spin		1
1.2 Review of the gravity theory and Palatini formalism		2
1.3 Higher spin gravity and AdS/CFT		4
1.4 Higher spin Chern-Simons theory in three dimensional spacetime		9
1.5 Boundary charge algebra		12
<b>2 Introduction to the KdV hierarchies</b>		<b>14</b>
2.1 KdV hierarchies		14
2.2 Examples		19
2.3 Drinfeld-Sokolov reduction		22
<b>3 Lifshitz Chern-Simons theory and the asymptotic symmetry</b>		<b>24</b>
3.1 Lifshitz spacetime and Lifshitz field theories		24
3.2 Asymptotic Lifshitz solution of Chern-Simons theory		25
3.3 Boundary charge algebra of Lifshitz Chern-Simons theory		30
<b>4 Lifshitz Chern-Simons theory and KdV hierarchies</b>		<b>34</b>
4.1 Lifshitz Chern-Simons theory and KdV hierarchies identified		34
4.2 A proof by Drinfeld-Sokolov formalism		36

<b>5</b>	<b>The supersymmetric Lifshitz Chern-Simons theory and super KdV hier-</b>	
	<b>archies</b> . . . . .	<b>43</b>
5.1	Super KdV hierarchies . . . . .	43
5.2	$sl(3 2)$ , $z = 2$ Lifshitz Chern-Simons theory and its integrability . . . . .	46
<b>6</b>	<b>Discussion</b> . . . . .	<b>55</b>
<b>A</b>	<b><math>sl(3, \mathbb{R})</math>, <math>sl(4, \mathbb{R})</math> and <math>hs(\lambda)</math> conventions</b> . . . . .	<b>58</b>
A.1	$sl(3, \mathbb{R})$ . . . . .	58
A.2	$sl(4, \mathbb{R})$ . . . . .	59
A.3	$hs(\lambda)$ conventions . . . . .	60
<b>B</b>	<b>A review of <math>sl(3 2)</math></b> . . . . .	<b>62</b>
<b>C</b>	<b>Chern-Simons KdV map</b> . . . . .	<b>67</b>
<b>D</b>	<b>Proofs of statements used in the Drinfeld-Sokolov formalism</b> . . . . .	<b>79</b>
D.1	Gauge transformation of PDOs . . . . .	79
D.2	Scalar coefficient form and conserved quantities . . . . .	80
D.3	Matrices that commute with $L_0$ . . . . .	82
D.4	The Lax equation preserves gauge equivalence . . . . .	82
D.5	Equivalent evolution equations of gauge equivalent classes . . . . .	83
	<b>References</b> . . . . .	<b>84</b>



## ACKNOWLEDGMENTS

First I would like to thank my advisor Michael Gutperle for his effort and patience in guiding me in the graduate research. I also want to thank Joshua Samani, Matteo Becarria, Guido Macorini for the valuable discussion in the research. In addition, I am grateful to Eric D'Hoker, Zvi Bern, Per Kraus, Roberto Peccei, Gang Liu, Peter Petersen and Terence Tao for their excellent teaching at UCLA. As for my peers, I benefited a lot from the discussion with Allic Sivaramakrishnan, Louis Yang, Shan Jiang, Yao Wang, Ashwin Hedge, Mert Besken, Andrea Trivella, Justin Kaidi and Julio Parra Martinez. Last but not least, I want to thank the Department of Physics and Astronomy and NSF for their financial support to my graduate study.

## VITA

- 2007-2011 Undergraduate student, Peking University, China
- 2011-2018 Graduate student, University of California, Los Angeles, United States of America

## PUBLICATIONS

- “Higher spin Lifshitz theory and integrable systems” *Phys. Rev. D* **91** (2015), 046012, M. Gutperle, Y. Li
- “Higher spin Lifshitz theories and the Korteweg-de Vries hierarchy” *Phys. Rev. D* **92** (2015), 085005, M. Beccaria, M. Gutperle, Y. Li, and G. Macorini
- “Higher spin Chern-Simons theory and the super Boussinesq hierarchy” *arXiv:1709.02345 [hep-th]*, M. Gutperle, Y. Li

# CHAPTER 1

## Introduction to the higher spin Chern-Simons theory

### 1.1 Poincare symmetry and spin

As the core of modern physics, the relativistic field theory is the field theory with Poincare symmetry, which is generated by translations  $P_\alpha$  and rotations  $M_{\mu\nu}$  that satisfy the commutation relation

$$[P_\alpha, P_\beta] = 0 \tag{1.1}$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\mu\sigma}M_{\nu\rho} + \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\sigma}M_{\mu\rho}) \tag{1.2}$$

$$[M_{\mu\nu}, P_\alpha] = i(\eta_{\nu\alpha}P_\mu - \eta_{\mu\alpha}P_\nu) \tag{1.3}$$

The Poincare symmetry classifies the fields as (single or double valued) representations of the Poincare group, in particular, representations of the double cover of the Lorentz group, the spin group, labeled by the spin. After quantization of fields, the one particle states form an unitary irreducible representation of the Poincare group, with the quadratic Casimir  $P = P_\alpha P^\alpha$  and  $W = \frac{1}{2}M_{\mu\nu}M^{\mu\nu}P_\alpha P^\alpha - M_{\mu\nu}M^{\rho\nu}P^\mu P_\rho$  corresponding to the mass and spin of the particle

$$P = m^2 \tag{1.4}$$

$$W = m^2 s(s + 1) \tag{1.5}$$

Moreover, particles with integer spin are Bosons while the particles with half integer spin are Fermions by the spin-statistics theorem[1]. In the most microscopic level of physics verified by experiment with high precision, we have the well-established standard model of particle physics, which includes the Higgs Boson with spin zero, the leptons and quarks with spin

one half, and vector gauge Bosons with spin one. In the physics of the largest scale like cosmology, the gravity theory is an essential part. As a manifestation of curved spacetime, it is formulated by (pseudo) Riemannian geometry which is described by the spin two metric. The classical gravity theory is successfully described by the Einstein-Hilbert action and its variants, while a complete and consistent quantum theory is still elusive, with string theory being the most promising candidate.

## 1.2 Review of the gravity theory and Palatini formalism

In (pseudo) Riemannian geometry we start with a metric and an affine connection which defines the covariant derivative of tensors, it takes the form in the coordinate basis

$$\nabla_\mu \partial_\nu = \Gamma_{\mu\nu}^\rho \partial_\rho \quad (1.6)$$

where  $\Gamma_{\mu\nu}^\rho$ s are also called "Christoffel symbols". The torsion tensor  $T$  is defined as

$$T[X, Y] = \nabla_X Y - \nabla_Y X - [X, Y] \quad (1.7)$$

where  $X, Y$  are vector fields. It takes the form in the coordinate basis

$$T_{\mu\nu}^\rho = \Gamma_{\mu\nu}^\rho - \Gamma_{\nu\mu}^\rho \quad (1.8)$$

In the gravity theory we usually consider the Levi-Civita connection, that is, torsion free  $T = 0$  and metric compatible  $\nabla g = 0$ . The Levi-Civita connection takes the explicit form that only depends on the metric

$$\Gamma_{\mu\nu}^\rho = \frac{1}{2} g^{\rho\lambda} (\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}) \quad (1.9)$$

The curvature tensor  $R$  is defined as

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z \quad (1.10)$$

and takes the form

$$R_{\sigma\mu\nu}^\rho = \partial_\mu \Gamma_{\nu\sigma}^\rho - \partial_\nu \Gamma_{\mu\sigma}^\rho + \Gamma_{\mu\lambda}^\rho \Gamma_{\nu\sigma}^\lambda - \Gamma_{\nu\lambda}^\rho \Gamma_{\mu\sigma}^\lambda \quad (1.11)$$

in the coordinate basis. The Ricci tensor is defined as the contraction of the curvature tensor  $R_{\mu\nu} = R^{\rho}_{\mu\rho\nu}$  and the scalar curvature is defined as the contraction of the Ricci tensor  $R = R^{\rho}_{\rho}$ . The action of Einstein's gravity theory, the Einstein-Hilbert action is

$$S = \frac{1}{16\pi G} \int dV (R - 2\Lambda) + S_M \quad (1.12)$$

where  $\Lambda$  is the cosmological constant,  $dV$  is the volume form,  $S_M$  is the action of matter fields and  $G$  is the gravitational constant. When the action is viewed as a functional of the metric assuming the connection is Levi-Civita, the action principle yields the Einstein's equation

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \quad (1.13)$$

where the energy momentum tensor  $T_{\mu\nu}$  is defined as  $-2\frac{\delta S_M}{\delta g^{\mu\nu}}$ . In the Palatini formalism however, we view the action as a functional of the metric  $g_{\mu\nu}$  and the connection  $\Gamma^{\mu}_{\nu\rho}$  which is independent on the metric. Then the equation of motion by the variation of the action with respect to the connection sets the connection to be Levi-Civita, and the second equation of motion by the variation of the action with respect to the metric yields the Einstein's equation. We can also formulate the gravity in non-coordinate basis, in which the gravity manifests itself as a gauge theory. We are particularly interested in orthonormal frame (Lorentz frame) in which spinors could be defined. We introduce the vielbein as the basis of the local Lorentz frame  $e_a$  and its dual one-form  $e^a$

$$g_{\mu\nu} e_a^{\mu} e_b^{\nu} = \eta_{ab} \quad \eta_{ab} e_{\mu}^a e_{\nu}^b = g_{\mu\nu} \quad (1.14)$$

The covariant derivative of the vielbein takes the form

$$\nabla_{\mu} e_a = \omega_{\mu a}^b e^b \quad (1.15)$$

where  $\omega_{\mu b}^a$  is the spin connection, which is essentially the affine connection in a special basis. It transforms as

$$\omega_{\mu b'}^{a'} = \Lambda_a^{a'} \Lambda_{b'}^b \omega_{\mu b}^a + \Lambda_c^{a'} \partial_{\mu} \Lambda_{b'}^c \quad (1.16)$$

under local Lorentz rotation. The spin connection can also be viewed as a one-form  $\omega_a^b = \omega_{\mu a}^b dx^\mu$ . The torsion and curvature form are defined as

$$T^a = de^a + \omega_b^a \wedge e^b \quad (1.17)$$

$$R_b^a = d\omega_b^a + \omega_c^a \wedge \omega_b^c \quad (1.18)$$

and are related to the torsion and curvature in the coordinate basis in the simple way  $T_{\mu\nu}^a = e_\rho^a T_{\mu\nu}^\rho$   $R_{b\mu\nu}^a = e_\rho^a e_b^\lambda R_{\lambda\mu\nu}^\rho$ . We can also define the generalized exterior differential  $D$  for tensor-valued forms  $\Omega$  as

$$D\Omega(X_1, \dots, X_{n+1}) = \sum_i \nabla_{X_i} \Omega(X_1, \dots, \hat{X}_i, \dots, X_{n+1}) - \sum_{i < j} \Omega([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{n+1}) \quad (1.19)$$

Then the torsion takes the simple form  $T = De$ . In addition,  $DT = 0$  and  $DR = 0$  represents the first and the second Bianchi identity for the curvature tensor, respectively. The Einstein-Hilbert action in D-dimensional spacetime can also be rewritten in terms of the vielbein and the spin connection

$$S_G = \frac{1}{16\pi G} \int \epsilon_{a_1 \dots a_D} R^{a_1 a_2} \wedge e^{a_3} \wedge \dots \wedge e^{a_D} - 2\Lambda e^1 \wedge \dots \wedge e^D \quad (1.20)$$

and the Palatini formalism remains working, that is, we get Levi-Civita connection if we set the variation of the action with respect to the spin connection to zero, then we can get Einstein's equation by setting the variation of the action with respect to the vielbein to zero.

### 1.3 Higher spin gravity and AdS/CFT

Fields of spin higher than two are called higher spin fields. The study of free higher spin fields was initiated by Fierz and Pauli[2]. A little bit later Wigner classified the irreducible unitary representations of the Poincare group in four dimensional spacetime [3](which was generalized to spacetime with arbitrary dimension in [4]) and proposed with Bargmann the dynamic equation of free massive higher spin fields. The spin  $j$  fields are realized by  $2j$ -fold symmetric tensor product of Dirac spinors  $\psi_{\alpha_1 \dots \alpha_{2j}}$  and the dynamic equation, known as the

Bargmann-Wigner equation, reads

$$(i\gamma^\mu \partial_\mu - m)_{\alpha_r \alpha_{r'}} \psi_{\alpha_1 \dots \alpha_{r'} \dots \alpha_{2j}} = 0 \quad (1.21)$$

that is, a Dirac equation for each Dirac spinor component in the tensor product. The equation can be recasted in terms of symmetric tensors in the case of integer spin, and symmetric tensor spinors in the case of half integer spin, with transeverse and trace condition to ensure the irreducibility of the representation

$$(\partial^2 - m^2)\Phi_{\mu_1 \dots \mu_s} = 0 \quad (1.22)$$

$$\partial^{\mu_1} \Phi_{\mu_1 \dots \mu_s} = 0$$

$$\eta^{\mu_1 \mu_2} \Phi_{\mu_1 \mu_2 \dots \mu_s} = 0$$

$$(i\gamma^\mu \partial_\mu - m)\Psi_{\mu_1 \dots \mu_s} = 0 \quad (1.23)$$

$$\partial^{\mu_1} \Phi_{\mu_1 \dots \mu_s} = 0$$

$$\gamma^{\mu_1} \Phi_{\mu_1 \mu_2 \dots \mu_s} = 0$$

In the case of spin  $0, \frac{1}{2}, 1, \frac{3}{2}$ , one can verify that the Klein-Gordon, Dirac, Proca and Rarita-Schwinger (in the massless limit) equations are recovered. A Lagrangian formalism of the massive free higher spin fields was, however, absent for decades until the work by Singh and Hagen [6, 7], in which auxiliary fields were introduced to impose the constraints. A few years later Fronsdal and Fang studied the massless limits[8, 9], where auxiliary fields can be absorbed by field redefinition and higher spin gauge symmetries emerge. The Fronsdal equation for massless fields with integer spin  $s$  is

$$F_{\mu_1 \dots \mu_s} \equiv \partial^2 \Phi_{\mu_1 \dots \mu_s} - \sum_{\sigma} \partial_{\sigma(\mu_1)} \partial^\rho \Phi_{\rho\sigma(\mu_2) \dots \sigma(\mu_s)} + \sum_{\sigma} \partial_{\sigma(\mu_1)} \partial_{\sigma(\mu_2)} \Phi_{\rho\sigma(\mu_3) \dots \sigma(\mu_s)} = 0 \quad (1.24)$$

where  $\sigma$  is permutation of the indices  $\mu_1 \dots \mu_s$ . A higher spin gauge transformation is defined as

$$\delta \Phi_{\mu_1 \dots \mu_s} = \sum_{\sigma} \partial_{\sigma(\mu_1)} \Lambda_{\sigma(\mu_2) \dots \sigma(\mu_s)} \quad (1.25)$$

and the corresponding variation of the Fronsdal tensor is

$$\delta F_{\mu_1 \dots \mu_s} = 3 \sum_{\sigma} \partial_{\sigma(\mu_1)} \partial_{\sigma(\mu_2)} \partial_{\sigma(\mu_3)} \Lambda_{\rho\sigma(\mu_4) \dots \sigma(\mu_s)}^{\rho} \quad (1.26)$$

So the gauge symmetry can only be realized with traceless gauge parameters. The Fronsdal-Fang equation for massless fields with half integer spin  $s + \frac{1}{2}$  is

$$S_{\mu_1 \dots \mu_s} \equiv \sum_{\sigma} i(\gamma^{\rho} \partial_{\rho} \Psi_{\sigma(\mu_1) \dots \sigma(\mu_s)} - \gamma^{\rho} \partial_{\sigma(\mu_1)} \Psi_{\rho\sigma(\mu_2) \dots \sigma(\mu_s)}) = 0 \quad (1.27)$$

Under a higher spin gauge transformation

$$\delta \Psi_{\mu_1 \dots \mu_s} = \sum_{\sigma} \partial_{\sigma(\mu_1)} \epsilon_{\sigma(\mu_2) \dots \sigma(\mu_s)} \quad (1.28)$$

the variation of the Fronsdal tensor spinor is

$$\delta S_{\mu_1 \dots \mu_s} = - \sum_{\sigma} 2i\gamma^{\rho} \partial_{\sigma(\mu_1)} \partial_{\sigma(\mu_2)} \epsilon_{\rho\sigma(\mu_3) \dots \sigma(\mu_s)} \quad (1.29)$$

So the gauge symmetry is realized with  $\gamma$ -traceless gauge parameters. A Lagrangian formalism for Fronsdal's theory free from constraint was found at the beginning of the new century at the price of introducing non-local terms [10, 11].

A interacting theory always has much more physical significance than a free theory. It is more important to find a consistent interacting theory of higher spin massless fields. It turned out to be impossible in flat spacetime due to the no-go theorems [12, 13, 14], which exclude the existence of higher spin conserved currents by analyzing the symmetries of the S matrix. The no-go theorems can be circumvented in curved spacetime where S matrix is not well-defined. It is Vasiliev who proposed such a higher spin gravity theory, known as Vasiliev theory, in any dimension [15, 16, 17, 18]. The theory contains an infinite tower of fields of all spins, and all of them must be included for consistent interaction for generic spacetime dimensions. It's perturbed around (Anti) de-Sitter vacuum and the flat limit doesn't exist. In general it's very complicated and only known at the level of equation of motion.

One of the main purpose of studying higher spin gravity theories is, conjectured to be the tensionless limits of string theories (e.g.[19]), they provide useful playgrounds for the



AdS/CFT correspondence, which is probably the most important discovery in the past two decades in theoretical physics [20]. It states that quantum gravity with Anti-de Sitter (AdS) spacetime background in  $n + 1$  dimensional spacetime is equivalent to a conformal field theory (CFT) in  $n$  dimensional spacetime, in particular, the classical AdS gravity should be dual to a strongly coupled CFT. For higher spin gravity, we have the duality between  $AdS_4$  higher spin gravity and the  $O(N)$  vector model CFT in three dimensional spacetime [21, 22] and the duality between  $AdS_3$  higher spin gravity and the  $W_n$  minimal model CFT in two dimensional spacetime [23, 24]. For completeness we briefly review the Anti-de Sitter spacetime and the conformal field theory here.

The  $AdS_{n+1}$ , namely the  $n + 1$ -dimensional Anti-de Sitter spacetime is the maximally symmetry spacetime with negative constant curvature. It's a hyperboloid in the  $n + 2$  dimensional Minkowski spacetime

$$-t_1^2 - t_2^2 + \sum_{i=1}^n x^{i^2} = -l^2 \quad (1.30)$$

It's a maximally symmetric spacetime with isometry group  $SO(2, n)$ . The most frequently used coordinate patch for the half spacetime is

$$ds^2 = \frac{l^2}{y^2} (dy^2 - dt^2 + \sum_{i=1}^n x^{i^2}) \quad (1.31)$$

or

$$ds^2 = l^2 d\rho^2 + e^{2\rho} (-dt^2 + \sum_{i=1}^{n-1} dx^{i^2}) \quad (1.32)$$

$\rho$  is called holographic radial coordinate. In the context of holography the boundary is at  $\rho \rightarrow \infty$ .

Conformal field theories are field theories with conformal symmetry. The conformal transformations of a manifold are diffeomorphisms that rescale the metric. For the  $n$  dimensional Minkowski spacetime, the conformal transformation group contains the Poincare group as the isometry subgroup, and the scaling transformation and special conformal transformations

$$x'^{\mu} = \alpha x^{\mu} \quad (1.33)$$

$$x'^{\mu} = \frac{x^{\mu} - b^{\mu} x^2}{1 - 2x \cdot b + b^2 x^2} \quad (1.34)$$

The conformal algebra is

$$\begin{aligned}
[D, P_\mu] &= iP_\mu \\
[D, K_\mu] &= -iK_\mu \\
[K_\mu, P_\nu] &= 2i(\eta_{\mu\nu}D - L_{\mu\nu}) \\
[K_\rho, L_{\mu\nu}] &= i(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu) \\
[P_\rho, L_{\mu\nu}] &= i(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu) \\
[L_{\mu\nu}, L_{\rho\sigma}] &= i(\eta_{\nu\rho}L_{\mu\sigma} + \eta_{\mu\sigma}L_{\nu\rho} - \eta_{\nu\sigma}L_{\mu\rho} - \eta_{\mu\rho}L_{\nu\sigma})
\end{aligned} \tag{1.35}$$

It's isomorphic to  $so(2, n)$ , the Lie algebra of the isometry group of  $\text{AdS}_{n+1}$ . The conformal symmetry is so powerful such that the form of the two points and three points correlators are determined by the symmetry. In addition, the energy momentum tensor of a conformal field theory is traceless due to the scale invariance, and scale invariance often implies full conformal invariance[25]. In two dimensional spacetime the conformal group becomes infinite dimensional and the theory is much more restrictive[26]. The conformal generators form the Virasoro algebra

$$\begin{aligned}
[L_n, L_m] &= (n - m)L_{n+m} + \frac{c}{12}n(n + 1)\delta_{m+n,0} \\
[\bar{L}_n, \bar{L}_m] &= (n - m)\bar{L}_{n+m} + \frac{c}{12}n(n + 1)\delta_{m+n,0} \\
[L_n, \bar{L}_m] &= 0
\end{aligned} \tag{1.36}$$

where  $c$  is the central charge of the conformal field theory, which can also be defined by the operator product expansion of two energy momentum tensors. For some CFTs, there is the  $W$  symmetry as an extension of the conformal symmetry [27], which corresponds to higher spin current in the theory.

## 1.4 Higher spin Chern-Simons theory in three dimensional space-time

In general Vasiliev theory is very complicated. However, in three dimensional spacetime the higher spin gravity theory can be realized by the relatively simple Chern-Simons theory with the higher spin gauge algebra  $hs(\lambda)$ . Moreover, the Chern-Simons theory with gauge algebra  $sl(N, \mathbb{R})$  provides a consistent truncation of the Vasiliev theory at a finite spin  $N$ , which is specific to the spacetime dimension three. The Chern-Simons action at level  $k$  in three dimensional space-time is given by the following

$$S_{CS}[A] = \frac{k}{4\pi} \int_{\mathcal{M}} \text{tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \quad (1.37)$$

where  $A = A^a T_a$  is the connection valued in the Lie gauge algebra,  $\text{tr}$  is a symmetric nondegenerate bilinear form on the Lie algebra, and  $\mathcal{M} = \mathbb{R} \times \Sigma$  is the spacetime manifold with  $\mathbb{R}$  being the time direction. A gauge transformation is defined to be

$$A' = g^{-1} A g + g^{-1} dg \quad (1.38)$$

or in the infinitesimal form

$$\delta A = d\Lambda + [A, \Lambda] \quad (1.39)$$

Defined by the commutator of covariant derivatives, the curvature is

$$F = dA + A \wedge A \quad (1.40)$$

The variation of the Chern-Simons action by a variation of the connection  $\delta A$  is

$$\delta S_{CS} = \frac{k}{2\pi} \int_{\mathcal{M}} \text{tr} F \wedge \delta A - \frac{k}{4\pi} \int_{\partial\mathcal{M}} \text{tr} A \wedge \delta A \quad (1.41)$$

and for an infinitesimal gauge transformation

$$\delta S_{CS} = \frac{k}{2\pi} \int_{\partial\mathcal{M}} \text{tr} dA \Lambda \quad (1.42)$$

If the boundary term is negligible, then the action is differentiable (that is, the variation of the action is linear in the variation of the connection field) and yields the equation of motion which is also known as the flatness condition

$$F = dA + A \wedge A = 0 \tag{1.43}$$

and the action is invariant under a gauge transformation.

The Chern-Simons theory consists of two copies of the Chern-Simons action

$$S = S_{CS}[A] - S_{CS}[\bar{A}] \tag{1.44}$$

with the gauge algebra usually chosen to be  $sl(N, \mathbb{R})$  or  $hs(\lambda)$  to correspond to higher spin gravity. As a special case, the Chern-Simons theory with gauge algebra  $sl(2, \mathbb{R})$  recovers Einstein's gravity with negative cosmological constant in three dimensional spacetime [28] and the Chern-Simons theories with gauge Lie superalgebras realize super gravity theories [29]. The gravity in three dimensional spacetime has some interesting features [30]. The curvature tensor has the same number of independent components as the Ricci tensor, so it comes with no surprise that the curvature tensor can be expressed by the metric and the Ricci tensor

$$R_{\rho\sigma\mu\nu} = g_{\rho\mu}R_{\sigma\nu} - g_{\rho\nu}R_{\sigma\mu} + g_{\sigma\nu}R_{\rho\mu} - g_{\sigma\mu}R_{\rho\nu} - \frac{1}{2}(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}) \tag{1.45}$$

With this relation, it's easy to see that the local vacuum solution to the Einstein's equation must be spacetime with constant curvature. In other words, the spacetime is non-trivially curved only at points with presence of matter, there is no propagating degrees of freedom. Dynamical degrees of freedom are on the boundary. In addition, it has the AdS<sub>3</sub>, the three dimensional Anti-de Sitter space as the vacuum solution when the cosmological constant is negative  $\Lambda = -\frac{1}{l^2}$ . The action of the three dimensional Einstein's gravity in terms of the vielbein and spin connection is

$$S = \frac{1}{8\pi G} \int R_a \wedge e^a - \Lambda e^1 \wedge e^2 \wedge e^3 \tag{1.46}$$

where

$$\begin{aligned} R_a &= d\omega_a + \frac{1}{2}\epsilon_{abc}\omega^b \wedge \omega^c \\ \omega_a &= \frac{1}{2}\epsilon_{abc}\omega^{bc} \end{aligned} \quad (1.47)$$

Viewed as a functional of the vielbein and spin connection, the action yields the torsion free condition and Einstein's equation as expected. In addition, the action is invariant under two types of gauge transformations, the local Lorentz frame rotation

$$\delta e^a = \epsilon^{abc}e_b\tau_c \quad (1.48)$$

$$\delta\omega^a = d\tau^a + \epsilon^{abc}\omega_b\tau_c \quad (1.49)$$

and the local translation

$$\delta e^a = d\rho^a + \epsilon^{abc}\omega_b\rho_c \quad (1.50)$$

$$\delta\omega^a = -\Lambda\epsilon^{abc}e_b\rho_c \quad (1.51)$$

where  $\tau$  and  $\rho$  are arbitrary functions as gauge transformation parameters. Diffeomorphism, as the basic symmetry of a gravity theory, is included as a combination of the two gauge transformations given above [28]. Now we define the Lie algebra valued vielbein and spin connection for Chern-Simons theory

$$e = \frac{l}{2}(A - \bar{A}), \quad \omega = \frac{1}{2}(A + \bar{A}) \quad (1.52)$$

with the group index in the connection corresponding to the vielbein frame index. The generators of  $sl(2, \mathbb{R})$   $T_0 = L_0$ ,  $T_1 = \frac{1}{\sqrt{2}}(L_1 + L_{-1})$ ,  $T_2 = \frac{1}{\sqrt{2}}(L_1 - L_{-1})$  satisfy (see Appendix A)

$$[T_a, T_b] = \epsilon_{abc}T^c \quad (1.53)$$

$$\text{tr}(T_a T_b) = \frac{1}{2}\eta_{ab} \quad (1.54)$$

It's straightforward to verify that the Chern-Simons theory with gauge algebra  $sl(2, \mathbb{R})$  is identical to the Einstein-Hilbert action with negative cosmological constant  $\Lambda = -\frac{1}{l^2}$  in the

Palatini formalism, with the identification  $k = \frac{l}{4G}$ . Moreover, the gauge transformation of Chern-Simons theory

$$\begin{aligned}\delta A &= d\Lambda + [A, \Lambda] \\ \delta \bar{A} &= d\Lambda + [\bar{A}, \Lambda]\end{aligned}\tag{1.55}$$

translates to the local Lorentz rotation of the Palatini formalism of gravity. When we choose gauge algebra  $sl(N, \mathbb{R})$  or  $hs(\lambda)$  with the "gravity section"  $sl(2, \mathbb{R})$  embedded in, the generalized vielbein and spin connection are

$$A = \left(\omega + \frac{e}{l}\right)^a T_a + \left(\omega + \frac{e}{l}\right)^{a_1 \dots a_s} T_{a_1 \dots a_s}\tag{1.56}$$

$$\bar{A} = \left(\omega - \frac{e}{l}\right)^a T_a + \left(\omega - \frac{e}{l}\right)^{a_1 \dots a_s} T_{a_1 \dots a_s}\tag{1.57}$$

where  $T_a$ 's are  $sl(2, \mathbb{R})$  subalgebra generators and  $T_{a_1 \dots a_s}$ 's are higher spin generators. The metric and the higher spin fields can be expressed in terms of traces of symmetrized products of the generalized vielbein  $e$

$$g_{\mu\nu} = \eta_{ab} e_\mu^a e_\nu^b = \frac{1}{2} \text{tr}(e_\mu e_\nu)\tag{1.58}$$

The higher spin fields, for example spin 3 fields can be written as

$$\phi_{\mu_1 \dots \mu_s} = \frac{1}{6} \text{tr}(e_{(\mu_1} \dots e_{\mu_s)})\tag{1.59}$$

The Chern-Simons theory with gauge algebra  $hs(\lambda)$  realizes Vasiliev theory and Chern-Simons theory with gauge algebra  $sl(N, \mathbb{R})$  is a consistent truncation of the higher spin gravity theory of fields with integer spin up to  $N$ [31, 32, 33].

## 1.5 Boundary charge algebra

Now we briefly review the boundary charge algebra following [33, 34]. We start at the Hamiltonian formalism of the Chern-Simons theory in three dimensional spacetime. We label the time coordinate as  $t$  and two space coordinate as  $x^i$ ,  $i = 1, 2$ . The Chern-Simons

action then reads

$$S_{CS} = \frac{k}{4\pi} \int_{\mathcal{M}} dt \wedge dx^i \wedge dx^j \text{tr}(A_0 F_{ij} - A_i \dot{A}_j) + \frac{k}{4\pi} \int_{\mathbb{R} \times \partial\Sigma} dt \wedge dx^i \text{tr}(A_0 A_i) \quad (1.60)$$

$A_0$  is Lagrange multiplier and  $A_i$ 's are dynamical fields, and the Poisson bracket is

$$\{F, H\} = \frac{2\pi}{k} \int_{\Sigma} dx^i \wedge dx^j \text{tr} \left( \frac{\delta F}{\delta A_i(x)} \frac{\delta H}{\delta A_j(x)} \right) \quad (1.61)$$

for two phase space functionals  $F[A_i]$  and  $H[A_i]$ . If the boundary terms can be neglected, an infinitesimal gauge transformation with parameter  $\Lambda$  can be generated by the charge  $G^0(\Lambda) = \frac{k}{4\pi} \int_{\Sigma} dx^i \wedge dx^j \text{tr}(\Lambda F_{ij})$ . To include the boundary terms, we need to add a boundary term to the charge to make it differentiable in the fields

$$G(\Lambda) = G^0(\Lambda) + Q(\Lambda) \quad (1.62)$$

with the boundary term, known as the boundary charge, satisfying the relation

$$\delta Q(\Lambda) = -\frac{k}{2\pi} \int_{\partial\Sigma} \text{tr}(\Lambda \delta A) \quad (1.63)$$

and it takes the simple form

$$Q(\Lambda) = -\frac{k}{2\pi} \int_{\partial\Sigma} \text{tr}(\Lambda A) \quad (1.64)$$

if the gauge transformation parameter  $\Lambda$  is independent on the fields. After gauge fixing, the  $Q(\Lambda)$ 's define the global charges of the Chern-Simons theory, and the boundary charge algebra is given by

$$\delta_{\Lambda} F = \{Q(\Lambda), F\} \quad (1.65)$$

here the brackets are Dirac brackets on the reduced phase space, and can be calculated by choosing field-dependent gauge parameters.

# CHAPTER 2

## Introduction to the KdV hierarchies

### 2.1 KdV hierarchies

Symmetry is probably the most important concept in modern physics. It is not surprising that integrable systems, in some sense the dynamical systems with "maximal symmetry", have long played an important role in physics. In particular, we have found that the Chern-Simons theory with gauge symmetry, or in a different perspective, as a higher spin gravity theory with higher spin symmetry is closely related to the KdV hierarchies, a series of integrable system hierarchies. In this chapter we review the KdV hierarchies to lay the foundation of discussion of the relation.

The KdV equation is a partial differential equation firstly proposed to describe the propagation of shallow water waves in channels to explain the soliton wave observed

$$4\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} + 6u\frac{\partial u}{\partial x} \quad (2.1)$$

The KdV equation is an integrable system, with a Lax pair description, infinitely many conserved and commuting charges, soliton solutions with dispersion-free scattering, and compatible with Painleve test. Later it is found to be a particular member of a series of integrable system hierarchies, the so called KdV hierarchies, where an integrable system hierarchy is defined as a system with infinitely many commuting Hamiltonian flows that each defines an integrable system.

We review the KdV hierarchies now using the formulation by Lax pairs of pseudo-differential operators, which has been discussed extensively in the review [35]. Pseudo-differential operators are extension of differential operators to include negative powers of



differentiations  $\partial$  retaining the rules of differentiation, that is, the linearity and the Leibniz rule. It has the form

$$F = \sum_{k=-\infty}^n f_k \partial^k \quad (2.2)$$

and the Leibniz rule translates to

$$\partial^i f = \sum_0^{\infty} C_i^k f^{(k)} \partial^{i-k} \quad (2.3)$$

where the combinatorial factor satisfies the recursive relation  $C_i^k + C_i^{k+1} = C_{i+1}^{k+1}$  for  $k \in \mathbb{N}, i \in \mathbb{Z}$  which is essentially the Leibniz rule and the boundary condition  $C_0^k = 0, C_i^0 = 1$ . It has the explicit formula

$$C_i^k = \frac{i(i-1)\dots(i-k+1)}{k!} \quad (2.4)$$

We don't have to define how the negative powers of differentiations act on a function, it's only the algebraic structure of the pseudo-differential operators that is of interest and generates the integrable system. We denote the subspace of pseudo-differential operators of non-negative powers of differentiation by  $\mathcal{R}_+$  and the subspace of pseudo-differential operators of negative powers of differentiation by  $\mathcal{R}_-$ , and the subspace of  $\mathcal{R}_+$  with powers of differentiation lower than  $n$  by  $\mathcal{R}_n, n \in \mathbb{N}$ . Furthermore, we define the residue of a pseudo-differential operator, denoted by  $\text{res}$ , as the coefficient of  $\partial^{-1}$ . It can be shown that the residue of a commutator of two pseudo-differential operators is a total derivative, therefore we have

$$\int \text{res}(XY) dx = \int \text{res}(YX) dx \quad (2.5)$$

The  $n$ -th KdV hierarchy is formulated by the pseudo-differential operator  $L$  of  $n$  fields

$$L = \partial^n + u_1 \partial^{n-1} + u_2 \partial^{n-2} + \dots + u_{n-1} \partial + u_n \quad (2.6)$$

where  $\partial = \frac{\partial}{\partial x}$  and  $u_k = u_k(x, t)$ . A partial differentiation with respect to time is denoted by a dot above. The formalism of pseudo-differential operators allows us to define  $L^{1/n}$ , by putting the ansatz  $L^{\frac{1}{n}} = \partial + \sum_{k=1}^{\infty} v_k \partial^{1-k}$  in the defining equation  $(L^{\frac{1}{n}})^n = L$ , which are

essentially differential-algebraic equations of the coefficients that can be solved iteratively. We define  $L^{-1}$  analogously and  $L^{\frac{m}{n}}$  is defined as  $(L^{\frac{1}{n}})^m$ . For the m-th dynamical equation one defines

$$P_m = (L^{m/n})_+ \quad (2.7)$$

where the subscript  $+$  denotes the projection to  $\mathcal{R}_+$ , and subscript  $-$  will also be used later to denote the projection to  $\mathcal{R}_-$ . An integrable system is constructed by the Lax pair  $P_m, L$ , i.e. the evolution equation

$$\dot{L} = [P_m, L] \quad (2.8)$$

gives a system of partial differential equations for  $u_i(x, t)$ . As a consequence, a similar equation holds for  $L^{\frac{k}{n}}, k \in \mathbb{N}$

$$\dot{L}^{\frac{k}{n}} = [P_m, L^{\frac{k}{n}}] \quad (2.9)$$

The equation yields  $u_1 = 0$ , hence we usually set  $u_1 = 0$  from the very beginning. For the KdV hierarchy an infinite set of conserved quantities can be obtained by

$$q^k = \int \text{res}(L^{\frac{k}{n}}) dx \quad (2.10)$$

It's easy to check they are conserved by the equation of motion

$$\dot{q}^k = \int \text{res}(\dot{L}^{\frac{k}{n}}) dx = \int \text{res}([P_m, L^{\frac{k}{n}}]) dx = 0 \quad (2.11)$$

Like many other integrable systems, the n-th KdV hierarchy is bi-Hamiltonian, that is, each equation of motion comes from two Hamiltonian structures, and the second Hamiltonian structure is the  $W_n$  algebra, which, as an extension of Virasoro algebra, is closely related to the conformal field theory [27]. To look at the bi-Hamiltonian structure of the KdV hierarchies, we briefly review the Hamiltonian formalism first, and the simpler case is to consider a dynamical system on a finite dimensional manifold  $\mathcal{M}$ . The Poisson bracket is the essential part of a Hamiltonian formalism. To define a Poisson bracket, we start with a symplectic form  $\omega$ , which is defined as a non-degenerate closed two form. We define a

map from the tangent space to the cotangent space  $\xi \rightarrow -i_\xi\omega$ , it's an isomorphism since  $\omega$  is non-degenerate. The Hamiltonian map  $H$  is defined as the inverse of this map, which satisfies

$$\alpha(\eta) = \omega(\eta, H\alpha), \eta \in T\mathcal{M}, \alpha \in T^*\mathcal{M} \quad (2.12)$$

The image of the differential of functions by the Hamiltonian map generates symplectomorphisms, that is, diffeomorphisms that preserves the symplectic form

$$L_{Hdf}\omega = (i_{Hdf}d + di_{Hdf})\omega = -d^2f = 0 \quad (2.13)$$

The Poisson bracket is defined as

$$\{f, g\} = (Hdf)g = dg(Hdf) = \omega(df, dg) \quad (2.14)$$

It's obviously antisymmetric, and the Jacobi identity comes from the closedness of  $\omega$ . A Hamiltonian flow generated by a Hamiltonian  $h$  is a flow of symplectomorphism generated by  $Hdh$

$$\dot{f} = (Hdh)f = \{h, f\} \quad (2.15)$$

which is the familiar Hamilton equation (with a different sign convention). The Hamiltonian formalism of infinite dimensional dynamical systems is a bit more complicated since we don't have a manifold to define the symplectic form and Hamiltonian map in a regular way. Nonetheless we can still make analogous definitions retaining most of the algebraic properties.

The n-th KdV hierarchy is formulated by the pseudo-differential operator  $L = \partial^n + \sum_{k=1}^n u_k \partial^{n-k}$ . A function is considered to be a function of  $u_k$ 's and their derivatives of all orders and possibly  $x$ . The tangent space is identified with  $\mathcal{R}_n$ , and a tangent vector  $\partial_a$ , where  $a = \sum_{k=1}^n a_k \partial^{n-k} \in \mathcal{R}_n$ , acts on a function by taking the functional derivative in the direction of  $a$ , explicitly

$$\partial_a f = \sum_{k=1}^n \sum_{i=0}^{\infty} a_k^{(i)} \frac{\partial f}{\partial u_k^{(i)}} = \sum_{k=1}^n \int a_k \frac{\delta f}{\delta u_k} dx \quad (2.16)$$

The dual cotangent space is identified with  $\mathcal{R}_-/\partial^{-n}\mathcal{R}_-$ . A cotangent vector  $X = \sum_{k=1}^n X_k \partial^{-k}$  acts on a tangent vector  $a = \sum_{k=1}^n a_k \partial^{n-k}$  as

$$X(a) = \int \text{res}(Xa)dx = \int \sum_{i=1}^n a_i X_{n+1-i} dx \quad (2.17)$$

We define a cotangent vector  $\frac{\delta f}{\delta L} = \sum_{k=1}^n \partial^{k-n-1} \frac{\delta f}{\delta u_k}$  for a function  $f$ .

$$\frac{\delta f}{\delta L}(\partial_a) = \int \text{res}(a \frac{\delta f}{\delta L})dx = \int \sum_{k=1}^n a_k \frac{\delta f}{\delta u_k} dx = \sum_{k=1}^n \sum_{i=0}^{\infty} a_k^{(i)} \frac{\partial f}{\partial u_k^{(i)}} = \partial_a f \quad (2.18)$$

Therefore  $\frac{\delta f}{\delta L}$  is just  $df$ , the differential of the function  $f$ . Now it comes to the crucial part of defining a Hamiltonian map. We define the Adler map from  $\mathcal{R}_-/\partial^{-n}\mathcal{R}_-$  to  $R_n$  depending on a parameter  $z$  as

$$A_z(X) = (L_z X)_+ L_z - L_z (X L_z)_+ \quad (2.19)$$

where  $L_z = L - z^n$ . The dependence on  $z$  can be singled out as

$$A_z(X) = A_0(X) + z^n A_\infty(X) \quad (2.20)$$

where  $A_0(X) = (LX)_+ L - L(XL)_+$  and  $A_\infty(X) = [X, L]_+$ . It can be shown that the map  $H_z : X \rightarrow \partial_{A_z}(x)$  is a Hamiltonian map for any  $z$ , that is, it's antisymmetric and induces a closed two form on its image, which is a subalgebra of the tangent space. The Hamiltonian map induces a Poisson bracket

$$\{f, g\} = (H_z \frac{\delta f}{\delta L})h = \int \text{res}(A_z(\frac{\delta f}{\delta L}) \frac{\delta g}{\delta L})dx \quad (2.21)$$

In particular, it's called the first Hamiltonian structure when  $z = \infty$  and the second Hamiltonian structure when  $z = 0$ . From the previously constructed conserved quantities of the  $n$ -th KdV hierarchy, we define

$$h_k = -\frac{n}{k} \int \text{res} L^{\frac{k}{n}} dx \quad (2.22)$$

It can be verified that the  $m$ -th Hamiltonian flow is generated by the Hamiltonian  $h_{m+n}$  in the first Hamiltonian structure and by the Hamiltonian  $h_m$  in the second Hamiltonian structure. Therefore KdV hierarchies are bi-Hamiltonian.

## 2.2 Examples

In the following we will present some members of the KdV hierarchies, for which we will show that they are related to the Lifshitz Chern-Simons theories.

The  $n = 2$  KdV hierarchy is the one which contains the original KdV equation as the  $m = 3$  member. It's formulated by the pseudo-differential operator

$$L = \partial^2 + u_2 \quad (2.23)$$

After putting the ansatz  $L^{\frac{1}{2}} = \partial + \sum_{i=1}^{\infty} f_i \partial^{-i}$  into the defining equation  $(L^{\frac{1}{2}})^2 = L$ , a set of differential-algebraic equation is obtained

$$\begin{aligned} 2f_1 &= u_2 \\ f_1' + 2f_2 &= 0 \\ f_1^2 + f_2' + f_3 &= 0 \\ \dots & \end{aligned} \quad (2.24)$$

and we get

$$L^{\frac{1}{2}} = \partial + \frac{u_2}{2} \partial^{-1} - \frac{u_2'}{4} \partial^{-2} + \frac{u_2'' - u_2^2}{4} \partial^{-3} + O(\partial^{-4}) \quad (2.25)$$

For the KdV equation  $m = 3$ ,

$$P_3 = (L^{\frac{3}{2}})_+ = \partial^3 + \frac{3}{2} u_2 \partial + \frac{3}{4} u_2' \quad (2.26)$$

The Lax pair commutator is

$$[P_3, L] = \frac{1}{4} u_2''' + \frac{3}{2} u_2 u_2' \quad (2.27)$$

and the Lax equation  $\dot{L} = [P_3, L]$  takes the following form

$$4\dot{u}_2 = u_2''' + 6u_2 u_2' \quad (2.28)$$

which reproduces the KdV equation. The Poisson brackets can be computed by (2.21). For the first Hamiltonian structure

$$\{f, g\}_{z=\infty} = -2 \int \left( \frac{\delta f}{\delta u_2} \right)' \frac{\delta g}{\delta u_2} dx \quad (2.29)$$

so

$$\{u_2(x), u_2(y)\}_{z=\infty} = 2\delta'(x - y) \quad (2.30)$$

Similarly, the second Hamiltonian structure is computed to be

$$\{f, g\} = \int \left( \frac{\delta f}{\delta u} \right)''' \frac{\delta g}{\delta u} + 2u \left( \frac{\delta f}{\delta u} \right)' \frac{\delta g}{\delta u} + u' \frac{\delta f}{\delta g} \frac{\delta g}{\delta u} dx \quad (2.31)$$

$$\{u_2(x), u_2(y)\}_{z=0} = -\delta'''(x - y) - u'(x)\delta(x - y) - 2u(x)\delta'(x - y) \quad (2.32)$$

If the space is a circle  $|x| = 1$  in the complex plane, we have the Hamiltonian structure expressed in the Fourier components  $u_{2,k} = \frac{1}{2\pi i} \int x^{k+1} u(x) dx$

$$\{u_{2,k}, u_{2,l}\}_{z=0} = \frac{1}{2\pi i} (k(k-1)\delta_{k+l,0} + (k-l)u_{2,k+l}) \quad (2.33)$$

It is the  $W_2$  algebra (up to rescaling of the Poisson bracket and the field), also known as Virasoro algebra. According to (2.10) the conserved quantities are

$$\begin{aligned} q^1 &= \frac{1}{2} \int u_2 dx \\ q^2 &= 0 \\ q^3 &= \frac{1}{4} \int u_2^2 dx \\ &\dots \end{aligned} \quad (2.34)$$

The next example is the  $n = 3$  KdV hierarchy, also known as the Boussinesq hierarchy because it contains the Boussinesq equation as the  $m = 2$  member. The pseudo-differential operator  $L$  is now of third order and contains two independent fields  $u_2$  and  $u_3$

$$L = \partial^3 + u_2 \partial + u_3 \quad (2.35)$$

and

$$\begin{aligned} L^{1/3} &= \partial + \frac{1}{3} u_2 \partial^{-1} + \frac{1}{3} (u_3 - u_2') \partial^{-2} + o(\partial^{-3}) \\ P_2 &= (L^{2/3})_+ = \partial^2 + \frac{2}{3} u_2 \end{aligned} \quad (2.36)$$

The Lax equation  $\dot{L} = [P_2, L]$  takes the form

$$\begin{aligned} \dot{u}_2 &= 2u_3' - u_2'' \\ \dot{u}_3 &= u_3'' - \frac{2}{3}u_2''' - \frac{2}{3}u_2u_2' \end{aligned} \quad (2.37)$$

Eliminating  $u_3$ , an equation for  $u_2$  alone was obtained

$$\ddot{u}_2 = -\frac{1}{3}u_2''' - \frac{4}{3}(u_2u_2')' \quad (2.38)$$

This is the Boussinesq equation, which has been studied in the context of propagation of waves. The second Hamiltonian structure is the  $W_3$  algebra. The conserved quantities are

$$\begin{aligned} q^1 &= \int \text{res}(L^{\frac{1}{3}}) = \frac{1}{3} \int u_2 dx \\ q^2 &= \int \text{res}(L^{\frac{2}{3}}) = \int \left( \frac{2}{3}u_3 - \frac{1}{3}u_2' \right) = \frac{2}{3} \int u_3 dx \\ &\dots \end{aligned} \quad (2.39)$$

Now let's consider the  $n = 4$  KdV hierarchy and its  $m = 3$  member. The pseudo-differential operator  $L$  is now of fourth order and contains three fields  $u_2, u_3$  and  $u_4$

$$L = \partial^4 + u_2\partial^2 + u_3\partial + u_4 \quad (2.40)$$

and

$$\begin{aligned} L^{1/4} &= \partial + \frac{u_2}{4}\partial^{-1} + \frac{1}{4}(u_3 - \frac{3}{2}u_2')\partial^{-2} + \left( \frac{1}{4}u_4 - \frac{3}{8}u_3' + \frac{5}{16}u_2'' - \frac{3}{32}u_2^2 \right)\partial^{-3} + o(\partial^{-4}) \\ P_3 = L_+^{3/4} &= \partial^3 + \frac{3}{4}u_2\partial + \frac{3}{4}u_3 - \frac{3}{8}u_2' \end{aligned} \quad (2.41)$$

The Lax equation  $\dot{L} = [P_3, L]$  takes the form

$$\begin{aligned} \dot{u}_2 &= \frac{1}{4}u_2''' - \frac{3}{2}u_3'' + 3u_4' - \frac{3}{4}u_2u_2' \\ \dot{u}_3 &= -2u_3''' + 3u_4'' + \frac{3}{4}u_2'''' - \frac{3}{4}u_2u_3' - \frac{3}{4}u_3u_2' \\ \dot{u}_4 &= u_4'''' + \frac{3}{8}u_2'''' - \frac{3}{4}u_3'''' + \frac{3}{4}u_2u_4' \\ &\quad - \frac{3}{4}u_2u_3'' + \frac{3}{8}u_2u_2''' - \frac{3}{4}u_3u_3' + \frac{3}{8}u_3u_2'' \end{aligned} \quad (2.42)$$

and the conserved quantities are

$$\begin{aligned}
q^{(1)} &= \int \text{res}(L^{\frac{1}{4}}) = \frac{1}{4} \int u_2 dx \\
q^{(2)} &= \int \text{res}(L^{\frac{2}{4}}) = \int \left( \frac{1}{2} u_3 - \frac{1}{2} u_2' \right) = \frac{1}{2} \int u_3 dx \\
q^{(3)} &= \int \text{res}(L^{\frac{3}{4}}) = \int \left( \frac{3}{4} u_4 - \frac{3}{8} u_3' + \frac{1}{16} u_2'' - \frac{3}{32} u_2^2 \right) = \int \left( \frac{3}{4} u_4 - \frac{3}{32} u_2^2 \right) dx \\
&\dots
\end{aligned} \tag{2.43}$$

### 2.3 Drinfeld-Sokolov reduction

It's well known that we can reduce the order of differential equations by introducing more variables. It turns out the  $n$ -th KdV hierarchy, which we formulated by the differential operator  $L = \partial^n + \sum_{k=1}^n u_k \partial^{n-k}$  of order  $n$ , can be formulated by matrix-valued first order differential operator instead, this is often called the Drinfeld-Sokolov reduction named after its inventors[36]. We consider the  $n \times n$  matrix-valued first order differential operator of the form

$$l_q = \partial - J + q \tag{2.44}$$

Here  $\partial$  is short for the matrix with the diagonal elements being  $\partial$  and everything else zero.  $J = \sum_{i=1}^{n-1} e_{i,i+1}$  where  $e_{i,j}$  is the matrix with the  $i, j$  entry being 1 and everything else zero.  $J$  is the  $V_1^2$  element in the usual matrix representation of  $sl(N, \mathbb{R})$ .  $q$  is a  $n \times n$  matrix in general, with its entries being functions of  $x, t$ . Let  $F$  be a column vector with entries being the functions of  $x, t$

$$F = \begin{pmatrix} f \\ f_2 \\ \dots \\ f_n \end{pmatrix} \tag{2.45}$$

If we set  $q$  to be the matrix with the bottom row being  $u_n, u_{n-1}, \dots, u_1$  and everything else zero, then the equation  $l_q F = 0$  is equivalent to  $n$  differential equations, with  $n - 1$  of them



defining  $f_i$  as  $f^{(i-1)}$  and the last one reduced to  $Lf = 0$ . In fact for any lower triangular matrix  $q$ ,  $l_q F = 0$  corresponds to a differential equation of  $f$  of order  $n$ . That is, the  $n \times n$  matrix-valued first order differential operator  $l_q$  corresponds to a  $n$ -th order differential operator, and it can be shown that the correspondence is one-to-one for equivalent classes of  $l_q$  under similarity transformations by matrices of the form  $S = I + \nu$ , where  $\nu$  is a strictly lower triangular matrix, and we call that kind of transformation a gauge transformation for  $l_q$ . The Hamiltonian structure of  $l_q$  can be defined in a similar way to what we did for  $L$ , and it can be shown that it's equivalent to the Hamiltonian structure of  $L$  generated by the Adler map[35].

In fact, we can formulate KdV hierarchies by Lax pairs of matrix valued pseudo-differential operators. We will defer its discussion to Chapter 4 since it directly helps us to establish the relation between KdV hierarchies and the Lifshitz Chern-Simons theory.

## CHAPTER 3

# Lifshitz Chern-Simons theory and the asymptotic symmetry

### 3.1 Lifshitz spacetime and Lifshitz field theories

The asymptotic AdS solution in the Chern-Simons theory, which is dual to a CFT on the boundary, has been extensively studied (e.g.[33]). However, Chern-Simons theory also allows for the construction of non-AdS solutions [37, 38, 39, 40], such as asymptotically Lobachevsky, Schrödinger, warped AdS and Lifshitz spacetimes. The three dimensional Lifshitz spacetime takes the form

$$ds^2 = d\rho^2 - e^{2z\rho} dt^2 + e^{2\rho} dx^2 \tag{3.1}$$

It recovers the AdS spacetime when  $z = 1$ . Lifshitz spacetimes are not solutions to Einstein's gravity and nontrivial matter interactions must be added. The first Lifshitz spacetime solution was found in four-dimensional gravity coupled to antisymmetric tensor fields [41]. The shift  $\rho \rightarrow \rho + \ln \lambda$  in the holographic radial coordinate induces Lifshitz scaling on  $t, x$ , that is, an anisotropic scaling transformation on the spacetime coordinate

$$t \rightarrow \lambda^z t, \quad x \rightarrow \lambda x \tag{3.2}$$

where  $z$  is called Lifshitz scaling exponent. An asymptotic Lifshitz solution in the Chern-Simons theory is a solution to the flatness condition which yields an asymptotic Lifshitz spacetime. It is dual to a field theory of Lifshitz scaling symmetry on the boundary. Unlike the isotropic scaling symmetry, Lifshitz symmetry is not compatible with the Poincare symmetry hence it cannot be found in relativistic field theories, which laid the foundation for the

particle physics. However, it's ubiquitous and important in condensed matter systems near quantum critical points (see e.g. [41]). The Lifshitz symmetry in two dimensional spacetime is given by the time translation  $H$ , the space translation  $P$  and the Lifshitz scaling  $D$ , they satisfy the Lifshitz algebra

$$\begin{aligned} [P, H] &= 0 \\ [D, H] &= zH \\ [D, P] &= P \end{aligned} \tag{3.3}$$

The energy momentum tensor contains four components, the energy density  $\mathcal{E}$ , the energy flux  $\mathcal{E}^x$ , the momentum density  $\mathcal{P}_x$  and the stress density  $\Pi_x^x$ . These components satisfy the conservation law

$$\begin{aligned} \partial_t \mathcal{E} + \partial_x \mathcal{E}^x &= 0 \\ \partial_t \mathcal{P}_x + \partial_x \Pi_x^x &= 0 \end{aligned} \tag{3.4}$$

In addition, the scaling invariance imposes a modified traceless condition for the energy momentum tensor

$$z\mathcal{E} + \Pi_x^x = 0 \tag{3.5}$$

as compared the traceless condition of the energy momentum tensor of conformal field theories.

## 3.2 Asymptotic Lifshitz solution of Chern-Simons theory

In the following we will focus on the construction of asymptotically Lifshitz solutions that is, the solutions that yield asymptotic Lifshitz spacetimes, in the Chern-Simons theory with gauge algebra  $hs(\lambda)$ . We following the approach developed in [42], with detailed results in my paper [43] [44]. First of all, we choose the "radial gauge" as in [33] to specify the

$\rho$ -dependence of the connection

$$\begin{aligned} A_\mu(x, t, \rho) &= b^{-1}a_\mu(x, t)b + b^{-1}\partial_\mu b \\ \bar{A}_\mu(x, t, \rho) &= b\bar{a}_\mu(x, t)b^{-1} + b\partial_\mu(b^{-1}) \end{aligned} \quad (3.6)$$

where  $b = \exp(\rho L_0)$ . Here  $L_0$  is a Cartan generator of a  $sl(2, \mathbb{R})$  sub-algebra of  $sl(N, \mathbb{R})$ , or its correspondent  $V_0^2$  in  $hs(\lambda)$ . In this "radial gauge" the flatness condition (1.43) reduces to equations of the  $\rho$ -independent fields  $a_t$  and  $a_x$

$$\partial_t a_x - \partial_x a_t + [a_t, a_x] = 0, \quad \partial_t \bar{a}_x - \partial_x \bar{a}_t + [\bar{a}_t, \bar{a}_x] = 0 \quad (3.7)$$

To preserve the radial gauge under a gauge transformation, the gauge transformation parameter must take the form

$$\Lambda(\rho, x, t) = b^{-1}\lambda(x, t)b \quad (3.8)$$

and the gauge transformation on  $A_\mu$  translates to the gauge transformation on  $a_\mu$

$$\delta a_\mu = \partial_\mu \lambda + [a_\mu, \lambda] \quad (3.9)$$

Therefore the time evolution of  $a_x$  and  $a_t$  can be viewed as a gauge transformation generated by gauge transformation parameter  $a_t$ .

The solutions which produce an exact Lifshitz metric with Lifshitz scaling exponent  $z$  can be easily found, the unbarred connection is given by

$$a = V_z^{z+1} dt + V_1^2 dx, \quad A = V_z^{z+1} e^{z\rho} dt + V_1^2 e^\rho dx + V_0^2 d\rho \quad (3.10)$$

and the barred connection is given by

$$\bar{a} = V_{-z}^{z+1} dt + V_{-1}^2 dx, \quad \bar{A} = V_{-z}^{z+1} e^{z\rho} dt + V_{-1}^2 e^\rho dx - V_0^2 d\rho \quad (3.11)$$

One can verify by (1.58) that these connections yield a Lifshitz spacetime with an arbitrary integer  $z$ . For an integral value  $\lambda = N$ , the algebra  $hs(\lambda)$  is truncated to  $sl(N, \mathbb{R})$ . For example in the  $z = 2$  case, one reproduces the  $sl(3, \mathbb{R})$  Lifshitz connections studied in [42] with the identification  $V_{\pm 2}^3 = W_{\pm 2}$ ,  $V_{\pm 1}^2 = L_{\pm 1}$  and  $V_0^2 = L_0$ .

In the following we will consider connections where the barred sector is determined in terms of the unbarred sector. This is possible due to an automorphism of  $hs(\lambda)$  algebra, which is obtained from a conjugation  $(V_m^s)^c = (-1)^{s+m+1}V_{-m}^s$  (see Appendix A). In particular the generator  $V_0^2$  used in constructing the radial gauge transformations is self conjugate up to a sign, i.e.  $(V_0^2)^c = -V_0^2$ . Consequently, if  $A$  solves the flatness condition  $F = 0$  in the radial gauge, the barred connection is chosen to be the conjugate  $\bar{A} = A^c$ , which is automatically in the radial gauge and satisfies the flatness condition  $\bar{F} = 0$ . From now on we will leave out the barred sector as it is determined from the un-barred sector. Though we have explicit expression for Lifshitz connections (3.10), they are static solutions without any dynamics. Here we want to construct asymptotic Lifshitz, in which leading terms are Lifshitz connections given by (3.10) where additional terms are present with sub-leading powers  $e^\rho$ . Consequently such connections will lead to asymptotic Lifshitz spacetimes where the metric and tensor fields have additional terms which become negligible as  $\rho \rightarrow \infty$  compared to the Lifshitz vacuum.

To further simplify the theory, we choose the "lowest weight gauge", that is,  $a_x$  only contains lowest weight terms except for  $V_1^2$

$$a_x = V_1^2 + \sum_{i=2}^{\infty} \alpha_i V_{-i+1}^i \quad (3.12)$$

In general we can transform away all non lowest weight terms in  $a_x$  step by step. Under an infinitesimal gauge transformation, the variation of  $a_x$  is  $\delta a_x = [a_x, \lambda] + \partial_x \lambda$ . We have  $V_1^2$  in  $a_x$ , so we can put  $V_{s-2}^s$  in the gauge parameter  $\lambda$  to gain a highest weight term  $V_{s-1}^s$  in  $\delta a_x$  from the commutator. We can exponentiate this infinitesimal transformation to cancel the highest weight term in the original  $a_x$ . After eliminating all highest weight terms, we use  $V_{s-3}^s$  in  $\lambda$  to cancel  $V_{s-2}^s$  terms. Do this recursively we get to the lowest weight gauge.

With the lowest weight gauge chosen for  $a_x$ ,  $a_t$  should start with  $V_z^{z+1}$  to yield asymptotic Lifshitz spacetime with scaling exponent  $z$ , and its non-highest weight terms should be completely determined by highest weight terms because it must preserve the lowest weight gauge of  $a_x$  in time evolution. Hence the asymptotic Lifshitz connection is determined up to the free choice of the highest weight terms in  $a_t$ , which we call a gauge freedom of  $a_t$ , if

any abuse of terminology, because the time evolution of  $a_x$  is a gauge transformation with gauge parameter  $a_t$  anyway. We will choose the highest weight terms in  $a_t$  to be differential polynomials of the fields in  $a_x$ , hence all fields in  $a_t$  will be differential polynomials of the fields in  $a_x$ . We will see some gauge choice is of particular interests because it's related to the KdV hierarchy, which we call "KdV gauge".

Needless to say the Lifshitz scaling symmetry plays a fundamental role in the construction of asymptotic Lifshitz connection and in fact it allows us to assign a scaling dimension to each field. The pedestrian way to see it is that the weight of  $hs(\lambda)$  elements is additive under multiplication of elements hence the fields as the coefficients of the elements in  $a_x$  gain a scaling dimension. By the flatness condition (3.7) we have

$$\begin{aligned} [\partial_x] &= 1 & [\partial_t] &= z \\ [\alpha_i] &= i \end{aligned} \tag{3.13}$$

In fact, the scaling dimensions correspond to the scaling of the fields demanding the invariance of the connections  $A_t dt$  and  $A_x dx$  under the Lifshitz scaling

$$\begin{aligned} \rho' &= \rho + \log \lambda \\ x' &= \lambda^{-1} x \\ t' &= \lambda^{-3} t \end{aligned} \tag{3.14}$$

Now we begin our explicit construction of asymptotic Lifshitz connection. The infinite dimensional gauge algebra  $hs(\lambda)$  is hard to work with in a explicit computation, so we usually work with  $sl(N, \mathbb{R})$  as the truncation of  $hs(\lambda)$  when  $\lambda = N$  to obtain concrete results. Here we present a few examples we have worked out. The first is the  $z = 2$  asymptotic Lifshitz solution to the Chern-Simons theory with gauge algebra  $sl(3, \mathbb{R})$ . The asymptotic Lifshitz

connection is<sup>1</sup>

$$\begin{aligned} a_x &= V_1^2 + \alpha_2 V_{-1}^2 + \alpha_3 V_{-2}^3 \\ a_t &= V_2^3 + 2\alpha_2 V_0^3 - \frac{2}{3}\alpha_2' V_{-1}^3 - 2\alpha_3 V_{-1}^2 + \left(\alpha_2^2 + \frac{1}{6}\alpha_2''\right) V_{-2}^3 \end{aligned} \quad (3.15)$$

The flatness condition yields the equations of motion

$$\begin{aligned} \dot{\alpha}_2 &= -2\alpha_3' \\ \dot{\alpha}_3 &= \frac{4}{3}(\alpha_2^2)' + \frac{1}{6}\alpha_2''' \end{aligned} \quad (3.16)$$

Note that we actually can't add a  $V_1^2$  term to  $a_t$  because that would require a field of scaling dimension one to be the coefficient, which cannot be a differential polynomial of fields in  $a_x$ . So in this case we pretty much don't have the gauge freedom of  $a_t$  (if we insist it should be a differential polynomial of  $a_x$ ). In addition, the Lifshitz scaling symmetry of the equation of motion is obvious.

Our next example is  $z = 3$  asymptotic Lifshitz solution to the Chern-Simons theory with gauge algebra  $sl(4, \mathbb{R})$ .

$$\begin{aligned} a_x &= V_1^2 + \alpha_2 V_{-1}^2 + \alpha_3 V_{-2}^3 + \alpha_4 V_{-3}^4 \\ a_t &= V_3^4 + \left(c - \frac{41}{5}\right)\alpha_2 V_1^2 + \dots \end{aligned} \quad (3.17)$$

Here the coefficient of  $V_1^2$  in  $a_t$  has to be a constant multiple of  $\alpha_2$  by dimensional analysis and the gauge freedom of  $a_t$  is characterized by a single parameter  $c$ . The flatness condition yields the equations of motion

$$\begin{aligned} \dot{\alpha}_2 &= -\left(\frac{41}{10} - \frac{1}{2}c\right)\alpha_2''' - \left(\frac{123}{5} - 3c\right)\alpha_2'\alpha_2 + \frac{54}{5}\alpha_4' \\ \dot{\alpha}_3 &= -\frac{1}{2}\alpha_3''' - (15 - c)\alpha_3'\alpha_2 - (30 - 3c)\alpha_2'\alpha_3 \\ \dot{\alpha}_4 &= \frac{1}{10}\alpha_4''' + \frac{1}{120}\alpha_2'''' - (30 - 4c)\alpha_2'\alpha_4 - \left(\frac{27}{5} - c\right)\alpha_2\alpha_4' - 12\alpha_3'\alpha_3 + \frac{13}{30}\alpha_2\alpha_2''' \\ &\quad + \frac{59}{60}\alpha_2'\alpha_2'' + \frac{24}{5}\alpha_2^2\alpha_2' \end{aligned} \quad (3.18)$$

---

<sup>1</sup>Here we have adapted the general notation, with  $-\mathcal{L}$  in the paper [42] replaced by  $\alpha_2$  and  $\mathcal{W}$  replaced by  $\alpha_3$ .

Again the Lifshitz scaling symmetry of the equation of motion is obvious. If we want a conserved quantity with scaling dimension two, it must be the integral of  $\alpha_3$ , therefore  $\dot{\alpha}_3$  must be a total derivative and we have to choose  $c = \frac{15}{2}$ . In the next chapter we will see the theory corresponds to  $m = 3$  member of the  $n = 4$  KdV hierarchy with the choice  $c = \frac{15}{2}$ .

### 3.3 Boundary charge algebra of Lifshitz Chern-Simons theory

The Lifshitz symmetry is not only realized on the level of equation of motion, it's also realized as Lifshitz subalgebra in the boundary charge algebra. In the radial gauge the defining equation of the boundary charge simplifies to

$$\delta Q_\lambda = -\frac{k}{2\pi} \int dx \text{tr} \Lambda \delta A_x = -\frac{k}{2\pi} \int dx \text{tr} \lambda \delta a_x \quad (3.19)$$

The gauge transformation of the Chern-Simons theory induces a Poisson structure on the Lifshitz field theory on the boundary as follows

$$\delta_\lambda \phi = \{Q_\lambda, \phi\} \quad (3.20)$$

The Poisson brackets of all fields can be computed by choosing field-dependent gauge parameters, and the algebra of boundary charges is called the boundary charge algebra. The three gauge parameters generating time translation, space translation and Lifshitz scaling are

$$\begin{aligned} \lambda_H &= -a_t \\ \lambda_P &= -a_x \\ \lambda_D &= xa_x + zta_t - V_0^2 \end{aligned} \quad (3.21)$$

We can verify directly by the flatness condition that these gauge parameters generate the desired transformations and preserve lowest weight gauge for  $a_x$ . Now we calculate the corresponding boundary charges  $Q(\Lambda_H)$ ,  $Q(\Lambda_P)$  and  $Q(\Lambda_D)$ , that is, the Hamiltonian, the momentum, and the charge which corresponds to Lifshitz scaling, to verify they satisfy the commutation relation of Lifshitz algebra (3.3). In addition, we also want to check the energy



momentum conservation and modified traceless condition of the energy momentum tensor for the Lifshitz Chern-Simons theory.

For  $sl(3, \mathbb{R})$ ,  $z = 2$  Lifshitz Chern-Simons theory, the generic gauge transformation parameter is

$$\lambda = \sum_{i=-1}^1 \epsilon_i V_i^2 + \sum_{j=-2}^2 \chi_j V_j^3. \quad (3.22)$$

To keep  $a_x$  in the lowest weight gauge, only the coefficient of highest weight terms  $\epsilon_1, \chi_2$  are free and all the other variables in the gauge parameter are expressed in terms of them. We can assign specific values to  $\epsilon_1, \chi_2$  to get gauge parameters generating time translation, space translation and Lifshitz scaling

$$\begin{aligned} \epsilon_1 = 0, \chi_2 = -1, \lambda = \lambda_H \\ \epsilon_1 = -1, \chi_2 = 0, \lambda = \lambda_P \\ \epsilon_1 = x, \chi_2 = 2t, \lambda = \lambda_D \end{aligned} \quad (3.23)$$

Using (3.19) we get the corresponding boundary charges

$$\begin{aligned} Q(\Lambda_H) &= \frac{2k}{\pi} \int dx \alpha_3 \\ Q(\Lambda_P) &= -\frac{2k}{\pi} \int dx \alpha_2 \\ Q(\Lambda_D) &= \frac{2k}{\pi} \int dx (x\alpha_2 - 2t\alpha_3) \end{aligned} \quad (3.24)$$

Using (1.65) we can verify they form a Lifshitz subalgebra

$$\begin{aligned} \{Q(\Lambda_H), Q(\Lambda_P)\} &= 0 \\ \{Q(\Lambda_D), Q(\Lambda_H)\} &= 2Q(\Lambda_H) \\ \{Q(\Lambda_D), Q(\Lambda_P)\} &= Q(\Lambda_P) \end{aligned} \quad (3.25)$$

In addition, we identify the density of  $Q(\Lambda_H)$  as the energy density, the density of  $Q(\Lambda_P)$  as the momentum density

$$\begin{aligned} \mathcal{E} &= \frac{2k}{\pi} \alpha_3 \\ \mathcal{P}_x &= -\frac{2k}{\pi} \alpha_2. \end{aligned} \quad (3.26)$$

Use the traceless condition of the energy-momentum tensor of Lifshitz field theory  $2\mathcal{E} + \Pi_x^x = 0$  to get  $\Pi_x^x = -\frac{4k}{\pi}\alpha_3$ , we can verify that the conservation of momentum

$$\partial_t \mathcal{P}_x + \partial_x \Pi_x^x = 0 \quad (3.27)$$

is guaranteed by the equations of motion. Plugging the expression for  $\mathcal{E}$  into the equation of conservation of energy  $\partial_t \mathcal{P} + \partial_x \mathcal{E}^x = 0$  one obtains the expression for energy flow

$$\mathcal{E}^x = -\frac{2k}{\pi} \left( \frac{2}{3} \alpha_2^2 + \frac{1}{6} \alpha_2'' \right) \quad (3.28)$$

For  $sl(4, \mathbb{R})$ ,  $z = 3$  Lifshitz Chern-Simons theory with  $c = \frac{15}{2}$ , the generic gauge transformation parameter is

$$\lambda = \sum_{i=-1}^1 \epsilon_i V_i^2 + \sum_{j=-2}^2 \chi_j V_j^3 + \sum_{k=-3}^3 \mu_k V_k^4. \quad (3.29)$$

Again to keep  $a_x$  in the lowest weight gauge, only the highest weight terms  $\epsilon_1, \chi_2, \mu_3$  are free. By appropriately choosing values for these three variables, we get the desired gauge parameters

$$\begin{aligned} \epsilon_1 &= \frac{7}{10} \alpha_2, \chi_2 = 0, \mu_3 = -1, \lambda = \lambda_H \\ \epsilon_1 &= -1, \chi_2 = 0, \mu_3 = 0, \lambda = \lambda_P \\ \epsilon_1 &= x - \frac{21}{10} \alpha_2 t, \chi_2 = 0, \mu_3 = 3t, \lambda = \lambda_D \end{aligned} \quad (3.30)$$

For these three gauge parameters, we use (3.19) to calculate the boundary charges

$$\begin{aligned} Q(\Lambda_H) &= \frac{k}{2\pi} \int dx \left( -36\alpha_4 + \frac{7}{2} \alpha_2^2 \right) \\ Q(\Lambda_P) &= \frac{k}{2\pi} \int dx \left( -10\alpha_2 \right) \\ Q(\Lambda_D) &= \frac{k}{2\pi} \int dx \left( 10x\alpha_2 + 108t\alpha_4 + 21t\alpha_2^2 \right) \end{aligned} \quad (3.31)$$

Using (1.65) we can verify the Lifshitz symmetry algebra. The density of  $Q(\Lambda_H)$  is identified with the energy density up to a total derivative and the density of  $Q(\Lambda_P)$  is identified with the momentum density

$$\begin{aligned} \mathcal{E} &= \frac{k}{2\pi} \left( -36\alpha_4 + \frac{7}{2} \alpha_2^2 + \frac{7}{6} \alpha_2'' \right) \\ \mathcal{P}_x &= -\frac{k}{2\pi} 10\alpha_2 \end{aligned} \quad (3.32)$$

Using the traceless condition of the energy-momentum tensor  $3\mathcal{E} + \Pi_x^x = 0$  to get  $\Pi_x^x = -3\mathcal{E}$ , we can show that

$$\partial_t \mathcal{P}_x + \partial_x \Pi_x^x = 0. \tag{3.33}$$

## CHAPTER 4

### Lifshitz Chern-Simons theory and KdV hierarchies

#### 4.1 Lifshitz Chern-Simons theory and KdV hierarchies identified

In this chapter I demonstrate that there is a one to one correspondence between the Lifshitz Chern-Simons theories and members of the KdV hierarchies. It's mainly inspired by two facts, first, both theories possess Lifshitz scaling symmetries, and the second, the flatness condition very much resembles a Lax type equation. We begin by discussing two examples  $z = 2, sl(3, \mathbb{R})$  and  $z = 3, sl(4, \mathbb{R})$  Lifshitz Chern-Simons theory. Then we propose a conjecture that Lifshitz Chern-Simons theory with Lifshitz scaling exponent  $z$  and gauge algebra  $sl(N, \mathbb{R})$  corresponds to the  $m$ -th member of the  $n$ -th KdV hierarchy, with  $m = z$  and  $N = n$ . Then we prove the correspondence by Drinfeld-Sokolov formalism of KdV hierarchies.

We recall that for  $z = 2, sl(3, \mathbb{R})$  Lifshitz Chern-Simons theory the equations of motion is

$$\dot{\alpha}_2 = -2\alpha'_3 \quad (4.1)$$

$$\dot{\alpha}_3 = \frac{4}{3}(\alpha_2^2)' + \frac{1}{6}\alpha_2''' \quad (4.2)$$

Compared to the Boussinesq equations ( $n = 3, m = 2$  member in the KdV hierarchies)

$$\begin{aligned} \dot{u}_2 &= 2u'_3 - u_2'' \\ \dot{u}_3 &= u_3'' - \frac{2}{3}u_2''' - \frac{2}{3}u_2u_2' \end{aligned} \quad (4.3)$$

we find the following map identifies two sets of equations

$$\begin{aligned} u_2 &= 4\alpha_2 \\ u_3 &= -4\alpha_3 + 2\alpha_2' \end{aligned} \quad (4.4)$$

or conversely

$$\begin{aligned}\alpha_2 &= \frac{1}{4}u_2 \\ \alpha_3 &= -\frac{1}{4}u_3 + \frac{1}{8}u_2'.\end{aligned}\tag{4.5}$$

In addition, the conserved boundary charges energy and momentum of the Chern-Simons theory are proportional to the conserved quantities in the KdV hierarchy

$$\begin{aligned}Q(\Lambda_H) &= \frac{2k}{\pi} \int dx \alpha_3 \sim q^2 = \frac{2}{3} \int u_3 dx \\ Q(\Lambda_P) &= -\frac{2k}{\pi} \int dx \alpha_2 \sim q^1 = \frac{1}{3} \int u_2 dx\end{aligned}\tag{4.6}$$

For  $sl(4, \mathbb{R})$ ,  $z = 3$  case, it has the novelty of gauge dependence described by the parameter  $c$ . We want to find a map from Chern-Simons connection variables to KdV variables such that the equations of motion of Chern-Simons theory are equivalent to the  $m = 3$  member of  $n = 4$  KdV hierarchy. The Chern-Simons variables have scaling dimensions from the Lifshitz isometry as discussed in Chapter 3. The KdV variables also have scaling dimensions by the formulation of pseudo-differential operators. The fact that the scaling dimensions on both sides have to agree puts strong restrictions on the map of the variables. Hence we must use the ansatz  $u_2 = k\alpha_2$ ,  $u_3 = a\alpha_2' + b\alpha_3$ . For the second KdV equation

$$\dot{u}_3 = -2u_3''' + 3u_4'' + \frac{3}{4}u_2'''' - \frac{3}{4}u_2u_3' - \frac{3}{4}u_3u_2'\tag{4.7}$$

on the right hand side  $\alpha_2\alpha_3'$  and  $\alpha_2'\alpha_3$  have the same coefficients, on the left hand side the same kind of terms come from  $\dot{\alpha}_3 = -\frac{1}{2}\alpha_3''' - (15 - c)\alpha_3'\alpha_2 - (30 - 3c)\alpha_2'\alpha_3$ , so we must have  $(15 - c) = (30 - 3c)$ , obtaining  $c = \frac{15}{2}$ . Comparing terms and check integrability condition recursively one can obtain  $k = 10$ ,  $a = 10$ ,  $b = 24$  and the full map

$$\begin{aligned}u_2 &= 10\alpha_2 \\ u_3 &= 10\alpha_2' + 24\alpha_3 \\ u_4 &= 3\alpha_2'' + 9\alpha_2'^2 + 12\alpha_3' + 36\alpha_4\end{aligned}\tag{4.8}$$

establish the correspondence. With  $c = \frac{15}{2}$ , the equation of motion of  $sl(4, \mathbb{R})$ ,  $z = 3$  Lifshitz Chern-Simons theory reads

$$\begin{aligned}\dot{\alpha}_2 &= -\frac{7}{20}\alpha_2''' - \frac{21}{10}\alpha_2'\alpha_2 + \frac{54}{5}\alpha_4' \\ \dot{\alpha}_3 &= -\frac{1}{2}\alpha_3''' - \frac{15}{2}\alpha_3'\alpha_2 - \frac{15}{2}\alpha_2'\alpha_3 \\ \dot{\alpha}_4 &= \frac{1}{10}\alpha_4''' + \frac{1}{120}\alpha_2'''' + \frac{21}{10}\alpha_2\alpha_4' - 12\alpha_3'\alpha_3 + \frac{13}{30}\alpha_2\alpha_2''' + \frac{59}{60}\alpha_2'\alpha_2'' + \frac{24}{5}\alpha_2^2\alpha_2'\end{aligned}\quad (4.9)$$

In addition, the momentum and energy in the Chern-Simons theory are proportional to the conserved quantities in the KdV hierarchy

$$\begin{aligned}Q(\Lambda_H) &= \frac{k}{2\pi} \int dx (-36\alpha_4 + \frac{7}{2}\alpha_2^2) \sim q^3 = \int (\frac{3}{4}u_4 - \frac{3}{32}u_2^2) dx \\ Q(\Lambda_P) &= \frac{k}{2\pi} \int dx (-10\alpha_2) \sim q^1 = \frac{1}{4} \int u_2 dx\end{aligned}\quad (4.10)$$

In the two previous examples we have mapped the equations of motion for asymptotic Lifshitz connections to member of the KdV hierarchy in two particular cases, namely the  $z = 2$ ,  $sl(3, \mathbb{R})$  to the  $n = 3, m = 2$  element of the KdV hierarchies and  $z = 3$ ,  $sl(4, \mathbb{R})$  to the  $n = 4, m = 3$  element of the KdV hierarchies. These results inspire us to propose the general conjecture: the Lifshitz Chern-Simons theory with gauge algebra  $sl(N, \mathbb{R})$  and an integer Lifshitz scaling exponent  $z$  corresponds to the member of the KdV hierarchy with  $n = N, m = z$ . Many more examples have been worked out to verify this conjecture in [44], which we include in (Appendix C).

## 4.2 A proof by Drinfeld-Sokolov formalism

In fact, we can prove it by the Drinfeld-Sokolov formalism of KdV hierarchies[36], where the it is formulated in Lax type equation of matrix-valued PDOs, which is very similar to the construction of asymptotic Lifshitz connection in Chern-Simons theory. To begin with, we rewrite the flatness condition in a Lax form

$$\frac{d}{dt}D_x + [a_t, D_x] = 0, \quad (4.11)$$

where the covariant derivative  $D_x = \partial_x + a_x$  is regarded as a Lie algebra valued differential operator (and hence it can be regarded as a PDO without any negative powers of  $\partial$ ). For the gauge algebra  $sl(N, \mathbb{R})$ , we can use the matrix representation and the flatness condition becomes a Lax equation the of matrix valued PDO. Our main result is that both the Lifshitz Chern-Simons theory and the KdV hierarchy can be deduced from the Drinfeld-Sokolov formalism and are related by making two different gauge choices for the PDO.

The Drinfeld-Sokolov formalism starts by defining the PDO valued in  $sl(N, \mathbb{R})$

$$L_q = \partial_x + q(x, t) + \Lambda, \quad (4.12)$$

where  $q$  is a lower triangular matrix (or non-positive weight element, if we use the terminology in  $hs(\lambda)$  and view  $sl(N, \mathbb{R})$  as a truncation of it) and

$$\Lambda = V_1^2 + \lambda e. \quad (4.13)$$

The parameter  $\lambda$  should not be confused with the deformation parameter in the gauge algebra  $hs(\lambda)$ . In fact the construction in the present section is limited to  $sl(N, \mathbb{R})$  and it is an interesting open question how to generalize the present construction to  $hs(\lambda)$ .

Here  $e_{i,j}$  denotes the matrix with a single one in the  $i$ 'th row and  $j$ 'th column, and zeros elsewhere. In the matrix representation we use,  $V_1^2 = J = \sum_{i=1}^{N-1} e_{i,i+1}$ , and  $e = e_{N,1}$  is proportional to  $V_{-N+1}^N$ . The Lax equation is defined as

$$\frac{d}{dt}L = [P, L], \quad (4.14)$$

where  $P$  is some differential polynomial in  $q$  that has to be carefully chosen. The left hand side of the Lax equation is independent on  $\lambda$  and lower triangular, so we want the commutator on the right hand side to be also independent on  $\lambda$  and lower triangular. Suppose  $M = \sum_{i=-\infty}^n m_i \lambda^i$  is a matrix that commutes with  $L$  where  $m_i$ 's are matrix valued coefficients (i.e. matrices multiplied by powers in  $\lambda$ ), then we can set  $P = M_+$ , the part of  $M$  with non-negative powers in  $\lambda$ . From  $[M, L] = 0$  it follows  $[M_+, L] = -[M_-, L]$ . Since the left hand side only contains non-negative powers in  $\lambda$  but the right hand side only contains non-positive powers in  $\lambda$ , they should be both independent on  $\lambda$  and  $-[M_-, L] = [m_{-1}, e]$  is

necessarily lower triangular. Now we have  $[P, L] = [M_+, L] = [m_0, \partial_x + V_1^2 + q]$ . We identify  $V_1^2 + q$  as  $a_x$ , so we have  $L = D_x + \lambda e$ . We furthermore identify  $-m_0 = -\text{Zero}(P)$  as  $a_t$ , where symbolically  $\text{Zero}$  means to take the  $\lambda^0$  part. Then the Lax equation is reduced to our flatness condition in Chern-Simons Lifshitz theory. It should be noted that the parameter  $\lambda$  is used in setting up the PDOs, the actual equations of motion and the conserved charges are all independent on  $\lambda$ .

As we discussed in Chapter 2, the gauge equivalence classes of the matrix valued PDO  $l_q$  are in one-to-one correspondence to the ordinary PDO  $L$ . Therefore we want to impose the restriction on the Lax equation that it must preserve gauge equivalence. Furthermore it will be shown that the Lifshitz Chern-Simons theory and the KdV hierarchy are just reduction of Drinfeld-Sokolov formalism by special gauge choices. The gauge transformation, is defined for a PDO as

$$L'_q = S^{-1}L_qS, \quad (4.15)$$

where  $S$  is a  $\lambda$ -independent lower triangular matrix with ones in the diagonal, or in the higher spin algebra language,  $S$  is  $V_0^1$  plus negative weight terms. Define  $L'_q = \partial_x + a'_x + \lambda e = \partial_x + V_1^2 + q' + \lambda e$ , then this PDO gauge transformation induces a gauge transformation of  $a_x$  (or  $q$ )

$$\begin{aligned} a'_x &= S^{-1}a_xS + S^{-1}\partial_xS, \\ q' &= S^{-1}V_1^2S - V_1^2 + S^{-1}\partial_xS, \end{aligned} \quad (4.16)$$

where we used the fact that  $e$  commutes with  $S$  in the calculation. By the explicit construction specified later  $P$  is a differential polynomial in  $q$  and so is the commutator  $[P, L]$ . Hence the Lax equation is essentially a evolution equation for  $q$

$$\partial_t q = p(q), \quad (4.17)$$

where  $p(q)$  means a differential polynomial in  $q$ . We require the evolution equation to preserve gauge equivalence, that is, when starting with two initial conditions for  $q$  which are connected by a gauge transformation, the two solutions should be also connected by a



(time-dependent) gauge transformation at any time. The Lax equation preserving gauge equivalence is actually an evolution equation of gauge equivalent classes. Needless to say, we can choose representatives of some special form to specify the time evolution of the gauge equivalent classes. This motivates the definition of the canonical form of  $L$ , or  $q$ . We denote the part of  $q$  with weight  $-i$  by  $q_i$ . In principle  $q_i$  lies in the  $N - |i|$  dimensional linear space spanned by  $V_i^{|i|+1}, \dots, V_i^N$ . By restricting  $q_i$  to be in a one dimensional subspace, that is, a specific linear combination, we define a canonical form for  $q$ . For technical reasons, we also require the one dimensional subspace has a nonzero lowest weight projection. The name canonical form is justified by the following theorem, for any  $q$  there is a unique gauge transformation to transform it into the canonical form, and the expression in the canonical form is unique. See Appendix D.1 for a proof. The choice of the one dimensional subspaces that  $q'_i$  lie in defines the specific canonical form. Two choices are of particular importance in our discussion. The first one, we restrict  $q'_i$  to be lowest weight, if not an abuse of language, we call this the lowest weight canonical form. The second one, we restrict  $q'_i$  to be multiple of  $e_{1,i+1}$ , which we call the KdV canonical form. In the lowest weight canonical form,

$$q = \sum_{i=1}^N \alpha_i V_{-i+1}^i, \quad (4.18)$$

the Lax equation  $\frac{d}{dt}L = [P, L]$  gives us the flatness condition of Chern-Simons theory in the lowest weight gauge (by appropriately choosing  $a_t$ ). In the KdV canonical form

$$q = - \sum_{i=1}^N u_i e_{1,i}, \quad (4.19)$$

the Lax equation  $\frac{d}{dt}L = [P, L]$  gives us KdV, as proved in the paper by Drinfeld and Sokolov. The evolution equation in the lowest weight canonical form and that in the KdV canonical form are just two special explicit forms of the same equation. There is a unique gauge transformation that transforms between these two canonical forms, which establish the one-to-one correspondence between Lifshitz Chern-Simons theory with  $sl(N, \mathbb{R}), z$  and KdV with  $n = N, m = z$ , and explicitly the map from  $\alpha_i$ 's to  $u_i$ 's. From the relation

$$\text{Tr}[P, L] = -\text{Tr}[m_{-1}, e] = 0, \quad (4.20)$$

it follows that the trace part of  $L$  must be constant by the equation of motion. In the following we set to be zero for simplicity. For example, we can set  $\alpha_1 = 0$  for the  $q$  in the lowest weight canonical form.

Now let's construct the conserved quantities from the Lax equation. In general, a general matrix  $A$  whose elements are power series in  $\lambda$  (both positive and negative) can be uniquely expanded in the form

$$A = \sum_i a_i \Lambda^i, \quad (4.21)$$

where  $a_i$ 's denote diagonal matrices which are independent of  $\lambda$ .

Here  $q$  is lower triangular, so it has the expansion  $\sum_{i=0}^{N-1} d_i \Lambda^{-i}$ , or equivalently

$$L = \partial_x + \Lambda + \sum_{i=0}^{N-1} d_i \Lambda^{-i}. \quad (4.22)$$

There is a similarity transformation to transform  $L$  into a scalar coefficient form, that is, there is a formal series

$$T = E + \sum_{i=1}^{\infty} h_i \Lambda^{-i}, \quad (4.23)$$

where  $h_i$ 's are diagonal matrices, such that

$$L_0 = TLT^{-1} = \partial_x + \Lambda + \sum_{i=0}^{\infty} f_i \Lambda^{-i}, \quad (4.24)$$

where  $f_i$ 's are scalar functions, as opposed to matrices multiplied to the left.  $T$  is determined up to multiplication by series of the form  $E + \sum_{i=1}^{\infty} t_i \Lambda^i$  where  $t_i$ 's are scalar functions, and  $f_i$ 's are determined up to a total derivative. Most importantly

$$q^i = \int f_i, \quad (4.25)$$

are conserved by the Lax equation. See Appendix D.2 for the proof.

The scalar coefficient form  $L_0 = \partial_x + \Lambda + \sum_{i=0}^{\infty} f_i \Lambda^{-i}$  not only gives us the conserved quantities, but also can help us to determine the form of the matrices that commute with  $L$ , and ultimately the form of  $P$ . Matrices that commute with  $L_0$  must take the form

$\sum_{i=-\infty}^n c_i \Lambda^i$  with  $c_i$ 's as constant coefficients, see Appendix D.3 for a proof. Therefore matrices that commute with  $L$  must have the form

$$M = T^{-1} \left( \sum_{i=-\infty}^n c_i \Lambda^i \right) T. \quad (4.26)$$

because  $[M, L] = 0$  is equivalent to  $[TMT^{-1}, L_0] = 0$ . Setting  $P = M_+$ , we get the consistent Lax equation  $\frac{d}{dt}L = [P, L]$ . Despite the simple appearance, several remarks about this equation are necessary. First,  $T$  is the series that transforms  $L$  into a form with scalar coefficients  $L_0$  and it's in general a differential polynomial in  $q$ , hence  $P$  is a differential polynomial in  $q$  and so is the commutator  $[P, L]$ . Second, though  $T$  has the indeterminacy of a multiplicative series  $E + \sum_{i=1}^{\infty} t_i \Lambda^{-i}$  where  $t_i$ 's are scalar functions,  $P$  is uniquely defined because  $\sum_{i=-\infty}^n c_i \Lambda^i$  commute with this series. Last but the most important, this Lax equation preserves gauge equivalence, a proof of this statement will be given in the Appendix D.4.

As a evolution equation of gauge equivalent classes, the explicit form of the Lax equation  $\frac{d}{dt}L = [P, L]$  is certainly not unique and different explicit forms correspond to choice of different representatives in gauge equivalent classes. We have the following theorem, if the difference between  $P_1$  and  $P_2$  is a negative weight matrix with no time or  $\lambda$  dependence, then  $\frac{d}{dt}L = [P_1, L]$  and  $\frac{d}{dt}L = [P_2, L]$  give the same evolution equations of gauge equivalent classes. See Appendix D.5 for a proof. Applying this theorem, we can add a negative weight matrix both independent on time and  $\lambda$  to  $P$  without actually changing the evolution equation of gauge equivalent classes. We do need to do so when we want to obtain the Lax equation in certain canonical form, because the commutator  $[P, L]$  is guaranteed to be negative weight, but not necessarily in the specific canonical form. The correction added to  $P$  can be uniquely determined. The proof of this statement will be omitted because it's structurally the same as the proof of existence and uniqueness of the gauge transformation that transforms  $L$  into a canonical form.

At last we have enough ingredients to explain how the integrable Lifshitz Chern-Simons theory for  $sl(N, \mathbb{R})$  and  $z$  emerges from the Drinfeld-Sokolov formalism. First the Lax equation  $\frac{d}{dt}L = [P, L]$  is equivalent to the flatness condition  $\frac{d}{dt}D_x + [a_t, D_x] = 0$  with the

identification  $a_x = V_1^2 + q$  and  $a_t = -Zero(P)$ . Second, the Lax equation, viewed as evolution equation of gauge equivalent classes, can be put in the lowest weight canonical form, which corresponds to lowest weight gauge choice in the Chern-Simons theory. Then, considering the Lifshitz exponent is  $z$ , we set  $P = (T^{-1}\Lambda^z T)_+$  up to a multiplicative constant. At last we add a correction to  $P$  to make  $[P, L]$  lowest weight. From  $P$  obtained in this way,  $a_t = -Zero(P)$  coincides with  $a_t$  in "KdV gauge" in the previous section. If we choose the KdV canonical form for  $L$ , we get KdV hierarchy as proved in the paper by Drinfeld and Sokolov. The gauge transformation between the two canonical forms gives us the explicit map between the Lifshitz Chern-Simons theory and the KdV hierarchy. This map is  $z$  independent simply because  $z$  doesn't involve in the construction of gauge transformation between the two canonical forms.

## CHAPTER 5

# The supersymmetric Lifshitz Chern-Simons theory and super KdV hierarchies

The main goal of this chapter is to construct asymptotic Lifshitz solution to the supersymmetric Chern-Simons theory, that is, Chern-Simons theory with a Lie superalgebra as the gauge algebra. What's more important, we want to extend the correspondence between the Lifshitz Chern-Simons theory and the KdV hierarchy to the supersymmetric case. We choose to work on a specific example, the Lifshitz Chern-Simons theory with gauge algebra  $sl(3|2)$  and relate it to one of the supersymmetric extensions of the Boussinesq hierarchy.

### 5.1 Super KdV hierarchies

Now we briefly review the supersymmetric extension of the KdV hierarchies. Based on superspace formalism of supersymmetry, The basic way of supersymmetric extension is to introduce fermionic fields and to combine it with the original bosonic fields to form superfields, and then rewrite the equation of motion in terms of superfields and their covariant derivatives. For example, in the  $N = 1$  supersymmetric extension of the KdV equation[45], by adding a fermionic field  $\xi(x)$  we introduce the fermionic superfield

$$\Phi(x, \theta) = \xi(x) + \theta u(x) \tag{5.1}$$

in the  $(x, \theta)$  superspace. The supersymmetry transformation is generated by the operator

$$Q = \partial_\theta - \theta \partial_x \tag{5.2}$$

The covariant derivative is defined as

$$D = \partial_\theta + \theta\partial_x \quad (5.3)$$

It satisfies  $D^2 = \partial_x$  and  $\{D, Q\} = 0$ , so equations of motion of the superfields involving covariant derivatives are automatically supersymmetric. By inspection, the equation of motion must take the form

$$\dot{\Phi} = -\partial^3\Phi + a\partial(\Phi D\Phi) + (6 - 2a)D\Phi\partial\Phi \quad (5.4)$$

to give the correct bosonic limit when  $\xi$  is set to zero. It was also shown that the parameter  $a$  must take the value 3 to allow a supersymmetric extension of the first Hamiltonian structure. Alternatively, we can require existence of higher order conserved quantities for the system to be integrable to fix the free parameter.

Because  $sl(3|2)$  has two sets of fermionic generators, we should look for  $N = 2$  supersymmetric extension of  $n = 3$  KdV hierarchy, that is,  $N = 2$  super Boussinesq hierarchy. However brute force supersymmetric extension following the method above is not quite workable because large amount of undetermined coefficients. Instead, since the second Hamiltonian structure of the Boussinesq hierarchy is the  $W_3$  algebra, one should expect  $N = 2$  super Boussinesq hierarchy to possess  $N = 2$  super  $W_3$  algebra as the second Hamiltonian structure. Guided by this principle,  $N = 2$  super Boussinesq hierarchy was constructed in [46, 47] in terms of two bosonic superfields  $J$  and  $T$  in the superspace coordinates  $(x, \theta, \bar{\theta})$

$$\begin{aligned} J(x, \theta, \bar{\theta}) &= \bar{\theta}\theta u(x) + \theta\xi(x) + \bar{\theta}\bar{\xi}(x) + y(x) \\ T(x, \theta, \bar{\theta}) &= \bar{\theta}\theta z(x) + \theta\eta(x) + \bar{\theta}\bar{\eta}(x) + v(x) \end{aligned} \quad (5.5)$$

with two free parameters  $c$  and  $\alpha$ , where  $c$  is a free constant in the  $N = 2$  super  $W_3$  algebra realized by  $J, T$  that corresponds to rescaling freedom of  $J, T$ , and  $\alpha$  is a free constant in the Hamiltonian  $H = \int dx d\theta d\bar{\theta} (T + \alpha J^2)$  that generates the time evolution

$$\dot{J} = \{J, H\}, \quad \dot{T} = \{T, H\} \quad (5.6)$$

The super Boussinesq equation should reduce to the Boussinesq equation when  $sl(3|2)$  reduces to  $sl(3, R)$ , and that's possible only when the parameter  $\alpha$  takes the following value

$\alpha = -\frac{4}{c}$  [47]. After setting  $c = -\frac{4}{\alpha}$ , the  $N = 2$  super Boussinesq equation reads in terms of superfields

$$\begin{aligned} \dot{J} &= 2T' - \delta J' + 4\alpha J J' \\ \dot{T} &= -2J''' + \delta T' - 20\alpha \partial(\bar{D} J D J) + 8\alpha J' \delta J + 4\alpha J \delta J' \\ &\quad + 16\alpha^2 J^2 J' - 12\alpha \bar{D} J D T - 12\alpha D J \bar{D} T - 12\alpha J' T - 4\alpha J T' \end{aligned} \quad (5.7)$$

where

$$D = \partial_\theta - \frac{1}{2}\bar{\theta}\partial, \quad \bar{D} = \partial_{\bar{\theta}} - \frac{1}{2}\theta\partial \quad (5.8)$$

$$\delta = [\bar{D}, D] \quad (5.9)$$

It was shown in [46] that if we choose  $c = 8$  the parameter  $\alpha$  must take one of these three values  $-2, -\frac{1}{2}, \frac{5}{2}$  for the equation to be integrable in the sense that higher order conserved charges exist. We see  $\alpha = -\frac{4}{c} = -\frac{1}{2}$  is indeed one of them, and later an elegant Lax pair formulation of this case was given in [48]. In the form in components the time evolution equations (5.7) read

$$\begin{aligned} \dot{y} &= 2(u+v)' + 4\alpha y y' \\ \dot{\xi} &= \xi'' + 2\eta' + 4\alpha(y\xi)' \\ \dot{\bar{\xi}} &= -\bar{\xi}'' + 2\bar{\eta}' + 4\alpha(y\bar{\xi})' \\ \dot{u} &= 2z' + \frac{1}{2}y''' + 4\alpha(yu)' + 4\alpha(\xi\bar{\xi})' \\ \dot{v} &= -2z' - 16\alpha u y' - 8\alpha u' y - 4\alpha y v' - 12\alpha y' v + 12\alpha(\eta\bar{\xi} + \bar{\eta}\xi) + 20\alpha(\xi\bar{\xi})' - 2y''' + 16\alpha^2 y^2 y' \\ \dot{\eta} &= -\eta'' - 2\xi''' - 28\alpha u' \xi - 36\alpha u \xi' - 10\alpha v' \xi - 12\alpha v \xi' - 12\alpha u \eta + 12\alpha z \xi \\ &\quad + 10\alpha y'' \xi + 32\alpha^2 y y' \xi + 2\alpha y' \xi' + 16\alpha^2 y^2 \xi' - 4\alpha y \xi'' - 6\alpha y' \eta - 4\alpha y \eta' \\ \dot{\bar{\eta}} &= \bar{\eta}'' - 2\bar{\xi}''' - 28\alpha u' \bar{\xi} - 36\alpha u \bar{\xi}' - 10\alpha v' \bar{\xi} - 12\alpha v \bar{\xi}' + 12\alpha u \bar{\eta} - 12\alpha z \bar{\xi} \\ &\quad - 10\alpha y'' \bar{\xi} + 32\alpha^2 y y' \bar{\xi} - 2\alpha y' \bar{\xi}' + 16\alpha^2 y^2 \bar{\xi}' + 4\alpha y \bar{\xi}'' - 6\alpha y' \bar{\eta} - 4\alpha y \bar{\eta}' \\ \dot{z} &= -2u''' - \frac{1}{2}v''' - 64\alpha u u' - 16\alpha u v' - 12\alpha u' v + 32\alpha^2 y y' u + 16\alpha^2 y^2 u' - 4\alpha y z' - 2\alpha y y'' \\ &\quad + 6\alpha y' y'' + 10\alpha \bar{\xi} \eta' + 6\alpha \bar{\xi}' \eta - 10\alpha \xi \bar{\eta}' - 6\alpha \xi' \bar{\eta} + 14\alpha \xi \bar{\xi}'' - 14\alpha \xi'' \bar{\xi} + 32\alpha^2 (y\xi\bar{\xi})' \end{aligned} \quad (5.10)$$

## 5.2 $sl(3|2)$ , $z = 2$ Lifshitz Chern-Simons theory and its integrability

Now we look at the Chern-Simons theory. When we take the gauge algebra to be Lie superalgebra, most notably  $sl(p|q)$ , the Chern-Simons action takes the form

$$S_{CS}[A] = \frac{k}{4\pi} \int \text{str} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \quad (5.11)$$

where  $\text{str}$  denotes the supertrace. In complete analogy to the non-supersymmetric case, higher spin supergravity can be formulated by two copies of Chern-Simons actions, with the vielbein and spin connection expressed in terms of the gauge connection

$$e_\mu = \frac{1}{2}(A_\mu - \bar{A}_\mu), \quad \omega_\mu = \frac{1}{2}(A_\mu + \bar{A}_\mu) \quad (5.12)$$

and the metric is given by

$$g_{\mu\nu} = \frac{1}{\text{str}(L_0)^2} \text{str}(e_\mu e_\nu) = \frac{1}{4\text{str}(L_0)^2} \text{str}((A_\mu - \bar{A}_\mu)(A_\nu - \bar{A}_\nu)) \quad (5.13)$$

Now we focus on  $sl(3|2)$  Lifshitz Chern-Simons theory, that is, Chern-Simons theory with  $sl(3|2)$  gauge algebra that gives asymptotic Lifshitz spacetime. It should be noted that  $sl(3|2)$  Chern-Simons theory has been studied in the past, see for example [49, 50]. We follow the notation of generators of  $sl(3|2)$  in [50] and the super matrix representation which we include for completeness can also be found in the Appendix B. We adopt the radial gauge as we did in the non-supersymmetric case

$$A_\mu(\rho, x, t) = b(\rho)^{-1} a_\mu(x, t) b(\rho) + b(\rho)^{-1} \partial_\mu b(\rho), \quad \bar{A}_\mu(\rho, x, t) = b(\rho) \bar{a}_\mu(x, t) b(\rho)^{-1} + b(\rho) \partial_\mu b(\rho)^{-1} \quad (5.14)$$

where  $b(\rho) = e^{\rho L_0}$  and  $a_\rho = \bar{a}_\rho = 0$ . Clearly the weight of terms in  $a_\mu$  will translate to growth rate with  $\rho$  in  $A_\mu$  because the weight is the eigenvalue of the commutator with  $L_0$ . An exact Lifshitz spacetime can be obtained by setting

$$a_x = L_1, \quad a_t = \frac{\sqrt{3}}{4} W_2 \quad (5.15)$$

$$\bar{a}_x = L_{-1}, \quad \bar{a}_t = \frac{\sqrt{3}}{4} W_{-2} \quad (5.16)$$



that is

$$A = L_0 d\rho + L_1 e^\rho dx + \frac{\sqrt{3}}{4} W_2 e^{2\rho} dt \quad (5.17)$$

$$\bar{A} = -L_0 d\rho + L_{-1} e^\rho dx + \frac{\sqrt{3}}{4} W_{-2} e^{2\rho} dt \quad (5.18)$$

One can verify that the connection yields Lifshitz spacetime  $ds^2 = d\rho^2 + e^{2\rho} dx^2 - e^{4\rho} dt^2$  with Lifshitz scaling exponent  $z = 2$ . Now we add dynamical terms to the connection but keeping the leading term fixed to get asymptotic Lifshitz spacetime. We will focus on the unbarred sector here, the barred sector can be worked out by the same algorithm thanks to the weight flipping automorphism of  $sl(3|2)$ . The ansatz of  $a_x$  in the lowest weight gauge is

$$a_x = L_1 + jJ + aA_{-1} + lL_{-1} + wW_{-2} + gG_{-\frac{1}{2}} + hH_{-\frac{1}{2}} + sS_{-\frac{3}{2}} + tT_{-\frac{3}{2}} \quad (5.19)$$

with all the dynamical terms being the lowest weight elements in  $sl(3|2)$ . The component  $a_t$  should start with  $\frac{\sqrt{3}}{4}W_2$ , and its non-highest weight terms are completely determined by highest weight terms because it must preserve the lowest weight gauge of  $a_x$  in time evolution. We take the highest weight terms to be differential polynomials of fields in  $a_x$  of the correct dimension, so  $a_t$  must take the form

$$a_t = \frac{\sqrt{3}}{2} \left( \frac{1}{2} W_2 + (d_1 a + d_2 l + d_3 j^2 + d_4 j') J + c_1 j A_1 + c_2 j L_1 + c_3 g G_{\frac{1}{2}} + c_4 h H_{\frac{1}{2}} + \dots \right) \quad (5.20)$$

with eight free constants  $c_1, c_2, c_3, c_4, d_1, d_2, d_3, d_4$ , and with non-highest weight terms omitted. We factor out  $\frac{\sqrt{3}}{2}$  for calculational simplicity. Now we deal with the problem of fixing  $a_t$  to map the time evolution equation of  $a_x$  to the  $N = 2$  super Boussinesq equation. In order to have the lowest dimensional conserved bosonic charge and fermionic quantities,  $\dot{j}$  and  $\dot{g}$  must be total derivatives.

This condition fixes  $a_t$  up to only one free constant  $c_3$

$$\begin{aligned}
c_1 &= -1 \\
c_2 &= -2c_3 + \frac{5}{3} \\
c_4 &= -c_3 \\
d_1 &= \frac{1}{9}(8 - 15c_3) \\
d_2 &= -c_3 \\
d_3 &= -3c_3 \\
d_4 &= 0
\end{aligned} \tag{5.21}$$

It turns out if we set  $c_3 = \frac{1}{3}$ , the time evolution equation of  $a_x$  can be identified with the  $N = 2$  super Boussinesq equation after we rescale the time evolution by a factor  $-2\sqrt{3}$ , that is equivalent to replacing  $a_t$  by  $-2\sqrt{3}a_t$ . The time evolution equation of  $a_x$  we get after replacing  $a_t$  by  $-2\sqrt{3}a_t$  is

$$\begin{aligned}
\dot{j} &= (l - a + 3j^2)' \\
\dot{g} &= g'' - 6s' + 4(jg)' \\
\dot{h} &= -h'' - 6t' + 4(jh)' \\
\dot{a} &= \frac{3}{2}j''' + 6w' + 3jl' + 6lj' - 6aj' - 3ja' + \frac{15}{2}(gh)' + \frac{27}{2}(gt + hs) \\
\dot{l} &= -\frac{3}{2}j''' + 6w' - 3jl' - 6lj' + 6aj' + 3ja' - \frac{33}{2}(gh)' - \frac{45}{2}(gt + hs) \\
\dot{s} &= -s'' + \frac{2}{3}g''' - (10a + 6l + 6j^2 + 6j')s - 4js' + \left(\frac{10}{3}a' + \frac{14}{3}l' - \frac{10}{3}j'' - 16w + \frac{32}{3}aj - \frac{4}{3}jj'\right)g \\
&\quad + \left(\frac{14}{3}a + \frac{2}{3}j^2 + 6l - \frac{2}{3}j'\right)g' + \frac{4}{3}jg'' \\
\dot{t} &= t'' + \frac{2}{3}h''' + (10a + 6l + 6j^2 - 6j')t - 4jt' + \left(\frac{10}{3}a' + \frac{14}{3}l' + \frac{10}{3}j'' + 16w - \frac{32}{3}aj - \frac{4}{3}jj'\right)h \\
&\quad + \left(\frac{14}{3}a + \frac{2}{3}j^2 + 6l + \frac{2}{3}j'\right)h' - \frac{4}{3}jh'' \\
\dot{w} &= -\frac{1}{4}(a + l)''' - 2[(a + l)^2]' - 4j'gh + j(gh)' + 9j(gt + hs) - \frac{7}{4}(gh'' + hg'') \\
&\quad - \frac{9}{4}g't - \frac{15}{4}gt' + \frac{9}{4}h's + \frac{15}{4}hs'
\end{aligned} \tag{5.22}$$

which can be identified with the  $N = 2$  super Boussinesq equation via the explicit map

$$\begin{aligned}
j &= \alpha y \\
g &= k\xi, \quad h = -k\bar{\xi} \\
a &= -\frac{3}{4}\alpha v, \quad l = -\alpha^2 y^2 + \frac{5}{4}\alpha v + 2\alpha u \\
s &= -\frac{1}{3}k\eta, \quad t = \frac{1}{3}k\bar{\eta} \\
w &= \frac{\alpha}{4}z - \frac{\alpha^2}{2}yv
\end{aligned} \tag{5.23}$$

with  $k^2 = 2\alpha^2$ , no matter which root  $k$  takes.

We have worked out a specific example of the relation between supersymmetric Chern-Simons Lifshitz theory and super Boussinesq hierarchy, that is, we established the map between  $sl(3|2)$  Lifshitz Chern-Simons theory and  $N = 2$  super Boussinesq hierarchy such that the time evolution equations of the two theories coincide. Now we show that there is a structurally deeper connection of the two theories, the Poisson structure of  $sl(3|2)$  Lifshitz Chern-Simons theory induced by gauge transformation is identical to the second Hamiltonian structure of  $N = 2$  super Boussinesq hierarchy.

The time evolution of Chern-Simons is essentially a gauge transformation with gauge transformation parameter  $a_t$ , that is  $\dot{a}_x = \delta_{a_t} a_x = \partial_x a_t + [a_x, a_t]$ . Fixed in the lowest weight gauge, the gauge transformation induces a Poisson structure of the fields in the reduced phase space [33]. That is, the gauge transformation of a field  $\phi$  with gauge parameter  $\lambda$  is regarded as a Poisson bracket between the field and the charge associated with the gauge transformation parameter

$$\delta_\lambda \phi = \{Q_\lambda, \phi\} \tag{5.24}$$

where the charge is given by

$$\delta Q_\lambda = C \int dx \operatorname{str} \lambda \delta a_x \tag{5.25}$$

with  $C = -\frac{k}{2\pi}$ . The Poisson brackets of all fields can be computed by choosing different gauge parameters, and it's used to calculate the boundary charge algebra in the context of

holography, see for example [49]. Now the time evolution equation of Chern-Simons theory can be recast in a form resembling the Hamiltonian dynamics

$$\dot{a}_x = \{Q_{a_t}, a_x\} \quad (5.26)$$

On the other hand, the time evolution of the  $N = 2$  super Boussinesq hierarchy is generated by its Hamiltonian structure

$$\dot{T} = \{T, H\}, \quad \dot{J} = \{J, H\} \quad (5.27)$$

Since we have a map between the two theories that identifies the time evolution equation, it's natural to conjecture the Poisson structure of the  $sl(3|2)$  Chern-Simons theory is identical to the second Hamiltonian structure of  $N = 2$  super Boussinesq hierarchy via the established map. Note we have replaced  $a_t$  by  $-2\sqrt{3}a_t$  to make the map, it's actually

$$\dot{a}_x = \{-2\sqrt{3}Q_{a_t}, a_x\} \quad (5.28)$$

that is identified with the  $N = 2$  super Boussinesq equation, therefore we must have  $2\sqrt{3}Q_{a_t} = H$ . Straightforward computation yields

$$2\sqrt{3}Q_{a_t} = 2C \int dx j(l - a) + 4w + j^3 - gh = 6\alpha C \int dx z + 2\alpha(yu + \xi\bar{\xi}) \quad (5.29)$$

where we have used the map between two theories. On the other hand, the Hamiltonian of the second Hamiltonian structure of  $N = 2$  super Boussinesq hierarchy is given as

$$H = \int dx d\theta d\bar{\theta} (T + \alpha J^2) = \int dx z + 2\alpha(uy + \xi\bar{\xi}) \quad (5.30)$$

We see that  $2\sqrt{3}Q_{a_t}$  is equal to the Hamiltonian in the second Hamiltonian structure of the  $N = 2$  super Boussinesq hierarchy with the choice  $C = \frac{1}{6\alpha}$ . We have computed the Poisson structure of the  $sl(3|2)$  Chern-Simons theory with  $C = \frac{1}{6\alpha}$  in the below. One can verify it's indeed identical to the second Hamiltonian structure of the  $N = 2$  super Boussinesq hierarchy given in [46].

As an example, we show how to calculate the Poisson bracket  $\{h(x'), g(x)\}$ . Clearly we need to find a gauge transformation parameter  $\lambda$  which is associated with the charge  $Q_\lambda$

that takes the form of an integral of the product of  $h$  and an arbitrary fermionic function.  $\text{str}(G_{\frac{1}{2}}H_{-\frac{1}{2}})$  is nonzero so we want  $\lambda$  to start with  $\gamma G_{\frac{1}{2}}$ , where  $\gamma$  is an arbitrary fermionic function. The other non-highest weight terms in  $\lambda$  are determined by requiring the gauge transformation preserves the lowest weight gauge of  $a_x$  and we find

$$\lambda = \gamma G_{\frac{1}{2}} - (\gamma' + \gamma j)G_{-\frac{1}{2}} - \frac{9}{4}\gamma a S_{-\frac{3}{2}} + \gamma h L_{-1} - \frac{3}{8}\gamma t W_{-2} \quad (5.31)$$

The associated charge is calculated as

$$\begin{aligned} \delta Q_\lambda &= \frac{1}{6\alpha} \int dx \text{str}(\lambda \delta a_x) = -\frac{1}{\alpha} \int dx \gamma \delta \\ Q_\lambda &= -\frac{1}{\alpha} \int dx \gamma h \end{aligned} \quad (5.32)$$

The gauge transformation on  $g$  is calculated to be

$$\begin{aligned} \delta_\lambda g(x) &= -\gamma(x) \left( \frac{5}{3} a(x) + j(x)^2 + l(x) + j'(x) \right) - 2\gamma'(x)j(x) - \gamma''(x) \\ &= \{Q_\lambda, g(x)\} = -\frac{1}{\alpha} \int dx' \gamma(x') \{h(x'), g(x)\} \end{aligned} \quad (5.33)$$

Therefore

$$\{h(x'), g(x)\} = \alpha \left( \frac{5}{3} a(x) + j(x)^2 + l(x) + j'(x) \right) \delta(x' - x) - 2\alpha j(x) \delta'(x' - x) + \alpha \delta''(x' - x) \quad (5.34)$$

We list the Poisson brackets of all the fields here

$$\begin{aligned}
\{j, j\} &= \alpha\delta' \\
\{j, g\} &= \alpha g\delta \\
\{j, h\} &= -\alpha h\delta \\
\{j, s\} &= \alpha s\delta \\
\{j, t\} &= -\alpha t\delta \\
\{h, g\} &= \alpha\left(\frac{5}{3}a + j^2 + l + j'\right)\delta - 2\alpha j\delta' + \alpha\delta'' \\
\{h, s\} &= \frac{4}{9}\alpha(-6w + a' + 4aj)\delta - \frac{16}{9}\alpha a\delta' \\
\{h, a\} &= \frac{9}{4}\alpha t\delta \\
\{h, l\} &= -\alpha\left(2jh + h' + \frac{15}{4}t\right)\delta + 3\alpha h\delta' \\
\{h, w\} &= \alpha\left(\frac{2}{3}ah + \frac{3}{2}jt + \frac{3}{8}t'\right)\delta + \frac{15}{8}\alpha t\delta' \\
\{g, t\} &= -\frac{4}{9}\alpha(6w + a' - 4aj)\delta + \frac{16}{9}\alpha a\delta' \\
\{g, a\} &= -\frac{9}{4}\alpha s\delta \\
\{g, l\} &= -\alpha\left(-2jg + g' - \frac{15}{4}t\right)\delta + 3\alpha g\delta' \\
\{g, w\} &= -\alpha\left(\frac{2}{3}ag + \frac{3}{2}js - \frac{3}{8}s'\right)\delta - \frac{15}{8}\alpha s\delta' \\
\frac{1}{6\alpha}\{5a + 3l + 3j^2, g\} &= -g'\delta + \frac{3}{2}g\delta' \\
\frac{1}{6\alpha}\{5a + 3l + 3j^2, h\} &= -h'\delta + \frac{3}{2}h\delta' \\
\frac{1}{6\alpha}\{5a + 3l + 3j^2, a\} &= -a'\delta + 2a\delta' \\
\frac{1}{6\alpha}\{5a + 3l + 3j^2, l\} &= -l'\delta + 2l\delta' + \frac{1}{2}\delta''' \\
\frac{1}{6\alpha}\{5a + 3l + 3j^2, s\} &= -s'\delta + \frac{5}{2}s\delta' \\
\frac{1}{6\alpha}\{5a + 3l + 3j^2, t\} &= -t'\delta + \frac{5}{2}t\delta' \\
\frac{1}{6\alpha}\{5a + 3l + 3j^2, w\} &= -w'\delta + 3w\delta'
\end{aligned}$$

$$\begin{aligned}
\{a, a\} &= -\frac{3}{8}\alpha(5a' - 3l')\delta + \frac{3}{8}\alpha(10a - 6l)\delta' - \frac{9}{16}\alpha\delta''' \\
\{a, s\} &= -\frac{\alpha}{4}(g(9l - 9a + j^2 + j') + 6js + 2jg' + 6s' + g'')\delta + \frac{\alpha}{4}(4jg + 15s + 4g')\delta' - \frac{3\alpha}{2}g\delta'' \\
\{a, t\} &= -\frac{\alpha}{4}(h(9a - 9l - j^2 + j') - 6jt + 2jh' + 6t' - h'')\delta + \frac{\alpha}{4}(4jh + 15t - 4h')\delta' + \frac{3\alpha}{2}h\delta'' \\
\{a, w\} &= -\frac{3\alpha}{16}(-15(gt + hs) + 5(gh)' + 4w')\delta + \frac{3\alpha}{16}(15gh + 12w)\delta' \\
\{t, t\} &= \frac{2\alpha}{3}(10ht + \frac{4}{3}hh')\delta \\
\{s, s\} &= -\frac{2\alpha}{3}(10gs - \frac{4}{3}gg')\delta \\
\{t, s\} &= -\frac{2\alpha}{3}(-\frac{5}{2}a^2 + \frac{5}{9}aj^2 + \frac{1}{6}j^4 + al + \frac{5}{3}j^2l + \frac{3}{2}l^2 + \frac{8}{3}jgh + 4gt + 3gh' - 4hs - \frac{4}{3}hg' \\
&\quad - \frac{8}{3}jw + \frac{5}{9}(ja)' + \frac{5}{3}(jl)' + j^2j' + \frac{1}{2}(j')^2 - \frac{4}{3}w' + \frac{1}{6}a'' + \frac{2}{3}jj'' + \frac{1}{2}l'' + \frac{1}{6}j''')\delta \\
&\quad + \frac{2\alpha}{3}(\frac{10}{9}aj + \frac{2}{3}j^3 + \frac{10}{3}jl + \frac{13}{3}gh) - \frac{8}{3}w + \frac{5}{9}a + 2jj' + \frac{5}{3}l' + \frac{2}{3}j''\delta' \\
&\quad - \frac{2\alpha}{3}(\frac{5}{9}a + j^2 + \frac{5}{3}l + j')\delta'' + \frac{4\alpha}{9}\delta''' - \frac{\alpha}{9}\delta^{(4)} \\
\{t, w\} &= -\frac{2\alpha}{3}(\frac{13}{12}ajh + \frac{1}{4}j^3h + \frac{9}{4}jlh - \frac{33}{8}at - \frac{3}{8}j^2t - \frac{15}{8}lt - 5wh + \frac{13}{16}a'h + \frac{13}{16}ah' \\
&\quad + \frac{3}{16}j^2h' + \frac{19}{16}lh' + \frac{9}{8}jj'h - \frac{3}{4}j't + \frac{7}{16}j'h' + \frac{27}{16}l'h - \frac{3}{8}jt' + \frac{1}{8}jh'' + \frac{9}{16}j''h - \frac{3}{16}t'' + \frac{1}{16}h''')\delta \\
&\quad + \frac{2\alpha}{3}(\frac{91}{48}ah + \frac{15}{16}j^2h + \frac{55}{16}lh - \frac{9}{8}jt + \frac{5}{8}jh' + \frac{25}{16}j'h - \frac{3}{4}t' + \frac{5}{16}h'')\delta' \\
&\quad - \frac{2\alpha}{3}(\frac{5}{4}jh - \frac{15}{16}t + \frac{5}{8}h')\delta'' + \frac{5}{12}\alpha h\delta''' \\
\{s, w\} &= -\frac{2\alpha}{3}(-\frac{13}{12}ajg - \frac{1}{4}j^3g - \frac{9}{4}jlg + \frac{33}{8}as + \frac{3}{8}j^2s + \frac{15}{8}ls + 5wg + \frac{13}{16}a'g + \frac{13}{16}ag' \\
&\quad + \frac{3}{16}j^2g' + \frac{19}{16}lg' + \frac{9}{8}jj'g - \frac{3}{4}j's - \frac{7}{16}j'g' + \frac{27}{16}l'g - \frac{3}{8}js' - \frac{1}{8}jg'' - \frac{9}{16}j''g + \frac{3}{16}s'' + \frac{1}{16}g''')\delta \\
&\quad + \frac{2\alpha}{3}(\frac{91}{48}ag + \frac{15}{16}j^2g + \frac{55}{16}lg - \frac{9}{8}js - \frac{5}{8}jg' - \frac{25}{16}j'g + \frac{3}{4}s' + \frac{5}{16}g'')\delta' \\
&\quad - \frac{2\alpha}{3}(-\frac{5}{4}jg + \frac{15}{16}s + \frac{5}{8}g')\delta'' + \frac{5}{12}\alpha g\delta''' \\
\{w, w\} &= \frac{\alpha}{32}(33(gt)' - 33(hs)' - 28(jgh)' + 14(gh'' - g''h) + 16((a+l)')^2 + 2(a+l)''')\delta \\
&\quad - \frac{\alpha}{32}(32(a+l)^2 - 56jgh + 66(gt - hs) + 28gh' - 28g'h + 9(a+l)''')\delta' \\
&\quad + \frac{15\alpha}{32}(a+l)'\delta'' - \frac{5\alpha}{16}(a+l)\delta''' - \frac{\alpha}{64}\delta^{(5)} \tag{5.35}
\end{aligned}$$

where the first field in the bracket is at  $x'$  and the second is at  $x$ , all the fields on the right hand side are at  $x$ , and  $\delta$  is short for  $\delta(x' - x)$ . Brackets of fields not listed above

are either zero or can be inferred from the brackets listed by simple principle, for example, antisymmetry of Poisson brackets.



# CHAPTER 6

## Discussion

In Chapter 4 we have shown that the asymptotic Lifshitz solution for  $sl(N, \mathbb{R})$  and an arbitrary integer Lifshitz scaling exponent  $z$  can be mapped to the member of the KdV hierarchies with  $n = N, m = z$ . Now a natural question rises for the Lifshitz connection in the algebra  $hs(\lambda)$  where  $\lambda$  is not an integer: Can the equations of motion, which involve infinite number of fields be mapped to some integrable hierarchy? Note that such an integrable system should reduce to the KdV hierarchy when  $hs(\lambda)$  is truncated to  $sl(N, \mathbb{R})$  upon setting  $\lambda = N$ . A candidate for such integrable systems is the KP hierarchy which we briefly review here.

The starting point is the following pseudo differential operator which contains infinitely many fields  $v_i, i = 2, 3, \dots$ .

$$S = \partial + v_2\partial^{-1} + v_3\partial^{-2} + v_4\partial^{-3} + \dots \quad (6.1)$$

The Lax equation for the  $m$ -th element of the hierarchy<sup>1</sup> is defined by

$$\frac{\partial}{\partial t} S = [S_+^m, S] \quad (6.2)$$

The Lax equation gives equations of motion of the KP variables  $v$ 's.

The connection of the KP hierarchy to the KdV hierarchy is obtained as follows: Note that the Lax equation above implies the following equation for the  $n$ -th power of the operator  $S$

$$\dot{S}^n = [S_+^m, S^n] \quad (6.3)$$

---

<sup>1</sup>Note that the name “KP hierarchy” is usually reserved for the system of equations for all  $m$  where a different time variable  $t_m$  is associated with each element. We are interested in the a specific element of the hierarchy and denote the time simply by  $t$ .

With the definitions  $L = S^n$  and  $P_m = S_+^{\frac{m}{n}}$  we get the Lax equation of KdV defined in (2.8). At this point the pseudo differential operator  $L$  contains all possible powers of  $\partial$ , down to  $\partial^{-\infty}$ . It is possible to consistently restrict  $L$  to only non-negative powers of differentiation, which implies that the dynamics of the first  $n - 1$  variables is decoupled from the other, and they are just KdV hierarchy with the same values of  $m$  and  $n$ . Consequently, it is possible to perform a field redefinition to truncate KP to KdV. The map from  $sl(N, \mathbb{R}), z$  Chern-Simons Lifshitz theory to KdV with  $m = z, n = N$  can be regarded as a part of the whole map from  $hs(N), z$  Chern-Simons Lifshitz theory to KP with  $m = z$ , with  $N$  being the parameter of the map.

In general it is possible to define powers of the pseudo differential operator  $S$  to for non integer exponents [51, 52]. We conjecture that by choosing  $N$  as a real number  $\lambda$  we will be able to construct a map between Chern-Simons Lifshitz theory with generic  $hs(\lambda)$  and KP. We leave the explicit construction of this map for future work, but observe that there are several arguments that indicate that this correspondence indeed exists. First, finding the maps involves solving algebraic equations, as in the case of  $\lambda = N$ , but the recursive solution in general does not require  $N$  to be an integer. Second, the  $hs(\lambda)$  Chern-Simons for a conformal theory provides a realization of the  $W_\infty$  nonlinear extension of the  $W_N$  algebras [24, 53, 54]. While the construction is slightly different many of the features of the relation such as the relation of the gauge transformations which preserve the lowest weight gauge of  $a_x$  to the  $W$ -algebra transformation, carry over. When  $W$  algebras were first investigated in the early '90 a relation of the  $W_\infty$  algebra to the KP hierarchy was proposed in several papers [51, 55, 52, 56, 57, 58]

The supersymmetric extension of the correspondence between the Lifshitz Chern-Simons theory and the KdV hierarchies is worth some discussion too. In Chapter 5 we worked out that  $sl(3|2)$  Lifshitz Chern-Simons theory, as the supersymmetric extension of  $sl(3, \mathbb{R})$  Lifshitz Chern-Simons theory, corresponds to  $N = 2$  super Boussinesq hierarchy constructed in [46] with the appropriate choice of parameters  $\alpha = -\frac{4}{c}$ . It was found in [46] that for  $c = 8$  there are three values of  $\alpha$  (including the one we choose  $\alpha = -\frac{1}{2}$ ) such that the

equation obtained is an integrable system, in the sense that an infinite tower of higher order conserved quantities exist. In addition for some of them a Lax pair formulation exists or a bi-Hamiltonian structure exists [46, 48]. It is a natural question to ask if the super Boussinesq hierarchy with other values of the parameter  $\alpha$  also corresponds to Lifshitz Chern-Simons theory with other gauge algebra different from  $sl(3|2)$ . In fact, in almost all the cases of supersymmetric extension of KdV hierarchies, it turns out we have to choose a discrete set of values of the parameters to make the theory integrable [59, 60, 61]. If we can formulate all these supersymmetric extensions of KdV by Lifshitz Chern-Simons theory with different Lie superalgebras, we may be able to explain the choices of discrete values of parameters in the perspective of the theory of Lie superalgebras.

Another possible direction for research lies in the the construction of blackhole solutions in supersymmetric Lifshitz Chern-Simons theories following the work on the bosonic case [42, 62]. In particular, the integrability may enable some analytic calculation of the thermodynamic properties of the blackhole.

## APPENDIX A

### $sl(3, \mathbb{R})$ , $sl(4, \mathbb{R})$ and $hs(\lambda)$ conventions

In this appendix we present a realization of the  $sl(N, \mathbb{R})$  algebra which are used for calculations in the main body of the text.

#### A.1 $sl(3, \mathbb{R})$

The  $sl(2, \mathbb{R})$  generators of the principal embedding are given by the following matrices

$$L_{-1} = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}, \quad L_1 = \begin{pmatrix} 0 & 0 & 0 \\ -\sqrt{2} & 0 & 0 \\ 0 & -\sqrt{2} & 0 \end{pmatrix}, \quad L_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad (\text{A.1})$$

and the spin 3 generators, on which we omit the superscript <sup>(3)</sup> for notational simplicity, are as follows:

$$W_{-2} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad W_{-1} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \end{pmatrix}, \quad W_0 = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & -\frac{2}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} \quad (\text{A.2})$$

$$W_1 = \begin{pmatrix} 0 & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \quad W_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} \quad (\text{A.3})$$

If we define  $(T_1, T_2, \dots, T_8) = (L_1, L_0, L_{-1}, W_2, \dots, W_{-2})$ , then traces of all pairs of generators are given by

$$\text{tr}(T_i T_j) = \left( \begin{array}{ccc|ccc} & & -4 & 0 & \cdots & 0 \\ & & 2 & \vdots & \ddots & \vdots \\ -4 & & & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & & & 4 \\ & & & & & -1 \\ \vdots & \ddots & \vdots & & \frac{2}{3} & \\ & & & -1 & & \\ 0 & \cdots & 0 & 4 & & \end{array} \right) \quad (\text{A.4})$$

## A.2 $sl(4, \mathbb{R})$

The  $sl(4, \mathbb{R})$  matrix representation we use is the following. The  $sl(2, \mathbb{R})$  sub algebra given by

$$l_0 = \begin{pmatrix} -\frac{3}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{3}{2} \end{pmatrix} \quad l_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad l_{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -3 & 0 \end{pmatrix} \quad (\text{A.5})$$

$w_i, i = +2, +1, \dots, -2$  form a spin 2 representation, whereas the  $u_i, i = +3, +3, \dots, -3$  form a spin 3 representation of the  $sl(2, \mathbb{R})$  sub algebra.

$$\begin{aligned}
w_2 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & w_1 &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} & w_0 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
w_{-1} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 \end{pmatrix} & w_{-2} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 12 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 \end{pmatrix} & u_3 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
u_2 &= \begin{pmatrix} 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & u_1 &= \begin{pmatrix} 0 & \frac{2}{5} & 0 & 0 \\ 0 & 0 & -\frac{3}{5} & 0 \\ 0 & 0 & 0 & \frac{2}{5} \\ 0 & 0 & 0 & 0 \end{pmatrix} & u_0 &= \begin{pmatrix} -\frac{3}{10} & 0 & 0 & 0 \\ 0 & \frac{9}{10} & 0 & 0 \\ 0 & 0 & -\frac{9}{10} & 0 \\ 0 & 0 & 0 & \frac{3}{10} \end{pmatrix} \\
u_{-1} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\frac{6}{5} & 0 & 0 & 0 \\ 0 & \frac{12}{5} & 0 & 0 \\ 0 & 0 & -\frac{6}{5} & 0 \end{pmatrix} & u_{-2} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -6 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \end{pmatrix} & u_{-3} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -36 & 0 & 0 & 0 \end{pmatrix}
\end{aligned} \tag{A.6}$$

The  $w_i, i = +2, +1, \dots, -2$  form a spin 2 representation, whereas the  $u_i, i = +3, +3, \dots, -3$  form a spin 3 representation of the  $sl(2, \mathbb{R})$  sub algebra.

### A.3 $hs(\lambda)$ conventions

Higher spin algebra elements  $V_m^s, s = 1, 2, 3, \dots$  and  $m = -s + 1, -s + 2, \dots, s - 1$ . We call  $s$  the spin and  $m$  the weight.

The lone star product is defined as

$$V_m^s * V_n^t = \frac{1}{2} \sum_{u=1}^{s+t-|s-t|-1} g_u^{st}(m, n, \lambda) V_{m+n}^{s+t-u} \quad (\text{A.7})$$

The structure constants of the  $hs(\lambda)$  algebra were defined in [?] and can be represented as follows

$$g_u^{st}(m, n; \lambda) = \frac{q^{u-2}}{2(u-1)!} \phi_u^{st}(\lambda) N_u^{st}(m, n) \quad (\text{A.8})$$

$q$  is a normalization constant which can be eliminated by a rescaling on the generators, we choose  $q = 1/4$  to agree with the literature. The other terms in (A.8) are given by

$$N_u^{st}(m, n) = \sum_{k=0}^{u-1} (-1)^k \binom{u-1}{k} [s-1+m]_{u-1-k} [s-1-m]_k [t-1+n]_k [t-1-n]_{u-1-k}$$

$$\phi_u^{st}(\lambda) = F_3 \left[ \begin{array}{cccc} \frac{1}{2} + \lambda & \frac{1}{2} - \lambda & \frac{2-u}{2} & \frac{1-u}{2} \\ \frac{3}{2} - s & \frac{3}{2} - t & \frac{1}{2} + s + t - u & 1 \end{array} \right] \quad (\text{A.9})$$

The descending Pochhammer symbol  $[a]_n$  is defined as,

$$[a]_n = a(a-1)\dots(a-n+1) \quad (\text{A.10})$$

The commutator is defined as

$$[V_m^s, V_n^t] = V_m^s * V_n^t - V_n^t * V_m^s \quad (\text{A.11})$$

$V_0^1$  is the unit element. The trace of a  $hs(\lambda)$  element is defined as the coefficient of  $V_0^1$  up to a multiplicative constant  $\text{tr}(V_0^1)$ . When  $\lambda = N$  where  $N$  is a positive integer,  $hs(\lambda)$  is truncated to  $sl(N, \mathbb{R})$ . That means, we can consistently set  $V_m^s$  to be zero if  $s > N$ , and the remaining elements form  $sl(N, \mathbb{R})$  with star product identified as matrix multiplication and trace identified as matrix trace.

## APPENDIX B

### A review of $sl(3|2)$

The bosonic part of  $sl(3|2)$  is  $U(1) \oplus sl(2, \mathbb{R}) \oplus sl(3, \mathbb{R})$ , it's generated by spin 2 generators  $L_i, A_i$ ,  $i = -1, 0, 1$ , spin 3 generators  $W_i$ ,  $i = -2, -1, 0, 1, 2$ , and  $J$  the generator of  $u(1)$ . The fermionic part of  $sl(3|2)$  is generated by spin  $\frac{3}{2}$  generators  $G_r, H_r$ ,  $r = -\frac{1}{2}, \frac{1}{2}$  and spin  $\frac{5}{2}$  generators  $S_r, T_r$ ,  $r = -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$ .  $L_i$  generate the  $sl(2, \mathbb{R})$  subalgebra and the  $L_0$  is the Cartan generator. The non-zero commutation relations are

$$\begin{aligned}
 [L_i, L_j] &= (i - j)L_{i+j} & [A_i, A_j] &= (i - j)L_{i+j} & [L_i, A_j] &= (i - j)A_{i+j} \\
 [L_i, W_j] &= (2i - j)W_{i+j} & [A_i, W_j] &= (2i - j)W_{i+j} \\
 [W_i, W_j] &= \frac{1}{6}(j - i)(2i^2 + 2j^2 - ij - 8)(L_{i+j} + A_{i+j}) \\
 [L_i, G_r] &= \left(\frac{i}{2} - r\right)G_{i+r} & [L_i, H_r] &= \left(\frac{i}{2} - r\right)H_{i+r} \\
 [L_i, S_r] &= \left(\frac{3i}{2} - r\right)S_{i+r} & [L_i, T_r] &= \left(\frac{3i}{2} - r\right)T_{i+r} \\
 [A_i, G_r] &= \frac{4}{3}S_{i+r} + \frac{5}{3}\left(\frac{i}{2} - r\right)G_{i+r} & [A_i, H_r] &= -\frac{4}{3}T_{i+r} + \frac{5}{3}\left(\frac{i}{2} - r\right)H_{i+r} \\
 [A_i, S_r] &= \frac{1}{3}\left(\frac{3i}{2} - r\right)S_{i+r} - \frac{1}{3}\left(3i^2 - 2ir + r^2 - \frac{9}{4}\right)G_{i+r} \\
 [A_i, T_r] &= \frac{1}{3}\left(\frac{3i}{2} - r\right)T_{i+r} + \frac{1}{3}\left(3i^2 - 2ir + r^2 - \frac{9}{4}\right)H_{i+r}
 \end{aligned}$$



$$\begin{aligned}
[W_i, G_r] &= -\frac{4}{3}\left(\frac{i}{2} - 2r\right)S_{i+r} & [W_i, H_r] &= -\frac{4}{3}\left(\frac{i}{2} - 2r\right)T_{i+r} \\
[W_i, S_r] &= -\frac{1}{3}(2r^2 - 2ir + i^2 - \frac{5}{2})S_{i+r} - \frac{1}{6}(4r^3 - 3ir^2 + 2i^2r - i^3 - 9r - \frac{19}{4}i)G_{i+r} \\
[W_i, T_r] &= -\frac{1}{3}(2r^2 - 2ir + i^2 - \frac{5}{2})T_{i+r} - \frac{1}{6}(4r^3 - 3ir^2 + 2i^2r - i^3 - 9r - \frac{19}{4}i)H_{i+r} \\
[J, G_r] &= G_r & [J, H_r] &= -H_r & [J, S_r] &= S_r & [J, T_r] &= -T_r \\
\{G_r, H_s\} &= 2L_{r+s} + (r-s)J \\
\{S_r, T_s\} &= -\frac{3}{4}(r-s)W_{r+s} + \frac{1}{8}(3s^2 - 4rs + 3r^2 - \frac{9}{2})(L_{r+s} - 3A_{r+s}) - \frac{1}{4}(r-s)(r^2 + s^2 - \frac{5}{2})J \\
\{G_r, T_s\} &= -\frac{3}{2}W_{r+s} + \frac{3}{4}(3r-s)A_{r+s} - \frac{5}{4}(3r-s)L_{r+s} \\
\{H_r, S_s\} &= -\frac{3}{2}W_{r+s} - \frac{3}{4}(3r-s)A_{r+s} + \frac{5}{4}(3r-s)L_{r+s}
\end{aligned} \tag{B.1}$$

The subindex is the weight of the element, it's the eigenvalue of the commutator with  $L_0$ , and can be raised (lowered) by  $L_1$  ( $L_{-1}$ ). A weight-flipping automorphism exists

$$\begin{aligned}
J &\rightarrow -J \\
L_0 &\rightarrow -L_0, & L_1 &\rightarrow L_{-1}, & L_{-1} &\rightarrow L_1 \\
A_0 &\rightarrow -A_0, & A_1 &\rightarrow A_{-1}, & A_{-1} &\rightarrow A_1 \\
W_2 &\rightarrow -W_{-2}, & W_1 &\rightarrow W_{-1}, & W_0 &\rightarrow -W_0, & W_{-1} &\rightarrow W_1, & W_{-2} &\rightarrow -W_2 \\
G_{\frac{1}{2}} &\rightarrow H_{-\frac{1}{2}}, & G_{-\frac{1}{2}} &\rightarrow -H_{\frac{1}{2}} \\
H_{\frac{1}{2}} &\rightarrow -G_{-\frac{1}{2}}, & H_{-\frac{1}{2}} &\rightarrow G_{\frac{1}{2}} \\
S_{\frac{3}{2}} &\rightarrow -T_{-\frac{3}{2}}, & S_{\frac{1}{2}} &\rightarrow T_{-\frac{1}{2}}, & S_{-\frac{1}{2}} &\rightarrow -T_{\frac{3}{2}}, & S_{-\frac{3}{2}} &\rightarrow T_{\frac{3}{2}} \\
T_{\frac{3}{2}} &\rightarrow S_{-\frac{3}{2}}, & T_{\frac{1}{2}} &\rightarrow -S_{-\frac{1}{2}}, & T_{-\frac{1}{2}} &\rightarrow S_{\frac{1}{2}}, & T_{-\frac{3}{2}} &\rightarrow -S_{\frac{3}{2}}
\end{aligned} \tag{B.2}$$

The defining representation by super matrix is given by the following expressions

$$J = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

$$L_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix}, L_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, L_{-1} = \begin{pmatrix} 0 & -\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}, A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}, A_{-1} = \begin{pmatrix} 0 & -\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$W_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, W_{-2} = \begin{pmatrix} 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$W_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & -\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, W_{-1} = \begin{pmatrix} 0 & -\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, W_0 = \begin{pmatrix} \frac{2}{3} & 0 & 0 & 0 & 0 \\ 0 & -\frac{4}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$G_{\frac{1}{2}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 \end{pmatrix}, G_{-\frac{1}{2}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \end{pmatrix}$$

$$H_{\frac{1}{2}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, H_{-\frac{1}{2}} = \begin{pmatrix} 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned}
S_{\frac{3}{2}} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 & 0 \end{pmatrix}, S_{-\frac{3}{2}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
S_{\frac{1}{2}} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 \end{pmatrix}, S_{-\frac{1}{2}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix} \\
T_{\frac{3}{2}} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, T_{-\frac{3}{2}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
T_{\frac{1}{2}} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, T_{-\frac{1}{2}} = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{B.3}
\end{aligned}$$

They are all super-traceless, and closed under multiplication with the identity super matrix added. In addition, the weight is additive under super matrix multiplication if we count the weight of the identity super matrix as zero. In this super matrix representation, the weight-flipping automorphism is simply given by taking the negative of the transposition of the bosonic elements and the transposition of the fermionic elements.

## APPENDIX C

### Chern-Simons KdV map

In this appendix we exhibit explicit results for the map between  $sl(N, \mathbb{R})$  Lifshitz Chern-Simons theory with scaling exponent  $z$  and the  $m = z, n = N$  member of KdV hierarchy, for a few values of  $N$  and  $z$ .

#### $N = 3$

CS-KdV map:

$$\begin{aligned}u_2 &= 4\alpha_2, \\u_3 &= 2\alpha'_2 - 4\alpha_3.\end{aligned}\tag{C.1}$$

KdV equations of motion at  $z = 2$ :

$$\begin{aligned}\dot{u}_2 &= 2u'_3 - u''_2, \\ \dot{u}_3 &= -\frac{2}{3}u_2u'_2 + u''_3 - \frac{2}{3}u'''_2.\end{aligned}\tag{C.2}$$

CS equations of motion at  $z = 2$ :

$$\begin{aligned}\dot{\alpha}_2 &= -2\alpha'_3, \\ \dot{\alpha}_3 &= \frac{8}{3}\alpha_2\alpha'_2 + \frac{1}{6}\alpha'''_2.\end{aligned}\tag{C.3}$$

#### $N = 4$

CS-KdV map:

$$\begin{aligned}u_2 &= 10\alpha_2, \\u_3 &= 10\alpha'_2 - 24\alpha_3, \\u_4 &= -12\alpha'_3 + 3\alpha''_2 + 9\alpha_2^2 + 36\alpha_4.\end{aligned}\tag{C.4}$$

KdV equations of motion at  $z = 2$ :

$$\begin{aligned}
\dot{u}_2 &= 2u'_3 - 2u''_2, \\
\dot{u}_3 &= -u_2 u'_2 + 2u'_4 + u''_3 - 2u'''_2, \\
\dot{u}_4 &= -\frac{1}{2}u_3 u'_2 - \frac{1}{2}u_2 u''_2 + u''_4 - \frac{1}{2}u_2^{(4)}.
\end{aligned} \tag{C.5}$$

CS equations of motion at  $z = 2$ :

$$\begin{aligned}
\dot{\alpha}_2 &= -\frac{24}{5}\alpha'_3, \\
\dot{\alpha}_3 &= \frac{8}{3}\alpha_2 \alpha'_2 - 3\alpha'_4 + \frac{1}{6}\alpha_2''', \\
\dot{\alpha}_4 &= \frac{10}{3}\alpha_3 \alpha'_2 + \frac{12}{5}\alpha_2 \alpha'_3 + \frac{1}{15}\alpha_3'''.
\end{aligned} \tag{C.6}$$

KdV equations of motion at  $z = 3$ :

$$\begin{aligned}
\dot{u}_2 &= -\frac{3}{4}u_2 u'_2 + 3u'_4 - \frac{3}{2}u''_3 + \frac{1}{4}u_2''', \\
\dot{u}_3 &= -\frac{3}{4}u_3 u'_2 - \frac{3}{4}u_2 u'_3 + 3u''_4 - 2u'''_3 + \frac{3}{4}u_2^{(4)}, \\
\dot{u}_4 &= -\frac{3}{4}u_3 u'_3 + \frac{3}{4}u_2 u'_4 + \frac{3}{8}u_3 u''_2 - \frac{3}{4}u_2 u''_3 + \frac{3}{8}u_2 u_2''' + u_4''' - \frac{3}{4}u_3^{(4)} + \frac{3}{8}u_2^{(5)}.
\end{aligned} \tag{C.7}$$

CS equations of motion at  $z = 3$ :

$$\begin{aligned}
\dot{\alpha}_2 &= -\frac{21}{10}\alpha_2 \alpha'_2 + \frac{54}{5}\alpha'_4 - \frac{7}{20}\alpha_2''', \\
\dot{\alpha}_3 &= -\frac{15}{2}\alpha_3 \alpha'_2 - \frac{15}{2}\alpha_2 \alpha'_3 - \frac{1}{2}\alpha_3''', \\
\dot{\alpha}_4 &= \frac{59}{60}\alpha'_2 \alpha_2'' + \frac{24}{5}\alpha_2^2 \alpha'_2 + \frac{21}{10}\alpha_2 \alpha'_4 - 12\alpha_3 \alpha'_3 + \frac{13}{30}\alpha_2 \alpha_2''' + \frac{1}{10}\alpha_4''' + \frac{1}{120}\alpha_2^{(5)}.
\end{aligned} \tag{C.8}$$

## $N = 5$

CS-KdV map:

$$\begin{aligned}
u_2 &= 20\alpha_2, \\
u_3 &= 30\alpha'_2 - 84\alpha_3, \\
u_4 &= -84\alpha'_3 + 18\alpha_2'' + 64\alpha_2^2 + 288\alpha_4, \\
u_5 &= 64\alpha_2 \alpha'_2 + 144\alpha'_4 - 24\alpha_3'' + 4\alpha_2''' - 192\alpha_2 \alpha_3 - 576\alpha_5.
\end{aligned} \tag{C.9}$$

KdV equations of motion at  $z = 2$ :

$$\begin{aligned}
\dot{u}_2 &= 2u'_3 - 3u''_2, \\
\dot{u}_3 &= -\frac{6}{5}u_2u'_2 + 2u'_4 + u''_3 - 4u'''_2, \\
\dot{u}_4 &= -\frac{4}{5}u_3u'_2 + 2u'_5 - \frac{6}{5}u_2u''_2 + u''_4 - 2u_2^{(4)}, \\
\dot{u}_5 &= -\frac{2}{5}u_4u'_2 - \frac{2}{5}u_3u''_2 + u''_5 - \frac{2}{5}u_2u'''_2 - \frac{2}{5}u_2^{(5)}.
\end{aligned} \tag{C.10}$$

CS equations of motion at  $z = 2$ :

$$\begin{aligned}
\dot{\alpha}_2 &= -\frac{42}{5}\alpha'_3, \\
\dot{\alpha}_3 &= \frac{8}{3}\alpha_2\alpha'_2 - \frac{48}{7}\alpha'_4 + \frac{1}{6}\alpha'''_2, \\
\dot{\alpha}_4 &= \frac{10}{3}\alpha_3\alpha'_2 + \frac{12}{5}\alpha_2\alpha'_3 - 4\alpha'_5 + \frac{1}{15}\alpha'''_3, \\
\dot{\alpha}_5 &= \frac{14}{5}\alpha_3\alpha'_3 + 4\alpha_4\alpha'_2 + \frac{16}{7}\alpha_2\alpha'_4 + \frac{1}{28}\alpha'''_4.
\end{aligned} \tag{C.11}$$

KdV equations of motion at  $z = 3$ :

$$\begin{aligned}
\dot{u}_2 &= -\frac{6}{5}u_2u'_2 + 3u'_4 - 3u''_3 + u'''_2, \\
\dot{u}_3 &= -\frac{6}{5}u_3u'_2 - \frac{6}{5}u_2u'_3 + 3u'_5 + 3u''_4 - 5u'''_3 + 3u_2^{(4)}, \\
\dot{u}_4 &= -\frac{6}{5}u_3u'_3 - \frac{3}{5}u_4u'_2 + \frac{3}{5}u_2u'_4 + \frac{3}{5}u_3u''_2 - \frac{9}{5}u_2u''_3 + 3u''_5 + \frac{6}{5}u_2u'''_2 + \\
&\quad u'''_4 - 3u_3^{(4)} + \frac{12}{5}u_2^{(5)}, \\
\dot{u}_5 &= -\frac{3}{5}u_4u'_3 + \frac{3}{5}u_2u'_5 - \frac{3}{5}u_3u''_3 + \frac{3}{5}u_4u''_2 + \frac{3}{5}u_3u'''_2 - \frac{3}{5}u_2u'''_3 + u'''_5 + \\
&\quad \frac{3}{5}u_2u_2^{(4)} - \frac{3}{5}u_3^{(5)} + \frac{3}{5}u_2^{(6)}.
\end{aligned} \tag{C.12}$$

CS equations of motion at  $z = 3$ :

$$\begin{aligned}
\dot{\alpha}_2 &= -\frac{24}{5} \alpha_2 \alpha'_2 + \frac{216}{5} \alpha'_4 - \frac{4}{5} \alpha_2''', \\
\dot{\alpha}_3 &= -\frac{120}{7} \alpha_3 \alpha'_2 - \frac{120}{7} \alpha_2 \alpha'_3 + \frac{144}{7} \alpha'_5 - \frac{8}{7} \alpha_3''', \\
\dot{\alpha}_4 &= \frac{59}{60} \alpha'_2 \alpha_2'' + \frac{24}{5} \alpha_2^2 \alpha'_2 - \frac{36}{5} \alpha_2 \alpha'_4 - \frac{147}{5} \alpha_3 \alpha'_3 - 12 \alpha_4 \alpha'_2 + \frac{13}{30} \alpha_2 \alpha_2''' - \\
&\quad \frac{1}{5} \alpha_4''' + \frac{1}{120} \alpha_2^{(5)}, \\
\dot{\alpha}_5 &= \frac{97}{140} \alpha'_3 \alpha_2'' + \frac{29}{56} \alpha'_2 \alpha_3'' + \frac{144}{35} \alpha_2^2 \alpha'_3 + \frac{396}{35} \alpha_3 \alpha_2 \alpha'_2 + \frac{36}{7} \alpha_2 \alpha'_5 - \frac{126}{5} \alpha_4 \alpha'_3 - \frac{72}{5} \alpha_3 \alpha'_4 + \\
&\quad \frac{5}{28} \alpha_2 \alpha_3''' + \frac{123}{280} \alpha_3 \alpha_2''' + \frac{1}{7} \alpha_5''' + \frac{1}{560} \alpha_3^{(5)}.
\end{aligned} \tag{C.13}$$

KdV equations of motion at  $z = 4$ :

$$\begin{aligned}
\dot{u}_2 &= \frac{6}{5} u_2'^2 - \frac{4}{5} u_3 u_2' - \frac{4}{5} u_2 u_3' + 4 u_5' + \frac{6}{5} u_2 u_2'' - 2 u_4'' + u_2^{(4)}, \\
\dot{u}_3 &= \frac{24}{5} u_2' u_2'' + \frac{12}{25} u_2^2 u_2'' - \frac{4}{5} u_2 u_4' - \frac{2}{5} u_2' u_3' - \frac{4}{5} u_3 u_3' - \frac{4}{5} u_4 u_2' - \frac{2}{5} u_2 u_3'' + \\
&\quad 6 u_5'' + 2 u_2 u_2''' - 4 u_4''' + u_3^{(4)} + \frac{6}{5} u_2^{(5)}, \\
\dot{u}_4 &= \frac{16}{5} u_2' u_2''' + \frac{12}{25} u_2 u_2'^2 + \frac{8}{25} u_3 u_2 u_2' + \frac{8}{5} u_2 u_5' - \frac{4}{5} u_4 u_3' - \frac{2}{5} u_2' u_4' - \\
&\quad \frac{4}{5} u_3 u_4' + \frac{12}{25} u_2^2 u_2'' - \frac{8}{5} u_2 u_4'' + \frac{12}{5} u_2''^2 + \frac{2}{5} u_4 u_2'' + \frac{2}{5} u_2 u_3''' + \frac{2}{5} u_3 u_2''' + \\
&\quad 4 u_5''' + \frac{6}{5} u_2 u_2^{(4)} - 3 u_4^{(4)} + \frac{6}{5} u_3^{(5)} + \frac{2}{5} u_2^{(6)}, \\
\dot{u}_5 &= \frac{12}{25} u_2 u_2' u_2'' + \frac{4}{5} u_2' u_2^{(4)} + \frac{4}{25} u_4 u_2 u_2' + \frac{4}{25} u_3 u_2'^2 - \frac{4}{5} u_4 u_4' - \frac{2}{5} u_2' u_5' + \frac{4}{5} u_3 u_5' + \\
&\quad \frac{8}{5} u_2'' u_2''' + \frac{4}{25} u_3 u_2 u_2'' + \frac{4}{5} u_2 u_5'' + \frac{2}{5} u_4 u_3'' - \frac{4}{5} u_3 u_4'' + \frac{4}{25} u_2^2 u_2''' - \frac{4}{5} u_2 u_4''' + \\
&\quad \frac{2}{5} u_3 u_3''' + \frac{2}{5} u_2 u_3^{(4)} + u_5^{(4)} + \frac{4}{25} u_2 u_2^{(5)} - \frac{4}{5} u_4^{(5)} + \frac{2}{5} u_3^{(6)}.
\end{aligned} \tag{C.14}$$



CS equations of motion at  $z = 4$

$$\begin{aligned}
\dot{\alpha}_2 &= \frac{144}{5} \alpha_3 \alpha'_2 + \frac{144}{5} \alpha_2 \alpha'_3 - \frac{576}{5} \alpha'_5 + \frac{18}{5} \alpha_3''', \\
\dot{\alpha}_3 &= -\frac{24}{7} \alpha'_2 \alpha_2'' - \frac{64}{7} \alpha_2^2 \alpha'_2 + \frac{384}{7} \alpha_2 \alpha'_4 + \frac{336}{5} \alpha_3 \alpha'_3 + \frac{384}{7} \alpha_4 \alpha'_2 - \\
&\quad \frac{12}{7} \alpha_2 \alpha_2''' + \frac{24}{7} \alpha_4''' - \frac{1}{14} \alpha_2^{(5)}, \\
\dot{\alpha}_4 &= -\frac{68}{15} \alpha'_3 \alpha_2'' - \frac{61}{15} \alpha'_2 \alpha_3'' - \frac{96}{5} \alpha_2^2 \alpha'_3 - \frac{208}{5} \alpha_3 \alpha_2 \alpha'_2 - \frac{64}{5} \alpha_2 \alpha'_5 + \frac{336}{5} \alpha_4 \alpha'_3 + \\
&\quad \frac{336}{5} \alpha_3 \alpha'_4 - \frac{26}{15} \alpha_2 \alpha_3''' - \frac{13}{5} \alpha_3 \alpha_2''' - \frac{4}{5} \alpha_5''' - \frac{1}{30} \alpha_3^{(5)}, \\
\dot{\alpha}_5 &= \frac{1108}{315} \alpha_2 \alpha'_2 \alpha_2'' - \frac{7}{2} \alpha'_3 \alpha_3'' + \frac{12}{35} \alpha'_4 \alpha_2'' + \frac{8}{7} \alpha'_2 \alpha_4'' + \frac{13}{168} \alpha'_2 \alpha_2^{(4)} + \frac{256}{35} \alpha_2^3 \alpha'_2 + \\
&\quad \frac{256}{35} \alpha_2^2 \alpha'_4 - \frac{272}{5} \alpha_3 \alpha_2 \alpha'_3 + \frac{32}{35} \alpha_4 \alpha_2 \alpha'_2 + \frac{62}{63} \alpha_2'^3 - 32 \alpha_3^2 \alpha'_2 + \frac{576}{5} \alpha_4 \alpha'_4 - \\
&\quad \frac{144}{5} \alpha_3 \alpha'_5 + \frac{47}{360} \alpha_2'' \alpha_2''' + \frac{244}{315} \alpha_2^2 \alpha_2''' + \frac{4}{7} \alpha_2 \alpha_4''' - \frac{19}{10} \alpha_3 \alpha_3''' - \frac{38}{35} \alpha_4 \alpha_2''' + \\
&\quad \frac{29 \alpha_2 \alpha_2^{(5)}}{1260} + \frac{1}{140} \alpha_4^{(5)} + \frac{\alpha_2^{(7)}}{5040}.
\end{aligned} \tag{C.15}$$

## $N = 6$

CS-KdV map:

$$\begin{aligned}
u_2 &= 35 \alpha_2, \\
u_3 &= 70 \alpha'_2 - 224 \alpha_3, \\
u_4 &= -336 \alpha'_3 + 63 \alpha_2'' + 259 \alpha_2^2 + 1296 \alpha_4, \\
u_5 &= 518 \alpha_2 \alpha'_2 + 1296 \alpha'_4 - 192 \alpha_3'' + 28 \alpha_2''' - 1760 \alpha_2 \alpha_3 - 5760 \alpha_5, \\
u_6 &= -880 \alpha_2 \alpha'_3 + 130 \alpha_2'^2 - 880 \alpha_3 \alpha'_2 - 2880 \alpha'_5 + 155 \alpha_2 \alpha_2'' + 360 \alpha_4'' - 40 \alpha_3''' + 5 \alpha_2^{(4)} \\
&\quad + 225 \alpha_2^3 + 3600 \alpha_4 \alpha_2 + 1600 \alpha_3^2 + 14400 \alpha_6.
\end{aligned} \tag{C.16}$$

KdV equations of motion at  $z = 2$ :

$$\begin{aligned}
\dot{u}_2 &= 2u'_3 - 4u''_2 \\
\dot{u}_3 &= -\frac{4}{3}u_2u'_2 + 2u'_4 + u''_3 - \frac{20}{3}u'''_2, \\
\dot{u}_4 &= -u_3u'_2 + 2u'_5 - 2u_2u''_2 + u''_4 - 5u_2^{(4)}, \\
\dot{u}_5 &= -\frac{2}{3}u_4u'_2 + 2u_6' - u_3u''_2 + u''_5 - \frac{4}{3}u_2u'''_2 - 2u_2^{(5)}, \\
\dot{u}_6 &= -\frac{1}{3}u_5u'_2 - \frac{1}{3}u_4u''_2 + u_6'' - \frac{1}{3}u_3u'''_2 - \frac{1}{3}u_2u_2^{(4)} - \frac{1}{3}u_2^{(6)}.
\end{aligned} \tag{C.17}$$

CS equations of motion at  $z = 2$ :

$$\begin{aligned}
\dot{\alpha}_2 &= -\frac{64}{5}\alpha'_3, \\
\dot{\alpha}_3 &= \frac{8}{3}\alpha_2\alpha'_2 - \frac{81}{7}\alpha'_4 + \frac{1}{6}\alpha_2''', \\
\dot{\alpha}_4 &= \frac{10}{3}\alpha_3\alpha'_2 + \frac{12}{5}\alpha_2\alpha'_3 - \frac{80}{9}\alpha'_5 + \frac{1}{15}\alpha_3''', \\
\dot{\alpha}_5 &= \frac{14}{5}\alpha_3\alpha'_3 + 4\alpha_4\alpha'_2 + \frac{16}{7}\alpha_2\alpha'_4 - 5\alpha'_6 + \frac{1}{28}\alpha_4''', \\
\dot{\alpha}_6 &= \frac{16}{5}\alpha_4\alpha'_3 + \frac{18}{7}\alpha_3\alpha'_4 + \frac{14}{3}\alpha_5\alpha'_2 + \frac{20}{9}\alpha_2\alpha'_5 + \frac{1}{45}\alpha_5'''.
\end{aligned} \tag{C.18}$$

KdV equations of motion at  $z = 3$ :

$$\begin{aligned}
\dot{u}_2 &= -\frac{3}{2}u_2u'_2 + 3u'_4 - \frac{9}{2}u''_3 + \frac{9}{4}u'''_2, \\
\dot{u}_3 &= -\frac{3}{2}u_3u'_2 - \frac{3}{2}u_2u'_3 + 3u'_5 + 3u''_4 - 9u'''_3 + \frac{15}{2}u_2^{(4)}, \\
\dot{u}_4 &= -\frac{3}{2}u_3u'_3 - u_4u'_2 + \frac{1}{2}u_2u'_4 + 3u_6' + \frac{3}{4}u_3u''_2 - 3u_2u''_3 + 3u'_5 + \frac{5}{2}u_2u'''_2 + u_4''' - \\
&\quad \frac{15}{2}u_3^{(4)} + \frac{33}{4}u_2^{(5)}, \\
\dot{u}_5 &= -u_4u'_3 - \frac{1}{2}u_5u'_2 + \frac{1}{2}u_2u'_5 - \frac{3}{2}u_3u''_3 + u_4u''_2 + 3u_6'' + \frac{7}{4}u_3u'''_2 - 2u_2u'''_3 + u_5''' + \\
&\quad \frac{5}{2}u_2u_2^{(4)} - 3u_3^{(5)} + 4u_2^{(6)}, \\
\dot{u}_6 &= \frac{1}{2}u_2u_6' - \frac{1}{2}u_5u'_3 - \frac{1}{2}u_4u''_3 + \frac{3}{4}u_5u''_2 - \frac{1}{2}u_3u'''_3 + \frac{3}{4}u_4u'''_2 + u_6''' + \frac{3}{4}u_3u_2^{(4)} - \\
&\quad \frac{1}{2}u_2u_3^{(4)} + \frac{3}{4}u_2u_2^{(5)} - \frac{1}{2}u_3^{(6)} + \frac{3}{4}u_2^{(7)}.
\end{aligned} \tag{C.19}$$

CS equations of motion at  $z = 3$ :

$$\begin{aligned}
\dot{\alpha}_2 &= -\frac{81}{10} \alpha_2 \alpha'_2 + \frac{3888}{35} \alpha'_4 - \frac{27}{20} \alpha_2''', \\
\dot{\alpha}_3 &= -\frac{405}{14} \alpha_3 \alpha'_2 - \frac{405}{14} \alpha_2 \alpha'_3 + \frac{540}{7} \alpha'_5 - \frac{27}{14} \alpha_3''', \\
\dot{\alpha}_4 &= \frac{59}{60} \alpha'_2 \alpha_2'' + \frac{24}{5} \alpha_2^2 \alpha'_2 - \frac{557}{30} \alpha_2 \alpha'_4 - \frac{152}{3} \alpha_3 \alpha'_3 - \frac{80}{3} \alpha_4 \alpha'_2 + \frac{100}{3} \alpha'_6 + \\
&\quad \frac{13}{30} \alpha_2 \alpha_2''' - \frac{17}{30} \alpha_4''' + \frac{1}{120} \alpha_2^{(5)}, \\
\dot{\alpha}_5 &= \frac{97}{140} \alpha'_3 \alpha_2'' + \frac{29}{56} \alpha'_2 \alpha_3'' + \frac{144}{35} \alpha_2^2 \alpha'_3 + \frac{396}{35} \alpha_3 \alpha_2 \alpha'_2 - \frac{85}{14} \alpha_2 \alpha'_5 - \frac{252}{5} \alpha_4 \alpha'_3 - \\
&\quad \frac{1188}{35} \alpha_3 \alpha'_4 - \frac{35}{2} \alpha_5 \alpha'_2 + \frac{5}{28} \alpha_2 \alpha_3''' + \frac{123}{280} \alpha_3 \alpha_2''' - \frac{1}{14} \alpha_5''' + \frac{1}{560} \alpha_3^{(5)}, \\
\dot{\alpha}_6 &= \frac{9}{25} \alpha'_3 \alpha_3'' + \frac{79}{140} \alpha'_4 \alpha_2'' + \frac{19}{56} \alpha'_2 \alpha_4'' + \frac{80}{21} \alpha_2^2 \alpha'_4 + \frac{976}{105} \alpha_3 \alpha_2 \alpha'_3 + \frac{196}{15} \alpha_4 \alpha_2 \alpha'_2 + \\
&\quad \frac{55}{6} \alpha_2 \alpha'_6 + \frac{45}{7} \alpha_3^2 \alpha'_2 - \frac{972}{35} \alpha_4 \alpha'_4 - \frac{224}{5} \alpha_5 \alpha'_3 - \frac{120}{7} \alpha_3 \alpha'_5 + \frac{41}{420} \alpha_2 \alpha_4''' + \\
&\quad \frac{92}{525} \alpha_3 \alpha_3''' + \frac{7}{15} \alpha_4 \alpha_2''' + \frac{1}{6} \alpha_6''' + \frac{\alpha_4^{(5)}}{1680}.
\end{aligned} \tag{C.20}$$

KdV equations of motion at  $z = 4$ :

$$\begin{aligned}
\dot{u}_2 &= \frac{8}{3} u_2'^2 - \frac{4}{3} u_3 u_2' - \frac{4}{3} u_2 u_3' + 4 u_5' + \frac{8}{3} u_2 u_2'' - 4 u_4'' + \frac{2}{3} u_3''' + \frac{8}{3} u_2^{(4)}, \\
\dot{u}_3 &= \frac{40}{3} u_2' u_2'' + \frac{8}{9} u_2^2 u_2' - \frac{4}{3} u_2 u_4' - \frac{2}{3} u_2' u_3' - \frac{4}{3} u_3 u_3' - \frac{4}{3} u_4 u_2' + 4 u_6' - \\
&\quad \frac{2}{3} u_2 u_3'' + 6 u_5'' + \frac{16}{3} u_2 u_2''' - \frac{28}{3} u_4''' + \frac{13}{3} u_3^{(4)} + \frac{34}{9} u_2^{(5)}, \\
\dot{u}_4 &= \frac{40}{3} u_2' u_2''' + \frac{4}{3} u_2 u_2'^2 + \frac{2}{3} u_3 u_2 u_2' + \frac{4}{3} u_2 u_5' - \frac{4}{3} u_4 u_3' - \frac{2}{3} u_2' u_4' - \frac{4}{3} u_3 u_4' - \\
&\quad \frac{2}{3} u_5 u_2' + \frac{4}{3} u_2^2 u_2'' - \frac{10}{3} u_2 u_4'' + 10 u_2'^2 + \frac{2}{3} u_4 u_2'' + 6 u_6'' + \frac{4}{3} u_2 u_3''' + u_3 u_2''' + \\
&\quad 4 u_5''' + \frac{14}{3} u_2 u_2^{(4)} - 9 u_4^{(4)} + 6 u_3^{(5)} + \frac{5}{3} u_2^{(6)}, \tag{C.21} \\
\dot{u}_5 &= \frac{8}{3} u_2 u_2' u_2'' + \frac{20}{3} u_2' u_2^{(4)} + \frac{4}{9} u_4 u_2 u_2' + \frac{4}{3} u_2 u_6' + \frac{2}{3} u_3 u_2'^2 - \frac{4}{3} u_4 u_4' - \frac{2}{3} u_5 u_3' - \\
&\quad \frac{2}{3} u_2' u_5' + \frac{2}{3} u_3 u_5' + \frac{40}{3} u_2'' u_2''' + \frac{2}{3} u_3 u_2 u_2'' + \frac{2}{3} u_2 u_5'' + \frac{2}{3} u_4 u_3'' - 2 u_3 u_4'' + \frac{2}{3} u_5 u_2'' + \\
&\quad \frac{8}{9} u_2^2 u_2''' - \frac{8}{3} u_2 u_4''' + \frac{4}{3} u_3 u_3''' + \frac{4}{9} u_4 u_2''' + 4 u_6''' + 2 u_2 u_3^{(4)} + \frac{1}{3} u_3 u_2^{(4)} + u_5^{(4)} + \\
&\quad \frac{14}{9} u_2 u_2^{(5)} - 4 u_4^{(5)} + \frac{10}{3} u_3^{(6)}, \\
\dot{u}_6 &= \frac{2}{3} u_3 u_2' u_2'' + \frac{8}{9} u_2 u_2' u_2''' + \frac{4}{3} u_2' u_2^{(5)} + \frac{2}{9} u_5 u_2 u_2' + \frac{2}{9} u_4 u_2'^2 - \frac{2}{3} u_5 u_4' - \\
&\quad \frac{2}{3} u_2' u_6' + \frac{2}{3} u_3 u_6' + \frac{10}{3} u_2'' u_2^{(4)} + \frac{2}{3} u_2 u_2''^2 + \frac{2}{9} u_4 u_2 u_2'' + \frac{2}{3} u_2 u_6'' - \frac{2}{3} u_4 u_4'' + \\
&\quad \frac{2}{3} u_5 u_3'' + \frac{2}{9} u_3 u_2 u_2''' + \frac{20}{9} u_2''^2 + \frac{2}{3} u_4 u_3''' - \frac{2}{3} u_3 u_4''' - \frac{1}{9} u_5 u_2''' + \frac{2}{9} u_2^2 u_2^{(4)} - \frac{2}{3} u_2 u_4^{(4)} + \\
&\quad \frac{2}{3} u_3 u_3^{(4)} - \frac{1}{9} u_4 u_2^{(4)} + u_6^{(4)} + \frac{2}{3} u_2 u_3^{(5)} - \frac{1}{9} u_3 u_2^{(5)} + \frac{1}{9} u_2 u_2^{(6)} - \frac{2}{3} u_4^{(6)} + \frac{2}{3} u_3^{(7)} - \frac{1}{9} u_2^{(8)}.
\end{aligned}$$

CS equations of motion at  $z = 4$ :

$$\begin{aligned}
\dot{\alpha}_2 &= \frac{2048}{21} \alpha_3 \alpha'_2 + \frac{2048}{21} \alpha_2 \alpha'_3 - \frac{4608}{7} \alpha'_5 + \frac{256}{21} \alpha_3''', \\
\dot{\alpha}_3 &= -\frac{160}{21} \alpha'_2 \alpha_2'' - \frac{1280}{63} \alpha_2^2 \alpha'_2 + \frac{1440}{7} \alpha_2 \alpha'_4 + \frac{5072}{21} \alpha_3 \alpha'_3 + \frac{1440}{7} \alpha_4 \alpha'_2 - \\
&\quad \frac{1800}{7} \alpha'_6 - \frac{80}{21} \alpha_2 \alpha_2''' + \frac{90}{7} \alpha_4''' - \frac{10}{63} \alpha_2^{(5)}, \\
\dot{\alpha}_4 &= -\frac{272}{27} \alpha'_3 \alpha_2'' - \frac{244}{27} \alpha'_2 \alpha_3'' - \frac{128}{3} \alpha_2^2 \alpha'_3 - \frac{832}{9} \alpha_3 \alpha_2 \alpha'_2 + \frac{1504}{27} \alpha_2 \alpha'_5 + \frac{896}{3} \alpha_4 \alpha'_3 + \\
&\quad \frac{896}{3} \alpha_3 \alpha'_4 + \frac{2800}{27} \alpha_5 \alpha'_2 - \frac{104}{27} \alpha_2 \alpha_3''' - \frac{52}{9} \alpha_3 \alpha_2''' + \frac{8}{9} \alpha_5''' - \frac{2}{27} \alpha_3^{(5)}, \tag{C.22} \\
\dot{\alpha}_5 &= \frac{1108}{315} \alpha_2 \alpha'_2 \alpha_2'' - \frac{884}{105} \alpha'_3 \alpha_3'' - \frac{549}{140} \alpha'_4 \alpha_2'' - \frac{51}{28} \alpha'_2 \alpha_4'' + \frac{13}{168} \alpha'_2 \alpha_2^{(4)} + \frac{256}{35} \alpha_2^3 \alpha'_2 - \\
&\quad \frac{1504}{105} \alpha_2^2 \alpha'_4 - \frac{2720}{21} \alpha_3 \alpha_2 \alpha'_3 - \frac{6208}{105} \alpha_4 \alpha_2 \alpha'_2 - \frac{800}{21} \alpha_2 \alpha'_6 + \frac{62}{63} \alpha_2'^3 - \\
&\quad \frac{528}{7} \alpha_3^2 \alpha'_2 + \frac{1944}{5} \alpha_4 \alpha'_4 + \frac{448}{3} \alpha_5 \alpha'_3 + \frac{1088}{21} \alpha_3 \alpha'_5 + \frac{47}{360} \alpha_2'' \alpha_2''' + \frac{244}{315} \alpha_2^2 \alpha_2''' - \\
&\quad \frac{19}{42} \alpha_2 \alpha_4''' - \frac{156}{35} \alpha_3 \alpha_3''' - \frac{156}{35} \alpha_4 \alpha_2''' - \frac{10}{7} \alpha_6''' + \frac{29 \alpha_2 \alpha_2^{(5)}}{1260} - \frac{1}{280} \alpha_4^{(5)} + \frac{\alpha_2^{(7)}}{5040}, \\
\dot{\alpha}_6 &= \frac{11828 \alpha_2 \alpha'_3 \alpha_2''}{4725} + \frac{8902 \alpha_2 \alpha'_2 \alpha_3''}{4725} + \frac{3673 \alpha_3 \alpha'_2 \alpha_2''}{1050} - \frac{104}{25} \alpha'_4 \alpha_3'' - \frac{56}{25} \alpha'_3 \alpha_4'' + \\
&\quad \frac{32}{63} \alpha'_5 \alpha_2'' + \frac{116}{63} \alpha'_2 \alpha_5'' + \frac{451 \alpha'_3 \alpha_2^{(4)}}{9450} + \frac{41 \alpha'_2 \alpha_3^{(4)}}{1890} + \frac{128}{21} \alpha_2^3 \alpha'_3 + \frac{2624}{105} \alpha_3 \alpha_2^2 \alpha'_2 + \\
&\quad \frac{3200}{189} \alpha_2^2 \alpha'_5 - \frac{2080}{21} \alpha_4 \alpha_2 \alpha'_3 - \frac{1824}{35} \alpha_3 \alpha_2 \alpha'_4 + \frac{448}{135} \alpha_5 \alpha_2 \alpha'_2 + \frac{6577 \alpha_2'^2 \alpha'_3}{3150} - \\
&\quad \frac{608}{7} \alpha_3^2 \alpha'_3 - \frac{752}{7} \alpha_3 \alpha_4 \alpha'_2 + \frac{1728}{5} \alpha_5 \alpha'_4 + \frac{1152}{7} \alpha_4 \alpha'_5 - \frac{1936}{21} \alpha_3 \alpha'_6 + \\
&\quad \frac{559 \alpha_3'' \alpha_2'''}{9450} + \frac{8}{175} \alpha_2'' \alpha_3''' + \frac{1538 \alpha_2^2 \alpha_3'''}{4725} + \frac{2459 \alpha_3 \alpha_2 \alpha_2'''}{1575} + \frac{152}{189} \alpha_2 \alpha_5''' - \\
&\quad \frac{664}{175} \alpha_4 \alpha_3''' - \frac{108}{175} \alpha_3 \alpha_4''' - \frac{392}{135} \alpha_5 \alpha_2''' + \frac{1}{189} \alpha_2 \alpha_3^{(5)} + \frac{131 \alpha_3 \alpha_2^{(5)}}{6300} + \frac{2}{315} \alpha_5^{(5)} + \frac{\alpha_3^{(7)}}{37800}.
\end{aligned}$$

$N = 7$

CS-KdV map:

$$\begin{aligned}
u_2 &= 56 \alpha_2, \\
u_3 &= 140 \alpha_2' - 504 \alpha_3, \\
u_4 &= -1008 \alpha_3' + 168 \alpha_2'' + 784 \alpha_2^2 + 4320 \alpha_4, \\
u_5 &= 2352 \alpha_2 \alpha_2' + 6480 \alpha_4' - 864 \alpha_3'' + 112 \alpha_2''' - 8928 \alpha_2 \alpha_3 - 31680 \alpha_5, \\
u_6 &= -8928 \alpha_2 \alpha_3' + 1180 \alpha_2'^2 - 8928 \alpha_3 \alpha_2' - 31680 \alpha_5' + 1408 \alpha_2 \alpha_2'' + 3600 \alpha_4''' - \\
&\quad 360 \alpha_3''' + 40 \alpha_2^{(4)} + 2304 \alpha_2^3 + 40320 \alpha_4 \alpha_2 + 18000 \alpha_3^2 + 172800 \alpha_6, \\
u_7 &= 708 \alpha_2' \alpha_2'' + 3456 \alpha_2^2 \alpha_2' + 20160 \alpha_2 \alpha_4' - 4488 \alpha_2' \alpha_3' + 18000 \alpha_3 \alpha_3' + \\
&\quad 20160 \alpha_4 \alpha_2' + 86400 \alpha_6' - 2544 \alpha_2 \alpha_3'' - 2664 \alpha_3 \alpha_2'' - 8640 \alpha_5'' + 312 \alpha_2 \alpha_2''' + \\
&\quad 720 \alpha_4''' - 60 \alpha_3^{(4)} + 6 \alpha_2^{(5)} - 13824 \alpha_3 \alpha_2^2 - 103680 \alpha_5 \alpha_2 - 86400 \alpha_3 \alpha_4 - \\
&\quad 518400 \alpha_7.
\end{aligned} \tag{C.23}$$

KdV equations of motion at  $z = 2$ :

$$\begin{aligned}
\dot{u}_2 &= 2 u_3' - 5 u_2'', \\
\dot{u}_3 &= -\frac{10}{7} u_2 u_2' + 2 u_4' + u_3'' - 10 u_2''', \\
\dot{u}_4 &= -\frac{8}{7} u_3 u_2' + 2 u_5' - \frac{20}{7} u_2 u_2'' + u_4'' - 10 u_2^{(4)}, \\
\dot{u}_5 &= -\frac{6}{7} u_4 u_2' + 2 u_6' - \frac{12}{7} u_3 u_2'' + u_5'' - \frac{20}{7} u_2 u_2''' - 6 u_2^{(5)}, \\
\dot{u}_6 &= -\frac{4}{7} u_5 u_2' + 2 u_7' - \frac{6}{7} u_4 u_2'' + u_6'' - \frac{8}{7} u_3 u_2''' - \frac{10}{7} u_2 u_2^{(4)} - 2 u_2^{(6)}, \\
\dot{u}_7 &= -\frac{2}{7} u_6 u_2' - \frac{2}{7} u_5 u_2'' + u_7'' - \frac{2}{7} u_4 u_2''' - \frac{2}{7} u_3 u_2^{(4)} - \frac{2}{7} u_2 u_2^{(5)} - \frac{2}{7} u_2^{(7)}.
\end{aligned} \tag{C.24}$$

CS equations of motion at  $z = 2$ :

$$\begin{aligned}
\dot{\alpha}_2 &= -18 \alpha'_3, \\
\dot{\alpha}_3 &= \frac{8}{3} \alpha_2 \alpha'_2 - \frac{120}{7} \alpha'_4 + \frac{1}{6} \alpha_2''', \\
\dot{\alpha}_4 &= \frac{10}{3} \alpha_3 \alpha'_2 + \frac{12}{5} \alpha_2 \alpha'_3 - \frac{44}{3} \alpha'_5 + \frac{1}{15} \alpha_3''', \\
\dot{\alpha}_5 &= \frac{14}{5} \alpha_3 \alpha'_3 + 4 \alpha_4 \alpha'_2 + \frac{16}{7} \alpha_2 \alpha'_4 - \frac{120}{11} \alpha'_6 + \frac{1}{28} \alpha_4''', \\
\dot{\alpha}_6 &= \frac{16}{5} \alpha_4 \alpha'_3 + \frac{18}{7} \alpha_3 \alpha'_4 + \frac{14}{3} \alpha_5 \alpha'_2 + \frac{20}{9} \alpha_2 \alpha'_5 - 6 \alpha'_7 + \frac{1}{45} \alpha_5''', \\
\dot{\alpha}_7 &= \frac{20}{7} \alpha_4 \alpha'_4 + \frac{18}{5} \alpha_5 \alpha'_3 + \frac{22}{9} \alpha_3 \alpha'_5 + \frac{16}{3} \alpha_6 \alpha'_2 + \frac{24}{11} \alpha_2 \alpha'_6 + \frac{1}{66} \alpha_6'''.
\end{aligned} \tag{C.25}$$

KdV equations of motion at  $z = 3$ :

$$\begin{aligned}
\dot{u}_2 &= -\frac{12}{7} u_2 u'_2 + 3 u'_4 - 6 u_3'' + 4 u_2''', \\
\dot{u}_3 &= -\frac{12}{7} u_3 u'_2 - \frac{12}{7} u_2 u'_3 + 3 u'_5 + 3 u_4'' - 14 u_3''' + 15 u_2^{(4)}, \\
\dot{u}_4 &= -\frac{12}{7} u_3 u'_3 - \frac{9}{7} u_4 u'_2 + \frac{3}{7} u_2 u'_4 + 3 u'_6 + \frac{6}{7} u_3 u_2'' - \frac{30}{7} u_2 u_3'' + 3 u_5'' + \\
&\quad \frac{30}{7} u_2 u_2''' + u_4''' - 15 u_3^{(4)} + 21 u_2^{(5)}, \\
\dot{u}_5 &= -\frac{9}{7} u_4 u'_3 - \frac{6}{7} u_5 u'_2 + \frac{3}{7} u_2 u'_5 + 3 u'_7 - \frac{18}{7} u_3 u_3'' + \frac{9}{7} u_4 u_2'' + 3 u_6'' + \frac{24}{7} u_3 u_2''' - \\
&\quad \frac{30}{7} u_2 u_3''' + u_5''' + \frac{45}{7} u_2 u_2^{(4)} - 9 u_3^{(5)} + 15 u_2^{(6)}, \\
\dot{u}_6 &= -\frac{6}{7} u_5 u'_3 - \frac{3}{7} u_6 u'_2 + \frac{3}{7} u_2 u'_6 - \frac{9}{7} u_4 u_3'' + \frac{9}{7} u_5 u_2'' + 3 u_7'' - \frac{12}{7} u_3 u_3''' + \frac{15}{7} u_4 u_2''' + \\
&\quad u_6''' + 3 u_3 u_2^{(4)} - \frac{15}{7} u_2 u_3^{(4)} + \frac{27}{7} u_2 u_2^{(5)} - 3 u_3^{(6)} + \frac{39}{7} u_2^{(7)}, \\
\dot{u}_7 &= -\frac{3}{7} u_6 u'_3 + \frac{3}{7} u_2 u'_7 - \frac{3}{7} u_5 u_3'' + \frac{6}{7} u_6 u_2'' - \frac{3}{7} u_4 u_3''' + \frac{6}{7} u_5 u_2''' + u_7''' - \frac{3}{7} u_3 u_3^{(4)} + \frac{6}{7} u_4 u_2^{(4)} + \\
&\quad \frac{6}{7} u_3 u_2^{(5)} - \frac{3}{7} u_2 u_3^{(5)} + \frac{6}{7} u_2 u_2^{(6)} - \frac{3}{7} u_3^{(7)} + \frac{6}{7} u_2^{(8)}.
\end{aligned} \tag{C.26}$$

CS equations of motion at  $z = 3$ :

$$\begin{aligned}
\dot{\alpha}_2 &= -12 \alpha_2 \alpha'_2 + \frac{1620}{7} \alpha'_4 - 2 \alpha_2''', \\
\dot{\alpha}_3 &= -\frac{300}{7} \alpha_3 \alpha'_2 - \frac{300}{7} \alpha_2 \alpha'_3 + \frac{1320}{7} \alpha'_5 - \frac{20}{7} \alpha_3''', \\
\dot{\alpha}_4 &= \frac{24}{5} \alpha'_2 \alpha_2^2 - 32 \alpha'_4 \alpha_2 + \frac{13}{30} \alpha_2''' \alpha_2 - 44 \alpha_4 \alpha'_2 - \frac{379}{5} \alpha_3 \alpha'_3 + 120 \alpha'_6 + \frac{59}{60} \alpha'_2 \alpha_2'' - \\
&\quad \alpha_4''' + \frac{1}{120} \alpha_2^{(5)}, \\
\dot{\alpha}_5 &= \frac{144}{35} \alpha'_3 \alpha_2^2 + \frac{396}{35} \alpha_3 \alpha'_2 \alpha_2 - \frac{1488}{77} \alpha'_5 \alpha_2 + \frac{5}{28} \alpha_3''' \alpha_2 - \frac{420}{11} \alpha_5 \alpha'_2 - \frac{882}{11} \alpha_4 \alpha'_3 - \\
&\quad \frac{4392}{77} \alpha_3 \alpha'_4 + \frac{540}{11} \alpha'_7 + \frac{97}{140} \alpha'_3 \alpha_2'' + \frac{29}{56} \alpha'_2 \alpha_3'' + \frac{123}{280} \alpha_3 \alpha_2''' - \frac{25}{77} \alpha_5''' + \frac{1}{560} \alpha_3^{(5)}, \quad (\text{C.27}) \\
\dot{\alpha}_6 &= \frac{80}{21} \alpha'_4 \alpha_2^2 + \frac{196}{15} \alpha_4 \alpha'_2 \alpha_2 + \frac{976}{105} \alpha_3 \alpha'_3 \alpha_2 - 4 \alpha'_6 \alpha_2 + \frac{41}{420} \alpha_4''' \alpha_2 + \frac{45}{7} \alpha_3^2 \alpha'_2 - \\
&\quad 24 \alpha_6 \alpha'_2 - \frac{396}{5} \alpha_5 \alpha'_3 - 54 \alpha_4 \alpha'_4 - \frac{275}{7} \alpha_3 \alpha'_5 + \frac{79}{140} \alpha'_4 \alpha_2'' + \frac{9}{25} \alpha'_3 \alpha_3'' + \frac{19}{56} \alpha'_2 \alpha_4'' + \\
&\quad \frac{7}{15} \alpha_4 \alpha_2''' + \frac{92}{525} \alpha_3 \alpha_3''' + \frac{\alpha_4^{(5)}}{1680}, \\
\dot{\alpha}_7 &= \frac{40}{11} \alpha'_5 \alpha_2^2 + \frac{816}{55} \alpha_5 \alpha'_2 \alpha_2 + \frac{3996}{385} \alpha_4 \alpha'_3 \alpha_2 + \frac{1940}{231} \alpha_3 \alpha'_4 \alpha_2 + \frac{156}{11} \alpha'_7 \alpha_2 + \frac{61}{990} \alpha_5''' \alpha_2 + \\
&\quad \frac{304}{21} \alpha_3 \alpha_4 \alpha'_2 + \frac{77}{15} \alpha_3^2 \alpha'_3 - 72 \alpha_6 \alpha'_3 - \frac{324}{7} \alpha_5 \alpha'_4 - \frac{220}{7} \alpha_4 \alpha'_5 - 20 \alpha_3 \alpha'_6 + \\
&\quad \frac{65}{132} \alpha'_5 \alpha_2'' + \frac{443 \alpha'_4 \alpha_3''}{1540} + \frac{103}{440} \alpha'_3 \alpha_4'' + \frac{491 \alpha'_2 \alpha_5''}{1980} + \frac{83}{165} \alpha_5 \alpha_2''' + \frac{69}{385} \alpha_4 \alpha_3''' + \\
&\quad \frac{25}{264} \alpha_3 \alpha_4''' + \frac{2}{11} \alpha_7''' + \frac{\alpha_5^{(5)}}{3960}.
\end{aligned}$$



# APPENDIX D

## Proofs of statements used in the Drinfeld-Sokolov formalism

In this part of appendix we give the proofs to the theorems used in Drinfeld-Sokolov formalism. Most of them are essentially contained in the original paper by Drinfeld and Sokolov. However, the original paper is a little bit condensed, so we add details to the proofs to make them easier to follow.

### D.1 Gauge transformation of PDOs

Here we give the proof of the following statement: For any  $q$  and any canonical form, there exist a unique gauge transformation  $S$  to transform  $q$  into  $q' = S^{-1}V_1^2S - V_1^2 + S^{-1}\partial_x S$  in the canonical form chosen.

The proof proceeds as follows: We rewrite the gauge transformation as

$$Sq' = qS + [V_1^2, S] + \partial_x S \quad (\text{D.1})$$

and then by comparing the weight  $-i$  part we get

$$\sum_{j=0}^i S_{i-j}q'_j = \sum_{j=0}^i q_j S_{i-j} + [V_1^2, S_{i+1}] + \partial_x S_i \quad (\text{D.2})$$

which holds for all  $i$ 's. Using the fact  $S_0$  is the identity matrix  $E$ , we put it in a recursive form

$$q'_i - [V_1^2, S_{i+1}] = q_i + \partial_x S_i - \sum_{j=0}^{i-1} S_{i-j}q'_j + \sum_{j=0}^{i-1} q_j S_{i-j}. \quad (\text{D.3})$$

Given  $q$ , and suppose  $q'_j$  and  $S_{j+1}$  are known for all  $j < i$ , from the lowest weight projection of the right hand side we can find  $q'_i$  if we restrict it to be in a one dimensional subspace of weight  $-i$  elements which has nonzero lowest weight projection. Then  $S_{i+1}$  is also determined by equating non lowest weight terms on both sides. The initial conditions, needless to say, are  $q'_0 = q_0$  and  $S_0 = E$ .

## D.2 Scalar coefficient form and conserved quantities

Here we proof the following statement: For generic  $L$ , there is a formal series

$$T = E + \sum_{i=1}^{\infty} h_i \Lambda^{-i}, \quad (\text{D.4})$$

where  $h_i$ 's are diagonal matrices, such that

$$L_0 = TLT^{-1} = \partial_x + \Lambda + \sum_{i=0}^{\infty} f_i \Lambda^{-i}, \quad (\text{D.5})$$

where  $f_i$ 's are scalar functions.  $T$  is determined up to multiplication by series of the form  $E + \sum_{i=1}^{\infty} t_i \Lambda^i$  where  $t_i$ 's are scalar functions, and  $f_i$ 's are determined up to a total derivative. Furthermore  $q^i = \int f_i$  are conserved by the Lax equation.

The proof proceeds as follows: By equating the coefficients of the same powers of  $\Lambda$  in the equality  $TL = L_0T$  we get

$$d_i + h_{i+1} + \sum_{j=0}^{i-1} h_{i-j} d_j^{\sigma^{-(i-j)}} = f_i E + \partial_x h_i + h_{i+1}^{\sigma} + \sum_{j=1}^i f_{i-j} h_j^{\sigma^{-(i-j)}}. \quad (\text{D.6})$$

Here the notation  $A^{\sigma^i}$  means  $\Lambda^i A \Lambda^{-i}$ , which is  $i$  times cyclic permutation of the diagonal elements for a diagonal matrix  $A$ . For example if  $A = \text{Diag}\{a_1, a_2, a_3, a_4\}$  then  $A^{\sigma} = \text{Diag}\{a_2, a_3, a_4, a_1\}$ . We rewrite the equation above as

$$h_{i+1} - h_{i+1}^{\sigma} - f_i E = -d_i + \partial_x h_i - \sum_{j=0}^{i-1} h_{i-j} d_j^{\sigma^{-(i-j)}} + \sum_{j=1}^i f_{i-j} h_j^{\sigma^{-(i-j)}}. \quad (\text{D.7})$$

$f_i$  is obtained by taking the trace on both sides, then  $h_{i+1}$  is determined up to an additive multiple of identity. Now suppose  $T'$  transforms  $L$  to

$$L'_0 = T'LT'^{-1} = \partial_x + \Lambda + \sum_{i=0}^{\infty} f'_i \Lambda^{-i}. \quad (\text{D.8})$$

Define  $TT'^{-1} = A = E + \sum_{i=1}^{\infty} a_i \Lambda^i$  where  $a_i$ 's are diagonal matrices. We have  $A^{-1}L_0A = L'_0$  or  $L_0A = AL'_0$ . By equating the coefficients of the same power in  $\Lambda$  we get

$$a_{i+1} - a_{i+1}^{\sigma} + f'_i E - f_i E = \partial_x a_i + \sum_{j=0}^{i-1} f_j a_{i-j}^{\sigma^{-i}} - \sum_{j=0}^{i-1} f'_j a_{i-j} \quad (\text{D.9})$$

with the initial conditions

$$\begin{aligned} a_1 - a_1^{\sigma} + f'_0 E - f_0 E &= 0, \\ a_2 - a_2^{\sigma} + f'_1 E - f_1 E &= \partial_x a_1. \end{aligned} \quad (\text{D.10})$$

From this recursive formula it's easy to see  $a_i - a_i^{\sigma} = 0$  for all  $i$ , that is  $a_i$ 's are all multiples of identity, say,  $a_i = t_i E$ . Plug this back into the recursive formula we have

$$f'_i - f_i = \partial_x t_i - \sum_{j=0}^{i-1} t_{i-j} (f'_j - f_j) \quad (\text{D.11})$$

with the initial condition

$$\begin{aligned} f'_0 - f_0 &= 0, \\ f'_1 - f_1 &= \partial_x t_1. \end{aligned} \quad (\text{D.12})$$

One can prove by induction that  $f'_i - f_i$  is a total derivative.

The evolution equation of  $L_0$  is

$$\frac{d}{dt} L_0 = [P_0, L_0], \quad (\text{D.13})$$

where  $P_0 = \frac{dT}{dt} T^{-1} + T P T^{-1}$ . Expand  $P_0$  as  $\sum_{i=-\infty}^n p_i \Lambda^i$ , then the Lax equation above gives us

$$\begin{aligned} 0 &= p_n - p_n^{\sigma}, \\ 0 &= -\partial_x p_i + p_{i-1} - p_{i-1}^{\sigma} + \sum_{j=i}^n f_{j-i} (p_j - p_j^{\sigma^{j-i}}), \quad 0 < i \leq n, \\ \dot{f}_{-i} &= -\partial_x p_i + p_{i-1} - p_{i-1}^{\sigma} + \sum_{j=i}^n f_{j-i} (p_j - p_j^{\sigma^{j-i}}), \quad i \leq 0. \end{aligned} \quad (\text{D.14})$$

This recursive formula demands all  $p_i$ 's to be multiples of identity. From this, in turn, the commutator simplifies to  $-\partial_x P_0$ , hence  $\dot{f}_i$ 's are equal to total derivatives and  $\int f_i$ 's are conserved.

### D.3 Matrices that commute with $L_0$

Here we would like to show that All matrices that commute with  $L_0 = \partial_x + \Lambda + \sum_{i=0}^{\infty} f_i \Lambda^{-i}$  have the form  $\sum_{i=-\infty}^n c_i \Lambda^i$  with  $c_i$ 's as constant coefficients.

This follows from letting  $M = \sum_{i=-\infty}^n m_i \Lambda^i$  be a matrix commuting with  $L_0$ . By equating coefficients of the same power in  $\Lambda$  in the equation  $ML_0 = L_0M$  we get

$$\begin{aligned} m_n - m_n^\sigma &= 0, \\ -\partial_x m_i + m_{i-1} - m_{i-1}^\sigma + \sum_{j=i}^n f_{j-i} (m_j - m_j^{\sigma^{j-i}}) &= 0, \quad i \leq n. \end{aligned} \quad (\text{D.15})$$

Therefore all  $m_i$ 's are constants times identity matrix.

### D.4 The Lax equation preserves gauge equivalence

In this subsection we prove the statement that by choosing  $P = (T^{-1}(\sum_{i=-\infty}^n c_i \Lambda^i)T)_+$  the Lax equation preserves gauge equivalence.

This can be shown as follows: It suffices to prove if  $L$  satisfies the Lax equation, then so does  $L' = S^{-1}LS$  where  $S$  is a gauge transformation matrix that only depends on  $x$ . In other words  $\partial_t q = p(q)$  implies  $\partial_t q' = p(q')$ . Using the original Lax equation, it's straightforward to get

$$\frac{d}{dt} L' = [S^{-1}PS, L']. \quad (\text{D.16})$$

So we want  $S^{-1}PS = P'$ , which means,  $S^{-1}PS$  is the same differential polynomial in  $q'$  as  $P$  in  $q$ . Explicitly we have

$$S^{-1}PS = S^{-1}(T^{-1}(\sum_{i=-\infty}^n c_i \Lambda^i)T)_+ S = ((TS)^{-1}(\sum_{i=-\infty}^n c_i \Lambda^i)(TS))_+. \quad (\text{D.17})$$

Suppose  $T'$  transforms  $L'$  into the form of scalar coefficients, that is  $T'L'T'^{-1} = L'_0$ , so  $T'$  is the same differential polynomial in  $q'$  as  $T$  in  $q$ . Plug in  $L' = S^{-1}LS$  we get  $(T'S^{-1})L(T'S^{-1})^{-1} = L'_0 = L_0 = TLT^{-1}$ . Hence  $T'S^{-1} = T$  or  $TS = T'$ , and at last

we get

$$S^{-1}PS = (T'^{-1}(\sum_{i=-\infty}^n c_i \Lambda^i)T')_+ = P'. \quad (\text{D.18})$$

## D.5 Equivalent evolution equations of gauge equivalent classes

We want to prove the following statement: Given that the difference between  $P_1$  and  $P_2$  is a negative weight matrix with no time or  $\lambda$  dependence, then  $\frac{d}{dt}L = [P_1, L]$  and  $\frac{d}{dt}L = [P_2, L]$  give the same evolution equations of gauge equivalent classes.

The proof proceeds as follows: Let's  $R$  denote the ring of scalar differential polynomials in  $q$  which are invariant under gauge transformation. For any  $f \in R$  the time derivative of  $f$  by the Lax equation also belongs to  $R$ , and the form of time derivatives of all  $f \in R$  uniquely specify the evolution equation of gauge equivalent classes. Now for any  $f \in R$ , let  $g$  be the difference of the time derivative of  $f$  by the above two Lax equations, then  $g$  is actually the time derivative of  $f$  by the Lax equation  $\frac{d}{dt}L = [P_1 - P_2, L]$ . Formally

$$g(L) = \frac{d}{dt}f(\mathcal{L}(t))|_{t=0}, \quad (\text{D.19})$$

where  $\mathcal{L}(t)$  satisfies

$$\begin{aligned} \mathcal{L}(0) &= L, \\ \frac{d}{dt}\mathcal{L}(t)|_{t=0} &= [P_1 - P_2, L]. \end{aligned} \quad (\text{D.20})$$

Apparently  $\mathcal{L}(t) = SLS^{-1}$  where  $S = E + t(P_1 - P_2)$  satisfies these conditions, and its time evolution is just a gauge transformation. Therefore we have  $g = 0$  because  $g \in R$ .

## REFERENCES

- [1] W. Pauli, *Phys. Rev.* **58** (1940) 716. doi:10.1103/PhysRev.58.716
- [2] M. Fierz and W. Pauli, *Proc. Roy. Soc. Lond. A* **173** (1939) 211. doi:10.1098/rspa.1939.0140
- [3] E. P. Wigner, *Annals Math.* **40** (1939) 149 [*Nucl. Phys. Proc. Suppl.* **6** (1989) 9]. doi:10.2307/1968551
- [4] X. Bekaert and N. Boulanger, hep-th/0611263.
- [5] V. Bargmann and E. P. Wigner, *Proc. Nat. Acad. Sci.* **34** (1948) 211. doi:10.1073/pnas.34.5.211
- [6] L. P. S. Singh and C. R. Hagen, *Phys. Rev. D* **9** (1974) 898. doi:10.1103/PhysRevD.9.898
- [7] L. P. S. Singh and C. R. Hagen, *Phys. Rev. D* **9** (1974) 910. doi:10.1103/PhysRevD.9.910
- [8] C. Fronsdal, *Phys. Rev. D* **18** (1978) 3624. doi:10.1103/PhysRevD.18.3624
- [9] J. Fang and C. Fronsdal, *Phys. Rev. D* **18** (1978) 3630. doi:10.1103/PhysRevD.18.3630
- [10] D. Francia and A. Sagnotti, *Phys. Lett. B* **543** (2002) 303 doi:10.1016/S0370-2693(02)02449-8 [hep-th/0207002].
- [11] D. Francia and A. Sagnotti, *Class. Quant. Grav.* **20** (2003) S473 [*Comment. Phys. Math. Soc. Sci. Fenn.* **166** (2004) 165] [*PoS JHW 2003* (2003) 005] doi:10.1088/0264-9381/20/12/313 [hep-th/0212185].
- [12] S. R. Coleman and J. Mandula, *Phys. Rev.* **159** (1967) 1251. doi:10.1103/PhysRev.159.1251
- [13] R. Haag, J. T. Lopuszanski and M. Sohnius, *Nucl. Phys. B* **88** (1975) 257. doi:10.1016/0550-3213(75)90279-5
- [14] X. Bekaert, N. Boulanger and P. Sundell, *Rev. Mod. Phys.* **84** (2012) 987 doi:10.1103/RevModPhys.84.987 [arXiv:1007.0435 [hep-th]].
- [15] M. A. Vasiliev, hep-th/0002183.
- [16] M. A. Vasiliev, “Higher spin gauge theories in various dimensions,” *Fortsch. Phys.* **52** (2004) 702 [*PoS jhw 2003* (2003) 003] [hep-th/0401177].
- [17] X. Bekaert, S. Cnockaert, C. Iazeolla and M. A. Vasiliev, “Nonlinear higher spin theories in various dimensions,” hep-th/0503128.
- [18] V. E. Didenko and E. D. Skvortsov, “Elements of Vasiliev theory,” arXiv:1401.2975 [hep-th].

- [19] A. Sagnotti and M. Tsulaia, Nucl. Phys. B **682** (2004) 83 doi:10.1016/j.nuclphysb.2004.01.024 [hep-th/0311257].
- [20] J. M. Maldacena, Int. J. Theor. Phys. **38** (1999) 1113 [Adv. Theor. Math. Phys. **2** (1998) 231] doi:10.1023/A:1026654312961, 10.4310/ATMP.1998.v2.n2.a1 [hep-th/9711200].
- [21] I. R. Klebanov and A. M. Polyakov, Phys. Lett. B **550** (2002) 213 doi:10.1016/S0370-2693(02)02980-5 [hep-th/0210114].
- [22] S. Giombi and X. Yin, J. Phys. A **46** (2013) 214003 doi:10.1088/1751-8113/46/21/214003 [arXiv:1208.4036 [hep-th]].
- [23] M. R. Gaberdiel and R. Gopakumar, “An  $AdS_3$  Dual for Minimal Model CFTs,” Phys. Rev. D **83** (2011) 066007 [arXiv:1011.2986 [hep-th]].
- [24] M. R. Gaberdiel and R. Gopakumar, “Minimal Model Holography,” J. Phys. A **46** (2013) 214002 [arXiv:1207.6697 [hep-th]].
- [25] Y. Nakayama, Phys. Rept. **569** (2015) 1 doi:10.1016/j.phys.rep.2014.12.003 [arXiv:1302.0884 [hep-th]].
- [26] A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, Nucl. Phys. B **241** (1984) 333. doi:10.1016/0550-3213(84)90052-X
- [27] P. Bouwknegt and K. Schoutens, Phys. Rept. **223** (1993) 183 doi:10.1016/0370-1573(93)90111-P [hep-th/9210010].
- [28] E. Witten, “(2+1)-Dimensional Gravity as an Exactly Soluble System,” Nucl. Phys. B **311** (1988) 46.
- [29] A. Achucarro and P. K. Townsend, “A Chern-Simons Action for Three-Dimensional anti-De Sitter Supergravity Theories” Phys. Lett. B **180** (1986) 89.
- [30] S. Carlip, J. Korean Phys. Soc. **28** (1995) S447 [gr-qc/9503024].
- [31] M. P. Blencowe, “A Consistent Interacting Massless Higher Spin Field Theory in  $D = (2+1)$ ,” Class. Quant. Grav. **6** (1989) 443.
- [32] E. Bergshoeff, M. P. Blencowe and K. S. Stelle, “Area Preserving Diffeomorphisms and Higher Spin Algebra,” Commun. Math. Phys. **128** (1990) 213.
- [33] A. Campoleoni, S. Fredenhagen, S. Pfenninger and S. Theisen, “Asymptotic symmetries of three-dimensional gravity coupled to higher-spin fields,” JHEP **1011** (2010) 007 [arXiv:1008.4744 [hep-th]].
- [34] M. Banados, Phys. Rev. D **52** (1996) 5816 doi:10.1103/PhysRevD.52.5816 [hep-th/9405171].
- [35] L. A. Dickey, Acta Appl. Math. **47** (1997) no.3, 243. doi:10.1023/A:1017903416906

- [36] V. G. Drinfeld and V. V. Sokolov, *J. Sov. Math.* **30**, 1975 (1984). doi:10.1007/BF02105860
- [37] M. Gary, D. Grumiller and R. Rashkov, “Towards non-AdS holography in 3-dimensional higher spin gravity,” *JHEP* **1203** (2012) 022 [arXiv:1201.0013 [hep-th]].
- [38] H. Afshar, M. Gary, D. Grumiller, R. Rashkov and M. Riegler, “Non-AdS holography in 3-dimensional higher spin gravity - General recipe and example,” *JHEP* **1211** (2012) 099 [arXiv:1209.2860 [hep-th]].
- [39] H. A. Gonzalez, J. Matulich, M. Pino and R. Troncoso, “Asymptotically flat spacetimes in three-dimensional higher spin gravity,” *JHEP* **1309** (2013) 016 [arXiv:1307.5651 [hep-th]].
- [40] M. Gary, D. Grumiller, S. Prohazka and S. J. Rey, “Lifshitz Holography with Isotropic Scale Invariance,” *JHEP* **1408** (2014) 001 [arXiv:1406.1468 [hep-th]].
- [41] S. Kachru, X. Liu and M. Mulligan, “Gravity duals of Lifshitz-like fixed points,” *Phys. Rev. D* **78** (2008) 106005 [arXiv:0808.1725 [hep-th]].
- [42] M. Gutperle, E. Hijano and J. Samani, *JHEP* **1404** (2014) 020 doi:10.1007/JHEP04(2014)020 [arXiv:1310.0837 [hep-th]].
- [43] M. Gutperle and Y. Li, “Higher Spin Lifshitz Theory and Integrable Systems,” *Phys. Rev. D* **91** (2015) no.4, 046012 doi:10.1103/PhysRevD.91.046012 [arXiv:1412.7085 [hep-th]].
- [44] M. Beccaria, M. Gutperle, Y. Li and G. Macorini, “Higher spin Lifshitz theories and the Korteweg-de Vries hierarchy,” *Phys. Rev. D* **92** (2015) no.8, 085005 doi:10.1103/PhysRevD.92.085005 [arXiv:1504.06555 [hep-th]].
- [45] P. Mathieu, *J. Math. Phys.* **29** (1988) 2499. doi:10.1063/1.528090
- [46] S. Bellucci, E. Ivanov, S. Krivonos and A. Pichugin, “N=2 superBoussinesq hierarchy: Lax pairs and conservation laws,” *Phys. Lett. B* **312** (1993) 463 doi:10.1016/0370-2693(93)90983-O [hep-th/9305078].
- [47] E. Ivanov and S. Krivonos, “Superfield realizations of N=2 superW(3),” *Phys. Lett. B* **291** (1992) 63 Erratum: [*Phys. Lett. B* **301** (1993) 454] doi:10.1016/0370-2693(92)90119-O [hep-th/9204023].
- [48] F. Delduc and L. Gallot, “N=2 KP and KdV hierarchies in extended superspace,” *Commun. Math. Phys.* **190** (1997) 395 doi:10.1007/s002200050246 [solv-int/9609008].
- [49] C. Peng, “Dualities from higher-spin supPeng:2012aeergravity,” *JHEP* **1303** (2013) 054 doi:10.1007/JHEP03(2013)054 [arXiv:1211.6748 [hep-th]].



- [50] B. Chen, J. Long and Y. N. Wang, “Conical Defects, Black Holes and Higher Spin (Super-)Symmetry,” *JHEP* **1306** (2013) 025 doi:10.1007/JHEP06(2013)025 [arXiv:1303.0109 [hep-th]].
- [51] J. M. Figueroa-O’Farrill, J. Mas and E. Ramos, “A One parameter family of Hamiltonian structures for the KP hierarchy and a continuous deformation of the nonlinear  $W(KP)$  algebra,” *Commun. Math. Phys.* **158** (1993) 17 [hep-th/9207092].
- [52] B. Khesin and I. Zakharevich, “Poisson Lie group of pseudodifferential symbols and fractional KP - KdV hierarchies,” hep-th/9311125.
- [53] M. R. Gaberdiel and T. Hartman, *JHEP* **1105** (2011) 031 doi:10.1007/JHEP05(2011)031 [arXiv:1101.2910 [hep-th]].
- [54] A. Campoleoni, S. Fredenhagen and S. Pfenninger, *JHEP* **1109** (2011) 113 doi:10.1007/JHEP09(2011)113 [arXiv:1107.0290 [hep-th]].
- [55] J. M. Figueroa-O’Farrill, J. Mas and E. Ramos, “The Topography of  $W(\infty)$  type algebras,” *Phys. Lett. B* **299** (1993) 41 [hep-th/9208077].
- [56] B. Khesin and I. Zakharevich, “Poisson - Lie group of pseudodifferential symbols,” *Commun. Math. Phys.* **171** (1995) 475 [hep-th/9312088].
- [57] F. Yu and Y. S. Wu, “Nonlinearly deformed  $W(\infty)$  algebra and second Hamiltonian structure of KP hierarchy,” *Nucl. Phys. B* **373** (1992) 713.
- [58] F. Yu and Y. S. Wu, “Hamiltonian structure, (anti)selfadjoint flows in KP hierarchy and the  $W(1+\infty)$  and  $W(\infty)$  algebras,” *Phys. Lett. B* **263** (1991) 220.
- [59] P. Labelle and P. Mathieu, “A New  $N=2$  supersymmetric Korteweg-De Vries equation,” *J. Math. Phys.* **32** (1991) 923. doi:10.1063/1.529351
- [60] F. Delduc and E. Ivanov, “ $N=4$  super KdV equation” *Phys. Lett. B* **309** (1993) 312
- [61] F. Delduc, E. Ivanov and S. Krivonos, “ $N=4$  superKdV hierarchy in  $N=4$  and  $N=2$  superspaces,” *J. Math. Phys.* **37** (1996) 1356 Erratum: [*J. Math. Phys.* **38** (1997) 1224] doi:10.1063/1.531796 [hep-th/9510033].
- [62] A. Perez, D. Tempo and R. Troncoso, “Higher Spin Black Holes,” *Lect. Notes Phys.* **892** (2015) 265 [arXiv:1402.1465 [hep-th]].